L^p-OPERATOR ALGEBRAS WITH
APPROXIMATE IDENTITIES, I

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We initiate an investigation into how much the existing theory of (nonselfad-joint) operator algebras on a Hilbert space generalizes to algebras acting on L^p-spaces. In particular we investigate the applicability of the theory of real positivity, which has recently been useful in the study of L^2-operator algebras and Banach algebras, to algebras of bounded operators on L^p-spaces. In the process we answer some open questions on real positivity in Banach algebras from work of Blecher and Ozawa.

1. Introduction

In a series of recent papers (see, e.g., [Phillips 2012; 2013a; 2013b; 2014a; 2014b]) the second author has pointed out that the study of algebras of bounded operators on L^p-spaces, henceforth, L^p-*operator algebras*, has been somewhat overlooked, and has initiated the study of these objects. Subsequently others have followed him into this inquiry (for example, Gardella, Thiel, Lupini, and Viola; see, e.g., [Gardella and Thiel 2015a; 2015b; 2019; Gardella and Lupini 2017; Phillips and Viola 2017]). However, as he has frequently stated, these investigations have been very largely focused on examples; one still lacks an abstract general theory in this setting.

   Here and in a sequel in preparation we initiate an investigation into how much the existing theory of (nonselfadjoint) L^2-operator algebras (see, e.g., [Arveson 1969; Blecher and Le Merdy 2004; Blecher and Read 2011]) generalizes to the L^p case. We restrict ourselves almost exclusively to the "isometric theory"; we may pursue the isomorphic theory elsewhere. In addition to establishing some general facts about L^p-operator algebras, the main goal of the present paper is to investigate to what extent the first author's theory of real positivity (developed with Read, Neal, Ozawa, and others; see, e.g., [Blecher and Read 2011; 2013; 2014; Blecher and Ozawa 2015]), is applicable to L^p-operator algebras, particularly those which are approximately unital, that is, have contractive approximate identities. As an easy motivation, notice that the canonical approximate identity for the compact operators $\mathbb{K}(l^p)$ is real positive, and the real positive elements span $B(L^p([0, 1]))$ (as they do any unital Banach algebra).

   The theory of real positivity was developed as a tool for generalizing certain parts of C^*-algebra theory to more general algebras. In [Blecher and Ozawa 2015] this was extended to Banach algebras (see also [Blecher 2016] for a survey and some additional results). All this theory of course therefore applies to L^p-operator algebras. We refer to the first of these papers frequently, although most of our paper may be read without a deep familiarity with that paper.

   Some parts of [Blecher and Ozawa 2015] applied only to certain classes of Banach algebras defined there, which were shown to behave in some respects similarly to L^2-operator algebras. For example, a nonunital approximately unital Banach algebra A was defined there to be *scaled* if the set of restrictions to A of states on the multiplier unitization A^1 equals the quasistate space $Q(A)$ of A (that is, the set of $\lambda\varphi$ for $\lambda \in [0, 1]$ and φ a norm 1 functional on A that extends to a state on A^1). All unital Banach algebras are scaled. In [loc. cit.], there are several pretty equivalent conditions for a Banach algebra to be scaled (see the start of our Section 6 for some of these), and this class of Banach algebras was shown to have several nice theoretical features, such as a Kaplansky density type theorem. Thus it is natural to ask the following:

(1) To which of the classes defined in [Blecher and Ozawa 2015] do L^p-operator algebras belong?

(2) For those classes in [loc. cit.] to which they do not belong, to what extent do the theorems for those classes from that paper still hold for L^p-operator algebras?

(3) To what extent do other parts of the theory of L^2-operator algebras hold for L^p-operator algebras?

We focus mostly here on the parts of the theory of Blecher with Read, Neal, and others referred to above, that were not already extended to the general classes considered in [Blecher and Ozawa 2015]. For example, one may ask if the material in Section 4 of that paper, and in particular the theory of hereditary subalgebras, improves (that is, becomes closer to the L^2-operator case) for L^p-operator algebras. Similarly, one may ask about the *noncommutative topology* (in the sense of Akemann, Pedersen, L. G. Brown, and others) of L^p-operator algebras. In papers of Blecher with Read, Neal, and others referred to above, Akemann's noncommutative topology of C^*-algebras was fused with the classical theory of (generalized) peak sets of function algebras to create a *relative* noncommutative topology for closed subalgebras of C^*-algebras that has proved to have many applications. Examples given in [Blecher and Ozawa 2015] show that not much of this will extend to general Banach algebras, and it is natural to ask if L^p-operator algebras are better in this regard. Most of the present paper and the sequel in preparation is devoted to answering these questions. In the process we also answer some open questions from [loc. cit.].

We admit from the outset that for $p \neq 2$, and for some significant part of the theory, the answer to question (2) above is, so far, in the negative. This may change somewhat in the future, for example if we were able to solve some of the open problems listed at the end of this paper. It should also be admitted that for $p \neq 2$, the "projection lattice" of $B(L^p(X, \mu))$ is problematic from our perspective (see Example 3.2 and the sequel paper), in contrast to the projection lattices of von Neumann algebras and L^2-operator algebras.

Concerning question (1), several classes of Banach algebras introduced and considered in [Blecher and Ozawa 2015] coincide for approximately unital L^p-operator algebras. Indeed the classes of *scaled* and *M-approximately unital* Banach algebras defined in that paper coincide for L^p-operator algebras, and these also turn out to be the approximately unital L^p-operator algebras which satisfy the aforementioned Kaplansky density property. (We remark that the usual Kaplansky density theorem variants for C^*-algebras can be shown to follow easily from the weak* density of the subset of interest in A within the matching set in A^{**}. Our Kaplansky density theorems have the latter flavor.) We show that some approximately unital L^p-operator algebras are scaled and others are not. This answers the questions from [loc. cit.] as to whether every approximately unital Banach algebra is scaled, or has a Kaplansky density property. Also, nonscaled approximately unital L^p-operator algebras may contain no real positive elements (whereas it was shown in [loc. cit.]

that if they are scaled then there is an abundance of real positive elements, e.g., every element in A is a difference of two real positive elements).

Concerning question (3) above, indeed, some aspects of the theory improve. For example, Section 4 of [loc. cit.] improves drastically in our setting, and indeed L^p-operator algebras do support a basic theory of noncommutative topology and hereditary subalgebras, unlike general Banach algebras. This is worked out in the sequel paper in preparation, where the reader will find many more positive results than in the present paper. It is worth remarking that the methods used here do not seem to extend far beyond the class of L^p-operator algebras as we will discuss elsewhere. However, most of our results for L^p-operator algebras in Sections 2 and 4 do generalize to the class of SQ$_p$-*operator algebras*, by which we mean closed algebras of operators on an SQ$_p$-space, that is, a quotient of a subspace of an L^p-space. (See, e.g., [Le Merdy 1996] and [Junge 1996]. We thank Eusebio Gardella for suggesting SQ$_p$-spaces after we listed in a talk the properties needed for our results to work.)

On the other hand, except cosmetically, not much to speak of in Section 3 of [Blecher and Ozawa 2015] improves for L^p-operator algebras, in the sense of becoming significantly more like the L^2-operator algebra case. However several of the concepts appearing throughout the latter paper become much simpler in our setting. For example as we said above, three of the main classes of Banach algebras considered there coincide. Also as we shall see the subscript and superscript $\mathfrak{e}$ which appear often in that paper may be erased in our setting, since we are able to show that all L^p-operator algebras are Hahn–Banach smooth. Then of course the Arens regularity of L^p-operator algebras means that many irritating features of the bidual appearing in that paper disappear, such as mixed identities in A^{**}.

We now describe the contents of the present paper.

We will be assuming that $p \in (1, \infty) \setminus \{2\}$ in all results in the paper unless stated to the contrary. As usual $\frac{1}{p} + \frac{1}{q} = 1$. In the remainder of Section 1 we give some notation and basic definitions. In Section 2 we discuss further notation and background. We also collect a large number of useful general facts, many of which are well known. They concern topics such as duals, bidual algebras, the multiplier unitization, states and real positivity, hermitian elements, representations, etc. We just mention one sample result from this section: if A is an approximately unital L^p-operator algebra with $p \in (1, \infty)$, then there exists a measure space (X, μ) and a unital isometric representation $\theta : A^{**} \to B(L^p(X, \mu))$ which is a weak* homeomorphism onto its range, and such that $\theta(A)$ acts nondegenerately on $L^p(X, \mu)$.

In Section 3 we list the main examples of L^p-operator algebras which we use in this paper for counterexamples, as well as some other basic examples not in the literature. Some of these have real positive approximate identities, and others do not. We also expose some of the aforementioned bad properties of the "projection lattice" of $B(L^p(X, \mu))$.

Section 4 contains many miscellaneous results. Here is a sample of these. We show that the quotient of an L^p-operator algebra by an approximately unital closed ideal satisfying a simple extra condition is again (isometrically) an L^p-operator algebra. An example is presented to prove that this can fail if the ideal is only assumed to be closed and approximately unital. We show that an L^p-operator algebra A need not have a unique unitization, unlike in the case $p = 2$ (Meyer's unitization theorem). However there is a unique unitization if we restrict attention to nondegenerately represented approximately unital L^p-operator algebras. The nonuniqueness above is related to the fact that when $p \neq 2$ the Cayley transform can take a real positive element of A to an element of norm greater than 1. We study support idempotents of elements of A and their properties. We also give some important consequences of the strict convexity of L^p-spaces. For example, a state on a unital L^p-operator algebra that takes the value zero at a real positive idempotent e is zero on the left or right ideal generated by e. We also deduce that an L^p-operator algebra is Hahn–Banach smooth in its multiplier unitization. These results have several significant applications in this paper and its sequel. For example they yield in Section 4 several foundational properties of states and state extensions.

In Section 5 and Section 6 we discuss M-ideals and scaled Banach algebras. Our main result here is that in the setting of approximately unital L^p-operator algebras, the classes of scaled algebras and M-approximately unital algebras coincide. These are also the algebras which satisfy the aforementioned Kaplansky density property, as we show in Section 7. We will see for example that the algebra $\mathbb{K}(L^p(X, \mu))$ of compact operators is in this class if and only if μ is purely atomic (Proposition 5.2). The L^p-operator algebras with a hermitian contractive identity are also in this class. We also show for example in these sections that every M-ideal J in an approximately unital L^p-operator algebra A is an approximately unital closed ideal. Moreover, if in addition A is scaled then so is J (this follows from Theorem 5.4 (3)(a)).

At the end of the paper we provide an index listing some of the main definitions in this paper and where they may be found.

In the sequel paper in preparation we show that the theory of one-sided ideals, hereditary subalgebras, open projections, etc. for L^p-operator algebras is quite similar to the (nonselfadjoint) L^2-operator algebra case. This is particularly so for certain large classes of L^p-operator algebras. We feel that this is important, since hereditary subalgebras play a large role in modern C^*-algebra theory, and thus hopefully will be important for L^p-operator algebras too.

We end our introduction with a few definitions and basic lemmas.

We set $\mathbb{R}_+ = [0, \infty)$.

Notation 1.1. Let E be a normed vector space. Then $\mathrm{Ball}(E)$ is the closed unit ball of E, that is,

$$\mathrm{Ball}(E) = \{\xi \in E : \|\xi\| \leq 1\}.$$

Notation 1.2. Let $p \in [1, \infty]$. Let E and F be normed vector spaces. We denote by $E \oplus^p F$ their L^p direct sum, that is, the algebraic direct sum $E \oplus F$ with the norm given for $\xi \in E$ and $\eta \in F$ by $\|(\xi, \eta)\| = (\|\xi\|^p + \|\eta\|^p)^{1/p}$ if $p < \infty$ and $\|(\xi, \eta)\| = \max(\|\xi\|, \|\eta\|)$ if $p = \infty$.

Although many of our Banach algebras have identities of norm greater than 1, the adjectives "unital" or "approximately unital" for a Banach algebra will carry a norm 1 requirement.

Definition 1.3. A *unital* Banach algebra is a Banach algebra with an identity 1 such that $\|1\| = 1$.

Definition 1.4. A *cai* in a Banach algebra is a contractive approximate identity, that is, an approximate identity $(e_t)_{t \in \Lambda}$ such that $\|e_t\| \le 1$ for all $t \in \Lambda$. An *approximately unital* Banach algebra is a Banach algebra which has a cai.

When we write L^p or $L^p(X)$ we mean the L^p-space of some measure space (X, μ).

Definition 1.5. Recall that a Banach space E is *strictly convex* if whenever $\xi, \eta \in E \setminus \{0\}$ satisfy
$$\|\xi + \eta\| = \|\xi\| + \|\eta\|,$$
then there is $\lambda \in (0, \infty)$ such that $\xi = \lambda \eta$, and *smooth* if for given $\xi \in E$ with $\|\xi\| = 1$, there is a unique $\eta \in \text{Ball}(E^*)$ with $\langle \xi, \eta \rangle = 1$.

If $1 < p < \infty$, then $L^p(X)$ is strictly convex (by the converse to Minkowski's inequality). Moreover, still assuming $1 < p < \infty$, the space $L^p(X)$ is smooth, with η above given by the function
$$\eta(x) = \begin{cases} \overline{\xi(x)}|\xi(x)|^{p-2}, & \xi(x) \neq 0 \\ 0, & \xi(x) = 0 \end{cases}$$
in $L^q(X)$. We will frequently use the fact that $L^p(X)$ is smooth and strictly convex if $1 < p < \infty$.

Definition 1.6. Let $p \in [1, \infty)$. An L^p-*operator algebra* is a Banach algebra which is isometrically isomorphic to a norm closed subalgebra of the algebra of bounded operators on $L^p(X, \mu)$ for some measure space (X, μ). When $p = 2$ we simply refer to an *operator algebra*. (See the beginning of Section 2.1 of [Blecher and Le Merdy 2004], except that we do not consider matrix norms in the present paper.)

Definition 1.7. Let A be an L^p-operator algebra (not necessarily approximately unital). We say that an L^p-operator algebra B is an L^p-*operator unitization* of A if either A is unital and $B = A$, or if A is nonunital, B is unital (in particular, by our convention, $\|1\| = 1$), and A is a codimension one ideal in B.

Definition 1.8 [Blecher and Le Merdy 2004, (A.9) on p. 364]. Let A be a nonunital approximately unital Banach algebra (as in Definition 1.4). We define its *multiplier unitization* A^1 to be the usual unitization $A + \mathbb{C} \cdot 1$ with the norm

$$\|a + \lambda 1\|_{A^1} = \sup(\{\|ac + \lambda c\| : c \in \mathrm{Ball}(A)\})$$

for $a \in A$ and $\lambda \in \mathbb{C}$. If A is already unital then we set $A^1 = A$.

Remark 1.9. We recall the following easy standard facts.

(1) If A is an approximately unital Banach algebra, then the standard inclusion of A in A^1 is isometric.

(2) Let A be a Banach algebra, and let $(e_t)_{t \in \Lambda}$ be any cai in A. Then

$$\|a + \lambda 1\|_{A^1} = \lim_t \|ae_t + \lambda e_t\| = \sup_t \|ae_t + \lambda e_t\|.$$

(3) If A is any nonunital Banach algebra, and B is a unital Banach algebra which contains A as a codimension 1 subalgebra, then the map $\chi_0 : B \to \mathbb{C}$, given by $\chi_0(a + \lambda 1_B) = \lambda$ for $a \in A$ and $\lambda \in \mathbb{C}$, is contractive.

(4) If A is any nonunital Banach algebra with a cai, and B is a unital Banach algebra which contains A as a codimension 1 subalgebra, then the map $\psi : B \to A^1$, given by $\psi(a + \lambda 1_B) = a + \lambda 1_{A^1}$ for $a \in A$ and $\lambda \in \mathbb{C}$, is a contractive homomorphism. Thus A^1 has the smallest norm of any unitization. This follows, e.g., by a small variant of the proof of Lemma 1.10.

Lemma 1.10. *Suppose that A is a closed subalgebra of a nonunital approximately unital Banach algebra B, and suppose that A has a cai but is not unital. Then for all $a \in A$ and $\lambda \in \mathbb{C}$ we have $\|a + \lambda 1\|_{A^1} \le \|a + \lambda 1\|_{B^1}$.*

Proof. Clearly

$$\sup(\{\|ac + \lambda c\| : c \in \mathrm{Ball}(A)\}) \le \sup(\{\|ac + \lambda c\| : c \in \mathrm{Ball}(B)\}),$$

as desired. □

It is easy to find examples showing that the homomorphism above need not be isometric, for example, with notation as in Example 3.2 (or Example 3.5) below, $\mathbb{C} \, e_2 \otimes c_0 \subseteq M_2^p \otimes c_0$. However we have the following result.

Lemma 1.11. *Let A and B be nonunital approximately unital Banach algebras. Let $\varphi : A \to B$ be a contractive (resp. isometric) homomorphism. Suppose that there is a cai $(e_t)_{t \in \Lambda}$ for A such that $(\varphi(e_t))_{t \in \Lambda}$ is a cai for B. Then the obvious unital homomorphism $A^1 \to B^1$ between the multiplier unitizations is contractive (resp. isometric).*

Proof. If $a \in A$ and $\lambda \in \mathbb{C}$ then

$$\|\varphi(a)\varphi(e_t) + \lambda \varphi(e_t)\| \le \|ae_t + \lambda e_t\|.$$

In the isometric case this is an equality. Taking limits over t and using Remark 1.9 (2) gives the result.   □

We recall two further standard facts. The first is that the relation $\mathbb{K}(L^2(X))^{**} = B(L^2(X))$ is true with 2 replaced by any $p \in (1, \infty)$.

Definition 1.12. We recall that the bidual A^{**} of a Banach algebra has in general two canonical products, called the left and right Arens products [Palmer 1994, Definition 1.4.1]. We say that A is *Arens regular* if these two products coincide.

Theorem 1.13. *Let $p \in (1, \infty)$ and let (X, μ) be a measure space. Let $q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

(1) *There is an isometric isomorphism $\mathbb{K}(L^p(X, \mu))^* \to L^q(X, \mu) \widehat{\otimes} L^p(X, \mu)$ (projective tensor product) which for $\rho \in L^p(X, \mu)$ and $\eta \in L^q(X, \mu)$ sends $\eta \otimes \rho$ to the operator $\xi \mapsto \langle \xi, \eta \rangle \rho$.*

(2) *There is an isometric algebra isomorphism from $\mathbb{K}(L^p(X, \mu))^{**}$ (with either Arens product) onto $B(L^p(X, \mu))$ which extends the inclusion $\mathbb{K}(L^p(X, \mu)) \subseteq B(L^p(X, \mu))$.*

Proof. This follows from results of Grothendieck, as described in the theorem on page 828 of [Palmer 1985], the discussion after that, and Theorems 1–3 there. It is stated there that any Banach space X such that X and X^* have the Radon–Nikodym property and the approximation property, satisfies [Palmer 1985, Theorem 1] and the aforementioned theorem of Grothendieck, giving (1), and also the case of (2) for the first Arens product. By [Palmer 1985, Theorem 2], if X is also reflexive then $\mathbb{K}(X)$ is Arens regular, so (2) holds as stated. See also the discussion on page 24, Corollary 4.13, and Theorem 5.33 of [Ryan 2002] (and we thank M. Mazowita for this reference). The explicit formulas there are useful to check directly the Arens product assertion. One needs to know that $L^p(X, \mu)$ has the Radon–Nikodym property and the approximation property, and this follows, e.g., from [Ryan 2002, Example 4.5 and Corollary 5.45].   □

We remark that the last result and proof works with L^p replaced by any reflexive space with the approximation property, since reflexive spaces have the Radon–Nikodym property, and indeed [Ryan 2002, Corollary 4.7] implies that if E is reflexive and has the approximation property, then so does E^*.

By Theorem 1.13, a net $(x_t)_{t \in \Lambda}$ in $B(L^p(X))$ converges weak* to x if and only if, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum_{k=1}^{\infty} \langle x_t \xi_k, \eta_k \rangle \to \sum_{k=1}^{\infty} \langle x \xi_k, \eta_k \rangle$$

for all $\xi_1, \xi_2, \ldots \in L^p(X)$ and $\eta_1, \eta_2, \ldots \in L^q(X)$ with $\sum_{k=1}^{\infty} \|\xi_k\|_p \|\eta_k\|_q < \infty$ (or equivalently, by the usual trick, with $\sum_{k=1}^{\infty} \|\xi_k\|_p^p < \infty$ and $\sum_{k=1}^{\infty} \|\eta_k\|_q^q < \infty$).

If $(x_t)_{t \in \Lambda}$ is bounded then by Banach duality principles this is equivalent to $x_t \to x$ in the weak operator topology, that is, $\langle x_t \xi, \eta \rangle \to \langle x\xi, \eta \rangle$ for all $\xi \in L^p(X)$ and $\eta \in L^q(X)$. We will not use this here but it is well known that essentially the usual L^2-operator proof shows that the weak operator closure of a convex set in $B(L^p([0,1]))$ equals the strong operator closure. Indeed, for a Banach space E, the strong operator continuous linear functionals on $B(E)$ are the same as those that are weak operator continuous.

The argument for the following well known lemma will be reused several times, once in the form of an approximate identity bounded by M converging weak* to an identity in A^{**} of norm at most M.

