Pacific Journal of Mathematics

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Volume 303 No. 2 December 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 303, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.303.491

ON THE BOUNDEDNESS OF MULTILINEAR FRACTIONAL STRONG MAXIMAL OPERATORS WITH MULTIPLE WEIGHTS

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We investigate the boundedness of multilinear fractional strong maximal operator $\mathcal{M}_{\mathcal{R},\alpha}$ associated with rectangles or related to more general basis with multiple weights $A_{(\vec{p},q),\mathcal{R}}$. In the rectangular setting, we first give an end-point estimate of $\mathcal{M}_{\mathcal{R},\alpha}$, which not only extends the famous linear result of Jessen, Marcinkiewicz and Zygmund, but also extends the multilinear result of Grafakos, Liu, Pérez and Torres ($\alpha = 0$) to the case $0 < \alpha < mn$. Then, in the one weight case, we give several equivalent characterizations between $\mathcal{M}_{\mathcal{R},\alpha}$ and $A_{(\vec{p},q),\mathcal{R}}$. Based on the Carleson embedding theorem regarding dyadic rectangles, we obtain a multilinear Fefferman–Stein type inequality, which is new even in the linear case. We present a sufficient condition for the two weighted norm inequality of $\mathcal{M}_{\mathcal{R},\alpha}$ and establish a version of the vector-valued two weighted inequality for the strong maximal operator when $m = 1$. In the general basis setting, we study the properties of the multiple weight $A_{(\vec{p},q),\mathcal{R}}$ conditions, including the equivalent characterizations and monotonic properties, which essentially extends previous understanding. Finally, a survey on multiple strong Muckenhoupt weights is given, which demonstrates the properties of multiple weights related to rectangles systematically.

1. Introduction

The study of multiparameter operators originated in the works of Fefferman and Stein [1982] on two-parameter singular integral operators. Journé [1985] gave a multiparameter version of the $T1$ theorem on product spaces. A new type of the $T1$ theorem on product spaces was formulated by Pott and Villarroya [2011].

Xue was supported partly by NSFC (Nos. 11671039,11871101) and NSFC-DFG (No. 11761131002). Yabuta was supported partly by Grant-in-Aid for Scientific Research (C) Nr. 15K04942, Japan Society for the Promotion of Science. Xue is the corresponding author.

MSC2010: primary 42B25; secondary 47G10.

Keywords: multilinear, strong maximal operator, multiple weights, two-weight inequalities, endpoint estimate.

Martikainen [2012] demonstrated a two-parameter representation of singular integrals in expression of the dyadic shifts, which was extended in the famous result of Hytönen [2017] for the one-parameter case. More recently, using the probabilistic methods and the techniques of dyadic analysis, Hytönen and Martikainen [2014] gave a two-parameter version of the $T1$ theorem in spaces of nonhomogeneous type. A two-parameter version of the Tb theorem on product Lebesgue spaces was obtained by Ou [2015], where b is a tensor product of two pseudoaccretive functions.

It is also well known that the most prototypical representative of the multiparameter operators is the following strong maximal operator $M_{\mathcal{R}}$:

$$M_{\mathcal{R}} f(x) := \sup_{\substack{R \ni x \\ R \in \mathcal{R}}} \frac{1}{|R|} \int_R |f(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where $\mathcal{R}$ is the collection of all rectangles $R \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. It can be seen as a geometric maximal operator which commutes with a full n-parameter group of dilations $(x_1, \dots, x_n) \to (\delta_1 x_1, \dots, \delta_n x_n)$. The strong $L^p(\mathbb{R}^n)(1 < p < \infty)$ boundedness of $M_{\mathcal{R}}$ was given by García-Cuerva and Rubio de Francia [1985, p.456]. A maximal theorem was given by Jessen, Marcinkiewicz and Zygmund in [Jessen et al. 1935]. They pointed out that unlike the classical Hardy–Littlewood maximal operator, the strong maximal function is not of weak type (1, 1). Moreover, they studied the end-point behavior of $M_{\mathcal{R}}$ and obtained the inequality

$$|\{x \in \mathbb{R}^n : M_{\mathcal{R}} f(x) > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) dx. \tag{1-1}$$

Córdoba and Fefferman [1975] gave a geometric proof of (1-1) and established a covering lemma for rectangles. Their covering lemma is quite useful because it overcomes the failure of the Besicovitch covering argument for rectangles with arbitrary eccentricities. The selection algorithm given by Córdoba and Fefferman was used many times to gain end-point estimates for $M_{\mathcal{R}}$, as demonstrated in [Córdoba 1976; Fefferman 1981; Grafakos et al. 2011; Hagelstein and Parissis 2018; Liu and Luque 2014; Long and Shen 1988; Luque and Parissis 2014; Mitsis 2006].

The corresponding weighted version of (1-1) with $w \in A_{1,\mathcal{R}}$ was shown by Bagby and Kurtz [1984]. In addition, the weighted weak type and strong type norm inequalities for vector-valued strong maximal operators were obtained in [Capri and Gutiérrez 1988]. It is worth pointing out that this was the first time that the Córdoba–Fefferman covering lemma was not used in obtaining the end-point estimate of $M_{\mathcal{R}}$. Subsequently, the above weighted results were improved by enlarging the range of weights class in [Luque and Parissis 2014; Mitsis 2006]. Luque and Parissis [2014] formulated a weighted version of the Córdoba–Fefferman covering lemma and showed the weighted version of (1-1) for any $n \geq 2$ and $w \in A_{\infty,\mathcal{R}}$. For $n = 2$, the weighted endpoint estimate was first proved in [Mitsis 2006] for $w \in A_{p,\mathcal{R}}$ and $1 < p < \infty$. Unfortunately, the combinatorics of two-dimensional rectangles used there are not available in higher dimensions. To overcome this obstacle, Luque and Parissis [2014] adopted a different approach, relying heavily on the best constant of the weighted estimates of the strong maximal operator [Long and Shen 1988].

Grafakos et al. [2011] first introduced the multilinear version of the strong maximal operator $\mathcal{M}_{\mathcal{R}}$. Later, it was improved by Cao, Xue and Yabuta [Cao et al. 2017] to the multilinear fractional strong maximal operator $\mathcal{M}_{\mathcal{R},\alpha}$

$$\mathcal{M}_{\mathcal{R},\alpha}(\vec{f})(x) := \sup_{\substack{R\ni x\\ R\in\mathcal{R}}} \prod_{i=1}^{m} \frac{1}{|R|^{1-\alpha/(mn)}} \int_R |f_i(y)|\,dy, \tag{1-2}$$

where $0 \le \alpha < mn$. Similarly, one can define the multilinear maximal function $\mathcal{M}_{\mathscr{B}}$ on a general basis $\mathscr{B}$ if $\mathcal{R}$ is replaced by $\mathscr{B}$ in (1-2). In [Grafakos et al. 2011], it is also proved that for a Muckenhoupt basis $\mathscr{B}$, the multilinear maximal operator $\mathcal{M}_{\mathscr{B}}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p,\infty}(v)$ provided that $(\vec{w}, v)$ are weights satisfying $v \in A_{\infty,\mathscr{B}}$ and the power bump condition for some $r > 1$,

$$\sup_{B\in\mathscr{B}} \left(\frac{1}{|B|}\int_B v\,dx\right) \prod_{i=1}^{m} \left(\frac{1}{|B|}\int_B w_i^{(1-p_i')r}\,dx\right)^{p/p_i' r} < \infty. \tag{1-3}$$

It is also worth mentioning that the authors of [Grafakos et al. 2011] established the sharp multilinear version of the endpoint inequality for $\mathcal{M}_{\mathcal{R}}$. Subsequently, under a weaker condition (Tauberian condition) than $v \in A_{\infty,\mathscr{B}}$, Liu and Luque [2014] investigated the strong boundedness of the two-weighted inequality for the maximal operator $\mathcal{M}_{\mathscr{B}}$. They showed that if the maximal operator $M_{\mathscr{B}}$ satisfies the Tauberian condition (called condition (A) in [Hagelstein et al. 2015; Jawerth 1986; Pérez 1993]) then $\mathcal{M}_{\mathscr{B}}$ enjoys the strong-type boundedness. Recently, Hagelstein et al. [2015] discussed the relationship between the boundedness of $M_{\mathscr{B}}$, the Tauberian condition $(A_{\mathscr{B},\gamma,\mu})$ and the weighted Tauberian condition. Furthermore, Hagelstein and Parissis [2018] proved that the asymptotic estimate for weighted Tauberian constant associated to rectangles is equivalent to $w \in A_{\infty,\mathcal{R}}$, which gives a new characterization of the class $A_{\infty,\mathcal{R}}$.

Inspired by [Grafakos et al. 2011], the authors [Cao et al. 2017] studied the relationship between the multilinear fractional strong maximal operator $\mathcal{M}_{\mathcal{R},\alpha}$ and multiple weights $A_{(\vec{p},q),\mathcal{R}}$ associated with rectangles defined by

$$[\vec{w}, v]_{A_{(\vec{p},q),\mathcal{R}}} := \sup_{R\in\mathcal{R}} |R|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|R|}\int_R v\,dx\right)^{\frac{1}{q}} \prod_{i=1}^{m} \left(\frac{1}{|R|}\int_R w_i^{1-p_i'}\,dx\right)^{\frac{1}{p_i'}} < \infty.$$

The dyadic reverse doubling condition associated with rectangles, which is weaker than $A_{\infty,\mathcal{R}}$, was also introduced. It was shown that if each $w_i^{1-p_i'}$ satisfies the dyadic reverse doubling condition, then the two-weight boundedness of $\mathcal{M}_{\mathcal{R},\alpha}$ is equivalent to $(\vec{w}, v) \in A_{(\vec{p},q),\mathcal{R}}$. Significantly, a Carleson embedding theorem regarding dyadic rectangles was established and was the core of the proof.

Motivated by [Cao et al. 2017; Grafakos et al. 2011; Liu and Luque 2014], here we continue to investigate the boundedness of multilinear strong and fractional strong maximal operators in the setting of rectangles and in the setting of a more general basis. We are mainly concerned with the end-point behavior, characterizations of two weighted norm inequalities and vector-valued norm inequalities. We will also give a survey on multiple strong Muckenhoupt weights, which demonstrates the properties of multiple weights associated with rectangles systematically.

2. Definitions and main results

Rectangular setting. We now formulate the main results of the maximal operators related to rectangles. The first result is concerned with the end-point behavior of $\mathcal{M}_{\mathcal{R},\alpha}$.

