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EMBEDDING AND COMPACT EMBEDDING FOR WEIGHTED AND ABSTRACT SOBOLEV SPACES

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Let Ω be an open set in a metric space H, $1 \le p_0, p \le q < \infty$, $a, b, \gamma \in \mathbb{R}$, $a \ge 0$. Suppose σ, μ, w are Borel measures. Combining results from earlier work (2009) with those obtained in work with Wheeden (2011) and with Rodney and Wheeden (2013), we study embedding and compact embedding theorems of sets $\mathfrak{S} \subset L^1_{\sigma,\mathrm{loc}}(\Omega) \times L^p_w(\Omega)$ to $L^q_\mu(\Omega)$ (projection to the first component) where $\mathfrak{S}$ (abstract Sobolev space) satisfies a Poincaré-type inequality, σ satisfies certain weak doubling property and μ is absolutely continuous with respect to σ. In particular, when $H = \mathbb{R}^n$, w, μ, ρ are weights so that ρ is essentially constant on each ball deep inside in $\Omega \setminus F$, and F is a finite collection of points and hyperplanes. With the help of a simple observation, we apply our result to the study of embedding and compact embedding of $L^{p_0}_{\rho^\gamma}(\Omega) \cap E^p_{w\rho^b}(\Omega)$ and weighted fractional Sobolev spaces to $L^q_{\mu\rho^a}(\Omega)$, where $E^p_{w\rho^b}(\Omega)$ is the space of locally integrable functions in Ω such that their weak derivatives are in $L^p_{w\rho^b}(\Omega)$. In $\mathbb{R}^n$, our assumptions are mostly sharp. Besides extending numerous results in the literature, we also extend a result of Bourgain et al. (2002) on cubes to John domains.

1. Introduction

Sobolev embedding, compact embedding and Poincaré inequalities are essential tools in the study of elliptic partial differential equations (including Yamabe-type problems)

$$\nabla \cdot (A(x)|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = |u|^{q-2}u \quad (q > p > 1), \tag{1-1}$$

where q is less than the critical exponent in the Sobolev embedding and $A(x)$ is a uniformly (or at least locally) positive definite matrix valued function. However, stronger (for example weighted) Sobolev (and compact) embedding is needed if $A(x)$ fails to be uniformly positive definite or degenerate. In this direction, Caffarelli,

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Kohn and Nirenberg [Caffarelli et al. 1984] studied the following weighted Sobolev interpolation inequalities

$$\||x|^{\alpha}u\|_{L^q(\mathbb{R}^n)} \le C\big(\||x|^{\gamma}u\|_{L^r(\mathbb{R}^n)} + \||x|^{\beta}\nabla u\|_{L^p(\mathbb{R}^n)}\big).$$

It has been extended to Lipschitz (and $C^{0,\lambda}$) domains and various distant weights by Gurka and Opic [1988; 1989; 1991] and Kufner [1985] (see also Brown and Hinton [1988]). More recently, it has also been generalized to domains satisfying chain conditions (such as John domains and generalized John domains [Hajłasz and Koskela 1998; Chua 2005; 2009]).

Together with the Poincaré inequality, embedding and compact embedding on Sobolev spaces are used in the studies of elliptic [Saloff-Coste 2002; Brezis and Nirenberg 1983] and degenerate elliptic partial differential equations [Chua and Wheeden 2017; Rodney 2010; Sawyer and Wheeden 2006]. For example, boundedness and regularity of solutions can be obtained if the associate operator of equations satisfies some structure conditions [Monticelli et al. 2012; 2015] while existence of solutions can be assured by embedding and compact embedding [Chua and Wheeden 2017]. Indeed, just Sobolev embedding alone (for the associated operator) will lead to boundedness of solutions of degenerate equations [Chua 2017a]. We will study the counterpart of embedding and compact embedding on abstract Sobolev spaces which include degenerate Sobolev spaces (including weighted fractional Sobolev spaces) on irregular domains. We are able to obtain such embeddings for (Borel) measures that need not be doubling nor reverse doubling (on Ω). We will always assume a simple Poincaré-type inequality (1-4) and use it to obtain various Poincaré inequalities via a standard technique of self improving [Franchi et al. 2003; Chua and Wheeden 2008] on (weak) John domains and balls without any chain or geodesic path condition (see Remark 2.8(3)). Such inequalities are then used to obtain embedding and compact embedding on domains which are a countable union of bounded overlapping (weak) John domains with the same parameters (for example, a generalized John domain). We further provide a unified approach for weights that are essentially constant (1-21) on δ-balls (balls that are "deep" inside the domain). In particular, in case of Euclidean spaces, our assumptions turn out to be simple (and sharp) for such an embedding to hold. As applications, we extend many known results in the literature; for example, [Chanillo and Wheeden 1992; Gatto and Wheeden 1989] (see Corollary 1.6, Remark 1.7); Bourgain, Brezis and Mironescu [Bourgain et al. 2002] (that has been improved by Mazya and Shaposhnikova [2002]). For the latter, we extend it to weighted fractional Sobolev inequalities on John domains in Remark 1.7(3). Furthermore, we extend a weighted Sobolev interpolation inequality by Caffarelli, Kohn and Nirenberg [Caffarelli et al. 1984] to a weighted fractional interpolation inequality with much more complicated weights that may not be doubling (see Theorem 1.14).

In what follows, C will denote a generic positive constant while $C(\alpha, \beta, \gamma, \dots)$ will denote a constant that is depending only on $\alpha, \beta, \gamma, \dots$. When μ and w are weights (nonnegative locally integrable Borel measurable functions), by abusing the notation, $d\mu$ and dw will denote the measure $\mu\, dx$ and $w\, dx$ respectively. When Ω is a domain in the Euclidean space, $E^p_w(\Omega)$ will denote the class of locally Lebesgue integrable functions on Ω with weak derivatives in $L^p_w(\Omega)$. We will write $W^{1,p}_w(\Omega) = L^p_w(\Omega) \cap E^p_w(\Omega)$. This space could be just a normed space (it is a Banach space if $w^{-1/(p-1)}$ is locally integrable in Ω [Kufner and Opic 1984]). We will also work on (weighted) fractional Sobolev spaces $(0 < \alpha < 1)$

$$\widehat{W}^{\alpha,p}_w(\Omega) = \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{\widehat{W}^{\alpha,p}_w(\Omega)} = \left(\int_\Omega \int_{B(x,\rho_\Omega(x)/2)} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\, dy\, w(x) dx \right)^{1/p} < \infty \right\},$$

where $\rho_\Omega(x) = \inf\{|x-y| : y \in \Omega^c\}$ $(\rho_\Omega(x) = \infty$ if $\Omega^c = \varnothing)$. Note that a more common (weighted) fractional Sobolev space is usually defined as

$$W^{\alpha,p}_w(\Omega) = \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{W^{\alpha,p}_w(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\, dy\, w(x) dx \right)^{1/p} < \infty \right\}.$$

While it is clear that $W^{\alpha,p}_w(\Omega) \subset \widehat{W}^{\alpha,p}_w(\Omega)$, the converse is in general not true even when $w = 1$ [Dyda et al. 2016]. In Euclidean spaces, we usually assume (Q is any ball in $\mathbb{R}^n$)

$$\frac{1}{|Q|} \|f - f_Q\|_{L^1(Q)} \le a(Q) \|\nabla^\Omega_{\alpha,p} f\|_{L^p_w(Q)}, \quad \text{where } f_Q = \int_Q f\, dx/|Q|, \tag{1-2}$$

$a(Q)$ is a ball set function and $\nabla^\Omega_{\alpha,p} f$ $(0 < \alpha \le 1)$ could be either the usual gradient $|\nabla f|$ (when $\alpha = 1$) or the "fractional derivative," that is

$$\nabla_\alpha f(x) = \nabla^\Omega_{\alpha,p} f(x) = \left(\int_{B(x,\rho_\Omega(x)/2)} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\, dy \right)^{1/p}.$$

For example, when $5Q \subset \Omega$, (1-2) is known to hold for $w = 1$ with $a(Q) = C|Q|^{\alpha/n - 1/p}$ and hence also holds for any weight w with

$$a(Q) = C|Q|^{\alpha/n} \|w^{-1/p}\|_{L^{p'}(Q)}.$$

For $0 < \alpha < 1$, see Remark 1.2(5). The case where $\alpha = 1$ is well-known (for all balls Q).

Equation (1-2) can be used to obtain

$$\|f - f_{B'}\|_{L^q_\mu(\Omega)} \le C\|\nabla_\alpha f\|_{L^p_w(\Omega)}, \tag{1-3}$$

where B' is a "central ball" in Ω (see Theorem 1.8). Hence (if $\mu(\Omega) < \infty$)

$$\|f\|_{L^q_\mu(\Omega)} \le C\big(\|f\|_{L^1(\Omega)} + \|\nabla_\alpha f\|_{L^p_w(\Omega)}\big).$$

Moreover, as (1-3) will always imply the inequality (1-3) with $f_{B'}$ being replaced by $\int_\Omega f\, d\mu/\mu(\Omega)$, we also have

$$\|f\|_{L^q_\mu(\Omega)} \le C\big(\|f\|_{L^1_\mu(\Omega)} + \|\nabla_\alpha f\|_{L^p_w(\Omega)}\big).$$

The case where $d\mu = \operatorname{dist}(x, \Omega_0)^a\, dx$ and $dw = \operatorname{dist}(x, \Omega_0)^b\, dx$, $a \ge 0, b \in \mathbb{R}$, $\alpha = 1$, $\Omega_0 \subset \Omega^c$ has been studied in [Chua and Wheeden 2011] when Ω is an s-John domain ($s \ge 1$). See [Gurka and Opic 1988; 1989; 1991] for some other weights on $C^{0,1/s}$ domains. For negative a and $0 < \alpha < 1$, see [Chua 2016; 2017b]. In this note, we will discuss the case where Ω (an open set in a metric space) is a bounded overlapping countable union of weak John domains (see (1-7)) with a fixed parameter. This includes generalized John domains [Chua 2009, Definition 1.2] which include bounded and unbounded John domains [Väisälä 1989]. We also allow $\Omega_0 \not\subset \Omega^c$ and more complicated weights which may degenerate (0 or ∞) in Ω. Most of the previous studies assumed μ to be doubling or at least reverse doubling; see [Chua and Wheeden 2011; Hajłasz and Koskela 1998; Hurri-Syrjänen 2004]. Indeed, they considered mostly the case $a \ge 0$. Even though there were studies for the case $a < 0$, the weight μ was known to be doubling (i.e., μdx is doubling) [Chua 2009; 1995; Chua and Wheeden 2011]. For simplicity, we discuss only a few typical applications that include the case where the power a may be negative and μ may neither be δ-doubling (see below) nor reverse doubling. In order to overcome this problem, we first observe that a John domain is still John domain after a finite number of points is removed. We then see that the Sobolev space on the resulting smaller domain contains the original Sobolev space.

