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THE CENTER OF A GREEN BISET FUNCTOR

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For a Green biset functor A, we define the commutant and the center of A and we study some of their properties and their relationship. This leads in particular to the main application of these constructions: the possibility of splitting the category of A-modules as a direct product of smaller abelian categories. We give explicit examples of such decompositions for some classical shifted representation functors. These constructions are inspired by similar ones for Mackey functors for a fixed finite group.

Introduction

This paper is devoted to the construction of two analogues of the center of a ring in the realm of Green biset functors, that is "biset functors with a compatible ring structure". For a Green biset functor A, we present *the commutant* CA of A, defined from a commutation property, and *the center* ZA of A, defined from the structure of the category of A-modules. Both CA and ZA are again Green biset functors. These constructions are inspired by similar ones for Mackey functors for a fixed finite group made in Chapter 12 of [Bouc 1997].

The commutant CA is always a Green biset subfunctor of A, and we say that A is *commutative* if $CA = A$. Most of the classical representation functors are commutative in that sense. One of them plays a fundamental — we should say *initial* — role, namely the Burnside biset functor B, as biset functors are nothing but *modules* over the Burnside functor. An important feature of the category B-Mod is its monoidal structure: given two biset functors M and N, one can build their tensor product $M \otimes N$, which is again a biset functor. For this tensor product, the category B-Mod becomes a symmetric monoidal category, and a Green biset functor A is a *monoid object* in B-Mod.

More generally, for any Green biset functor A, we consider the category A-Mod of A-modules. We will make a heavy use of the equivalence of categories between A-Mod and the category of linear representations of the category $\mathcal{P}_A$ introduced in Chapter 8 of [Bouc 2010] (see also Definition 9 below), which has finite groups as objects, and in which the set of morphisms from G to H is equal to $A(H \times G)$.

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The category $\mathcal{P}_B$ associated to the Burnside functor is precisely the *biset category* of finite groups. It is a symmetric monoidal category (for the product given by the direct product of groups), and this monoidal structure induces via Day convolution ([Day 1970]) the monoidal structure of B-Mod mentioned before.

A natural question is then to know when the cartesian product of groups endows the category $\mathcal{P}_A$ with a symmetric monoidal structure, and we show that this is the case precisely when A is commutative. In this case the category A-Mod also becomes a symmetric monoidal category.

Even though the definition of the center ZA of a Green biset functor A is fairly natural, showing that it is endowed with a Green biset functor structure (even showing that $ZA(G)$ is indeed a set!) is not an easy task; it requires several sometimes rather nasty computations. On the other hand, one of the rewarding consequences of this laborious process is that we obtain a description of $ZA(G)$ in terms also of a commutation condition, this time on the morphisms of $\mathcal{P}_A$. Once we have that ZA is indeed a Green biset functor, we show some nice properties of it, for instance that there is an injective morphism of Green biset functors from CA to ZA. This implies in particular that ZA is a CA-module. We show also that in the case where A is commutative, it is a direct summand of ZA as A-modules.

In the last section, we work within $ZA(1)$, which, as we will see, coincides with the center of the category A-Mod. Any decomposition of the identity element of $ZA(1)$ as a sum of orthogonal idempotents, fulfilling certain finiteness conditions, allows us to decompose A-Mod as a direct product of smaller abelian categories. Moreover, since $CA(1)$ is generally easier to compute than $ZA(1)$, we can also use similar decompositions of the identity element of $CA(1)$ instead, thanks to the inclusion $CA \hookrightarrow ZA$. We give then a series of explicit examples. The first one is the Burnside p-biset functor $A = RB_p$ over a ring R where the prime p is invertible. In this case, we obtain an infinite series of orthogonal idempotents in $ZA(1)$, and this shows in particular that ZA can be much bigger than CA. Next we consider some classical representation functors, shifted by some fixed finite group L via the *Yoneda–Dress functor*. In this series of examples, we will see that the smaller abelian categories obtained in the decomposition are also module categories for Green biset functors arising from the functor A, the shifting group L, and the above-mentioned idempotents.

1. Preliminaries

Throughout the paper, we fix a commutative unital ring R. All referred groups will be finite. The center of a ring S will be denoted by $Z(S)$.

1.1. ***Green biset functors.*** The biset category over R will be denoted by $R\mathcal{C}$. Recall that its objects are all finite groups, and that for finite groups G and H, the hom-set

$\mathrm{Hom}_{R\mathcal{C}}(G, H)$ is $RB(H, G) = R \otimes_{\mathbb{Z}} B(H, G)$, where $B(H, G)$ is the Grothendieck group of the category of finite (H, G)-bisets. The composition of morphisms in $R\mathcal{C}$ is induced by R-bilinearity from the composition of bisets, which will be denoted by $\circ$.

We fix a nonempty class $\mathcal{D}$ of finite groups closed under subquotients and cartesian products, and a set $\boldsymbol{D}$ of representatives of isomorphism classes of groups in $\mathcal{D}$. We denote by $R\mathcal{D}$ the full subcategory of $R\mathcal{C}$ consisting of groups in $\mathcal{D}$, so in particular $R\mathcal{D}$ is a *replete subcategory* of $R\mathcal{C}$, in the sense of [Bouc 2010, Definition 4.1.7]. The category of biset functors, i.e., the category of R-linear functors from $R\mathcal{C}$ to the category R-Mod of all R-modules, will be denoted by Fun_R. The category $\mathrm{Fun}_{\mathcal{D},R}$ of *$\mathcal{D}$-biset functors* is the category of R-linear functors from $R\mathcal{D}$ to R-Mod.

A Green $\mathcal{D}$-biset functor is defined as a monoid in $\mathrm{Fun}_{\mathcal{D},R}$ (see Definition 8.5.1 in [Bouc 2010]). This is equivalent to the following definition:

Definition 1. A $\mathcal{D}$-biset functor A is a Green $\mathcal{D}$-biset functor if it is equipped with bilinear products $A(G) \times A(H) \to A(G \times H)$ denoted by $(a, b) \mapsto a \times b$, for groups G, H in $\mathcal{D}$, and an identity element $\varepsilon_A \in A(1)$, satisfying the following conditions:

1. Associativity: Let G, H and K be groups in $\mathcal{D}$. If we consider the canonical isomorphism from $G \times (H \times K)$ to $(G \times H) \times K$, then for any $a \in A(G)$, $b \in A(H)$ and $c \in A(K)$,
$$(a \times b) \times c = A\big(\mathrm{Iso}_{G\times(H\times K)}^{(G\times H)\times K}\big)(a \times (b \times c)).$$
2. Identity element: Let G be a group in $\mathcal{D}$ and consider the canonical isomorphisms $1 \times G \to G$ and $G \times 1 \to G$. Then for any $a \in A(G)$,
$$a = A\big(\mathrm{Iso}_{1\times G}^{G}\big)(\varepsilon_A \times a) = A\big(\mathrm{Iso}_{G\times 1}^{G}\big)(a \times \varepsilon_A).$$
3. Functoriality: If $\varphi : G \to G'$ and $\psi : H \to H'$ are morphisms in $R\mathcal{D}$, then for any $a \in A(G)$ and $b \in A(H)$,
$$A(\varphi \times \psi)(a \times b) = A(\varphi)(a) \times A(\psi)(b).$$

The identity element of A will be denoted simply by ε if there is no risk of confusion.

If A and C are Green $\mathcal{D}$-biset functors, a morphism of Green $\mathcal{D}$-biset functors from A to C is a natural transformation $f : A \to C$ such that $f_{H\times K}(a\times b) = f_H(a)\times f_K(b)$ for any groups H and K in $\mathcal{D}$ and any $a \in A(H)$, $b \in A(K)$, and such that $f_1(\varepsilon_A) = \varepsilon_C$. We will denote by $\mathrm{Green}_{\mathcal{D},R}$ the category of Green $\mathcal{D}$-biset functors with morphisms given in this way.

There is an equivalent way of defining a Green biset functor; see Lemma 3.

Definition 2. A $\mathcal{D}$-biset functor A is a Green $\mathcal{D}$-biset functor provided that for each group H in $\mathcal{D}$, the R-module $A(H)$ is an R-algebra with unity that satisfies the following. If K and G are groups in $\mathcal{D}$ and $K \to G$ is a group homomorphism, then:

1. For the (K, G)-biset G, which we denote by G_r, the morphism $A(G_r)$ is a ring homomorphism.
2. For the (G, K)-biset G, denoted by G_l, the morphism $A(G_l)$ satisfies the Frobenius identities
$$A(G_l)(a) \cdot b = A(G_l)(a \cdot A(G_r)(b)),$$
$$b \cdot A(G_l)(a) = A(G_l)(A(G_r)(b) \cdot a)$$
for all $b \in A(G)$ and $a \in A(K)$, where $\cdot$ denotes the ring product on $A(G)$ and $A(K)$, respectively.

Lemma 3 [Romero 2011, Lema 4.2.3]. *Definitions 1 and 2 are equivalent. Starting with Definition 1, the ring structure of* $A(H)$ *is given by*
$$a \cdot b = A\big(\mathrm{Iso}^{H}_{\Delta(H)} \circ \mathrm{Res}^{H\times H}_{\Delta(H)}\big)(a \times b)$$
for a *and* b *in* $A(H)$, *with the unity given by* $A(\mathrm{Inf}^{H}_{1})(\varepsilon)$. *Conversely, starting with Definition 2, the product of* $A(G) \times A(H) \to A(G \times H)$ *is given by*
$$a \times b = A\big(\mathrm{Inf}^{G\times H}_{G}\big)(a) \cdot A\big(\mathrm{Inf}^{G\times H}_{H}\big)(b)$$
for $a \in A(G)$ *and* $b \in A(H)$, *with the identity element given by the unity of* $A(1)$.

In what follows, the ring structure on $A(G)$ will be understood as $(A(G), \cdot)$.

Observe that in the case of $A(1)$, the product $\times : A(1) \times A(1) \to A(1)$ coincides with the ring product $\cdot : A(1) \times A(1) \to A(1)$, up to identification of 1×1 with 1, and the unity coincides with the identity element.

Remark 4. A morphism of Green $\mathcal{D}$-biset functors $f : A \to C$ induces, in each component G, a unital ring homomorphism $f_G : A(G) \to C(G)$. Conversely, a morphism of biset functors $f : A \to C$ such that f_G is a unital ring homomorphism for every G in $\mathcal{D}$, is a morphism of Green $\mathcal{D}$-biset functors.

Example 5. Classical examples of Green biset functors are the following:

- The Burnside functor B. The Burnside group of a finite group G is known to define a biset functor. The cross product of sets defines the bilinear products $B(G) \times B(H) \to B(G \times H)$ that make B a Green biset functor. The functor B can also be considered with coefficients in R, and denoted by $RB = R \otimes_{\mathbb{Z}} B(\_)$. It is shown in Proposition 8.6.1 of [Bouc 2010] that RB is an initial object in $\mathrm{Green}_{\mathcal{D},R}$. More precisely, for a Green $\mathcal{D}$-biset functor A, the unique morphism of Green functors $\upsilon_A : RB \to A$ is defined at $G \in \mathcal{D}$ as the linear map $\upsilon_{A,G}$ sending a G-set X to $A({}_GX_1)(\varepsilon_A)$, where ${}_GX_1$ is the set X viewed as a $(G, 1)$-biset.

• The functor of $\mathbb{K}$-linear representations, $R_{\mathbb{K}}$, where $\mathbb{K}$ is a field of characteristic 0. That is, the functor which sends a finite group G to the Grothendieck group $R_{\mathbb{K}}(G)$ of the category of finitely generated $\mathbb{K}G$-modules. Also known to be a biset functor, it has a Green biset functor structure given by the tensor product over $\mathbb{K}$. We will consider the scalar extension $\mathbb{F}R_{\mathbb{K}} = \mathbb{F} \otimes_{\mathbb{Z}} R_{\mathbb{K}}(\_)$, where $\mathbb{F}$ is a field of characteristic 0.

• The functor of p-permutation representations pp_k, for k an algebraically closed field of positive characteristic p. This is the functor sending a finite group G to the Grothendieck group $pp_k(G)$ of the category of finitely generated p-permutation kG-modules (also known as trivial source modules), for relations given by direct sum decompositions. The biset functor pp_k is a Green biset functor with products given by the tensor product over the field k. When considering coefficients for this functor, we will assume that $\mathbb{F}$ is a field of characteristic 0 containing all the p'-roots of unity, and we write $\mathbb{F}pp_k = \mathbb{F} \otimes_{\mathbb{Z}} pp_k(\_)$.

In Section 5.2 we will focus on the above examples only, but there are many other important examples of Green biset functors, e.g., the monomial Burnside functor — also called the fibred Burnside functor — which gives rise to fibred biset functors (see [Barker 2004; Romero 2013; Boltje and Coşkun 2018]), or the slice Burnside functor (see [Bouc 2012; Tounkara 2018a; 2018b]).

When p is a prime number, and $\mathcal{D}$ is the full subcategory of $\mathcal{C}$ consisting of finite p-groups, the $\mathcal{D}$-biset functors are simply called *p-biset functors*, and their category is denoted by $\mathrm{Fun}_{p,R}$. Similarly, the Green $\mathcal{D}$-biset functors will be called *Green p-biset functors*, and their category will be denoted by $\mathrm{Green}_{p,R}$.

An important element in what follows will be the Yoneda–Dress construction. We recall some of the basic results about it, more details can be found in [Bouc 2010, Section 8.2]. If G is a fixed group in $\mathcal{D}$ and F is a $\mathcal{D}$-biset functor, then the Yoneda–Dress construction of F at G is the $\mathcal{D}$-biset functor F_G that sends each group K in $\mathcal{D}$ to $F(K \times G)$. The morphism $F_G(\varphi) : F(H \times G) \to F(K \times G)$ associated to an element φ in $RB(K, H)$ is defined as $F(\varphi \times G)$. In turn, $F(\varphi \times G)$ is defined by R-bilinearity from the case where φ is represented by a (K, H)-biset U: in this case, $\varphi \times G$ denotes the cartesian product $U \times G$, endowed with its obvious $(K \times G, H \times G)$-biset structure. We also call F_G the functor *shifted* by G.