Theorem 2.1. *Let $n \geq 1$, $m \geq 1$ and $0 \leq \alpha < mn$. Then for any $\lambda > 0$, the following endpoint estimate holds*:

$$\left|\left\{x \in \mathbb{R}^n ; \mathcal{M}_{\mathcal{R},\alpha}(\vec{f})(x) > \lambda^m\right\}\right|^{m-\alpha/n}$$
$$\lesssim_{m,n,\alpha} \prod_{i=1}^m \left[1+\left(\frac{\alpha}{mn} \log^+ \prod_{j=1}^m \int_{\mathbb{R}^n} \Phi_n^{(m)}\left(\frac{|f_j(y)|}{\lambda}\right) dy\right)^{n-1}\right]^m \int_{\mathbb{R}^n} \Phi_n^{(m)}\left(\frac{|f_i(y)|}{\lambda}\right) dy,$$

where $\Phi_n(t) := t[1 + (\log^+ t)^{n-1}]$ and $\Phi_n^{(m)} = \overbrace{\Phi_n \circ \cdots \circ \Phi_n}^{m}$. Moreover, the exponent is sharp in the sense that we cannot replace $\Phi_n^{(m)}$ by $\Phi_n^{(k)}$ for $k \leq m-1$.

Remark 2.2. If $m = 1$ and $\alpha = 0$, then the above inequality in Theorem 2.1 coincides with the inequality (1-1). In the multilinear setting, if $\alpha = 0$, Theorem 2.1 recovers the corresponding inequality in [Grafakos et al. 2011]. Therefore, Theorem 2.1 extends not only the linear result given by Jessen, Marcinkiewicz and Zygmund [Jessen et al. 1935] but also extends the multilinear result proved by Grafakos et al. [2011]. Even in the linear setting, Theorem 2.1 is completely new for $0 < \alpha < n$.

In order to state the other results, we need to introduce one more definition:

Definition 2.3 [Liu and Luque 2014]. Let $1 < p < \infty$. A Young function Φ is said to satisfy the B_p^* condition, written $\Phi \in B_p^*$, if there is a positive constant c such that

$$\int_c^\infty \frac{\Phi_n(\Phi(t))}{t^p} \frac{dt}{t} < \infty,$$

where $\Phi_n(t) := t[1 + (\log^+ t)^{n-1}]$ for all $t > 0$.

We obtain the two weighted, vector-valued estimate of $M_{\mathcal{R}}$ as follows:

Theorem 2.4. *Let $1 < q < p < \infty$, $r = p/q$. Assume that A and B are Young functions such that their complementary Young functions $\bar{A}$ and $\bar{B}$ satisfy $\bar{A} \in B^*_{r'}$ and $\bar{B} \in B^*_q$, respectively. Let (w, v) be a couple of weights such that*

$$\sup_{R\in\mathcal{R}} \|w^q\|^{1/q}_{A,R} \|v^{-1}\|_{B,R} < \infty. \tag{2-1}$$

For some fixed $\gamma \in (0, 1)$ and for any nonnegative function $h \in L^{r'}(\mathbb{R}^n)$ with $\|h\|_{L^{r'}(\mathbb{R}^n)} = 1$, assume that $M_{\mathcal{R}}$ satisfies the $(A_{\mathcal{R},\gamma,h})$ condition and the $(A_{\mathcal{R},\gamma,w^q h})$ condition. Then, the two weight vector-valued inequality holds for $M_{\mathcal{R}}$,

$$\|M_{\mathcal{R}} f\|_{L^p(\ell^q,w^p)} \lesssim \|f\|_{L^p(\ell^q,v^p)}.$$

Remark 2.5. Theorem 2.4 was shown by Pérez [2000], whenever the family of rectangles $\mathcal{R}$ is replaced by cubes. Moreover, in the scalar-valued case, Theorem 2.4 was proved by Liu and Luque [2014].

In order to establish the boundedness of the multilinear fractional strong maximal operator $\mathcal{M}_{\mathcal{R},\alpha}$, we give the definition of the corresponding multiple weights.

Definition 2.6 (class of $A_{(\vec{p},q),\mathcal{R}}$, [Cao et al. 2017]). Let $1 < p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $q > 0$. Suppose that $\vec{w} = (w_1, \dots, w_m)$ and each w_i is a nonnegative locally integrable function on $\mathbb{R}^n$. We say that $\vec{w}$ satisfies the $A_{(\vec{p},q),\mathcal{R}}$ condition or $\vec{w} \in A_{(\vec{p},q),\mathcal{R}}$ if it satisfies

$$[\vec{w}]_{A_{(\vec{p},q),\mathcal{R}}} := \sup_R \left(\frac{1}{|R|}\int_R \nu_{\vec{w}}^q \, dx\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|R|}\int_R w_i^{-p_i'} \, dx\right)^{1/p_i'} < \infty,$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$. If $p_i = 1$, $\left(\frac{1}{R}\int_R w_i^{1-p_i'}\right)^{1/p_i'}$ is understood as $(\inf_R w_i)^{-1}$.

We formulate the weighted results of $\mathcal{M}_{\mathcal{R},\alpha}$ in the following characterizations:

Theorem 2.7. *Let $k \in \mathbb{N}$, $0 \le \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \dots, p_m < \infty$, and $0 < p \le q < \infty$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then the following are equivalent:*

$$\vec{w} \in A_{(\vec{p},q),\mathcal{R}}; \tag{2-2}$$

$$\vec{w}^r \in A_{(\vec{p}/r,\,q/r),\mathcal{R}} \quad \textit{for some } r > 1; \tag{2-3}$$

$$\mathcal{M}_{\mathcal{R},\alpha} : L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m}) \to L^q(\nu_{\vec{w}}^q); \tag{2-4}$$

$$\mathcal{M}_{\mathcal{R},\alpha,\Phi_{k+1}} : L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m}) \to L^q(\nu_{\vec{w}}^q). \tag{2-5}$$

Remark 2.8. Although the fact that (2-2) is equivalent to (2-4) was given in [Cao et al. 2017], we here present some new ingredients. In addition, Theorem 2.7 tells us that the weight class $A_{(\vec{p},q),\mathcal{R}}$ not only implies the boundedness of $\mathcal{M}_{\mathcal{R},\alpha}$, but that it also characterizes much bigger operators $\mathcal{M}_{\mathcal{R},\alpha,\Phi_{k+1}}$.

Furthermore, we obtain the following result:

Theorem 2.9. *Let* $0 \le \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *with* $1 < p_1, \ldots, p_m < \infty$, *and* $0 < p \le q < \infty$. *If* $(\vec{w}, v)$ *are weights such that* $v \in A_{\infty,\mathcal{R}}$ *and the power bump condition holds for some* $r > 1$,

$$\sup_{R \in \mathcal{R}} |R|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|R|}\int_R v\,dx\right)^{\frac{1}{q}} \prod_{i=1}^{m}\left(\frac{1}{|R|}\int_R w_i^{(1-p_i')r}\,dx\right)^{\frac{1}{rp_i'}} < \infty, \tag{2-6}$$

then $\mathcal{M}_{\mathcal{R},\alpha} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^q(v)$.

Corollary 2.10. *Suppose that* $0 \le \alpha < mn$ *and that* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *with* $1 < p_1, \ldots, p_m < mn/\alpha$. *Let each* u_i *be a nonnegative locally integrable function. Then* $\vec{u} \in A_{\vec{p},\mathcal{R}}$ *implies that*

$$\|\mathcal{M}_{\mathcal{R},\alpha}(\vec{f})\|_{L^p(v^p)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i^{p_i})},$$

where $v = \prod_{i=1}^{m} u_i^{1/p_i}$ *and* $w_i = M_{\alpha p_i/m}(u_i)$.

Finally, we end this subsection with a multilinear Fefferman–Stein type inequality.

Theorem 2.11. *Let* $0 \le \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *with* $1 < p_1, \ldots, p_m < \infty$, *and* $0 < p \le q < \infty$ *satisfying* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then, for any weights* $\vec{\omega}$ *on* $\mathbb{R}^n$ *and* $\nu = \prod_{i=1}^{m} \omega_i^{1/m}$, *we have that*

$$\|\mathcal{M}_{\mathcal{R},\alpha}(\vec{f})\|_{L^q(\nu)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}((M_{\mathcal{R}} w_i)^{p_i/mq})},$$

where the constant C *is independent of the weights* $\vec{\omega}$ *and* $\vec{f}$.

The general basis and two weight norm inequalities. In this subsection, we will present some general results for the maximal operator defined on the general basis. We start by introducing some definitions and notations, which will be used later.

By a basis $\mathscr{B}$ in $\mathbb{R}^n$ we mean a collection of open sets in $\mathbb{R}^n$. We say that w is a weight associated with the basis $\mathscr{B}$ if w is a nonnegative measurable function in $\mathbb{R}^n$ such that $w(B) = \int_B w(x)\,dx < \infty$ for each $B \in \mathscr{B}$. Moreover, $w \in A_{p,\mathscr{B}}$ means that

$$\sup_{B \in \mathscr{B}} \left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B w^{1-p'}\,dx\right)^{p/p'} < \infty.$$

We say that $\mathscr{B}$ is a Muckenhoupt basis if $M_{\mathscr{B}} : L^p(w) \to L^p(w)$ for any $1 < p < \infty$ and for any $w \in A_{p,\mathscr{B}}$.

We also need some basic property of Orlicz spaces. More details can be found in [Rao and Ren 1991]. A Young function is a continuous, convex, increasing function

$\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and such that $\Phi(t)/t \to \infty$ as $t \to \infty$. The Φ-norm of a function f over a set E with finite measure is defined by

$$\|f\|_{\Phi,E} = \inf\left\{\lambda > 0;\ \frac{1}{|E|}\int_E \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}. \tag{2-7}$$

For a given Young function Φ, one can define a complementary function

$$\overline{\Phi}(s) = \sup_{t>0}\{st - \Phi(t)\}, \quad s \ge 0.$$

Moreover, the generalized Hölder inequality holds:

$$\frac{1}{|E|}\int_E |f(x)g(x)|\, dx \le 2\|f\|_{\Phi,E}\|g\|_{\overline{\Phi},E}. \tag{2-8}$$

Definition 2.12. Suppose that the function $\varphi : (0,\infty) \to (0,\infty)$ is essentially nondecreasing and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0$. Assume that $\mathscr{B}$ is a basis and that $\{\Psi_i\}_{i=1}^m$ is a sequence of Young functions. We define the multilinear Orlicz maximal operator associated with the function φ by

$$\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f})(x) = \sup_{\substack{B\ni x\\ B\in\mathscr{B}}} \varphi(|B|)\prod_{i=1}^m \|f_i\|_{\Psi_i,B}, \quad x \in \mathbb{R}^n.$$

In particular, if $\Psi_i(t) = t,\ i = 1,\dots,m$, we denote $\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}$ by $\mathcal{M}_{\mathscr{B},\varphi}$. If $\varphi(t) = t^{\alpha/n}$, we denote $\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}$ and $\mathcal{M}_{\mathscr{B},\varphi}$ by $\mathcal{M}_{\mathscr{B},\alpha,\vec{\Psi}}$ and $\mathcal{M}_{\mathscr{B},\alpha}$ respectively. When $\mathscr{B} = \mathcal{R}$, $\mathcal{M}_{\mathscr{B},\alpha}$ coincides with $\mathcal{M}_{\mathcal{R},\alpha}$.