For simplicity, we will consider mostly metric spaces where Sobolev spaces are well studied [Cheeger 1999; Heinonen 2001; Hajłasz 1996; Keith 2004; Keith and Zhong 2008] instead of quasimetric spaces even though the technique can be extended to quasimetric spaces as in [Chua and Wheeden 2011; Chua et al. 2013; Sawyer and Wheeden 2010]. Indeed, given any quasimetric d, there exists $\varepsilon > 0$ such that d^ε is bi-Lipschitz equivalent to a metric [Heinonen 2001, Proposition 14.5]. Note that our study will also include Alexandrov spaces and Carnot–Carathéodory metric spaces.

Let $0 < \delta \le \frac{1}{2}$ and Ω be an open set in a metric space. $B(x, r)$ or $B_r(x)$ will denote the metric (or quasimetric) ball with center x and radius $r(B) = r$. Furthermore, $CB = CB(x, r)$ will denote the ball $B(x, Cr)$. We say B is a δ-ball of Ω if $B/\delta \subset \Omega$.

We say σ is a δ-doubling measure on Ω if $\sigma(2^k B\cap\Omega)\le (D_\sigma)^k\sigma(B)$ for all δ-balls B of Ω and $k\in\mathbb{N}$. Moreover, we say it is doubling on Ω if the above holds for all balls with center in Ω. We say σ is doubling if it is doubling on the whole metric space. Let w be a Borel measure on Ω and $1\le\tau\le 1/(2\delta)$. We will be interested in (abstract Sobolev space) $\mathfrak{S}\subset L^1_{\sigma,\mathrm{loc}}(\Omega)\times L^p_{w,\mathrm{loc}}(\Omega)$ (or $L^1_{\sigma,\mathrm{loc}}(\Omega)\times L^p_{w,\mathrm{loc}}(\Omega)^n$) that satisfies the following Poincaré-type inequality:

$$\text{(1-4)}\qquad \frac{1}{\sigma(Q)}\|f-f_{Q,\sigma}\|_{L^1_\sigma(Q)}\le a(Q)\|g\|_{L^p_w(\tau Q)} \quad \text{for all } \delta\text{-balls } Q \text{ of } \Omega \text{ and } (f,g)\in\mathfrak{S},$$

where $f_{Q,\sigma}=\int_Q f\,d\sigma/\sigma(Q)$ and $a(Q)$ is a ball set function (independent of (f,g)). By $f\in L^1_{\sigma,\mathrm{loc}}(\Omega)$, we mean $f\in L^1_\sigma(B)$ for all δ-balls B. The definition will be independent of $\delta\le\frac{1}{2}$ as Ω in this note is assumed to be at most a countable union of bounded overlapping weak John domains Ω_j such that σ is δ-doubling on each Ω_j. Such a simple Poincaré inequality is known to hold in Riemannian manifolds with $g=|\nabla f|$ and Sobolev space on Carnot–Carathéodory metric spaces with Hörmander vector fields [Lu 1992b; 1996; Franchi et al. 1995] with $g=|Xf|$, where X is the "differential operator" associated to the vector field. Indeed, in the later case, it holds with $\sigma=w=1$ and $p=1$ on metric (associated to the vector field) balls. Furthermore, similar to [Chua and Wheeden 2011], for any function f, $b\in\mathbb{R}$ and $\omega>0$, we define (the truncation of $|f-b|$)

$$f_b^\omega=\min\bigl\{\max\{0,|f-b|-\omega\},\omega\bigr\}.$$

We say that $\mathfrak{S}$ satisfies (1-4) with the truncation property if for all $(f,g)\in\mathfrak{S}$, $b\in\mathbb{R}$ and $\omega>0$, there exists $g_b^\omega\in L^p_w(\Omega)$ such that (f_b^ω,g_b^ω) satisfies the inequality (1-4) and

$$\text{(1-5)}\qquad \sup_{\omega>0,b\in\mathbb{R}}\sum_{k=1}^{\infty}\|g_b^{2^k\omega}\|^p_{L^p_w(\Omega)}\le (c_T)^p\|g\|^p_{L^p_w(\Omega)}\quad (c_T\ge 1).$$

For example, if (1-4) holds for all Lipschitz functions u and their derivative $|\nabla u|$ on a Riemannian manifold, it will satisfy (1-4) with the truncation property. Similarly, when X is a "differential operator" and $g=|Xf|$, (1-4) also holds with the truncation property. A more subtle (and not obvious) example will be the fractional derivatives defined above; see Proposition 2.14. Note that our truncation property seems to be weaker than the truncation property introduced in [Hajłasz and Koskela 2000]. For example, fractional derivatives satisfy our truncation property while it is not clear that they satisfy that of [Hajłasz and Koskela 2000].

Following [Hajłasz and Koskela 2000, p. 39], given $0<c<1$, we say that a domain Ω in a metric space $\langle H,d\rangle$ (or quasimetric space) is a weak John domain if there is a fixed "center" $x'\in\Omega$ such that for any $x\ne x'$ in Ω, there exists

$\gamma : [0, l] \to \Omega$ such that $\gamma(0) = x,\ \gamma(l) = x'$ with

$$d(\gamma(t_1), \gamma(t_0)) \le |t_1 - t_0| \quad \text{for all } t_1, t_0 \in [0, l] \tag{1-6}$$

and γ satisfies the weak John condition

$$d(\gamma(t), \Omega^c) = \inf\{d(\gamma(t), y) : y \notin \Omega\} \ge c\, d(\gamma(t), x) \quad \text{for all } t. \tag{1-7}$$

We will write $\Omega \in J'(c)$. The corresponding definition in [Chua and Wheeden 2008; 2011] replaces (1-7) by $\rho(\gamma(t)) > ct$, which is nominally a stronger assumption since $d(x, \gamma(t)) = d(\gamma(0), \gamma(t)) \le t$ by (1-7). The weak version (1-7) was first given by Väisälä in $\mathbb{R}^n$ [Hajłasz and Koskela 2000, Theorem 9.6; Väisälä 1988, Theorem 2.18] and shown to be equivalent to the strong version in $\mathbb{R}^n$. It was extended to metric spaces in [Hajłasz and Koskela 2000; Chua and Wheeden 2015]. We do not know an example when the weak version is true and the strong version is false. In general, the weak version is easier to apply. See also [Martio and Sarvas 1979] for the definition and studies on John domains in Euclidean spaces. More properties of weak John domains can be found in Section 2 and [Chua and Wheeden 2015, Section 2]. Note also that Lipschitz continuity (1-6) could be replaced by just continuity. We now state the main theorem of this paper. The assumptions may look complicated on general metric spaces, but most of them become simple (and sharp) or redundant on Euclidean spaces.

Theorem 1.1. *Let $1 \le p < q < \infty$. Let Ω be an open set in a metric space H and let $0 < \delta \le \frac{1}{2}$, $1 \le \tau \le 1/(2\delta)$, μ, w, σ be Borel measures on H such that μ is absolutely continuous with respect to σ. Suppose there exists $0 < c < 1$ such that Ω is a countable union of sets $\Omega_j \in J'(c)$ with $\sum_j \chi_{\Omega_j} \le M$, $M \in \mathbb{N}$ and σ is δ-doubling on each Ω_j with doubling constant D_σ independent of j, i.e., $\sigma(2^k B \cap \Omega_j) \le (D_\sigma)^k \sigma(B)$ for all δ-balls B of Ω_j and $k \in \mathbb{N}$. Let $\mathfrak{S} \subset L^1_{\sigma,\mathrm{loc}}(\Omega) \times L^p_{w,\mathrm{loc}}(\Omega)$ satisfy the Poincaré inequality (1-4) with the truncation property (1-5). Suppose there exists a ball set function μ^* with $\mu(B \cap \Omega_j) \le \mu^*(B)$ for all balls B and Ω_j, and*

(i) *μ^* satisfies Condition (R) on each Ω_j* (with parameters independent of j):

Condition (R) *There exist $0 < \theta_1 < \theta_2 < 1$, $A_1, A_2 > 0$ such that for each $x \in \Omega_j$, there is a strictly decreasing sequence $\{r^x_m\}_{m\in\mathbb{N}}$ of positive real numbers such that $r^x_m \to 0$, $r^x_1 = \mathrm{diam}(\Omega_j)$, $r^x_m/2 \le r^x_{m+1} < r^x_m$ and*

$$A_1\theta_1^k \le \frac{\mu^*(B(x, r^x_{m+k}))}{\mu^*(B(x, r^x_m))} \le A_2\theta_2^k \quad \textit{for all } m, k \in \mathbb{N}. \tag{1-8}$$

(ii) *There exists $C_1 > 0$ such that for all j,*

$$\mu^*(B)^{1/q} a(Q) \le C_1 \quad \textit{for all balls } B \textit{ with center in } \Omega_j \textit{ and } Q \subset B,\ Q/\delta \subset \Omega_j \quad \textit{with } r(Q) \ge c\delta r(B)/(4\tau). \tag{1-9}$$

(iii) *There exists* $V_\mu \geq 1$ *such that for all* j, *given any collection of balls* $\mathcal{B}_E = \{B_{r_x}(x) : x \in E\}$ *with* $E \subset \Omega_j$, *it has a subfamily* $\mathcal{B}'_E$ *of pairwise disjoint balls such that*

$$\mu(E) \leq V_\mu \sum_{B \in \mathcal{B}'_E} \mu^*(B). \tag{1-10}$$

(We will say (μ, μ^*) satisfies the Vitali-type property on Ω_j with constant V_μ.)