Lemma 1.14. *Let A be an approximately unital Arens regular Banach algebra. Then A^{**} has an identity $1_{A^{**}}$ of norm 1, and any cai for A converges weak* to $1_{A^{**}}$.*

Proof. The argument follows the proof of [Blecher and Le Merdy 2004, Proposition 2.5.8]. Since identities are unique if they exist, it suffices to show that every subnet of any cai in A has in turn a subnet which converges to an identity for A^{**}. Using Alaoglu's theorem and since a subnet of a cai is a cai, one sees that it is enough to show that if $e \in A^{**}$ is the weak* limit of a cai, then e is an identity for A^{**}. Multiplication on A^{**} is separately weak* continuous by [Blecher and Le Merdy 2004, 2.5.3], so $ea = ae = a$ for all $a \in A$. A second application of separate weak* continuity of multiplication shows that this is true for all $a \in A^{**}$. $\square$

2. Notation, background, and general facts

2A. *Dual and bidual algebras.*

Lemma 2.1. *Let $p \in (1, \infty)$. Let A be an L^p-operator algebra (resp. SQ_p-operator algebra). Then*:

(1) *A is Arens regular.*

(2) *Multiplication on A^{**} is separately weak* continuous.*

(3) *A^{**} is an L^p-operator algebra (resp. SQ_p-operator algebra).*

Proof. We first recall (Theorem 3.3 (ii) of [Heinrich 1980], or [Le Merdy 1996], or the remarks above Theorem 4.1 in [Daws 2010]) that any ultrapower of L^p-spaces (resp. SQ_p-spaces) is again an L^p-space (resp. SQ_p-space). In the SQ_p-space case this uses the well known fact that ultrapowers behave well with respect to subspaces and quotients (this is obvious for subspaces, for quotients see, e.g., the proof of Proposition 6.5 in [Heinrich 1980]). In particular, such an ultrapower is reflexive, so every L^p-space (resp. SQ_p-space) is superreflexive. (See Proposition 1 of [Daws 2004].)

Now let E be an L^p-space (resp. SQ_p-space). Theorem 1 of [Daws 2004] implies that $B(E)$ is Arens regular. The proof of Theorem 1 of [Daws 2004] embeds

$B(E)^{**}$ isometrically as a subalgebra of $B(F)$ for a Banach space F obtained as an ultrapower of $l^r(E)$ for an arbitrarily chosen $r \in (1, \infty)$ (called p in [Daws 2004]). We may choose $r = p$. Then $l^r(E)$ is isometrically isomorphic to an L^p-space (resp. SQ_p-space). Since ultrapowers of L^p-spaces (resp. SQ_p-spaces) are L^p-spaces (resp. SQ_p-spaces) as we said at the start of this proof, we have shown that $B(E)^{**}$ is an L^p- (resp. SQ_p-) operator algebra.

Now suppose that $A \subseteq B(E)$ is a norm closed subalgebra. Since $B(E)$ is Arens regular, A^{**} is a subalgebra of $B(E)^{**}$ and A is Arens regular by [Blecher and Le Merdy 2004, 2.5.2]. It is now immediate that A^{**} is an L^p- (resp. SQ_p-) operator algebra. It also follows from [Blecher and Le Merdy 2004, 2.5.3] that multiplication on A^{**} is separately weak* continuous.   □

It follows from [Daws 2004, Proposition 8] that $B(L^1(X, \mu))$ is not Arens regular unless $L^1(X, \mu)$ is finite-dimensional.

Corollary 2.2. *Let $p \in (1, \infty)$ and let (X, μ) be a measure space. Then multiplication on $B(L^p(X, \mu))$ is separately weak* continuous.*

Proof. We have $\mathbb{K}(L^p(X, \mu))^{**} \cong B(L^p(X, \mu))$ by Theorem 1.13 (2).   □

Definition 2.3. Let $p \in (1, \infty)$. A *dual L^p-operator algebra* is a Banach algebra A with a predual such that there is a measure space (X, μ) and an isometric and weak* homeomorphic isomorphism from A to a weak* closed subalgebra of $B(L^p(X, \mu))$.

By Corollary 2.2, the multiplication on a dual L^p-operator algebra is separately weak* continuous.

Corollary 2.4. *Let $p \in (1, \infty)$ and let A be an L^p-operator algebra. Then A^{**} is a dual L^p-operator algebra.*

Proof. The embedding of $B(L^p(X, \mu))^{**}$ in Lemma 2.1 coming from the proof from [Daws 2004] is easily checked to be weak* continuous, hence a weak* homeomorphism onto its range by the Krein–Smulian theorem. Hence $B(L^p(X, \mu))^{**}$ is a dual L^p-operator algebra. It easily follows that A^{**} is too.   □

Lemma 2.5. *Let $p \in (1, \infty)$ and let A be a dual L^p-operator algebra. Then*:

(1) *The weak* closure of any subalgebra of A is a dual L^p-operator algebra.*

(2) *If A is approximately unital then A is unital.*

Proof. The proofs are essentially the same as in the case $p = 2$, as done in the proof of Proposition 2.7.4 of [Blecher and Le Merdy 2004].   □

2B. *States, hermitian elements, and real positivity.* We take states to be as at the beginning of Section 2 of [Blecher and Ozawa 2015].

Definition 2.6. If A is a unital Banach algebra, then a *state* on A is a linear functional $\omega : A \to \mathbb{C}$ such that $\|\omega\| = \omega(1) = 1$. If A is an approximately unital Banach algebra, we define a *state* on A to be a linear functional $\omega : A \to \mathbb{C}$ such that $\|\omega\| = 1$ and ω is the restriction to A of a state on the multiplier unitization A^1 (Definition 1.8).

We denote by $S(A)$ the set of all states on A, and write $Q(A)$ for the quasistate space (that is, the set of $\lambda\varphi$ for $\lambda \in [0, 1]$ and $\varphi \in S(A)$).

If $\mathfrak{e} = (e_t)_{t \in \Lambda}$ is a cai for A, define

$$S_{\mathfrak{e}}(A) = \{\omega \in \text{Ball}(A^*) : \omega(e_t) \to 1\}$$

and define

$$Q_{\mathfrak{e}}(A) = \{\lambda\varphi : \lambda \in [0, 1] \text{ and } \varphi \in S_{\mathfrak{e}}(A)\}.$$

If A is a C^*-algebra (unital or not), this definition gives the usual states and quasistates on A.

The first part of the following definition is Definition 2.6.1 of [Palmer 1994].

Definition 2.7. Let A be a unital Banach algebra, and let $a \in A$. We define the *numerical range* of a to be $\{\varphi(a) : \varphi \in S(A)\}$.

If E is a Banach space and $a \in B(E)$, we define the *spatial numerical range of a* to be

$$\{\langle a\xi, \eta \rangle : \xi \in \text{Ball}(E) \text{ and } \eta \in \text{Ball}(E^*) \text{ with } \langle \xi, \eta \rangle = 1\}.$$

There are other definitions of the numerical range. For our purposes, only the convex hull is important, and by Theorem 14 of [Lumer 1961], the convex hulls of the numerical range and the spatial numerical range of an element in $B(E)$ are always the same.

Definition 2.8 (see Definition 2.6.5 of [Palmer 1994] and the preceding discussion). Let A be a unital Banach algebra, and let $a \in A$. We say that a is *hermitian* if $\|\exp(i\lambda a)\| = 1$ for all $\lambda \in \mathbb{R}$.

If A is approximately unital we define the hermitian elements of A to be the elements in A which are hermitian in the multiplier unitization A^1 (Definition 1.8).

Lemma 2.9 (see [Palmer 1994, Theorem 2.6.7]). *Let A be a unital Banach algebra, and let $a \in A$. Then a is hermitian if and only if $\varphi(a) \in \mathbb{R}$ for all states φ of A.*

Lemma 2.10. *Let A be an approximately unital Banach algebra, and let $B \subseteq A$ be a closed subalgebra which contains a cai for A. Let $a \in B$. Then a is hermitian as an element of B if and only if a is hermitian as an element of A.*

Proof. By definition, we work in the multiplier unitizations. By Lemma 1.11, B^1 is isometrically a unital subalgebra of A^1. The Hahn–Banach theorem now shows that states on B^1 are exactly the restrictions of states on A^1. So the conclusion follows from Lemma 2.9. $\qquad\square$

Definition 2.11. Let (X, μ) be a measure space that is not σ-finite. Recall that a function $f : X \to \mathbb{C}$ is locally measurable if $f^{-1}(E) \cap F$ is measurable for all Borel sets $E \subseteq \mathbb{C}$ and all subsets $F \subseteq X$ of finite measure. Two such functions are "locally a.e. equal" if they agree a.e. on any such set F. We interpret $L^\infty(X, \mu)$ as $L^\infty_{\text{loc}}(X, \mu)$, the Banach space of essentially bounded locally measurable scalar functions mod local a.e. equality.

Further recall that a measure space (X, μ) is decomposable if X may be partitioned into sets X_i of finite measure for $i \in I$ such that a set F in X is measurable if and only if $F \cap X_i$ is measurable for every $i \in I$, and then $\mu(F) = \sum_{i \in I} \mu(F \cap X_i)$.

By, e.g., the corollary on page 136 in [Lacey 1974], any abstract L^p-space "is" decomposable, indeed it is isometric to a direct sum of L^p-spaces of finite measures. Thus, we may assume that all measure spaces (X, μ) are decomposable.

The following result is in the literature with extra hypotheses, such as if μ is σ-finite [Gardella and Thiel 2019, Lemma 5.2]. (See also, e.g., Theorem 4 and the remark following it in [Tam 1969], when in addition p is not an even integer.) We are not aware of a reference for the general case, but it is probably folklore.

Proposition 2.12. *Let $p \in [1, \infty) \setminus \{2\}$. Let (X, μ) be a decomposable measure space, and let $a \in B(L^p(X, \mu))$ be hermitian. Then there is a real-valued function $f \in L^\infty(X, \mu)$ such that a is multiplication by f, and such that $|f(x)| \leq \|a\|$ for all $x \in X$.*

Proof. Let $X = \coprod_{i \in I} X_i$ be a partition of X into sets of finite measure as in the discussion of decomposability above. For $i \in I$ let $e_i \in B(L^p(X, \mu))$ be multiplication by χ_{X_i}. Since hermitian elements have numerical range contained in $\mathbb{R}$, we can apply Theorem 6 of [Payá-Albert 1982] (see the beginning of [Payá-Albert 1982] for the definitions and notation), to see that a commutes with e_i for all $i \in I$. One easily checks that $h = e_i a e_i$ is a hermitian element of $B(L^p(X_i, \mu))$. By the finite measure case of our result ([Gardella and Thiel 2019, Lemma 5.2]), there is a real-valued function $f_i \in L^\infty(X_i, \mu)$ such that h is multiplication by f_i.

We can clearly assume that f_i is bounded by $\|e_i a e_i\| \leq \|a\|$. Now define $f : X \to \mathbb{R}$ by $f(x) = f_i(x)$ when $i \in I$ and $x \in X_i$. Then f is bounded by $\|a\|$, and is measurable by the choice of the partition of X. For $i \in I$ and $\xi \in L^p(X_i, \mu)$ we clearly have $a\xi = f\xi$. It follows from density of the linear span of the subspaces $L^p(X_i, \mu)$ that a is multiplication by f.   □

The σ-finite case is deduced in [Gardella and Thiel 2019] from the finite measure case of Lamperti's theorem [Fleming and Jamison 2003, Theorem 3.2.5] by considering the invertible isometries e^{ith} for $t \in [0, 1]$. We mention another approach when p is not an even integer. It is known ([Delbaen et al. 1998, Corollary 1.8]; we thank Gideon Schechtman for this reference) that l^p doesn't contain a two-dimensional

Hilbert space, and so Theorem 4 of [Tam 1969] implies our conclusion. Lemma 11 of [Tam 1969] also proves the result in the case that μ has no atomic part in X_i.

Definition 2.13. Let A be a unital Banach algebra. Let $a \in A$. We say that a is *accretive* or *real positive* if the numerical range of a is contained in the closed right half plane. That is, $\mathrm{Re}(\varphi(a)) \geq 0$ for all states φ of A.

If instead A is approximately unital, we define the real positive elements of A to be the elements in A which are real positive in the multiplier unitization A^1.

In both cases, we denote the set of real positive elements of A by $\mathfrak{r}_A$.

Following page 8 of [Blecher and Ozawa 2015], we further define

$$\mathfrak{c}_{A^*} = \{\varphi \in A^* : \mathrm{Re}(\varphi(a)) \geq 0 \text{ for all } a \in \mathfrak{r}_A\}.$$

The elements of $\mathfrak{c}_{A^*}$ are called *real positive functionals* on A.

For other equivalent conditions for real positivity, see for example [Blecher 2016, Lemma 2.4 and Proposition 6.6].

We warn the reader that $\mathfrak{r}_{A^{**}}$ is defined after Lemma 2.5 of [Blecher and Ozawa 2015] to be a proper subset of the real positive elements in A^{**}, the set of elements of A^{**} which are real positive with respect to $(A^1)^{**}$. One should be careful with this ambiguity; fortunately it only pertains to second duals and seldom arises. (Also see Proposition 4.26.)

Lemma 2.14. *Let A be an approximately unital Banach algebra, and let $B \subseteq A$ be a closed subalgebra which contains a cai for A. Let $a \in B$. Then a is real positive as an element of B if and only if a is real positive as an element of A.*

Proof. The proof is the same as that of Lemma 2.10, using Definition 2.13 in place of Lemma 2.9. □

Lemma 2.15. *Let $p \in [1, \infty) \setminus \{2\}$, let A be an approximately unital L^p-operator algebra, and assume that the multiplier unitization A^1 is again an L^p-operator algebra. Let $a \in A$ be hermitian. Then there exist $b, c \in A$, each of which is both hermitian and real positive, such that*

(2-1) $a = b - c, \quad bc = cb = 0, \quad \|b\| \leq \|a\|, \quad and \quad \|c\| \leq \|a\|.$

By Lemma 2.24, the hypothesis that A^1 be an L^p-operator algebra is automatic for $p \neq 1$.

It seems unlikely that Lemma 2.15 holds for a general Banach algebra.

Proof of Lemma 2.15. We may assume (using, e.g., the corollary on page 136 in [Lacey 1974]) that (X, μ) is a decomposable measure space and A^1 is a unital subalgebra of $B(L^p(X, \mu))$. Since a is hermitian in A^1, Lemma 2.10 implies that a is hermitian in $B(L^p(X, \mu))$. Proposition 2.12 provides $f \in L^\infty(X, \mu)$ such that a is multiplication by f, and such that $|f(x)| \leq \|a\|$ for all $x \in X$.

Choose a sequence $(r_n)_{n \in \mathbb{N}}$ of polynomials with real coefficients such that $r_n(\lambda) \to \lambda^{1/4}$ uniformly on $[0, \|a\|^2]$. Adjusting by constants and scaling, we may assume that $r_n(0) = 0$ and $|r_n(\lambda)| \leq \|a\|^{1/2}$ for $\lambda \in [0, \|a\|^2]$. Set $s_n(\lambda) = r_n(\lambda^2)^2$ for $\lambda \in [-\|a\|, \|a\|]$. Then $(s_n)_{n \in \mathbb{N}}$ is a sequence of polynomials with real coefficients such that $r_n(\lambda) \to |\lambda|$ uniformly on $[-\|a\|, \|a\|]$. Moreover, for all $n \in \mathbb{N}$ we have $s_n(0) = 0$ and $0 \leq s_n(\lambda) \leq \|a\|$ for all $\lambda \in [-\|a\|, \|a\|]$. In particular, $s_n \circ f \to |f|$ uniformly on X.

For $n \in \mathbb{N}$, define $d_n = s_n(a)$, which is the multiplication operator by the function $s_n \circ f$, and let d be the multiplication operator by $|f|$. Then $d_n \in A$ for all $n \in \mathbb{N}$ and $\|d_n - d\| \to 0$, so $d \in A$ and $\|d\| \leq \|a\|$. Therefore also

$$b = \tfrac{1}{2}(d + a) \quad \text{and} \quad c = \tfrac{1}{2}(d - a)$$

are in A. The conditions (2-1) are clearly satisfied.

The multiplication operator map from $L^\infty(X, \mu)$ to $B(L^p(X, \mu))$ is an isometric unital homomorphism. (Recall the convention that we are using $L_{\text{loc}}^\infty(X, \mu)$ here.) The functions $\tfrac{1}{2}(|f| + f)$ and $\tfrac{1}{2}(|f| - f)$ are nonnegative, hence both hermitian and real positive in $L^\infty(X, \mu)$ (because $L^\infty(X, \mu)$ is a C^*-algebra). Lemmas 2.10 and 2.14 therefore imply that their multiplication operators b and c are both hermitian and real positive in $B(L^p(X, \mu))$. A second application of these lemmas shows that the same holds in A^1. By definition, this is also true in A. □

Definition 2.16. Let A be a unital or approximately unital Banach algebra. Taking 1 to be the identity of A^1 in the approximately unital case, we define

$$\mathfrak{F}_A = \{a \in A : \|1 - a\| \leq 1\}.$$

Proposition 2.17 [Blecher and Ozawa 2015, Proposition 3.5]. *Let A be a unital or approximately unital Banach algebra. Then, in the notation of Definitions 2.13 and 2.16, we have $\mathfrak{r}_A = \overline{\mathbb{R}_+ \mathfrak{F}_A}$.*

We recall some facts about roots of elements of $\mathfrak{r}_A$.

Definition 2.18. Let A be a unital or approximately unital Banach algebra, let $b \in \mathfrak{r}_A$, and let $t \in (0, 1)$. If A is unital, we denote by b^t the element b_t constructed in [Li et al. 2003, Theorem 1.2]. If A is approximately unital, let A^1 be the multiplier unitization, recall that $b \in \mathfrak{r}_{A^1}$ by definition, and define b^t to be as above but evaluated in A^1.

The conditions required in [Li et al. 2003, Theorem 1.2] for the existence of b^t are weaker than what we require in Definition 2.18, but the case in Definition 2.18, is all we need. Such noninteger powers, for the special case $\|b - 1\| < 1$ and when A is commutative, seem to have first appeared in Definition 2.3 of [Esterle 1978]. A discussion relating this definition to others, and giving a number of properties, is contained in [Blecher and Ozawa 2015], from Proposition 3.3 through Lemma 3.8 there. In particular, $(b^{1/n})^n = b$ and $t \mapsto b^t$ is continuous. For later use, we recall

several of these properties and state a few other facts not given explicitly in [Blecher and Ozawa 2015].

Proposition 2.19. *Let A be a unital or approximately unital Banach algebra, and let* $a \in \mathfrak{r}_A$.

(1) *If* $t \in (0, 1)$ *and* $\|b - 1\| \leq 1$ *(that is,* $b \in \mathfrak{F}_A$*), then*

$$b^t = 1 + \sum_{k=1}^{\infty} \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!}(-1)^k(1-b)^k,$$

with absolute convergence.

(2) *If* $t \in (0, 1)$ *and* $\lambda \in (0, \infty)$ *then* $(\lambda x)^t = \lambda^t x^t$.

(3) *For all* $t \in (0, 1)$, $\|a^t\| \leq 2\|a\|^t/(1 - t)$.

(4) *For all* $t \in (0, 1)$, a^t *is a norm limit of polynomials in* a *with no constant term.*

(5) *For all* $t \in (0, 1)$, $a^t a = a a^t$.

(6) $\lim_{n \to 0} \|a^{1/n} a - a\| = \lim_{n \to 0} \|a a^{1/n} - a\| = 0$.

(7) *If* $a \in \mathfrak{F}_A$ *and* $t \in (0, 1)$, *then* $\|1 - a^t\| \leq 1$.

Proof. For part (1), see the proof of [Blecher and Ozawa 2015, Proposition 3.3] and the discussion in and before the remark before [Blecher and Ozawa 2015, Lemma 3.6].

For (2), see the discussion after [Blecher and Ozawa 2015, Proposition 3.5].

Part (3) is a slight weakening of the second estimate in Lemma 3.6 of [Blecher and Ozawa 2015].

Part (4) holds for $a \in \mathfrak{F}_A$ by the proof of Proposition 3.3 of [Blecher and Ozawa 2015]. By (2), it holds for $a \in \mathbb{R}_+ \mathfrak{F}_A$. By continuity (Corollary 1.3 of [Li et al. 2003]), it holds for $a \in \overline{\mathbb{R}_+ \mathfrak{F}_A}$. Apply Proposition 2.17.

Part (5) is immediate from Part (4). Part (6) is Lemma 3.7 of [Blecher and Ozawa 2015].

For (7), use (1), together with

$$\frac{t(t-1)(t-2)\cdots(t-k+1)}{n!}(-1)^k < 0$$

for $k = 1, 2, \ldots$ and

$$\sum_{k=1}^{\infty} \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!}(-1)^k = -1.$$

This completes the proof. □

Lemma 2.20. *Suppose that A is a closed subalgebra of an approximately unital Banach algebra B, and suppose that A has a cai. Then* $\mathfrak{F}_B \cap A \subseteq \mathfrak{F}_A$ *and* $\mathfrak{r}_B \cap A \subset \mathfrak{r}_A$.

