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COMPACTNESS THEOREMS FOR 4-DIMENSIONAL GRADIENT RICCI SOLITONS

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We prove compactness theorems for noncompact 4-dimensional shrinking and steady gradient Ricci solitons, respectively, satisfying: (1) every bounded open subset can be embedded in a closed 4-manifold with vanishing second homology group, and (2) are strongly κ-noncollapsed on all scales with respect to a uniform κ. These solitons are of interest because they are the only ones that can arise as finite-time singularity models for a Ricci flow on a closed 4-manifold with vanishing second homology group.

1. Introduction

Since the works of Cheeger and Gromov, compactness and precompactness theorems have played a fundamental role in understanding the geometry and topology of Riemannian manifolds. In the setting of the Ricci flow, Shi's local derivative of curvature estimates [1989] enabled Hamilton [1995a] to improve the convergence to C^∞-convergence of solutions. In dimension 3, in the setting of ancient noncollapsed Ricci flow, this was remarkably strengthened by Perelman [2002] who proved that the global curvature and bound follows from a curvature bound only at a single point. In dimensions 4 and above, this is no longer possible because of the existence of asymptotically conical singularity models, and in particular, asymptotically conical shrinking gradient Ricci solitons. Besides the weakness of the hypotheses, one of the strengths of Perelman's compactness theorem is that it is indeed a compactness result, not just a precompactness result. So the limit extracted from a subsequence is in the same class of objects as the original sequence of objects, in Perelman's case, 3-dimensional ancient κ-solutions.

In this paper we consider 4-dimensional Ricci solitons satisfying a certain topological condition which is of interest in the study of the Ricci flow on closed 4-manifolds with vanishing second homology group, which include homotopy 4-spheres. In singularity analysis of the Ricci flow in relation to developing a theory of Ricci flow with surgery, one considers the case where the underlying manifold of the Ricci shrinker is noncompact. In view of this, we seek a *pointed* compactness result.

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The large body of works by Munteanu and Wang on gradient Ricci solitons [2011; 2012; 2014; 2015; 2016; 2017a] led them to conjecture that 4-dimensional Ricci shrinkers may be classifiable. Indeed, this classification is completed under the condition of nonnegative isotropic curvature, or nonnegative sectional curvature, or nonnegative curvature operator; see [Li et al. 2018; Munteanu and Wang 2017b; Naber 2010]. In the more general case, Munteanu and Wang have made substantial progress towards their conjecture that such objects either are the quotients of splitting Ricci shrinkers or are asymptotically conical Ricci shrinkers. In the most optimistic version of their conjecture, one would expect that a *generic* Ricci flow with surgery on a closed 4-manifold would only produce a quotient 2-surgery, a quotient 3-surgery, or a smooth blow down, all in the case of a type I singularity. More conservatively, one may not wish to rule out Ricci flat ALE spaces and cohomegeneity-one steady gradient Ricci solitons forming generically as singularity models in dimension 4. Returning to dimension 3, paradoxically Perelman's theory of the space of non-compact ancient κ-solutions with positive sectional curvature, which builds on the work of Hamilton [1995b] and which is one of the deepest in the subject, is about a space conjectured by Perelman to be only a single point, namely the Bryant soliton. Brendle's proof of the uniqueness of the Bryant soliton in the class of nonflat 3-dimensional κ-noncollapsed steady Ricci solitons is also a deep result; see [Brendle 2013]. For these reasons, one may expect that a 4-dimensional theory of singularity models for Ricci flow may be related to the prototypical cases (perhaps more so than in dimension 3), which are the shrinking and steady Ricci solitons.

A triple (M^n, g, f), where (M^n, g) is a Riemannian manifold and f is a function on M^n, is called a gradient Ricci soliton, if

$$\mathrm{Ric} + \nabla^2 f = \frac{\lambda}{2} g,$$

where λ is a constant and when $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ the soliton is called shrinking, steady or expanding, respectively. In this paper we focus on shrinking and steady gradient Ricci solitons, or Ricci shrinkers and Ricci steadies for short, respectively. In other words, we always let $\lambda \geq 0$. By scaling the metric and adding a constant to the potential function f, a Ricci shrinker can always be normalized in the following way:

$$\begin{aligned} \mathrm{Ric} + \nabla^2 f &= \tfrac{1}{2} g, \\ |\nabla f|^2 + R &= f, \end{aligned} \tag{1-1}$$

and a non-Ricci-flat Ricci steady can be normalized in the following way:

$$\begin{aligned} \mathrm{Ric} + \nabla^2 f &= 0, \\ |\nabla f|^2 + R &= 1. \end{aligned} \tag{1-2}$$

Shrinking and steady gradient Ricci solitons are of great interest in the study of the singularity formation for the Ricci flow. For instance, they arise as blow-up limits of finite-time singularities in Ricci flows; see [Enders et al. 2011; Gu and Zhu 2008; Hamilton 1995b], and Ricci shrinkers are also blow-down limits of ancient solutions with nonnegative curvature operator [Perelman 2002]. In this paper, we restrict our attention to the shrinking and steady gradient Ricci solitons satisfying a topological assumption, that is, every bounded open subset can be embedded in a closed 4-manifold with vanishing second homology group. This condition was previously considered by Bamler and Zhang [2017]. Besides that, we impose a uniform strong noncollapsing assumption, which fortunately holds for singularity models; see below. We define the following space of Ricci shrinkers and Ricci steadies.

Definition 1.1. Given $\kappa > 0$, $\mathcal{M}^4(\kappa)$ is the collection of all the 4-dimensional noncompact shrinking gradient Ricci solitons (M^4, g, f, p), where p is the point at which f attains its minimum, satisfying:

(a) (M^4, g) is nonflat.

(b) Every bounded open subset of M^4 can be embedded in a closed 4-manifold N^4 with $H_2(N) = 0$, where H_2 is the second homology group with coefficients in $\mathbb{Z}$.

(c) (M^4, g) is strongly κ-noncollapsed on all scales.

Definition 1.2. Given $\kappa > 0$, $\mathcal{N}^4(\kappa)$ is the collection of all the 4-dimensional noncompact steady gradient Ricci solitons (M^4, g, f, p), where $p \in M$ is such that $f(p) = 0$, satisfying:

(a) (M^4, g) is nonflat.

(b) Every bounded open subset of M^4 can be embedded in a closed 4-manifold N^4 with $H_2(N) = 0$, where H_2 is the second homology group with coefficients in $\mathbb{Z}$.

(c) (M^4, g) is strongly κ-noncollapsed on all scales.

Remarks: (1) In item (b) of Definition 1.1 and 1.2, one may simply assume that M^4 can be embedded in N^4 and the same curvature estimates in Section 4 still hold. However, our assumption is more natural in view of singularity models; see below for more details.

(2) The closed 4-manifold N^4 mentioned in item (b) of both Definition 1.1 and 1.2 may depend on the soliton (M^4, g, f, p) or even the open subset, we do not need to assume that every soliton in $\mathcal{M}^4(\kappa)$ or $\mathcal{N}^4(\kappa)$ satisfies this property for the same N^4.

(3) In Definition 1.1 the base point p is the always the minimum point of the potential function f, whereas in Definition 1.2 the base point p can be fixed at any point in M, since one can always replace f by $f - f(p)$, and this does not affect the normalization (1-2)

(4) Since Ricci-flatness and strong noncollapsing condition implies ALE (Corollary 8.86 in [Cheeger and Naber 2015]), by Theorem 6.1 such ALE manifold, when regarded as Ricci steadies, cannot be included in $\mathcal{N}^4(\kappa)$. Henceforth, unless otherwise stated, we always work on non-Ricci-flat Ricci steadies.

(5) There are only a few examples for simply connected 4-dimensional Ricci shrinkers: $\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{R}^2$, $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^2$, and the FIK shrinker (see [Feldman et al. 2003]). Noncollapsed simply connected 4-dimensional Ricci steady has more examples, except for the Bryant soliton [2005], there is a family of Ricci steadies discovered by Appleton [2017]. However, Appleton's solitons are not κ-noncollapsed with respect to a uniform κ.

By strong noncollapsing we mean the following:

Definition 1.3. A Riemannian manifold (M^n, g) is strongly κ-noncollapsed on all scales, where $\kappa > 0$, if the following holds. For all $x \in M$ and $r > 0$, if $R < r^{-2}$ on $B(x, r)$, then $\mathrm{Vol}(B(x, r)) \geq \kappa r^n$. Here we use R to denote the scalar curvature.