If $f : F \to T$ is a morphism of $\mathcal{D}$-biset functors, then $f_G : F_G \to T_G$ is defined in its component K as $(f_G)_K = f_{K\times G}$. It is shown in Proposition 8.2.7 of [Bouc 2010] that this construction is a self-adjoint exact R-linear endofunctor of $\mathrm{Fun}_{D,R}$.

When A is a Green $\mathcal{D}$-biset functor, the particular shifted functor A_G is also a Green $\mathcal{D}$-biset functor (Lemma 4.4 in [Romero 2012]) with product given by

$$A_G(H) \times A_G(K) \to A_G(H \times K), \quad (a, b) \mapsto A(\alpha)(a \times b),$$

where α is the biset $\mathrm{Iso}_D^{H\times K\times G}\mathrm{Res}_D^{H\times G\times K\times G}$ and $D \cong H \times K \times G$ is the subgroup of $H \times G \times K \times G$ consisting of elements of the form (h, g, k, g). Usually, by an abuse of notation, we will denote this biset simply by $\mathrm{Res}^{H\times G\times K\times G}_{H\times K\times\Delta(G)}$. To avoid confusion with the product $\times$ of A we denote the product of A_G by $\times^d$, where the exponent d stands for *diagonal*.

Remark 6. It is not hard to show that the ring structure of Lemma 3 in $A_G(H)$ induced by the product $\times^d$ of A_G coincides with the ring structure of $A(H \times G)$ induced by the product $\times$ of A. So there is no risk of confusion when talking about *the ring* $A_G(H)$, since the ring structure we are considering is unique. In particular, the isomorphism $A_G(1) \cong A(G)$ is an isomorphism of rings.

1.2. *A-modules.*

Definition 7 [Bouc 2010, Definition 8.5.5]. Given a Green $\mathcal{D}$-biset functor A, a left A-module M is defined as a $\mathcal{D}$-biset functor, together with bilinear products

$$\_ \times \_ : A(G) \times M(H) \to M(G \times H)$$

for every pair of groups G and H in $\mathcal{D}$, that satisfy analogous conditions to those of Definition 1. The notion of right A-module is defined similarly, from bilinear products $M(G) \times A(H) \to M(G \times H)$.

We use the same notation $\times$ for the product of A and the action of A on A-modules, as long as there is no risk of confusion.

If M and N are A-modules, a *morphism of A-modules* is defined as a morphism of $\mathcal{D}$-biset functors $f : M \to N$ such that $f_{G\times H}(a \times m) = a \times f_H(m)$ for all groups G and H in $\mathcal{D}$, $a \in A(G)$ and $m \in M(H)$. With these morphisms, the A-modules form a category, denoted by A-Mod. The category A-Mod is an abelian subcategory of $\mathrm{Fun}_{\mathcal{D},R}$. Actually, the direct sum of biset functors is also the direct sum of A-modules. Furthermore, the kernel, the image and the cokernel of a morphism of A-modules are A-modules. Basic results on modules over a ring can be stated for A-modules.

In particular, a left (resp. right) ideal of a Green $\mathcal{D}$-biset functor A is an A-submodule of the left (resp. right) A-module A. A two-sided ideal of A is a left ideal which is also a right ideal.

Example 8. If A is the Burnside functor RB, then an A-module is nothing but a biset functor with values in R-Mod.

From Proposition 8.6.1 of [Bouc 2010], or Proposition 2.11 of [Romero 2012], an equivalent way of defining an A-module is as an R-linear functor from the category $\mathcal{P}_A$ to R-Mod, where the category $\mathcal{P}_A$ is defined next.

Definition 9. Let A be a Green $\mathcal{D}$-biset functor over R. The category $\mathcal{P}_A$ is defined in the following way:

- The objects of $\mathcal{P}_A$ are all finite groups in $\mathcal{D}$.
- If G and H are groups in $\mathcal{D}$, then $\mathrm{Hom}_{\mathcal{P}_A}(H, G) = A(G \times H)$.
- Let H, G and K be groups in $\mathcal{D}$. The composition of $\beta \in A(H \times G)$ and $\alpha \in A(G \times K)$ in $\mathcal{P}_A$ is

$$\beta \circ \alpha = A\big(\mathrm{Def}^{H\times\Delta(G)\times K}_{H\times K} \circ \mathrm{Res}^{H\times G\times G\times K}_{H\times\Delta(G)\times K}\big)(\beta \times \alpha).$$

- For a group G in $\mathcal{D}$, the identity morphism ε_G of G in $\mathcal{P}_A$ is

$$A\big(\mathrm{Ind}^{G\times G}_{\Delta(G)} \circ \mathrm{Inf}^{\Delta(G)}_{1}\big)(\varepsilon).$$

Observe that the biset $\mathrm{Def}^{H\times\Delta(G)\times K}_{H\times K} \circ \mathrm{Res}^{H\times G\times G\times K}_{H\times\Delta(G)\times K}$ can also be written as

$$H \times \big(\mathrm{Def}^{\Delta(G)}_{1} \circ \mathrm{Res}^{G\times G}_{\Delta(G)}\big) \times K.$$

Another way of denoting the $(1, G \times G)$-biset $\mathrm{Def}^{\Delta(G)}_{1} \circ \mathrm{Res}^{G\times G}_{\Delta(G)}$ is as $\overleftarrow{G}$. In some cases it will be more convenient to use this notation.

The category $\mathcal{P}_A$ is essentially small, as it has a skeleton consisting of our chosen set $\boldsymbol{D}$ of representatives of isomorphism classes of groups in $\mathcal{D}$. Hence, the category $\mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod})$ of R-linear functors is an abelian category. The above-mentioned equivalence of categories between A-Mod and $\mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod})$ is built as follows:

- If M is an A-module, let $\widetilde{M} \in \mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod})$ be the functor defined by:
  (1) For $G \in \mathcal{D}$, we have $\widetilde{M}(G) = M(G)$.
  (2) For $G, H \in \mathcal{D}$ and a morphism $\alpha \in A(H \times G)$ from G to H in $\mathcal{P}_A$, the map $\tilde{\alpha} : \widetilde{M}(G) \to \widetilde{M}(H)$ is the map sending

$$m \in M(G) \mapsto M(H \times \overleftarrow{G})(\alpha \times m).$$

- Conversely if $F \in \mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod})$, let $\widehat{F}$ be the A-module defined by:
  (1) If $G \in \mathcal{D}$, then $\widehat{F}(G) = F(G)$.
  (2) For $G, H \in \mathcal{D}$, $a \in A(G)$ and $m \in F(H)$, set

$$a \times m = F\big(A\big(\mathrm{Ind}^{G\times H\times H}_{G\times\Delta(H)}\,\mathrm{Inf}^{G\times H}_{G}\big)(a)\big)(m) \in F(G \times H),$$

  where $A\big(\mathrm{Ind}^{G\times H\times H}_{G\times\Delta(H)}\,\mathrm{Inf}^{G\times H}_{G}\big)(a) \in A(G\times H\times H)$ is viewed as a morphism from H to $G \times H$ in the category $\mathcal{P}_A$.

Then $M \mapsto \widetilde{M}$ and $F \mapsto \widehat{F}$ are well-defined equivalences of categories between A-Mod and $\mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod})$, inverse to each other.

Finally, we extend to A-modules our previous definition of the Yoneda–Dress construction.

Definition 10. Let A be a Green $\mathcal{D}$-biset functor. For $L\in\mathcal{D}$, consider the assignment $\rho_L=-\times L$ defined for objects G, H of $\mathcal{P}_A$ and morphisms $\alpha\in\mathrm{Hom}_{\mathcal{P}_A}(G,H)=A(H\times G)$ by

$$\begin{cases}\rho_L(G)=G\times L,\\ \rho_L(\alpha)=\alpha\times L:=\mathrm{Iso}_{H\times G\times L\times L}^{H\times L\times G\times L}(\alpha\times \upsilon_{A,L\times L}(L)),\end{cases}$$

where $\upsilon_{A,L\times L}(L)$ is the image in $A(L\times L)$ of the identity (L,L)-biset L under the canonical morphism $\upsilon_{A,L\times L}$, and the isomorphism $H\times G\times L\times L\to H\times L\times G\times L$ maps (h,g,l_1,l_2) to (h,l_1,g,l_2).

A straightforward computation shows that

$$\rho_L(\alpha)=A\big(\mathrm{Ind}_{H\times G\times L}^{H\times L\times G\times L}\,\mathrm{Inf}_{H\times G}^{H\times G\times L}\big)(\alpha),$$

and this form may be more convenient for calculations. Here $H\times G\times L$ embeds into $H\times L\times G\times L$ via the map $(h,g,l)\mapsto(h,l,g,l)$, and maps surjectively onto $H\times G$ via $(h,g,l)\mapsto(h,g)$.

It is easy to check that ρ_L is in fact an endofunctor of $\mathcal{P}_A$, called the (*right*) *L-shift*. It induces by precomposition an endofunctor of the category $\mathrm{Fun}_R(\mathcal{P}_A,R\text{-Mod})$, that is, up to the above equivalence of categories, an endofunctor of the category A-Mod, which can be described as follows. It maps an A-module M to the shifted $\mathcal{D}$-biset functor M_L, endowed with the following product: for $G,H\in\mathcal{D}$, $\alpha\in A(H)$ and $m\in M_L(G)=M(G\times L)$, the element $\alpha\times m$ of $M_L(H\times G)=M(H\times G\times L)$ is simply the element $\alpha\times m$ obtained from the A-module structure of M.

This endofunctor $M\mapsto M_L$ of the category A-Mod will be denoted by Id_L. It is the Yoneda–Dress construction for A-modules.

Remark 11. For $L\in\mathcal{D}$, there is another obvious endofunctor $\lambda_L=L\times -$ of $\mathcal{P}_A$ defined for objects G, H of $\mathcal{P}_A$ and morphisms $\alpha\in\mathrm{Hom}_{\mathcal{P}_A}(G,H)=A(H\times G)$ by

$$\begin{cases}\lambda_L(G)=L\times G,\\ \lambda_L(\alpha)=L\times\alpha:=\mathrm{Iso}_{L\times L\times H\times G}^{L\times H\times L\times G}(\upsilon_{A,L\times L}(L)\times\alpha),\end{cases}$$

where the isomorphism $L\times L\times H\times G\to L\times H\times L\times G$ maps (l_1,l_2,h,g) to (l_1,h,l_2,g). As before, it is easy to see that $L\times\alpha=A\big(\mathrm{Ind}_{L\times H\times G}^{L\times H\times L\times G}\,\mathrm{Inf}_{H\times G}^{L\times H\times G}\big)(\alpha)$.

It is then natural to ask if the assignment $\times:\mathcal{P}_A\times\mathcal{P}_A\to\mathcal{P}_A$ sending (G,K) to $G\times K$ and $(\alpha,\beta)\in A(H\times G)\times A(L\times K)$ to $(\alpha\times L)\circ(G\times\beta)\in A(H\times L\times G\times K)$ is a functor. We will answer this question at the end of Section 3 (Corollary 26).

2. Adjoint functors

Let A and C be Green $\mathcal{D}$-biset functors. A morphism $f:A\to C$ of Green $\mathcal{D}$-biset functors induces an obvious functor $\mathcal{P}_f:\mathcal{P}_A\to\mathcal{P}_C$, which is the identity on objects, and maps $\alpha \in \mathrm{Hom}_{\mathcal{P}_A}(G, H) = A(H \times G)$ to $f_{H\times G}(\alpha) \in C(H \times G) = \mathrm{Hom}_{\mathcal{P}_C}(G, H)$.

Let L be a fixed group in $\mathcal{D}$. The *inflation* morphism $\mathrm{Inf}_L : A \to A_L$, introduced in [García 2018], is the morphism of Green biset functors defined for each $G \in \mathcal{D}$ and each $\alpha \in A(G)$ by $\mathrm{Inf}_L(\alpha) = A(\mathrm{Inf}_G^{G\times L})(\alpha) \in A(G \times L) = A_L(G)$, where G is identified with $(G \times L)/(\{1\} \times L)$. The corresponding functor $\mathcal{P}_A \to \mathcal{P}_{A_L}$ will be denoted by ψ_L. Explicitly, for each $G \in \mathcal{D}$, we have $\psi_L(G) = G$, and for a morphism $\alpha \in A(H \times G)$, we have

$$\begin{aligned}\psi_L(\alpha) &= A\big(\mathrm{Inf}_{H\times G}^{H\times G\times L}\big)(\alpha) \in A(H \times G \times L)\\ &= A_L(H \times G) = \mathrm{Hom}_{\mathcal{P}_{A_L}}(\psi_L(G), \psi_L(H)).\end{aligned}$$

We introduce another functor $\theta_L : \mathcal{P}_{A_L} \to \mathcal{P}_A$, defined as follows: for an object G of $\mathcal{P}_{A_L}$, we set $\theta_L(G) = G \times L$, viewed as an object of $\mathcal{P}_A$. For a morphism $\alpha \in \mathrm{Hom}_{\mathcal{P}_{A_L}}(G, H) = A_L(H \times G) = A(H \times G \times L)$, we define

$$\theta_L(\alpha) = A\big(\mathrm{Ind}_{H\times G\times L}^{H\times L\times G\times L}\big)(\alpha) \in A(H \times L \times G \times L) = \mathrm{Hom}_{\mathcal{P}_A}(\theta_L(G), \theta_L(H)),$$

where $H \times G \times L$ is viewed as a subgroup of $H \times L \times G \times L$ via the injective group homomorphism $(h, g, l) \in H \times G \times L \mapsto (h, l, g, l) \in H \times L \times G \times L$.