Proof. The first statement follows easily from Lemma 1.10. The second follows from the first and the relations $\mathfrak{r}_A = \overline{\mathbb{R}_+ \, \mathfrak{F}_A}$ and $\mathfrak{r}_B = \overline{\mathbb{R}_+ \, \mathfrak{F}_B}$ (Proposition 2.17). □

Proposition 2.21. *Let B be a nonunital approximately unital Banach algebra, and let $A \subseteq B$ be a closed subalgebra which contains a cai for B. Then*:

(1) $A^1 \subseteq B^1$ *isometrically.*

(2) $\mathfrak{F}_A = \mathfrak{F}_B \cap A$ *and* $\mathfrak{r}_A = \mathfrak{r}_B \cap A$.

(3) *Every state or quasistate on A may be extended to a state or quasistate on B.*

Proof. Part (1) is Lemma 1.11. That $\mathfrak{F}_A = \mathfrak{F}_B \cap A$ is immediate from (1), and now $\mathfrak{r}_A = \mathfrak{r}_B \cap A$ by, e.g., Proposition 2.17. Part (3) is obvious from (1), Definition 2.6, and the Hahn–Banach theorem. □

Lemma 2.22. *Suppose that an Arens regular Banach algebra A has a cai and also has a real positive approximate identity. Then A has a cai in $\mathfrak{F}_A$. If in addition A has a countable bounded approximate identity, then A has a cai in $\mathfrak{F}_A$ which is a sequence.*

Proof. Corollary 3.9 of [Blecher and Ozawa 2015] implies that A has an approximate identity in $\mathfrak{F}_A$. Since $\mathfrak{F}_A$ is bounded, one may then use the argument in the second paragraph of the proof of [Blecher 2016, Proposition 6.13] to see that A has a cai $(e_t)_{t \in \Lambda}$ in $\mathfrak{F}_A$. If in addition A has a countable bounded approximate identity, then one can use Corollary 32.24 of [Hewitt and Ross 1970] and its analog on the right (see also Theorem 4.4 of [Blecher and Ozawa 2015]) to find $x, y \in A$ with $A = \overline{xA} = \overline{Ay}$. Choose $t_1, t_2, \ldots \in \Lambda$ with $t_1 < t_2 < \cdots$ and $\|f_{t_k}x - x\| + \|y f_{t_k} - y\| < 2^{-k}$; then (f_{t_k}) is a countable cai in $\mathfrak{F}_A$. □

Corollary 2.23. *Suppose that A is an approximately unital Arens regular Banach algebra. If $1_{A^{**}}$ is a weak* limit of a bounded net of real positive elements in A, then A has a real positive cai.*

Proof. By a standard convexity argument, or, e.g., [Blecher and Ozawa 2015, Lemma 2.1], A has a real positive bounded approximate identity. It follows from Lemma 2.22 that A has a cai in $\mathfrak{F}_A$. □

The hypothesis in the last result about $1_{A^{**}}$ being a weak* limit holds if A has one of the Kaplansky density type properties, e.g., properties (1)–(3) in Proposition 7.1. See also the proof of Proposition 6.4 of [Blecher and Ozawa 2015].

2C. *More on the multiplier unitization.* The multiplier unitization was defined in Definition 1.8.

Lemma 2.24. *Let E be a Banach space. Suppose that A is a nonunital closed approximately unital subalgebra of $B(E)$ which acts nondegenerately on E. Then the multiplier unitization of A is isometrically isomorphic to $A + \mathbb{C} 1_E$, where 1_E is the identity operator on E.*

Proof. For $a, c \in A$ and $\lambda \in \mathbb{C}$, we clearly have

$$\|ac + \lambda c\| = \|(a + \lambda 1_E)c\| \leq \|a + \lambda 1_E\| \|c\|.$$

So $\|a + \lambda 1\|_{A^1} \leq \|a + \lambda 1_E\|$. The reverse inequality follows from the fact that if $(e_t)_{t \in \Lambda}$ is a cai for A, then $ae_t + \lambda e_t \to a + \lambda 1_E$ in the strong operator topology on $B(E)$. □

Lemma 2.25. *Suppose that A is an approximately unital Arens regular Banach algebra, and let $\mathfrak{e} = (e_t)_{t \in \Lambda}$ be a cai for A. Then:*

(1) *The multiplier unitization of A is isometrically isomorphic to $A + \mathbb{C}\, 1_{A^{**}}$ in A^{**}.*

(2) *With $S_{\mathfrak{e}}(A)$ as defined in Definition 2.6, and identifying A^* with the weak* continuous functionals on A^{**}, we have*

$$S_{\mathfrak{e}}(A) = \{\omega \in S(A^{**}) : \omega \text{ is weak* continuous}\}$$

(the normal state space of A^{**}).

(3) *$S_{\mathfrak{e}}(A)$ and $S(A)$ both span A^*, and both separate the points of A.*

(4) *In the notation found before Lemma 2.6 of [Blecher and Ozawa 2015] and in Definition 2.13, we have*

$$\mathfrak{r}_A^{\mathfrak{e}} = \mathfrak{r}_A \quad \text{and} \quad \mathfrak{c}_{A^*}^{\mathfrak{e}} = \mathfrak{c}_{A^*}.$$

(5) *If A is also nonunital then $\{\varphi|_A : \varphi \in S(A^1)\}$ is the weak* closure in A^* of any one of the following sets in Definition 2.6: $S(A)$, $S_{\mathfrak{e}}(A)$, $Q(A)$, or $Q_{\mathfrak{e}}(A)$.*

Proof. The proof of (1) is essentially the same as the proof of Lemma 2.24: for $a, c \in A$ and $\lambda \in \mathbb{C}$, clearly

$$\|ac + \lambda c\| = \|(a + \lambda 1_{A^{**}})c\| \leq \|a + \lambda 1_{A^{**}}\| \|c\|.$$

So $\|a + \lambda 1\|_{A^1} \leq \|a + \lambda 1_{A^{**}}\|$. The reverse inequality follows from the fact that if $(e_t)_{t \in \Lambda}$ is a cai, then Lemma 1.14 implies that $ae_t + \lambda e_t \to a + \lambda 1_{A^{**}}$ weak*.

For (2), since $e_t \to 1$ weak* in A^{**} by Lemma 1.14, it is clear that weak* continuous states on A^{**} restrict to elements of $S_{\mathfrak{e}}(A)$. For the reverse inclusion, let $\omega \in S_{\mathfrak{e}}(A)$. Then ω^{**} is a weak* continuous functional on A^{**} and $\|\omega^{**}\| = 1$. That $\omega^{**}(1) = 1$ follows from weak* continuity of ω^{**} and the weak* convergence $e_t \to 1$.

The assertion about $S_{\mathfrak{e}}(A)$ in (3) follows from part (2) and Theorem 2.2 of [Magajna 2009], according to which the normal state space of A^{**} spans A^* and separates the points of A. The second assertion in (3) follows from the first assertion and the inclusion $S_{\mathfrak{e}}(A) \subseteq S(A)$, which is in Lemma 2.2 of [Blecher and Ozawa 2015].

We prove (4). We need only prove $\mathfrak{r}_A^{\mathfrak{e}} \subseteq \mathfrak{r}_A$, since the reverse inclusion holds by definition, and equality implies $\mathfrak{c}_{A^*} = \mathfrak{c}_{A^*}^{\mathfrak{e}}$ by definition. So let $a \in \mathfrak{r}_A^{\mathfrak{e}}$ and let $\omega \in S(A)$.

By definition, ω extends to a state ω^1 on A^1. By part (1) we have $A^1 \subseteq A^{**}$, so the Hahn–Banach theorem provides an extension of ω^1 to a state φ on A^{**}. Use weak* density of the normal states in $S(A^{**})$ (which follows from Theorem 2.2 of [Magajna 2009]) to find a net $(\varphi_t)_{t \in \Lambda}$ in the normal state space of A^{**} which converges weak* to φ. Now $\mathrm{Re}(\omega(a)) = \lim_t \mathrm{Re}(\varphi_t(a)) \geq 0$. So $a \in \mathfrak{r}_A$.

Finally, we prove (5).

It follows from [Blecher and Ozawa 2015, Lemma 2.6] that, with overlines denoting weak* closures, we have

$$\overline{S(A)} = \overline{Q(A)} \subseteq \overline{\{\varphi|_A : \varphi \in S(A^1)\}}.$$

Also, $\{\varphi|_A : \varphi \in S(A^1)\}$ is shown to be weak* closed in the proof of that lemma.

Now suppose that $\varphi \in S(A^1)$ and set $\psi = \varphi|_A$. Use the Hahn–Banach theorem to extend φ to a state ρ on A^{**}. Use again weak* density of the normal states in $S(A^{**})$ to find a net $(\psi_t)_{t \in \Lambda}$ in the normal state space of A^{**} which converges weak* to ρ. Set $\varphi_t = \psi_t|_A$ for $t \in \Lambda$. For $a \in A$ we then have

$$\varphi_t(a) = \psi_t(a) \to \psi(a) = \varphi(a).$$

By part (2), this shows that ψ is in the weak* closure of $S_{\mathfrak{e}}(A)$. Since

$$S_{\mathfrak{e}}(A) \subseteq S(A) \subseteq Q(A) \quad \text{and} \quad S_{\mathfrak{e}}(A) \subseteq Q_{\mathfrak{e}}(A),$$

the assertion follows. $\square$

The set $\mathfrak{r}_{A^{**}}$, as defined on page 11 of [Blecher and Ozawa 2015], may be a proper subset of the accretive elements in A^{**}, even for approximately unital L^p-operator algebras. In fact, the identity e of A^{**} is certainly accretive in A^{**}, but need not be accretive in $(A^1)^{**}$. (Equivalently, by Lemma 2.29 (4), we need not have $\|1 - e\| \leq 1$.) This happens for $A = \mathbb{K}(L^p([0, 1]))$, by Proposition 3.10. However, it follows from the later result Proposition 4.26 (and Proposition 4.24 (2)) that $\mathfrak{r}_{A^{**}}$, as defined on page 11 of [Blecher and Ozawa 2015], equals the accretive elements in A^{**} if A is a scaled approximately unital L^p-operator algebra.

Remark 2.26. The sets $S_{\mathfrak{e}}(A)$ and $Q_{\mathfrak{e}}(A)$ are easily seen to be convex in A^*. We do not know whether $S(A)$ and $Q(A)$ are necessarily convex if A is a general approximately unital Arens regular Banach algebra, since convex combinations of norm 1 functionals may have norm strictly less than 1. However they are convex if A is an approximately unital L^p-operator algebra, since Corollary 4.25 (1) implies convexity of $S(A)$, and this implies convexity of $Q(A)$.

Proposition 2.27. *Let $p \in (1, \infty)$. The multiplier unitization of an approximately unital L^p-operator algebra is an L^p-operator algebra.*

Proof. This follows from Lemma 2.25 (1) and the fact (Lemma 2.1 (3)) that biduals of L^p-operator algebras are L^p-operator algebras (or from Lemmas 2.24 and 2.33). $\square$

Similarly, for $p \in (1, \infty)$ the multiplier unitization of an approximately unital SQ$_p$-operator algebra is an SQ$_p$-operator algebra.

The multiplier algebra M(A), and the left and right multiplier algebras LM(A) and RM(A), of an approximately L^p-operator algebra may be defined to be subsets of A^{**} just as in the operator algebra case. Then the multiplier unitization A^1 is contained in M(A) isometrically and unitally. If A is represented isometrically and nondegenerately on $L^p(X)$ then, just as in the operator algebra case, M(A), LM(A), and RM(A) may be identified isometrically as Banach algebras with the usual subalgebras of $B(L^p(X))$. See Theorem 3.19 of [Gardella and Thiel 2019], and the discussion in that paper. One can also, for example, copy the proof of Theorem 2.6.2 of [Blecher and Le Merdy 2004] for LM(A), and later results in Section 2.6 of [Blecher and Le Merdy 2004] for RM(A) and M(A).

In particular, M(A), LM(A), and RM(A) are all unital L^p-operator algebras. Similarly, LM(A) can be identified with the algebra of bounded right A-module endomorphisms of A, as usual. One may also check that the useful principle in [Blecher and Le Merdy 2004, Proposition 2.6.12] holds for approximately L^p-operator algebras, with the same proof. (Also see Theorem 3.17 of [Gardella and Thiel 2019].)

2D. *Idempotents.*

Definition 2.28. We recall that if A is a unital Banach algebra, then an idempotent $e \in A$ is called *bicontractive* if $\|e\| \leq 1$ and $\|1 - e\| \leq 1$. We say that an element s of a unital Banach algebra A is an *invertible isometry* if s is invertible, $\|s\| = 1$, and $\|s^{-1}\| = 1$.

We collect some standard facts related to bicontractive idempotents.

Lemma 2.29. (1) *Let A be a unital Banach algebra and let $e \in A$ be a hermitian idempotent. Then $1 - 2e$ is an invertible isometry of order 2.*

(2) *Let A be a unital Banach algebra. Then every hermitian idempotent in A is bicontractive.*

(3) *Let $p \in [1, \infty)$, let (X, μ) be a measure space, and let $e \in B(L^p(X, \mu))$ be an idempotent. Then e is bicontractive if and only if $1 - 2e$ is an invertible isometry.*

(4) *Let A be a unital Banach algebra and let $e \in A$ be an idempotent. Then e is real positive if and only if $1 - e$ is contractive ($\|1 - e\| \leq 1$).*

The converse of (2) is false, even in L^p-operator algebras. See Lemma 6.11 of [Phillips and Viola 2017], which is just the idempotent e_2 of Example 3.2 for $p \neq 2$.

Part (3) fails in general unital Banach algebras. This failure is well known, and our Example 4.7 contains an explicit counterexample.

Proof of Lemma 2.29. For (1), by definition we have

$$\|1 + [\exp(i\lambda) - 1]e\| = \|\exp(i\lambda e)\| \leq 1$$

for all $\lambda \in \mathbb{R}$. Setting $\lambda = \pi$ gives $\|1 - 2e\| \leq 1$. One checks immediately that $(1 - 2e)^2 = 1$, so in fact $\|1 - 2e\| = 1$. The rest of (1) follows easily.

Part (2) follows from Lemma 6.6 of [Phillips and Viola 2017].

We prove (3). The forward direction follows from [Bernau and Lacey 1977, Theorem 2.1] (or, when $\mu(X) = 1$, from the corollary on page 11 of [Byrne and Sullivan 1972]). Conversely, if $\|1 - 2e\| \leq 1$ then

$$\|e\| = \left\| \tfrac{1}{2}[1 - (1 - 2e)] \right\| \leq \tfrac{1}{2}(\|1\| + \|1 - 2e\|) \leq 1,$$

and the proof that $\|1 - e\| \leq 1$ is similar.

Part (4) is [Blecher and Ozawa 2015, Lemma 3.12]. $\qquad\qquad\square$

Definition 2.30. We define two order relations on idempotents e, f in a Banach algebra A. We write $e \leq_{\mathrm{r}} f$ if $fe = e$ and $e \leq f$ if $ef = fe = e$.

If A is a subalgebra of $B(E)$ then, viewing these idempotents as operators on E, then $e \leq_{\mathrm{r}} f$ simply says that $\mathrm{Ran}(e) \subseteq \mathrm{Ran}(f)$. The second relation is the ordering considered in, e.g., [Phillips and Viola 2017, Section 6].

Clearly $e \leq f$ and $f \leq e$ imply $e = f$. This isn't true for the relation $\leq_{\mathrm{r}}$.

Lemma 2.31. *Let $p \in (1, \infty)$, and let A be an approximately unital L^p-operator algebra. Let e, $f \in A$ be idempotents. Assume that e and f are both contractive or both real positive. Then*:

(1) *$fe = e$ if and only if $ef = e$.*

(2) *$e \leq_{\mathrm{r}} f$ if and only if $e \leq f$.*

Proof. Part (2) is immediate from part (1), so we just prove part (1).

By definition (see Definition 2.13), we may work in the multiplier unitization A^1, which is a unital L^p-operator algebra by Proposition 2.27. So we can assume that there is a measure space (X, μ) such that A is a unital subalgebra of $B(L^p(X, \mu))$.

First suppose that e and f are contractive. Assume that $fe = e$. Then ef is necessarily an idempotent, and is clearly contractive. Clearly $\mathrm{Ran}(ef) \subseteq \mathrm{Ran}(e)$. Since $efe = e^2 = e$, we have $\mathrm{Ran}(e) \subseteq \mathrm{Ran}(ef)$. By [Cohen and Sullivan 1970, Theorem 6], the range of a contractive idempotent on a smooth space determines the idempotent. So $ef = e$, as desired.

Next assume that $ef = e$. Let $q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then e^*, $f^* \in B(L^q(X, \mu))$ are contractive idempotents such that $f^*e^* = e^*$. The case already considered implies $e^*f^* = e^*$, whence $fe = e$.

Now suppose that e and f are real positive. Then $1 - e$ and $1 - f$ are contractive idempotents by Lemma 2.29 (4). So $(1 - e)(1 - f) = 1 - f$ if and only if

$(1 - f)(1 - e) = 1 - f$ by the contractive case. Expanding and rearranging, we get $fe = e$ if and only if $ef = e$. □

2E. *Representations.* We say a few words on representations.

Lemma 2.32. *Let $p \in (1, \infty)$, let A be an L^p-operator algebra, let X be a measure space, let M be a weak* closed subalgebra of $B(L^p(X))$, and let $\pi : A \to M$ be a contractive homomorphism. Then there exists a unique weak* continuous contractive homomorphism $\widetilde{\pi} : A^{**} \to M$ which extends π.*

Proof. The proof is the same as for the operator algebra case (2.5.5 in [Blecher and Le Merdy 2004], but without the matrix norms) and using Lemma 2.1. □

Let $\pi : A \to B(L^p(X))$ be a contractive representation of an approximately unital L^p-operator algebra. Then $E = \overline{\text{span}}(\pi(A)(L^p(X)))$ may not be an L^p-space on a subset of X. Indeed, in Example 3.2, $\text{Ran}(e_2)$ is not an L^p-space on a subset. However it is isometric to an L^p-space, as we will see next.

Some of the following follows from [Johnson 1972, Proposition 1.8] (we thank Eusebio Gardella for this reference) and [Gardella and Thiel 2019, Theorem 3.12, Corollary 3.13] (see also [Phillips and Viola 2017, Section 2]), but for completeness we give a self-contained proof.

Lemma 2.33. *Let $p \in (1, \infty)$, let A be an approximately unital Banach algebra, and let $\pi : A \to B(L^p(X))$ be a contractive representation. Set $E = \overline{\text{span}}(\pi(A)(L^p(X)))$. Then there exists a unique contractive idempotent $f \in B(L^p(X))$ whose range is E. Moreover, E and f have the following properties.*

(1) *For every cai $(e_t)_{t \in \Lambda}$ for A, the net $(\pi(e_t))_{t \in \Lambda}$ converges to f in both the weak* topology and the strong operator topology on $B(L^p(X))$.*

(2) *For all $a \in A$ we have $\pi(a) = f\pi(a)f$.*

(3) *The compression of π to E is a contractive representation, which is isometric if π is isometric.*

(4) *The compression of π to E is nondegenerate.*

(5) *E is linearly isometric to an L^p-space.*

Proof. Let $q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

We claim that if $(e_t)_{t \in \Lambda}$ is a cai in A such that $(\pi(e_t))_{t \in \Lambda}$ converges weak* to some $f \in B(L^p(X))$, then f is a contractive idempotent whose range is E. Assume the claim has been proved. Since $L^p(X)$ is a smooth space, such an idempotent is unique by [Cohen and Sullivan 1970, Theorem 6]. The argument of Lemma 1.14, with this uniqueness statement in place of uniqueness of the identity in A^{**}, shows that such an idempotent f exists and that for any cai $(e_t)_{t \in \Lambda}$ in A, we have $\pi(e_t) \to f$ weak*.

We prove the claim. We have $\|f\| \leq 1$ and $\langle f\pi(a)\xi, \eta\rangle = \langle \pi(a)\xi, \eta\rangle$ for all $a \in A$, $\xi \in L^p(X)$, and $\eta \in L^q(X)$. It follows that $f\xi = \xi$ for all $\xi \in E$. So $E \subseteq \mathrm{Ran}(f)$. Also, if $\eta \in E^\perp \subseteq L^q(X)$, then $\langle f\xi, \eta\rangle = \lim_t \langle \pi(e_t)\xi, \eta\rangle = 0$. Thus $E^\perp \subseteq \mathrm{Ran}(f)^\perp$, whence $\mathrm{Ran}(f) \subseteq E$. The claim is proved. We now have the main statement, and weak* convergence in (1).

Part (5) follows from the fact (Theorem 3 in Section 17 of [Lacey 1974]; see also Theorem 4 of [Ando 1966]) that the range of a contractive idempotent on an L^p-space is isometrically isomorphic to an L^p-space.

