

Pacific Journal of Mathematics

**POLARIZATION, SIGN SEQUENCES
AND ISOTROPIC VECTOR SYSTEMS**

GERGELY AMBRUS AND SLOAN NIETERT

POLARIZATION, SIGN SEQUENCES AND ISOTROPIC VECTOR SYSTEMS

GERGELY AMBRUS AND SLOAN NIETERT

We determine the order of magnitude of the n -th ℓ_p -polarization constant of the unit sphere S^{d-1} for every $n, d \geq 1$ and $p > 0$. For $p = 2$, we prove that extremizers are isotropic vector sets, whereas for $p = 1$, we show that the polarization problem is equivalent to that of maximizing the norm of signed vector sums. Finally, for $d = 2$, we discuss the optimality of equally spaced configurations on the unit circle.

1. Introduction

Let $\omega_n = \{u_1, \dots, u_n\}$ be a multiset of n unit vectors in \mathbb{R}^d , and set $p > 0$. The ℓ_p -potential of ω_n at the unit vector $v \in S^{d-1}$ is defined as

$$U^p(\omega_n, v) = \sum_{i=1}^n |\langle v, u_i \rangle|^p,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. This is an analogue of the classical Riesz potential for inner products. The ℓ_p -polarization of ω_n is given by

$$M^p(\omega_n) = \max_{v \in S^{d-1}} U^p(\omega_n, v).$$

We are interested in finding the minimum ℓ_p -polarization of $\omega_n \subset S^{d-1}$, for fixed d and n , that is,

$$M_n^p(S^{d-1}) = \min_{\omega_n \subset S^{d-1}} M^p(\omega_n) = \min_{u_1, \dots, u_n \in S^{d-1}} \max_{v \in S^{d-1}} \sum_{i=1}^n |\langle v, u_i \rangle|^p.$$

The quantity $M_n^p(S^{d-1})$ is called the n -th ℓ_p -polarization (or Chebyshev) constant of S^{d-1} .

Ambrus' research was supported by NKFIH grants PD125502 and K116451 and by the Bolyai Research Scholarship of the Hungarian Academy of Sciences. Nietert's research was supported by Budapest Semesters in Mathematics and the Hungarian-American Fulbright Commission.

MSC2010: primary 52A40; secondary 41A17.

Keywords: polarization problems, discrete potentials, Chebyshev constants, isotropic vectors sets, tight frames, vector sums.

Related questions for $p \leq 0$ have been studied extensively; see, e.g., the recent article of Hardin, Petrache and Saff [Hardin et al. 2019] about general polarization problems. In the planar case, $M_n^p(S^1)$ has a direct connection to the classical notions of Riesz potentials and Chebyshev constants. This connection is described in Section 5. Polarization problems have been subject to very active research in the last 15 years, although their study dates back to at least 1967 [Ohtsuka 1967]. The most relevant results to our present problem are discussed in [Ambrus 2009; Nikolov and Rafailov 2011; 2013; Stolarsky 1975a; 1975b].

Determining the exact value of $M_n^p(S^{d-1})$ is hopeless in general, except for certain cases. Therefore, our first result provides asymptotic bounds. For brevity, we introduce the quantity

$$\mu_{d,p} = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d+p}{2}\right)}.$$

Clearly, $\mu_{d,p} = \Theta(d^{-p/2})$. Here, and throughout the paper, we are going to use the standard asymptotic notations following Knuth [1997]: given two positive-valued functions $f(n) > 0$ and $g(n) > 0$, $n \in \mathbb{N}$, we write

$$\begin{aligned} f(n) &= \mathcal{O}(g(n)) && \text{if } \limsup_{n \rightarrow \infty} f(n)/g(n) < \infty; \\ f(n) &= o(g(n)) && \text{if } \lim_{n \rightarrow \infty} f(n)/g(n) = 0; \\ f(n) &= \Omega(g(n)) && \text{if } \liminf_{n \rightarrow \infty} f(n)/g(n) > 0; \\ f(n) &= \omega(g(n)) && \text{if } \lim_{n \rightarrow \infty} f(n)/g(n) = \infty; \\ f(n) &= \Theta(g(n)) && \text{if } f(n) = \mathcal{O}(g(n)) \text{ and } f(n) = \Omega(g(n)). \end{aligned}$$

Depending on the number of points compared to the dimension, we derive different estimates.

Theorem 1. *For every $p > 0$,*

$$M_n^p(S^{d-1}) = n\mu_{d,p} + o(nd^{-p/2})$$

as $d, n \rightarrow \infty$ and $n = \omega(d^{1+p} \log d)$.

Furthermore, for $0 < p \leq 2$,

$$M_n^p(S^{d-1}) = \Theta(nd^{-p/2}),$$

while for $p > 2$,

$$M_n^p(S^{d-1}) = \Omega(n d^{-p/2}) \quad \text{and} \quad M_n^p(S^{d-1}) = \mathcal{O}(n d^{-1})$$

holds, as $d, n \rightarrow \infty$ and $n \geq d$.

For special values of p and d , stronger results may be proved. In order to discuss the case $p = 2$, we introduce the following notion: $\omega_n = \{u_1, \dots, u_n\} \subset S^{d-1}$ is an *isotropic set of unit vectors* if

$$\sum_{i=1}^n u_i \otimes u_i = \frac{n}{d} I_d,$$

where I_d is the identity operator on \mathbb{R}^d . Isotropic sets of unit vectors are also called *unit norm tight frames*; see, e.g., [Benedetto and Fickus 2003].