Definition 2.13. We say that the maximal operator $M_{\mathscr{B}}$ satisfies the $(A_{\mathscr{B},\gamma,\mu})$ condition with respect to some $\gamma \in (0,1)$ and a weight μ, if there exists a positive constant $C_{\mathscr{B},\gamma,\mu}$ such that, for all measurable sets E, it holds that

$$\mu\big(\{x \in \mathbb{R}^n : M_{\mathscr{B}}(\mathbf{1}_E)(x) > \gamma\}\big) \le C_{\mathscr{B},\gamma,\mu}\,\mu(E).$$

We summarize the main results as follows:

Theorem 2.14. *Let* $0 < p \le q < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *with* $1 < p_1,\dots,p_m < \infty$. *Let* $\mathcal{A}_i$, $\mathcal{B}_i$ *and* $\mathcal{C}_i$ $(i = 1,\dots,m)$ *be Young functions such that* $\mathcal{A}_i^{-1}(t)\mathcal{C}_i^{-1}(t) \le \mathcal{B}_i^{-1}(t),\ t > 0$ *for each* $i = 1,\dots,m$. *Assume that* $\mathscr{B}$ *is a basis and* $\{\mathcal{C}_i\}_{i=1}^m$ *is a sequence of Young functions satisfying*

$$\mathcal{M}_{\mathscr{B},\vec{\mathcal{C}}} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

If $(\vec{w}, v)$ *are weights such that* $\mathcal{M}_{\mathscr{B},\varphi,\vec{\mathcal{B}}}$ *satisfies the* $(A_{\mathscr{B},\gamma,v^q})$ *condition and*

$$\sup_{B\in\mathscr{B}} \varphi(|B|)|B|^{\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|B|}\int_B v^q\, dx\right)^{\frac{1}{q}} \prod_{i=1}^m \|w_i^{-1}\|_{\mathcal{A}_i,B} < \infty, \tag{2-9}$$

then $\mathcal{M}_{\mathscr{B},\varphi,\vec{\mathcal{B}}} : L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m}) \to L^q(v^q)$.

Corollary 2.15. *Let $0 \le \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \dots, p_m < \infty$, and $0 < p \le q < \infty$. Assume that $\mathscr{B}$ is a Muckenhoupt basis. If $(\vec{w}, v)$ are weights such that $\mathcal{M}_{\mathscr{B},\alpha}$ satisfies the $(A_{\mathscr{B},\gamma,v})$ condition and the power bump condition*

$$\sup_{B\in\mathscr{B}} |B|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|B|}\int_B v\,dx\right)^{\frac{1}{q}} \times \prod_{i=1}^m \left(\frac{1}{|B|}\int_B w_i^{(1-p_i')r}\,dx\right)^{\frac{1}{rp_i'}} < \infty \quad \text{for some } r > 1, \tag{2-10}$$

then $\mathcal{M}_{\mathscr{B},\alpha} : L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^q(v)$.

Remark 2.16. It is easy to see that our Corollary 2.15 extends Theorem 2.3 of [Grafakos et al. 2011]. Indeed, under the same assumptions, the authors [Grafakos et al. 2011] only achieved boundedness from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p,\infty}(v)$. On the other hand, we enlarge the range of α from $\alpha = 0$ to $0 \le \alpha < mn$.

Finally, we present a two weighted norm inequality in the more general context of Banach function spaces.

Theorem 2.17. *Let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \dots, p_m < \infty$, and $0 < p \le q < \infty$. Let φ be a function as in Definition 2.12. Suppose that $Y_1, \dots, Y_m$ are Banach function spaces such that*

$$\mathcal{M}_{\vec{Y}'} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

If $(\vec{w}, v)$ are weights such that $\mathcal{M}_{\vec{Y}'}$ satisfies the $(A_{\mathscr{B},\gamma,v^q})$ condition and

$$\sup_{B\in\mathscr{B}} \varphi(|B|)|B|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|B|}\int_B v^q\,dx\right)^{\frac{1}{q}} \prod_{i=1}^m \|w_i^{-1}\|_{Y_i,B} < \infty, \tag{2-11}$$

then $\mathcal{M}_{\mathscr{B},\varphi} : L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m}) \to L^q(v^q)$.

This article is organized as follows. In Section 3, some important properties of multiple weight $A_{(\vec{p},q),\mathcal{R}}$ will be given. In Section 4, we shall prove Theorems 2.1 and 2.7. Section 5 is devoted to proving Theorem 2.11. As for the rest of the theorems, we will complete their proofs in Section 6.

3. A survey on multiple strong Muckenhoupt weights

In this section, our goal is to study the properties of multiple weights related to rectangles systematically. We first recall the definition of $A_{\vec{p},\mathcal{R}}$ which was introduced in [Grafakos et al. 2011].

Definition 3.1. Let $1 \le p_1, \dots, p_m < \infty$. We say that m-tuple of weights $\vec{w}$ satisfies the $A_{\vec{p},\mathcal{R}}$ condition (or $\vec{w} \in A_{\vec{p},\mathcal{R}}$) if

$$[\vec{w}]_{A_{\vec{p},\mathcal{R}}} := \sup_{R\in\mathcal{R}} \left(\frac{1}{|R|} \int_R \hat{\nu}_{\vec{w}}\, dx \right) \prod_{i=1}^m \left(\frac{1}{|R|} \int_R w_i^{1-p_i'}\, dx \right)^{p/p_i'} < \infty,$$

where $\hat{\nu}_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$. If $p_i = 1$, $\left(\frac{1}{R}\int_R w_i^{1-p_i'}\right)^{1/p_i'}$ is understood as $(\inf_R w_i)^{-1}$.

The characterizations of multiple weights are as follows.

Theorem 3.2. *Let* $1 \le p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $p_0 = \min\{p_i\}_i$. *Then the following statements hold*:

(1) $A_{r_1\vec{p},\mathcal{R}} \subsetneqq A_{r_2\vec{p},\mathcal{R}}$ *for any* $1/p_0 \le r_1 < r_2 < \infty$.

(2) $A_{\vec{p},\mathcal{R}} = \bigcup_{1/p_0 \le r < 1} A_{r\vec{p},\mathcal{R}}$.

(3) $\vec{w} \in A_{\vec{p},\mathcal{R}}$ *if and only if*

$$\hat{\nu}_{\vec{w}} \in A_{mp,\mathcal{R}} \quad \text{and} \quad w_i^{1-p_i'} \in A_{mp_i',\mathcal{R}},\ i = 1, \dots, m,$$

where $w_i^{1-p_i'} \in A_{mp_i',\mathcal{R}}$ *is understood as* $w_i^{1/m} \in A_{1,\mathcal{R}}$ *if* $p_i = 1$.

Theorem 3.3. *Let* $1 \le p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $\frac{1}{m} \le p \le q < \infty$. *Then, it holds that*

(i) $\vec{w} \in A_{(\vec{p},q),\mathcal{R}}$ *if and only if*

$$\nu_{\vec{w}}{}^q \subset A_{mq,\mathcal{R}} \quad \text{and} \quad w_i^{-p_i'} \in A_{mp_i',\mathcal{R}},\ i = 1, \dots, m.$$

When $p_i = 1$, $w_i^{-p_i'}$ *is understood as* $w_i^{1/m} \in A_{1,\mathcal{R}}$.

(ii) *Assume that* $0 < \alpha < mn$, $p_1, \dots, p_m < \frac{mn}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then* $\vec{w} \in A_{(\vec{p},q),\mathcal{R}}$ *if and only if*

$$\nu_{\vec{w}}{}^q \in A_{q(m-\alpha/n),\mathcal{R}} \quad \text{and} \quad w_i^{-p_i'} \in A_{p_i'(m-\alpha/n),\mathcal{R}},\ i = 1, \dots, m.$$

When $p_i = 1$, $w_i^{-p_i'} \in A_{p_i'(m-\alpha/n),\mathcal{R}}$ *is understood as* $w_i^{n/(mn-\alpha)} \in A_{1,\mathcal{R}}$.

Theorem 3.4. *Let* $1 < p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{m} < p \le q < \infty$ *and* $p_0 = \min\{p_i\}_i$. *It holds that*

(a) $A_{(\vec{p},q,r_2),\mathcal{R}} \subsetneqq A_{(\vec{p},q,r_1),\mathcal{R}}$, *whenever* $1 \le r_1 < r_2 < p_0$.

(b) *For any* $1 \le r_1 < p_0$,

$$A_{(\vec{p},q,r_1),\mathcal{R}} = \bigcup_{r_1 < r < p_0} A_{(\vec{p},q,r),\mathcal{R}},$$

where $A_{(\vec{p},q,s),\mathcal{R}} := \left\{\vec{w};\ \vec{w}^s = (w_1^s, \dots, w_m^s) \in A_{(\vec{p}/s,\, q/s),\mathcal{R}}\right\}$ *for any* $s \ge 1$.

Proofs of Theorems 3.2–3.4. The argument used in [Chen et al. 2014, Theorems 2.4 and 3.11] relies only on the use of Hölder's inequality, and doesn't involve any geometric property of cubes or rectangles. Hence we may also use the method in [Chen et al. 2014] to complete our proof. Since the main ideas are almost the same, we omit the proof here. It is worth mentioning that when considering the strict inclusion relation in Theorem 3.2 (1) and Theorem 3.4 (a), we need the characterization of $|x|^\alpha \in A_{p,\mathcal{R}}$, which will be shown in Proposition 3.7 below. □

Definition 3.5. We say that a nonnegative measurable function ω satisfies the dyadic reverse doubling condition, or $\omega \in RD^{(\beta)}$, if ω is locally integrable on $\mathbb{R}^n$ and there is a constant $\beta > 1$ such that $\beta\omega(I) \le \omega(J)$ for any $I, J \in \mathcal{DR}$, where $I \subset J$ and $|I| = 2^{-1}|J|$.

Proposition 3.6. $A_{\infty,\mathcal{R}}(\mathbb{R}^n) \subsetneqq RD^{(\beta)}(\mathbb{R}^n)$, *for any* $\beta > 1$ *and* $n \ge 2$.

Proof. The inclusion relation $A_{\infty,\mathcal{R}}(\mathbb{R}^n) \subset RD^{(\beta)}(\mathbb{R}^n)$ has been proved in [Cao et al. 2017, Proposition 4.2]. Thus, it suffices to show that there exists some weight $w \in RD^{(\beta)}(\mathbb{R}^n) \setminus \bigcup_{1\le p<\infty} A_{p,\mathcal{R}}$. This follows from the following fact.