Notation 12. In what follows, we will use a convenient abuse of notation, and generally drop the symbols $\times$ of cartesian products of groups, writing, e.g., $HLGL$ instead of $H \times L \times G \times L$.

Theorem 13. (1) *ψ_L is an R-linear functor from $\mathcal{P}_A$ to $\mathcal{P}_{A_L}$.*

(2) *θ_L is an R-linear functor from $\mathcal{P}_{A_L}$ to $\mathcal{P}_A$.*

(3) *The functors ψ_L and θ_L are left- and right-adjoint to one another. In other words, for any G and H in $\mathcal{D}$, there are R-module isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}_{A_L}}(G, \psi_L(H)) &\cong \mathrm{Hom}_{\mathcal{P}_A}(\theta_L(G), H),\\ \mathrm{Hom}_{\mathcal{P}_{A_L}}(\psi_L(G), H) &\cong \mathrm{Hom}_{\mathcal{P}_A}(G, \theta_L(H))\end{aligned}$$

which are natural in G and H.

Proof. Assertion (1) is clear, since the functor ψ_L is built from a morphism of Green biset functors $\mathrm{Inf}_L : A \to A_L$.

To prove assertion (2), let $G, H, K \in \mathcal{D}$. If $\alpha \in A_L(HG)$ and $\beta \in A_L(KH)$, then

$$\begin{aligned}&\theta_L(\beta) \circ \theta_L(\alpha)\\ &\qquad = A\big(\mathrm{Def}_{KLGL}^{KLHLGL}\,\mathrm{Res}_{KLHLGL}^{KLHLHLGL}\big)\big(A\big(\mathrm{Ind}_{KHL}^{KLHL}\big)(\beta) \times A\big(\mathrm{Ind}_{HGL}^{HLGL}\big)(\alpha)\big)\\ &\qquad = A\big(\mathrm{Def}_{KLGL}^{KLHLGL}\,\mathrm{Res}_{KLHLGL}^{KLHLHLGL}\mathrm{Ind}_{KHLHGL}^{KLHLHLGL}\big)(\beta \times \alpha).\end{aligned}$$

In the restriction $\mathrm{Res}_{KLHLGL}^{KLHLHLGL}$, the group $KLHLGL$ maps into $KLHLHLGL$ via

$$f : (k, l_1, h, l_2, g, l_3) \in KLHLGL \mapsto (k, l_1, h, l_2, h, l_2, g, l_3) \in KLHLHLGL,$$

and in the induction $\mathrm{Ind}_{KHLHGL}^{KLHLHLGL}$, the group $KHLHGL$ maps into $KLHLHLGL$ via

$$f' : (k', h'_1, l'_1, h'_2, g, l'_2) \in KHLHGL \mapsto (k', l'_1, h'_1, l'_1, h'_2, l'_2, g', l'_2) \in KLHLHLGL.$$

Then one checks easily that $\mathrm{Im}(f)\mathrm{Im}(f') = KLHLHLGL$, and that $\mathrm{Im}(f) \cap \mathrm{Im}(f')$ is isomorphic to $KHGL$. Hence by the Mackey formula, there is an isomorphism of bisets

$$\mathrm{Res}_{KLHLGL}^{KLHLHLGL} \mathrm{Ind}_{KHLHGL}^{KLHLHLGL} \cong \mathrm{Ind}_{KHGL}^{KLHLGL} \mathrm{Res}_{KHGL}^{KHLHGL},$$

where in $\mathrm{Ind}_{KHGL}^{KLHLGL}$, the inclusion $KHGL \hookrightarrow KLHLGL$ is

$$(k, h, g, l) \mapsto (k, l, h, l, g, l),$$

and in $\mathrm{Res}_{KHGL}^{KHLHGL}$, the inclusion $KHGL \hookrightarrow KHLHGL$ is

$$(k, h, g, l) \mapsto (k, h, l, h, g, l).$$

Now in the deflation $\mathrm{Def}_{KLGL}^{KLHLGL}$, the group $KLHLGL$ maps onto $KLGL$ via $(k, l_1, h, l_2, g, l_3) \mapsto (k, l_1, g, l_3)$. It follows that there is an isomorphism of bisets

$$\mathrm{Def}_{KLGL}^{KLHLGL} \mathrm{Ind}_{KHGL}^{KLHLGL} \cong \mathrm{Ind}_{KGL}^{KLGL} \mathrm{Def}_{KGL}^{KHGL},$$

which gives

$$\begin{aligned}
\theta_L(\beta) \circ \theta_L(\alpha) &= A\big(\mathrm{Ind}_{KGL}^{KLGL}\big) A\big(\mathrm{Def}_{KGL}^{KHGL} \mathrm{Res}_{KHGL}^{KHLHGL}\big)(\beta \times \alpha) \\
&= A\big(\mathrm{Ind}_{KGL}^{KLGL}\big) A_L\big(\mathrm{Def}_{KG}^{KHG} \mathrm{Res}_{KHG}^{KHHG}\big) A\big(\mathrm{Res}_{KHHGL}^{KHLHGL}\big)(\beta \times \alpha) \\
&= A\big(\mathrm{Ind}_{KGL}^{KLGL}\big) A_L\big(\mathrm{Def}_{KG}^{KHG} \mathrm{Res}_{KHG}^{KHHG}\big)(\beta \times^d \alpha) \\
&= A\big(\mathrm{Ind}_{KGL}^{KLGL}\big)(\beta \circ^d \alpha) \\
&= \theta_L(\beta \circ^d \alpha),
\end{aligned}$$

where $\circ^d$ denotes the composition in the category $\mathcal{P}_{A_L}$. This shows that θ_L is compatible with composition of morphisms. A straightforward computation shows that it maps identity morphisms to identity morphisms. This completes the proof of assertion (2), since θ_L is obviously R-linear.

The complete proof of assertion (3) demands the verification of many technical details, so we only include the full proof that θ_L is left-adjoint to ψ_L. We next simply give the description of the bijection involved in the other direction, and leave the corresponding verifications to the reader.

For G and H in $\mathcal{D}$, we have

$$\mathrm{Hom}_{\mathcal{P}_{A_L}}(G, \psi_L(H)) = A_L(\psi_L(H)G) = A(HGL),$$

and

$$\mathrm{Hom}_{\mathcal{P}_A}(\theta_L(G), H) = A(HGL),$$

so the identity map of $A(HGL)$ is an obvious candidate for an isomorphism $\mathrm{Hom}_{\mathcal{P}_{A_L}}(G, \psi_L(H)) \to \mathrm{Hom}_{\mathcal{P}_A}(\theta_L(G), H)$. For $\alpha \in \mathrm{Hom}_{\mathcal{P}_{A_L}}(G, \psi_L(H))$, we denote by $\tilde{\alpha}$ the element α viewed as an element of $\mathrm{Hom}_{\mathcal{P}_A}(\theta_L(G), H)$, to avoid confusion.

We now check that the map $\alpha \mapsto \tilde{\alpha}$ is natural in G and H. For naturality in G, if $G' \in \mathcal{D}$ and $u \in \mathrm{Hom}_{A_L}(G', G)$, we have the diagrams

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & \psi_L(H) \\ {\scriptstyle u}\uparrow & \nearrow{\scriptstyle \alpha u} & \\ G' & & \end{array} \qquad \begin{array}{ccc} \theta_L(G) & \xrightarrow{\tilde{\alpha}} & H \\ {\scriptstyle \theta_L(u)}\uparrow & \nearrow{\scriptstyle \widetilde{\alpha \circ^d u}} & \\ \theta_L(G') & & \end{array}$$

and we have to show that the right-hand side diagram is commutative, i.e., that

$$\widetilde{\alpha \circ^d u} = \tilde{\alpha} \circ \theta_L(u).$$

But

$$\begin{aligned} \tilde{\alpha} \circ \theta_L(u) &= \alpha \circ A\big(\mathrm{Ind}_{GG'L}^{GLG'L}\big)(u) \\ &= A\big(\mathrm{Def}_{HG'L}^{HGLG'L}\,\mathrm{Res}_{HGLG'L}^{HGLGLG'L}\big)\big(\alpha \times A\big(\mathrm{Ind}_{GG'L}^{GLG'L}\big)(u)\big) \\ &= A\big(\mathrm{Def}_{HG'L}^{HGLG'L}\,\mathrm{Res}_{HGLG'L}^{HGLGLG'L}\mathrm{Ind}_{HGLGG'L}^{HGLGLG'L}\big)(\alpha \times u). \end{aligned}$$

In the restriction $\mathrm{Res}_{HGLG'L}^{HGLGLG'L}$, the inclusion $HGLG'L \hookrightarrow HGLGLG'L$ is the map

$$f : (h, g, l_1, g', l_2) \in HGLG'L \mapsto (h, g, l_1, g, l_1, g', l_2) \in HGLGLG'L,$$

and in the induction $\mathrm{Ind}_{HGLGG'L}^{HGLGLG'L}$, the inclusion $HGLGG'L \hookrightarrow HGLGLG'L$ is the map

$$f' : (\eta, \gamma_1, \lambda_1, \gamma_2, \gamma', \lambda_2) \in HGLGG'L \mapsto (\eta, \gamma_1, \lambda_1, \gamma_2, \lambda_2, \gamma', \lambda_2) \in HGLGLG'L.$$

Then clearly $\mathrm{Im}(f)\mathrm{Im}(f') = HGLGLG'L$, and $\mathrm{Im}(f) \cap \mathrm{Im}(f') \cong HGG'L$. By the Mackey formula, this gives an isomorphism of bisets

$$\mathrm{Res}_{HGLG'L}^{HGLGLG'L}\mathrm{Ind}_{HGLGG'L}^{HGLGLG'L} \cong \mathrm{Ind}_{HGG'L}^{HGLG'L}\mathrm{Res}_{HGG'L}^{HGLGG'L},$$

where, in $\mathrm{Ind}_{HGG'L}^{HGLG'L}$, the inclusion $HGG'L \hookrightarrow HGLG'L$ is

$$(h, g, g', l) \mapsto (h, g, l, g', l),$$

and in $\mathrm{Res}_{HGG'L}^{HGLGG'L}$, the inclusion $HGG'L \hookrightarrow HGLGG'L$ is

$$(h, g, g', l) \mapsto (h, g, l, g, g', l).$$

Now in $\mathrm{Def}_{HG'L}^{HGLG'L}$, the quotient map $HGLG'L \to HG'L$ sends (h, g, l_1, g', l_2) to (h, g', l_2), so the image of the subgroup $HGG'L$ is the whole of $HG'L$. It follows that there is an isomorphism of bisets

$$\mathrm{Def}_{HG'L}^{HGLG'L} \,\mathrm{Ind}_{HGG'L}^{HGLG'L} \cong \mathrm{Def}_{HG'L}^{HGG'L},$$

which gives finally

$$\begin{aligned}
\tilde{\alpha} \circ \theta_L(u) &= A\big(\mathrm{Def}_{HG'L}^{HGG'L} \,\mathrm{Res}_{HGG'L}^{HGLGG'L}\big)(\alpha \times u) \\
&= A_L\big(\mathrm{Def}_{HG'}^{HGG'}\big) A_L\big(\mathrm{Res}_{HGG'}^{HGGG'}\big) A\big(\mathrm{Res}_{HGGG'L}^{HGLGG'L}\big)(\alpha \times u) \\
&= A_L\big(\mathrm{Def}_{HG'}^{HGG'}\big) A_L\big(\mathrm{Res}_{HGG'}^{HGGG'}\big)(\alpha \times^d u) \\
&= \alpha \circ^d u,
\end{aligned}$$

as was to be shown.

We check that the map $\alpha \mapsto \tilde{\alpha}$ is natural in H. If $H' \in \mathcal{D}$ and $v \in \mathrm{Hom}_{\mathcal{P}_A}(H, H') = A(H'H)$, we have the diagrams

$$\begin{array}{ccc}
A & \xrightarrow{\ \alpha\ } & \psi_L(H) \\
& {\scriptstyle \psi_L(v)\circ^d \alpha} \searrow & \downarrow {\scriptstyle \psi_L(v)} \\
& & \psi_L(H')
\end{array}
\qquad\qquad
\begin{array}{ccc}
\theta_L(G) & \xrightarrow{\ \tilde{\alpha}\ } & H \\
& {\scriptstyle \widetilde{\psi_L(v)\circ^d \alpha}} \searrow & \downarrow {\scriptstyle v} \\
& & H'
\end{array}$$

and we have to show that the right-hand side diagram is commutative, i.e., that

$$\widetilde{\psi_L(v) \circ^d \alpha} = v \circ \tilde{\alpha}.$$

But

$$\begin{aligned}
\psi_L(v) \circ^d \alpha &= A\big(\mathrm{Inf}_{H'H}^{H'HL}\big)(v) \circ^d \alpha \\
&= A_L\big(\mathrm{Def}_{H'G}^{H'HG} \,\mathrm{Res}_{H'HG}^{H'HHG}\big)\big(A\big(\mathrm{Inf}_{H'H}^{H'HL}\big)(v) \times^d \alpha\big) \\
&= A\big(\mathrm{Def}_{H'GL}^{H'HGL} \,\mathrm{Res}_{H'HGL}^{H'HHGL}\big) A\big(\mathrm{Res}_{H'HHGL}^{H'HLHGL} \,\mathrm{Inf}_{H'HHGL}^{H'HLHGL}\big)(v \times \alpha) \\
&= A\big(\mathrm{Def}_{H'GL}^{H'HGL} \,\mathrm{Res}_{H'HGL}^{H'HLHGL} \,\mathrm{Inf}_{H'HHGL}^{H'HLHGL}\big)(v \times \alpha).
\end{aligned}$$

In $\mathrm{Res}_{H'HGL}^{H'HLHGL}$, the inclusion $H'HGL \hookrightarrow H'HLHGL$ is

$$(h', h, g, l) \mapsto (h', h, l, h, g, l),$$

and in $\mathrm{Inf}_{H'HHGL}^{H'HLHGL}$, the quotient map $H'HLHGL \to H'HHGL$ is

$$(h', h_1, l_1, h_2, g, l_2) \mapsto (h', h_1, h_2, g, l_2).$$

The composition of these two maps sends (h', h, g, l) to (h', h, h, g, l), hence it is injective. This gives an isomorphism of bisets

$$\mathrm{Res}^{H'HLHGL}_{H'HGL}\,\mathrm{Inf}^{H'HLHGL}_{H'HHGL} \cong \mathrm{Res}^{H'HHGL}_{H'HGL},$$

from which we get

$$\psi_L(v)\circ^d \alpha = A\big(\mathrm{Def}^{H'HGL}_{H'GL}\,\mathrm{Res}^{H'HHGL}_{H'HGL}\big)(v\times\alpha) = v\circ\tilde{\alpha},$$

as was to be shown.