Our main theorems are the following:

Theorem 1.4. *$\mathcal{M}^4(\kappa)$ is compact in the smooth pointed Cheeger–Gromov sense, where each $(M^4, g, f, p) \in \mathcal{M}^4(\kappa)$ is normalized as in (1-1).*

Theorem 1.5. *$\mathcal{N}^4(\kappa)$ is precompact in the smooth pointed Cheeger–Gromov sense, where each $(M^4, g, f, p) \in \mathcal{N}^4(\kappa)$ is normalized as in (1-2). Furthermore, for any convergent sequence in $\mathcal{N}^4(\kappa)$, the limit is either the Euclidean space or still lies in $\mathcal{N}^4(\kappa)$.*

Here by saying that $\mathcal{N}^4(\kappa)$ is precompact we mean that for every sequence $\{(M^4_k, g_k, f_k, p_k)\}_{k=1}^{\infty}$ contained in $\mathcal{N}^4(\kappa)$, there exists a subsequence that converges in the pointed smooth Cheeger–Gromov sense to a Ricci steady $(M^4_\infty, g_\infty, f_\infty, p_\infty)$; by saying that $\mathcal{M}^4(\kappa)$ is compact we mean that first of all it is precompact, and furthermore, the limit of every convergent sequence in $\mathcal{M}^4(\kappa)$ also lies in $\mathcal{M}^4(\kappa)$.

A homotopy four-sphere, as a particular example, has vanishing second homology group. When approaching the 4-dimensional smooth Poincaré conjecture using the Ricci flow with surgery, the Ricci solitons that may arise in the analysis of the first singularities, being the blow-up Cheeger–Gromov–Hamilton limit of the homotopy four-sphere, satisfies the property that every open bounded subset can be embedded in the original homotopy four-sphere. Furthermore, according to Perelman [2002], every Ricci flow on closed manifold forming a finite-time singularity is strongly κ-noncollapsed on some fixed finite positive scale, where $\kappa > 0$ depends only on the initial data, the length of the time interval of the Ricci flow, and the scale (see Theorem 6.74 in [Chow et al. 2007]). Thus any blow-up limit at the singular time must be strongly κ-noncollapsed on all scales. Therefore, all Ricci shrinkers or Ricci steadies that arise from such singularity analysis must lie in $\mathcal{M}^4(\kappa)$ or $\mathcal{N}^4(\kappa)$, respectively. We hope that our result will be helpful to the finite-time singularity analysis for the Ricci flow on 4-dimensional closed manifolds. We mention here that Hamilton [1995b] classified finite-time singularities as type I and type II, while Ricci shrinkers and Ricci steadies, being singularity models, are correspondent to these two singularity types, respectively. It is known that the fixed-point blow-up limit of a type I singularity is always a nonflat Ricci shrinker [Enders et al. 2011; Naber 2010], but it remains open whether a similar result is true for type II singularities, that is, *is the blow-up limit of a type II singularity, obtained by some careful point picking, always a Ricci steady?* Hamilton answered this question positively under the assumption that the blow-up limit, obtained by some careful point-picking, has nonnegative curvature operator; see [Hamilton 1993; 1995b].

Condition (b) in both Definition 1.1 and 1.2 plays a very important role in ruling out the Ricci-flat limits. By the strong noncollapsing property, a Ricci-flat blow-up limit of a Ricci flow at a finite-time singularity must have Euclidean volume growth, and, according to Cheeger and Naber [2015], must be *asymptotically locally Euclidean* (ALE for short), which cannot be embedded in any closed 4-manifold with vanishing second homology group (see Corollary 5.8 in [Anderson 2010]; an alternative proof by Richard Bamler is provided in Section 6). This idea gives a uniform curvature growth estimate for every element in the space $\mathcal{M}^4(\kappa)$ and a uniform curvature bound for every element in the space $\mathcal{N}^4(\kappa)$; see Theorems 4.1 and 4.2 below, from which we obtain the compactness results. This argument is in the spirit of Perelman's bounded curvature at bounded distance result for κ-solutions with nonnegative curvature operator (see section 11 of [Perelman 2002]). Perelman also assumes a uniform κ, which is motivated by the reason that all these κ-solutions arise from the same Ricci flow that forms a finite-time singularity. However, there is always a universal κ for all the 3-dimensional κ-solutions that is not a shrinking space form because of the classification of 3-dimensional Ricci shrinkers.

In their papers, Haslhofer and Müller [2011; 2015] have proved a compactness theorem for 4-dimensional Ricci shrinkers, where they only assume a uniform lower bound of the entropy, but where the limit could possibly be an orbifold shrinker. In comparison, the strong noncollapsing assumption in our theorem is correspondent to their bounded entropy assumption (indeed, it is clear that a uniform lower bound of entropy implies κ-noncollapsing with respect to a universal κ; see [Carrillo and Ni 2009; Yokota 2012], yet we do not know how it is related to our strong noncollapsing assumption); in addition, we have a topological restriction. What is novel in our work is that the orbifold Ricci shrinkers will never show up as limits.

From the proof of Theorem 1.4, we also get the following property of the space $\mathcal{M}^4(\kappa)$:

Corollary 1.6. *There exist $C_1 > 0$, $C_2 > 0$, and $C_3 < \infty$ depending only on κ, such that for every $(M^4, g, f, p) \in \mathcal{M}^4(\kappa)$ the following hold*:

(a) $R(p) > C_1$.

(b) $R(x) > C_2 f^{-1}(x)$, *for all* $x \in M$.

(c) $|\pi_1(M)| < C_3$, *where* $\pi_1(M)$ *is the fundamental group of* M.

This paper is organized as follows. In Section 2 we collect some known results for Ricci solitons, which are used in our arguments. In Section 3 we carry out some a priori estimates. In Section 4 we prove curvature estimates for the Ricci shrinkers in $\mathcal{M}^4(\kappa)$ and for the Ricci steadies in $\mathcal{N}^4(\kappa)$. In Section 5 we prove Theorems 1.4, 1.5, and Corollary 1.6. In Section 6 we provide an alternative proof of Anderson's theorem [2010].

2. Preliminaries

In this section we collect some well-known results that are used in our proof. Notice that in this paper the Ricci soliton equations that we work with may not be normalized as (1-1) or (1-2), since sometimes scaling is necessary. Hence we will specify the Ricci soliton equations in every statement. We start with the following differential equations for the geometric quantities on Ricci shrinkers and steadies.

Proposition 2.1. *Let* (M, g, f) *be a shrinking or steady gradient Ricci soliton satisfying*

$$\operatorname{Ric} + \nabla^2 f = \frac{\lambda}{2} g,$$

where $\lambda \geq 0$. *Then the following hold.*

$$\Delta_f R = \lambda R - 2|\operatorname{Ric}|^2, \tag{2-1}$$

$$\Delta_f \operatorname{Ric} = \lambda \operatorname{Ric} + \operatorname{Rm} * \operatorname{Ric}, \tag{2-2}$$

$$\Delta_f \operatorname{Rm} = \lambda \operatorname{Rm} + \operatorname{Rm} * \operatorname{Rm}, \tag{2-3}$$

$$\Delta_f \nabla^k \operatorname{Rm} = \lambda\Big(\frac{k}{2} + 1\Big)\nabla^k \operatorname{Rm} + \sum_{j=0}^{k} \nabla^j \operatorname{Rm} * \nabla^{k-j} \operatorname{Rm}, \tag{2-4}$$

where $*$ *stands for some contraction and* $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ *is the* f*-Laplacian operator.*

Proof. Since every reference on these differential equations we can find deals only with the case $\lambda = 1$ or $\lambda = 0$, we take (2-3) as an example to quickly sketch how these formulae can be carried out; other equations can be proved in the same way. Recall that the canonical form of a Ricci soliton $g(t) = \tau(t)\varphi_t^*(g)$ evolves by the Ricci flow (see Theorem 4.1 of [Chow et al. 2006]), where

$$\tau(t) = 1 - \lambda t, \quad \frac{d}{dt}\varphi_t = \frac{1}{\tau}\nabla f \circ \varphi_t, \quad \varphi_0 = \mathrm{id}.$$