(I) *Then*

$$\|f - f_{B'_j,\sigma}\|_{L^q_\mu(\Omega_j)} \leq C c_T C_1 V_\mu^{1/q} \|g\|_{L^p_w(\Omega_j)} \quad \text{for all } j \text{ and } (f, g) \in \mathfrak{S} \tag{1-11}$$

where $B'_j = B(x'_j, \delta d(x'_j, \Omega^c_j))$, x'_j *is the center of* Ω_j *and* C *depends on* q, p, θ_1, θ_2, A_1, A_2, c, δ, τ *and* D_σ.

(II)(a) *If in addition* $1 \leq p_0 \leq q$ *and there exists* $C_2 > 0$ *such that*

$$\mu(\Omega_j)^{1/q} \leq C_2 \sigma(\Omega_j)^{1/p_0} \quad \text{for all } j, \tag{1-12}$$

then

$$\|f\|_{L^q_\mu(\Omega)} \leq C\big(C_2 M^{1/p_0} \|f\|_{L^{p_0}_\sigma(\Omega)} + C_1 c_T V_\mu^{1/q} M^{1/p} \|g\|_{L^p_w(\Omega)}\big) \tag{1-13}$$

for all $(f, g) \in \mathfrak{S}$ *where* C *depends on* p_0 *and all those parameters given in* (I).

(b) *Furthermore, if* $\mu(\Omega) < \infty$, *then for every sequence* $\{(f_n, g_n)\}$ *in* $\mathfrak{S}$ *such that* $\{f_n\}$ *and* $\{g_n\}$ *are bounded in* $L^{p_0}_\sigma(\Omega)$ *and* $L^p_w(\Omega)$ *respectively*, $\{f_n\}$ *has a subsequence that converges in* $L^{\tilde{q}}_\mu(\Omega)$ *for* $1 \leq \tilde{q} < q$ *to a function in* $L^q_\mu(\Omega)$.

(c) *If* $p_0 < q$ *and instead of* (1-12), *we have*

$$\mu(\Omega_j)^{1/q - 1/p_0} \leq C_2 \quad \text{for all } j, \tag{1-14}$$

then (1-13) *and the conclusion in* (*a*) *will hold with* $L^{p_0}_\sigma(\Omega)$ *being replaced by* $L^{p_0}_\mu(\Omega)$. *Moreover, conclusion in* (*b*) *will hold with* σ *being replaced by* μ (*if* $\mu(\Omega) < \infty$).

Remark 1.2. (1) If $\mathfrak{S}$ only satisfies (1-4) without the truncation property and $\mu(\Omega_j) \leq C_3 < \infty$ for all j, then (1-13) holds with $\|f\|_{L^q_\mu(\Omega)}$ being replaced by $\|f\|_{L^{\tilde{q}}_\mu(\Omega)}$, $p, p_0 \leq \tilde{q} < q$. Thus, (II)(b) remains true if $\mu(\Omega) < \infty$, i.e., for every sequence $\{(f_n, g_n)\}$ in $\mathfrak{S}$ such that $\{f_n\}$ and $\{g_n\}$ are bounded in $L^{p_0}_\sigma(\Omega)$ and $L^p_w(\Omega)$ respectively, $\{f_n\}$ has a subsequence that converges in $L^{\tilde{q}}_\mu(\Omega)$. A similar conclusion holds for the case $p_0 < q$ under the assumption (1-14) for (II)(c).

(2) The case where μ is reverse doubling on $\Omega \subset \mathbb{R}^n$ has been discussed in [Chua and Wheeden 2011, Remark 1.7(3)] when Ω is an s-John domain; see also [Drelichman and Durán 2008] for 1-John domains.

(3) Condition (1-9) can often be simplified. For example, it can be simplified to

$$\mu^*(Q)^{1/q}a(Q) \le C_1 \quad \text{for all } \delta\text{-balls } Q \tag{1-15}$$

when μ^* is doubling. For more discussion, see [Chua and Wheeden 2011, Remark 1.7(4)].

(4) In particular, (1-11) holds with $\Omega_j = \Omega \in J'(c)$ when μ is δ-doubling, under the assumption (1-4) with the truncation property and $\mu(Q)^{1/q}a(Q) \le C_1$ for all δ-balls Q. Note that in this case μ will satisfy Condition (R) and (μ, μ) will satisfy Vitali-type property (1-10).

(5) In case $\Omega \subset H = \mathbb{R}^n$, $\mathfrak{S} = \{(f, \nabla^{\Omega}_{\alpha,p} f) : f \in \mathfrak{S}_\alpha(\Omega), \nabla^{\Omega}_{\alpha,p} f \in L^p_w(\Omega)\}$, where $\mathfrak{S}_\alpha(\Omega) = L^1_{\text{loc}}(\Omega)$ for $0 < \alpha < 1$ and $\mathfrak{S}_1(\Omega) = \text{Lip}_{\text{loc}}(\Omega)$ the space of locally Lipschitz continuous functions on Ω, then (1-4) is known to hold with $d\sigma = dx$, $g = \nabla^{\Omega}_{\alpha,p} f$ (see (1-2)), w a Muckenhoupt A_p weight ($w \in A_p$) and $a(Q) = C_w r(Q)^\alpha w(Q)^{-1/p}$ ($C_w = C(w)$). Indeed, it holds for general weight w with

$$a(Q) = Cr(Q)^{\alpha-n} \|w^{-1/p}\|_{L^{p'}(Q)} \quad (\text{where } 1/p + 1/p' = 1)$$

provided $\|w^{-1/p}\|_{L^{p'}(Q)} < \infty$. The case $\alpha = 1$ is well-known. For $0 < \alpha < 1$, first observe that

$$\begin{aligned} \frac{1}{|Q|}\|f - f_Q\|_{L^1(Q)} &\le \frac{1}{|Q|}\int_Q \frac{1}{|Q|}\int_Q |f(x) - f(y)|\,dy\,dx \\ &\le |Q|^{-1-1/p}\int_Q \biggl(\int_Q |f(x) - f(y)|^p\,dy\biggr)^{1/p} dx \\ &\le C|Q|^{\alpha/n-1}\int_Q \biggl(\int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha p}}\,dy\biggr)^{1/p} dx. \end{aligned} \tag{1-16}$$

Now if Q is any ball with $5Q \subset \Omega$, then $Q \subset B(x, \rho(x)/2)$ for all $x \in Q$. Finally, just apply Hölder's inequality again. Some other discussion on fractional Poincaré inequalities can be found in [Chua 2016; Mazya and Shaposhnikova 2002; Bourgain et al. 2002]. Moreover, if $\alpha = 1$, $w = |J_\phi|^{1-p/n}$, $1 < p < n$, where J_ϕ is the Jacobian of a quasiconformal map ϕ, (1-4) is known to be true with $d\sigma = dw$ and $a(Q) = Cr(Q)w(Q)^{-1/p}$ [Heinonen et al. 1993, p. 10].

In case where $\mu = w \in A_p$ and $\alpha = 1$, compact embedding has already been discussed in [Chua et al. 2013, Theorem 2.2] when $\Omega \subset \mathbb{R}^n$ is a John domain.

(6) When X is a "differential operator" such that

$$\frac{1}{\sigma(Q)}\|f - f_{Q,\sigma}\|_{L^1_\sigma(Q)} \le a(Q)\|Xf\|_{L^p_w(\tau Q)} \tag{1-17}$$

for all $f \in \text{Lip}_{\text{loc}}(\Omega)$ and δ-balls Q, then Theorem 1.1 applies to any doubling measure μ such that $\mu(Q)^{1/q}a(Q) \le C$. For example, when a domain is equipped with Carnot–Carathéodory metric and X associated with a Hölmander's vector field, (1-17) is known to hold with $w = \sigma = 1$ and $p = 1$ by [Jerison 1986]; see [Franchi et al. 1995] for more literature review. Indeed, a complete study can be found in [Franchi et al. 1995] when weights are in (Muckenhaupt) A_p. We are able to reproduce all results in [Franchi et al. 1995] by Theorem 1.1 since the Carnot–Carathéodory metric balls are known to be Boman domains [Lu 1994, Lemma 3.1] and (1-17) is known to hold when $\sigma = 1$ and $w \in A_p$ with $a(Q) = Cr(Q)w(Q)^{-1/p}$. In particular, we obtain [Franchi et al. 1995, Theorem 2] with $\mu = w_2$ and $w_1 = w$. However, instead of assuming $w \in A_p$ together with the balance condition [Franchi et al. 1995, (1.5)], we only need to assume that μ is doubling and

$$\mu(Q)^{1/q} r(Q) \|w^{-1/p}\|_{L^{p'}(Q)} \le C \quad \text{for all } \delta\text{-balls in a given ball } B.$$