Hence the isomorphism $\alpha\in \mathrm{Hom}_{\mathcal{P}_{A_L}}(G,\psi_L(H)) \mapsto \tilde{\alpha}\in \mathrm{Hom}_{\mathcal{P}_A}(\theta_L(G),H)$ is natural in G and H, so θ_L is left-adjoint to ψ_L.

We now describe the bijection implying that θ_L is also right-adjoint to ψ_L. So, for $G, H\in\mathcal{D}$, we have to build an isomorphism

$$\alpha\in \mathrm{Hom}_{\mathcal{P}_{A_L}}(\psi_L(G),H) \mapsto \widehat{\alpha}\in \mathrm{Hom}_{\mathcal{P}_A}(G,\theta_L(H))$$

of R-modules, natural in G and H. But

$$\mathrm{Hom}_{\mathcal{P}_{A_L}}(\psi_L(G),H) = A_L(HG) = A(HGL) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{P}_A}(G,\theta_L(H)) = A(HLG),$$

so an obvious candidate for the above isomorphism is to set $\widehat{\alpha} = A(\mathrm{Iso}^{HLG}_{HGL})(\alpha)$. The verification that this isomorphism is functorial in G and H is similar to the proof of the first adjunction, and we omit it. □

Definition 14. Let A be a Green $\mathcal{D}$-biset functor and $L\in\mathcal{D}$. We denote by

$$\Psi_L : \mathrm{Fun}_R(\mathcal{P}_{A_L}, R\text{-Mod}) \to \mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod})$$

the functor induced by precomposition with ψ_L, and by

$$\Theta_L : \mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod}) \to \mathrm{Fun}_R(\mathcal{P}_{A_L}, R\text{-Mod})$$

the functor induced by precomposition with θ_L.

Proposition 15. *The functors Ψ_L and Θ_L are mutual left- and right-adjoint functors between* $\mathrm{Fun}_R(\mathcal{P}_{A_L}, R\text{-Mod})$ *and* $\mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod})$.

Proof. This follows from Theorem 13, by standard category theory. □

Remark 16. Using the above equivalences of categories between $\mathrm{Fun}_R(\mathcal{P}_A, R\text{-Mod})$ and A-Mod, and $\mathrm{Fun}_R(\mathcal{P}_{A_L}, R\text{-Mod})$ and A_L-Mod, we will consider Ψ_L as a functor from A_L-Mod to A-Mod and Θ_L as a functor from A-Mod to A_L-Mod. One can check that, from this point of view, if N is an A_L-module, then $\Psi_L(N)$ is the A-module defined as follows:

- If $G\in\mathcal{D}$, then $\Psi_L(N)(G) = N(G)$.

- If $G, H \in \mathcal{D}$, $a \in A(G)$ and $v \in N(H)$, then
$$a \times v = A\big(\mathrm{Inf}_G^{G\times L}\big)(a) \times^d v$$
where $\times^d$ denotes the action of A_L on N, and $A(\mathrm{Inf}_G^{G\times L})(a) \in A(G \times L)$ is viewed as an element of $A_L(G)$.

Conversely, if M is an A-module, then $\Theta_L(M)$ is the A_L-module defined as follows:

- If $G \in \mathcal{D}$, then $\Theta_L(M)(G) = M(G \times L)$.
- If $G, H \in \mathcal{D}$, $a \in A_L(G)$ and $m \in M(H \times L)$, then
$$a \times^d m = M\big(\mathrm{Res}_{G\times H\times L}^{G\times L\times H\times L}\big)(a \times m),$$
where $a \times m$ is the product of $a \in A(G\times L)$ and $m \in M(H \times L)$, and $H \times G \times L$ is viewed as a subgroup of $G \times L \times H \times L$ via the map $(g, h, l) \mapsto (g, l, h, l)$.

Theorem 17. *Let A be a Green $\mathcal{D}$-biset functor, and $L \in \mathcal{D}$. The endofunctor ρ_L of $\mathcal{P}_A$ is isomorphic to $\theta_L \circ \psi_L$ and so the endofunctor $\Psi_L \circ \Theta_L$ of A-*Mod *is isomorphic to the Yoneda–Dress functor* Id_L. *In particular*, Id_L *is self-adjoint.*

Proof. One checks readily that ρ_L is isomorphic to the composition $\theta_L \circ \psi_L$. The other assertions follow by Theorem 13, as the Yoneda–Dress functor Id_L is obtained by precomposition with $\rho_L = - \times L$. □

We observe that the L-shift of the A-module A is the representable functor $A(-, L)$ of the category $\mathcal{P}_A$, so it is projective. More generally, the L-shift of the representable functor $A(-, X)$ is the representable functor $A(-, L \times X)$. Hence the Yoneda–Dress construction maps a representable functor to a representable functor.

3. The commutant

Definition 18. Let A be a Green $\mathcal{D}$-biset functor.

(1) For $G, H \in \mathcal{D}$, we say that an element $a \in A(G)$ and an element $b \in A(H)$ *commute* if
$$a \times b = A\big(\mathrm{Iso}_{H\times G}^{G\times H}\big)(b \times a).$$

(2) For a group G in $\mathcal{D}$, we denote by $CA(G)$ the set of elements of $A(G)$ which commute with any element of $A(H)$, for any $H \in \mathcal{D}$, i.e.,
$$\big\{a \in A(G) \mid \text{for all } H \in \mathcal{D} \text{ and all } b \in A(H),\ a \times b = A\big(\mathrm{Iso}_{H\times G}^{G\times H}\big)(b \times a)\big\},$$
and call it *the commutant* of A at G.

Observe that $CA(G)$ is an R-submodule of $A(G)$, since the product $\times$ is bilinear.

Lemma 19. *Let A be a Green $\mathcal{D}$-biset functor. Then the commutant of A is a Green $\mathcal{D}$-biset subfunctor of A.*

Proof. To see it is a biset functor, let Y be a (K, G)-biset for groups K and G in $\mathcal{D}$, and let a be in $CA(G)$. If b is in $A(H)$ for a given group H in $\mathcal{D}$, we have

$$A(Y)(a) \times b = A\big((Y \times H) \circ \mathrm{Iso}_{H\times G}^{G\times H}\big)(b \times a),$$

where $Y \times H$ is seen as a $(K \times H, G \times H)$-biset. If we show that $(Y \times H) \circ \mathrm{Iso}_{H\times G}^{G\times H}$ is isomorphic to $\mathrm{Iso}_{H\times K}^{K\times H} \circ (H \times Y)$, where $H \times Y$ is seen as a $(H \times K, H \times G)$-biset, the right-hand side of the equality above will be equal to

$$A\big(\mathrm{Iso}_{H\times K}^{K\times H}\big)(b \times A(Y)(a)),$$

which is what we want. Now, $\mathrm{Iso}_{H\times G}^{G\times H}$ is the group $H \times G$, seen as a $(G \times H, H \times G)$-biset, and $\mathrm{Iso}_{H\times K}^{K\times H}$ is the group $H \times K$, seen as a $(K \times H, H \times K)$-biset. So, it is not hard to see that $(Y \times H) \circ \mathrm{Iso}_{H\times G}^{G\times H}$ is isomorphic to $Y \times H$ as a $(K \times H, H \times G)$-biset, where the right action of $H \times G$ is given by $(y, h)(h_1, g_1) = (yg_1, hh_1)$. Similarly, $\mathrm{Iso}_{H\times K}^{K\times H} \circ (H \times Y)$ is isomorphic to $H \times Y$ as a $(K \times H, H \times G)$-set, where the left action of $K \times H$ is given by $(k_1, h_1)(h, y) = (h_1 h, k_1 y)$. Hence, it is easy to verify that the map $Y \times H \to H \times Y$ sending (y, h) to (h, y) defines an isomorphism between these two bisets.

To see that CA is closed under the product $\times$, let a be in $CA(G)$, b be in $CA(H)$ and c be in $A(K)$. We have

$$a \times (b \times c) = a \times A\big(\mathrm{Iso}_{K\times H}^{H\times K}\big)(c \times b),$$

which is clearly equal to $A\big(\mathrm{Iso}_{G\times K\times H}^{G\times H\times K}\big)(a \times c \times b)$. Similarly,

$$(a \times c) \times b = A\big(\mathrm{Iso}_{K\times G\times H}^{G\times K\times H}\big)(c \times a \times b).$$

Finally, clearly we have

$$\mathrm{Iso}_{G\times K\times H}^{G\times H\times K} \circ \mathrm{Iso}_{K\times G\times H}^{G\times K\times H} = \mathrm{Iso}_{K\times G\times H}^{G\times H\times K},$$

which yields the first equality

$$(a \times b) \times c = A\big(\mathrm{Iso}_{K\times G\times H}^{G\times H\times K}\big)(c \times (a \times b)).$$

To finish the proof, it is clear that the identity element $\varepsilon \in A(1)$ belongs to $CA(1)$. □

Corollary 20. *Let A be a Green $\mathcal{D}$-biset functor. Then the image of the unique Green biset functor morphism $\upsilon_A : RB \to A$ is contained in CA.*

Proof. Indeed, by uniqueness of υ_A and υ_{CA}, the diagram

$$\begin{array}{ccc} & CA & \\ {}^{\upsilon_{CA}}\nearrow & & \searrow \\ RB & \xrightarrow[\upsilon_A]{} & A \end{array}$$

is commutative. □

Definition 21. We will say that a Green $\mathcal{D}$-biset functor A is *commutative* if $A = CA$.

It is easy to see that CA is commutative. All the examples considered in Example 5 are commutative Green biset functors.

If A is commutative, then clearly A_G is commutative for any G. More generally, we have the following result.

Proposition 22. *Let A be a Green $\mathcal{D}$-biset functor and $G \in \mathcal{D}$. Then $CA_G = (CA)_G$.*

Proof. Observe that CA_G and $(CA)_G$ are both Green $\mathcal{D}$-biset subfunctors of A_G, so to prove they are equal as Green $\mathcal{D}$-biset functors, it suffices to prove that for every group $H \in \mathcal{D}$, we have $(CA)_G(H)=CA_G(H)$.

To prove that $(CA)_G(H) \subseteq CA_G(H)$, we choose a group K in $\mathcal{D}$, and elements $a \in (CA)_G(H)$ and $b \in A_G(K)$. We must prove that

$$a \times^d b = A_G\big(\mathrm{Iso}_{K\times H}^{H\times K}\big)(b \times^d a).$$

We have

$$a \times^d b = A\big(\mathrm{Res}_{H\times K\times\Delta(G)}^{H\times G\times K\times G}\big)(a \times b) \quad \text{and} \quad b \times^d a = A\big(\mathrm{Res}_{K\times H\times\Delta(G)}^{K\times G\times H\times G}\big)(b \times a).$$

Now, by definition $(CA)_G(H) = CA(H \times G)$, so the element a satisfies

$$a \times b = A\big(\mathrm{Iso}_{K\times G\times H\times G}^{H\times G\times K\times G}\big)(b \times a).$$

Substituting this in the above equation on the left we easily obtain what we wanted.

To prove the reverse inclusion $CA_G(H) \subseteq (CA)_G(H)$, we now let $a \in CA_G(H)$ and $b \in A(K)$, and consider $c = A(\mathrm{Inf}_K^{K\times G})(b)$. Then we have

$$a \times^d c = A_G\big(\mathrm{Iso}_{K\times H}^{H\times K}\big)(c \times^d a),$$

and clearly

$$a \times^d c = A\big(\mathrm{Res}_{H\times K\times\Delta(G)}^{H\times G\times K\times G} \circ \mathrm{Inf}_{H\times G\times K}^{H\times G\times K\times G}\big)(a \times b).$$

But it is easy to see (for example from Section 1.1.3 of [Bouc 2010]) that

$$\mathrm{Res}_{H\times K\times\Delta(G)}^{H\times G\times K\times G} \circ \mathrm{Inf}_{H\times G\times K}^{H\times G\times K\times G} \cong \mathrm{Iso}_{H\times G\times K}^{H\times K\times\Delta(G)}.$$

By doing a similar transformation with $c \times^d a$, and applying the corresponding isomorphisms, we easily obtain what we wanted. □

Lemma 23. *Let A be a Green $\mathcal{D}$-biset functor. Then for any group G in $\mathcal{D}$, the commutant $CA(G)$ is a subring of $Z(A(G))$.*

Proof. Take $a \in CA(G)$ and $b \in A(G)$, then

$$\begin{aligned} a \cdot b &= A\big(\mathrm{Iso}_{\Delta(G)}^{G} \circ \mathrm{Res}_{\Delta(G)}^{G\times G}\big)(a \times b) = A\big(\mathrm{Iso}_{\Delta(G)}^{G} \circ \mathrm{Res}_{\Delta(G)}^{G\times G} \circ \mathrm{Iso}(\sigma_G)\big)(b \times a) \\ &= A\big(\mathrm{Iso}_{\Delta(G)}^{G} \circ \mathrm{Res}_{\Delta(G)}^{G\times G}\big)(b \times a) = b \cdot a, \end{aligned}$$

where σ_G is the automorphism of $G \times G$ switching the components. Since $CA(G)$

and $Z(A(G))$ have the same ring structure, inherited from the Green $\mathcal{D}$-biset functor structure of A, this shows that $CA(G)$ is a subring of $Z(A(G))$. □

Remark 24. It is not hard to see then that A is a commutative Green biset functor if and only if for every group G, the ring $A(G)$ is a commutative ring.