Theorem 2. *For every $d \geq 1$ and $n \geq d$,*

$$M_n^2(S^{d-1}) = \frac{n}{d},$$

and the extremal ω_n configurations are exactly the isotropic sets of unit vectors.

For $p = 1$, [Theorem 1](#) provides the exact asymptotics: $M_n^1(S^{d-1}) = \Theta(nd^{-1/2})$. By the following fact, this also provides an estimate to a quantity involving sign sequences:

Proposition 3. *For any set of unit vectors $\omega_n = \{u_1, \dots, u_n\} \subset S^{d-1}$,*

$$\max_{\varepsilon \in \{-1, 1\}^n} \left| \sum_{i=1}^n \varepsilon_i u_i \right| = M^1(\omega_n).$$

As a consequence of [Proposition 3](#) and [Theorem 1](#), we immediately obtain:

Theorem 4. *For every $d \geq 1$ and $n \geq d$,*

$$(1) \quad \min_{(u_i)_i^n \subset S^{d-1}} \max_{\varepsilon \in \{\pm 1\}^n} \left| \sum_{i=1}^n \varepsilon_i u_i \right| = \Theta\left(\frac{n}{\sqrt{d}}\right).$$

Finally, we discuss the ℓ_p -polarization constants of the unit circle.

Proposition 5. *For $d = 2$ and $0 < p \leq 1$ as well as for $p = 2, 4, \dots, 2n - 2$, $M^p(\omega_n)$ is minimized by vector sets which are equally distributed on the half-circle. For $p = 2, 4, \dots, 2n - 2$, the potential function of any extremal configuration is constant on T , whereas for $0 < p \leq 1$,*

$$M_n^p(S^1) = \sum_{k=1}^n \left| \cos\left(\frac{k\pi}{n} - \frac{\pi}{2n}\right) \right|^p$$

for even values of n , and

$$M_n^p(S^1) = \sum_{k=1}^n \left| \cos\left(\frac{k\pi}{n}\right) \right|^p$$

for odd n .

2. General asymptotics

Proof of Theorem 1. We start with the lower bound, which holds for every $p > 0$ and $n \geq d$. Let $\omega_n = \{u_1, \dots, u_n\} \subset S^{d-1}$ be fixed. Note that

$$\begin{aligned} M^p(\omega_n) &= \max_{v \in S^{d-1}} \sum_{i=1}^n |\langle v, u_i \rangle|^p \geq \mathbb{E}_v \left[\sum_{i=1}^n |\langle v, u_i \rangle|^p \right] \\ &= \sum_{i=1}^n \mathbb{E}_v |\langle v, u_i \rangle|^p = n \mathbb{E}_v |\langle v, u_1 \rangle|^p, \end{aligned}$$

where the expectation is taken as v being selected uniformly at random from the sphere. By a standard calculation, we obtain that

$$\mathbb{E}_v |\langle v, u_1 \rangle|^p = \frac{2}{B(\frac{1}{2}, \frac{d-1}{2})} \int_0^1 t^p (1-t^2)^{(d-3)/2} dt = \mu_{d,p}$$

and by the previous arguments,

$$(2) \quad M_n^p(S^{d-1}) \geq n \mu_{d,p}.$$

Next, we show that this bound is asymptotically correct when n is large, or when $0 < p \leq 2$. First, assume that $n = \Omega(d^{1+p} \log d)$, and select an independent, random uniform sample $\omega_n = \{u_1, \dots, u_n\}$ from S^{d-1} . We will show that with positive probability, $M^p(\omega_n)$ is of order $O(n d^{-p/2})$.

For conciseness, let

$$(3) \quad f(v) = \sum_{i=1}^n |\langle v, u_i \rangle|^p.$$

Observe that f is np -Lipschitz, since

$$\begin{aligned} |f(v) - f(w)| &= \sum_{i=1}^n (|\langle v, u_i \rangle|^p - |\langle w, u_i \rangle|^p) \\ &\leq \sum_{i=1}^n ||\langle v, u_i \rangle|^p - |\langle w, u_i \rangle|^p| \\ &\leq p \sum_{i=1}^n ||\langle v, u_i \rangle| - |\langle w, u_i \rangle|| \leq p \sum_{i=1}^n |\langle v - w, u_i \rangle| \\ &\leq p \sum_{i=1}^n |v - w| = np |v - w|. \end{aligned}$$

On the other hand, for any fixed $v \in S^{d-1}$,

$$(4) \quad \mathbb{E} f(v) = n\mu_{d,p},$$

where the expectation refers to the choice of the random base system ω_n . Moreover, since $0 \leq |\langle v, u_i \rangle|^p \leq 1$, Hoeffding's inequality and (4) yields that for any fixed $v \in S^{d-1}$ and $t > 0$,

$$(5) \quad \mathbb{P}(|f(v) - n\mu_{d,p}| > t) < 2e^{-2t^2/n}.$$

We are going to bound the maximum of $f(v)$ on S^{d-1} by pinning it down at the points of a δ -net and then exploiting the Lipschitz property. It is well known (see, e.g., [Ball 1997]) that there exists a δ -net (with respect to the Euclidean metric) in S^{d-1} with at most $(4/\delta)^d$ points. Let D be such a δ -net. Choose $v^* \in S^{d-1}$ such that

$$f(v^*) = M^P(\omega_n) = \max_{v \in S^{d-1}} f(v).$$

Since v^* must be within δ of some $w \in D$ and f is np -Lipschitz, we have that $|f(w) - M^P(\omega_n)| \leq \delta np$. Then, the union bound and (5) gives that for every $\lambda > 0$,

$$\begin{aligned} \mathbb{P}(|M^P(\omega_n) - n\mu_{p,d}| > \lambda) &\leq \sum_{w \in D} \mathbb{P}(|f(w) - n\mu_{p,d}| > \lambda - \delta np) \\ &\leq 2\left(\frac{4}{\delta}\right)^d e^{-2(\lambda - \delta np)^2/n}. \end{aligned}$$