On the other hand, if we take $\mu = w \in A_p$, we then obtain the compact embedding given in [Lu 1992a, Lemmas 2.6, 2.9 and Corollary 2.10], where it uses a quite complicated method involving lifting and the Ascoli theorem. Indeed, it will follow from our theorem that if $\mathcal{D}$ is a finite union of sets $\Omega_j \in J'(c)$, then the embedding $W^{1,p}_w(\mathcal{D})$ to $L^q_w(\mathcal{D})$ is compact for $1 \le q < d/(d-1)$ (where d is the homogeneous dimension of the Carnot–Carathéodory metric) when $p = 1$ and $1 \le q \le dp/(d-1) + \varepsilon$ for some $\varepsilon > 0$ depending on w when $1 < p < d$. Furthermore, when $w \in A_1$, the embedding of $W^{1,p}_w(\mathcal{D})$ to $L^q_w(\mathcal{D})$ is compact for $1 \le q < dp/(d-p)$. To see that, it suffices to note that $w \in A_1$ implies $w \in A_p$ and

$$w(\tau B) \le C\tau^d w(B) \quad \text{for any ball } B \text{ and } \tau > 1, \tag{1-18}$$

and hence if $B_r(x)$ is a δ-ball in a John domain $\mathcal{D}$ and $R = \mathrm{diam}(\mathcal{D})$, then

$$w(B_r(x))^{1/q} r w(B_r(x))^{-1/p} \le C w(B_R(x))^{1/q-1/p} (r/R)^{d/q-d/p} r.$$

The above is bounded if $d/q - d/p + 1 \ge 0$. The claim will now follow from Theorem 1.1. The rest of our observations can be done similarly. Furthermore, in view of our theorem, we only need w to be A_p restricted to just δ-balls in the domain and (1-18) instead of assuming $w \in A_1$. A similar conclusion can be extended to weighted fractional Poincaré inequalities (see Theorem 1.8). In particular in $\mathbb{R}^n$, taking $w = 1$, we have the classical Rellich compact embedding. Note that some studies on certain nonsmooth domains using a quasi-isometrical homeomorphism can be found in [Goldshtein and Ukhlov 2009] for A_p weights.

(7) A not so refined Condition (R) was introduced in [Chua and Wheeden 2011, (1-5)] where it was assumed without constants A_1, A_2. The present Condition (R) appears to be weaker and easier to verify than that of [Chua and Wheeden 2011]. It is easy to see that a "reverse doubling weight" (on $\mathbb{R}^n$) will induce a ball set

function that satisfies Condition (R). For more discussion, see [Chua and Wheeden 2011, Remark 1.7(2)]. Indeed, in general, μ^* satisfies Condition (R) on $\Omega \subset \mathbb{R}^n$ if $\mu^* : \Omega \times \operatorname{diam}(\Omega) \to \mathbb{R}$ (written usually as $\mu^*(B_r(x))$) is positive continuous and reverse doubling (i.e., there exists $R_C > 1$ such that $\mu^*(B_{2r}(x)) \geq R_C \mu^*(B_r(x))$ for all $r \leq \operatorname{diam}(\Omega)/2$ and $x \in \Omega$); see [Chua and Wheeden 2011, Remark 1.7(2)] when μ^* is a measure. The assumption $r_1^x = \operatorname{diam}(\Omega)$ is not essential. Indeed we need only that $\mu(\Omega) \leq C\mu^*(B(x, r_1^x))$ for all $x \in \Omega$.

(8) If μ^* satisfies Condition (R) on Ω, then for any fixed $0 < \delta \leq \frac{1}{2}$,

$$\lim_{r\to 0} \sup\big\{\mu^*(B_r(x)) : x \in \Omega,\, B_r(x) \text{ is a } \delta\text{-ball of } \Omega\big\} = 0.$$

(9) If either the Besicovitch covering property holds or μ (or μ^*) is doubling, then (μ, μ^*) will satisfy the Vitali-type property. In particular, in $\mathbb{R}^n$, (μ, μ) (and hence (μ, μ^*)) will always satisfy the Vitali-type property (with parameter depending only on n) by Besicovitch covering.

(10) In general, for any Borel measure μ, by the triangle inequality and Hölder's inequality, if $\mathcal{D}' \subset \mathcal{D}$ with $\mu(\mathcal{D}') > 0$, then (for any constant C)

$$\begin{aligned} \|f - f_{\mathcal{D}',\mu}\|_{L^q_\mu(\mathcal{D})} &\leq \|f - C\|_{L^q_\mu(\mathcal{D})} + \mu(\mathcal{D})^{1/q} |f_{\mathcal{D}',\mu} - C| \\ &\leq \|f - C\|_{L^q_\mu(\mathcal{D})} + \frac{\mu(\mathcal{D})^{1/q}}{\mu(\mathcal{D}')^{1/q}} \left(\int_{\mathcal{D}'} |f - C|^q \, d\mu\right)^{1/q} \\ &\leq \big(1 + (\mu(\mathcal{D})/\mu(\mathcal{D}'))^{1/q}\big) \|f - C\|_{L^q_\mu(\mathcal{D})}. \end{aligned} \tag{1-19}$$

Applying the above to (1-11) with $\mathcal{D} = \Omega_j$ and $C = f_{B'_j,\sigma}$, we have

$$\|f - f_{\mathcal{D}',\mu}\|_{L^q_\mu(\mathcal{D})} \leq C c_T C_1 V_\mu^{1/q} \big(1 + (\mu(\mathcal{D})/\mu(\mathcal{D}'))^{1/q}\big) \|g\|_{L^p_w(\mathcal{D})}, \tag{1-20}$$

for all $(f, g) \in \mathfrak{S}$ and $\mathcal{D}' \subset \mathcal{D}$ with $\mu(\mathcal{D}') > 0$.

Next we will consider weighted versions of Theorem 1.1. We will be interested in weights ρ being essentially constant on δ-balls of Ω, i.e., for all δ-balls of Ω,

$$\bar\rho(B) = \sup\{\rho(y) : y \in B\} \leq C(\rho, \delta)\rho(x) \quad \text{for all } x \in B.$$

Furthermore, as we always assume $\delta \leq \frac{1}{2}$, we have

$$\bar\rho(B) \leq e_\rho \rho(x) \quad \text{for all } x \in B \text{ with } 2B \subset \Omega. \tag{1-21}$$

Indeed, many weights that have been studied in the literature satisfy (1-21). Let us look at some examples.

Example 1.3. (i) (1-21) holds if $\rho(x) = \inf\{d(x, y) : y \in \Omega_0\}$, with $\Omega_0 \subset \Omega^c$. In general, it holds if

$$\rho(x) = \prod_{i=1}^{l} \eta_i(x)^{\alpha_i} \prod_{i=l+1}^{l'} \left(\frac{\eta_i(x)}{1+\eta_i(x)}\right)^{\alpha_i} \prod_{i=l'+1}^{l''} (1+\eta_i(x))^{\alpha_i}, \tag{1-22}$$

where $\eta_i(x) = d(x, S_i) = \inf\{d(x, y) : y \in S_i\}$ with $S_i \subset \Omega^c$. A special case in $\mathbb{R}^n$,

$$(1+|x|)^{\alpha_0} \prod_{i=1}^{l} \left(\frac{|x-z_i|}{1+|x-z_i|}\right)^{\alpha_i}, \quad l \in \mathbb{N},\ \alpha_i \geq 0,\ z_i \in \mathbb{R}^n, \tag{1-23}$$

has been considered in [Chanillo and Wheeden 1992; Gatto and Wheeden 1989]; see Remark 1.7.

(ii) We say $\Psi : [0, \infty) \to [0, \infty)$ is doubling if it is a monotone increasing continuous function such that there exists $C_\Psi > 1$ with $\Psi(2t) \leq C_\Psi \Psi(t)$ for all $t > 0$. Then

$$\rho(x) = \prod_{i=1}^{l} \Psi_i(\eta_i(x))^{\alpha_i}, \quad \alpha_i \in \mathbb{R},\ \alpha_i \neq 0, \tag{1-24}$$

will satisfy (1-21) if all Ψ_i are doubling and η_i are as in (1-22). In particular, when $\alpha_i > 0$ for all i in (1-24), if μ is δ-doubling on Ω with doubling constant D_μ, then $\rho d\mu$ is δ-doubling on Ω with doubling constant $C\big(\{\alpha_i, C_{\Psi_i}\}_{i=1}^{l}\big) D_\mu$. Thus, weights in (1-23) are clearly δ-doubling (indeed, they are doubling on $\mathbb{R}^n$). In general, we will let $I^- = \{i : \alpha_i < 0\}$ and I^+ be its complement. Then we know $\prod_{i \in I^+} \Psi_i(\eta_i(x))^{\alpha_i}$ is doubling on Ω.

(iii) In case $H = \mathbb{R}^n$ and S_i's are finite and disjoint (i.e., $S_i \cap S_j = \varnothing$ for $i \neq j$). Let

$$\rho(x) = \prod_{i=1}^{l} d(x, S_i)^{a_i}, \quad -n < a_i < 0 \quad \text{for all } i.$$

Then $\rho\, dx$ is δ-doubling on any bounded domain; see Proposition A.4. However, this weight is neither doubling nor reverse doubling on any unbounded domain when $\sum a_i < -n$ as $\rho(\mathbb{R}^n) = \int_{\mathbb{R}^n} \rho(x)\, dx < \infty$.

(iv) In $\mathbb{R}^n$ (or other "nice" metric spaces), we do not need to assume $\bigcup_{i=1}^{l} S_i \subset \Omega^c$ (if S_i's are finite) since we can consider $\Omega \setminus \bigcup_{i=1}^{l} S_i$ in view of the fact that a weak John domain with finitely many points being removed is still a weak John domain by Proposition 2.9.