We prove (2). We know that $f\pi(a) = \pi(a)$ for all $a \in A$, so we prove that $\pi(a)f = \pi(a)$. For $\xi \in L^p(X)$ and $\eta \in L^q(X)$, we have

$$\langle \pi(a)f\xi, \eta\rangle = \langle f\xi, \pi(a)^*\eta\rangle = \lim_t \langle \pi(e_t)\xi, \pi(a)^*\eta\rangle$$

$$= \lim_t \langle \pi(ae_t)\xi, \eta\rangle = \langle \pi(a)\xi, \eta\rangle.$$

Thus $\pi(a)f = \pi(a)$.

Part (3) is now immediate, as is (4) since $\pi(e_t)\pi(a)\xi \to \pi(a)\xi$ for $a \in A$ and $\xi \in L^p(X)$.

We prove strong operator convergence in (1). It suffices to prove that $\pi(e_t)\xi \to f\xi$ for $\xi \in fL^p(X)$ and for $\xi \in (1-f)L^p(X)$. The first of these follows from (4). The second case is trivial: $\pi(e_t)\xi = 0$ by (2), and $f\xi = 0$. $\square$

Remark 2.34. The last result also holds with L^p-spaces replaced by the SQ_p-spaces mentioned in the introduction, although (5) would then say that E is an SQ_p-space. The proof is essentially the same, except that (5) becomes trivial. We also need to use the fact that SQ_p-spaces are smooth for $p \in (1, \infty)$. In fact, they are also strictly convex. To see this, first observe that reflexivity of L^p-spaces implies reflexivity of SQ_p-spaces. Next, L^p-spaces are both smooth and strictly convex, so their subspaces are as well. So the duals of subspaces are both strictly convex and smooth. By reflexivity, the quotient of a subspace is the dual of a subspace of the dual, so both smooth and strictly convex.

If A is unital as a Banach algebra and also is an L^p-operator algebra then it follows that A may be viewed as a subalgebra of $B(L^p(X))$ containing the identity operator on $L^p(X)$, for some measure space X. This was proved first in Section 2 of [Phillips and Viola 2017].

Corollary 2.35. *Let $p \in (1, \infty)$. Let A be a dual unital L^p-operator algebra (Definition 2.3). Then A has an isometric unital representation on an L^p-space which is a weak* homeomorphism onto its range.*

Proof. Let $\pi : A \to B(L^p(X))$ be an isometric representation which is a weak* homeomorphism onto its range. As in Lemma 2.33, let $E = \overline{\mathrm{span}}(\pi(A)(L^p(X))$, and let f be as there. Clearly $f = \pi(1_A)$. Define $\sigma : A \to B(E) = fB(L^p(X))f$ by $\sigma(a) = f\pi(a)f$ for $a \in A$. Lemma 2.33 implies that σ is an isometric unital

representation on an L^p-space. In light of the Krein–Smulian theorem, all we need to show is that the weak* topology on $B(E)$ is the same as the restriction to $f B(L^p(X)) f$ of the weak* topology on $B(L^p(X))$. The inclusion of E in $L^p(X)$ as a complemented subspace gives an inclusion of $\mathbb{K}(E)$ in $\mathbb{K}(L^p(X))$, and by Theorem 1.13 (2) the second dual of this inclusion is $B(E) \hookrightarrow B(L^p(X))$, which is therefore a weak* homeomorphism onto its image.   □

In particular, applying this principle to the bidual of an approximately unital L^p-operator algebra A, we obtain a faithful normal isometric representation of A^{**} that can to some extent play the role of the enveloping von Neumann algebra coming from the universal representation of a C^*-algebra.

Corollary 2.36. *Let $p \in (1, \infty)$, and let A be an approximately unital L^p-operator algebra. Then there exists a measure space (X, μ) and a unital isometric representation $\theta : A^{**} \to B(L^p(X, \mu))$ such that*:

(1) *θ is a weak* homeomorphism onto its range.*

(2) *$\theta|_A$ acts nondegenerately on $L^p(X, \mu)$.*

(3) *For any cai $(e_t)_{t \in \Lambda}$ in A, we have $\theta(e_t) \to 1$ in the strong operator topology on $B(L^p(X, \mu))$.*

Proof. This is clear from Corollary 2.35 and Lemma 2.33.   □

3. Examples

As we mentioned in the introduction, so far the study of L^p-operator algebra has been very largely example driven. Thus there is a wealth of examples in the literature, or in preprint form. (See the works of Phillips, Viola, Gardella and Thiel, and others referred to earlier.) In this section we discuss the main examples which we have used, or which seem useful but are not in the literature. We recall again that, as always, in this section $p \in (1, \infty) \setminus \{2\}$ unless stated to the contrary.

Notation 3.1. As in, for example, [Phillips and Viola 2017, Lemma 6.11], for $n \in \mathbb{N}$ and $p \in [1, \infty]$ we write l_n^p for L^p of an n point space with counting measure, and define $M_n^p = B(l_n^p)$.

Example 3.2. Let $p \in [1, \infty)$. Let $e_n \in M_n^p$ be the $n \times n$ matrix whose entries are all $\frac{1}{n}$. We will use e_n several times in this paper and so the calculations that follow will be important for us. If $p = 2$ then e_n is a rank one projection. For the rest of this example, assume $p \neq 2$, and let $q \in (1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Suppose $n = 2$. We have

$$1 - 2e_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

which is an invertible isometry. So $\|e_2\| = \|1 - e_2\| = 1$ by Lemma 2.29 (3), and e_2 is

real positive by Lemma 2.29 (4). However, e_2 is not hermitian, by Proposition 2.12, or by Lemma 6.11 of [Phillips and Viola 2017].

For the rest of this example, assume $n \geq 3$ (as well as $p \neq 2$). We claim that $\|e_n\| = 1$ but $\|1 - e_n\| > 1$, so that e_n is not bicontractive. Then Lemma 2.29 (4) implies that e_n is not real positive.

To see that e_n is contractive, set

$$\eta = (1, 1, \ldots, 1) \in l_n^p \quad \text{and} \quad \mu = \tfrac{1}{n}(1, 1, \ldots, 1) \in l_n^q.$$

Then one easily checks that for all $\xi \in l_n^p$ we have $e_n \xi = \langle \mu, \xi \rangle \eta$, so $\|e_n\| \leq \|\mu\|_q \|\eta\|_p = 1$.

To show that $\|1 - e_n\| > 1$, by Lemma 2.29 (3) it is enough to prove that $1 - 2e_n$ is not isometric. As pointed out to us by Eusebio Gardella, Lamperti's theorem [Fleming and Jamison 2003, Theorem 3.2.5] implies that the only matrices which are isometries in the L^p-operator norm are the complex permutation matrices, and clearly $1 - 2e_n$ is not of this form. However, we can give a direct proof.

Define $g : [1, \infty) \to [0, \infty)$ by

$$g(p) = \|(1 - 2e_n)(1, 0, 0, \ldots, 0)\|_p^p.$$

We have

$$(1 - 2e_n)(1, 0, 0, \ldots, 0) = \left(1 - \frac{2}{n}, -\frac{2}{n}, -\frac{2}{n}, \ldots, -\frac{2}{n}\right),$$

so

$$g(p) = \left(1 - \frac{2}{n}\right)^p + (n - 1)\left(\frac{2}{n}\right)^p$$

for $p \in [1, \infty)$. One further has $g(2) = 1$ and

$$g'(p) = \left(1 - \frac{2}{n}\right)^p \log\left(1 - \frac{2}{n}\right) + (n - 1)\left(\frac{2}{n}\right)^p \log\left(\frac{2}{n}\right)$$

for all $p \in [1, \infty)$. Both the logarithm terms are strictly negative, so $g'(p) < 0$. Therefore,

$$\|(1 - 2e_n)(1, 0, 0, \ldots, 0)\|_p \neq 1$$

for all $p \in [1, \infty) \setminus 2$. Thus $\|1 - e_n\| > 1$.

One can see easily that $\|1 - e_n\| < 2$ (this follows for example from a lemma in the sequel paper), but we will not use this here.

Lemma 2.29 (4) implies that $1 - e_n$ is real positive. It follows also that the "support idempotent" $s(1 - e_n)$ of $1 - e_n$ (see Definition 4.12) is not contractive, unlike support idempotents for real positive Hilbert space operators (see, e.g., Corollary 3.4 of [Blecher and Read 2013]). In turn this shows that, unlike the Hilbert space operator case, the limit $\lim_{m \to \infty} \|x^{1/m}\|$ need not equal 1 for real positive elements in an L^p-operator algebra A (or even for elements of $\mathfrak{F}_A$). We are using the m-th root in Definition 2.18 and the discussions after it. We also see

that, unlike the Hilbert space operator case in Proposition 2.3 of [Blecher and Read 2011], $\frac{1}{2}\mathfrak{F}_A$ is not closed under n-th roots. Indeed,

$$\tfrac{1}{2}(1 - e_n) \in \tfrac{1}{2}\mathfrak{F}_A \subseteq \text{Ball}(A)$$

but

$$\lim_{m \to \infty} \left(\tfrac{1}{2}(1 - e_n)\right)^{1/m} = s(1 - e_n) = 1 - e_n \notin \text{Ball}(A).$$

Nonetheless it is true that $\mathfrak{F}_A$ is closed under n-th roots, by Proposition 2.19 (7).

Another example of bicontractive idempotents, related to the case of M_2^p discussed above, appears in the group L^p-operator algebra of a discrete group containing elements of order 2. (See, e.g., [Phillips 2013a; Gardella and Thiel 2015a; 2015b].) These elements give projections in the group C^*-algebra, which are actually in the purely algebraic group algebra. The corresponding idempotents in the group L^p-operator algebra are bicontractive, and "look like" the bicontractive idempotents in M_2^p. Since we make little use of group L^p-operator algebras in this paper, we omit the details. As described below, however, they motivate Example 3.3.

Let E be a Banach space of the form $L^p(X, \mu)$ for some measure space (X, μ) and some $p \in (1, \infty)$. Let $e, f \in B(E)$ be commuting contractive idempotents. It is very tempting to conjecture that, as in the Hilbert space operator case, $e + f - ef$, which is an idempotent with range $\text{Ran}(e) + \text{Ran}(f)$, is also contractive. This conjecture is false, as we will see in Example 3.3, even if e and f are bicontractive. Thus, the lattice theoretic properties of (even commuting) bicontractive idempotents on L^p-spaces are deficient. Indeed we shall see that there is a disappointing comparison between the structure of the lattice of idempotents in $B(L^p(X))$ and the beautiful and fundamental behavior of projections in von Neumann algebras. Our example also does two other things. It shows that the product of two commuting real positive idempotents need not be real positive. And it shows that on L^p, there are commuting accretive operators whose geometric mean exists but is not accretive. This shows that [Blecher and Wang 2016, Lemma 5.8] fails with Hilbert spaces replaced by L^p-spaces.

The construction of the example is motivated as follows. Fix $p \in (1, \infty) \setminus \{2\}$. By Lemma 2.29 (3), commuting pairs of bicontractive idempotents in $B(L^p(X, \mu))$ are in one-to-one correspondence with pairs of commuting invertible isometries of order 2 in $B(L^p(X, \mu))$, and therefore with representations of $(\mathbb{Z}/2\mathbb{Z})^2$ on $L^p(X, \mu)$ via isometries. In particular, the conjecture in the previous paragraph holds for all (X, μ) (for our given value of p) if and only if it holds for the pair of bicontractive idempotents coming from the universal isometric L^p representation of $(\mathbb{Z}/2\mathbb{Z})^2$. Since $(\mathbb{Z}/2\mathbb{Z})^2$ is amenable, this will be true if and only if it holds for the left regular representation of $(\mathbb{Z}/2\mathbb{Z})^2$ on

$$l^p((\mathbb{Z}/2\mathbb{Z})^2) \cong l_4^p.$$

Example 3.3. Fix $p \in (1, \infty) \setminus \{2\}$. There is a finite-dimensional unital L^p-operator algebra (specifically M_4^p) which contains the following:

(1) Two commuting bicontractive idempotents whose product is not bicontractive.

(2) Two commuting real positive idempotents whose product is not real positive.

(3) Two commuting accretive operators whose geometric mean exists but is not accretive.

We work throughout in M_4^p. Define

$$s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in M_4^p \quad \text{and} \quad t = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in M_4^p.$$

One checks that these are commuting isometries of order 2. Next, define

$$e = \tfrac{1}{2}(1 + s) \quad \text{and} \quad f = \tfrac{1}{2}(1 + t).$$

These are commuting idempotents, and they are bicontractive by Lemma 2.29 (3). Then one checks that ef is the idempotent e_4 of Example 3.2, and that $e + f - ef$ is an idempotent. We claim that it is not contractive. First, we look at $1 - (e + f - ef)$, getting

$$1 - (e + f - ef) = \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Define $w = \operatorname{diag}(1, -1, -1, 1)$, which is an invertible isometry in M_4^p. Then one checks that $w[1 - (e + f - ef)]w^{-1} = e_4$ in the language of Example 3.2. In that example we showed that this idempotent is contractive, and also showed that $1 - w[1 - (e + f - ef)]w^{-1}$ is not contractive. Therefore also

$$e + f - ef = w^{-1}(1 - w[1 - (e + f - ef)]w^{-1})w$$

is not contractive. This yields (1) for the bicontractive idempotents $1 - e$ and $1 - f$.

Now define $e_0 = 1 - e$ and $f_0 = 1 - f$. We have seen that e and f are contractive, so e_0 and f_0 are real positive by Lemma 2.29 (4). However, $1 - e_0 f_0 = e + f - ef$ is not contractive, so $e_0 f_0$ is not real positive, again by Lemma 2.29 (4). This is (2).

We turn to (3). We want invertible elements. Neither e nor f is invertible, but this is easily fixed by adding $\varepsilon 1$ to each of them, which does not change the fact that they commute. We recall the well-known Ando et al. list of properties that a "good" geometric mean should possess (see, e.g., page 306 of [Ando et al. 2004]). One of these is that the geometric mean of a and b should be $a^{1/2} b^{1/2}$ (as

in Definition 2.18) whenever a and b commute. One also needs to assume in our case that these principal square roots exist.

Suppose that $(\varepsilon 1 + e)^{1/2}(\varepsilon 1 + f)^{1/2}$ is accretive for all $\varepsilon > 0$. Using the Macaev–Palant formula $\|a^{1/2} - b^{1/2}\| \leq K\|a - b\|^{1/2}$ (see Lemma 2.4 and the discussion which precedes it in [Blecher and Wang 2016]; the proof is the same as Theorem 1 of [Macaev and Palant 1962], which is referenced there), letting $\varepsilon \to 0$ implies that $e^{1/2}f^{1/2}$ is accretive. We have $e^{1/2} = e$ and $f^{1/2} = f$ by, e.g., Proposition 2.19 (1). So ef is accretive, a contradiction.

Example 3.4. Let $p \in [1, \infty)$. Given a closed linear subspace $E \subseteq B(L^p(X))$, define $\mathcal{U}(E) \subseteq B(L^p(X) \oplus^p L^p(X))$ to be the set of operators which have the 2×2 matrix form

$$(3\text{-}1) \qquad \begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix}$$

with $\lambda, \mu \in \mathbb{C}$ and $x \in E$. Then $\mathcal{U}(E)$ is a unital L^p-operator algebra. Moreover, if $F \subseteq L^p(Y)$ and $u : E \to F$ is linear, then the map $\mathcal{U}(u) : \mathcal{U}(E) \to \mathcal{U}(F)$, defined by

$$\mathcal{U}(u)\left(\begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix}\right) = \begin{bmatrix} \lambda & u(x) \\ 0 & \mu \end{bmatrix}$$

for $\lambda, \mu \in \mathbb{C}$ and $x \in E$, is a unital homomorphism. We will show that if u is contractive or isometric, then so is $\mathcal{U}(u)$.

To begin, we claim that if $\lambda, \mu \in \mathbb{C}$ and $x \in B(L^p(X))$, then

$$(3\text{-}2) \qquad \left\|\begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix}\right\| = \left\|\begin{bmatrix} |\lambda| & \|x\| \\ 0 & |\mu| \end{bmatrix}\right\|,$$

with the norm on the right-hand side being taken in M_2^p. Hence the norm on $\mathcal{U}(E)$ only depends on the norms of elements in E, not the elements themselves.

We prove the claim. Let $\lambda, \mu \in \mathbb{C}$ and let $x \in B(L^p(X))$. Define

$$a = \begin{bmatrix} \lambda & x \\ 0 & \mu \end{bmatrix} \in B(L^p(X) \oplus^p L^p(X)) \quad \text{and} \quad c = \begin{bmatrix} |\lambda| & \|x\| \\ 0 & |\mu| \end{bmatrix} \in M_2^p.$$

We have

$$(3\text{-}3) \quad \|a\| = \sup\left(\left\{(\|\lambda\eta + x\xi\|_p^p + \|\mu\xi\|_p^p)^{1/p} : \eta, \xi \in L^p(X), \ \|\eta\|_p^p + \|\xi\|_p^p \leq 1\right\}\right).$$

The quantity inside the supremum is dominated by

$$\left[(|\lambda|\|\eta\|_p + \|x\|\|\xi\|_p)^p + (|\mu|\|\xi\|_p)^p\right]^{1/p} = \|c(\|\eta\|_p, \|\xi\|_p)\|_p \leq \|c\|.$$

So $\|a\| \leq \|c\|$. To see the other direction we may assume that $x \neq 0$. Choose scalars α, β with $|\alpha|^p + |\beta|^p \leq 1$ such that the norm of c is achieved at (α, β). Multiplying α and β by a complex number of absolute value 1, we may assume that $\beta \geq 0$.

Since $c(\alpha, \beta) = (\alpha|\lambda| + \beta\|x\|, \ \beta|\mu|)$, we see that $\|c(\alpha, \beta)\|_p \le \|c(|\alpha|, \beta)\|_p$, so we may also assume that $\alpha \ge 0$. If $\beta = 0$ then

$$\|c\| = \|c(\alpha, \beta)\|_p = |\alpha\lambda| \le |\lambda| \le \|a\|.$$

Otherwise, let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\delta < \beta\|x\| \quad \text{and} \quad \big(\big|\,|\lambda|\alpha + \beta\|x\|\big| - \delta\big)^p > \big|\,|\lambda|\alpha + \beta\|x\|\big|^p - \varepsilon.$$

Choose $\xi \in L^p(X)$ of norm β so that $\big|\,\|x\xi\|_p - \beta\|x\|\big| < \delta$. Then $x\xi \ne 0$. Choose $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$ and $\zeta\lambda = |\lambda|$. Define $\eta = \zeta\alpha\|x\xi\|_p^{-1}x\xi \in L^p(X)$. Then η has norm α, so that $\|\eta\|_p^p + \|\xi\|_p^p \le 1$. Now

$$\begin{aligned}
\|a(\eta, \xi)\|^p = \|\lambda\eta + x\xi\|_p^p + \|\mu\xi\|_p^p &= \left(\left|\frac{\lambda\zeta\alpha}{\|x\xi\|_p} + 1\right|\|x\xi\|_p\right)^p + |\mu\beta|^p \\
&= \big|\,|\lambda|\alpha + \|x\xi\|_p\big|^p + |\mu\beta|^p > \big(\big|\,|\lambda|\alpha + \beta\|x\|\big| - \delta\big)^p + |\mu\beta|^p \\
&> \big|\,|\lambda|\alpha + \beta\|x\|\big|^p - \varepsilon + |\mu\beta|^p \\
&= \|c(\alpha, \beta)\|_p^p - \varepsilon = \|c\|^p - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the claim follows.

It follows that if $u : E \to F$ as above is isometric, then so is $\mathcal{U}(u)$.

We claim that if $u : E \to F$ is a linear contraction, then $\mathcal{U}(u)$ is also contractive. By the previous claim, it suffices to prove that if $\lambda, \mu, \rho, \sigma \in [0, \infty)$ and $\rho \le \sigma$, then

$$(3\text{-}4) \qquad\qquad \left\|\begin{bmatrix} \lambda & \rho \\ 0 & \mu \end{bmatrix}\right\| \le \left\|\begin{bmatrix} \lambda & \sigma \\ 0 & \mu \end{bmatrix}\right\|.$$

We apply (3-3) to these matrices. For $\alpha, \beta \in \mathbb{C}$ we have $\|(|\alpha|, |\beta|)\|_p = \|(\alpha, \beta)\|_p$. Since $\lambda, \rho \ge 0$, the expression $|\lambda\alpha + \rho\beta|^p + |\mu\beta|^p$ becomes no smaller if α and β are replaced by $|\alpha|$ and $|\beta|$, and similarly with σ in place of ρ. Therefore the norms of the matrices in (3-4) are $N(\rho)$ and $N(\sigma)$, with N given by

$$N(\tau) = \sup\big(\big\{\big((\lambda\alpha + \tau\beta)^p + (\mu\beta)^p\big)^{1/p} : \alpha, \beta \in [0, \infty) \text{ satisfy } \alpha^p + \beta^p \le 1\big\}\big)$$

for $\tau \in [0, \infty)$. Since all the variables are nonnegative, clearly $\rho \le \sigma$ implies $N(\rho) \le N(\sigma)$. This yields (3-4). The claim is proved.