Setting $\delta = \lambda/(2np)$, the bound simplifies to

$$\mathbb{P}(|M^P(\omega_n) - n\mu_{p,d}| > \lambda) \leq 2\left(\frac{8np}{\lambda}\right)^d e^{-\lambda^2/(2n)}.$$

Take $\lambda = cnd^{-p/2}$ with some constant $c > 0$. Then

$$\mathbb{P}(|M^P(\omega_n) - n\mu_{p,d}| > cnd^{-p/2}) \leq c'^d d^{dp/2} e^{-(c^2/2)nd^{-p}}$$

with $c' = 10/c$. Taking logarithm shows that the above probability is guaranteed to be less than one if

$$n > \frac{2 \log c'}{c^2} d^{1+p} + \frac{p}{c^2} d^{1+p} \log d.$$

Therefore, when $n = \omega(d^{1+p} \log d)$, we obtain that

$$M_n^P(S^{d-1}) = n\mu_{d,p} + o(nd^{-p/2}) = \Theta(nd^{-p/2}).$$

Let us turn to the estimates valid for smaller values of n . The lower bound (2) still holds, so we only have to prove the upper estimates. Without changing the

asymptotic bounds, we may assume that $n = kd$. Let ω_n consist of k copies of an orthonormal basis of \mathbb{R}^d . Then,

$$\max_{v \in S^{d-1}} U^p(\omega_n, v) = k \max_{|v|=1} \sum_{i=1}^d |v_i|^p = \begin{cases} kd^{1-p/2} & \text{for } 0 < p \leq 2, \\ k & \text{for } 2 < p, \end{cases}$$

which implies the upper bounds for arbitrary $n \geq d$. \square

Remark. The following construction gives a slightly stronger estimate for a small number of points, when $0 < p \leq 2$. Let $c_{n,d,p}$ be the infimum of all constants $c \in \mathbb{R}$ satisfying

$$M_n^p(S^{d-1}) \leq c n d^{-p/2}.$$

Let H and H^\perp be two orthogonal, d -dimensional linear subspaces in \mathbb{R}^{2d} . Take ω_n and ω_n^\perp to be n -element vector sets in H and H^\perp , respectively, with $M_n^p(S^{d-1}) = M^p(\omega_n) = M^p(\omega_n^\perp)$. Let $\omega_{2n} = \omega_n \cup \omega_n^\perp \subset \mathbb{R}^{2d}$. Then

$$\begin{aligned} M^p(\omega_{2n}) &= \max_{\substack{v \in H, v^\perp \in H^\perp \\ |v|^2 + |v^\perp|^2 = 1}} (U^p(\omega_n, v) + U^p(\omega_n^\perp, v^\perp)) \\ &\leq \max_{|v|^2 + |v^\perp|^2 = 1} (|v|^p + |v^\perp|^p) c_{n,d,p} n d^{-p/2} \\ &= \frac{2}{2^{p/2}} c_{n,d,p} n d^{-p/2} = c_{n,d,p} (2n)(2d)^{-p/2}. \end{aligned}$$

Thus, $c_{2n,2d,p} \leq c_{n,d,p}$. Using the fact that for $d = 0$, $M_n^p(S^0) = n = n d^{-p/2}$, it follows that for $a, b \in \mathbb{N}$, $a \geq b$, we have $c_{2^a, 2^b, p} \leq 1$. Moreover, for $2^a < n < 2^{a+1}$, it is easy to see that $M_n^p(S^{d-1}) \leq 2M_{2^a}^p(S^{d-1})$ (by taking the vectors of ω_n once or twice). Likewise, for $2^b < d < 2^{b+1}$, we know that $M_n^p(S^{d-1}) \leq 2^{p/2} M_n^p(S^{2^b-1})$ by keeping the optimal vectors from the $d = 2^b$ case. Therefore, $c_{n,d,p} \leq 2^{p/2}$ for all $n \geq d$.

3. Isotropic vector sets: $p = 2$

Proof of Theorem 2. Let $\omega_n = \{u_1, \dots, u_n\} \subset S^{d-1}$. Introduce the frame operator

$$A = \sum_{i=1}^n u_i \otimes u_i,$$

where $u \otimes v = uv^\top$ denotes the tensor product of the two vectors. Then for any vector $v \in S^{d-1}$,

$$v^\top A v = \sum_{i=1}^n \langle v, u_i \rangle^2.$$

Therefore, $\max_{v \in S^{d-1}} \sum \langle v, u_i \rangle^2$ is attained at the eigenvector of norm 1 of A belonging to the maximal eigenvalue. Since $\text{tr } A = n$, we obtain that

$$M^2(\omega_n) \geq \frac{n}{d},$$

and equality holds if and only if $\sum_{i=1}^n u_i \otimes u_i = (n/d)I_d$, that is, if ω_n is an isotropic vector system. \square

Isotropic vector sets also arise in different contexts: in frame theory, they are called *unit norm tight frames* or UNTFs, while in the context of John's theorem, their rescaled copies provide a *decomposition of the identity*. A characterization of them was first given by Benedetto and Fickus [2003] (for a simplified proof, see [Ambrus 2014]): they showed that a set of n unit vectors form an isotropic set if and only if they are the minimizer of the *frame potential* among n -element vector sets in S^{d-1} , defined by

$$\text{FP}(\omega_n) = \sum_{i,j} |\langle u_i, u_j \rangle|^2.$$

In particular, it follows that n -element isotropic sets of d -dimensional unit vectors exist for every $n \geq d$.