Theorem 1.4. *Let Ψ_i be as in Example 1.3*(ii) *and η_i be as in (1-22). Let $\bar{\eta}_i(B) = \sup\{\eta_i(x) : x \in B\}$ and*

$$\rho_1(x) = \prod_{i=1}^{l} \Psi_i(\eta_i(x))^{a_i}, \quad \rho_2(x) = \prod_{i=1}^{l} \Psi_i(\eta_i(x))^{b_i}, \quad \rho_0(x) = \prod_{i=1}^{l} \Psi_i(\eta_i(x))^{\gamma_i}$$

with $a_i, b_i, \gamma_i \in \mathbb{R}$, $a_i > 0$ *for all* i. *Under the assumption of Theorem 1.1*(I), *except that (1-9) in condition* (ii) *is being replaced by*

$$\text{(1-25)}\quad \text{for each } j,\quad \mu^*(B)^{1/q} a(Q) \prod_{i=1}^{l} \Psi_i(\bar{\eta}_i(Q))^{a_i/q - b_i/p} \le C_1$$
$$\text{for all balls } B \text{ with center in } \Omega_j$$
$$\text{and balls } Q \subset B,\ Q/\delta \subset \Omega_j \text{ with } r(Q) \ge c\delta r(B)/(4\tau);$$

and the Vitali-type property holds for $(\rho_1\mu, \mu_a^*)$ *on each* Ω_j (where $\mu_a^*(B) = \mu^*(B)\prod_{i=1}^{l}\Psi_i(\bar{\eta}_i(B))^{a_i}$) *instead of (1-10).*

(I) *Then* (denoting $\rho_1\, d\mu$ by $\rho_1\mu$ and similarly for $\rho_0\, d\sigma$ and $\rho_2\, dw$)

$$\text{(1-26)}\qquad \|f - f_{B_j',\sigma}\|_{L^q_{\rho_1\mu}(\Omega_j)} \le C c_T C_1 V_\mu^{1/q} \|g\|_{L^p_{\rho_2 w}(\Omega_j)} \quad \text{for all } j,$$

where the constant C *depends also on* $\{a_i, b_i, C_{\Psi_i}\}_{i=1}^{l}$ *besides those listed in Theorem 1.1 for (1-11).*

(II)(a) *If (1-12) is being replaced by* (*again* $1 \le p_0 \le q$)

$$\text{(1-27)}\qquad \mu(\Omega_j)^{1/q}\sigma(\Omega_j)^{-1/p_0} \prod_{i=1}^{l} \Psi_i(\bar{\eta}_i(B_j'))^{a_i/q - \gamma_i/p_0} \le C_2 \quad \text{for all } j,$$

then

$$\text{(1-28)}\qquad \|f\|_{L^q_{\rho_1\mu}(\Omega)} \le C\big(C_2 M^{1/p_0}\|f\|_{L^{p_0}_{\rho_0\sigma}(\Omega)} + C_1 c_T V_\mu^{1/q} M^{1/p}\|g\|_{L^p_{\rho_2 w}(\Omega)}\big)$$

for all $(f, g) \in \mathfrak{S}$ *where* C *depends on* $\{C_{\Psi_i}, a_i, b_i, \gamma_i\}_{i=1}^{l}$ *besides those parameters listed in Theorem 1.1.*

(b) *Furthermore, if* $\rho_1\mu(\Omega) < \infty$, *then for every sequence* $\{(f_n, g_n)\}$ *in* $\mathfrak{S}$ *such that* $\{f_n\}$ *and* $\{g_n\}$ *are bounded in* $L^{p_0}_{\rho_0\sigma}(\Omega)$ *and* $L^p_{\rho_2 w}(\Omega)$ *respectively,* $\{f_n\}$ *has a subsequence that converges in* $L^{q_0}_{\rho_1\mu}(\Omega)$ *for* $1 \le q_0 < q$ *to a function in* $L^q_{\rho_1\mu}(\Omega)$.

(c) *If* $\rho_1\mu(\Omega_j)^{1/q-1/p_0} \le C_2$ *instead of (1-27)* (*and* $1 \le p_0 < q$), *then similar conclusions hold as in part* (*a*) *and* (*b*) *with* $L^{p_0}_{\rho_0\sigma}(\Omega)$ *being replaced by* $L^{p_0}_{\rho_1\mu}(\Omega)$.

Remark 1.5. Similarly, if we only assume (1-4) holds without the truncation property, then for any $1 \le q_0 < q$, any $L^{p_0}_{\rho_0\sigma}(\Omega) \times L^p_{\rho_2 w}(\Omega)$ bounded sequence $\{(f_n, g_n)\}$ in $\mathfrak{S}$ has a subsequence $\{f_{n_k}\}$ that converges in $L^{q_0}_{\rho_1\mu}(\Omega)$ provided $\rho_1\mu(\Omega) < \infty$; see Remark 1.2(1).

As mentioned earlier, assumptions become simpler and sharp in $\mathbb{R}^n$. In particular, the following is an extension of [Chanillo and Wheeden 1992, Theorem 1; Gatto and Wheeden 1989, Corollary 1.4].

Corollary 1.6. *Let* $\Omega \in J'(c)$ $(0 < c < 1)$, $\Omega \subset \mathbb{R}^n$, $1 \le p < q < \infty$, $w \in A_p$ *and* $v = \rho w$ *such that* ρ *is essentially constant on* δ*-balls of* Ω *(1-21). Suppose* μ *is* *any Borel measure such that there is a doubling ball set function* μ^* (with doubling constant D^*_μ) *with* $\mu(B\cap\Omega)\le\mu^*(B)$ *for all balls* B *with center in* Ω. *If*

$$\mu^*(B)^{1/q}|B|^{\alpha/n}\le C_1^* v(B)^{1/p} \quad \textit{for all } \delta\textit{-balls in } \Omega, \tag{1-29}$$

then for all $f\in\widehat{W}_v^{\alpha,p}(\Omega)$ *when* $0<\alpha<1$ $(f\in E_v^p(\Omega)$ *when* $\alpha=1)$, *we have*

$$\|f-f_{B'}\|_{L^q_\mu(\Omega)}\le C(c,p,q,n,D^*_\mu)e_\rho^{1/p}C_1^*C_w\|\nabla^\Omega_{\alpha,p}f\|_{L^p_v(\Omega)}, \tag{1-30}$$

where B' *is the "central" ball of* Ω (see Remark 1.2(5) for C_w) *and hence*

$$\|f-f_{\Omega,\mu}\|_{L^q_\mu(\Omega)}\le C(c,p,q,n,D^*_\mu)e_\rho^{1/p}C_1^*C_w\|\nabla^\Omega_{\alpha,p}f\|_{L^p_v(\Omega)}. \tag{1-31}$$

Moreover, *if* $\mathcal{D}\in J(c,\infty)$ *is a generalized John domain* [Chua 2009, Definition 1.2] *such that* (1-29) *holds for all* δ*-balls in* $\mathcal{D}$, *and*

$$\lim_{r\to\infty}\inf\{\mu(B_r(x)):x\in\mathcal{D}\}=\infty, \tag{1-32}$$

then for all $1\le p_0<q$, $f\in L^{p_0}(\mathcal{D})\cap E_v^p(\mathcal{D})$ *if* $\alpha=1$, $(L^{p_0}(\mathcal{D})\cap\widehat{W}_v^{\alpha,p}(\mathcal{D})$ if $0<\alpha<1)$,

$$\|f\|_{L^q_\mu(\mathcal{D})}\le C(c,p,q,n)e_\rho^{1/p}C_1^*C_w\|\nabla^{\mathcal{D}}_{\alpha,p}f\|_{L^p_v(\mathcal{D})}. \tag{1-33}$$

Proof. We will use Theorem 1.1 with $d\sigma=dx$, the Lebesgue measure and $\delta=\frac{1}{5}$. It is clear that (μ,μ^*) satisfies the Vitali-type property (1-10). Next, since $w\in A_p$, we have the Poincaré inequality (1-2) with $a(Q)=C_w|Q|^{\alpha/n}w(Q)^{-1/p}$ $(C_w=C(w))$ for all balls Q with $5Q\subset\Omega$; see Remark 1.2(5). Next, since ρ is essentially constant on δ-balls of Ω, we have for all δ-balls Q of Ω,

$$\frac{1}{|Q|}\|f-f_Q\|_{L^1(Q)}\le C(n)e_\rho^{1/p}C_w|Q|^{\alpha/n}v(Q)^{-1/p}\|\nabla^\Omega_{\alpha,p}f\|_{L^p_v(Q)}. \tag{1-34}$$

Let $\mathfrak{S}=\{(f,g):f\in\mathfrak{S}_\alpha(\Omega),\nabla^\Omega_{\alpha,p}f\in L^p_v(\Omega)\}$ (see Remark 1.2(5)). Then $\mathfrak{S}$ satisfies (1-4) (with $d\sigma=dx$, $\tau=1$, $w=v$) with the truncation property by Proposition 2.14. Next, (1-29) implies (1-9) with $C_1=C(D^*_\mu)C_we_\rho^{1/p}C_1^*$ as μ^* is doubling. Indeed,

$$\mu^*(Q)^{1/q}|Q|^{\alpha/n}v(Q)^{-1/p}\le C(D^*_\mu)C_1^*.$$

Moreover, (1-8) holds with $A_1,A_2,\theta_1,\theta_2$ depending on D^*_μ $(r_m^x=\operatorname{diam}(\Omega)/2^{m-1})$. Furthermore, (1-10) holds with $V_\mu=C(D^*_\mu)$. We can then conclude (1-30) for $f\in\mathfrak{S}_\alpha(\Omega)$ by Theorem 1.1(I). For $\alpha<1$, it is then clear that (1-30) holds for $f\in\widehat{W}_v^{\alpha,p}(\Omega)$. For $\alpha=1$, first recall that for any ball B,

$$\|f-f_{B,w}\|_{L^p_w(B)}\le C_w|B|^{1/n}\|\nabla f\|_{L^p_w(B)} \quad \text{for } f\in E^p_w(\Omega) \text{ as } w\in A_p.$$