Example 3.4 is often useful for counterexamples because it can convert a bad linear subspace of $B(L^p(X))$ into a suitably badly behaved L^p-operator algebra. Note that if E is weak* closed in $B(L^p(X))$ then $\mathcal{U}(E)$ is a dual L^p-operator algebra in the sense of Definition 2.3. This follows just as in Lemma 2.7.7 (1) of [Blecher and Le Merdy 2004], but using the characterization of weak* convergent nets in $B(L^p(X))$ given after Corollary 2.2.

Example 3.5. Let $p \in [1, \infty)$. The set of continuous functions $f : [0, 1] \to M_2^p$ is a unital L^p-operator algebra. We may view this as the canonical copy of $C([0, 1]) \otimes M_2^p$ inside the bounded operators on

$$L^p([0, 1]) \otimes l_2^p \cong l_2^p(L^p([0, 1])) \cong L^p([0, 1]) \oplus^p L^p([0, 1]).$$

The subalgebra consisting of functions with $f(1)$ diagonal is also a unital L^p-operator algebra. The subalgebras consisting of functions f with $f(0) = 0$, or with $f(0) = 0$ and $f(1)$ diagonal, are approximately unital L^p-operator algebras. Indeed, if $(e_n)_{n \in \mathbb{N}}$ is a cai for $C_0((0, 1])$, then, using tensor notation, $(e_n \otimes 1_2)_{n \in \mathbb{N}}$ is a cai for these algebras.

Example 3.6. Let $p \in (1, \infty)$. Let (X, μ) be a measure space, and, to avoid trivialities, assume that $L^p(X, \mu)$ is not separable. Let $A \subseteq B(L^p(X, \mu))$ be the ideal of operators on $L^p(X)$ with separable range, which is known to be a closed ideal. We claim that A is an L^p-operator algebra with a cai, and, if X is a discrete space with counting measure, even a cai consisting of hermitian and real positive idempotents.

We prove the first part of the claim. If $E \subseteq L^p(X)$ is any separable subspace, it follows by Theorem 6 in Section 16 on page 146 of [Lacey 1974] and Lemma 2 in Section 17 on page 153 of [Lacey 1974] (see also Proposition 1.25 in [Phillips 2013a]), that E is contained in the range of a contractive idempotent with separable range. (Spaces are assumed to be real in [Lacey 1974, Section 16], however the complex case is no doubt well known to Banach space experts. Indeed by the just cited results or their proofs, a separable subspace of $L^p(X)$ is contained in a separable closed sublattice F. Since the norm on F is p-additive, F is an abstract L^p-space (see page 131 of [Lacey 1974] for definitions of these terms), so by Theorem 3 in both Sections 15 and 17 of [Lacey 1974], F is contractively complemented.)

Also, it is well known and an exercise that an operator x on a reflexive space has separable range if and only if x^* has separable range. Taking $q \in (1, \infty)$ to satisfy $\frac{1}{p} + \frac{1}{q} = 1$, we see that A^* is the collection of operators on $L^q(X, \mu)$ with separable range. For any $x_1, x_2, \ldots, x_n \in A$, the closure of the linear span of their ranges is separable by standard arguments. It follows that there exist contractive idempotents e and f with separable ranges such that $x_k = ex_k = x_k f$ for $k = 1, 2, \ldots, n$. Thus A has a cai $(e_t)_{t \in \Lambda}$, indeed a cai consisting of contractive idempotents and such that for any finite set $F \subseteq A$ there is $t \in \Lambda$ such that $e_t x = x e_t = x$ for all $x \in F$. Indeed take Λ to be the collection of such finite subsets of A.

Now take X to be a set I with counting measure, so $L^p(X) = l^p(I)$. For any $J \subseteq I$ let e_J be the canonical hermitian (diagonal) projection e_J onto the image of $l^p(J)$ in $l^p(I)$. Suppose $x_1, x_2, \ldots, x_n \in B(l^p(I))$ have separable ranges. Then, as above, the closure E of the joint span of their ranges is separable. So there exists a countable subset J of I (the union of the supports of elements in a countable dense

set in E) such that all elements of E are supported on J. As in the last paragraph, the net (e_J), indexed by the countable subsets J of I ordered by inclusion, is a real positive hermitian cai consisting of bicontractive idempotents (since $1 - e_J = e_{I \setminus J}$ is contractive).

Example 3.7. Let G be a locally compact group which is not discrete, with Haar measure μ. Then $L^1(G)$ is approximately unital, and by Wendel's theorem, its multiplier algebra is $M(G)$, the measure algebra on G. In particular, $M(G)$ in an L^1-operator algebra. The identity of $M(G)$ is δ_1, the Dirac measure at 1_G. Hence the multiplier unitization of $L^1(G)$ is $L^1(G) + \mathbb{C}\delta_1 \subseteq M(G) \subseteq B(L^1(G))$. If $f \in L^1(G)$ and $\lambda \in \mathbb{C}$ then

$$\|f + \lambda\delta_1\| = \sup\left(\left\{\left|\int_G fg\,d\mu + \lambda g(1)\right| : g \in \mathrm{Ball}(C_0(G))\right\}\right).$$

We claim that the multiplier unitization of $L^1(G)$ is $L^1(G) \oplus^1 \mathbb{C}$. Fix $f \in L^1(G)$ and $\lambda \in \mathbb{C}$; it is enough to prove that $\|f + \lambda\delta_1\|_{M(G)} \geq \|f\|_1 + |\lambda|$. Given $\varepsilon > 0$, choose $h \in \mathrm{Ball}(C_0(G))$ with $\left|\int_G fh\,d\mu\right| > \|f\|_1 - \varepsilon$. Replacing h by $e^{i\beta}h$ for suitable $\beta \in \mathbb{R}$, we may assume that $\int_G fh\,d\mu \geq 0$. We have $\mu(\{1\}) = 0$ since G is not discrete. Choose by regularity a neighborhood U of 1 such that $\int_U |f|\,d\mu < \varepsilon$. By Urysohn's lemma there is a continuous function $k_1 : G \to [0, 1]$ with compact support K contained in U and taking the value 1 at 1_G. There is also a continuous function $k_2 : G \to [0, 1]$ which is 1 on $G \setminus U$ and is 0 on K. Choose $\theta \in \mathbb{R}$ such that $e^{i\theta}\lambda = |\lambda|$, and let $g = hk_2 + e^{i\theta}k_1$. Thus we have $g \in \mathrm{Ball}(C_0(G))$ with $\lambda g(1) = |\lambda|$, and $g = h$ on $G \setminus U$. So

$$\left|\int_G fg\,d\mu - \int_G fh\,d\mu\right| \leq 2\int_U |f|\,d\mu < 2\varepsilon.$$

Using $\int_G fh\,d\mu \geq 0$ and $\lambda g(1) = |\lambda| \geq 0$, we have

$$\|f + \lambda\delta_1\| \geq \left|\int_G fg\,d\mu + \lambda g(1)\right| > \left|\int_G fh\,d\mu + \lambda g(1)\right| - 2\varepsilon$$

$$= \int_G fh\,d\mu + |\lambda| - 2\varepsilon > \|f\|_1 + |\lambda| - 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the claim is proved.

It follows (see Definition 2.16 and Proposition 2.17) that $\mathfrak{F}_{L^1(G)} = \mathfrak{r}_{L^1(G)} = \{0\}$. By Lemma 2.15, $L^1(G)$ also has no nonzero hermitian elements. In particular, $L^1(G)$ has no hermitian or real positive cai.

Example 3.8. A good example of an L^p-operator algebra with a real positive cai but no hermitian cai is the set A of functions in the disk algebra vanishing at 1, represented on L^p of the circle as multiplication operators. The disk algebra contains no nontrivial hermitian elements, since the latter would be real-valued

functions. However, A is approximately unital. One way to see this is to combine Example I.1.4 (b) of [Harmand et al. 1993] (after Lemma I.1.5 there) with Theorem 4.8.5 (1) of [Blecher and Le Merdy 2004], realizing the disk algebra as an operator algebra by representing it on L^2 of the circle (instead of L^p) as multiplication operators.

Example 3.9. Let $p \in [1, \infty) \setminus \{2\}$. We consider the algebras $\mathbb{K}(L^p(X, \mu))$ for $X = \mathbb{N}$ with counting measure and $X = [0, 1]$ with Lebesgue measure. The first has a cai consisting of real positive, in fact, hermitian, idempotents. The second has a cai, but contains no nonzero real positive elements, and in particular no nonzero hermitian elements.

A hermitian element in $B(L^p(X, \mu))$ is "multiplication by an essentially bounded real-valued locally measurable function" (Proposition 2.12). Thus the hermitian elements in $B(l^p)$ are the infinite diagonal matrices with bounded real entries. Therefore the canonical approximate identity in $\mathbb{K}(l^p)$ is a cai consisting of real positive and hermitian elements. (Also see the discussion in [Phillips and Viola 2017, Section 6].)

Abbreviate $A = \mathbb{K}(L^p([0, 1]))$. This algebra is approximately unital by, e.g., Theorem 2 of [Palmer 1985]. We can in fact give a formula for cai $(e_n)_{n=0,1,\dots}$ consisting of contractive finite rank idempotents which is increasing in the order $\leq$ in Definition 2.30. For $n = 0, 1, \dots$, for

$$\xi \in L^p([0, 1]), \qquad k = 1, 2, \dots, 2^n, \quad \text{and} \quad x \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right),$$

define

$$(e_n \xi)(x) = 2^n \int_{(k-1)/2^n}^{k/2^n} \xi(t) \, dt.$$

One easily checks that $(e_n)_{n=1,2,\dots}$ has the properties claimed for it.

Assume now that $p \in (1, \infty) \setminus \{2\}$. It is known (see Theorem 2 of [Benyamini and Lin 1985]) there is no nonzero $a \in A$ with $\|1 - a\| \leq 1$. It follows from Proposition 2.17 that $\mathfrak{r}_A = \{0\}$. That is, for $p \in (1, \infty) \setminus \{2\}$, there are no nonzero real positive elements in A in the main sense of [Blecher and Ozawa 2015]. Hence by Lemma 2.25 (4) and Lemma 2.1 (1), for every cai $\mathfrak{e}$, we have $\mathfrak{r}_A^{\mathfrak{e}} = \{0\}$. (This set was defined before Lemma 2.6 in [Blecher and Ozawa 2015]. In our present case, by Lemma 2.25 (4) and Definition 2.13, $\mathfrak{r}_A^{\mathfrak{e}}$ is the set of elements $x \in A$ with $\text{Re}(\varphi(x)) \geq 0$ for all $\varphi \in S(A)$.) In particular, for $p \in (1, \infty) \setminus \{2\}$, A has no real positive cai. So, by Lemma 2.15 and Proposition 2.27, A has no hermitian cai.

It is easy to see directly that $\mathbb{K}(L^p([0, 1]))$ has no nonzero hermitian elements. Indeed, Proposition 2.12 implies that a hermitian element in $B(L^p([0, 1]))$ is the multiplication operator M_f by a bounded measurable real-valued function f. If the range of such an operator M_f is nonzero then it contains $L^p(E)$ for some

non-null $E \subseteq [0, 1]$. Indeed there is $\varepsilon > 0$ such that $E = \{x \in [0, 1] : |f(x)| > \varepsilon\}$ has strictly positive measure. So $L^p(E)$ is contained in the range of multiplication by f. Since the measure has no atoms, $L^p(E)$ is infinite-dimensional. This cannot be if M_f is compact, since in that case its restriction to $L^p(E)$ is compact and bounded below.

Proposition 3.10. *Let $p \in (1, \infty) \setminus \{2\}$. Set $A = \mathbb{K}(L^p([0, 1]))$. If e is the identity of A^{**}, viewed as an element of $(A^1)^{**}$, then $\|1 - e\| > 1$.*

Proof. Suppose that $\|1 - e\| \leq 1$. Then by Goldstine's theorem there are nets $(a_t)_{t \in \Lambda}$ in A and $(\lambda_t)_{t \in \Lambda}$ in $\mathbb{C}$ such that $\|\lambda_t 1 + a_t\| \leq 1$ for all $t \in \Lambda$ and $\lambda_t 1 + a_t \to 1 - e$ weak*. Applying the character annihilating A we see that $\lambda_t \to 1$. Hence $a_t \to -e$ weak*. Theorem 2 of [Benyamini and Lin 1985] provides $\delta > 0$ such that whenever $b \in A$ satisfies $\|b\| \geq \frac{1}{2}$ then $\|1 - b\| > 1 + \delta$. Choose $t_0 \in \Lambda$ such that $|1 - \lambda_t| < \frac{\delta}{2}$ for $t \in \Lambda$ with $t \geq t_0$. There is $t_1 \in \Lambda$ such that $t_1 \geq t_0$ and $\|a_{t_1}\| > \| - e\| - \frac{1}{2}$ (for otherwise $\|a_t\| \leq \| - e\| - \frac{1}{2}$ for $t \geq t_0$, giving the contradiction $\| - e\| \leq \| - e\| - \frac{1}{2}$). Clearly $\| - e\| \geq 1$. So $\|a_{t_1}\| > \frac{1}{2}$, whence $\|1 + a_{t_1}\| > 1 + \delta$. But

$$\|1 + a_{t_1}\| \leq |1 - \lambda_{t_1}| + \|\lambda_{t_1} + a_{t_1}\| < \frac{\delta}{2} + 1.$$

This contradiction shows that $\|1 - e\| \leq 1$ is impossible. $\qquad\square$

4. Miscellaneous results on L^p-operator algebras

4A. *Quotients and bi-approximately unital algebras.*

Definition 4.1. Let A be an L^p-operator algebra and let $J \subseteq A$ be a closed ideal. We say that J is a *bi-approximately unital ideal* in A (or is *bi-approximately unital in A*) if J is approximately unital and if there is an L^p-operator unitization B of A (as in Definition 1.7) such that identity e of the bidual J^{**} is a bicontractive idempotent in B^{**}.

Definition 4.2. Let A be an approximately unital Arens regular Banach algebra. We say that A is *bi-approximately unital* if in the bidual $(A^1)^{**}$ of its multiplier unitization A^1 the identity e of A^{**} is a bicontractive idempotent.

The next lemma shows that the terminology is consistent.

Lemma 4.3. *Let A be an approximately unital L^p-operator algebra. Then A is bi-approximately unital in the sense of Definition 4.2 if and only if A is bi-approximately unital as an ideal in itself in the sense of Definition 4.1.*

Recall from Lemma 2.1 (1) that L^p-operator algebras are automatically Arens regular.

Proof of Lemma 4.3. If A is bi-approximately unital in the sense of Definition 4.2, we can take the L^p-operator unitization required in Definition 4.1 to be A^1, recalling

from Proposition 2.27 that A^1 is an L^p-operator algebra. If A is bi-approximately unital as an ideal in itself, let B be an L^p-operator unitization as required in Definition 4.1, and let e be as there. The obvious homomorphism $\varphi : B \to A^1$ is contractive, by Remark 1.9 (4), so $\varphi^{**} : B^{**} \to (A^1)^{**}$ is contractive. Thus $\|\varphi^{**}(e)\| \le \|e\| \le 1$ and $\|1 - \varphi^{**}(e)\| \le \|1 - e\| \le 1$. Since $\varphi^{**}(e)$ is the identity of A^{**} as in Definition 4.2, we have shown that A is bi-approximately unital.   □

The algebra $\mathbb{K}(L^p([0, 1]))$ is an approximately unital L^p-operator algebra which is not bi-approximately unital. See Example 3.9 and Proposition 3.10.

By Lemma 2.29 (3), A is bi-approximately unital if and only if A is a u-ideal in A^1 as defined at the beginning of Section 3 of [Godefroy et al. 1993], that is, that $\|1 - 2e\| \le 1$ where e is the identity of $A^{\perp\perp}$ in $(A^1)^{**}$.

Lemma 4.4. *Let A be an approximately unital Arens regular Banach algebra. If A has a real positive bounded approximate identity, then A is bi-approximately unital in the sense of Definition 4.2.*

Proof. Lemma 2.22 implies that A has a cai in $\mathfrak{F}_A$. This cai converges weak* to the identity e of A^{**} by Lemma 1.14. Since norm closed balls are weak* closed, we get $\|e\| \le 1$ and $\|1 - e\| \le 1$. Hence e is bicontractive.   □

We conjecture that the converse of Lemma 4.4 is always true for L^p-operator algebras, namely that a bi-approximately unital L^p-operator algebra A has a real positive cai. Corollary 2.23 may be helpful for this question.

In [Gardella and Thiel 2016] it is shown that quotients of L^p-operator algebras by closed ideals need not be L^p-operator algebras, giving a negative solution to Problem 3.8 in [Le Merdy 1996]. Things are better if the ideal is approximately unital.

Lemma 4.5. *Let $p \in (1, \infty)$, let A be an L^p-operator algebra, and let $J \subseteq A$ be a closed ideal.*

(1) *If J is a bi-approximately unital ideal in A then A/J is an L^p-operator algebra.*

(2) *If J is approximately unital then there is a continuous bijective homomorphism from A/J to an L^p-operator algebra whose inverse is also continuous.*

Proof. We may suppose that A is unital with identity 1. Recall from Lemma 2.1 (2) that multiplication on A^{**} is separately weak* continuous. Also, the weak* closure of J in A^{**} is $J^{\perp\perp}$.

Let $(e_t)_{t \in \Lambda}$ be a cai for J. Since J is Arens regular (Lemma 2.1 (2)), Lemma 1.14 shows that there is $e \in J^{**}$ which is an identity for J^{**} and such that $(e_t)_{t \in \Lambda}$ converges weak* to e. Clearly $\|e\| \le 1$.

We claim that $eA^{**} = J^{\perp\perp}$ and $A^{**}e = J^{\perp\perp}$. The proofs are the same, so we do only the first. We have $J^{\perp\perp} \subseteq eA^{**}$ since e is an identity for $J^{\perp\perp}$. Also, if $a \in A$

then $e_t a \in J$ for all $t \in \Lambda$, and $e_t a \to ea$ weak*, so ea is in the weak* closure of J in A^{**}, which is $J^{\perp\perp}$. Thus $eA \subseteq J^{\perp\perp}$. Since multiplication on A^{**} is separately weak* continuous, it follows that $eA^{**} \subseteq J^{\perp\perp}$. The claim is proved.

For $a \in A^{**}$, since $ae, ea \in J^{\perp\perp}$ and e is an identity for $J^{\perp\perp}$, we get $(ea)e = ea$ and $e(ae) = ae$. So e is central in A^{**}.

Since e is an idempotent, we have an algebra homeomorphism (not necessarily isometric) $A^{**}/eA^{**} \cong (1-e)A^{**}$. If J is bi-approximately unital, then $\|1-e\| = 1$, and this isomorphism is isometric. Therefore we have algebras homomorphisms

$$A/J \hookrightarrow A^{**}/J^{\perp\perp} = A^{**}/eA^{**} \to (1-e)A^{**} \hookrightarrow A^{**}.$$

All maps are isometric except possibly the third, which is a homeomorphism in general and is isometric if J is bi-approximately unital. Since A^{**} is an L^p-operator algebra by Lemma 2.1 (2), we are done. □

Remark 4.6. (1) Using an ultrapower argument, Charles Read showed in an unfinished personal communication that the quotient $B(l^p)/\mathbb{K}(l^p)$ is isometrically an L^p-operator algebra. This fact is also contained in Theorem 2.1 and Remark 2 of [Boedihardjo and Johnson 2015], combined with the fact (Theorem 3.3 (ii) of [Heinrich 1980]) that ultrapowers of L^p-spaces are L^p-spaces. This result also follows from Lemma 4.5 (1), since the canonical cai for $\mathbb{K}(l^p)$ is bicontractive and hence so is its weak* limit.

Read was also working on whether $B(L^p)/\mathbb{K}(L^p)$ is an L^p-operator algebra. The results of [Boedihardjo and Johnson 2015] quoted above show that it is at least isomorphic to one, a fact which also follows from Lemma 4.5 (2). We are studying Read's unfinished proof of the latter in hopes of answering this question.

(2) We remind the reader of an example from [Gardella and Thiel 2016]: the p variant of the Toeplitz algebra quotiented by $\mathbb{K}(l^p)$ is isomorphic to $F^p(\mathbb{Z})$, the norm closed subalgebra of $B(l^l(\mathbb{Z}))$ generated by the bilateral shift and its inverse. (This is the full group L^p-operator algebra of $\mathbb{Z}$ as defined in [Phillips 2013a]; see Definition 3.3 and the discussion before Proposition 3.14 there.) In particular, it is not $C(\mathbb{T})$.

Example 4.7. We exhibit $p \in (1, \infty) \setminus \{2\}$ and an L^p-operator algebra A with a closed approximately unital ideal J such that A/J is not isometrically isomorphic to an L^p-operator algebra. This shows that Lemma 4.5 (2) can't be improved. In our example, A is commutative and three dimensional, and J has an identity e which is central in A and with $\|e\| = 1$ (but $\|1-e\| > 1$).

Fix $n \in \{2, 3, \ldots\}$. (We will later take $n = 3$.) Let e_n be as in Example 3.2. Define $\zeta = e^{2\pi i/n}$ and $s = \mathrm{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$. For $k = 0, 1, \ldots, n-1$ set $f_k = s^k e_n s^{-k}$. We claim that:

(1) f_k is a contractive idempotent for $k = 0, 1, \ldots, n-1$.