Taking Rm as a (4, 0)-tensor, we have $\mathrm{Rm}(g(t)) = \tau(t)\,\mathrm{Rm}(\varphi_t^*(g)) = \tau(t)\varphi_t^*(\mathrm{Rm})$. Hence by the standard curvature evolution equation (see Theorem 7.1 of [Hamilton 1982]) we have

$$\Delta\,\mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} = \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Rm}(g(t)) = -\lambda\,\mathrm{Rm} + \mathcal{L}_{\nabla f}\,\mathrm{Rm},$$

where $\mathcal{L}$ stands for the Lie derivative. Let Y_1, Y_2, Y_3, Y_4 be four arbitrary vector fields. Then

$$\begin{aligned}
\mathcal{L}_{\nabla f}\,\mathrm{Rm}(Y_1, Y_2, Y_3, Y_4) &= \nabla_{\nabla f}(\mathrm{Rm}(Y_1, Y_2, Y_3, Y_4)) - \mathrm{Rm}(\mathcal{L}_{\nabla f} Y_1, Y_2, Y_3, Y_4)\\
&\qquad - \mathrm{Rm}(Y_1, \mathcal{L}_{\nabla f} Y_2, Y_3, Y_4) - \mathrm{Rm}(Y_1, Y_2, \mathcal{L}_{\nabla f} Y_3, Y_4)\\
&\qquad - \mathrm{Rm}(Y_1, Y_2, Y_3, \mathcal{L}_{\nabla f} Y_4)\\
&= \nabla_{\nabla f}\,\mathrm{Rm}(Y_1, Y_2, Y_3, Y_4) + \mathrm{Rm}(\nabla_{Y_1}\nabla f, Y_2, Y_3, Y_4)\\
&\qquad + \mathrm{Rm}(Y_1, \nabla_{Y_2}\nabla f, Y_3, Y_4) + \mathrm{Rm}(Y_1, Y_2, \nabla_{Y_3}\nabla f, Y_4)\\
&\qquad + \mathrm{Rm}(Y_1, Y_2, Y_3, \nabla_{Y_4}\nabla f).
\end{aligned}$$

Taking into account that $\nabla^2 f = \frac{\lambda}{2} g - \mathrm{Ric}$ we obtain the conclusion. □

The following two propositions for the potential function growth rate and the volume growth rate for Ricci shrinkers were proved by Cao and Zhou [2010], with an observation of Munteanu [2009]. We use its sharpened version of Haslhofer and Müller [2011]. Besides that, Munteanu and Wang [2014] proved a volume growth estimate with an improved constant.

Proposition 2.2. *Let (M^n, g, f) be a noncompact shrinking gradient Ricci soliton normalized as in (1-1). Let p be a point where f attains its minimum. Then the following holds*:

$$\tag{2-5} \tfrac{1}{4}(d(x, p) - 5n)_+^2 \le f(x) \le \tfrac{1}{4}(d(x, p) + \sqrt{2n})^2,$$

where $u_+ := \max\{u, 0\}$ denotes the positive part of a function.

Proposition 2.3. *There exists $C < \infty$ depending only on the dimension n, such that under the same assumption of Proposition 2.2 the following holds*:

$$\tag{2-6} \mathrm{Vol}(B(p, r)) \le C r^n,$$

for all $r > 0$.

To locally estimate the Ricci curvature, we need the following local Sobolev inequality, whose constant depends only on the local geometry.

Proposition 2.4. *For all $\kappa > 0$, there exists $C < \infty$ and $\delta \in (0, 2)$, depending only on κ and the dimension $n \ge 3$ such that the following holds. Let (M^n, g) be* *a Riemannian manifold and* $x_0 \in M$, *and assume that* $|\mathrm{Rm}| \leq 2$ *on* $B(x_0, 2)$ *and* $\mathrm{Vol}(B(x_0, 2)) \geq \kappa$. *Then*

$$\|u\|_{L^{2n/(n-2)}} \leq C\|\nabla u\|_{L^2}, \tag{2-7}$$

for all $u \in C_0^\infty(B(x_0, \delta))$.

Proof. This is a standard result; for the convenience of the readers we sketch the proof. We follow the lines of reasoning of Lemma 3.2 of [Haslhofer and Müller 2011]. We need only to prove an L^1 Sobolev inequality

$$\|u\|_{L^{n/(n-1)}} \leq C_1\|\nabla u\|_{L^1}, \tag{2-8}$$

for all $u \in C_0^\infty(B(x_0, \delta))$, where δ and C_1 depend only on κ and the dimension n. Then (2-7) follows from (2-8). Indeed, C_1 is equal to the isoperimetric constant of $B(x_0, \delta)$,

$$C_1 = C_I = \sup |\Omega|^{n/(n-1)}/|\partial\Omega|,$$

where the supremum is taken over all the open sets $\Omega \subset B(x_0, \delta)$ with smooth boundary. By a theorem of Croke [1980, Theorem 11], C_I can be estimated by

$$C_I \leq C(n)\omega^{-(n+1)/n},$$

where $C(n)$ is a constant depending only on the dimension and ω is the visibility angle defined by

$$\omega = \inf_{y \in B(x_0,\delta)} |U_y|/|\mathbb{S}^{n-1}|,$$

where $U_y = \{v \in T_yB(x_0, \delta) : |v| = 1, \text{ the geodesic } \gamma_v \text{ minimizes up to } \partial B(x_0, \delta)\}$.

We restrict δ in $\left(0, \frac{1}{2}\right)$ and let y be an arbitrary point in $B(x_0, \delta)$. Let

$$J(r, \theta)dr \wedge d\theta, \quad \bar{J}(r, \theta)dr \wedge d\theta$$

be the volume elements in terms of spherical normal coordinates around the point y and in the hyperbolic space with constant sectional curvature -2, respectively. By the relative volume comparison theorem, we have

$$\begin{aligned} c_2\kappa - C_3\delta^n \leq |B(x_0, 1)| - |B(x_0, \delta)| &\leq \int_{U_y}\int_0^{1+\delta} J(r, \theta)\, dr\, d\theta \\ &\leq \int_{U_y}\int_0^{1+\delta} \bar{J}(r, \theta)\, dr\, d\theta \leq C_4|U_y|\left(\tfrac{3}{2}\right)^n, \end{aligned}$$

where c_2, C_3, and C_4 are constants depending only on the dimension n. Taking $\delta = (c_2\kappa/(2C_3))^{1/n}$, we have that $|U_y|$ is bounded from below by a constant depending only on κ and the dimension n, for all $y \in B(x_0, \delta)$, and the conclusion follows. □

We conclude this section with the following gap theorem of Yokota [2009; 2012], which is used in the proof of Theorem 1.4 to show that the limit shrinker is nonflat.

Proposition 2.5. *There exists $\varepsilon > 0$ depending only on the dimension n such that the following holds. Let (M^n, g, f) be a shrinking gradient Ricci soliton, which is normalized as in (1-1). If*

$$\frac{1}{(4\pi)^{n/2}} \int_M e^{-f}\, dg > 1 - \varepsilon,$$

then (M^n, g, f) is the Gaussian shrinker, that is, (M^n, g) is the Euclidian space.

3. A priori estimates

The a priori estimates in this section hold for any dimension $n \geq 3$. We start with a localized derivative estimate for the Riemann curvature tensor.

Proposition 3.1. *There exists $C < \infty$ depending only on the dimension n such that the following holds. Let (M^n, g, f) be a shrinking or steady gradient Ricci soliton such that*

$$\mathrm{Ric} + \nabla^2 f = \frac{\lambda}{2} g,$$

where $\lambda \geq 0$. Let $x_0 \in M$ and $r > 0$. If $|\mathrm{Rm}| \leq r^{-2}$ and $|\nabla f| \leq r^{-1}$ on $B(x_0, 2r)$, then

$$|\nabla \mathrm{Rm}| \leq C r^{-3} \text{ on } B(x_0, r). \tag{3-1}$$

More generally, there exist C_l depending only on $l \geq 0$ and the dimension n, such that under the above assumptions, it holds that

$$|\nabla^l \mathrm{Rm}| \leq C_l r^{-2-l} \text{ on } B(x_0, r). \tag{3-2}$$

Proof. The proof is a standard elliptic modification of Shi's estimates [1989]. One can also combine [Shi 1989] with the canonical form to obtain this result. For the readers' convenience we will give a proof for (3-1). The higher derivative estimates (3-2) follow in a standard way by induction. We compute using (2-3)

$$\begin{aligned} \Delta_f |\mathrm{Rm}|^2 &= 2\langle \mathrm{Rm}, \Delta_f \mathrm{Rm}\rangle + 2|\nabla \mathrm{Rm}|^2 \\ &= 2|\nabla \mathrm{Rm}|^2 + 2\lambda |\mathrm{Rm}|^2 + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm} \\ &\geq 2|\nabla \mathrm{Rm}|^2 - C_1 |\mathrm{Rm}|^3, \end{aligned}$$

where $C_1 < \infty$ depends only on the dimension n. By (2-4), we have

$$\begin{aligned} \Delta_f |\nabla \mathrm{Rm}|^2 &= 2\langle \nabla \mathrm{Rm}, \Delta_f \nabla \mathrm{Rm}\rangle + 2|\nabla^2 \mathrm{Rm}|^2 \\ &= 2|\nabla^2 \mathrm{Rm}|^2 + 3\lambda |\nabla \mathrm{Rm}|^2 + \mathrm{Rm} * \nabla \mathrm{Rm} * \nabla \mathrm{Rm} \\ &\geq 2|\nabla^2 \mathrm{Rm}|^2 - C_2 |\mathrm{Rm}||\nabla \mathrm{Rm}|^2, \end{aligned}$$

where $C_2 < \infty$ depends only on the dimension n.