By Propositions 2.12 and 2.11, we conclude by a density argument that (1-30) holds for all $f\in E^p_v(\Omega)$. Next, (1-30) implies (1-31) by Remark 1.2(10). For the second assertion, note that as $\mathcal{D} \in J(c, \infty)$, for all $K > 0$, there exists $\{\Omega_j^K\} \subset J'(c)$ such that $\operatorname{diam}(\Omega_j^K) \sim K$, "center ball" B_j^K of Ω_j^K with $r(B_j^K) \sim K$, $\bigcup \Omega_j^K = \mathcal{D}$ and $\sum \chi_{\Omega_j^K} \le M = C(n)$. From the first part, we have (1-31) for $\Omega = \Omega_j^K$ for each j. Hence by the triangle inequality and Hölder's inequality, (and $\nabla_{\alpha,p}^{\Omega_j^K} f \le \nabla_{\alpha,p}^{\mathcal{D}} f$)

$$\|f\|_{L^q_\mu(\Omega_j^K)} \le \mu(\Omega_j^K)^{1/q-1/p_0} \|f\|_{L^{p_0}_\mu(\Omega_j^K)} + C(c, n, p, q, D^*_\mu) e_\rho^{1/p} C_w C_1^* \|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_v(\Omega_j^K)}.$$

Now using the fact that $q \ge p_0$, p and summing up the above with respect to j, we have

$$\|f\|^q_{L^q_\mu(\mathcal{D})} \le 2^{q-1} M \biggl(\sup_j \mu(\Omega_j^K)^{1-q/p_0} \|f\|^q_{L^{p_0}_\mu(\mathcal{D})} + C \|\nabla^{\mathcal{D}}_{\alpha,p} f\|^q_{L^p_v(\mathcal{D})} \biggr).$$

Letting $K \to \infty$, we conclude (1-33) by (1-32). □

Remark 1.7. (1) In [Chanillo and Wheeden 1992; Gatto and Wheeden 1989], ρ has been assumed to be a very special case (1-23) while we allow any general weight that is essentially constant on δ-balls and we only assume (1-29) for δ-balls. Moreover, [Chanillo and Wheeden 1992] only consider Ω to be balls, $p > 1$, $\alpha = 1$ and μ is doubling. By using Corollary 1.6, we are able to extend the weight ρ to (1-22) with each S_i consisting of finitely many points. However, we need to observe that $\Omega = B \setminus \bigl(\bigcup_{i=1}^{l''} S_i\bigr) \in J'(c)$ for some fixed constant c depending only the total number of distinct points in $\bigcup_{i=1}^{l''} S_i$ (Proposition 2.9) and the fact that ρ is essentially constant on δ-balls of Ω. Hence (1-30) will hold with $\alpha = 1$ for balls B if we assume the following balanced condition (given in [Chanillo and Wheeden 1992]):

$$\left(\frac{|Q|}{|B|}\right)^{1/n} \left(\frac{\mu(Q)}{\mu(B)}\right)^{1/q} \le C \left(\frac{v(Q)}{v(B)}\right)^{1/p} \tag{1-35}$$

for all δ-balls Q in $B \setminus \bigl(\bigcup_{i=1}^{l''} S_i\bigr)$ (instead of all balls Q in B given in [Chanillo and Wheeden 1992]). Next, as $\mathbb{R}^n \setminus \bigl(\bigcup_{i=1}^{l''} S_i\bigr) \in J(c, \infty)$ [Chua 2009, Proposition 2.24], we obtain [Gatto and Wheeden 1989, Corollary 1.4].

(2) It has been observed that if both μ and v are doubling, $\alpha = 1$ and Ω is a ball, then (1-35) is indeed necessary for (1-31) to hold for all Lipschitz continuous functions [Chanillo and Wheeden 1992]; see also [Chanillo and Wheeden 1985, p. 1192]. Note that (for $\alpha = 1$) it is enough to assume only $\mu = \mu^*$ is doubling without assuming v be doubling so that (1-29) is necessary for (1-31) to hold for all Lipschitz continuous functions. To this end, first observe that when μ is doubling, suppose f is a Lipschitz function that vanishes on a δ-ball $B_0 \subset \Omega$, by (1-19),

taking $\mathcal{D}' = B_0$ and $\mathcal{D} = \Omega$, (1-31) will imply
(1-36)

$$\|f\|_{L^q_\mu(\Omega)} \le C(c, p, q, n, D^*_\mu) e_\rho^{1/p} C_1^* C_w \left(1 + \left(\frac{\mu(\Omega)}{\mu(B_0)}\right)^{1/q}\right) \|\nabla^\Omega_{\alpha,p} f\|_{L^p_v(\Omega)}.$$

We now fix a Lipschitz function $\phi(x)$ on $[0, \infty)$ such that $\chi_{[0,1/2]} \le \phi \le \chi_{[0,1]}$ with $\phi(x) = 2 - 2x$ on $\left[\frac{1}{2}, 1\right]$. Given any δ-ball B in Ω, by translation, we may assume 0 is the center of B. Let $f(x) = \phi(|x|/r)$, where $r = r(B)$. Then f vanishes outside B. In particular, it vanishes on a ball $\widetilde{B}$ such that $\Omega \subset C(c, \delta)\widetilde{B}$. Hence by (1-36) and (1-31), we have

$$\mu(B/2)^{1/p} \le C r^{-1} v(B)^{1/q} \quad (\text{if } \alpha = 1).$$

Since μ is doubling, we have (1-29) with $\mu^* = \mu$. Unfortunately, for $0 < \alpha < 1$, the same method only produces (1-29) for δ-balls with radius comparable to $\rho_\Omega(x_B)$ (x_B is the center of B). This is not really surprising in view of the definition of our fractional Sobolev norm.

(3) For $0 < \alpha < 1$, we can extend a result of Bourgain, Brezis, and Mironescu. In [Bourgain et al. 2002], they discuss what happens in the fractional Poincaré inequality on unit cubes when $\alpha \to 1$. Recall that in [Mazya and Shaposhnikova 2002, Corollary 2 (see also the Erratum)] when $\alpha p < n$, $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$, Q is a unit cube in $\mathbb{R}^n$ and $f \in L^1(Q)$,

$$\text{(1-37)} \quad \|f - f_Q\|^p_{L^{p^*}(Q)} \le C(n, p) \frac{1-\alpha}{(n-\alpha p)^{p-1}} \|f\|^p_{W^{\alpha,p}(Q)} = C(n, p) \frac{1-\alpha}{(n-\alpha p)^{p-1}} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha p}}\, dy\, dx.$$

Hence by dilation, for any cube Q, we have by Jensen's inequality,

$$\text{(1-38)} \quad \left(\frac{1}{|Q|}\|f - f_Q\|_{L^1(Q)}\right)^p \le \left(\frac{1}{|Q|}\right)^{p/p^*} \|f - f_Q\|^p_{L^{p^*}(Q)} \le C(n, p)|Q|^{\frac{\alpha p}{n} - 1} \frac{1-\alpha}{(n-\alpha p)^{p-1}} \|f\|^p_{W^{\alpha,p}(Q)}.$$

Now let $\Omega \in J'(c)$. and suppose ρ is a weight that is essentially constant on δ-balls of Ω (1-21). As cubes are metric balls under the metric

$$d_\infty(x, y) = \max_{1 \le i \le n} \{|x_i - y_i|\},$$

for easy computation, we will use this metric instead of the Euclidean metric. Then $Q \subset B(x, \rho_\Omega(x)/2)$ for all $x \in Q$ whenever $5Q \subset \Omega$. Using (1-21) for ρ, we have

$$\text{(1-39)} \quad \frac{1}{|Q|}\|f - f_Q\|_{L^1(Q)} \le \left(C(n)|Q|^{\alpha p/n} \rho(Q)^{-1} e_\rho \frac{1-\alpha}{(n-\alpha p)^{p-1}}\right)^{1/p} \|\nabla^\Omega_{\alpha,p} f\|_{L^p_\rho(Q)}.$$

If we assume that $\mu(Q)^{1/q}\rho(Q)^{-1/p}|Q|^{\alpha/n} \le C_*$ (with $q > p$) for all cubes Q with $5Q \subset \Omega$ and μ is doubling with doubling constant D_μ, then we can use Theorem 1.4 (by similar argument as in the proof of Corollary 1.6) to get

$$\begin{aligned}(1\text{-}40)\quad \|f - f_{Q'}\|^p_{L^q_\mu(\Omega)} &\le C(n, c, p, q, D_\mu)e_\rho(C_*C_w)^p \frac{1-\alpha}{(n-\alpha p)^{p-1}}\|f\|^p_{\widehat{W}^{\alpha,p}_\rho(\Omega)} \\ &= C(n, c, p, q, D_\mu)e_\rho(C_wC_*)^p \frac{1-\alpha}{(n-\alpha p)^{p-1}} \\ &\quad \times \int_\Omega \int_{B(x,\rho_\Omega(x)/2)} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\,dy\,\rho(x)dx,\end{aligned}$$

where Q' is the "central cube" in Ω. Thus, we have extended the results of [Mazya and Shaposhnikova 2002; Bourgain et al. 2002] to weighted fractional Sobolev inequalities on John domains. Again, by Remark 1.2(10) we can replace $f_{Q'}$ in (1-40) by $f_{\Omega,\mu}$.

We now discuss applications on $\mathbb{R}^n$. As mentioned earlier, conditions are now simpler and mostly sharp.