(2) $f_0, f_1, \ldots, f_{n-1}$ are orthogonal, that is, $f_j f_k = 0$ if $j \neq k$.

(3) $\sum_{k=0}^{n-1} f_k = 1_{M_n}$, the $n \times n$ identity matrix.

For (1), recall from Example 3.2 that $\|e_n\| = 1$, and use $\|f_k\| \leq \|s\|^k \|e_n\| \|s^{-1}\|^k$. For (2) and (3), let $u \in M_n$ be the matrix whose k-th column (starting the count at 0 instead of 1) is

$$\frac{1}{\sqrt{n}} s^k (1, 1, \ldots, 1) = \frac{1}{\sqrt{n}} (1, \zeta^k, \zeta^{2k}, \ldots, \zeta^{(n-1)k}).$$

Computations show that u is unitary (in the $p = 2$ sense), and that

$$u^* e_n u = \mathrm{diag}(1, 0, 0, \ldots, 0) \quad \text{and} \quad u^* s u = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

For $k = 0, 1, \ldots, n - 1$, it follows that $u^* f_k u = (u^* s u)^k (u^* e_n u)(u^* s u)^{-k}$ is the orthogonal projection (in the $p = 2$ sense) to the span of the k-th standard basis vector (starting the count at 0 instead of 1). Parts (2) and (3) of the claim now follow immediately.

Set $n = 3$ and let A be the subalgebra of M_3^p spanned by f_0, f_1, and f_2. This contains 1_{M_3}. Let $J = \mathbb{C} e_3$, an ideal in A with an identity of norm 1. We claim that if

$$(4\text{-}1) \qquad\qquad p > \frac{\log(4)}{\log(4) - \log(3)}$$

then A/J is not isometric to an L^p-operator algebra. (This is presumably true for all $p \in [1, \infty) \setminus \{2\}$.) Indeed, the image f of f_1 in A/J is a contractive idempotent. It is actually bicontractive since

$$\|1 - f\| = \inf\{\|1 - f_1 + \lambda f_0\| : \lambda \in \mathbb{C}\}$$
$$\leq \|1 - f_1 - f_0\| = \|f_2\| \leq 1.$$

We claim that if p is as in (4-1) then $\|1 - 2f\| > 1$. If we can prove this claim then A/J cannot be an L^p-operator algebra by Lemma 2.29 (3).

To prove the last claim note first that since

$$1 - 2f_1 + \lambda f_0 = s_1(1 - 2e_3 + \lambda s_1^{-1} f_0 s_1)s_1^{-1},$$

we have

$$\|1 - 2f\| = \inf\{\|1 - 2f_1 + \lambda f_0\| : \lambda \in \mathbb{C}\}$$
$$= \inf\{\|1 - 2e_3 + \lambda s_1^{-1} f_0 s_1\| : \lambda \in \mathbb{C}\}.$$

With $\frac{1}{p} + \frac{1}{q} = 1$, the norm of $1 - 2e_3 + \lambda s_1^{-1} f_0 s_1$ dominates the q-norm of the first row of $1 - 2e_3 + \lambda s_1^{-1} f_0 s_1$. This first row is

(4-2) $\qquad \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right) - \frac{1}{3}\lambda\left(1, \zeta, \zeta^2\right) = \frac{1}{3}\left(1 - \lambda, -2 - \lambda\zeta, -2 - \lambda\zeta^2\right).$

We estimate the minimum of

$$|1 - \lambda|^q + |2 + \lambda\zeta|^q + |2 + \lambda\zeta^2|^q = |1 - \lambda|^q + |2\bar{\zeta} + \lambda|^q + |2\bar{\zeta}^2 + \lambda|^q.$$

Write $\lambda = x + iy$ for real x and y. Then

$$2\bar{\zeta} + \lambda = -1 + x + i(y - \sqrt{3}) \quad \text{and} \quad 2\bar{\zeta}^2 + \lambda = -1 + x + i(y + \sqrt{3}).$$

Thus we are minimizing

$$G(x, y) = ((1-x)^2 + y^2)^{q/2} + ((1-x)^2 + (y - \sqrt{3})^2)^{q/2} + ((1-x)^2 + (y + \sqrt{3})^2)^{q/2}.$$

Clearly $G(x, y) \geq G(1, y)$ for all $x, y \in \mathbb{R}$. So we must minimize the function

$$g_c(y) = |y|^q + |y - c|^q + |y + c|^q$$

for $c = \sqrt{3}$. For any $c > 0$, this function is continuous, even, and clearly strictly increasing on $[c, \infty)$. For $y \in (0, c)$ we have

$$g_c'(y) = q(y^{q-1} + (c + y)^{q-1} - (c - y)^{q-1}).$$

Since $q - 1 \geq 0$ and $c + y > c - y > 0$, it follows that $g_c'(y) > 0$. By symmetry, the minimum value of g_c occurs at $y = 0$. So, for all $x, y \in \mathbb{R}$, we have $G(x, y) \geq G(1, 0) = 2 \cdot 3^{q/2}$.

Applying this estimate to the q-norm of the right-hand side of (4-2), we get

$$\|1 - 2f\|^q \geq \frac{2 \cdot 3^{q/2}}{3^q} = 2 \cdot 3^{-q/2}.$$

If $q < 2\log(2)/\log(3)$, this quantity is greater than 1, and this happens exactly when (4-1) holds. The claim is proved.

4B. Unitization of nonunital L^p-operator algebras. Unfortunately Meyer's beautiful unitization theorem (see [Blecher and Le Merdy 2004, Corollary 2.1.15]) for operator algebras on Hilbert spaces fails badly for L^p-operator algebras. That is, unitizations of nonunital L^p-operator algebras are not unique isometrically (Examples 4.8 and 4.9 below). However if two approximately unital L^p-operator algebras A_1 and A_2 are isometrically isomorphic and they each act nondegenerately on the L^p-spaces on which they act, then $A_1 + \mathbb{C} 1$ is isometrically isomorphic to $A_2 + \mathbb{C} 1$. Indeed, for $j = 1, 2$, the algebra $A_j + \mathbb{C} 1$ is isometrically isomorphic to the multiplier unitization of A_j by Lemma 2.24.

We now illustrate the failure of Meyer's theorem, even in the case of approximately unital L^p-operator algebras. We give two versions. In the first, the algebras are finite-dimensional and already unital, but degenerately represented. In the second, the algebras are genuinely nonunital.

Example 4.8. Let $M_2^p = B(l_2^p)$ be as in Notation 3.1. Let $e = e_2$ be as in Example 3.2, and let $f = e_{1,1}$, the $(1, 1)$ standard matrix unit. Let $1 = 1_{M_2}$ be the 2×2 identity matrix. Then $\mathbb{C}\,e \cong \mathbb{C}\,f$ isometrically. We claim that $\mathbb{C}\,e + \mathbb{C}\,1$ is not isometric to $\mathbb{C}\,f + \mathbb{C}\,1$, so that Meyer's unitization theorem fails. The idempotents in $\mathbb{C}\,f + \mathbb{C}\,1$ are 0, f, $1 - f$, and 1, all of which are clearly hermitian. By Example 3.2, however, e is a nonhermitian idempotent in $\mathbb{C}\,e + \mathbb{C}\,1$. The claim follows.

Example 4.9. We continue with the notation in Example 4.8, to produce a nonunital example where Meyer's unitization theorem fails. Set $A = c_0 \oplus \mathbb{C}\,e$ and $B = c_0 \oplus \mathbb{C}\,f$, both viewed as subalgebras of $B(l^p(\mathbb{N}) \oplus^p l_2^p)$. These are isometrically isomorphic L^p-operator algebras, which are approximately unital. Indeed, they have obvious increasing approximate identities consisting of hermitian idempotents. Write 1 for the identity of $B(l^p(\mathbb{N}) \oplus^p l_2^p)$. We claim that $A + \mathbb{C}\,1$ is not isometrically isomorphic to $B + \mathbb{C}\,1$. To see this, first observe that all elements of $B + \mathbb{C}\,1$ are multiplication operators on $l^p(\mathbb{N}) \oplus^p l_2^p = l^p(\mathbb{N} \amalg \{0, 1\})$. It is immediate that all idempotents in this algebra are hermitian. On the other hand, there is a canonical restriction homomorphism $\rho : A + \mathbb{C}\,1 \to B(l_2^p)$, which is a unital contractive surjection to $\mathbb{C}\,e + \mathbb{C}\,1_{M_2}$, namely "compression" to the subspace l_2^p of $l^p(\mathbb{N}) \oplus^p l_2^p$. As we said in Example 4.8, $e \in \mathbb{C}\,e + \mathbb{C}\,1_{M_2}$ is a nonhermitian idempotent. However, $g = (0, e) \in A \subseteq A + \mathbb{C}\,1$ is an idempotent such that $\rho(g) = e$. If g were hermitian, then e would be too, by [Phillips and Viola 2017, Lemma 6.7]. So $A + \mathbb{C}\,1$ contains a nonhermitian idempotent.

In Example 4.9, one can show that the algebra $B + \mathbb{C}\,1$ is a spatial L^p AF algebra in the sense of Definition 9.1 of [Phillips and Viola 2017], while $A + \mathbb{C}\,1$ isn't.

We remark that [Phillips and Viola 2017, Proposition 9.9] gives conditions which force uniqueness of the unitization of an L^p-operator algebra. The fact that Meyer's theorem fails for $\mathbb{C}\,e$ and $\mathbb{C}\,f$ in Example 4.8 shows, by Meyer's proof (see [Blecher and Le Merdy 2004, 2.1.14]), that, even in $M_2^p = B(l_2^p)$, some of the basic properties of the Cayley transform for Hilbert space operators must fail for $p \neq 2$. We turn to this next.

4C. *The Cayley and $\mathfrak{F}$ transforms.* The Cayley transform $\kappa(x) = (x - 1)(x + 1)^{-1}$ is an important tool for operator algebras on a Hilbert space, as is the fact that in that setting $\kappa(x)$ is a contraction for accretive x. In [Blecher and Read 2013; 2014] the associated transform

$$\mathfrak{F}(x) = x(x + 1)^{-1} = \tfrac{1}{2}(1 + \kappa(x))$$

is used. For L^2-operator algebras it takes $\mathfrak{r}_A$ onto the strict contractions in $\frac{1}{2}\mathfrak{F}_A$.

This all fails in full generality for L^p-operator algebras, which means that many of the general results in [Blecher and Ozawa 2015] do not improve for L^p-operator algebras.

Here are two things which do work. First, if A is an approximately unital L^p-operator algebra then the $\mathfrak{F}$ transform does map $\mathfrak{r}_A$ into $\mathfrak{F}_A$. (By Lemma 3.4 of [Blecher and Ozawa 2015], this is true for arbitrary approximately unital Banach algebras.) Second, if A is any unital Banach algebra and $x \in \mathfrak{F}_A$, then $\|\kappa(x)\| = \|1 - 2\mathfrak{F}(x)\| \leq 1$. Indeed, with $y = x - 1$, we have $\|y\| \leq 1$, so that

$$\|\kappa(x)\| = \left\|\left(1 + \tfrac{1}{2}y\right)^{-1}\left(\tfrac{1}{2}y\right)\right\| \leq \left\|\tfrac{1}{2}y\right\| \sum_{k=0}^{\infty} \left\|\tfrac{1}{2}y\right\|^k \leq 1.$$

Example 4.10. We prove the existence of $\delta > 0$ such that for all $p \in [1, 1 + \delta)$ there is a unital finite dimensional L^p-operator algebra containing a real positive element x for which $\|\kappa(x)\| > 1$. Presumably this happens for all $p \in [1, \infty) \setminus \{2\}$, but proving this may require more work.

Indeed, in M_2^p (Notation 3.1) consider

$$x = \begin{bmatrix} 1 - i & 1 \\ 1 & 1 - i \end{bmatrix} \quad \text{and} \quad \kappa(x) = \frac{1}{5}\begin{bmatrix} 1 - 3i & 1 + 2i \\ 1 + 2i & 1 - 3i \end{bmatrix}.$$

Since $x = 2e_2 - i1_{M_2}$ in the notation of Example 3.2, it follows from considerations in that example that x is real positive in M_2^p. However $\kappa(x)$ applied to the unit vector $(1, 0)$ has p-norm $\frac{1}{5}(10^{p/2} + 5^{p/2})^{1/p}$, which exceeds 1 for $p \in [1, \delta)$, for some fixed $\delta > 0$.

One may also arrive at this same example by modifying the L^1-operator algebra example given in Example 3.14 in [Blecher and Ozawa 2015]. It was stated there that the convolution algebra $l^1(\mathbb{Z}_2)$ contains real positive elements x for which $\|\kappa(x)\| > 1$. An explicit example of such an element was not given there though. Let $F_r^p(\mathbb{Z}_2)$ be the reduced group L^p-operator algebra of the two element group (as defined in [Phillips 2013a]). This is isometric, via the regular representation of $\mathbb{Z}_2$ on $l^p(\mathbb{Z}_2)$, to the unital subalgebra of M_2^p generated by the idempotent

$$e = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(called e_2 in Example 3.2). This latter algebra contains our element x above. The regular representation of $\mathbb{Z}_2$ on $l^p(\mathbb{Z}_2)$ sends the nontrivial group element to

$$s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and we have the relations $e = \frac{1}{2}(s + 1)$ and $s = 2e - 1$.

Moreover, $F_r^1(\mathbb{Z}_2) \cong l^1(\mathbb{Z}_2)$ isometrically. Via these considerations, a real positive element w in Example 3.14 in [Blecher and Ozawa 2015] corresponds to a real positive element a in $F_r^1(\mathbb{Z}_2)$ and a real positive matrix x in M_2^1. Moreover, $\|\kappa(w)\| > 1$ if and only if $\|\kappa(a)\| > 1$, in turn if and only if $\|\kappa(x)\| > 1$. Since the map $F_r^1(\mathbb{Z}_2) \to F_r^p(\mathbb{Z}_2)$ is unital and contractive for $p \in [1, \infty)$, it follows easily that a (resp. x) is also real positive in $F_r^p(\mathbb{Z}_2)$ (resp. M_2^p). By "continuity in p", the Cayley transform of x in M_2^p is not contractive for p close to 1. A specific example of this of course is the matrix x in the second paragraph of the present example.

4D. *Support idempotents.* There is some improvement over [Blecher and Ozawa 2015] in the theory of support idempotents.

Proposition 4.11. *Let A be an approximately unital Arens regular Banach algebra, and let $x \in \mathfrak{r}_A$. Then, using the notation of Definition 2.18, the sequence $(x^{1/n})_{n \in \mathbb{N}}$ has a weak* limit $s(x) \in A^{**}$. Moreover*:

(1) *$s(x)$ is an idempotent.*

(2) *$s(x)$ is an identity for the second dual of the closed subalgebra of A generated by x.*

(3) *$s(x)x = xs(x) = x$.*

(4) *With $\mathfrak{F}$ as in Section 4C, we have $s(\mathfrak{F}(x)) = s(x)$.*

(5) *$\|1 - s(x)\| \leq 1$.*

(6) *$s(x)$ is a real positive idempotent in A^{**}.*

Definition 4.12. Let A and $x \in A$ be as in Proposition 4.11. We call $s(x)$ the *support idempotent* of x.

Proposition 4.11 is proved in the discussion after Proposition 3.17 of [Blecher and Ozawa 2015] (see also the discussion after Corollary 6.20 in [Blecher 2016]). Our advantage here over the situation in those papers is that the weak* limit of $x^{1/n}$ exists (it equals the identity for the second dual in (2) above), and so the support idempotent $s(x)$ is unique.

The support idempotent of x is minimal in several senses related to the orderings in Definition 2.30.

Corollary 4.13. *Under the hypotheses of Proposition 4.11, we furthermore have*:

(1) *If $f \in A^{**}$ is any idempotent with $fx = x$, then $fs(x) = s(x)$, that is, $s(x) \leq_r f$ in the sense of Definition 2.30.*

(2) *If $f \in A^{**}$ is any idempotent with $xf = x$, then $s(x)f = s(x)$.*

(3) *If $f \in A^{**}$ is any idempotent with $fx = x$ and $xf = x$, then $s(x) \leq f$ in the sense of Definition 2.30.*

Proof. By Proposition 2.19 (4), in part (1) we have $fx^{1/n} = x^{1/n}$. Hence (1) follows from $x^{1/n} \to s(x)$ weak* and separate weak* continuity of multiplication ([Blecher and Le Merdy 2004, 2.5.3]). Similarly for (2). Part (3) is now obvious. $\qquad\square$

Thus $s(x)$ is the smallest idempotent in A^{**} with $fx = x$, in the ordering $\leq_r$ (or with $fx = x$ and $xf = x$, in the ordering $\leq$). Recall from Corollary 2.4 that if A is an L^p-operator algebra then so is A^{**}, and so by Lemma 2.31 (2) we see that $\leq$ coincides with $\leq_r$ on real positive idempotents in A^{**}. Hence in this case $s(x)$ is the smallest idempotent in A^{**} with $fx = x$ (or with $xf = x$), in the ordering $\leq$.

In the case of a subalgebra of $B(L^p(X))$, we also get a support idempotent acting on $L^p(X)$.

Proposition 4.14. *Let $p \in (1, \infty)$, let $A \subseteq B(L^p(X))$ be an approximately unital closed subalgebra, and let $x \in \mathfrak{r}_A$. Let $s(x)$ be as in Proposition 4.11. Let $\varphi : A^{**} \to B(L^p(X))$ be the contractive homomorphism obtained from the identity representation of A as in Lemma 2.32, and set $e = \varphi(s(x))$. Then:*

(1) *e is an idempotent with range $\overline{xL^p(X)}$, and e is real positive if A is nondegenerate.*

(2) *$ex = xe = x$.*

(3) *$\overline{xL^p(X)}$ is a complemented subspace of $L^p(X)$.*

(4) *Using the notation of Definition 2.18, $x^{1/n} \to e$ in the strong operator topology on $B(L^p(X))$.*

(5) *If A is nondegenerate and $f \in B(L^p(X))$ is any real positive idempotent with $fx = x$ or $xf = x$, then $e \leq f$ in the sense of Definition 2.30.*

Nondegeneracy is probably needed for real positivity in (1) and for (5). Otherwise, letting f be as in Lemma 2.33, our proof below only yields a real positive idempotent in $B(fL^p(X))$.

Proof of Proposition 4.14. Let $E \subseteq L^p(X)$ and the idempotent $f \in B(L^p(X))$ be as in Lemma 2.33. We first claim that $\varphi(1) = f$. Indeed, this is always the case in the situation of Lemma 2.33 provided that A is Arens regular, by the following simple argument. Let $(e_t)_{t \in \Lambda}$ be a cai for A. Then $e_t \to 1$ weak* in A^{**} by Lemma 1.14. Therefore $e_t \to \varphi(1)$ weak* in $B(L^p(X))$ by weak* to weak* continuity of φ. Also $e_t \to f$ weak* in $B(L^p(X))$ by Lemma 2.33 (1). The claim is proved.

We have $x^{1/n} \to e$ weak* in $B(L^p(X))$. Since φ is a homomorphism, e is an idempotent satisfying $ex = xe = x$, which is (2). Using Proposition 4.11 (5) and $\varphi(1) = f$, we get $\|f - e\| \leq \|\varphi\|\|f - s(x)\| \leq 1$. If A is nondegenerate, then $f = 1$, so e is real positive by Lemma 2.29 (4). Since $ex = x$, we clearly have $xL^p(X) \subseteq eL^p(X)$. So $\overline{xL^p(X)} \subseteq eL^p(X)$. Since $x^{1/n}\eta \to e\eta$ weakly for $\eta \in L^p(X)$, it follows that $xL^p(X)$ is weakly, hence norm, dense in $eL^p(X)$. Thus $eL^p(X) = \overline{xL^p(X)}$ and we now have all of (1), as well as (3).

For $\eta \in L^p(X)$ we have $x^{1/n}x\eta \to x\eta$ in norm. Since $(x^{1/n})_{n\in\mathbb{N}}$ is a bounded sequence (using Proposition 2.19 (3)), it follows that $x^{1/n}e\xi \to e\xi$ for all $\xi \in L^p(X)$. Clearly $x^{1/n}(1-e)\xi = 0 \to e(1-e)\xi$ also, using (2) and Proposition 2.19 (4). Thus we have (4).

For (5), note that $fx = x$ if and only if $fe = e$ as in the proof of Corollary 4.13, and similarly $xf = x$ if and only if $ef = e$. Since $fe = e$ if and only if $ef = e$ by Lemma 2.31 (1), the proof of (5) is clear (as in the proof of Corollary 4.13). □

It is shown in [Blecher and Ozawa 2015, Corollary 3.19] that if x, $y \in \mathfrak{r}_A$ then $\overline{xA} \subseteq \overline{yA}$ if and only if $s(y)s(x) = s(x)$.