For $d = 2$ and $d = 3$, the characterization may be simplified by utilizing the connection with complex numbers. Goyal et al. [2001] showed that in \mathbb{R}^2 , isotropic sets of unit vectors correspond to sequences $\{z_i\}_{i=1}^n \subset \mathbb{C}$ satisfying $|z_i| = 1$ and

$$\sum_{i=1}^n z_i^2 = 0,$$

where the unit circle S^1 of \mathbb{R}^2 is identified with the complex unit circle T . For $d = 3$, Benedetto and Fickus [2003] provide a correspondence between isotropic vector sets and sequences $\{z_i\}_{i=1}^n \subset \mathbb{C}$ satisfying $|z_i| \leq 1$ and

$$\sum_{i=1}^n |z_i|^2 = \frac{2}{3}n, \quad \sum_{i=1}^n z_i^2 = 0, \quad \sum_{i=1}^n z_i \sqrt{1 - |z_i|^2} = 0.$$

Here, each point in S^2 is identified with its projection onto the unit disc of the complex plane.

4. Sign sequences: $p = 1$

Proof of Proposition 3. First, we show that $\max_{\varepsilon \in \{-1, 1\}^n} \left| \sum_{i=1}^n \varepsilon_i u_i \right| \leq M^1(\omega_n)$. Indeed, let ε be an arbitrary sign sequence, and define

$$z = \sum_{i=1}^n \varepsilon_i u_i.$$

Then

$$|z|^2 = \left| \sum \varepsilon_i u_i \right|^2 = \sum_{i,j} \varepsilon_i \varepsilon_j \langle u_i, u_j \rangle = \sum \varepsilon_i \langle z, u_i \rangle \leq \sum_{i=1}^n |\langle z, u_i \rangle|$$

which shows that

$$\left| \sum \varepsilon_i u_i \right| \leq U^1 \left(\omega_n, \frac{z}{|z|} \right) \leq M^1(\omega_n).$$

For the reverse direction, introduce the function $f(v) = \sum_{i=1}^n |\langle v, u_i \rangle|$ defined on S^{d-1} as in (3). Applying Lagrange multipliers implies that those critical points of f on S^{d-1} where f is differentiable satisfy

$$v = \frac{\sum \varepsilon_i u_i}{\left| \sum \varepsilon_i u_i \right|}$$

with $\varepsilon_i = \operatorname{sgn} \langle v, u_i \rangle$. By taking inner products of both sides with v we obtain that $\left| \sum \varepsilon_i u_i \right| = U^1(\omega_n, v)$.

Therefore, we only have to rule out the existence of maximizers of f at non-differentiable points. Assume on the contrary that $v \in S^{d-1}$ is a maximizer with $\langle v, u_j \rangle = 0$, where $1 \leq j \leq k$, and $|\langle v, u_j \rangle| > 0$ for $k < j \leq n$. Then for $\delta \in \mathbb{R}$ with sufficiently small absolute value,

$$\begin{aligned} f(v + \delta u_1) &= \sum_{i=1}^n |\langle v + \delta u_1, u_i \rangle| = |\delta| \sum_{i=1}^k |\langle u_1, u_k \rangle| + \sum_{i=k+1}^n |\langle v, u_i \rangle + \delta \langle u_1, u_i \rangle| \\ &= f(v) + \delta \sum_{i=k+1}^n \operatorname{sgn} \langle v, u_i \rangle \cdot \langle u_1, u_i \rangle + |\delta| \sum_{i=1}^k |\langle u_1, u_k \rangle|. \end{aligned}$$

Here, $\sum_{i=1}^k |\langle u_1, u_k \rangle| \geq 1$, and thus, for sufficiently small but nonzero δ whose sign agrees with that of $\sum_{i=k+1}^n \operatorname{sgn} \langle v, u_i \rangle \cdot \langle u_1, u_i \rangle$, we obtain that

$$f\left(\frac{v + \delta u_j}{|v + \delta u_j|}\right) \geq \frac{f(v) + |\delta|}{\sqrt{1 + \delta^2}} > f(v),$$

which contradicts the maximality of v . □

Sign sequences arise in several topics, most prominently in the context of discrepancy theory; see, for example, the famous conjecture of Komlós [Spencer 1987]. Note, however, a fundamental difference: in that setting, one would like to *minimize* the norm of $\sum \varepsilon_i u_i$, whereas here, the goal is to find the *maximizers*. The “dual” question of Theorem 4 was asked by Dvoretzky [1963]: Determine

$$\max_{(u_i)_1^n \in S^{d-1}} \min_{\varepsilon \in \{\pm 1\}^n} \left| \sum_{i=1}^n \varepsilon_i u_i \right|.$$

Various related games were studied by Spencer [1977]. Bárány and Grinberg [1981] proved a stronger result which implies an $O(d)$ upper bound on the above quantity.

More related to the present question is Bang's lemma [1951], which arose in the context of the well-known plank problem. Its simplest form [Ball 2001] states the following: *If u_1, \dots, u_n are unit vectors in \mathbb{R}^d , and the signs $\varepsilon_i = \pm 1$ are chosen so as to maximize the norm $|\sum_1^n \varepsilon_i u_i|$, then $|\langle u_k, \sum_1^n \varepsilon_i u_i \rangle| \geq 1$ holds for every k .* Note, however, that this only implies

$$\min \max_1 \left| \sum_1^n \varepsilon_i u_i \right| \geq \sqrt{n}.$$

The same estimate follows by taking the average of $|\sum \varepsilon_i u_i|^2$ over all possible sign sequences.