We now answer the question raised in Remark 11.

Proposition 25. *Let A be a Green $\mathcal{D}$-biset functor, and $G, H, K, L \in \mathcal{D}$. Let $\alpha \in A(HG)$ and $\beta \in A(LK)$. Then the square*

$$\begin{array}{ccc} G \times K & \xrightarrow{G\times\beta} & G \times L \\ {\scriptstyle \alpha\times K}\downarrow & & \downarrow{\scriptstyle \alpha\times L} \\ H \times K & \xrightarrow[H\times\beta]{} & H \times L \end{array}$$

commutes in $\mathcal{P}_A$ if and only if α and β commute.

Proof. Let $u = (\alpha \times L) \circ (G \times \beta)$. By definition

$$\begin{aligned} u &= A\big(\mathrm{Ind}_{HGL}^{HLGL}\,\mathrm{Inf}_{HG}^{HGL}\big)(\alpha) \circ A\big(\mathrm{Ind}_{GLK}^{GLGK}\,\mathrm{Inf}_{LK}^{GLK}\big)(\beta) \\ &= A\big(\mathrm{Def}_{HLGK}^{HL\underline{GL}GK}\,\mathrm{Res}_{HL\underline{GL}GK}^{HL\underline{GLGL}GK}\big) \\ &\qquad\qquad \times\big(A\big(\mathrm{Ind}_{HG\underline{L}}^{H\underline{L}G\underline{L}}\,\mathrm{Inf}_{HG}^{HG\underline{L}}\big)(\alpha) \times A\big(\mathrm{Ind}_{\underline{G}LK}^{\underline{G}L\underline{G}K}\,\mathrm{Inf}_{LK}^{\underline{G}LK}\big)(\beta)\big), \end{aligned}$$

where the notation $\mathrm{Def}_{HLGK}^{HL\underline{GL}GK}$ means the deflation with respect to the underlined normal subgroup, and $\mathrm{Res}_{HL\underline{GL}GK}^{HL\underline{GLGL}GK}$ means that the underlined GL in the subscript embeds diagonally in the underlined $GLGL$ in the superscript. Similarly, in $\mathrm{Ind}_{HG\underline{L}}^{H\underline{L}G\underline{L}}$, the group L in the subscript embeds diagonally in the two underlined copies of L in the superscript, and in $\mathrm{Inf}_{HG}^{HG\underline{L}}$, inflation is relative to the underlined L in the superscript. Thus,

$$u = A\big(\mathrm{Def}_{HLGK}^{HL\underline{GL}GK}\,\mathrm{Res}_{HL\underline{GL}GK}^{HL\underline{GLGL}GK}\,\mathrm{Ind}_{HG\underline{LG}LK}^{H\underline{L}G\underline{LG}L\underline{G}K}\,\mathrm{Inf}_{HGLK}^{HG\underline{LG}LK}\big)(\alpha \times \beta).$$

Standard relations in the composition of bisets (see Section 1.1.3 and Lemma 2.3.26 of [Bouc 2010]) and some tedious but straightforward calculations finally give

$$u = (\alpha \times L) \circ (G \times \beta) = A\big(\mathrm{Iso}_{HGLK}^{HLGK}\big)(\alpha \times \beta).$$

Similar calculations show that

$$(H \times \beta) \circ (\alpha \times K) = A\big(\mathrm{Iso}_{LKHG}^{HLGK}\big)(\beta \times \alpha).$$

So $(H \times \beta) \circ (\alpha \times K) = (\alpha \times L) \circ (G \times \beta)$ if and only if

$$\beta \times \alpha = A\big(\mathrm{Iso}_{HLGK}^{LKHG}\mathrm{Iso}_{HGLK}^{HLGK}\big)(\alpha \times \beta) = A\big(\mathrm{Iso}_{HGLK}^{LKHG}\big)(\alpha \times \beta),$$

that is, if α and β commute. □

Corollary 26. *The assignment* $\times : \mathcal{P}_A \times \mathcal{P}_A \to \mathcal{P}_A$ *sending* (G, K) *to* $G \times K$ *and* $(\alpha, \beta) \in A(H \times G) \times A(L \times K)$ *to* $(\alpha \times L) \circ (G \times \beta) \in A(H \times L \times G \times K)$ *is a functor if and only if* A *is commutative. In particular, when* A *is commutative, this functor* $\times$ *endows* $\mathcal{P}_A$ *with a structure of a* ***symmetric monoidal category****.*

4. The center

Definition 27. Let A be a Green $\mathcal{D}$-biset functor. For a group L in $\mathcal{D}$, we denote by $ZA(L)$ the family of all natural transformations $\mathrm{Id} \to \mathrm{Id}_L$ from the identity functor $\mathrm{Id} : A\text{-Mod} \to A\text{-Mod}$ to the functor Id_L. We call it *the center* of A at L.

When L is trivial, the functor Id_L is isomorphic to the identity functor, hence $ZA(1)$ is the family of natural endotransformations of the identity functor. So our definition is analogous to that of the center of a category (see for example [Hoffmann 1975] for arbitrary categories, or Section 19 of [Butler and Horrocks 1961] for abelian categories). Nonetheless, we want to regard this center as a Green $\mathcal{D}$-biset functor, and see its relation with the commutant CA. Our construction is inspired by an analogous construction for Green functors over a fixed finite group in [Bouc 1997, Section 12.2].

4.1. ***The center as a Green biset functor.*** Our goal is to show that for each Green $\mathcal{D}$-biset functor A, the assignment $L \mapsto ZA(L)$ is itself a Green $\mathcal{D}$-biset functor. For this, we will first give an equivalent description of $ZA(L)$, and then build a Green functor structure on ZA.

Proposition 28. *Let* A *be a Green* $\mathcal{D}$*-biset functor, and* $L \in \mathcal{D}$*. Then* $ZA(L)$ *is isomorphic to the family* $ZA'(L)$ *of natural transformations from the identity functor of* $\mathcal{P}_A$ *to* ρ_L*.*

Proof. Consider the Yoneda embedding $\mathcal{Y}_A : \mathcal{P}_A \to A\text{-Mod}$ sending $L \in \mathcal{D}$ to the functor $A(-, L)$. Since Id_L preserves the image of $\mathcal{Y}_A$, which is a fully faithful functor, we have $\mathrm{Id}_L \circ \mathcal{Y}_A = \mathcal{Y}_A \circ \rho_L$, and it follows that each element of $ZA(L)$ induces a natural transformation from the identity functor of $\mathcal{P}_A$, denoted by ρ_1, to ρ_L. In this way, we get a linear map $f_L : ZA(L) \to ZA'(L)$. Conversely, each natural transformation $\rho_1 \to \rho_L$ induces a natural transformation $\mathcal{Y}_A \to \mathrm{Id}_L \circ \mathcal{Y}_A$. Since the image of $\mathcal{Y}_A$ generates A-Mod, such a natural transformation extends to a natural transformation from the identity functor of A-Mod to Id_L. This gives a linear map $g_L : ZA'(L) \to ZA(L)$. Clearly f_L and g_L are inverse to one another. □

We will now use the previous identification to get a better understanding of $ZA(L)$. Indeed, a natural transformation t from the identity functor of $\mathcal{P}_A$ to the functor $\rho_L = - \times L = \theta_L \psi_L$ consists, for each $G \in \mathcal{D}$, of a morphism $t_G : G \to G \times L$ in $\mathcal{P}_A$, i.e., $t_G \in A(G \times L \times G)$, such that for any $H \in \mathcal{D}$ and any $\alpha \in A(H \times G)$, the diagram

(1)
$$\begin{array}{ccc} G & \xrightarrow{t_G} & G \times L \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle \alpha\times L = \theta_L\psi_L(\alpha)} \\ H & \xrightarrow{t_H} & H \times L \end{array}$$

is commutative in $\mathcal{P}_A$.

Lemma 29. *Let $G, H \in \mathcal{D}$, and $\alpha \in A(H \times G) = \mathrm{Hom}_{\mathcal{P}_A}(G, H)$. For an element u of $A(H \times L \times G) = \mathrm{Hom}_{\mathcal{P}_A}(G, H \times L)$, let $u^\natural$ denote the element u, viewed as a morphism from $L \times G$ to H in $\mathcal{P}_A$. Then, for any $t \in A(G \times L \times G)$,*

$$(\theta_L \psi_L(\alpha) \circ t)^\natural = \alpha \circ t^\natural \quad \textit{in } A(H \times L \times G).$$

Proof. The functor ρ_L is a self-adjoint R-linear endofunctor of $\mathcal{P}_A$. It follows from the proof of Theorem 13 that for any $G, H \in \mathcal{P}_A$, the natural bijection given by this adjunction

$$v \in \mathrm{Hom}_{\mathcal{P}_A}(G, \rho_L(H)) = A(HLG) \to v^\sharp \in \mathrm{Hom}_{\mathcal{P}_A}(\rho_L(G), H) = A(HGL)$$

is induced by the isomorphism $HLG \to HGL$ switching the components L and G. By adjunction we have commutative diagrams

$$\begin{array}{ccc} G & \xrightarrow{t} & \rho_L(G) \\ & {\scriptstyle \rho_L(\alpha)\circ t}\searrow & \downarrow{\scriptstyle \rho_L(\alpha)} \\ & & \rho_L(H) \end{array} \qquad \begin{array}{ccc} \rho_L(G) & \xrightarrow{t^\sharp} & G \\ & {\scriptstyle (\rho_L(\alpha)\circ t)^\sharp}\searrow & \downarrow{\scriptstyle \alpha} \\ & & H \end{array}$$

so $(\rho_L(\alpha) \circ t)^\sharp = \alpha \circ t^\sharp$. Since $t^\natural = t^\sharp \circ \tau_{L,G}$, where $\tau_{G,L} : LG \to GL$ is the isomorphism switching G and L, the lemma follows by right composition of the previous equality with $\tau_{G,L}$. □

Since v and $v^\natural$ are actually the same element of $A(HLG)$, for any $v \in A(HLG)$, the commutativity in diagram (1) can be simply written as

(2)
$$\alpha \circ_G t_G = t_H \circ_H \alpha,$$

where $\circ_G$ is the composition $A(HG) \times A(GLG) \to A(HLG)$, and $\circ_H$ is the composition $A(HLH) \times A(HG) \to A(HLG)$. Thus:

Proposition 30. *Let A be a Green $\mathcal{D}$-biset functor, and $L \in \mathcal{D}$. Then an element t of $ZA(L)$ consists of a family of elements $t_G \in A(GLG)$, for every $G \in \mathcal{D}$, such that $\alpha \circ_G t_G = t_H \circ_H \alpha$, for any G, H in $\mathcal{D}$ and $\alpha \in A(HG)$. In particular $ZA(L)$ is a set.*

Proof. It remains to see that $ZA(L)$ is a set. This is clear, since an element t of $ZA(L)$ is determined by its components t_G, where G runs through our chosen set $\boldsymbol{D}$ of representatives of isomorphism classes of groups in $\mathcal{D}$. More precisely, $ZA(L)$ is in one-to-one correspondence with the set $Cr_A(L)$ of sequences of elements $(t_G)_{G\in\boldsymbol{D}} \in \prod_{G\in\boldsymbol{D}} A(GLG)$ such that the above condition (2) holds for any $G, H \in \boldsymbol{D}$ and any $\alpha \in A(HG)$. □

Proposition 31. (1) *Let $K, L \in \mathcal{D}$, and $\beta \in CA(LK)$. Then the family of morphisms $\lambda_G(\beta) = G \times \beta : G \times K \to G \times L$, for $G \in \mathcal{D}$, define a natural transformation of functors ρ_β from ρ_K to ρ_L.*

(2) *Let $\mathrm{End}_R(\mathcal{P}_A)$ denote the category of R-linear endofunctors of $\mathcal{P}_A$, where morphisms are natural transformations of functors. Then the assignment*

$$\begin{cases} K \in \mathcal{D} \mapsto \rho_K \in \mathrm{End}_R(\mathcal{P}_A), \\ \beta \in CA(LK) \mapsto (\rho_\beta : \rho_K \to \rho_L) \end{cases}$$

is a faithful R-linear functor ρ_{CA} from $\mathcal{P}_{CA}$ to $\mathrm{End}_R(\mathcal{P}_A)$.

Proof. (1) This follows from Proposition 25.

(2) We have to check that if $G, J, K, L \in \mathcal{D}$, if $\alpha \in A(KJ)$ and $\beta \in A(LK)$, then $(G\times\beta)\circ(G\times\alpha) = G\times(\beta\circ\alpha)$ in $A(GLGJ)$, and that if β is the identity element of $CA(KK)$, then $G \times \beta$ is the identity morphism of $G \times K$ in $\mathcal{P}_A$. This follows from the fact that λ_G is a functor.