Let $\omega_0(t)$ be an even function on $(-\infty,\infty)$, which is defined for $t > 0$ by

$$\omega_0(t) = (1-t)\mathbf{1}_{[0,1)} + \sum_{k=1}^{\infty}[1 - 2^{-k+1}(t-2^{k-1})]\,\mathbf{1}_{[2^{k-1},2^k)}.$$

Then ω_0 satisfies the dyadic reverse doubling condition with $\beta = \frac{4}{3}$, but $\omega_0 \notin A_\infty(\mathbb{R})$. Moreover, if we define

$$\omega_j(x) := \omega_0(x_j)dx_1\cdots dx_n, \qquad j = 1,\dots,n,$$

then it holds that $\omega_j \in RD^{(\beta^*)} \setminus A_{\infty,\mathcal{R}}(\mathbb{R}^n)$, where $\beta^* = \max\{\beta, 2\}$.

Let us begin by showing $\omega_0 \notin A_\infty(\mathbb{R})$. For $j \in \mathbb{N}$, we get

$$\int_{1-j^{-2}}^{1+j^{-3}} \omega_0(t)\,dt = \int_{1-j^{-2}}^{1} \omega_0(t)\,dt + \int_{1}^{1+j^{-3}} \omega_0(t)\,dt = \frac{1}{2j^4} + \left(\frac{1}{j^3} - \frac{1}{2j^6}\right),$$

and so

$$\frac{\omega_0([1,1+j^{-3}))}{\omega_0([1-j^{-2},1+j^{-3}))} = \frac{\frac{1}{j^3}-\frac{1}{2j^6}}{\frac{1}{2j^4}+\frac{1}{j^3}-\frac{1}{2j^6}} = \frac{1-\frac{1}{2j^3}}{1+\frac{1}{2j}-\frac{1}{2j^3}} \to 1 \text{ as } j\to\infty,$$

and

$$\frac{|[1,1+j^{-3})|}{|[1-j^{-2},1+j^{-3})|} = \frac{1}{j+1} \to 0 \text{ as } j\to\infty.$$

From this we see that $\omega_0 \notin A_\infty(\mathbb{R})$.

A direct proof that $\omega_0 \notin A_\infty(\mathbb{R})$. For $0 < a < 1$ and $p > 2$ we have

$$\begin{aligned}\int_{1-a^2}^{1+a^3} \omega_0(t)\,dt &= \int_{1-a^2}^{1} (1-t)\ dt + \int_1^{1+a^3} (2-t)\,dt \\ &= \frac{a^4}{2} + a^3 - \frac{a^6}{2} \geq \frac{a^3}{2} + \frac{a^4}{2},\end{aligned}$$

$$\begin{aligned}\int_{1-a^2}^{1+a^3} \omega_0(t)^{1-p'}\,dt &= \int_{1-a^2}^{1} (1-t)^{1-p'}\,dt + \int_1^{1+a^3} (2-t)^{1-p'}\,dt \\ &= \frac{1}{2-p'}[a^{2(2-p')} + 1 - (1-a^3)^{2-p'}] \geq \frac{a^{2(2-p')}}{2-p'}.\end{aligned}$$

Hence for $I_a = [1-a^2, 1+a^3)$ we get

$$\begin{aligned}&\left(\frac{1}{|I_a|}\int_{-a^2}^{a^3} \omega_0(t)\,dt\right)\left(\frac{1}{|I_a|}\int_{-a^2}^{a^3} \omega_0(t)^{1-p'}\,dt\right)^{p-1} \\ &\qquad \geq \frac{1}{a^2+a^3}\left(\frac{a^3}{2}+\frac{a^4}{2}\right)\left(\frac{1}{a^2+a^3}\,\frac{a^{2(2-p')}}{2-p'}\right)^{p-1} \\ &\qquad \gtrsim a \times \left(\frac{a^{2(2-p')}}{a^2}\right)^{p-1} = a \times a^{2(1-p')(p-1)} = a^{-1}.\end{aligned}$$

This shows

$$\sup_{I:\text{intervals}} \left(\frac{1}{|I|}\int_I \omega_0(t)\,dt\right)\left(\frac{1}{|I|}\int_I \omega_0(t)^{1-p'}\,dt\right)^{p-1} = \infty,$$

Hence $\omega_0 \notin A_\infty(\mathbb{R})$.

Next, we demonstrate $\omega_0 \in RD^{(\beta)}$ with $\beta = \frac{4}{3}$.

Let $I \subset \mathbb{R}$ be a dyadic interval, with I_- and I_+ the left and right children of I, respectively. Set $I = [m2^k, (m+1)2^k),\ m, k \in \mathbb{Z}$. Since ω_0 is even, it suffices to consider $m \geq 0$.

Case 1: $m = 0$, $k \geq 1$. In this case, we have

$$\begin{aligned}\omega_0(I) &= \int_0^{2^k} \omega_0(t)\,dt = \int_0^1 \omega_0(t)\,dt + \sum_{j=1}^{k} \int_{2^{j-1}}^{2^j} \omega_0(t)\,dt \\ &= \frac{1}{2} + \sum_{j=1}^{k} 2^{j-2} = 2^{k-1},\end{aligned} \tag{3-1}$$

and

$$\omega_0(I_-) = \int_0^{2^{k-1}} \omega_0(t)\,dt = 2^{k-2}, \quad \omega_0(I_+) = \int_{2^{k-1}}^{2^k} \omega_0(t)\,dt = 2^{k-2}. \tag{3-2}$$

Thus, it holds that

$$2\omega_0(I_+) = 2\omega_0(I_-) = \omega_0(I). \tag{3-3}$$

Case 2: $m = 0$, $k \le 0$. It is easy to get $I = [0, 2^k) \subset [0, 1)$. Then we obtain

$$\frac{1}{4} \le \frac{\omega_0(I_-)}{\omega_0(I)} = \frac{2^{k-1}(1-2^{k-2})}{2^k(1-2^{k-1})} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2-2^k}\right) \le \frac{1}{2} \times \left(\frac{1}{2}+1\right) = \frac{3}{4},$$

and hence,

$$\tfrac{4}{3}\omega_0(I_-) \le \omega_0(I), \quad \tfrac{4}{3}\omega_0(I_+) \le \omega_0(I). \tag{3-4}$$

Case 3: $m \ge 1$, $m \cdot 2^k < 1$. We have $0 < m < 2^{-k} \in \mathbb{Z}_+$, and so $0 < m \le 2^{-k} - 1$. Hence $I = [m2^k, (m+1)2^k) \subset (0, 1)$. So, we also have (3-4)

Case 4: $m \ge 1$, $m \cdot 2^k \ge 1$. There exists some $\ell \in \{0, 1, 2, 3, \dots\}$ such that $m2^k \in [2^\ell, 2^{\ell+1})$. Then it follows that $2^{\ell-k} \le m < 2^{\ell-k+1}$, which, together with the fact that $m \in \mathbb{Z}_+$, implies that $\ell \ge k$. From this, we have $m+1 \le 2^{\ell-k+1}$, and so $(m+1)2^k \le 2^{\ell+1}$. This means that

$$I = [m2^k, (m+1)2^k) \subset [2^\ell, 2^{\ell+1}).$$

Therefore, we deduce that

$$\begin{aligned}\frac{1}{2} \le \frac{\omega_0(I_-)}{\omega_0(I)} &= \frac{2^{k-1}\big[1-\big(m+\frac{1}{4}\big)2^{k-\ell}\big]}{2^k\big[1-\big(m+\frac{1}{2}\big)2^{k-\ell}\big]} \\ &= \frac{1}{2}\left(1 + \frac{2^{k-\ell}/4}{1-\big(m+\frac{1}{2}\big)2^{k-\ell}}\right) \\ &\le \frac{1}{2} \times \left(1+\frac{1}{2}\right) = \frac{3}{4},\end{aligned}$$

and

$$\frac{1}{4} \le \frac{\omega_0(I_+)}{\omega_0(I)} = 1 - \frac{\omega_0(I_-)}{\omega_0(I)} \le \frac{1}{2}.$$

This implies that in this setting, the inequality (3-4) holds as well.

From Cases 1–4, we see that ω_0 satisfies the dyadic reverse doubling condition. The proof in higher dimensions follows from the one-dimension result. □

Proposition 3.7. *Let $1 < p < \infty$. The strong Muckenhoupt weight has the characterization*: $|x|^\alpha \in A_{p,\mathcal{R}}(\mathbb{R}^n)$ *if and only if* $-1 < \alpha < p - 1$.

Although this proposition is contained in [Kurtz 1980], we here present a new proof.

Proof. The "only if" part follows from Lemma 2.2 in [Kurtz 1980, p. 239], and the following fact:

$$w(t) = (1+|t|)^\alpha \in A_p(\mathbb{R}) \ \text{ if and only if } \ -1 < \alpha < p-1. \tag{3-5}$$

Conversely, in the case $-1 < \alpha \le 0$, we see that $t^\alpha \in A_1(\mathbb{R}_+)$ and is decreasing. So, $|x|^\alpha \in \widetilde{A}_1(\mathbb{R}_+)$, and hence by Theorem 4.4 in [Yabuta 2011] it belongs to $A_{1,\mathcal{R}}(\mathbb{R}^n) \subset A_{p,\mathcal{R}}(\mathbb{R}^n)$.

In the case $0 < \alpha < p-1$, we have $-1 < \alpha/(1-p) < 0$, and so $t^{\alpha/(1-p)} \in A_1(\mathbb{R}_+)$ and is decreasing. Hence $|x|^\alpha = (|x|^{\alpha/(1-p)})^{1-p} \in \widetilde{A}_p(\mathbb{R}_+)$, and so, as before, it belongs to $A_{p,\mathcal{R}}(\mathbb{R}^n)$.