Defining $u = (\beta r^{-4} + |\mathrm{Rm}|^2)|\nabla \mathrm{Rm}|^2$, where $\beta > 0$ is a constant that we will specify later, we have

$$\begin{aligned}\Delta_f u &= |\nabla \mathrm{Rm}|^2 \Delta_f |\mathrm{Rm}|^2 + (\beta r^{-4} + |\mathrm{Rm}|^2)\Delta_f |\nabla \mathrm{Rm}|^2 + 2\langle \nabla |\mathrm{Rm}|^2, \nabla |\nabla \mathrm{Rm}|^2\rangle \\ &\geq 2|\nabla \mathrm{Rm}|^4 - C_1 |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}|^3 \\ &\qquad + (\beta r^{-4} + |\mathrm{Rm}|^2)(2|\nabla^2 \mathrm{Rm}|^2 - C_2 |\mathrm{Rm}||\nabla \mathrm{Rm}|^2) \\ &\qquad - 8|\nabla \mathrm{Rm}| \cdot \big|\nabla |\mathrm{Rm}|\big| \cdot |\nabla^2 \mathrm{Rm}| \cdot |\mathrm{Rm}| \\ &\geq 2|\nabla \mathrm{Rm}|^4 - C_1 |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}|^3 \\ &\qquad + (\beta r^{-4} + |\mathrm{Rm}|^2)(2|\nabla^2 \mathrm{Rm}|^2 - C_2 |\mathrm{Rm}||\nabla \mathrm{Rm}|^2) \\ &\qquad - \tfrac{1}{2}|\nabla \mathrm{Rm}|^4 - 32|\nabla^2 \mathrm{Rm}|^2 |\mathrm{Rm}|^2,\end{aligned}$$

where we have used Kato's inequality as well as the Cauchy–Schwarz inequality. Letting $\beta = 16$ and taking into account that $|\mathrm{Rm}| \leq r^{-2}$ in $B(x_0, 2r)$, we have

$$\Delta_f u \geq \tfrac{3}{2}|\nabla \mathrm{Rm}|^4 - C_3 r^{-6} |\nabla \mathrm{Rm}|^2 \geq |\nabla \mathrm{Rm}|^4 - C_4 r^{-12},$$

where we have used the Cauchy–Schwarz inequality, and C_3 and C_4 are constants depending only on n. By the definition of u we have $|\nabla \mathrm{Rm}|^4 \geq (r^8/289)u^2$; hence

$$\Delta_f u \geq c_5 r^8 u^2 - C_5 r^{-12}, \tag{3-3}$$

where c_5 and C_5 are constants depending only on n.

We let $\phi(x) = \varphi(d(x_0, x))$ be the cut-off function, where $\varphi(s) = 0$ for $s \geq 2r$, $\varphi(s) = 1$ for $s \in [0, r]$, and

$$0 \leq \varphi \leq 1, \quad -2r^{-1} \leq \varphi'(s) \leq 0, \quad |\varphi''(s)| \leq 2r^{-2} \tag{3-4}$$

for all $s \in [r, 2r]$. We compute

$$\begin{aligned}\Delta_f(u\phi^2) &= \phi^2 \Delta_f u + u\Delta_f \phi^2 + 2\langle \nabla u, \nabla \phi^2\rangle \\ &\geq c_5 r^8 u^2 \phi^2 - C_5 r^{-12}\phi^2 + 2\langle \nabla(u\phi^2), \nabla \log \phi^2\rangle - 8|\nabla \phi|^2 u + u\Delta_f \phi^2.\end{aligned} \tag{3-5}$$

The last two terms in (3-5) need to be estimated. We have

$$|\nabla \phi|^2 = \varphi'^2 |\nabla d|^2 \leq 4r^{-2},$$

and

$$\begin{aligned}\Delta_f \phi^2 &= 2\phi(\varphi' \Delta_f d + \varphi'' |\nabla d|^2) + 2\varphi'^2 |\nabla d|^2 \\ &= 2\phi(\varphi' \Delta d - \varphi' \langle \nabla f, \nabla d\rangle) + 2\phi\varphi'' |\nabla d|^2 + 2\varphi'^2 |\nabla d|^2 \\ &\geq 2\left(-\frac{2(n-1)\coth(1)}{r^2} - \frac{2}{r^2}\right) - \frac{4}{r^2} \geq -C_6 r^{-2},\end{aligned}$$

where C_6 is a positive constant depending only on the dimension n. In the above derivation we have used $|\nabla f| \leq r^{-1}$, the properties of φ (3-4), the Laplacian comparison theorem, and that $\varphi'(s) = 0$ for all $s \in [0, r]$. Inserting the above inequalities into (3-5), and defining $G = u\phi^2$, we have

$$\Delta_f G \geq c_5 r^8 \frac{G^2}{\phi^2} - C_5 r^{-12} \phi^2 + 2\langle \nabla G, \nabla \log \phi^2 \rangle - C_7 r^{-2} \frac{G}{\phi^2}. \tag{3-6}$$

Let $x_1 \in B(x_0, 2r)$ be a point where G attains its maximum. Taking into account that $0 \leq \phi \leq 1$, it follows from (3-6) that

$$c_5 r^8 G(x_1)^2 - C_7 r^{-2} G(x_1) - C_5 r^{-12} \leq 0,$$

which solves $G(x_1) \leq C_8 r^{-10}$, where C_7 and C_8 depend only on n. Therefore $u(x) \leq C_8 r^{-10}$ on $B(x_0, r)$, where $\phi = 1$ and $G = u$. It follows from the definition of the function u that

$$|\nabla \operatorname{Rm}|^2 \leq C r^{-6} \text{ on } B(x_0, r). \qquad \square$$

The following proposition says that the smallness of the scalar curvature on a ball implies the smallness of the Ricci curvature on a smaller ball. Our argument is inspired by Theorem 3.2 in [Wang 2012]. The same idea was implemented in [Bamler and Zhang 2017].

Proposition 3.2. *For any $\kappa > 0$, there exists $\delta \in (0, 2)$ and $C < \infty$, depending only on κ and the dimension n, such that the following holds. Let (M^n, g, f) be a shrinking or steady gradient Ricci soliton such that*

$$\operatorname{Ric} + \nabla^2 f = \frac{\lambda}{2} g,$$

where $\lambda \geq 0$. Let $x_0 \in M$ and $r \in (0, 1]$. If

$$|\operatorname{Rm}| \leq 2, \quad R \leq r^2, \quad |\nabla f| \leq r \ \text{ on } B(x_0, 2) \quad \text{and} \quad \operatorname{Vol}(B(x_0, 2)) \geq \kappa,$$

then

$$|\operatorname{Ric}| \leq C r \ \text{ on } B\left(x_0, \frac{\delta}{2}\right). \tag{3-7}$$

Proof. We define a cut-off function that is similar to the one that we have used in the proof of the last proposition. Let $\phi(x) = \varphi(d(x_0, x))$, where $\varphi(s) = 0$ for $s \geq 2$, $\varphi(s) = 1$ for $s \in [0, 1]$, and

$$0 \leq \varphi \leq 1, \quad -2 \leq \varphi'(s) \leq 0, \quad |\varphi''(s)| \leq 2, \tag{3-8}$$

for $s \in [1, 2]$. Integrating (2-1) against ϕ, we have

$$\begin{aligned} 2\int |\operatorname{Ric}|^2 \phi &= \lambda \int R\phi - \int \phi \Delta R + \int \langle \nabla f, \nabla R \rangle \phi \\ &= \lambda \int R\phi - \int R \Delta\phi - \int \phi R \Delta f - \int R \langle \nabla\phi, \nabla f \rangle \end{aligned}$$

$$
\begin{aligned}
&= \Big(1-\frac{n}{2}\Big)\lambda \int R\phi - \int R\Delta\phi + \int \phi R^2 - \int R\langle \nabla\phi, \nabla f\rangle \\
&\le -\int R\Delta\phi + \int \phi R^2 - \int R\langle \nabla\phi, \nabla f\rangle,
\end{aligned}
$$

where we have used $\Delta f = \frac{n}{2}\lambda - R$ and Chen's result [2009] that $R \ge 0$ on a Ricci shrinker or Ricci steady. By the Laplacian comparison theorem, the Bishop–Gromov volume comparison theorem, and the property of ϕ (3-8), we have

$$
-\Delta\phi \le C_1, \quad |\langle \nabla\phi, \nabla f\rangle| \le C_2 r, \quad \mathrm{Vol}(B(x_0, 2)) \le C_3,
$$

where C_1, C_2, and C_3 are positive constants depending only on the dimension n. It then follows that

$$
\int |\mathrm{Ric}|^2 \phi \le C_1 C_3 r^2 + C_3 r^4 + C_2 C_3 r^3,
$$

and that

$$
\|\mathrm{Ric}\|_{L^2(B(x_0,\delta))} \le C_4 r,
$$

where C_4 depends only on the dimension n, and $\delta \in (0, 2)$ is the positive number given by Proposition 2.4 that depends only on κ and the dimension n.