Theorem 1.8. (I) *Let* $\Omega \subset \mathbb{R}^n$, $\Omega \in J'(c)$ (hence a John domain), $0 < c < 1$. *Let* $1 \le p < q < \infty$. *Let* $\Omega_0 \subset \Omega^c$ *and define* $\rho(x) = d(x, \Omega_0) = \inf\{|x-y| : y \in \Omega_0\}$. *Let* $\mathfrak{S}_\alpha(\Omega)$ *be as in Remark 1.2*(5). *Let w be a weight on Ω and* $0 < \alpha \le 1$ *such that the Poincaré inequality (1-2) holds for all balls Q with $2Q \subset \Omega$ and $f \in \mathfrak{S}_\alpha(\Omega)$. Suppose $C_* > 0$, $\beta \in \mathbb{R}$ such that*

$$(1\text{-}41)\qquad a(Q) \le C_* r(Q)^\beta \quad \text{for all balls } Q \text{ with } 2Q \subset \Omega.$$

Suppose μ is another weight on $\mathbb{R}^n$ such that there exist $C_\mu, N > 0$ with

$$(1\text{-}42)\qquad \mu(B \cap \Omega) \le C_\mu r(B)^N \quad \text{for all balls } B.$$

Let $a \ge 0$, $b \in \mathbb{R}$. We define $\mu_a(E) = \int_E \rho(x)^a\,d\mu$ and w_b similarly. Suppose

$$(1\text{-}43)\qquad \beta + \frac{N}{q} + \min\Big\{0, \frac{a}{q} - \frac{b}{p}\Big\} \ge 0.$$

Then

$$(1\text{-}44)\qquad \|f - f_{B'}\|_{L^q_{\mu_a}(\Omega)} \le C C_* C_\mu^{1/q} \bar\rho(\Omega)^{\beta+(N+a)/q-b/p} \|\nabla^\Omega_{\alpha,p} f\|_{L^p_{w_b}(\Omega)}$$

for all $f \in \mathfrak{S}_\alpha(\Omega)$, where $f_{B'} = \int_{B'} f\,dx/|B'|$, $B' = B(x', d(x', \Omega^c)/4)$, x' is the center of Ω where C depends only on c, N, n, p, q, a, b and β.

(II) *Suppose $\mathcal{D}$ is a countable union of $\Omega_j \in J'(c)$ $(0 < c < 1$ is fixed) such that $\sum_j \chi_{\Omega_j} \le M$, $M \in \mathbb{N}$ and $M_1 \le |\Omega_j| \le M_2$ for all j, $M_1, M_2 > 0$. Assume $\Omega_0 \subset \mathcal{D}^c$*

and

$$\text{(1-45)}\quad \textit{for all } j, \quad \mu(B\cap\Omega_j) \le C_\mu \min\{r(B)^N, r(B)^{N_1}\bar\rho(B)^{N_2}\} \quad \textit{for all balls } B, r(B)\le \operatorname{diam}(\Omega_j),$$

where $\bar\rho(B) = \sup\{\rho(x) : x \in B\}$, $N_1, N_2 \in \mathbb{R}$, (usually $N \ge N_1 + N_2$, $N_1 > 0$, $N_2 < 0$) *and*

$$\text{(1-46)}\qquad a(Q) \le C_* r(Q)^{\beta_1}\bar\rho(Q)^{\beta_2} \quad \textit{for all balls } Q \textit{ such that } 2Q\subset\Omega_j,$$

where $\beta_1, \beta_2 \in \mathbb{R}$. *Moreover, for any* $\gamma \in \mathbb{R}$, *we use* ρ^γ *to denote the measure defined by* $\rho^\gamma(E) = \int_E \rho(x)^\gamma\,dx$. *Suppose* $1 \le p_0 \le q$ *such that*

(i) $\beta_1 + \frac{N}{q} + \min\{0, \beta_2 + \frac{a}{q} - \frac{b}{p}\} \ge 0$; *and*

(ii) *both* $\min\{a, N_2 + a\}/q \le \gamma/p_0$ *and* $\beta_2 + \frac{a}{q} - \frac{b}{p} \le 0$ *in case* ρ *is unbounded on* $\mathcal{D}$.

Then for all $f \in \mathfrak{S}_\alpha(\Omega)$,

$$\text{(1-47)}\qquad \|f\|_{L^q_{\mu_a}(\mathcal{D})} \le C C_\mu^{1/q}\big(M^{1/p_0}\|f\|_{L^{p_0}_{\rho^\gamma}(\mathcal{D})} + C_* M^{1/p}\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{w_b}(\mathcal{D})}\big),$$

where C *depends also on* $N_1, N_2, p_0, \gamma, M_1$ *and* M_2 *besides those listed above for* (1-44). *Furthermore, if we have strict inequalities in both* (i) *and* (ii) *and* $\mu_a\{x \in \mathcal{D} : \rho(x) < r\} < \infty$ *for any* $r > 0$, *then given any sequence* $\{f_k\} \subset \mathfrak{S}_\alpha(\Omega)$ *such that both* $\|f_k\|_{L^{p_0}_{\rho^\gamma}(\mathcal{D})}$ *and* $\|\nabla^{\mathcal{D}}_{\alpha,p} f_k\|_{L^p_{w_b}(\mathcal{D})}$ *are bounded, it has a subsequence that converges in* $L^q_{\mu_a}(\mathcal{D})$.

Remark 1.9. (1) In view of the fact that a John domain with finitely many points removed is still a John domain (see Proposition 2.9), instead of assuming $\Omega_0 \subset \Omega^c$, it suffices to assume $\Omega_0 \setminus F \subset \Omega^c$ (or $\mathcal{D}^c$), where F is a set of finite points. Note that $\mathrm{Lip}_{\mathrm{loc}}(\Omega) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega\setminus F)$ and $L^{p_0}_\sigma(\Omega)\cap E^p_w(\Omega) \subset L^{p_0}_\sigma(\Omega\setminus F)\cap E^p_w(\Omega\setminus F)$.

(2) Any finite union of John domains is an example of domain $\mathcal{D}$ for the above theorem. Indeed, $\mathcal{D}$ can be a generalized John domain [Chua 2009, Definition 1.2 and Proposition 2.21].

(3) Strict inequalities in conditions (i) and (ii) will ensure that (1-47) holds with some $\tilde q > q$ instead of q. Note that $L^{\tilde q}_{\mu_a}(\mathcal{D}) \subset L^q_{\mu_a}(\mathcal{D})$ when $\mu_a(\mathcal{D}) < \infty$ and $\tilde q > q$.

(4) Similar to the previous two theorems, we can replace $f_{B'}$ by f_{Ω,μ_a} in (1-44). Equation (1-47) will then also hold with $\|f\|_{L^{p_0}_{\rho^\gamma}(\mathcal{D})}$ being replaced by $\|f\|_{L^{p_0}_{\mu_a}(\mathcal{D})}$ if $1 \le p_0 < q$ and $\sup_j \mu_a(\Omega_j)^{1/q-1/p_0} < \infty$. Conditions involving γ will then be redundant.

(5) By a standard density argument, one could obtain compact embedding result for the closure of $\mathrm{Lip}_{\mathrm{loc}}(\mathcal{D})\cap L^{p_0}_{\rho^\gamma}(\mathcal{D})\cap E^p_{w_b}(\mathcal{D})$ in $L^{p_0}_{\rho^\gamma}(\mathcal{D})\cap E^p_{w_b}(\mathcal{D})$.

(6) Some discussions of power-type weights (including logarithm) on special union of $C^{0,s}$ domains (bounded and unbounded) can be found in [Gurka and Opic 1988; 1989; 1991]. Note that weights are assumed to be positive and continuous on the domain there.

(7) For the necessity of conditions, see Remark 1.11.

(8) If $d\mu = dw = dx$ is the Lebesgue measure, then $N = N_1 = n$, $\beta = \alpha - \frac{n}{p}$ and $N_2 = 0$. The case $\alpha = 1$ has already been studied in [Chua and Wheeden 2011; Hajłasz and Koskela 1998].

(9) In most cases, $N_1 = n$, $N_2 \le 0$ and $N \le n$ in the above theorem. A typical example (a special case of Example 1.3(iii)) of μ will be

$$\mu(E) = \int_E |x - z_1|^{a_1} |x - z_2|^{a_2}\, dx, \quad \text{where } -n < a_1, a_2 < 0,\ z_1 \neq z_2.$$

Note that $\mu(\mathbb{R}^n) < \infty$ (and hence μ cannot be doubling on $\mathbb{R}^n$) if $a_1 + a_2 < -n$. Indeed, if $\bar\rho(B) = \sup\bigl\{\min\{|x - z_1|, |x - z_2|\} : x \in B\bigr\}$, then for any ball B with $r(B) \le C_0$, we have

$$\mu(B) \le C(C_0) \min\{r(B)^N, r(B)^n \bar\rho(B)^{a_1+a_2}\} \quad \text{with } N = \min\{n + a_1, n + a_2\}.$$

For more details, see Proposition A.4.

(10) When Ω is a John domain, the case $d\mu = \rho_\Omega^a\, dx$, $a < 0$ such that $\rho_\Omega^a(\Omega) < \infty$ but ρ_Ω^a may not be doubling has been studied in [Chua 2016].

In particular, when $\Omega_0 \setminus F \subset \mathrm{G}$ where G is the graph of a Lipschitz function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ with F being a finite set of points and $\mathcal{D} = \mathbb{R}^n \setminus (\mathrm{G} \cup F)$, we have an extension of [Mazya 2011, Theorem 1.4.2.1]. Indeed, we use only the fact that $\mathbb{R}^n \setminus \mathrm{G} \in J(c, \infty)$ (generalized John domain). For example G can be a finite union of hyperplanes that pass through a fixed point.