Here,

$$\widetilde{A}_p(\mathbb{R}_+) := \{\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p} :\ \nu_1, \nu_2 \in A_1(\mathbb{R}_+) \ \text{ are decreasing or } \nu_1^2, \nu_2^2 \in A_1(\mathbb{R}_+)\}$$

and

$$\widetilde{A}_1(\mathbb{R}_+) := \{\omega(x) = \nu_1(|x|) :\ \nu_1 \in A_1(\mathbb{R}_+) \ \text{ is decreasing or } \nu_1^2 \in A_1(\mathbb{R}_+)\},$$

which are the weight classes introduced by Duoandikoetxea [1993]. □

4. Proofs of Theorem 2.1 and Theorem 2.7

To show the endpoint estimate of $\mathcal{M}_{\mathcal{R},\alpha}$, we need the following key lemma:

Lemma 4.1 [Grafakos et al. 2011]. *Let $m \in \mathbb{N}$, and E be any set. If Φ is a submultiplicative Young function, then there is a constant C such that whenever*

$$1 < \prod_{i=1}^m \|f_i\|_{\Phi,E}$$

holds, one can get

$$\prod_{i=1}^m \|f_i\|_{\Phi,E} \le C \prod_{i=1}^m \frac{1}{|E|}\int_E \Phi^{(m)}(|f_i(x)|)\,dx.$$

Proof of Theorem 2.1. Denote $E = \{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{R},\alpha} f(x) > \lambda^m\}$. Then there exists a compact set K such that $K \subset E$ and

$$|K| \le |E| \le 2|K|.$$

By the compactness of K, one can find a finite collection of rectangles $\{R_j\}_{j=1}^N$ such that

$$K \subset \bigcup_{j=1}^N R_j \quad \text{and} \quad \lambda^m < \prod_{i=1}^m \frac{1}{|R_j|^{1-\alpha/(mn)}}\int_{R_j} |f_i(y)|\,dy, \quad j = 1, \dots, N. \tag{4-1}$$

According to the Córdoba–Fefferman rectangle covering lemma [1975], there are positive constants δ, c depending only on n and a subfamily $\{\widetilde{R}_j\}_{j=1}^{\ell}$ of $\{R_j\}_{j=1}^{N}$ satisfying

$$\Big|\bigcup_{j=1}^{N} R_j\Big| \le c\Big|\bigcup_{j=1}^{\ell} \widetilde{R}_j\Big|, \tag{4-2}$$

and

$$\int_{\bigcup_{j=1}^{\ell} \widetilde{R}_j} \exp\bigg(\delta \sum_{j=1}^{\ell} \mathbf{1}_{\widetilde{R}_j}(x)\bigg)^{\frac{1}{n-1}} dx \le 2\Big|\bigcup_{j=1}^{\ell} \widetilde{R}_j\Big|. \tag{4-3}$$

For convenience, we introduce the following notation: $\widetilde{E} = \bigcup_{j=1}^{\ell} \widetilde{R}_j$ and $\Psi_n(t) = \exp(t^{1/(n-1)}) - 1$. Then the inequality (4-3) is the same as

$$\frac{1}{|\widetilde{E}|} \int_{\widetilde{E}} \Psi_n\bigg(\delta \sum_{j=1}^{\ell} \mathbf{1}_{\widetilde{R}_j}(x)\bigg)\, dx \le 1.$$

Furthermore, using the fact that

$$\|f\|_{\Phi,E} \le 1 \Leftrightarrow \frac{1}{|E|} \int_E \Phi(|f(x)|)\, dx \le 1, \ \text{ for any set } |E| < \infty, \tag{4-4}$$

one can obtain

$$\bigg\|\sum_{j=1}^{\ell} \mathbf{1}_{\widetilde{R}_j}\bigg\|_{\Psi_n,\widetilde{E}} \le \delta^{-1}. \tag{4-5}$$

Therefore, in all, combining the inequalities (4-1) and (4-2), we have

$$\begin{aligned}
|\widetilde{E}|^{1-\alpha/(mn)} &= \Big|\bigcup_{j=1}^{\ell} \widetilde{R}_j\Big|^{1-\alpha/(mn)} \\
&\le \sum_{j=1}^{\ell} |\widetilde{R}_j|^{1-\alpha/(mn)} \bigg(\frac{1}{\lambda^m} \prod_{i=1}^{m} \frac{1}{|\widetilde{R}_j|^{1-\alpha/(mn)}} \int_{\widetilde{R}_j} |f_i(y)|\, dy\bigg)^{1/m} \\
&= \sum_{j=1}^{\ell} \bigg(\prod_{i=1}^{m} \int_{\widetilde{R}_j} \frac{|f_i(y)|}{\lambda}\, dy\bigg)^{1/m} \\
&\le \bigg(\prod_{i=1}^{m} \sum_{j=1}^{\ell} \int_{\widetilde{R}_j} \frac{|f_i(y)|}{\lambda}\, dy\bigg)^{1/m} \\
&= \bigg(\prod_{i=1}^{m} \int_{\widetilde{E}} \sum_{j=1}^{\ell} \mathbf{1}_{\widetilde{R}_j}(y) \frac{|f_i(y)|}{\lambda}\, dy\bigg)^{1/m}.
\end{aligned}$$

Hence, from the Hölder's inequalities (2-8) and (4-5), it now follows that

$$
\begin{aligned}
1 &\le \prod_{i=1}^{m} \frac{1}{|\widetilde{E}|} \int_{\widetilde{E}} \sum_{j=1}^{\ell} \mathbf{1}_{\widetilde{R}_j}(y) \cdot |\widetilde{E}|^{\alpha/(mn)} \frac{|f_i(y)|}{\lambda}\, dy \\
&\le \prod_{i=1}^{m} \Big\| \sum_{j=1}^{\ell} \mathbf{1}_{\widetilde{R}_j} \Big\|_{\Psi_n, \widetilde{E}} \Big\| |\widetilde{E}|^{\alpha/(mn)} \frac{f_i}{\lambda} \Big\|_{\Phi_n, \widetilde{E}} \\
&\le \prod_{i=1}^{m} \delta^{-1} \Big\| |\widetilde{E}|^{\alpha/(mn)} \frac{f_i}{\lambda} \Big\|_{\Phi_n, \widetilde{E}} = \prod_{i=1}^{m} \Big\| \delta^{-1} |\widetilde{E}|^{\alpha/(mn)} \frac{f_i}{\lambda} \Big\|_{\Phi_n, \widetilde{E}}.
\end{aligned}
$$

Applying Lemma 4.1, we deduce that

$$
1 \le \prod_{i=1}^{m} \frac{1}{|\widetilde{E}|} \int_{\widetilde{E}} \Phi_n^{(m)} \Big(\delta^{-1} |\widetilde{E}|^{\alpha/(mn)} \frac{|f_i(y)|}{\lambda} \Big)\, dy.
$$

Notice that the function $\Phi_n^{(m)}$ is submultiplicative. Accordingly, we get

$$
\begin{aligned}
1 &\lesssim \prod_{i=1}^{m} \frac{1}{|\widetilde{E}|} \int_{\widetilde{E}} \Phi_n^{(m)}(|\widetilde{E}|^{\alpha/(mn)}) \Phi_n^{(m)} \Big(\frac{|f_i(y)|}{\lambda} \Big)\, dy \\
&\lesssim \prod_{i=1}^{m} \frac{1}{|\widetilde{E}|^{1-\alpha/(mn)}} \big[1 + (\log^+ |\widetilde{E}|^{\alpha/(mn)})^{n-1}\big]^m \int_{\widetilde{E}} \Phi_n^{(m)} \Big(\frac{|f_i(y)|}{\lambda} \Big)\, dy,
\end{aligned}
\tag{4-6}
$$

where we have used the fact that $\Phi_n^{(m)}(t) \lesssim t[1 + (\log^+ t)^{n-1}]^m$. Moreover, (4-6) implies that

$$
|\widetilde{E}|^{m-\alpha/n} \lesssim \prod_{i=1}^{m} \big[1 + (\log^+ |\widetilde{E}|^{\alpha/(mn)})^{n-1}\big]^m \int_{\mathbb{R}^n} \Phi_n^{(m)} \Big(\frac{|f_i(y)|}{\lambda} \Big)\, dy. \tag{4-7}
$$

In order to get a further estimate, we need a basic fact: if $\theta \in (0, 1)$, then there exists a constant $C_0 > 1$ and β small enough such that

$$
0 < \beta < \frac{1-\theta}{mn}, \quad 1 + \log^+ t^\theta \le t^\beta, \text{ if } t > C_0. \tag{4-8}
$$

If $|\widetilde{E}| > C_0$, then by the inequalities (4-7) and (4-8) we have

$$
|\widetilde{E}|^{m-\alpha/n} \lesssim |\widetilde{E}|^{m^2(n-1)\beta} \prod_{i=1}^{m} \int_{\mathbb{R}^n} \Phi_n^{(m)} \Big(\frac{|f_i(y)|}{\lambda} \Big)\, dy,
$$

and hence

$$
|\widetilde{E}|^{m-\alpha/n-m^2(n-1)\beta} \lesssim \prod_{i=1}^{m} \int_{\widetilde{E}} \Phi_n^{(m)} \Big(\frac{|f(x)|}{\lambda} \Big)\, dx.
$$

Therefore,

$$\log^+ |\widetilde{E}|^{\alpha/(mn)} \lesssim \frac{\alpha}{mn} \log^+ \prod_{i=1}^m \int_{\mathbb{R}^n} \Phi_n^{(m)}\left(\frac{|f_i(y)|}{\lambda}\right) dy.$$

From this inequality and (4-7), we obtain

$$|\widetilde{E}|^{m-\alpha/n} \lesssim \prod_{i=1}^m \left[1+\left(\frac{\alpha}{mn} \log^+ \prod_{j=1}^m \int_{\mathbb{R}^n} \Phi_n^{(m)}\left(\frac{|f_j(y)|}{\lambda}\right) dy\right)^{n-1}\right]^m \times \int_{\mathbb{R}^n} \Phi_n^{(m)}\left(\frac{|f_i(y)|}{\lambda}\right) dy. \tag{4-9}$$

On the other hand, if $|\widetilde{E}| \le C_0$, then

$$1+(\log^+|\widetilde{E}|^{\alpha/(mn)})^{n-1} \lesssim 1.$$

Hence,

$$|\widetilde{E}|^{m-\alpha/n} \lesssim \prod_{i=1}^m \int_{\mathbb{R}^n} \Phi_n^{(m)}\left(\frac{|f_i(y)|}{\lambda}\right) dy. \tag{4-10}$$

Consequently, combining (4-9), (4-10) with the fact that $|E| \lesssim |\widetilde{E}|$, we deduce the desired result. □

Next, we will demonstrate Theorem 2.7. The proof will be based on Theorem 2.14, which will be proved in Section 6. First we recall the definition of the generalized Hölder's inequality on Orlicz spaces due to O'Neil [1965].

Lemma 4.2. *If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are Young functions satisfying*

$$\mathcal{A}^{-1}(t)\mathcal{C}^{-1}(t) \le \mathcal{B}^{-1}(t) \quad \textit{for any } t>0,$$

then for all functions f, g and any measurable set $E \subset \mathbb{R}^n$, the following inequality holds:

$$\|fg\|_{\mathcal{B},E} \le 2\|f\|_{\mathcal{A},E}\|g\|_{\mathcal{C},E}. \tag{4-11}$$

Proof of Theorem 2.7. The process of our proof is (2-3) ⇔ (2-2) ⇒ (2-5) ⇒ (2-4) ⇒ (2-2). In fact, (2-3) ⇔ (2-2) is contained in [Cao et al. 2017, Theorem 2.2]. From Lemma 4.2, it follows that $\mathcal{M}_{\mathcal{R},\alpha}(\vec{f}) \le \mathcal{M}_{\mathcal{R},\alpha,\Phi_{k+1}}(\vec{f})$. This shows (2-5) ⇒ (2-4). Moreover, taking $f_i = w_i^{-p_i'}\chi_R$ for a given rectangle R, we may obtain (2-4) ⇒ (2-2). Hence, it remains to prove (2-2) ⇒ (2-5).