We have the following inequality satisfied by $|\mathrm{Ric}|$:

$$
\begin{aligned}
2|\mathrm{Ric}|\Delta_f |\mathrm{Ric}| + 2\big|\nabla|\mathrm{Ric}|\big|^2 &= \Delta_f |\mathrm{Ric}|^2 = 2\langle \mathrm{Ric}, \Delta_f \mathrm{Ric}\rangle + 2|\nabla \mathrm{Ric}|^2 \\
&= 2\lambda|\mathrm{Ric}|^2 + 2|\nabla \mathrm{Ric}|^2 - \mathrm{Rm} * \mathrm{Ric} * \mathrm{Ric} \\
&\ge 2|\nabla \mathrm{Ric}|^2 - \mathrm{Rm} * \mathrm{Ric} * \mathrm{Ric}\,.
\end{aligned}
$$

Taking into account that $|\mathrm{Rm}| \le 2$ on $B(x_0, 2)$ and Kato's inequality that $\big|\nabla|\mathrm{Ric}|\big|^2 \le |\nabla \mathrm{Ric}|^2$, we have

$$
\Delta_f |\mathrm{Ric}| \ge -C_5 |\mathrm{Ric}|, \tag{3-9}
$$

where C_5 depends on the dimension n. We use the local Sobolev inequality (2-7) to apply the standard Moser iteration to the inequality (3-9). Notice that we need to use $|\nabla f| \le r \le 1$ when performing the iteration. Indeed, this is the only reason why we have to put a restriction on the scale r. It follows that

$$
\sup_{B(x_0,\,\delta/2)} |\mathrm{Ric}| \le C_6 \|\mathrm{Ric}\|_{L^2(B(x,\delta))} \le Cr,
$$

where C depends only on κ and the dimension n. □

4. Curvature estimates

In this section we prove a bounded curvature at bounded distance theorem for Ricci shrinkers in the space $\mathcal{M}^4(\kappa)$ as well as a uniformly bounded curvature theorem for Ricci steadies in $\mathcal{N}^4(\kappa)$. These results are analogues to Perelman's bounded curvature at bounded distance result (see section 11 of [Perelman 2002]). The fact that the Ricci-flat limit does not appear in our argument plays a role as equally important as the fact that the asymptotic volume ratio equals zero in Perelman's argument. However, our results are somewhat weaker than Perelman's. In Theorem 4.1 we are only able to fix the base point where the potential function attains its minimum (or wherever is at a bounded distance to it), while in Theorem 4.2 the curvature bound is at a fixed scale instead of a relative scale, that is, the curvature bound is in terms of a fixed number instead of the curvature at an arbitrary base point. The reason in analysis is the following: to implement results in Section 3 in an argument of contradiction, the curvature largeness should be characterized by $|\nabla f|^2$. Suppose around a point the curvature is large in some relative sense but small compared to $|\nabla f|^2$. Then the a priori estimates we have established in Section 3 do not hold any more, since the assumption $|\nabla f| \leq r \leq 1$ made in Proposition 3.2 is no longer valid after scaling, and Moser iteration does not yield a nice bound for the Ricci curvature as in (3-7). To give a geometric understanding for the aforementioned weakness, we take an asymptotic conical shrinker as an example: one could take a sequence of points tending to infinity in an asymptotically conical Ricci shrinker, and the associated pointed limit is the asymptotic cone of the Ricci shrinker. Since this asymptotic cone is singular at its vertex, we have neither Perelman's bounded curvature at bounded distance nor compactness.

Theorem 4.1. *There exists $C < \infty$ and $D < \infty$ depending only on κ, such that the following holds. Let $(M^4, g, f, p) \in \mathcal{M}^4(\kappa)$ be normalized as in (1-1). Then*

$$\begin{aligned} |\mathrm{Rm}|(x) &\leq C \quad \text{if } x \in B(p, 200), \\ \frac{|\mathrm{Rm}|}{f}(x) &\leq D \quad \text{if } x \notin B(p, 200). \end{aligned}$$

Proof. We argue by contradiction. Suppose the statement is not true, then there exist a sequence of counterexamples $\{(M_k^4, f_k, g_k, p_k)\}_{k=1}^{\infty} \subset \mathcal{M}^4(\kappa)$ normalized as in (1-1), and $x_k \in M_k$, such that for all $k \geq 1$, either

(a) $x_k \in B_{g_k}(p_k, 200)$ and $|\mathrm{Rm}_k|(x_k) \geq k$, or

(b) $x_k \notin B_{g_k}(p_k, 200)$ and $\frac{|\mathrm{Rm}_k|}{f_k}(x_k) \geq k$.

Notice that by (2-5), we have $f_k(x) \geq 1000$ whenever $x \notin B(p_k, 200)$; hence $|\mathrm{Rm}_k|(x_k) \to \infty$.

The following standard point picking technique is due to Perelman [2002].

Claim 1. There exists $A_k \to \infty$ and $y_k \in B_{g_k}(x_k, 1)$, such that

$$|\mathrm{Rm}_k|(x) \leq 2Q_k \quad \text{for all } x \in B_{g_k}(y_k, A_k Q_k^{-1/2}) \subset B_{g_k}(x_k, 2), \tag{4-1}$$

where $Q_k = |\mathrm{Rm}_k|(y_k) \geq |\mathrm{Rm}_k|(x_k)$.

Proof. Denote $Q_k^{(0)} = |\mathrm{Rm}_k|(x_k)$ and $y_k^{(0)} = x_k$, let $A_k = \frac{1}{100}(Q_k^{(0)})^{1/2} \to \infty$. We start from $y_k^{(0)}$. Suppose that $y_k^{(j)}$ is chosen and cannot be taken as y_k. Let $|\mathrm{Rm}_k|(y_k^{(j)}) = Q_k^{(j)}$. Then there exists $y_k^{(j+1)} \in B_{g_k}(y_k^{(j)}, A_k(Q_k^{(j)})^{-1/2})$, such that $Q_k^{(j+1)} = |\mathrm{Rm}_k|(y_k^{(j+1)}) \geq 2Q_k^{(j)}$. Hence we have

$$\begin{aligned}\mathrm{dist}_{g_k}(y_k^{(0)}, y_k^{(j+1)}) &\leq A_k(Q_k^{(0)})^{-1/2} + A_k(Q_k^{(1)})^{-1/2} + \cdots + A_k(Q_k^{(j)})^{-1/2}\\ &\leq A_k(Q_k^{(0)})^{-1/2}\left(1 + \frac{1}{\sqrt{2}} + \cdots + \left(\frac{1}{\sqrt{2}}\right)^j + \cdots\right)\\ &\leq \frac{1}{100} \times 4,\end{aligned}$$

and it follows that $y_k^{(j)} \in B_{g_k}(x_k, 1)$ for all $j \geq 0$. This procedure must terminate in finite steps since the manifold M_k is smooth; then the last element chosen by this procedure can be taken as y_k. □

Since for any $k \geq 1$ there can be only two cases (a) or (b), then either for infinitely many k, (a) holds, or, for infinitely many k, (b) holds. By passing to a subsequence, we need only to deal with the following two cases.

Case I. $x_k \in B_{g_k}(p_k, 200)$ and $|\mathrm{Rm}_k|(x_k) \geq k$, for all $k \geq 1$.