Lemma 4.15. *Let $p \in (1, \infty)$. Let A be an approximately unital L^p-operator algebra, and let x, $y \in \mathfrak{r}_A$. Then $\overline{xA} = \overline{yA}$ if and only if $s(x) = s(y)$.*

Proof. If $s(x) = s(y)$ then $\overline{xA} = \overline{yA}$ by [Blecher and Ozawa 2015, Corollary 3.18]. Conversely, if $\overline{xA} = \overline{yA}$ then by [Blecher and Ozawa 2015, Corollary 3.18] we have $s(x)A^{**} = s(y)A^{**}$. It follows that $s(x)s(y) = s(y)$ and $s(y)s(x) = s(x)$. By Proposition 4.11 (6) and Lemma 2.31 (2), the second equation implies $s(x)s(y) = s(x)$. So $s(x) = s(y)$. □

Unlike the L^2-operator algebra case (see, for example, [Blecher and Read 2011, Lemma 2.5]), if $x \in \frac{1}{2}\mathfrak{F}_A$ (that is, if $\|1 - 2x\| \le 1$), then $s(x)$ need not be contractive. An example is $x = \frac{1}{2}(1 - e_3)$, for e_3 as in Example 3.2.

4E. *Some consequences of strict convexity of L^p-spaces.*

Lemma 4.16. *Let E be a strictly convex Banach space, and let $f \in B(E)$ be a contractive idempotent. Let $\xi \in E$ satisfy $\|f\xi\| = \|\xi\|$. Then $f\xi = \xi$.*

Proof. This is well known. Suppose that $\xi \ne f\xi$. Set $\eta = \frac{1}{2}(\xi + f\xi)$. Then $\|\eta\| < \|f\xi\|$, giving the contradiction $\|f\xi\| = \|f\eta\| \le \|\eta\| < \|f\xi\|$. □

Lemma 4.17. *Let $p \in (1, \infty)$, let E and F be Banach spaces, and let $S \subseteq B(E, F)$ be a linear subspace. Define matrix norms on $B(E, F)$ by interpreting elements of $M_n(B(E, F))$ as linear maps from the l^p direct sum of n copies of E to the l^p direct sum of n copies of F. Then any $\varphi \in \text{Ball}(S^*)$ is p-completely contractive in the sense of [Pisier 1990].*

Proof. This follows by essentially the argument in the L^2-operator space case, and no doubt this is well known. By the usual argument (see, e.g., the proof of [Daws 2010, Lemma 4.2]), we have to show that

$$\left\| \sum_{j,k=1}^{n} \beta_j x_{j,k} \alpha_k \right\| \le \sup\left(\left\{ \left(\sum_{j=1}^{n} \left\| \sum_{k=1}^{n} x_{j,k}\xi_k \right\|^p \right)^{1/p} : \sum_{k=1}^{n} \|\xi_k\|^p \le 1 \right\} \right),$$

where $n \in \mathbb{N}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \text{Ball}(l_n^q)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \text{Ball}(l_n^p)$,

$\xi_1, \xi_2, \ldots, \xi_n \in E,$ and $x_{j,k} \in S$ for $j, k = 1, 2, \ldots, n.$ However the latter supremum may be written as

$$\sup\left(\left\{\left|\sum_{j=1}^{n} \psi_j\left(\sum_{k=1}^{n} x_{j,k}\xi_k\right)\right| : \sum_{k} \|\xi_k\|^p \leq 1, \ \sum_{j=1}^{n} \|\psi_j\|^q \leq 1\right\}\right),$$

where $\xi_1, \xi_2, \ldots, \xi_n \in E$ and $\psi_1, \psi_2, \ldots, \psi_n \in F^*.$ This supremum clearly dominates

$$\sup\left(\left\{\left|\sum_{j,k=1}^{n} \beta_j \psi(x_{j,k}\alpha_k\xi)\right| : \psi \in \text{Ball}(F^*), \ \xi \in \text{Ball}(E)\right\}\right),$$

since $\sum_{j=1}^{n} \|\beta_j\psi\|^q \leq 1$ and $\sum_{k=1}^{n} \|\alpha_k\xi\|^p \leq 1.$ This last supremum is equal to $\left\|\sum_{j,k=1}^{n} \beta_j x_{j,k}\alpha_k\right\|.$ $\square$

Both the following lemmas apply in particular to hermitian idempotents, by parts (2) and (4) of Lemma 2.29.

Lemma 4.18. *Let E be a Banach space, let $\omega \in \text{Ball}(E^*),$ and let $\xi \in \text{Ball}(E).$ Let φ be the vector state on $B(E)$ given by $\varphi(a) = \langle \omega, a\xi \rangle$ for all $a \in B(E).$ Let $e \in B(E)$ be a real positive idempotent, and suppose $\varphi(e) = 0.$*

(1) *If E is strictly convex then $\varphi(ae) = 0$ for all $a \in A.$*

(2) *If E^* is strictly convex then $\varphi(ea) = 0$ for all $a \in A.$*

Proof. From $\varphi(e) = 0$ we get $\varphi(1 - e) = 1.$ Also $\|1 - e\| \leq 1$ by Lemma 2.29 (4). Suppose E is strictly convex. We have

$$\|(1 - e)\xi\| \geq |\varphi(1 - e)| = 1 = \|\xi\|.$$

So $\xi = (1 - e)\xi$ by Lemma 4.16. For $a \in B(E)$ we then have $\varphi(ae) = \langle \omega, ae\xi \rangle = 0.$

Now suppose E^* is strictly convex. We have $\|(1 - e)^*\| = \|1 - e\| \leq 1,$ and $\varphi(a) = \langle a^*\omega, \xi \rangle$ for all $a \in B(E),$ so

$$\|(1 - e)^*\omega\| \geq |\varphi(1 - e)| = 1 = \|\omega\|.$$

So $\omega = (1 - e)^*\omega$ by Lemma 4.16, and for $a \in B(E)$ we then have $\varphi(ea) = \langle e^*\omega, a\xi \rangle = 0.$ $\square$

Lemma 4.19. *Let $p \in (1, \infty),$ and let A be a unital L^p-operator algebra. Let φ be a state on A and let $e \in A$ be a real positive idempotent. If $\varphi(e) = 0$ then $\varphi(ae) = \varphi(ea) = 0$ for all $a \in A.$*

Proof. We may assume that A is a unital subalgebra of $B(L^p(X))$ for some $X.$ By Lemma 4.17, φ is p-completely contractive in the sense of [Pisier 1990]. So by Theorem 2.1 of that paper and the remark after it, and using the fact that ultraproducts of L^p-spaces are L^p-spaces (Theorem 3.3 (ii) of [Heinrich 1980]), there exist an SQ_p-space $E,$ $\xi \in \text{Ball}(E),$ $\omega \in \text{Ball}(E^*),$ and a p-completely

contractive map $\pi : A \to B(E)$ such that $\varphi(a) = \langle \omega, \pi(a)\xi \rangle$ for all $a \in A$. It is easy to see and no doubt well known that π may be taken to be a unital homomorphism. Then $\pi(e)$ is an idempotent, and $\|1 - \pi(e)\| \leq 1$. As explained in Remark 2.34, SQ_p-spaces are both smooth and strictly convex. So their duals are also strictly convex. We may therefore apply Lemma 4.18 to the vector state $\langle \omega, \cdot \xi \rangle$ on $B(E)$. Thus for all $a \in E$ we have

$$\varphi(ae) = \langle \omega, \pi(a)\pi(e)\xi \rangle = 0 \quad \text{and} \quad \varphi(ea) = \langle \omega, \pi(e)\pi(a)\xi \rangle = 0. \qquad \square$$

Remark 4.20. (1) Lemma 4.19 holds if A is a unital SQ_p-operator algebra. The proof is the same too, but with L^p replaced by SQ_p throughout. For further details on the construction of π in this case see [Pisier 1990, Theorem 2.1] and, e.g., [Daws 2010, Theorem 4.1].

(2) Lemma 4.19 holds for an approximately unital L^p- (or SQ_p-) operator algebra A, and indeed holds for restrictions of states on any L^p- (or SQ_p-) operator algebra unitization of A. This follows by applying the unital case to the extending state on the unitization of A.

Corollary 4.21. *Let* $p \in (1, \infty)$. *Let* A *be an approximately unital* L^p-*operator algebra. If* $x \in \mathfrak{r}_A$ *and* $\varphi \in S(A)$ *with* $\varphi(s(x)) = 0$, *then* $\varphi(x) = 0$. *Conversely, if further* $x \in \mathfrak{F}_A$ *and* $\varphi(x) = 0$ *then* $\varphi(s(x)) = 0$.

Proof. We may work in A^1 by extending φ to a state there, and we may thus assume that A is unital.

The idempotent $s(x)$ is real positive by Proposition 4.11 (6). Using Lemma 4.19, Proposition 4.11 (3), and $\varphi(s(x)) = 0$, we get $\varphi(x) = 0$.

On the other hand, if $x \in \mathfrak{F}_A$ and $\varphi(x) = 0$ then $\varphi(1 - x) = 1$. As in the proof of Lemma 4.19, there are an SQ_p-space F, a contractive unital homomorphism $\pi : A^1 \to B(F)$, $\xi \in \text{Ball}(F)$, and $\eta \in \text{Ball}(F^*)$, such that $\varphi = \langle \pi(\cdot)\xi, \eta \rangle$ for all $a \in A^1$. Then

$$1 = \varphi(1 - x) = \langle \omega, (1 - \pi(x))\xi \rangle \leq \|\pi(1 - x)\xi\| \leq 1.$$

Therefore, with $\frac{1}{p} + \frac{1}{q} = 1$, both ξ and $(1 - \pi(x))\xi$ define norm one linear functionals on $L^q(X)$ which take ω to 1. Strict convexity of $L^q(X)$ implies $(1 - \pi(x))\xi = \xi$. So $\pi(x)\xi = 0$. Proposition 2.19 (4) now implies that $\pi(x^{1/n})\xi = 0$ for all $n \in \mathbb{N}$. Hence $\varphi(x^{1/n}) = 0$. In the limit $\varphi(s(x)) = 0$. $\qquad \square$

Lemma 4.22 is a generalization of part of [Blecher and Read 2011, Lemma 2.10], with a similar proof but using Corollary 4.21.

Lemma 4.22. *Let* $p \in (1, \infty)$. *Let* A *be an approximately unital* L^p-*operator algebra, and let* $x \in \mathfrak{F}_A$. *The following are equivalent*:

(1) $s(x) = 1$.

(2) $\varphi(x) \neq 0$ *for all* $\varphi \in S(A)$.

(3) $\mathrm{Re}(\varphi(x)) > 0$ *for all* $\varphi \in S(A)$.

If $x \in \mathfrak{r}_A$ *then* (3) *implies* (2) *and* (2) *implies* (1).

Proof. Let $x \in \mathfrak{r}_A$. Then (3) implies (2) trivially. To show that (2) implies (1), suppose (1) fails. Represent A^{**} as a unital subalgebra of $B(L^p(X))$ for some X by Corollary 2.36. Choose $\xi \in \mathrm{Ball}(L^p(X))$ in the range of the idempotent $1 - s(x)$, and choose $\eta \in \mathrm{Ball}(L^p(X)^*)$ with $\langle \xi, \eta \rangle = 1$. Then $\varphi(x) = \langle x\xi, \eta \rangle$ defines a state on A with $\varphi(1 - s(x)) = 1$. Since $\varphi(s(x)) = 0$, Corollary 4.21 implies $\varphi(x) = 0$.

If $x \in \mathfrak{F}_A$ then (1) implies (2) by Corollary 4.21. For (2) implies (3), follow part of the proof of [Blecher and Read 2011, Lemma 2.10]: $|1 - \varphi(x)| \leq 1$ is not compatible with both $\varphi(x) \neq 0$ and $\mathrm{Re}(\varphi(x)) \leq 0$. □

4F. *Hahn–Banach smoothness of L^p-operator algebras.*

Definition 4.23. Let E be a Banach space and let $M \subseteq E$ be a closed subspace. We say that M is *Hahn–Banach smooth* in E if for every $\omega_0 \in M^*$ there is a unique $\omega \in E^*$ with $\|\omega\| = \|\omega_0\|$ and $\omega|_M = \omega_0$,

Existence of ω is just the Hahn–Banach theorem. When verifying this property, we need only consider the case $\|\omega_0\| = 1$.

Proposition 2.1.18 in [Blecher and Le Merdy 2004] works for L^p-operator algebras.

Proposition 4.24. *Let* $p \in (1, \infty)$. *Let* A *be an approximately unital* L^p-*operator algebra and denote the identity of* A^1 *by* 1.

(1) *Let* $(e_t)_{t \in \Lambda}$ *be a cai in* A. *If* $\psi : A^1 \to \mathbb{C}$ *is a functional on* A^1, *then* $\lim_t \psi(e_t) = \psi(1)$ *if and only if* $\|\psi\| = \|\psi|_A\|$.

(2) A *is Hahn–Banach smooth in* A^1 (*Definition 4.23*).

Proof. We may assume that A is nonunital (the case of unital algebras being easy).

The forward direction of (1) is just as in the proof of [Blecher and Le Merdy 2004, Proposition 2.1.18].

For the other direction suppose that $\psi : A^1 \to \mathbb{C}$ with $\|\psi\| = \|\psi|_A\| = 1$. As in the proof of Lemma 4.19, there are an SQ_p-space F, a contractive unital homomorphism $\pi : A^1 \to B(F)$, $\xi \in \mathrm{Ball}(F)$, and $\eta \in \mathrm{Ball}(F^*)$, such that $\psi = \langle \pi(\cdot)\xi, \eta \rangle$ for all $a \in A^1$.

Apply the extension of Lemma 2.33 given in Remark 2.34 to the representation $\pi|_A$. Let $E \subseteq F$ and the idempotent $f \in B(F)$ be as there. The extensions of parts (1) and (3) of Lemma 2.33 imply that $\pi(e_t) \to f$ weak* in $B(F)$ and $\pi(a) = \pi(a)f$ for all $a \in A$. Thus, for all $a \in A$,

$$\big|\langle \pi(a)\xi, \eta \rangle\big| = \big|\langle \pi(a)f\xi, \eta \rangle\big| \leq \|a\|\|f\xi\|.$$

This shows that $\|\psi|_A\| \leq \|f\xi\|$. Hence, by hypothesis, $\|f\xi\| = \|\xi\| = 1$. Since E is strictly convex (see Remark 2.34), Lemma 4.16 implies $f\xi = \xi$, that is, $\xi \in \overline{\mathrm{span}}(\pi(A)E)$. Now, since $\pi(e_t) \to f$ weak* we have

$$\langle \pi(e_t)\xi, \eta \rangle \to \langle f\xi, \eta \rangle = \langle \xi, \eta \rangle,$$

which says that $\psi(e_t) \to \psi(1)$.

For the deduction of (2) from (1), let $\varphi \in A^*$ satisfy $\|\varphi\| = 1$. Proceed as in the proof of [Blecher and Le Merdy 2004, Proposition 2.1.18], but beginning by writing $\varphi \in A^*$ as $\varphi = \langle \pi(\cdot)\zeta, \eta \rangle$ for E as above, and for a contractive homomorphism $\pi : A \to B(E)$, $\zeta \in \mathrm{Ball}(E)$, and $\eta \in \mathrm{Ball}(E^*)$. This may be done for example by considering a Hahn–Banach extension of φ to A^1 and using the unital case above. □

Corollary 4.25. *Let* $p \in (1, \infty)$, *and let* A *be a nonunital approximately unital* L^p-*operator algebra.*

(1) *Let* $\varphi \in A^*$ *satisfy* $\|\varphi\| = 1$. *Then the following are equivalent*:

    (a) *φ is a state on A, that is (see Definition 2.6), φ extends to a state on A^1.*

    (b) *$\varphi(e_t) \to 1$ for every cai $(e_t)_{t\in\Lambda}$ for A.*

    (c) *$\varphi(e_t) \to 1$ for some cai $(e_t)_{t\in\Lambda}$ for A.*

    (d) *$\varphi(1_{A^{**}}) = 1$.*

(2) *Every state on A has a unique extension to a state on A^1.*

Proof. Everything is immediate from Proposition 4.24. □

Part (1) says that states on such algebras may be defined by any one of the equivalent conditions in Lemma 2.2 of [Blecher and Ozawa 2015]. The change in the statement of the last condition is justified by Lemma 1.14.

In the notation of Definition 2.6 (taken from [Blecher and Ozawa 2015]), for any cai $\mathfrak{e} = (e_t)_{t\in\Lambda}$ of A we have $S_{\mathfrak{e}}(A) = S(A)$. That is, states on an approximately unital L^p-operator algebra are the contractive functionals φ with $\varphi(e_t) \to 1$, or equivalently have norm 1 and extend to a state on A^1 (or on A^{**}).

We remark that the last several results hold (beginning with Lemma 4.19) if A is an approximately unital SQ_p-operator algebra. The proofs are almost identical, but with the kinds of emendations prescribed in the proof of Lemma 2.1 for SQ_p-spaces, Remark 4.20, and Remark 2.34.

The definition of a scaled Banach algebra, used in the next proposition, is stated in the introduction (see also the beginning of Section 6 below).

Proposition 4.26. *Suppose that A is an approximately unital scaled Banach algebra, that A is Hahn–Banach smooth in A^1 (Definition 4.23), and that A^{**} is unital. Then $\mathfrak{r}_{A^{**}}$ as defined in [Blecher and Ozawa 2015] (after Lemma 2.5 there) agrees with the set of accretive elements of the unital Banach algebra A^{**}.*

We are ignoring the statement in [Blecher and Ozawa 2015] that the definition there is only to be applied when A^{**} is not unital.

To be explicit, let R_0 be the set of accretive elements of A^{**}, where A^{**} is thought of as a unital Banach algebra in its own right, and let R_1 be the analogous subset of $(A^1)^{**}$. Then the assertion of the proposition is that $R_1 \cap A^{**} = R_0$.

Proof of Proposition 4.26. We may assume that A is nonunital (the case of unital algebras being easy). To avoid confusion, we use the notation R_0 and R_1 above.

We show $R_1 \cap A^{**} \subseteq R_0$. Proposition 2.11 of [Blecher and Ozawa 2015] (which works also when A^{**} is unital) implies that the weak* closure of $\mathfrak{r}_A$ is $R_1 \cap A^{**}$. So we need to show that $\mathfrak{r}_A \subseteq R_0$ and that R_0 is weak* closed. The second part is, e.g., Theorem 2.2 of [Magajna 2009]; the set $D_{A^{**}}$ (following the notation there) is $\{-a : a \in R_0\}$. One way to see the first part is that part of the proof of Lemma 1.14 shows that every cai for A converges weak* to $1_{A^{**}}$. Given this, the argument for Lemma 2.25 (1) shows that the subalgebra $A + \mathbb{C} \cdot 1_{A^{**}} \subseteq A^{**}$ is isometrically isomorphic to A^1. Thus, if $a \in \mathfrak{r}_A$, then $a \in \mathfrak{r}_{A^1}$ by Definition 2.13, so $a \in R_0$ by Lemma 2.14.

It remains to show that $R_0 \subseteq R_1$. Let $a \in R_0$, and let φ be a state on $(A^1)^{**}$. By weak* density of the normal states in $S((A^1)^{**})$ (which follows from Theorem 2.2 of [Magajna 2009]) there is a net $(\psi_t)_{t \in \Lambda}$ in $S(A^1)$ such that $\psi_t \to \varphi$ weak*. For $t \in \Lambda$, since A is scaled, there are $\lambda \in [0, 1]$ and $\omega \in S(A)$ such that $\psi_t = \lambda \omega$. Since A is Hahn–Banach smooth in A^1, [Blecher and Ozawa 2015, Lemma 2.2] implies that the canonical weak* continuous extension of ω is a state on A^{**}. So $\mathrm{Re}(\omega(a)) \geq 0$, whence $\mathrm{Re}(\psi_t(a)) \geq 0$. Then $\mathrm{Re}(\varphi(a)) = \lim_t \mathrm{Re}(\psi_t(a)) \geq 0$. So $a \in R_1$. $\square$

5. M-ideals

We recall the definitions of M-ideals and M-summands, together with some elementary facts. See, for example, Definition I.1.1 of [Harmand et al. 1993] and the discussion afterwards. If E is a Banach space and $P \in B(E)$ is an idempotent, then P is called an *L-projection* if $\|\xi\| = \|P\xi\| + \|(1 - P)\xi\|$ for all $\xi \in E$, and an *M-projection* if $\|\xi\| = \max(\|P\xi\|, \|(1 - P)\xi\|)$ for all $\xi \in E$. The ranges of L-projections and M-projections are called *L-summands* and *M-summands*. The idempotent P is an M-projection if and only if P^* is an L-projection, and is an L-projection if and only if P^* is an M-projection. Finally, a subspace $J \subseteq E$ is an *M-ideal* if $J^\perp$ is an L-summand in E^*, equivalently (using [Harmand et al. 1993, Theorem I.1.9]), $J^{\perp\perp}$ is an M-summand in E^{**}. By [Harmand et al. 1993, Proposition I.1.2 (a)], if J is an M-summand, then there is exactly one contractive idempotent with range J, namely the M-projection used in the definition.

Smith and Ward [1978] showed that the M-ideals in a C^*-algebra are exactly the closed ideals in the usual sense (Theorem 5.3), that an M-ideal in a unital Banach

algebra must be a subalgebra (Theorem 3.6), and that M-ideals in Banach algebras are often ideals (see, for example, Theorem 3.8). Example 4.1 of [Smith and Ward 1978] shows that there are M-ideals in $B(l_2^1)$ which are subalgebras but not ideals and do not have cais.