So we get a functor $\rho_{CA} : \mathcal{P}_{CA} \to \mathrm{End}_R(\mathcal{P}_A)$. Seeing that this functor is faithful amounts to seeing that if $\beta \in CA(LK)$, then $\rho_\beta = 0$ if and only if $\beta = 0$. But the component $1 \times \beta$ of ρ_β is clearly equal to β, after identification of $1 \times K$ with K and $1 \times L$ with L. □

Remark 32. In particular, it follows from assertion (2) that an isomorphism of groups $K \to K'$ induces an isomorphism of functors $\rho_K \to \rho'_K$: indeed, a group isomorphism $\varphi : K \to K'$ is represented by a (K', K)-biset $U_\varphi \in RB(K'K)$, and hence by an element $\beta_\varphi = \upsilon_{K'K}(U_\varphi) \in CA(K'K)$, by Corollary 20. The corresponding natural transformation ρ_{β_φ} is an isomorphism $\rho_K \to \rho'_K$, with inverse $\rho_{\beta_{\varphi^{-1}}}$.

Lemma 33. *Let A be a Green $\mathcal{D}$-biset functor and $K, L \in \mathcal{D}$.*

(1) *The endofunctors $\rho_L \circ \rho_K$ and ρ_{KL} of $\mathcal{P}_A$ are naturally isomorphic.*

(2) *Let $s \in ZA(LK)$, given by the family of elements $s_G \in A(GLKG)$, for $G \in \mathcal{D}$. Then the natural transformation $s^o : \rho_K \to \rho_L$ deduced from $s : \mathrm{Id} \to \rho_K\rho_L$ by adjunction, is defined by the family of morphisms*

$$s^o_G = \mathrm{Iso}^{GLGK}_{GLKG}(s_G) \in A(GLGK) = \mathrm{Hom}_{\mathcal{P}_A}(GK, GL).$$

(3) *The map $s \mapsto s^o$ is an isomorphism of R-modules,*

$$ZA(LK) \to \mathrm{Hom}_{\mathrm{End}_R(\mathcal{P}_A)}(\rho_K, \rho_L).$$

Proof. (1) This follows from a straightforward verification.

(2) By the proof of Theorem 13, for each $G \in \mathcal{D}$, the morphism $s_G \in A(GLKG)$,

$$s_G : G \to GLK = \rho_K \rho_L(G) = \theta_K \psi_K \rho_L(G),$$

in $\mathcal{P}_A$ gives by adjunction the morphism

$$u : \psi_K(G) \to \psi_K \rho_L(G)$$

in $\mathcal{P}_{A_K}$, defined as the element

$$u = A\big(\mathrm{Iso}_{GLKG}^{GLGK}\big)(s_G) \in A_K(GLG) = A(GLGK).$$

This element u gives in turn the morphism

$$v : \theta_K \psi_K(G) = \rho_K(G) \to \rho_L(G)$$

equal to $u \in A(GLGK)$, but viewed as a morphism in $\mathcal{P}_A$ from GK to GL.

(3) This is clear, by adjunction. □

Proposition 34. *The center of A is a $\mathcal{D}$-biset functor.*

Proof. First, $ZA(L)$ is obviously an R-module, for any $L \in \mathcal{D}$. Let $K \in \mathcal{D}$ and $t \in ZA(K)$, i.e., let t be a natural transformation $\mathrm{Id} \to \rho_K$ of endofunctors of the category $\mathcal{P}_A$. If $L \in \mathcal{D}$ and $u \in RB(LK)$, let $u_A = \upsilon_{LK}(u) \in A(LK)$ be the image of u under the unique morphism of Green functor $\upsilon : RB \to A$. Since $u_A \in CA(LK)$, by Corollary 20, we can compose t with the natural transformation $\rho_{u_A} : \rho_K \to \rho_L$ from Proposition 31, to get a natural transformation $\rho_{u_A} \circ t : \mathrm{Id} \to \rho_L$, i.e., an element of $ZA(L)$. Hence we get a linear map

$$u \in RB(LK) \mapsto (t \mapsto \rho_{u_A} \circ t \in \mathrm{Hom}_R(ZA(K), ZA(L))),$$

and assertion (2) of Proposition 31 shows that this endows ZA with a structure of biset functor. □

We now build a product on ZA, to make it a Green biset functor. For $K, L \in \mathcal{D}$, let $s \in ZA(K)$ and $t \in ZA(L)$. Since s is a natural transformation $\mathrm{Id} \to \rho_K$, we get, by adjunction, a natural transformation $s^o : \rho_K \to \mathrm{Id}$. By composition with $t : \mathrm{Id} \to \rho_L$, we obtain a natural transformation $t \circ s^o : \rho_K \to \rho_L$, which in turn, by adjunction again, gives a natural transformation ${}^o(t \circ s^o) : \mathrm{Id} \to (\rho_L)_K \cong \rho_{LK}$, i.e., an element of $ZA(LK)$. So we set

(3) $\quad t \times s = {}^o(t \circ s^o) \in ZA(LK) \quad$ for all $s \in ZA(K)$ and all $t \in ZA(L)$.

Translating this in the terms of Proposition 30 gives:

Lemma 35. *Let $s \in ZA(K)$ and $t \in ZA(L)$ be defined, respectively, by families of elements $s_G \in A(GKG)$ and $t_G \in A(GLG)$, for $G \in \mathcal{D}$. Then $t \times s$ is the element of $ZA(LK)$ defined by the family $(t \times s)_G = t_G \circ s_G \in A(GLKG)$, for $G \in \mathcal{D}$.*

Proof. As the adjunction $s \mapsto s^o$ amounts to switching the last two components of GKG, the element $t \times s = {}^o(t \circ s^o)$ is defined by the family

$$\begin{aligned}(t \times s)_G &= A\big(\mathrm{Iso}_{GLGK}^{GLKG}\big)\big(t_G \circ A\big(\mathrm{Iso}_{GKG}^{GGK}\big)(s_G)\big)\\ &= A\big(\mathrm{Iso}_{GLGK}^{GLKG}\big)A\big(\mathrm{Def}_{GLGK}^{GL\underline{G}GK}\,\mathrm{Res}_{GL\underline{G}GK}^{GL\underline{GG}GK}\big)\big(t_G \times A\big(\mathrm{Iso}_{GKG}^{GGK}\big)(s_G)\big),\end{aligned}$$

where the notation $\mathrm{Def}_{GLGK}^{GL\underline{G}GK}$ indicates that we take deflation with respect to the underlined factor, and $\mathrm{Res}_{GL\underline{G}GK}^{GL\underline{GG}GK}$ means that the underlined G in the subscript embeds diagonally in the underlined group GG in the superscript. It follows that

$$\begin{aligned}(t \times s)_G &= A\big(\mathrm{Def}_{GLKG}^{GL\underline{G}KG}\,\mathrm{Iso}_{GLGGK}^{GLGKG}\mathrm{Res}_{GL\underline{G}GK}^{GL\underline{GG}GK}\,\mathrm{Iso}_{GLGGKG}^{GLGGGK}\big)(t_G \times s_G)\\ &= A\big(\mathrm{Def}_{GLKG}^{GL\underline{G}KG}\,\mathrm{Res}_{GL\underline{G}KG}^{GL\underline{GG}KG}\big)(t_G \times s_G)\\ &= t_G \circ s_G \in A(GLKG).\end{aligned}$$ □

Notation 36. Let A be a Green $\mathcal{D}$-biset functor, and $G, H, K, L \in \mathcal{D}$. For morphisms in $\mathcal{P}_A$, namely $\alpha : G \to H$ in $A(HG)$ and $\beta : K \to L$ in $A(LK)$, we denote by $\alpha \boxtimes \beta : GK \to HL$ the morphism defined by

$$\alpha \boxtimes \beta = A\big(\mathrm{Iso}_{HGLK}^{HLGK}\big)(\alpha \times \beta) \in A(HLGK).$$

Proposition 37. *Let A be a Green $\mathcal{D}$-biset functor, and $G, H, K, L \in \mathcal{D}$. Let, moreover, $\alpha \in CA(HG)$ and $\beta \in CA(LK)$. Then for any $s \in ZA(G)$ and $t \in ZA(K)$, and for any $X \in \mathcal{D}$,*

$$(\rho_\alpha \circ s)_X \circ (\rho_\beta \circ t)_X = (\rho_{\alpha\boxtimes\beta} \circ (s \times t))_X.$$

Proof. The proof amounts to rather lengthy but straightforward calculations on bisets, similar to those we have already done several times above, e.g., in the proof of Theorem 13. We leave it as an exercise. □

Theorem 38. *Let A be a Green $\mathcal{D}$-biset functor. Then ZA, endowed with the product defined in (3), is a Green $\mathcal{D}$-biset functor.*

Proof. It is clear from Lemma 35 and Proposition 30 that the product on ZA is associative. Moreover the identity transformation from the identity functor to $\rho_1 = \mathrm{Id}_{\mathcal{P}_A}$ is obviously an identity element for the product on ZA. This product is also R-bilinear by construction. Finally, the equality $ZA(U)(s) \times ZA(V)(t) = ZA(U \boxtimes V)(s \times t)$ for bisets U and V is a special case of Proposition 37. □

4.2. ***Relations between the commutant and the center.***

Proposition 39. *Let A be a Green $\mathcal{D}$-biset functor.*

(1) *The maps sending $\alpha \in CA(L)$ to $\rho_\alpha \in ZA(L)$, for $L \in \mathcal{D}$, define a morphism of Green biset functors $\iota_A : CA \to ZA$.*

(2) *The maps sending* $t \in Cr_A(L) \cong ZA(L)$ *to* $t_1 \in A(L)$, *for* $L \in \mathcal{D}$, *define a morphism of Green biset functors* $\pi_A : ZA \to A$. *The image of this morphism in the component* 1 *lies in* $Z(A(1))$, *hence there is a morphism of rings* $\pi_{A,1} : ZA(1) \to Z(A(1))$.

(3) *The composition*

$$CA \xhookrightarrow{\iota_A} ZA \xrightarrow{\pi_A} A$$

is equal to the inclusion $CA \hookrightarrow A$. *In particular*, ι_A *is injective.*

Proof. For assertion (1), let $\alpha \in CA(K)$, for $K \in \mathcal{D}$. Then the element ρ_α of $ZA(K)$ corresponds to the family of elements $\rho_{\alpha,G} \in A(GKG)$, for $G \in \mathcal{D}$, defined by

$$\rho_{\alpha,G} = A\big(\mathrm{Ind}_{KG}^{GKG}\,\mathrm{Inf}_K^{KG}\big)(\alpha).$$

Similarly, if $L \in \mathcal{D}$ and $\beta \in CA(L)$, the element ρ_β of $ZA(L)$ corresponds to the family $\rho_{\beta,G} = A\big(\mathrm{Ind}_{LG}^{GLG}\,\mathrm{Inf}_L^{LG}\big)(\beta)$. By Lemma 35, the product $q = \rho_\alpha \times \rho_\beta$ in $ZA(KL)$ corresponds to the family

$$\begin{aligned}
q_G &= \rho_{\alpha,G} \circ \rho_{\beta,G}\\
&= A\big(\mathrm{Ind}_{KG}^{GKG}\,\mathrm{Inf}_K^{KG}\big)(\alpha) \circ A\big(\mathrm{Ind}_{LG}^{GLG}\,\mathrm{Inf}_L^{LG}\big)(\beta)\\
&= A\big(\mathrm{Def}_{GKLG}^{GK\underline{G}LG}\,\mathrm{Res}_{GK\underline{G}LG}^{GK\underline{GG}LG}\big)\big(A\big(\mathrm{Ind}_{KG}^{GKG}\,\mathrm{Inf}_K^{KG}\big)(\alpha) \times A\big(\mathrm{Ind}_{LG}^{GLG}\,\mathrm{Inf}_L^{LG}\big)(\beta)\big)\\
&= A\big(\mathrm{Def}_{GKLG}^{GK\underline{G}LG}\,\mathrm{Res}_{GK\underline{G}LG}^{GK\underline{GG}LG}\mathrm{Ind}_{KGLG}^{GKGGLG}\,\mathrm{Inf}_{KL}^{KGLG}\big)(\alpha \times \beta).
\end{aligned}$$

Standard relations in the composition of bisets then show that

$$q_G = A\big(\mathrm{Ind}_{KLG}^{GKLG}\,\mathrm{Inf}_{KL}^{KLG}\big)(\alpha \times \beta),$$

and it follows that $q = \rho_{\alpha\times\beta}$. In other words $\iota_A(\alpha \times \beta) = \iota_A(\alpha) \times \iota_A(\beta)$. Moreover, the identity element $\varepsilon_A \in CA(1)$ is mapped by ι_A to the element of $ZA(1)$ defined by the family of elements $A\big(\mathrm{Ind}_G^{GG}\,\mathrm{Inf}_1^G\big)(\varepsilon_A)$, for $G \in \mathcal{D}$, that is, the identity element of ZA. So ι_A is a morphism of Green $\mathcal{D}$-biset functors.

The first part of assertion (2) is a consequence of Lemma 35. Indeed, if $K, L \in \mathcal{D}$, if $s \in ZA(K)$ corresponds to the family $s_G \in Cr_A(K)$, and if $t \in ZA(L)$ corresponds to the family $\beta_G \in Cr_A(L)$, for $G \in \mathcal{D}$, then the product $u = s \times t$ is the element of $ZA(KL)$ corresponding to the family $u_G = s_G \circ t_G$. In particular, for $G = 1$,

$$u_1 = s_1 \circ t_1 = s_1 \times t_1.$$

This shows that the maps sending $t \in ZA(L)$ to $t_1 \in A(L)$, for $L \in \mathcal{D}$, is a morphism of Green $\mathcal{D}$-biset functors $\pi : ZA \to A$.