Case II. $x_k \notin B_{g_k}(p_k, 200)$ and $\frac{|\mathrm{Rm}_k|}{f_k}(x_k) \geq k$, for all $k \geq 1$.

We first consider Case I. We use Claim 1 to find $y_k \in B_{g_k}(x_k, 1)$, $Q_k = |\mathrm{Rm}_k|(y_k) \geq |\mathrm{Rm}_k|(x_k) \to \infty$, and $A_k \to \infty$ such that (4-1) holds. By (2-5) we have

$$R_k + |\nabla f_k|^2 = f_k \leq 10^5$$

on $B_{g_k}(y_k, A_k Q_k^{-1/2}) \subset B_{g_k}(p_k, 202)$. We scale g_k with the factor Q_k and use the notations with overlines to denote the scaled geometric quantities, that is, $\bar{g}_k = Q_k g_k$, $\overline{\mathrm{Rm}}_k = \mathrm{Rm}(\bar{g}_k)$, etc. Then we have that

$$\overline{\mathrm{Ric}}_k + \overline{\nabla}^2 f_k = \frac{Q_k^{-1}}{2}\bar{g}_k, \tag{4-2}$$

and that

$$|\overline{\mathrm{Rm}}_k| \leq 2, \tag{4-3}$$

$$\bar{R}_k + |\overline{\nabla} f_k|^2 \leq \frac{10^5}{Q_k} := r_k^2 \to 0, \tag{4-4}$$

on $B_{\bar{g}_k}(y_k, A_k)$, and by Proposition 3.1 and Proposition 3.2 that

$$|\overline{\nabla}\, \overline{\mathrm{Rm}}_k| \leq C_1, \tag{4-5}$$

$$|\overline{\mathrm{Ric}}_k| \leq C_2 r_k, \tag{4-6}$$

on $B_{\bar{g}_k}(y_k, A_k - 2)$, where C_1 is a constant depending only on the dimension $n = 4$, and C_2 is a constant depending only on the dimension $n = 4$ and $\kappa > 0$. We can apply (4-3), (4-5), and the strong κ-noncollapsing assumption to extract from $\{(B_{\bar{g}_k}(y_k, A_k - 2), \bar{g}_k, y_k)\}_{k=1}^{\infty}$ a subsequence that converges in the pointed $C^{2,\alpha}$ Cheeger–Gromov sense to a complete nonflat Riemannian manifold $(M_\infty, g_\infty, y_\infty)$ with $|\mathrm{Rm}_\infty|(x_\infty) = 1$. By (4-6), (M_∞, g_∞) must be Ricci-flat and therefore has Euclidean volume growth, since it is also strongly κ-noncollapsed. By Corollary 8.86 of [Cheeger and Naber 2015], (M_∞, g_∞) is asymptotically locally Euclidean (ALE). By the definition of ALE, we have that outside a compact set M_∞ is diffeomorphic to a finite quotient of $\mathbb{R}^4 \setminus B(O, 1)$, it follows that there exists an open set $U_\infty \subset M_\infty$ containing the point y_∞, such that $\overline{U}_\infty$ is compact and that M_∞ is diffeomorphic to U_∞. By the definition of the pointed Cheeger–Gromov convergence, U_∞ can be embedded in infinitely many elements of the sequence $\{(M_k^4, f_k, g_k, p_k)\}_{k=1}^{\infty}$, and the images of the embeddings are bounded open sets. Furthermore, every one in the sequence of shrinkers satisfies (b) in Definition 1.1; it follows that U_∞ can also be embedded in a closed 4-manifold with vanishing second homology group, which is a contradiction against Theorem 6.1.

Case II is almost the same as Case I. By the same point picking and scaling method we also get (4-2), (4-3), (4-5), and (4-6). The only place where special care should be taken is (4-4). Notice that by (2-5), we have that $f_k(x) \geq 1000$ whenever $\mathrm{dist}_{g_k}(x, p_k) \geq 198$. Moreover, since $|\nabla\sqrt{f_k}| \leq \frac{1}{2}$, we have

$$\sqrt{f_k(x)} \leq \sqrt{f_k(x_k)} + 1 \leq \sqrt{\tfrac{10}{9} f_k(x_k)},$$

for all $x \in B_{g_k}(y_k, A_k Q_k^{-1/2}) \subset B_{g_k}(x_k, 2)$. It follows that

$$\bar{R}_k + |\overline{\nabla} f_k|^2 = \frac{f_k}{Q_k} \leq \frac{10}{9} \frac{f_k(x_k)}{|\mathrm{Rm}_k|(x_k)} := r_k^2 \to 0,$$

on $B_{\bar{g}_k}(y_k, A_k)$. Therefore (4-4) also holds in Case II and we obtain the same contradiction as in Case I. □

Theorem 4.2. *There exists $C < \infty$ depending only on κ, such that the following holds. Let $(M^4, g, f, p) \in \mathcal{N}^4(\kappa)$ be normalized as in (1-2). Then it holds that*

$$|\mathrm{Rm}|(x) \leq C \quad \textit{for all } x \in M.$$

Proof. We argue by contradiction. Suppose the statement is not true; then there exist a sequence of counterexamples $\{(M_k^4, f_k, g_k, p_k)\}_{k=1}^{\infty} \subset \mathcal{N}^4(\kappa)$ normalized as in (1-2), such that $\sup_{x \in M_k} |\mathrm{Rm}_k(x)| \to \infty$. By shifting the base points p_k and replacing f_k by $f_k - f_k(p_k)$, we may assume that for each k

$$Q_k := |\mathrm{Rm}_k(p_k)| \geq \tfrac{1}{2} \sup_{x \in M_k} |\mathrm{Rm}_k(x)| \to \infty.$$

Now we scale the sequence $\{(M_k^4, f_k, g_k, p_k)\}_{k=1}^{\infty}$ by the factors Q_k and use the notations with overlines to denote the scaled quantities as before, that is, $\bar{g}_k = Q_k g_k$,

$\overline{\mathrm{Rm}}_k = \mathrm{Rm}(\bar{g}_k)$, etc. Then we have

$$\overline{\mathrm{Ric}}_k + \overline{\nabla}^2 f_k = 0, \qquad |\overline{\nabla} f_k|^2 + \bar{R}_k = \frac{1}{Q_k} := r_k^2 \to 0,$$
$$|\overline{\mathrm{Rm}}_k|(x) \le 2 \quad \text{for all } x \in M, \qquad |\overline{\mathrm{Rm}}_k|(p_k) = 1.$$

Recall that all $(M_k, \bar{g}_k)$ are κ-noncollapsed on call scales with respect to a uniform $\kappa > 0$. It then follows from Propositions 3.1 and 3.2 that there exists $C < \infty$ and $C_l < \infty$ for each $l \in \mathbb{Z}_+$, where C, κ and C_l's depend only on the dimension, such that

$$|\overline{\nabla}^l \overline{\mathrm{Rm}}_k| \le C_l, \quad |\overline{\mathrm{Ric}}_k| \le C r_k \to 0.$$

Hence, by the noncollapsing condition, we can extract from $\{(M_k, \bar{g}_k, p_k)\}_{k=1}^{\infty}$ a subsequence that converges in the smooth pointed Cheeger–Gromov sense to a smooth manifold $(M_\infty, g_\infty, p_\infty)$. By the choice of p_k's, we have that $|\mathrm{Rm}_\infty|(p_\infty) = 1 > 0$, hence g_∞ is nonflat. Since $|\overline{\mathrm{Ric}}_k|$ converges to 0 uniformly, we have that g_∞ is Ricci flat. Finally, since (M_∞, g_∞) is also strongly κ-noncollapsed on all scales, it has Euclidean volume growth, and hence must be ALE by Corollary 8.86 in [Cheeger and Naber 2015]. The rest of the proof now follows similarly from Theorem 4.1. □

5. Proof of the main theorems

Proof of Theorem 1.4. By Theorem 4.1, Proposition 3.1, and (2-5), we obtain locally uniform bounds for the curvatures, the derivatives of the curvatures, and the potential functions for any sequence in the space $\mathcal{M}^4(\kappa)$. Applying the standard regularity theorem to the elliptic equation $\Delta f = \frac{n}{2} - R$, we also obtain locally uniform bounds for the derivatives of the potential functions. Hence we can extract from any sequence contained in $\mathcal{M}^4(\kappa)$ a subsequence that converges in the smooth pointed Cheeger–Gromov sense to a shrinking gradient Ricci soliton, also normalized as in (1-1). It remains to show that the limit Ricci shrinker is in $\mathcal{M}^4(\kappa)$. Item (c) in Definition 1.1 is obvious, we proceed to show (a) and (b).