It remains an open question to determine the extremal point configurations of (1). In general, we have very little information about the extremizers, and a complete description of them can only be hoped for in a few special cases. For $n = d$, the above averaging argument yields that the extremum is uniquely achieved by the vectors e_i of an orthonormal basis, which satisfy $\min \max |\sum_1^n \varepsilon_i e_i| = \sqrt{d}$. For $n = d + 1$, natural intuition and numerical experiments suggest that each extremal configuration is, up to sign changes, the union of the vertex set of an even dimensional regular simplex and an orthonormal basis of the orthogonal complement of its subspace. The following conjecture was stated in a slightly incorrect form in [Brugger et al. 2018] and has been corrected by [Polyanskii 2019].

Conjecture 1. *For any $d \geq 1$, and for any configuration of $d + 1$ unit vectors $u_i, \dots, u_{d+1} \in S^{d-1}$, there exists a sequence of signs $\varepsilon \in \{\pm 1\}^{d+1}$ so that*

$$\left| \sum_{i=1}^{d+1} \varepsilon_i u_i \right| \geq \sqrt{d+2}.$$

Moreover, the above estimate is sharp if and only if, up to sign changes, $(u_i)_1^{d+1}$ is the union of the vertex set of a regular simplex centered at the origin in a subspace H , and an orthonormal basis of H^\perp , where H is an even dimensional linear subspace of \mathbb{R}^d .

5. Planar case: equidistributed sets

In the plane, finding $M^p(\omega_n)$ is equivalent to maximizing the sum of the p -th powers of the Euclidean distances from a variable unit vector to n fixed unit vectors via the following transformation. Identify S^1 with the complex unit circle T , and let $u_i = e^{i\alpha_i}$, $v = e^{i\phi}$. Introduce $\tilde{u}_i = e^{i2\alpha_i} = u_i^2$ and $\tilde{v} = e^{i(2\phi+\pi)} = -e^{i2\phi}$. Then

$$(6) \quad |\tilde{v} - \tilde{u}_i| = 2 \left| \sin \frac{2(\phi + \pi/2) - 2\alpha_i}{2} \right| = 2 |\cos(\alpha_i - \phi)| = 2 |\langle v, u_i \rangle|.$$

Therefore, $M^p(\omega_n)$ may be obtained by finding the point $\tilde{v} \in S^1$ for which $\sum |\tilde{u}_i - \tilde{v}|^p$ is maximal.

Accordingly, we introduce the following quantities for an n -point configuration $\omega_n = \{z_1, \dots, z_n\} \subset T$:

$$\begin{aligned}\tilde{U}^p(\omega_n, z) &= \sum_{i=1}^n |z - z_i|^p \\ \tilde{M}^p(\omega_n) &= \max_{z \in T} \tilde{U}^p(\omega_n, z) \\ \tilde{M}_n^p &= \min_{\omega_n \in T^n} \tilde{M}^p(\omega_n).\end{aligned}$$

Analogues of the above notions with negative p are called the Riesz potential and polarization quantities, and have been extensively studied before; see, e.g., [Erdélyi and Saff 2013] for general results in that direction.

By (6), $\tilde{M}_n^p = 2^p M_n^p(S^1)$, and thus Theorem 1 implies the lower bound

$$(7) \quad \tilde{M}_n^p \geq 2^p \cdot n \mu_{2,p} = n \cdot \tilde{\mu}_p,$$

where

$$\tilde{\mu}_p = 2^p \mu_{2,p} = \frac{2^p \Gamma(\frac{p+1}{2})}{\sqrt{\pi} \Gamma(\frac{p}{2} + 1)} = \frac{\Gamma(p+1)}{\Gamma(\frac{p}{2} + 1)^2} = \binom{p}{p/2},$$

using the Legendre duplication formula and the natural extension of the binomial coefficient to nonintegers.

The above notions have been studied by Stolarsky [1975a; 1975b], who determined $\tilde{M}^p(\omega_n^*)$ for $0 < p < 2n$, where ω_n^* is an *equidistributed set* on T :

$$\omega_n^* = \{1, \xi, \xi^2, \dots, \xi^{n-1}\},$$

where $\xi = e^{i2\pi/n}$. He also determined \tilde{M}_n^p for $n = 3$ and $0 < p \leq 2$. Nikolov and Rafailov [2011] determined the value \tilde{M}_n^p for $n = 3$ and arbitrary $p > 0$ and also discussed the critical points of $\tilde{U}^p(\omega_n^*, z)$ on T . They showed that if p is an even integer with $2 \leq p \leq 2n - 2$, then $\tilde{U}^p(\omega_n^*, z)$ is constant on T . Moreover, they proved [Nikolov and Rafailov 2013] that this property (holding for all even integer exponents between 2 and $2n$) characterizes equidistributed sets. They conjectured that the condition holding solely for $p = 2n - 2$ is already sufficient for characterization. This was verified by Bosuwan and Ruengrot [2017] (for the case $\omega_n \subset T$, which we assumed anyway). The authors also proved that for $p = 2, 4, \dots, 2n - 2$, \tilde{M}_n^p is attained at the configurations ω_n which satisfy

$$\sum_{z \in \omega_n} z^j = 0$$

for every $j = 1, 2, \dots, n - 1$.