By Theorem 3.3 and [García-Cuerva and Rubio de Francia 1985, Theorem 6.7, p. 458], it is easy to see that $\nu_{\vec{w}}^q$ satisfies the condition (A) and $w_i^{-p_i'}$ satisfies the reverse Hölder inequality. Thus, there exist constants $c_i > 0,\ r_i > 1$ $(i = 1, \dots, m)$ such that

$$\left(\frac{1}{|R|}\int_R w_i^{-p_i' r_i}\,dx\right)^{\frac{1}{r_i}} \le \frac{c_i}{|R|}\int_R w_i^{-p_i'}\,dx \quad \text{for any rectangle } R. \tag{4-12}$$

For fixed $k \in \mathbb{N}$, we introduce the notation

$$\mathcal{A}_i(t) = t^{r_i p_i'}, \quad \mathcal{C}_i(t) = [t(1+\log^+ t)^k]^{(r_i p_i')'}.$$

Thus, we can obtain that

$$\mathcal{A}_i^{-1}(t) = t^{1/(r_i p_i')} \quad \text{and} \quad \mathcal{A}_i^{-1}(t)\mathcal{C}_i^{-1}(t) \approx \Phi_{k+1}^{-1}(t).$$

Notice that $\mathcal{C}_i \in B_{p_i}^*$ and $\mathcal{C}_i$ is submultiplicative. From [Liu and Luque 2014, Proposition 2.2], it now follows that

$$M_{\mathcal{R},\mathcal{C}_i} : L^{p_i}(\mathbb{R}^n) \to L^{p_i}(\mathbb{R}^n), \quad i = 1, \dots, m.$$

This yields immediately that

$$\mathcal{M}_{\mathcal{R},\vec{\mathcal{C}}} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

In addition, for a given rectangle R, (4-12) yields that

$$\begin{aligned}
|R|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}&\left(\frac{1}{|R|}\int_R \nu_{\vec{w}}^q\,dx\right)^{\frac{1}{q}}\prod_{i=1}^m \|w_i^{-1}\|_{\mathcal{A}_i,R} \\
&= \left(\frac{1}{|R|}\int_R \nu_{\vec{w}}^q\,dx\right)^{\frac{1}{q}}\prod_{i=1}^m\left(\frac{1}{|R|}\int_R w_i^{-r_i p_i'}\,dx\right)^{\frac{1}{r_i p_i'}} \\
&\lesssim \left(\frac{1}{|R|}\int_R \nu_{\vec{w}}^q\,dx\right)^{\frac{1}{q}}\prod_{i=1}^m\left(\frac{1}{|R|}\int_R w_i^{-p_i'}\,dx\right)^{\frac{1}{p_i'}} \\
&\le [\vec{w}]_{A_{(\vec{p},q),\mathcal{R}}} < \infty.
\end{aligned}$$

This implies that $(\vec{w}, \nu_{\vec{w}})$ satisfies the two weighted condition (2-9). By Theorem 2.14, we get

$$\mathcal{M}_{\mathcal{R},\alpha,\Phi_{k+1}} : L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m}) \to L^q(\nu_{\vec{w}}^q).$$

Therefore, in all, we have completed the proof of Theorem 2.7. □

5. The multilinear Fefferman–Stein inequality

Before showing our multilinear Fefferman–Stein inequality, we present a Carleson embedding theorem regarding dyadic rectangles.

Theorem 5.1 [Cao et al. 2017]. *Let $1 < p \le q < \infty$, ω be a nonnegative locally integrable function on $\mathbb{R}^n$. Assume that $\omega^{1-p'}$ satisfies the dyadic reverse doubling condition with $\beta > 1$. Then the inequality*

$$\sum_{R\in\mathcal{DR}}\Big(\int_R \omega^{1-p'}\,dx\Big)^{-q/p'}\Big(\int_R f(x)\,dx\Big)^q \le C\Big(\int_{\mathbb{R}^n} f(x)^p\omega\,dx\Big)^{q/p}$$

holds for all nonnegative $f \in L^p(\omega)$, where C depends on n, p, q and β.

Proof of Theorem 2.11. It suffices to show the above result for the dyadic version of the maximal operator,

$$\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x) = \sup_{\substack{R\ni x\\ R\in\mathcal{DR}}}\prod_{i=1}^m \frac{1}{|R|^{1-\alpha/(mn)}}\int_R |f_i(y_i)|\,dy_i, \quad x\in\mathbb{R}^n.$$

Adopting the policy in [Cao et al. 2017], we will obtain the general result from the dyadic setting.

Without loss of generality, we can assume that $\vec{f}$ is bounded, $\vec{f} \ge 0$ and has a compact support. Therefore, $\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x) < \infty$ for all $x \in \mathbb{R}^n$.

According to the definition of $\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x)$, we have that for any $x\in\mathbb{R}^n$, there exists a dyadic rectangle R such that $x \in R$ and

$$\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x) \le 2\prod_{i=1}^m \frac{1}{|R|^{1-\alpha/(mn)}}\int_R f_i(y_i)\,dy_i. \tag{5-1}$$

For any dyadic rectangle R, define

$$E(R) := \{x\in\mathbb{R}^n : \text{(5-1) holds for } R \text{ but not for any proper subset of it}\}.$$

From the definition of maximal operators and the inequality (5-1), it is obvious that

$$\mathbb{R}^n = \bigcup_{R\in\mathcal{DR}} E(R).$$

Then it follows that

$$\begin{aligned}\|\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})\|^q_{L^q(\nu)} &\le \sum_{R\in\mathcal{DR}}\int_{E(R)}\Big(\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x)\Big)^q\,\nu\,dx\\ &\lesssim \sum_{R\in\mathcal{DR}}\Big(\prod_{i=1}^m\frac{1}{|R|^{1-\alpha/(mn)}}\int_R f_i(y_i)\,dy_i\Big)^q\nu(R).\end{aligned}$$

Note that

$$\nu(R) = \int_R\prod_{i=1}^m \omega_i(x)^{1/m}\,dx \le \prod_{i=1}^m \omega_i(R)^{1/m}.$$

Thus we have

$$\|\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})\|^q_{L^q(\nu)} \lesssim \sum_{R\in\mathcal{DR}} \prod_{i=1}^m \left(\frac{1}{|R|} \int_R f_i(y_i)\, dy_i \cdot \langle \omega_i \rangle_R^{1/mq} \right)^q |R|^{q/p_i}$$
$$\leq \sum_{R\in\mathcal{DR}} \prod_{i=1}^m \left(\frac{1}{|R|} \int_R f_i(y_i) \cdot M^d_{\mathcal{R}}\omega_i(y_i)^{1/mq}\, dy_i \right)^q |R|^{q/p_i}.$$

Therefore, by Hölder's inequality $\sum_{j=1}^\infty \prod_{i=1}^m |a_{ij}| \leq \prod_{i=1}^m \big(\sum_{j=1}^\infty |a_{ij}|^{p_i/p}\big)^{p/p_i}$, we further deduce that

$$\|\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})\|^q_{L^q(\nu)} \leq \prod_{i=1}^m \left[\sum_{R\in\mathcal{DR}} |R|^{q/p} \left(\frac{1}{|R|} \int_R f_i(y_i) \cdot M^d_{\mathcal{R}}\omega_i(y_i)^{1/mq}\, dy_i \right)^{qp_i/p} \right]^{p/p_i}$$
$$\lesssim \prod_{i=1}^m \|f_i\|^q_{L^{p_i}((M^d_{\mathcal{R}}\omega_i)^{p_i/mq})},$$

where we used Theorem 5.1 with respect to the exponents $(p_i, qp_i/p)$ for $\omega \equiv 1$. □

6. Proofs of Theorems 2.4, 2.9, 2.14, 2.17 and Corollaries 2.10, 2.15

To prove Theorem 2.14, we first introduce the definition of the general basis and a key covering lemma.

Definition 6.1 [Jawerth 1986; Jawerth and Torchinsky 1984]. Let $\mathscr{B}$ be a basis and let $0 < \alpha < 1$. A finite sequence $\{A_i\}_{i=1}^N \subset \mathscr{B}$ of sets of finite dx-measure is called α-scattered with respect to the Lebesgue measure if

$$\Big| A_i \cap \bigcup_{s<i} A_s \Big| \leq \alpha |A_i| \quad \text{for all } 1 < i \leq N.$$

Lemma 6.2 [Grafakos et al. 2011; Jawerth 1986]. *Let $\mathscr{B}$ be a basis and let w be a weight associated to this basis. Suppose further that $M_{\mathscr{B}}$ satisfies the condition $(A_{\mathscr{B},\gamma,w})$ for some $0 < \gamma < 1$. Then, given any finite sequence $\{A_i\}_{i=1}^N$ of sets $A_i \in \mathscr{B}$, one can find a subsequence $\{\widetilde{A}_i\}_{i\in I}$ such that*:

(a) *$\{\widetilde{A}_i\}_{i\in I}$ is γ-scattered with respect to the Lebesgue measure.*

(b) $\widetilde{A}_i = A_i$, $i \in I$.

(c) *For any $1 \leq i < j \leq N+1$,*

$$w\Big(\bigcup_{s<j} A_s\Big) \lesssim w\Big(\bigcup_{s<i} A_s\Big) + w\Big(\bigcup_{i\leq s<j} \widetilde{A}_s\Big),$$

where $\widetilde{A}_s = \varnothing$ when $s \notin I$.

Proof of Theorem 2.14. The idea of the following arguments is essentially a combination of the ideas from [Grafakos et al. 2011; Jawerth 1986; Liu and Luque 2014]. Let $N > 0$ be a large integer. We will prove the required estimate for the quantity

$$\int_{\{2^{-N}<\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f})\leq 2^{N+1}\}} \mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f})(x)^q v^q\,dx,$$

with a bound independent of N. We begin with the following claim.