We let $\{(M_k, g_k, f_k, p_k)\}_{k=1}^{\infty} \subset \mathcal{M}^4(\kappa)$, all normalized as in (1-1), and we let $(M_\infty, g_\infty, f_\infty, p_\infty)$ be their limit Ricci shrinker in the smooth pointed Cheeger–Gromov sense, also normalized as in (1-1). By the definition of Cheeger–Gromov convergence, we have that every open bounded subset in (M_∞, g_∞) can be embedded in infinitely many (M_k, g_k)'s in the sequence, and the images of these embeddings are also bounded open sets, and therefore can be embedded in closed 4-manifolds with vanishing second homology group. To show that (M_∞, g_∞) is nonflat, we make the following observation.

Claim 2.
$$\mathrm{Vol}_f(g_\infty) = \lim_{k\to\infty} \mathrm{Vol}_f(g_k),$$
where Vol_f is the f-volume defined by $\mathrm{Vol}_f(g) = \int_M e^{-f}\, dg$.

Proof. By the uniform rapid decay of e^{-f} (2-5) and the uniform volume growth bound (2-6) we have that for any $\eta > 0$, there exists $A_0 < \infty$ such that for all $A > A_0$ it holds that

$$\mathrm{Vol}_f(g_k) - \eta < \int_{B_{g_k}(p_k, A)} e^{-f_k}\, dg_k \le \mathrm{Vol}_f(g_k),$$

for every $k \ge 1$. The conclusion follows from first taking $k \to \infty$, and then $A \to \infty$, and finally $\eta \to 0$. □

By Proposition 2.5 and Claim 2 we have

$$\mathrm{Vol}_f(g_\infty) = \lim_{k\to\infty} \mathrm{Vol}_f(g_k) \le (4\pi)^{n/2}(1-\varepsilon),$$

where $\varepsilon > 0$ is given by Proposition 2.5. Hence $(M_\infty, g_\infty, f_\infty, p_\infty)$ is not flat, because the f-volume of the Gaussian shrinker is $(4\pi)^{n/2}$. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. Combining Theorem 4.2, the fact that $|\nabla f| \le 1$ by (1-2), and Proposition 3.1, we have that any curvature derivative is uniformly bounded for all elements in $\mathcal{N}^4(\kappa)$. Furthermore, since

$$f(p) = 0, \quad |\nabla f| \le 1, \quad |\nabla^2 f| = |\mathrm{Ric}| \le C(\kappa),$$

we have a uniform growth estimate for $|f|$, and we can derive uniform higher derivative estimates for f by using the elliptic equation $\Delta f = -R$. Taking into account the noncollapsing condition, we immediately obtain the precompactness. By the same argument as in the proof of Theorem 1.4, we have that every possible limit of a convergent sequence in $\mathcal{N}^4(\kappa)$ must satisfy (b) and (c) in Definition 1.2. Such a limit can be either nonflat, hence lies in $\mathcal{N}^4(\kappa)$, or, flat, hence must be the Euclidean space because of its maximum volume growth by (c). This completes the proof of the theorem. □

Proof of Corollary 1.6. To prove (a) we argue by contradiction. Suppose there exists $\{(M_k, g_k, f_k, p_k)\}_{k=1}^{\infty} \subset \mathcal{M}^4(\kappa)$ such that $R_k(p_k) \to 0$. By Theorem 1.4 we can extract a subsequence that converges to a shrinking gradient Ricci soliton $(M_\infty, g_\infty, f_\infty, p_\infty)$ with $R_\infty(p_\infty) = 0$, which by Chen [2009] is flat; this is a contradiction.

To prove (b), we recall that by the proof of Chow, Lu, and Yang [Chow et al. 2011], we only need a uniform upper bound for f and a uniform lower bound for R on a sufficiently large ball, say $B(p, 1000)$, where the former is given by (2-5) and the latter is proved in the same way as for (a).

To prove (c), we claim that there exist $c > 0$, depending only on κ, such that $\mathrm{Vol}_f(g) > c$ for all $(M, g, f, p) \in \mathcal{M}^4(\kappa)$. Suppose this is not true. As in the proof of (a), we can find a sequence of counterexamples converging to a Ricci shrinker $(M_\infty, g_\infty, f_\infty, p_\infty)$ with $\mathrm{Vol}_f(g_\infty) = 0$, which is a contradiction. Hence we have

$\mathrm{Vol}_f(g) \in [c, (4\pi)^{n/2}(1-\varepsilon)]$ for all $(M, g, f, p) \in \mathcal{M}^4(\kappa)$. The conclusion follows from [Wylie 2008] and [Chow and Lu 2016]. $\square$

6. Excluding instantons by a topological condition

In this section we provide an alternative proof for Corollary 5.8 of [Anderson 2010]. This proof is based in essence altogether on the personal notes of Richard Bamler, to whom we are indebted for graciously providing them. However, any mistakes in transcription is solely due to the author. Forasmuch as Anderson's result is of fundamental importance to our main theorem, we include this section for the sake of completeness to help the readers to follow some details.

Theorem 6.1. *Let N be a smooth closed 4-dimensional manifold such that*

$$H_2(N) = 0, \tag{6-1}$$

where H_2 is the second homology group with coefficients in $\mathbb{Z}$. Then there is no open subset $U \subset N$ with the property that U admits an Einstein ALE metric.

We split the proof into the following lemmas.

Lemma 6.2. *Let N be the closed manifold in the statement of Theorem 6.1. Let $U \subset N$ be an connected open subset such that $\partial U \cong \mathbb{S}^3/\Gamma$ and $H_1(U, \partial U) = 0$, where Γ is a finite group. Then the following hold.*

$$H_1(\partial U) = H_1(U) \oplus H_1(U), \tag{6-2}$$

$$H_2(U) = 0. \tag{6-3}$$

Proof. By Poincaré duality, we have

$$H^2(N; \mathbb{Z}) \cong H_2(N) = 0.$$

By the universal coefficient theorem, we have

$$0 = H^2(N; \mathbb{Z}) \cong \mathrm{Hom}(H_2(N), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(N), \mathbb{Z}),$$

which implies that $H_1(N)$ is torsion free, so henceforth we may assume

$$H_1(N) \cong \mathbb{Z}^d, \tag{6-4}$$

where $d \geq 0$. By Poincaré duality and by the universal coefficient theorem, we have

$$H_2(\partial U) \cong H^1(\partial U) \cong \mathrm{Hom}(H_1(\partial U), \mathbb{Z}) = 0,$$

where the last equality is because $H_1(\partial U)$ is a finite abelian group and hence purely torsion. Then we have the Mayer–Vietoris sequence

$$0 = H_2(\partial U) \to H_2(U) \oplus H_2(N \backslash U) \to H_2(N) = 0,$$

from whence we obtain

$$H_2(U) = H_2(N \backslash U) = 0. \tag{6-5}$$

On the other hand, we consider the long exact sequence

$$0 = H_2(U) \to H_2(U, \partial U) \to H_1(\partial U) \to H_1(U) \to H_1(U, \partial U) = 0. \tag{6-6}$$

It follows that $H_1(U)$ is purely torsion since the third homomorphism above is surjective and $H_1(\partial U)$ is finite. Hence by the universal coefficient theorem and by Poincaré–Lefschetz duality we have

$$H_2(U, \partial U) \cong H^2(U) \cong \mathrm{Hom}(H_2(U), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(U), \mathbb{Z}) \cong H_1(U),$$

where the last isomorphism follows from the fact that $H_1(U)$ is purely torsion and (6-5). Hence (6-6) is simplified as

$$0 \to H_1(U) \to H_1(\partial U) \to H_1(U) \to 0. \tag{6-7}$$

To see (6-7) splits, we consider the following Mayer–Vietoris sequence

$$0 = H_2(N) \to H_1(\partial U) \to H_1(U) \oplus H_1(N \backslash U) \to H_1(N) \cong \mathbb{Z}^d,$$

where we have used (6-4). If we write $H_1(U) \oplus H_1(N \backslash U) \cong H_1(U) \oplus T \oplus \mathbb{Z}^e$, where T is the torsion part of $H_1(N \backslash U)$, since $H_1(\partial U)$ is purely torsion, we have that the image of the second homomorphism in the above sequence is in $H_1(U) \oplus T \oplus \{0\}$, whose image under the third homomorphism is 0. Hence we can simplify the above sequence as

$$0 \to H_1(\partial U) \to H_1(U) \oplus T \to 0.$$

The inclusion $H_1(U) \hookrightarrow H_1(U) \oplus T \cong H_1(\partial U)$ gives a homomorphism $H_1(U) \to H_1(\partial U)$, whose composition with the third homomorphism in (6-7) is the identity on $H_1(U)$. It follows that (6-7) splits and we have completed the proof. □

Lemma 6.3. *Let $\mathbb{S}^3/\Gamma$ be a round space form, where Γ is a finite group. If $H_1(\mathbb{S}^3/\Gamma) \cong G \oplus G$, for some group G, then either Γ is the binary dihedral group D_n^* with n being even, or Γ is the binary icosahedral group with order* 120.