On the other hand, Hardin, Kendall and Saff [Hardin et al. 2013], proving a conjecture in [Ambrus et al. 2013], proved the polarization optimality of equidistributed sets on the unit circle for convex potentials. Recently, their result has been extended to more general settings [Farkas et al. 2018].

Proof of Proposition 5. By (6), finding the polarization constants is equivalent to maximizing the quantity $\sum |\tilde{u}_i - \tilde{v}|^p$.

First, we assume $0 < p \leq 1$. Let $g(t) = -|\sin(t/2)|^p + 1$. Then g is nonnegative, nonincreasing and strictly convex on $[0, 2\pi]$. Moreover,

$$\frac{1}{2} \sum |\tilde{v} - \tilde{u}_i|^p = - \sum g(\psi - \beta_i) + n,$$

where $\tilde{v} = e^{i\psi}$ and $\tilde{u}_i = e^{i\beta_i}$. Therefore, $\tilde{M}^p(\omega_n)$ is attained when $\sum g(\psi - \beta_i)$ is minimized. Theorem 1 of [Hardin et al. 2013] implies that \tilde{M}_n^p is achieved by equidistributed points sets; moreover, these are the only optimizers. Accordingly, the lines spanned by an optimal configuration for $M_n^p(S^1)$ are evenly spaced. It is easy to check [Stolarsky 1975a] that for such a configuration, the maximum of the potential function $U^p(\omega_n, \cdot)$ is attained at one of the base points for odd n , and at the midpoint between two consecutive base points for even n .

The case $p = 2, 4, \dots, 2n - 2$ is discussed in [Bosuwana and Ruengrot 2017, Theorem 2]. We also give a short proof here. It was shown in [Nikolov and Rafailov 2011] that for these values of p , $\tilde{U}^p(\omega_n^*, z)$ is constant on T . Therefore, for any n -point configuration ω_n on T ,

$$\tilde{M}^p(\omega_n) \geq \frac{1}{n} \sum_{z \in \omega_n^*} \tilde{U}^p(\omega_n, z) = \frac{1}{n} \sum_{v \in \omega_n} \tilde{U}^p(\omega_n^*, v) = \tilde{M}^p(\omega_n^*). \quad \square$$

For equally distributed point sets, it was proven by Stolarsky [1975a] and by Nikolov and Rafailov [2011] that $\tilde{M}^p(\omega_n^*) = \max_{z \in T} \sum_{k=0}^{n-1} |z - \xi^k|^p$ is (not necessarily uniquely) attained at z which is, depending on p , either one of the base points ξ^k or is the midpoint between two consecutive base points. More precisely, introduce the *positive-exponent Riesz energy* of $\omega_n \subset T$ defined by

$$E^p(\omega_n) = \sum_{j,k=1}^n |z_j - z_k|^p$$

(note that in the previous articles related to Riesz energies, the exponent is taken to be $-p$, therefore the above quantity becomes the negative exponent Riesz energy). For brevity, let $E_n^p = E^p(\omega_n^*)$. Theorem 1.2 of [Stolarsky 1975a] states that for $0 < p < 2n$, taking $m = \lfloor p/2 \rfloor$,

$$(8) \quad \tilde{M}^p(\omega_n^*) = \begin{cases} \frac{E_n^p}{n}, & m \text{ odd}, \\ \frac{E_{2n}^p}{2n} - \frac{E_n^p}{n}, & m \text{ even}. \end{cases}$$

Furthermore, for $p \geq 2n$, Theorem 2 of [Nikolov and Rafailov 2011] implies that

$$(9) \quad \tilde{M}^p(\omega_n^*) = \begin{cases} \frac{E_n^p}{n}, & n \text{ even}, \\ \frac{E_{2n}^p}{2n} - \frac{E_n^p}{n}, & n \text{ odd}. \end{cases}$$

The asymptotic expansion of E_n^p was given by Brauchart, Hardin and Saff [Brauchart et al. 2009]:

$$E_n^p = n^2 \tilde{\mu}_p + \mathcal{O}(n^{1-p}), \quad n \rightarrow \infty.$$

This, along with (7)–(9), and the fact $\tilde{M}_n^p \leq \tilde{M}^p(\omega_n^*)$, implies that $\tilde{M}_n^p \sim n \tilde{\mu}_p = n \binom{p}{p/2}$ as $n \rightarrow \infty$.

Proposition 5 shows that for integer exponents p with $0 < p < 2n$, $\tilde{M}_n^p = \tilde{M}^p(\omega_n^*)$. For these exponents, we provide the explicit value of E_n^p (and, by (8) and (9), of $\tilde{M}^p(\omega_n^*)$) by a combinatorial argument. If $p = 2m$ for some integer $m < n$,

$$\begin{aligned} E_n^p &= \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} |\xi^j - \xi^k|^{2m} = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} |1 - \xi^{k-j}|^p \\ &= n \sum_{j=0}^{n-1} |1 - \xi^j|^p = n \sum_{j=0}^{n-1} (1 - \xi^j)^m (1 - \xi^{-j})^m. \end{aligned}$$

Using binomial expansion gives

$$\begin{aligned} E_n^p &= n \sum_{j=0}^{n-1} \sum_{r,s=0}^m \binom{m}{r} \binom{m}{s} (-1)^{r+s} (\xi^j)^{r-s} \\ &= n \sum_{r,s=0}^m \binom{m}{r} \binom{m}{s} (-1)^{r+s} \sum_{j=0}^{n-1} (\xi^{r-s})^j \\ &= n^2 \sum_r \binom{m}{r}^2 = n^2 \binom{2m}{m} = n^2 \tilde{\mu}_p. \end{aligned}$$