Since composition $\circ : A(1) \times A(1) \to A(1)$ coincides with the product of $A(1)$ as a ring, the commutativity property defining the series of $Cr_A(1)$ shows that $\pi_{A,1}$ has image in $Z(A(1))$. This completes the proof of assertion (2).

For assertion (3), we start with an element $\alpha \in CA(L)$, for $L \in \mathcal{D}$. It is sent by ι_A to the element $t \in ZA(L)$ corresponding to the family $t_G = A\big(\mathrm{Ind}_{LG}^{GLG} \mathrm{Inf}_L^{LG}\big)(\alpha)$, for $G \in \mathcal{D}$, in $Cr_A(L)$. In particular $t_1 = A\big(\mathrm{Ind}_L^L \mathrm{Inf}_L^L\big)(\alpha) = \alpha$, so $\pi_A \circ \iota_A$ is equal to the inclusion $CA \hookrightarrow A$. □

The morphism ι_A of the previous proposition allows us to give a CA-module structure to ZA. With this structure, (the image under ι_A of) CA is a CA-submodule of ZA. In the particular case where A is commutative, Proposition 39 tells us more.

Corollary 40. *If A is a commutative Green $\mathcal{D}$-biset functor, then A is isomorphic to a direct summand of ZA in the category A-Mod.*

Proof. This follows from the fact that ι_A and π_A are morphisms of Green $\mathcal{D}$-biset functors, and thus, in particular, are morphisms of A-modules. Moreover, the composition $\pi_A \circ \iota_A$ is equal to the identity when A is commutative. □

Proposition 41. *Let A be a Green $\mathcal{D}$-biset functor. Let $\mathrm{End}_R(\mathcal{P}_A)$ be the category of R-linear endofunctors of $\mathcal{P}_A$.*

(1) *The assignment*

$$\begin{cases} K \in \mathcal{D} \mapsto \rho_K \in \mathrm{End}_R(\mathcal{P}_A), \\ t \in ZA(LK) \mapsto t^o \in \mathrm{Hom}_{\mathrm{End}_R(\mathcal{P}_A)}(\rho_K, \rho_L) \end{cases}$$

is a fully faithful R-linear functor ρ_{ZA} from $\mathcal{P}_{ZA}$ to $\mathrm{End}_R(\mathcal{P}_A)$.

(2) *The assignment μ_A,*

$$\begin{cases} K \in \mathcal{D} \mapsto K \in \mathcal{D}, \\ \alpha \in CA(LK) \mapsto {}^o\rho_\alpha \in ZA(LK), \end{cases}$$

is equal to the functor $\mathcal{P}_{\iota_A}$ from $\mathcal{P}_{CA}$ to $\mathcal{P}_{ZA}$, induced by $\iota_A : CA \to ZA$. In particular μ_A is faithful, and such that

$$\rho_{ZA} \circ \mu_A = \rho_{CA}.$$

(3) *The assignment ν_A,*

$$\begin{cases} K \in \mathcal{D} \mapsto K \in \mathcal{D}, \\ s \in ZA(LK) \mapsto s_1 \in A(LK), \end{cases}$$

is equal to the functor $\mathcal{P}_{\pi_A}$ from $\mathcal{P}_{ZA}$ to $\mathcal{P}_A$ induced by the morphism of Green biset functors $\pi_A : ZA \to A$. The composition $\nu_A \circ \mu_A$ is equal to the inclusion functor $\mathcal{P}_{CA} \to \mathcal{P}_A$.

Proof. All the assertions are straightforward consequences of Proposition 39. □

To conclude this section, we will show that the isomorphism $CA_L \cong (CA)_L$ of Proposition 22 only extends to an injection $ZA_L \hookrightarrow (ZA)_L$.

Lemma 42. *Let A be a Green $\mathcal{D}$-biset functor. For $L \in \mathcal{D}$, let $\psi_L^A : \mathcal{P}_A \to \mathcal{P}_{A_L}$ be the functor ψ_L of Theorem 13. If $K \in \mathcal{D}$, let $\psi_K^{A_L} : \mathcal{P}_{A_L} \to \mathcal{P}_{(A_L)_K}$ be the similar functor built from A_L and K. Then the diagram*

$$\begin{array}{ccccc} \mathcal{P}_A & \xrightarrow{\psi_L^A} & \mathcal{P}_{A_L} & \xrightarrow{\psi_K^{A_L}} & \mathcal{P}_{(A_L)_K} \\ & \searrow^{\psi_{KL}^A} & & & \downarrow e_{K,L}\ \cong \\ & & & & \mathcal{P}_{A_{KL}} \end{array}$$

of categories and functors, is commutative, where $e_{K,L}$ is the natural equivalence of categories $\mathcal{P}_{(A_L)_K} \to \mathcal{P}_{A_{KL}}$ provided by the canonical isomorphism of Green $\mathcal{D}$-biset functors $(A_L)_K \cong A_{KL}$.

Proof. Indeed, all the functors involved are the identity on objects. And for a morphism $\alpha : G \to H$ in $\mathcal{P}_A$, i.e., an element α of $A(HG)$, we have

$$\begin{aligned} \psi_K^{A_L}\psi_L^A(\alpha) &= \psi_K^{A_L} A\big(\mathrm{Inf}_{HG}^{HGL}\big)(\alpha) = A_L\big(\mathrm{Inf}_{HG}^{HGK}\big)A\big(\mathrm{Inf}_{HG}^{HGL}\big)(\alpha) \\ &= A\big(\mathrm{Inf}_{HGL}^{HGKL}\big)A\big(\mathrm{Inf}_{HG}^{HGL}\big)(\alpha) \\ &= A\big(\mathrm{Inf}_{HG}^{HGKL}\big)(\alpha) = \psi_{KL}^A(\alpha). \end{aligned}$$

□

Proposition 43. *Let A be a Green biset functor and $L \in \mathcal{D}$. Then there is an injective morphism of Green $\mathcal{D}$-biset functors from ZA_L to $(ZA)_L$.*

Proof. Let $K, L \in \mathcal{D}$, and $t \in ZA_L(K)$, i.e., a natural transformation,

$$t : \mathrm{Id}_{\mathcal{P}_{A_L}} \to \rho_K^{A_L},$$

from the identity functor of $\mathcal{P}_{A_L}$ to the functor $\rho_K^{A_L} = \theta_K^{A_L}\psi_K^{A_L}$, where $\theta_K^{A_L}$ is the functor $\mathcal{P}_{(A_L)_K} \to \mathcal{P}_{A_L}$ of Theorem 13 built from A_L and K. By precomposition of this natural transformation with the functor ψ_L^A, we get a natural transformation,

$$\psi_L^A \to \theta_K^{A_L}\psi_K^{A_L}\psi_L^A,$$

which by adjunction, gives a natural transformation,

$$\mathrm{Id}_{\mathcal{P}_A} \to \theta_L^A\theta_K^{A_L}\psi_K^{A_L}\psi_L^A.$$

By Lemma 42, the functor $\psi_K^{A_L}\psi_L^A$ is isomorphic to ψ_{KL}^A. By Theorem 13, the functor $\theta_K^{A_L}$ is left- and right-adjoint to the functor $\psi_K^{A_L}$, and θ_L^A is left- and right-adjoint to ψ_L^A. It follows that the functor $\theta_L^A\theta_K^{A_L}$ is isomorphic to the adjoint θ_{KL}^A of ψ_{KL}^A. Hence we have a natural transformation,

$$T : \mathrm{Id}_{\mathcal{P}_A} \to \theta_{KL}^A\psi_{KL}^A = \rho_{KL}^A,$$

that is an element of $ZA(KL) = (ZA)_L(K)$.

So we have a map $j_{L,K} : t \in ZA_L(K) \mapsto T \in (ZA)_L(K)$, which is obviously R-linear. Lengthy but straightforward calculations show that the family of these maps, for $K \in \mathcal{D}$, form a morphism of Green biset functors from ZA_L to $(ZA)_L$. □

5. Application: some equivalences of categories

5.1. ***General setting.*** We begin by recalling some well-known folklore facts on the decomposition of a category $\mathcal{F}_\mathcal{P}$ of functors from a small R-linear category $\mathcal{P}$ to R-Mod, using an orthogonal decomposition of the identity in the center $Z\mathcal{P}$ of $\mathcal{P}$.

Since $\mathcal{P}$ is R-linear, its center $Z\mathcal{P}$ is a commutative R-algebra. Suppose we have a family $(\gamma_i)_{i\in I}$ of elements of $Z\mathcal{P}$ indexed by a set I, with the following properties:

(1) For $i, j \in I$, the product $\gamma_i\gamma_j$ is equal to 0 if $i \neq j$, and to γ_i if $i = j$.

(2) For any object G of $\mathcal{P}$, there is only a finite number of elements $i \in I$ such that $\gamma_{i,G} \neq 0$. Then, for each object $G \in \mathcal{P}$, we can consider the (finite) sum $\sum_{i\in I} \gamma_{i,G}$, which is a well-defined endomorphism of G. We assume that this endomorphism is the identity of G, for any $G \in \mathcal{P}$.

If F is an R-linear functor from $\mathcal{P}$ to R-Mod, and $i \in I$, we denote by $F\gamma_i$ the functor that in an object G of $\mathcal{P}$ is defined as the image of $F(\gamma_{i,G})$, that is,

$$(F\gamma_i)(G) = \mathrm{Im}(F(\gamma_{i,G}) : F(G) \to F(G)),$$

which is an R-submodule of $F(G)$. For a morphism $\alpha : G \to H$, we denote by $(F\gamma_i)(\alpha)$ the restriction of $F(\alpha)$ to $(F\gamma_i)(G)$. The image of $(F\gamma_i)(\alpha)$ is contained in $F\gamma_i(H)$, because the square

$$\begin{array}{ccc} G & \xrightarrow{\gamma_{i,G}} & G \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ H & \xrightarrow[\gamma_{i},H]{} & H \end{array}$$

is commutative in $\mathcal{P}$, and hence also its image by F.

It is easy to check that $F\gamma_i$ is an R-linear functor from $\mathcal{P}$ to R-Mod, which is a subfunctor of F. Moreover, the assignment $F \mapsto F\gamma_i$ is an endofunctor Γ_i of the category $\mathcal{F}_\mathcal{P}$. The image of this functor consists of those functors $F \in \mathcal{F}_\mathcal{P}$ such that the subfunctor $F\gamma_i$ is equal to F. Let $\mathcal{F}_\mathcal{P}\gamma_i$ be the full subcategory of $\mathcal{F}_\mathcal{P}$ consisting of such functors. It is an abelian subcategory of $\mathcal{F}_\mathcal{P}$.

For each $G \in \mathcal{P}$, the direct sum $\bigoplus_{i\in I} F\gamma_i(G)$ is actually finite, and our assumptions ensure that it is equal to $F(G)$. This shows that the functor sending $F \in \mathcal{F}_\mathcal{P}$ to the family of functors $F\gamma_i$ is an equivalence between $\mathcal{F}_\mathcal{P}$ and the product of the categories $\mathcal{F}_\mathcal{P}\gamma_i$.

A particular case of the previous situation is when the identity element $\varepsilon \in A(1)$ of a Green biset functor A has a decomposition in orthogonal idempotents $\varepsilon = \sum_{i=1}^{n} e_i$ in the ring $CA(1)$. Each e_i induces a natural transformation $E^i : \mathrm{Id} \to \mathrm{Id}_1$, defined at an A-module M and a group $H \in \mathcal{D}$ as

$$E^i_{M,H} : M(H) \to M_1(H), \quad m \mapsto M(\mathrm{Iso}_{1\times H}^{H\times 1})(e_i \times m).$$

For simplicity, we will think of this natural transformation as sending m simply to $e_i \times m$, and we will denote by $e_i M$ the A-submodule of M given by the image of E^i_M.

Since the morphism from $CA(1)$ to $ZA(1)$ is a ring homomorphism, we have that the natural transformations E^i satisfy $E^i \circ E^i = E^i$, $E^i \circ E^j = 0$ if $i \neq j$ and that the identity natural transformation, $\mathbf{1}$, is equal to $\sum_{i=1}^{n} E^i$. By Proposition 28, we have then the hypothesis assumed at the beginning of the section, and so we obtain the equivalence of categories mentioned above. In this case, we can give a more precise description of this equivalence.

Lemma 44. *The A-module $e_i A$ is a Green $\mathcal{D}$-biset functor, and for every A-module M, the functor $e_i M$ is an $e_i A$-module. Furthermore $A \cong \bigoplus_{i=1}^{n} e_i A$ as Green $\mathcal{D}$-biset functors.*

Proof. As we have said, $e_i A$ is an A-module, in particular it is a biset functor. We claim that it is a Green biset functor with the product

$$e_i A(G) \times e_i A(K) \to e_i A(G \times K), \quad (e_i \times a) \times (e_i \times b) = e_i \times a \times b.$$

Observe that since all the $\times$ represent the product of A, then $(e_i \times a) \times (e_i \times b)$ is isomorphic to $a \times e_i \times e_i \times b$, because $e_i \in CA(1)$. But the product $\times$ coincides with the ring product in $A(1)$, hence this element is isomorphic to $a \times e_i \times b$ and then to $e_i \times a \times b$. This implies immediately that the product is associative; the identity element in $e_i A(1)$ is of course $e_i \times \varepsilon$. Next, notice that since E^i_A is a morphism of A-modules, if $L, G \in \mathcal{D}$ and X is an (L, G)-biset, then $A(X)(e_i \times a) \cong e_i \times A(X)(a)$ for all $a \in A(G)$. With this, one can easily show the functoriality of the product.