The following definition is from the introduction to [Blecher and Ozawa 2015].

Definition 5.1. Let A be a Banach algebra. We say that A is *M-approximately unital* if A is an M-ideal in the multiplier unitization A^1.

As in the introduction to [Blecher and Ozawa 2015], an M-approximately unital Banach algebra is approximately unital. The papers [Blecher and Ozawa 2015; Blecher 2016] give a number of properties of M-approximately unital Banach algebras. For example, an M-approximately unital Banach algebra has a real positive cai $(e_t)_{t \in \Lambda}$ satisfying $\|1 - 2e_t\| \leq 1$ for all $t \in \Lambda$ ([Blecher and Ozawa 2015, Theorem 5.2]), is Hahn–Banach smooth in its multiplier unitization (Proposition I.1.12 of [Harmand et al. 1993]), and has the Kaplansky density properties given in [Blecher and Ozawa 2015, Theorem 5.2 and Proposition 6.4].

Proposition 5.2. *Let $p \in (1, \infty) \setminus \{2\}$ and let (X, μ) be a measure space. Then $\mathbb{K}(L^p(X, \mu))$ is M-approximately unital if and only if μ is purely atomic.*

Proof. Theorem 11 of [Lima 1979] states that $\mathbb{K}(L^p(X, \mu))$ is an M-ideal in $B(L^p(X, \mu))$ if and only if μ is purely atomic. By Theorem VI.4.17 in [Harmand et al. 1993], $\mathbb{K}(L^p(X, \mu))$ is an M-ideal in $B(L^p(X, \mu))$ if and only if it is an M-ideal in $\mathbb{K}(L^p(X, \mu)) + \mathbb{C} 1$, where 1 is the identity operator on $L^p(X, \mu)$. By Lemma 2.24, $\mathbb{K}(L^p(X, \mu)) + \mathbb{C} 1$ is the multiplier unitization of $\mathbb{K}(L^p(X, \mu))$. $\square$

Lemma 5.3. *Let A be an approximately unital Arens regular Banach algebra, and let $J \subseteq A$ be an M-ideal in A, with associated M-projection $P : A^{**} \to J^{\perp\perp}$. Then J is an approximately unital closed ideal if and only if $P(1)$ is central in A^{**}.*

Proof. By the discussion before Proposition 8.1 of [Blecher and Ozawa 2015], centrality of $P(1)$ implies that J is an approximately unital closed ideal.

If J is an approximately unital closed ideal then, as in the proof of Lemma 4.5, there is a central idempotent e such that $J^{\perp\perp} = eA^{**} = A^{**}e$. The uniqueness of projections onto an M-summand implies that P is multiplication by e. So $P(1) = e$ is central. $\square$

Theorem 5.4. *Let $p \in (1, \infty)$ and let A be an L^p-operator algebra.*

(1) *Suppose that A is approximately unital. Then every M-ideal in A is an approximately unital closed ideal.*

(2) *Suppose that A is unital. Then $J \subseteq A$ is an M-summand if and only if there is a central hermitian idempotent $z \in A$ such that $J = Az$. In this case, multiplication by z is an M-projection with range J.*

(3) *Suppose that A is M-approximately unital (Definition 5.1). Then*:

    (a) *Every M-ideal in A is M-approximately unital.*

    (b) *The intersection of finitely many M-ideals in A is an M-ideal in A.*

    (c) *The closed ideal generated by any collection of M-ideals in A is an M-ideal in A.*

Proof. We prove (2). We may assume (by the discussion above Proposition 2.12, or the corollary on page 136 in [Lacey 1974]) that there is a decomposable measure space X such that A is a unital subalgebra of $B(L^p(X, \mu))$.

Suppose that $z \in A$ is a central hermitian idempotent. Then z is a hermitian idempotent in $B(L^p(X, \mu))$. It follows from Proposition 2.12 that z is multiplication by the characteristic function of a locally measurable subset E of X. Thus for $x, y \in A$ (with suitable interpretation of the integrals below if E is only locally measurable),

$$\|zxz + (1-z)y(1-z)\|^p$$
$$= \sup\left(\left\{\int_E |xz\xi|^p \, d\mu + \int_{X \setminus E} |y(1-z)\xi|^p \, d\mu : \xi \in \mathrm{Ball}(L^p(X))\right\}\right)$$
$$\leq \max(\|x\|, \|y\|)^p \sup\left(\left\{\int_E |\xi|^p \, d\mu + \int_{X \setminus E} |\xi|^p \, d\mu : \xi \in \mathrm{Ball}(L^p(X))\right\}\right)$$
$$= \max(\|x\|, \|y\|)^p.$$

So multiplication by z is an M-projection on A and zA is an M-summand.

Conversely, let P be an M-projection on A, and let $z = P(1)$. By [Smith and Ward 1978, Proposition 3.1], z is a hermitian idempotent. Also, P^* is an L-projection, so for any state φ on A, if $P^*(\varphi) \neq 0$ then $\psi = \|P^*(\varphi)\|^{-1} P^*(\varphi)$ is a state with $\psi(z) = 1$ as in the proof of 4.8.5 in [Blecher and Le Merdy 2004]. It follows from Lemma 4.19 that

$$\psi((1-z)A) = \psi(A(1-z)) = 0.$$

So

$$\varphi(P((1-z)A)) = \varphi(P(A(1-z))) = 0$$

for any state φ on A. Thus

$$P((1-z)A) = P(A(1-z)) = 0$$

by Lemma 2.25 (3). A similar argument applied to $1 - P$ shows that

$$(1 - P)(zA) = (1 - P)(Az) = 0.$$

So $zA + Az \subseteq P(A)$. Thus

$$P(a) = P(za + (1-z)a) = P(za) = za,$$

for all $a \in A$, and similarly $P(a) = az$. So z is central and $P(A) = Az$. This completes the proof of (2).

We prove (1). Let $J \subseteq A$ be an M-ideal. Since $J^{\perp\perp}$ is an M-ideal in A^{**}, since A is Arens regular (Lemma 2.1 (1)), and since A^{**} is an L^p-operator algebra (Lemma 2.1 (3)), we can apply part (2) and Lemma 5.3.

Part (3)(a) follows from [Blecher and Ozawa 2015, Proposition 3.2(3)] and part (1), and (3)(b) and (3)(c) now follow from [Blecher and Ozawa 2015, Theorem 8.3]. □

Remark 5.5. Let A be an approximately unital L^p-operator algebra. The proof of Theorem 5.4 shows that the h-ideals, as defined at the beginning of Section 3 of [Godefroy et al. 1993], are exactly the M-ideals. One may ask if these are also the u-ideals as defined before our Lemma 4.4. This is not true: the idempotent e_2 in Example 3.2 gives a u-projection which is not an M-projection, since as we said there e_2 is not hermitian. Suppose that A is a u-ideal in its multiplier unitization A^1, or, equivalently, as pointed out before Lemma 4.4, that A is bi-approximately unital. One may ask whether it follows that A is an M-ideal in A^1. As we will see in Corollary 6.2, the latter is equivalent to being scaled. Recall from Lemma 4.4 that an approximately unital L^p-operator algebra A with a real positive bounded approximate identity is bi-approximately unital. (We conjectured after Lemma 4.4 that the converse is true.)

6. Scaled L^p-operator algebras

In the Introduction we said that an approximately unital Banach algebra A is *scaled* if the set of restrictions to A of states on A^1 equals the quasistate space $Q(A)$ of A. Equivalently, (see [Blecher and Ozawa 2015], before Lemma 2.7 there) an approximately unital Banach algebra is scaled if every real positive functional (see Definition 2.13) is a nonnegative multiple of a state. That is, in the notation of Definitions 2.6 and 2.13, we have $\mathfrak{c}_{A^*} = \mathbb{R}_+ \, S(A)$, or, equivalently, $\mathfrak{c}_{A^*} \cap \mathrm{Ball}(A^*) = Q(A)$.

Unital Banach algebras are scaled (this is a special case of [Blecher and Ozawa 2015, Proposition 6.2]), and all C^*-algebras are well known to be scaled.

If A is a nonunital approximately unital Arens regular Banach algebra, then the support idempotent of A in $(A^1)^{**}$ is the weak* limit in $(A^1)^{**}$ of any cai in A. This exists and is an identity for A^{**} by the argument of Lemma 1.14. Clearly it is central in $(A^1)^{**}$.

Lemma 6.1. *Suppose that A is a nonunital scaled approximately unital Arens regular Banach algebra. Then the support idempotent of A in $(A^1)^{**}$ is hermitian.*

Proof. Suppose that A is scaled and $(e_t)_{t \in \Lambda}$ is a cai for A. Then, as above, $(e_t)_{t \in \Lambda}$ converges weak* to a central idempotent $e \in (A^1)^{**}$ which is an identity for A^{**}.

If φ is a state on A^1 then $\varphi|_A$ is a nonnegative multiple, r say, of a state on A, so that $\varphi(e) = \lim_t \varphi(e_t) = r \geq 0$. So every weak* continuous state on $(A^1)^{**}$ is

nonnegative on e. Since the weak* continuous states on a dual Banach algebra are weak* dense in the states by [Magajna 2009, Theorem 2.2], it follows from Lemma 2.9 that e is hermitian. □

The last result says that scaled approximately unital Arens regular Banach algebras are h-ideals in their multiplier unitizations as defined at the beginning of Section 3 of [Godefroy et al. 1993].

Corollary 6.2. *Suppose that A is an approximately unital Arens regular Banach algebra with the property that whenever $e \in (A^1)^{**}$ is a hermitian idempotent and $x, y \in A$, then $\|exe + (1 - e)y(1 - e)\| \leq \max(\|x\|, \|y\|)$. Then A is scaled if and only if A is M-approximately unital.*

Proof. Since unital algebras are both scaled and approximately unital, we may assume that A is nonunital. If A is M-approximately unital then A is scaled by [Blecher and Ozawa 2015, Proposition 6.2]. For the other direction, by Lemma 6.1 and the hypothesis, the support idempotent e of A in $(A^1)^{**}$ satisfies

$$\|ex + (1 - e)y\| \leq \max(\|x\|, \|y\|) \quad \text{for } x, y \in A.$$

By Goldstine's theorem and separate weak* continuity of multiplication ([Blecher and Le Merdy 2004, 2.5.3]), this inequality holds for all $x, y \in A^{**}$. So multiplication by e is an M-projection on $(A^1)^{**}$. Therefore A is an M-ideal in A^1. □

Corollary 6.3. *Let A be an approximately unital L^p-operator algebra. Then A is scaled if and only if A is M-approximately unital.*

Proof. Use Corollary 6.2 and a computation in the proof of Theorem 5.4 (2). □

Corollary 6.4. *Let $p \in (1, \infty)$ and let A be a nonunital approximately unital L^p-operator algebra. Then the following are equivalent*:

(1) *A is scaled.*

(2) *The support idempotent e of A in $(A^1)^{**}$ (as defined at the beginning of the section) is hermitian.*

(3) *The quasistate space $Q(A)$ is weak* compact.*

If these hold then, by Corollary 6.3, A has all the properties of M-approximately unital algebras described after Definition 5.1.

Proof of Corollary 6.4. The implication from (1) to (2) is Lemma 6.1. For the reverse, if e is hermitian then multiplication by e is an M-projection from $(A^1)^{**}$ to A^{**} by Theorem 5.4 (2), so A is M-approximately unital. Apply Corollary 6.3.

For the equivalence with (3), first, $Q(A)$ is weak* compact if and only if it is weak* closed. Also, Corollary 4.25 (1) implies convexity of $Q(A)$, as explained in Remark 2.26. Apply Lemma 2.7 (2) in [Blecher and Ozawa 2015]. □

The following answers the open question from [Blecher and Ozawa 2015] as to whether all approximately unital Banach algebras are scaled.

Corollary 6.5. *If $p \in (1, \infty) \setminus \{2\}$ then $\mathbb{K}(L^p([0, 1]))$ is an approximately unital L^p-operator algebra which is not scaled.*

Proof. That $\mathbb{K}(L^p([0, 1]))$ has a cai is observed in Example 3.9. This algebra is not M-approximately unital by Proposition 5.2, and so is not scaled by Corollary 6.3. $\square$

Corollary 6.6. *The algebra $\mathbb{K}(l^p)$ is scaled.*

Proof. This algebra is M-approximately unital by Proposition 5.2, hence scaled by Corollary 6.3. $\square$

The last result can also be deduced from Proposition 6.7.

Proposition 6.7. *Let $p \in (1, \infty)$. Suppose that an L^p-operator algebra A has a cai $(e_t)_{t \in \Lambda}$ consisting of hermitian elements of A^1. Then A is scaled.*

Proof. Since unital algebras are scaled, we may assume that A is nonunital. With $e_t \to e$ as usual, it follows as in the proof of Lemma 2.25 (4) (using the fact that normal states are weak* dense) that e is hermitian and central in $(A^1)^{**}$. It follows from Theorem 5.4 or Corollary 6.4 that A is M-approximately unital and scaled. $\square$

Proposition 6.7 may suggest that one requirement for a nonunital L^p-operator algebra to be "C^*-like" is that it have a hermitian cai. The canonical cai for $\mathbb{K}(l^p)$ is a real positive hermitian cai as we said in Example 3.9. On the other hand the cai for $\mathbb{K}(L^p([0, 1]))$ in Example 3.9 seems, perhaps surprisingly, to have no good "positivity" properties. Indeed as we said in Example 3.9, $A = \mathbb{K}(L^p([0, 1]))$ has no real positive cai. Of course for any approximately unital L^p-operator algebra the identity e of A^{**} is real positive in A^{**}. However e need not be real positive (accretive) in $(A^1)^{**}$, and certainly is not hermitian, as we said after the proof of Lemma 2.25. Example 3.8 shows that the converse of Proposition 6.7 is false.

Proposition 6.8. *Let $p \in (1, \infty)$. Suppose that A is a closed subalgebra of a scaled L^p-operator algebra B, with a common cai. Then A is scaled.*

Proof. We may assume that A is nonunital (unital algebras are scaled). We may view $A^1 \subseteq B^1$. Any state φ of A^1 extends to a state of B^1, and the restriction of this extension to B equals $\lambda \psi$ for some $\lambda \in [0, 1]$ and $\psi \in S(B)$. However $\psi|_A \in S(A)$ since Corollary 4.25 (1) implies that $\psi(e_t) \to 1$, where $(e_t)_{t \in \Lambda}$ is the common cai. Since $\varphi|_A = \lambda \psi|_A$ we are done. $\square$

Remark 6.9. Approximately unital ideals in a scaled L^p-operator algebra A need not be scaled, for example $\mathbb{K}(L^p([0, 1]))$ in $B(L^p([0, 1]))$. (The latter is scaled as is any unital Banach algebra, and we showed above that $\mathbb{K}(L^p([0, 1]))$ is not scaled.) However if the approximately unital ideal is also an M-ideal in A, then it is scaled by Theorem 5.4.

7. Kaplansky density

One may ask if in an approximately unital L^p-operator algebra there are Kaplansky density theorems analogous to the ones established by Blecher and Read for approximately unital L^2-operator algebras. See, e.g., [Blecher and Ozawa 2015, Theorem 5.2 and Proposition 6.4] for a more general variant of the latter. As we said in the introduction, the usual Kaplansky density theorem variants for C^*-algebras can be shown to follow easily from the weak* density of the subset of interest in A within the matching set in A^{**}; and our Kaplansky density theorems have this flavor.

In the following result, for an approximately unital L^p-operator algebra A we take $\mathfrak{r}_{A^{**}}$ to be the accretive elements in the unital Banach algebra A^{**}. This is different from the definition after Lemma 2.5 in [Blecher and Ozawa 2015]. The two definitions do coincide if also A is scaled, by Proposition 4.26 and Proposition 4.24 (2).

Proposition 7.1. *Let $p \in (1, \infty)$ and let A be an approximately unital L^p-operator algebra. The following are equivalent*:

(1) $\mathfrak{r}_A$ *is weak* dense in* $\mathfrak{r}_{A^{**}}$.

(2) $\mathfrak{r}_A \cap \mathrm{Ball}(A)$ *is weak* dense in* $\mathfrak{r}_{A^{**}} \cap \mathrm{Ball}(A^{**})$.

(3) $\mathfrak{F}_A$ *is weak* dense in* $\mathfrak{F}_{A^{**}}$.

(4) *A is scaled.*

Proof. Since the definition of $\mathfrak{r}_{A^{**}}$ in [Blecher and Ozawa 2015] coincides with ours when A is scaled (as pointed out above), that (4) implies (1) follows from Proposition 2.11 of [Blecher and Ozawa 2015]. The proof of Lemma 6.4 of [Blecher and Ozawa 2015] works just as well for our version of $\mathfrak{r}_{A^{**}}$ as for the one there, and thus shows that (1) implies (2). By our Corollary 6.3 and by Theorem 5.2 of [Blecher and Ozawa 2015] it follows that (4) implies (3). That (3) implies (1) follows easily from Proposition 2.17.

Assuming (2) we will prove (4) by showing that every nontrivial real positive functional φ (see Definition 2.13) is a nonnegative multiple of a state. We may assume that A is nonunital. The canonical weak* continuous extension $\widetilde{\varphi}$ of φ to A^{**} is real positive by our assumption (2) and a standard approximation argument. Since A^{**} is unital it is scaled, so that $\widetilde{\varphi} = t\psi$ for a state ψ on A^{**} and some $t > 0$. Thus $\varphi = t\psi|_A$. The span of A and the identity of A^{**} is the multiplier unitization of A by the last paragraph of Section 1 of [Blecher and Ozawa 2015]. Hence $\psi|_A$ is a state on A. $\square$

These hold in particular if A is unital. Such results also hold if A has the following property: with 1 being the identity of some unitization of A, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in A$ with $\|1 - y\| < 1 + \delta$ then there is $z \in A$ with $\|1 - z\| = 1$ and $\|y - z\| < \varepsilon$. This follows from [Blecher and Ozawa 2015, Proposition 6.4] and the proof of [Blecher and Ozawa 2015, Theorem 5.2]. It may be interesting to ascertain which L^p-operator algebras have this property.

We end by mentioning some of what seem to us to be the most important open questions related to the approach of this paper. See the "Remarks added in proof" section below for the solution to several of these.

(1) Is there a Kaplansky density type theorem for a nonscaled approximately unital L^p-operator algebra A? (See Proposition 7.1 for the scaled case.) For example, one may ask if $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{(A^1)^{**}} \cap A^{**}$.

(2) Is every approximately unital subalgebra of $B(l^p)$ scaled?

(3) Let A be an approximately unital L^p-operator algebra. Is $\mathfrak{r}_A - \mathfrak{r}_A$ always a subalgebra?

(4) Is every bi-approximately unital L^p-operator algebra scaled? Does it have a real positive cai? More drastically, if an L^p-operator algebra possesses a real positive cai, then is it scaled?

8. Index

For the readers' convenience we list, alphabetically but compactly, some of the main definitions in this paper and where they may be found (the definition number, which is usually their first occurrence).

Accretive: 2.13; approximately unital Banach algebra: 1.4; Arens products, Arens regular: 1.12; bi-approximately unital algebra: 4.2; bi-approximately unital ideal: 4.1; bicontractive idempotent: 2.28; $\mathfrak{c}_{A^*}$: 2.13; cai: 1.4; decomposable: 2.11; dual L^p-operator algebra: 2.3; $\mathfrak{F}_A$: 2.16; Hahn–Banach smooth: 4.23; hermitian: 2.8; invertible isometry: 2.28; locally measurable, locally a.e.: 2.11; L^p-operator algebra: 1.6; M-approximately unital: 5.1; M-ideal, M-projection, M-summand: beginning of Section 5; multiplier unitization A^1: 1.8; order on idempotents $e \leq_r f$, $e \leq f$: 2.30; powers and roots b^t: 2.18; quasistate space $Q(A)$: 2.6; $\mathfrak{r}_A$: 2.13; real positive: 2.13; scaled: beginning of Section 6; smooth: 1.5; SQ_p-algebra, SQ_p-space: the introduction; state space $S(A)$: 2.6; strictly convex: 1.5; support idempotent $s(x)$: 4.12; unital Banach algebra: 1.3; unitization: 1.7.

Other definitions may be found in the introduction, or in the sections where they first appear (often at the start of the section), but are not specifically numbered.

Acknowledgements

Remarks added in proof

We are indebted to an anonymous reviewer of a different paper who referred us to the paper [Berkson 1972], which contains somewhat of a systematic study of hermitian idempotents on general Banach spaces, and which is well worth studying. There is not much overlap with that paper but some of the motifs are similar. For example Theorem 2.25 in Berkson's paper is our Lemma 4.16, but as we said there, this result is well known. Also, that paper has some nice examples in general Banach spaces complementing some of the examples which we give.

The paper [Phillips and Viola 2017] is in the process of revision, and this will change the section and result numbering. The references to that paper here refer to the arXiv version 2, as linked to in our bibliography.

Finally, question (2) at the end of Section 7 and the first and third questions stated in (4) there, have negative solutions. A counterexample to all three is the set of continuous functions from $[0, 1]$ to M_2^p with $f(0) \in \mathbb{C} e_2$. We give full details elsewhere.

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