On the other hand, assume that p is odd with $0 < p < 2n$. Noting that for $t \in [0, 2\pi)$,

$$|1 - e^{it}| = ie^{-it/2}(1 - e^{it}),$$

it follows that

$$\begin{aligned} E_n^p &= n \sum_{j=0}^{n-1} |1 - \xi^j|^p = n i^p \sum_{j=0}^{n-1} \xi^{-pj/2} (1 - \xi^j)^p \\ &= n i^p \sum_{j=0}^{n-1} \sum_{s=0}^p \binom{p}{s} (-1)^s \xi^{(s-p/2)j} = n i^p \sum_{s=0}^p \binom{p}{s} (-1)^s \sum_{j=0}^{n-1} \xi^{(s-p/2)j}. \end{aligned}$$

Now, using that p is odd, we have

$$\begin{aligned} E_n^p &= n i^p \sum_{s=0}^p \binom{p}{s} (-1)^s \frac{\xi^{n(s-p/2)} - 1}{\xi^{s-p/2} - 1} \\ &= 2n i^p \sum_{s=0}^p \binom{p}{s} \frac{(-1)^s}{1 - \xi^{s-p/2}}, \end{aligned}$$

since $\xi^{ns} = 1$ and $\xi^{-np/2} = -1$. Observing the symmetry of this sum about $p/2$, we can compute

$$\begin{aligned} E_n^p &= 4n i^{p+1} \operatorname{Im} \left(\sum_{s=0}^{\lfloor p/2 \rfloor} \binom{p}{s} \frac{(-1)^s}{1 - \xi^{s-p/2}} \right) \\ &= n (-1)^{(p-1)/2} \sum_{s=0}^p \binom{p}{s} (-1)^s \cot \left(\left(\frac{p}{2} - s \right) \frac{\pi}{n} \right). \end{aligned}$$

For $p = 1$, this gives $E_n^p = 2n \cot(\frac{\pi}{2n}) \sim \frac{4n^2}{\pi} = n^2 \tilde{\mu}_1$. In general, as $n \rightarrow \infty$, the Taylor expansion of the cotangent gives

$$E_n^p = n \left| \sum_{s=0}^p \binom{p}{s} (-1)^s \frac{n}{\pi(\frac{p}{2} - s)} + \mathcal{O}(1/n) \right| = n^2 \tilde{\mu}_p + \mathcal{O}(1),$$

where the second equality follows from a series computation described in Proposition 2.3 of [Garrappa 2007].

We conclude the paper by restating the following natural conjecture of Bosuwan and Ruengrot [2017], which is also supported by our numerical experiments:

Conjecture 2. *For any $n \geq 1$, the vector systems achieving \tilde{M}_n^p are equally distributed on the circle for every $p \in \mathbb{R}^+ \setminus \{2, 4, \dots, 2n - 2\}$.*

Acknowledgements

This research was done under the auspices of the Budapest Semesters in Mathematics program. The authors are grateful to I. Bárány for fruitful discussions and to A. Polyanskii for communicating the correct form of Conjecture 1 to us.

References

- [Ambrus 2009] G. Ambrus, *Analytic and probabilistic problems in discrete geometry*, Ph.D. thesis, University College London, 2009.
- [Ambrus 2014] G. Ambrus, “A short proof for the characterisation of tight frames”, preprint, 2014. [arXiv](#)
- [Ambrus et al. 2013] G. Ambrus, K. M. Ball, and T. Erdélyi, “Chebyshev constants for the unit circle”, *Bull. Lond. Math. Soc.* **45**:2 (2013), 236–248. [MR](#) [Zbl](#)