Similar arguments show that $e_i M$ is an $e_i A$-module with the product

$$e_i A(G) \times e_i M(K) \to e_i M(G \times K), \quad (e_i \times a) \times (e_i \times m) = e_i \times a \times m.$$

For the final statement, first it is an easy exercise to verify that given Green biset functors $A_1, \ldots, A_r$, then their direct sum $\bigoplus_{i=1}^{r} A_i$ in the category of biset functors is again a Green biset functor, with the product given component-wise. With this, it is straightforward to see that the isomorphism of biset functors $A \cong \bigoplus_{i=1}^{n} e_i A$ is an isomorphism of Green biset functors. □

All these observations give us the following result.

Theorem 45. *Let A be a Green $\mathcal{D}$-biset functor as above. Then the category A-*Mod *is equivalent to the product category*

$$\prod_{i=1}^{n} e_i A\text{-Mod}.$$

Moreover, for each indecomposable A-module M, there exists only one e_i such that $e_i M \neq 0$, and hence $e_i M \cong M$.

When considering the shifted functor A_H, if we have an idempotent $e \in CA_H(1)$ as before, then the evaluation of eA_H at a group G can be seen as follows. Since $eA_H(G) = e \times^d A_H(G)$, then for $a \in A_H(G)$ it is easy to see that

$$e \times^d a = A\big(\mathrm{Res}^{1\times H\times G\times H}_{G\times\Delta(H)}\big)(e \times a) = A\big(\mathrm{Inf}^{G\times H}_{H}\big)(e)\cdot a,$$

where the product $\cdot$ indicates the ring structure in $A(G \times H)$. The last equality follows from Lemma 3 and the properties of restriction and inflation. So, the evaluation of eA_H at a given group depends on how inflation of A acts on the idempotents of $CA(H)$.

5.2. *Some examples.*

5.2.1. *p-biset functors.* Let p be a prime, and RB_p denote the restriction to finite p-groups of the Burnside functor RB of Example 5. When p is invertible in the ring R, a family of orthogonal idempotents in the center of the Green biset functor RB_p of Example 5 was introduced in [Bouc 2018]. These idempotents $\widehat{b_L}$ are indexed by *atoric* p-groups L up to isomorphism, i.e., finite p-groups which cannot be decomposed as a direct product $Q \times C_p$ of a finite p-group Q and a group C_p of order p.

More precisely, for each such atoric p-group L and each finite p-group P, a specific idempotent b_L^P of $RB_p(P, P)$ was introduced (cf. [Bouc 2018, Theorem 7.4]), with the property that

$$a \circ b_L^P = b_L^Q \circ a$$

for any finite p-groups P and Q, and any $a \in RB(Q, P)$. In other words, the family $b_L = (b_L^P)_P$ is an element of the center of the biset category $R\mathcal{C}_p$ of finite p-groups. The elements $\widehat{b_L}$ of the center of the category of p-biset functors over R — i.e., the category RB_p-Mod — are deduced from the elements b_L in [Bouc 2018, Corollary 7.5].

Let $[\mathcal{A}t_p]$ denote a set of representatives of isomorphism classes of atoric p-groups. The idempotents b_L^P have the following additional properties:

(1) If L and L' are isomorphic atoric p-groups, then $b_L^P = b_{L'}^P$.

(2) If L and L' are nonisomorphic atoric p-groups, then $b_L^P b_{L'}^P = 0$.

(3) For a given finite p-group P, there are only a finite number of atoric p-groups L, up to isomorphism, such that $b_L^P \neq 0$.

(4) The sum $\sum_{L\in[\mathcal{A}t_p]} b_L^P$, which is a finite sum by the previous property, is equal to the identity element of $RB(P, P)$.

It follows that one can consider the sum $\sum_{L\in[\mathcal{A}t_p]} \widehat{b}_L$ in $Z(RB_p)(1)$, and that this sum is equal to the identity element of $Z(RB_p)(1)$. So we obtain a *locally finite* decomposition of the identity element of $Z(RB_p)(1)$ as a sum of orthogonal idempotents, which allows for a splitting of the category of p-biset functors over R as a direct product of abelian subcategories (cf. [Bouc 2018, Corollary 7.5]). As a consequence, for each indecomposable p-biset functor F over R, there is an atoric p-group L, unique up to isomorphism, such that $\widehat{b}_L$ acts as the identity of F (or equivalently, does not act by zero on F). This group L is called the *vertex* of F (cf. [Bouc 2018, Definition 9.2]).

Remark 46. This example shows in particular that ZA can be much bigger than CA: indeed, for $A = RB_p$, when R is a field of characteristic different from p, we see that $ZA(1)$ is an infinite-dimensional R-vector space, whereas $CA(1) \cong R$ is one-dimensional.

5.2.2. *Shifted representation functors.* Now we apply the results of Section 5.1 to some shifted classical representation functors, with coefficients in a field $\mathbb{F}$ of characteristic 0. In each case we will begin with a commutative Green biset functor C such that for each group H, the $\mathbb{F}$-algebra $C(H)$ is split semisimple. In particular, taking $A = C_H$, in $A(1) = C(H)$ we will have a family of orthogonal idempotents $\{e_i^H\}_{i=1}^{n_H}$ such that $\varepsilon = \sum_{i=1}^{n_H} e_i^H$. As we said in Section 5.1, the evaluation $e_i^H A(G)$ is given as

$$e_i^H \times^d a = A\big(\mathrm{Inf}_1^G\big)(e_i^H)\cdot a = C\big(\mathrm{Inf}_H^{G\times H}\big)(e_i^H)\cdot a$$

for $a \in A(G)$. Now, since inflation is a ring homomorphism, $A(\mathrm{Inf}_1^G)(e_i^H)$ is equal to $\sum_{j\in J} e_j^{G\times H}$ for some $J \subseteq \{1, \dots, n_{G\times H}\}$ depending on e_i^H and G. On the other hand, we also have

$$a = \sum_{i=1}^{n_{G\times H}} \alpha_i(a) e_i^{G\times H}$$

for some $\alpha_i(a) \in \mathbb{F}$. This implies that the idempotents appearing in the evaluation $e_i^H A(G)$ depend only on the set $\{e_j^{G\times H}\}_{j\in J}$.

Shifted Burnside functors. We consider the Burnside functor $\mathbb{F}B$ over $\mathbb{F}$. We fix a finite group H, and consider the shifted functor $A = \mathbb{F}B_H$. Then the algebra $A(1)$ is isomorphic to $\mathbb{F}B(H)$; hence it is split semisimple. Its primitive idempotents e_K^H are indexed by subgroups K of H, up to conjugation, and explicitly given (see [Gluck 1981; Yoshida 1983]) by

$$e_K^H = \frac{1}{|N_H(K)|}\sum_{L\leq K}|K|\mu(L,K)\,[H/L],$$

where μ is the Möbius function of the poset of subgroups of H and $[H/L]\in B(H)$ is the class of the transitive H-set H/L.

By Theorem 45, we get a decomposition of the category A-Mod as a product $\prod_{K\in[s_H]} e_K^H A$-Mod, where $[s_H]$ is a set of representatives of conjugacy classes of subgroups of H. From the action of inflation on the primitive idempotents of Burnside rings (see [Bouc 2010, Theorem 5.2.4]), it is easy to see that for $K\leq H$, the value $e_K^H A(G)$ of the Green functor $e_K^H A$ at a finite group G is equal to the set of linear combinations of idempotents $e_L^{G\times H}$ of $\mathbb{F}B(G\times H)$ indexed by subgroups L of $(G\times H)$ for which the second projection $p_2(L)$ is conjugate to K in H. Also, for each indecomposable A-module M, there exists a unique $K\leq H$, up to conjugation, such that $e_K^H M\neq 0$, and then $e_K^H M = M$.

Shifted functors of linear representations. Next, we consider the functor $\mathbb{F}R_{\mathbb{K}}$ of linear representations over $\mathbb{K}$, a field of characteristic 0. As before, we fix a finite group H and consider the shifted functor $A=(\mathbb{F}R_{\mathbb{K}})_H$. This is a commutative Green biset functor, and $A(1)$ is isomorphic to the split semisimple $\mathbb{F}$-algebra $\mathbb{F}R_{\mathbb{K}}(H)$. If $|H|=n$, it is shown in [García 2018, Section 3.3.1] (and slightly differently in [García 2019]) that $\mathbb{F}R_{\mathbb{K}}(H)$ has a complete family of orthogonal primitive idempotents e_D^H indexed by the E-conjugacy classes of H, where E is certain subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$. By E-conjugacy we mean that two elements $x,y\in H$ are E-conjugated if there exist $[i]\in E$ such that $x=_H y^i$. This defines an equivalence relation on H and the set of E-conjugacy classes is denoted by $\mathrm{Cl}_E(H)$. The group E is built in the following way: First we fix an algebraically closed field $\mathbb{L}$, which is an extension of $\mathbb{F}$ and $\mathbb{K}$, and then we take the intersection $\mathbb{E}=\mathbb{F}\cap\mathbb{K}$ in $\mathbb{L}$. By adding an n-th primitive root of unity, ω, to $\mathbb{E}$, we obtain E as the group isomorphic to $\mathrm{Gal}(\mathbb{E}[\omega]/\mathbb{E})$ in $(\mathbb{Z}/n\mathbb{Z})^\times$. Observe that, as a group, E depends only on $\mathbb{F}$, $\mathbb{K}$ and n, and not on the choice of $\mathbb{L}$. Then, by Theorem 45, we get a decomposition of the category A-Mod as a product

$$\prod_{D\in\mathrm{Cl}_E(H)} e_D^H A\text{-Mod}.$$

Also, for each indecomposable A-module M, there exists a unique E-conjugacy class D of H such that $e_D^H M\neq 0$ and so $e_D^H M = M$. On the other hand, in Corollary 3.3.14 of [García 2018] it is shown that $e_D^H A$ is a simple A-module and hence that A is a semisimple A-module, since $A=\sum_D e_D^H A$.

Finally, using Lemma 3.3.10 in [García 2018], we see that the idempotents $e_C^{G\times H}$, for C an E-class of $G\times H$, appearing in the evaluation $A(\mathrm{Inf}_1^G)(e_D^H)$ are those for which $\pi_H(C)$, the projection of C on H, is equal to D.

Shifted p-permutation functors. Let k be an algebraically closed field of positive characteristic p. In this case we assume also that $\mathbb{F}$ contains all the p'-roots of unity, and consider the functor $\mathbb{F}pp_k$. Then $\mathbb{F}pp_k$ is a commutative Green biset functor, and the category $\mathbb{F}pp_k$-Mod has been considered in particular in [Ducellier 2015] (when $\mathbb{F}$ is algebraically closed).

We fix a finite group H, and consider the shifted functor $A = (\mathbb{F}pp_k)_H$. Then the algebra $A(1)$ is isomorphic to the algebra $\mathbb{F}pp_k(H)$. This algebra is split semisimple, and its primitive idempotents $F^H_{Q,s}$ have been determined in [Bouc and Thévenaz 2010]: they are indexed by (conjugacy classes of) pairs (Q, s) consisting of a p-subgroup Q of H, and a p'-element s of $N_H(Q)/Q$. We denote by $\mathcal{Q}_{H,p}$ the set of such pairs, and by $[\mathcal{Q}_{H,p}]$ a set of representatives of orbits of H for its action on $\mathcal{Q}_{H,p}$ by conjugation.

If $(Q, s) \in \mathcal{Q}_{H,p}$ and $u \in \mathbb{F}pp_k(H)$, then $F^H_{Q,s}u = \tau^H_{Q,s}(u)F^H_{Q,s}$, where $\tau^H_{Q,s}(u) \in \mathbb{F}$. The maps $u \mapsto \tau^H_{Q,s}(u)$, for $(Q, s) \in [\mathcal{Q}_{H,p}]$ are the distinct algebra homomorphisms (the species) from $\mathbb{F}pp_k(H)$ to $\mathbb{F}$ (see, e.g., [Bouc and Thévenaz 2010, Proposition 2.18]). Moreover, the map $\tau^H_{Q,s}$ is determined by the fact that for any p-permutation kH-module M, the scalar $\tau^H_{Q,s}(M)$ is equal to the value at s of the Brauer character of the Brauer quotient $M[Q]$ of M at Q.

It follows that if $N \trianglelefteq H$, and $v \in \mathbb{F}pp_k(H/N)$, then $\tau^H_{Q,s}(\mathrm{Inf}^H_{H/N}\, v) = \tau^{H/N}_{\overline{Q},\overline{s}}(v)$, where $\overline{Q} = QN/N$, and $\overline{s} \in N_{H/N}(\overline{Q})/\overline{Q}$ is the projection of s to H/N. As a consequence, if $(R, t) \in \mathcal{Q}_{H/N,p}$, then $\mathrm{Inf}^H_{H/N}(F^{H/N}_{R,t})$ is equal to the sum of the idempotents $F^H_{Q,s}$ for those elements $(Q, s) \in [\mathcal{Q}_{H,p}]$ for which $(\overline{Q}, \overline{s})$ is conjugate to (R, t) in H/N.

Now by Theorem 45, we get a decomposition of the category A-Mod as a product $\prod_{(Q,s)\in[\mathcal{Q}_{H,p}]} F^H_{Q,s}A$-Mod. Let G be a finite group. It follows from the previous discussion on inflation that the evaluation $F^H_{Q,s}A(G)$ of A at G is equal to the set of linear combinations of primitive idempotents $F^{G\times H}_{L,t}$, for $(L, t) \in \mathcal{Q}_{G\times H,p}$, such that the pair $(p_2(L), p_2(t))$ is conjugate to (Q, s) in H, where $p_2 : G \times H \to H$ is the second projection. Also, for each indecomposable A-module M, there exists a unique $(Q, s) \in [\mathcal{Q}_{H,p}]$ such that $F^H_{Q,s}M \neq 0$, and then $F^H_{Q,s}M = M$.

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