Claim 6.3. *For each integer k with $|k| \leq N$, there exists a compact set K_k and a finite sequence $b_k = \{B_r^k\}_{r\geq 1}$ of sets $B_r^k \in \mathscr{B}$ such that*

$$v^q(K_k) \leq v^q(\{\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f}) > 2^k\}) \leq 2v^q(K_k).$$

Moreover, $\{\bigcup_{B\in b_k} B\}_{k=-N}^{N}$ is decreasing and therefore

$$\bigcup_{B\in b_k} B \subset K_k \subset \{\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f}) > 2^k\},$$

and

$$\varphi(|B_r^k|)\prod_{j=1}^{m}\|f_j\|_{\Psi_j,B_r^k} > 2^k. \tag{6-1}$$

Proof. To see the claim, for each k we choose a compact set $\widetilde{K}_k \subset \{\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f}) > 2^k\}$ such that

$$v^q(\widetilde{K}_k) \leq v^q(\{\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f}) > 2^k\})2 \leq v^q(\widetilde{K}_k).$$

For this $\widetilde{K}_k$, there exists a finite sequence $b_k = \{B_r^k\}_{r\geq 1}$ of sets $B_r^k \in \mathscr{B}$ such that every B_r^k satisfies (6-1) and such that $\widetilde{K}_k \subset \bigcup_{B\in b_k} B \subset \{\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f}) > 2^k\}$. Now, we take a compact set K_k such that

$$\bigcup_{B\in b_k} B \subset K_k \subset \{\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f}) > 2^k\}.$$

Finally, to ensure that $\{\bigcup_{B\in b_k} B\}_{k=-N}^{N}$ is decreasing, we begin the above selection from $k = N$ and once a selection is done for k we do the selection for $k-1$ with the additional requirement $\widetilde{K}_{k-1} \supset K_k$. This finishes the proof of the claim. □

We continue with the proof of Theorem 2.14. Since $\{\bigcup_{B\in b_k} B\}_{k=-N}^{N}$ is a sequence of decreasing sets, we set

$$\Omega_k = \begin{cases} \bigcup_r B_r^k = \bigcup_{B\in b_k} B & \text{when } |k| \leq N. \\ \varnothing & \text{otherwise.} \end{cases}$$

Observe that these sets are decreasing in k, i.e., $\Omega_{k+1} \subset \Omega_k$ when $-N < k \leq N$.

We now distribute the sets in $\bigcup_k b_k$ over μ sequences $\{A_i(\ell)\}_{i\geq 1}$, $0\leq \ell\leq \mu-1$, where μ will be chosen momentarily to be an appropriately large natural number. Set $i_0(0)=1$. In the first $i_1(0)-i_0(0)$ entries of $\{A_i(0)\}_{i\geq 1}$, i.e., for

$$i_1(0)\leq i < i_1(0),$$

we place the elements of the sequence $b_N=\{B_r^N\}_{r\geq 1}$ in the order indicated by the index r. For the next $i_2(0)-i_1(0)$ entries of $\{A_i(0)\}_{i\geq 1}$, i.e., for

$$i_1(0)\leq i < i_2(0),$$

we place the elements of the sequence $b_{N-\mu}$. We continue in this way until we reach the first integer m_0 such that $N-m_0\mu\geq -N$, when we stop. For indices i satisfying

$$i_{m_0}(0)\leq i < i_{m_0+1}(0),$$

we place in the sequence $\{A_i(0)\}_{i\geq 1}$ the elements of $b_{N-m_0\mu}$. The sequences $\{A_i(\ell)\}_{i\geq 1}$, $1\leq \ell\leq \mu-1$, are defined similarly, starting from $b_{N-\ell}$ and using the families $b_{N-\ell-s\mu}$, $s=0,1,\ldots,m_l$, where m_l is chosen to be the biggest integer such that $N-l-m_l\mu\geq -N$.

Since v^q is a weight associated to $\mathscr{B}$ and it satisfies the condition (A), we can apply Lemma 6.2 to each $\{A_i(\ell)\}_{i\geq 1}$ for some fixed $0<\lambda<1$. Then we obtain sequences

$$\{\widetilde{A}_i(\ell)\}_{i\geq 1}\subset \{A_i(\ell)\}_{i\geq 1},\quad 0\leq \ell\leq \mu-1,$$

which are λ-scattered with respect to the Lebesgue measure. In view of the definition of the set k and the construction of the families $\{A_i(\ell)\}_{i\geq 1}$, we may use assertion (c) of Lemma 6.2 to show that: for any $k=N-\ell-s\mu$ with $0\leq \ell\leq \mu-1$ and $1\leq s\leq m_\ell$,

$$\begin{aligned} v^q(\Omega_k)=v^q(\Omega_{N-\ell-s\mu})&\lesssim v^q(\Omega_{k+\mu})+v^q\Bigg(\bigcup_{i_s(\ell)\leq i\leq i_{s+1}(\ell)}\widetilde{A}_i(\ell)\Bigg)\\ &\leq v^q(\Omega_{k+\mu})+\sum_{i=i_s(\ell)}^{i_{s+1}(\ell)-1} v^q(\widetilde{A}_i(\ell)). \end{aligned}$$

For the case $s=0$, we have $k=N-\ell$ and

$$v^q(\Omega_k)=v^q(\Omega_{N-\ell})\lesssim \sum_{i=i_0(\ell)}^{i_1(\ell)-1} v^q(\widetilde{A}_i(\ell)).$$

Now, all these sets $\{\widetilde{A}_i(\ell)\}_{i=i_s(\ell)}^{i_{s+1}(\ell)}$ belong to b_k with $k=N-\ell-s\mu$ and so

$$\varphi(|\widetilde{A}_i(\ell)|)\prod_{j=1}^{m}\|f_j\|_{\Psi_j,\widetilde{A}_i(\ell)}>2^k. \tag{6-2}$$

Therefore, it now readily follows that

$$\int_{\{2^{-N}<\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f})\le 2^{N+1}\}} \mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f})(x)^q v^q\,dx \lesssim \sum_{k=-N}^{N-1} 2^{kq} v^q(\Omega_k) := \Delta_1,$$

and thus, we have

$$\begin{aligned}
\Delta_1 &= \sum_{\ell=0}^{\mu-1} \sum_{0\le s\le m_\ell} 2^{q(N-\ell-s\mu)} v^q(\Omega_{N-\ell-s\mu}) \\
&\lesssim \sum_{\ell=0}^{\mu-1} \sum_{0\le s\le m_\ell} 2^{q(N-\ell-s\mu)} v^q(\Omega_{N-\ell-s\mu+\mu}) \\
&\qquad + \sum_{\ell=0}^{\mu-1} \sum_{0\le s\le m_\ell} 2^{q(N-\ell-s\mu)} \sum_{i=i_s(\ell)}^{i_{s+1}(\ell)-1} v^q(\widetilde{A}_i(\ell)) \\
&:= \Delta_2+\Delta_3.
\end{aligned} \tag{6-3}$$

To analyze the contribution of Δ_2, we choose μ so large that $2^{-q\mu}\le\frac{1}{2}$. Therefore,

$$\begin{aligned}
\Delta_2 &= 2^{-q\mu} \sum_{\ell=0}^{\mu-1} \sum_{0\le s\le m_{\ell-1}} 2^{q(N-\ell-s\mu)} v^q(\Omega_{N-\ell-s\mu}) \\
&\le 2^{-q\mu} \sum_{k=-N}^{N-1} 2^{kq} v^q(\Omega_k) \le \tfrac{1}{2}\Delta_1.
\end{aligned} \tag{6-4}$$

Since everything involved is finite, Δ_2 can be subtracted from Δ_1. This yields that

$$\int_{\{2^{-N}<\mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f})\le 2^{N+1}\}} \mathcal{M}_{\mathscr{B},\varphi,\vec{\Psi}}(\vec{f})(x)^q v^q\,dx \lesssim \Delta_1 \lesssim \Delta_3.$$

Next we consider the contribution of Δ_3. For the sake of simplicity, for each ℓ we let $I(\ell)$ be the index set of $\{\widetilde{A}_i(\ell)\}_{0\le s\le m_\ell,\, i_s(\ell)\le i<i_{s+1}(\ell)}$. By (6-2) and the generalized Hölder's inequality (4-11), we obtain

$$\begin{aligned}
\Delta_3 &\lesssim \sum_{\ell=0}^{\mu-1} \sum_{i\in I(\ell)} v^q(\widetilde{A}_i(\ell)) \Big[\varphi(|\widetilde{A}_i(\ell)|) \prod_{i=1}^m \|f_i\|_{\Phi,\widetilde{A}_i(\ell)}\Big]^q \\
&\lesssim \sum_{l=0}^{\mu-1} \sum_{i\in I(\ell)} \Big[\prod_{j=1}^m \|f_j\|^p_{C_j,\widetilde{A}_i(\ell)} |\widetilde{A}_i(\ell)|\Big]^{q/p} \\
&\qquad \times \Big[\varphi(|\widetilde{A}_i(\ell)|)|\widetilde{A}_i(\ell)|^{\frac{1}{q}-\frac{1}{p}} \Big(\frac{1}{|\widetilde{A}_i(\ell)|}\int_{\widetilde{A}_i(\ell)} v^q\,dx\Big)^{\frac{1}{q}} \prod_{j=1}^m \|w_j^{-1}\|_{A_j,\widetilde{A}_i(\ell)}\Big]^q
\end{aligned} \tag{6-5}$$

$$\lesssim \sum_{\ell=0}^{\mu-1} \sum_{i\in I(\ell)} \left[\prod_{j=1}^{m} \|f_j w_j\|^p_{C_j,\widetilde{A}_i(\ell)} |\widetilde{A}_i(\ell)| \right]^{q/p}$$
$$\le \left[\sum_{\ell=0}^{\mu-1} \sum_{i\in I(\ell)} \prod_{j=1}^{m} \|f_j w_j\|^p_{C_j,\widetilde{A}_i(\ell)} |\widetilde{A}_i(\ell)| \right]^{q/p},$$

where in the third step we used the two-weight condition (2-9).