- [Ball 1997] K. Ball, “[An elementary introduction to modern convex geometry](#)”, pp. 1–58 in *Flavors of geometry*, edited by S. Levy, Math. Sci. Res. Inst. Publ. **31**, Cambridge Univ. Press, 1997. [MR](#) [Zbl](#)
- [Ball 2001] K. Ball, “[Convex geometry and functional analysis](#)”, pp. 161–194 in *Handbook of the geometry of Banach spaces*, vol. I, edited by W. B. Johnson and J. Lindenstrauss, North-Holland, Amsterdam, 2001. [MR](#) [Zbl](#)
- [Bang 1951] T. Bang, “[A solution of the “plank problem”](#)”, *Proc. Amer. Math. Soc.* **2** (1951), 990–993. [MR](#) [Zbl](#)
- [Bárány and Grinberg 1981] I. Bárány and V. S. Grinberg, “[On some combinatorial questions in finite-dimensional spaces](#)”, *Linear Algebra Appl.* **41** (1981), 1–9. [MR](#) [Zbl](#)
- [Benedetto and Fickus 2003] J. J. Benedetto and M. Fickus, “[Finite normalized tight frames](#)”, *Adv. Comput. Math.* **18**:2-4 (2003), 357–385. [MR](#) [Zbl](#)
- [Bosuwana and Ruengrot 2017] N. Bosuwana and P. Ruengrot, “[Constant Riesz potentials on a circle in a plane with an application to polarization optimality problems](#)”, *ScienceAsia* **43**:4 (2017), 267–274.
- [Brauchart et al. 2009] J. S. Brauchart, D. P. Hardin, and E. B. Saff, “[The Riesz energy of the \$N\$ th roots of unity: an asymptotic expansion for large \$N\$](#) ”, *Bull. Lond. Math. Soc.* **41**:4 (2009), 621–633. [MR](#) [Zbl](#)
- [Brugger et al. 2018] M. Brugger, M. Fiedler, B. González Merino, and A. Kirschbaum, “[Additive colourful Carathéodory type results with an application to radii](#)”, *Linear Algebra Appl.* **554** (2018), 342–357. [MR](#) [Zbl](#)
- [Dvoretzky 1963] A. Dvoretzky, “[Problem](#)”, pp. 496 in *Convexity* (Seattle, WA, 1961), edited by V. L. Klee, Proc. Sympos. Pure Math. **VII**, American Mathematical Society, Providence, RI, 1963. [MR](#) [Zbl](#)
- [Erdélyi and Saff 2013] T. Erdélyi and E. B. Saff, “[Riesz polarization inequalities in higher dimensions](#)”, *J. Approx. Theory* **171** (2013), 128–147. [MR](#) [Zbl](#)
- [Farkas et al. 2018] B. Farkas, B. Nagy, and S. G. Révész, “[A minimax problem for sums of translates on the torus](#)”, *Trans. London Math. Soc.* **5**:1 (2018), 1–46. [MR](#) [Zbl](#)
- [Garrappa 2007] R. Garrappa, “[Some formulas for sums of binomial coefficients and gamma functions](#)”, *Int. Math. Forum* **2**:13-16 (2007), 725–733. [MR](#) [Zbl](#)
- [Goyal et al. 2001] V. K. Goyal, J. Kovačević, and J. A. Kelner, “[Quantized frame expansions with erasures](#)”, *Appl. Comput. Harmon. Anal.* **10**:3 (2001), 203–233. [MR](#) [Zbl](#)
- [Hardin et al. 2013] D. P. Hardin, A. P. Kendall, and E. B. Saff, “[Polarization optimality of equally spaced points on the circle for discrete potentials](#)”, *Discrete Comput. Geom.* **50**:1 (2013), 236–243. [MR](#) [Zbl](#)
- [Hardin et al. 2019] D. P. Hardin, M. Petrache, and E. B. Saff, “[Unconstrained polarization \(Chebyshev\) problems: basic properties and Riesz kernel asymptotics](#)”, preprint, 2019. [arXiv](#)
- [Knuth 1997] D. E. Knuth, *The art of computer programming*, vol. 1: Fundamental algorithms, Addison-Wesley, Reading, MA, 1997. [MR](#) [Zbl](#)
- [Nikolov and Rafailov 2011] N. Nikolov and R. Rafailov, “[On the sum of powered distances to certain sets of points on the circle](#)”, *Pacific J. Math.* **253**:1 (2011), 157–168. [MR](#) [Zbl](#)
- [Nikolov and Rafailov 2013] N. Nikolov and R. Rafailov, “[On extremums of sums of powered distances to a finite set of points](#)”, *Geom. Dedicata* **167** (2013), 69–89. [MR](#) [Zbl](#)
- [Ohtsuka 1967] M. Ohtsuka, “[On various definitions of capacity and related notions](#)”, *Nagoya Math. J.* **30** (1967), 121–127. [MR](#) [Zbl](#)

[Polyanskii 2019] A. Polyanskii, private communication, 2019.

[Spencer 1977] J. Spencer, “Balancing games”, *J. Combinatorial Theory Ser. B* **23**:1 (1977), 68–74. [MR](#) [Zbl](#)

[Spencer 1987] J. Spencer, *Ten lectures on the probabilistic method*, CBMS-NSF Regional Conference Series in Applied Mathematics **52**, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1987. [MR](#) [Zbl](#)

[Stolarsky 1975a] K. B. Stolarsky, “The sum of the distances to certain pointsets on the unit circle”, *Pacific J. Math.* **59**:1 (1975), 241–251. [MR](#) [Zbl](#)

[Stolarsky 1975b] K. B. Stolarsky, “The sum of the distances to N points on a sphere”, *Pacific J. Math.* **57**:2 (1975), 563–573. [MR](#) [Zbl](#)

Received April 23, 2019. Revised June 19, 2019.

GERGELY AMBRUS

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS

HUNGARIAN ACADEMY OF SCIENCES

BUDAPEST

HUNGARY

ambrus.gergely@renyi.mta.hu

SLOAN NIETERT

DEPARTMENT OF COMPUTER SCIENCE

CORNELL UNIVERSITY

ITHACA, NY

UNITED STATES

nietert@cs.cornell.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2019 is US \$490/year for the electronic version, and \$665/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 303 No. 2 December 2019

Polarization, sign sequences and isotropic vector systems	385
GERGELY AMBRUS and SLOAN NIETERT	
L^p -operator algebras with approximate identities, I	401
DAVID P. BLECHER and N. CHRISTOPHER PHILLIPS	
The center of a Green biset functor	459
SERGE BOUC and NADIA ROMERO	
On the boundedness of multilinear fractional strong maximal operators with multiple weights	491
MINGMING CAO, QINGYING XUE and KÔZÔ YABUTA	
Embedding and compact embedding for weighted and abstract Sobolev spaces	519
SENG-KEE CHUA	
A pro- p group with infinite normal Hausdorff spectra	569
BENJAMIN KLOPSCH and ANITHA THILLAISUNDARAM	
Invariant connections and PBW theorem for Lie groupoid pairs	605
CAMILLE LAURENT-GENGOUX and YANNICK VOGLAIRE	
Random Möbius groups, I: Random subgroups of $\mathrm{PSL}(2, \mathbb{R})$	669
GAVEN MARTIN and GRAEME O'BRIEN	
Puzzles in K -homology of Grassmannians	703
PAVLO PYLYAVSKYY and JED YANG	
Linearly dependent powers of binary quadratic forms	729
BRUCE REZNICK	
Stability of the existence of a pseudo-Einstein contact form	757
YUYA TAKEUCHI	