Proof. We shall check every possible group Γ.

(a) **Lens space:** In this case $H_1(\mathbb{S}^3/\Gamma) \cong \Gamma = \mathbb{Z}_m$ with $m \geq 2$, which is not possible.

(b) **Prism manifold:** In this case the fundamental group has the presentation

$$\langle x, y \mid xyx^{-1} = y^{-1},\ x^{2^k} = y^n \rangle \times \mathbb{Z}_m,$$

where $k, m \geq 1$, $n \geq 2$, and m is coprime to $2n$. Its abelianization is

$$H_1(\mathbb{S}^3/\Gamma) \cong \langle x, y \mid y = y^{-1},\ x^{2^k} = y^n \rangle \times \mathbb{Z}_m,$$

where $y^2 = 1$. We have that $H_1(\mathbb{S}^3/\Gamma) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^k} \times \mathbb{Z}_m$ in the case n is even, and that $H_1(\mathbb{S}^3/\Gamma) \cong \mathbb{Z}_{2^{k+1}} \times \mathbb{Z}_m$ in the case n is odd. Since m is coprime to $2n$, we have that the only possible case is when $m = 1$, n is even, and $k = 1$.

(c) **Tetrahedral manifold:** In this case we have

$$\Gamma = \langle x, y, z \mid (xy)^2 = x^2 = y^2,\ zxz^{-1} = y,\ zyz^{-1} = xy,\ z^{3^k} = 1\rangle \times \mathbb{Z}_m,$$

where $k, m \geq 1$ and m is coprime to 6. Then we have

$$\begin{aligned} H_1(\mathbb{S}^3/\Gamma) &\cong \langle x, y, z \mid x^2 = y^2 = 1,\ x = y,\ y = xy,\ z^{3^k} = 1\rangle \times \mathbb{Z}_m \\ &= \langle x, z \mid x^2 = 1,\ x = x^2,\ z^{3^k} = 1\rangle \times \mathbb{Z}_m = \mathbb{Z}_{3^k} \times \mathbb{Z}_m. \end{aligned}$$

Since m is coprime to 6, this case is not possible.

(d) **Octahedral manifold:** In this case we have

$$\Gamma = \langle x, y \mid (xy)^2 = x^3 = y^4\rangle \times \mathbb{Z}_m,$$

where m is coprime to 6. Then we have

$$H_1(\mathbb{S}^3/\Gamma) \cong \langle x, y \mid x = y^2 = x^2\rangle \times \mathbb{Z}_m = \mathbb{Z}_2 \times \mathbb{Z}_m.$$

Since m is coprime to 6, this case is not possible.

(e) **Icosahedral manifold:** In this case we have

$$\Gamma = \langle x, y \mid (xy)^2 = x^3 = x^3y^5\rangle \times \mathbb{Z}_m,$$

where m is coprime to 30. Then we have

$$\begin{aligned} H_1(\mathbb{S}^3/\Gamma) &\cong \langle x, y \mid x = y^2,\ x^2 = y^3\rangle \times \mathbb{Z}_m \\ &= \langle x, y \mid x = y^2,\ x^2 = y^3,\ y = 1\rangle \times \mathbb{Z}_m = \mathbb{Z}_m. \end{aligned}$$

The only possibility is $m = 1$. □

We still need to consider the two cases when Γ is the binary dihedral group D^*_{2n} or the binary icosahedral group. In both cases Γ can be embedded in $SU(2)$. Indeed, it is well known that the binary dihedral, tetrahedral, octahedral, and icosahedral groups are all finite subgroups of $SU(2)$; see [Kronheimer 1989].

Lemma 6.4. *Let $\mathbb{S}^3/\Gamma$ be the spherical space form with $\Gamma < O(4)$ being either the binary dihedral group D^*_{2n} or the binary icosahedral group. Then there exists a complex structure on $\mathbb{R}^4$ such that $\Gamma < SU(2)$*

Lemma 6.5. *Let (U, g) be an Einstein ALE space which is asymptotic to $\mathbb{S}^3/\Gamma$, where $\Gamma < SU(2)$ is isomorphic to the binary dihedral group D^*_{2n} or to the binary icosahedral group. Then $b_2(U) \geq 1$.*

Proof. Assume that $b_2(U) = 0$. Then we have $\chi(U) = 1 - b_1(U) - b_3(U) \leq 1$ and $\tau(U) = 0$. Using the Chern–Gauss–Bonnet theorem and the Atiyah–Patodi–Singer index theorem, we have (see (4.4) and (4.5) in [Nakajima 1990])

$$1 \geq \chi(U) = \frac{1}{8\pi^2}\int_U |W|^2\,dg + \frac{1}{|\Gamma|},$$
$$0 = \tau(U) = \frac{1}{12\pi^2}\int_U (|W^+|^2 - |W^-|^2)\,dg - \eta_S(\mathbb{S}^3/\Gamma),$$

where η_S stands for the *eta invariant*. Hence we have

$$\frac{2}{3} \geq \frac{2}{12\pi^2}\int_U |W^-|^2\,dg + \frac{2}{3|\Gamma|} + \eta_S(\mathbb{S}^3/\Gamma),$$

which implies

$$\eta_S(\mathbb{S}^3/\Gamma) \leq \frac{2}{3}\Big(1 - \frac{1}{|\Gamma|}\Big) < \frac{2}{3}.$$

On the other hand, by [Nakajima 1990], if Γ is the binary dihedral group D^*_{2n}, we have

$$\eta_S(\mathbb{S}^3/\Gamma) = \frac{2(2n+2)^2 - 8(2n+2) + 9}{6\cdot 2n} = \frac{8n^2+1}{12n} > \frac{2}{3}.$$

Similarly, if Γ is the binary icosahedral group, then we have

$$\eta_S(\mathbb{S}^3/\Gamma) = \frac{361}{180} > \frac{2}{3}.$$

In either case we yield a contradiction. □

Proof of Theorem 6.1. Let N be the manifold in Theorem 6.1 and $U \subset N$ be a connected open subset that admits an Einstein ALE metric.

Claim 3. $$H_1(U, \partial U) = 0.$$

Proof. Suppose the claim does not hold. We first show that the boundary $\partial\widetilde{U}$ of the universal cover $\widetilde{U}$ has more than one component. Since ∂U is a deformation retraction of its collar neighbourhood, by excision we have

$$H_1(U/\partial U, \partial U/\partial U) = H_1(U, \partial U) \neq 0.$$

Hence we have $\pi_1(U/\partial U) \neq 0$. Let γ_0 be a loop in $U/\partial U$ based at $\partial U/\partial U$ that is not null-homotopic. Lifting this loop to U by the quotient map $q : U \to U/\partial U$, we obtain a curve γ in U, whose ends lie in ∂U. By using the universal covering map $p : \widetilde{U} \to U$, we can lift γ to $\widetilde{\gamma}$, a curve in $\widetilde{U}$ whose ends lie in $\partial\widetilde{U}$. If $\partial\widetilde{U}$ is connected, since $\widetilde{U}$ is simply connected, we have that $\widetilde{\gamma}$ is homotopic to a curve that lies in $\partial\widetilde{U}$. Composing this homotopy with $q \circ p$ we obtain a homotopy between γ_0 with a point; this is a contradiction. Hence $\partial\widetilde{U}$ has more than one component.

Next we observe that if U admits an Einstein ALE metric, then we can lift this metric to $\widetilde{U}$, which has more than one end. By Cheeger–Gromoll's splitting theorem, $\widetilde{U}$ splits as the product of a line and a Ricci flat 3-manifold; hence this metric is flat, which is a contradiction. □

We continue the proof of Theorem 6.1. By (6-3) we have that $b_2(U) = 0$. On the other hand, combining (6-2) and Lemma 6.3 we have that $\partial U \cong \mathbb{S}^3/\Gamma$, where Γ is either the binary dihedral group D^*_{2n} or the binary icosahedral group. It follows from Lemma 6.5 that $b_2(U) \geq 1$, and we obtain a contradiction. □

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