Now, we introduce the notations

$$E_1(\ell) = \widetilde{A}_i(\ell) \quad \text{and} \quad E_i(\ell) = \widetilde{A}_i(\ell) \setminus \bigcup_{s<i} \widetilde{A}_s(\ell) \quad \text{for all } i \in I(\ell). \tag{6-6}$$

Since the sequences $\{\widetilde{A}_i(\ell)\}_{i\in I(\ell)}$ are λ-scattered with respect to the Lebesgue measure, $|\widetilde{A}_i(\ell)| \le \frac{1}{1-\lambda}|E_i(\ell)|$ for each i. Then we have the following estimate for (6-5):

$$\Delta_3 \lesssim \left[\frac{1}{1-\lambda} \sum_{\ell=0}^{\mu-1} \sum_{i\in I(\ell)} \prod_{j=1}^{m} \|f_j w_j\|^p_{C_j,\widetilde{A}_i(\ell)} |E_i(\ell)| \right]^{q/p}. \tag{6-7}$$

The collection $\{E_i(\ell)\}_{i\in I(\ell)}$ is a disjoint family; we can therefore use the fact that $\mathcal{M}_{\mathscr{B},\vec{C}}$ is bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ to estimate (6-7). Then

$$\Delta_3 \lesssim \left[\sum_{\ell=0}^{\mu-1} \sum_{i\in I(\ell)} \int_{E_i(\ell)} \Big(\mathcal{M}_{\mathscr{B},\vec{C}}(f_1 w_1,\dots,f_m w_m)(x) \Big)^p dx \right]^{q/p}$$
$$\lesssim \left[\int_{\mathbb{R}^n} \Big(\mathcal{M}_{\mathscr{B},\vec{C}}(f_1 w_1,\dots,f_m w_m)(x) \Big)^p dx \right]^{q/p}$$
$$\lesssim \prod_{i=1}^{m} \|f_i w_i\|^q_{L^{p_i}(\mathbb{R}^n)} = \prod_{i=1}^{m} \|f_i\|^q_{L^{p_i}(w_i^{p_i})}.$$

Finally, letting $N\to\infty$, we finish the proof. □

Proof of Corollary 2.15. For each $i=1,\dots,m$, we set $\widetilde{w}_i := w_i^{1/p_i}$ and $\Psi_i(t) := t^{p_i{}' r}$ for any $t>0$. Set $\tilde{v} := v^{1/q}$. Then we can rewrite the power bump condition (2-10) as

$$\sup_{B\in\mathscr{B}} |B|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|B|} \int_B \tilde{v}^q \, dx \right)^{\frac{1}{q}} \prod_{i=1}^{m} \|\widetilde{w}_i^{-1}\|_{\Psi_i,B} < \infty.$$

In this case, for all $x\in\mathbb{R}^n$,

$$M_{\mathscr{B},\bar{\Phi}_i} f(x) = \sup_{\substack{B\ni x\\ B\in\mathscr{B}}} \left\{ \frac{1}{|B|} \int_B |f(y)|^{(p_i' r)'} dy \right\}^{1/(p_i' r)'}.$$

Since $\mathscr{B}$ is a Muckenhoupt basis and $(p_i'r)' < p_i$, every $M_{\mathscr{B},\overline{\Psi}_i}$ is bounded on $L^{p_i}(\mathbb{R}^n)$. It is easy to see that

$$\mathcal{M}_{\mathscr{B},\vec{\overline{\Psi}}}(\vec{f})(x) \leq \prod_{i=1}^{m} M_{\mathscr{B},\overline{\Psi}_i}(f_i)(x), \quad x \in \mathbb{R}^n,$$

which implies that $\mathcal{M}_{\mathscr{B},\vec{\overline{\Psi}}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Therefore, from Theorem 2.14, it follows that

$$\mathcal{M}_{\mathscr{B},\alpha} : L^{p_1}(\widetilde{w}_1^{p_1}) \times \cdots \times L^{p_m}(\widetilde{w}_m^{p_m}) \to L^q(\tilde{v}^q),$$

which completes the proof. □

Proof of Theorem 2.9. The fact that $\mathcal{R}$ is a Muckenhoupt basis can be found in [García-Cuerva and Rubio de Francia 1985, p. 454]. Moreover, for the rectangle family $\mathcal{R}$, the $A_{\infty,\mathcal{R}}$ condition is equivalent to Tauberian condition $(A_{\mathcal{R},\gamma,w})$, which was proved in [Hagelstein et al. 2015, Corollary 4.8]. Therefore, Theorem 2.9 follows from these facts and Corollary 2.15. □

Proof of Corollary 2.10. From Theorem 3.2, it follows that $v^p \in A_{mp,\mathcal{R}} \subset A_{\infty,\mathcal{R}}$. As for $v = \prod_{i=1}^m u_i^{1/p_i}$ and $w_i = M_{\alpha p_i/m}(u_i)$, it is easy to verify that $(\vec{w}, v)$ satisfies the power bump condition (2-6). Hence, it yields the desired result. □

Proof of Theorem 2.17. Theorem 2.17 follows by using similar arguments to those in the proof of Theorem 2.14. The difference lies in the boundedness of $\mathcal{M}_{\vec{Y}'}$ and the generalized Hölder's inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \leq \|f\|_X \|g\|_{X'}$$

for any Banach function space X. □

Proof of Theorem 2.4. It is well known that there exists some $h \in L^{r'}(\mathbb{R}^n)$ with norm $\|h\|_{L^{r'}(\mathbb{R}^n)} = 1$ such that

$$\begin{aligned}\|M_{\mathcal{R}} f\|^p_{L^p(\ell^q,w^p)} &= \int_{\mathbb{R}^n} \Big(\sum_j M_{\mathcal{R}} f_j(x)^q w(x)^q \Big)^r dx \\ &= \sum_j \int_{\mathbb{R}^n} M_{\mathcal{R}} f_j(x)^q w(x)^q h(x)\,dx.\end{aligned}$$

In order to estimate $\int_{\mathbb{R}^n} M_{\mathcal{R}} f_j(x)^q w(x)^q h(x)\,dx$ for fixed j, we adopt a similar method to that in the proof of Theorem 2.14. Since we obtained the inequality (6-4), we get for any fixed $N > 0$

$$\begin{aligned}\Lambda_{j,N} &:= \int_{\{2^{-N} < M_{\mathcal{R}} f_j(x) \le 2^{N+1}\}} M_{\mathcal{R}} f_j(x)^q w(x)^q h(x)\,dx \\ &\lesssim \sum_{\ell=0}^{\mu-1} \sum_{0\le s\le m_\ell} \sum_{i=i_s(\ell)}^{i_{s+1}(\ell)-1} (w^q h)(\widetilde{A}_i(\ell)) \left(\frac{1}{|\widetilde{A}_i(\ell)|} \int_{\widetilde{A}_i(\ell)} |f_j(x)|\,dx \right)^q.\end{aligned}$$

Making use of the Hölder inequality and two weight condition (2-1), we deduce

$$\begin{aligned}\Lambda_{j,N} &\lesssim \sum_{\ell,s,i} \|w^q\|_{A,\widetilde{A}_i(\ell)} \|h\|_{\bar{A},\widetilde{A}_i(\ell)} \|f_j v\|^q_{\bar{B},\widetilde{A}_i(\ell)} \|v^{-1}\|^q_{B,\widetilde{A}_i(\ell)} |\widetilde{A}_i(\ell)| \\ &\lesssim \sum_{\ell,s,i} \|f_j v\|^q_{\bar{B},\widetilde{A}_i(\ell)} \|h\|_{\bar{A},\widetilde{A}_i(\ell)} |\widetilde{A}_i(\ell)|.\end{aligned}$$

Using the same notations $\{E_i(\ell)\}$ as (6-6), we have

$$\begin{aligned}\Lambda_{j,N} &\lesssim \sum_{\ell,s,i} \|f_j v\|^q_{\bar{B},\widetilde{A}_i(\ell)} \|h\|_{\bar{A},\widetilde{A}_i(\ell)} |E_i(\ell)| \\ &\le \sum_{\ell,i} \int_{E_i(\ell)} M_{\mathcal{R},\bar{B}}(f_j v)(x)^q M_{\mathcal{R},\bar{A}} h(x)\,dx \\ &\lesssim \int_{\mathbb{R}^n} M_{\mathcal{R},\bar{B}}(f_j v)(x)^q M_{\mathcal{R},\bar{A}} h(x)\,dx.\end{aligned}$$

Letting $N \to \infty$, we have

$$\int_{\mathbb{R}^n} M_{\mathcal{R}} f_j(x)^q w(x)^q h(x)\,dx \lesssim \int_{\mathbb{R}^n} M_{\mathcal{R},\bar{B}}(f_j v)(x)^q M_{\mathcal{R},\bar{A}} h(x)\,dx.$$

Therefore, from the Hölder inequality and Proposition 6.4, it follows that

$$\begin{aligned}\|M_{\mathcal{R}} f\|^q_{L^p(\ell^q, w^p)} &\lesssim \left\| \left(\sum_{j=1} (M_{\mathcal{R},\bar{B}}(f_j v))^q \right)^{1/q} \right\|^q_{L^p(\mathbb{R}^n)} \|M_{\mathcal{R},\bar{A}} h\|_{L^{r'}(\mathbb{R}^n)} \\ &\lesssim \left\| \left(\sum_{j=1} (f_j v)^q \right)^{1/q} \right\|^q_{L^p(\mathbb{R}^n)} \|h\|_{L^{r'}(\mathbb{R}^n)} = \|f\|^q_{L^p(\ell^q, v^p)}.\end{aligned}$$

This completes the proof of Theorem 2.4. □

Proposition 6.4. *Let $1 < q < p < \infty$. Suppose Φ is a Young function such that $\Phi \in B_q^*$. If the $(A_{\mathcal{R},\gamma,g})$ condition holds for some fixed $\gamma \in (0,1)$ and any nonnegative function $g \in L^{r'}(\mathbb{R}^n)$ with $\|g\|_{L^{r'}(\mathbb{R}^n)} = 1$, then we have*

$$\|M_{\mathcal{R},\Phi} f\|_{L^p(\ell^q,\mathbb{R}^n)} \lesssim \|f\|_{L^p(\ell^q,\mathbb{R}^n)}.$$

Proof. Set $r = p/q$. Then, it holds that

$$\|M_{\mathcal{R},\Phi} f\|^q_{L^p(\ell^q,\mathbb{R}^n)} = \sup_{\|g\|_{L^{r'}(\mathbb{R}^n)}=1} \left| \int_{\mathbb{R}^n} \sum_j M_{\mathcal{R},\Phi} f_j(x)^q g(x) dx \right|.$$

For fixed $g \in L^{r'}(\mathbb{R}^n)$ with $\|g\|_{L^{r'}(\mathbb{R}^n)} = 1$, from the Fefferman–Stein inequality for the maximal operator $M_{\mathcal{R},\Phi}$ (see [Liu and Luque 2014, Theorem 2.1]), it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \sum_j M_{\mathcal{R},\Phi} f_j(x)^q g(x)\, dx \right| &\le \sum_j \int_{\mathbb{R}^n} M_{\mathcal{R},\Phi} f_j(x)^q |g(x)|\, dx \\ &\lesssim \sum_j \int_{\mathbb{R}^n} |f_j(x)|^q M_{\mathcal{R}} g(x)\, dx \\ &\le \Big\| \sum_j |f_j|^q \Big\|_{L^r(\mathbb{R}^n)} \|M_{\mathcal{R}} g\|_{L^{r'}(\mathbb{R}^n)} \\ &\lesssim \|f\|^q_{L^p(\ell^q,\mathbb{R}^n)} \|g\|_{L^{r'}(\mathbb{R}^n)} = \|f\|^q_{L^p(\ell^q,\mathbb{R}^n)}. \qquad \square \end{aligned}$$

Acknowledgements

The authors want to express their sincere thanks to the unknown referee for valuable remarks which made this paper more readable.

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Received October 5, 2017.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 303 No. 2 December 2019