Corollary 1.10. *Let $1 \le p$, $p_0 < q$ and $0 < \alpha \le 1$. Let F, G, Ω_0 be as above. Let $\rho(x) = \inf\{|x - z| : z \in \Omega_0\}$, $N > 0$, $a \ge 0$, $\gamma, b \in \mathbb{R}$ and μ be a weight on $\mathbb{R}^n$ such that*

$$\mu(B) \le C r(B)^N \quad \textit{for all balls } B.$$

Recall that $\mu_a(E) = \int_E \rho(x)^a\, d\mu$ and $\rho^\gamma(E) = \int_E \rho(x)^\gamma\, dx$. If $\frac{N+a}{q} - \frac{n+\gamma}{p_0} < 0$, and

$$\alpha + \frac{N+a}{q} - \frac{n+b}{p} = 0 \quad \textit{and} \quad \frac{a}{q} - \min\Bigl\{\frac{b}{p}, \frac{\gamma}{p_0}\Bigr\} \le 0, \tag{1-48}$$

then for all $f \in L^{p_0}_{\rho^\gamma}(\mathbb{R}^n) \cap E^p_{\rho^b}(\mathcal{D})$ when $\alpha = 1$ and $f \in L^{p_0}_{\rho^\gamma}(\mathbb{R}^n) \cap \widehat{W}^{\alpha,p}_{\rho^b}(\mathcal{D})$ when $\alpha < 1$, where $\mathcal{D} = \mathbb{R}^n \setminus (\mathrm{G} \cup F)$,

$$\|f\|_{L^q_{\mu_a}(\mathbb{R}^n)} \le C \|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho^b}(\mathcal{D})}. \tag{1-49}$$

Furthermore, (1-49) *also holds for* $f \in L^{p_0}_{\mu_a}(\mathbb{R}^n) \cap E^p_{\rho^b}(\mathcal{D})$ *(or* $L^{p_0}_{\mu_a}(\mathbb{R}^n) \cap \widehat{W}^{\alpha,p}_{\rho^b}(\mathcal{D})$ *when* $\alpha < 1$*) provided*

$$\alpha + \frac{N+a}{q} - \frac{n+b}{p} = 0 \quad \textit{and} \quad \frac{a}{q} - \frac{b}{p} \le 0 \tag{1-50}$$

$$\textit{and} \quad \lim_{r\to\infty} \inf\{\mu_a(B_r(x)) : x \in \mathbb{R}^n\} = \infty. \tag{1-51}$$

Remark 1.11. (1) Mazya [2011] considered the special case where $\alpha = 1$, $a = b = 0$ and $f \in C_0^\infty(\mathbb{R}^n)$. Of course, $C_0^\infty(\mathbb{R}^n) \subset L^{p_0}_{\rho^\gamma}(\mathbb{R}^n) \cap E^p_{\rho^b}(\mathbb{R}^n) \subset L^{p_0}_{\rho^\gamma}(\mathbb{R}^n) \cap E^p_{\rho^b}(\mathcal{D})$ when ρ^b and ρ^γ are both locally integrable. In general, $C_0^\infty(\mathcal{D}) \subset L^{p_0}_{\rho^\gamma}(\mathbb{R}^n) \cap E^p_{\rho^b}(\mathcal{D})$.

(2) The above result is sharp. For example, when $\mu(B) \ge Cr(B)^N$ for all δ-balls B of Ω, then (1-50) is indeed necessary. It can be done by a standard translation and dilation technique. We will only demonstrate the case where $0 < \alpha < 1$. Fix a C_0^∞ (or Lipschitz) function ϕ as in Remark 1.7(2). For simplicity, let us assume $\Omega_0 = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0 \text{ for some } i\}$ and F is a finite set. Suppose (1-49) holds. Then we have if B is any δ-ball in $\mathcal{D} = \mathbb{R}^n \setminus (G \cup F)$, as any appropriate translation and dilation of ϕ is C_0^∞ (or Lipschitz with compact support), we have

$$\mu_a(B/2)^{1/q} \le Cr(B)^{-\alpha}(\rho^b(B))^{1/p}.$$

Hence,

$$\bar{\rho}(B)^{a/q} r(B)^{N/q} \le Cr(B)^{-\alpha+n/p} \bar{\rho}(B)^{b/p}.$$

As we can take δ-balls B with $\bar{\rho}(B)$ comparable to $r(B)$, the first condition of (1-50) must hold. If we fix $r(B)$ but let $\bar{\rho}(B) \to \infty$, we see that $\frac{a}{q} - \frac{b}{p} \le 0$.

Next we have another application that extends a compact embedding result of [Xuan 2005, Theorem 2.1]. For simplicity, we shall only state that for Sobolev space (i.e., $\alpha = 1$).

Corollary 1.12. *Let* $1 \le p < q$, μ *and* ρ *be as in Corollary 1.10. Suppose* $\mathcal{D}$ *is a bounded domain. If*

$$1 + \frac{N}{q} - \frac{n}{p} + \min\left\{\frac{a}{q} - \frac{b}{p}, 0\right\} \ge 0, \tag{1-52}$$

then

$$\|f\|_{L^q_{\mu_a}(\mathcal{D})} \le C\|\nabla f\|_{L^p_{\rho^b}(\mathcal{D})} \tag{1-53}$$

for all $f \in C_0^\infty(\mathcal{D})$*. Furthermore, if in addition we have strict inequality in* (1-52)*, then the embedding of the closure of* $C_0^\infty(\mathcal{D})$ *in* $E^p_{\rho^b}(\mathcal{D})$ *to* $L^q_{\mu_a}(\mathcal{D})$ *is compact.*

Remark 1.13. (1) In particular, the above can be applied to compact embedding of $C_0^\infty(\mathcal{D}) \cap E^p_{\rho^\beta}(\mathcal{D})$ to $L^q_{\rho^b}(\mathcal{D})$ when $\Omega_0 = F = \{0\} \subset \mathcal{D}$. Note that $E^p_{\rho^\beta}(\mathcal{D}) \subset E^p_{\rho^\beta}(\mathcal{D} \setminus \{0\})$. To apply Corollary 1.12, we will take $d\mu = dx$ when $\beta \ge 0$ and $\mu(B) = \int_{B\cap\Omega} |x|^\beta \, dx$ when $-n < \beta < 0$. If B is any ball, it is clear that $\rho^\beta(B \cap \Omega) \le$

$Cr(B)^{n+\beta}$ when $-n < \beta < 0$ (and hence $N = n+\beta$ in (1-52)). We obtain the same conclusion as [Xuan 2005, Theorem 2.1] for $\beta > -n$. However, [Xuan 2005] further assumes that $\beta > p-n,\ p > 1$ and $\mathcal{D}$ has C^1 boundary.

(2) From the same construction as in Remark 1.11, (1-53) will imply (1-52) and thus (1-52) is necessary. Note that $\bar{\rho}(B)$ will be bounded when B is a ball inside $\mathcal{D}$.

Finally, we discuss an application related to Caffarelli, Kohn and Nirenberg-type inequalities [Caffarelli et al. 1984]. Instead of considering only powers of $|x|$ (i.e., $\Omega_0 = \{0\}$), we will consider more general power weights and include fractional derivatives. The next theorem allows the case $p = q$ as we will apply results from [Chua 2009] instead of Theorem 2.4. For a more general extension, see Remark 3.2.

Theorem 1.14. *Let $\mathcal{D} \subset \mathbb{R}^n$, $0 < \alpha \le 1$ and $1 \le p, p_0 \le q$. Suppose there exist $M > 0$, $0 < c < 1$ such that $\mathcal{D} = \bigcup_{j=1}^{\infty} \Omega_j$, $\Omega_j \in J'(c)$ with $\varepsilon_0/c_0 \le \operatorname{diam}(\Omega_j) \le c_0\varepsilon_0$, $(c_0, \varepsilon_0 > 0)$ for all j and $\sum \chi_{\Omega_j} \le M$. Let $\{z_i\}_{i=1}^{l} \subset \mathbb{R}^n$, $l \in \mathbb{N}$ $(z_i \ne z_m$ for $i \ne m)$ and*

$$(1\text{-}54)\quad \rho_1(x) = \prod_{i=1}^{l} |x - z_i|^{a_i}, \quad \rho_2(x) = \prod_{i=1}^{l} |x - z_i|^{b_i}, \quad \rho_0(x) = \prod_{i=1}^{l} |x - z_i|^{\gamma_i},$$

with $a_i, b_i, \gamma_i \in \mathbb{R}$ and $a_i > -n$ for all i. Let $I^- = \{i : a_i < 0\}$. Suppose further that

$$\text{(i)}\qquad b = \min\Big\{\alpha - \frac{n+b_i}{p} + \frac{n+a_i}{q} : i = 1, \dots, l\Big\} \ge 0 \quad \textit{and}$$

$$\text{(ii)}\qquad \sum \frac{a_i}{q} \le \min\Big\{\sum \frac{b_i}{p}, \sum \frac{\gamma_i}{p_0}\Big\} \quad \bigg(a = \frac{n + \sum_{i=1}^{l} \gamma_i}{p_0} - \frac{n + \sum_{i=1}^{l} a_i}{q}\bigg).$$

Then for all $f \in L^{p_0}_{\rho_0}(\mathcal{D}) \cap E^p_{\rho_2}(\mathcal{D})$ when $\alpha = 1$ $\big(f \in L^{p_0}_{\rho_0}(\mathcal{D}) \cap \widehat{W}^{\alpha,p}_{\rho_2}(\mathcal{D})$ when $\alpha < 1\big)$,

$$(1\text{-}55)\qquad \|f\|_{L^q_{\rho_1}(\mathcal{D})} \le C\big(M^{1/p_0}\varepsilon_0^{-a}\|f\|_{L^{p_0}_{\rho_0}(\mathcal{D})} + M^{1/p}\varepsilon_0^{b}\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho_2}(\mathcal{D})}\big),$$

where C depends only on

$$c,\ \{a_i, b_i, \gamma_i\}_{i=1}^{l},\ n,\ p,\ q,\ p_0,\ l \quad \textit{and} \quad \max\{\operatorname{diam}(\Omega_j) : j \in \mathbb{N}\}/\zeta$$

(where $\zeta = \min\{|z_i - z_m| : i \ne m, i, m \in I^-\}$, taking $\zeta = \infty$ when I^- has ≤ 1 element). *Furthermore, if we have strict inequalities in both* (i) *and* (ii), *then the natural embedding of $L^{p_0}_{\rho_0}(\mathcal{D}) \cap E^p_{\rho_2}(\mathcal{D})$)* (or $L^{p_0}_{\rho_0}(\mathcal{D}) \cap \widehat{W}^{\alpha,p}_{\rho_2}(\mathcal{D})$ when $\alpha < 1$) *to $L^q_{\rho_1}(\mathcal{D})$ is compact.*

Finally, if $\mathcal{D} \in J(c, \varepsilon_0)$ (generalized John domains [Chua 2009]), *then*

$$(1\text{-}56)\quad \|f\|_{L^q_{\rho_1}(\mathcal{D})} \le C\varepsilon^{-a}\|f\|_{L^{p_0}_{\rho_0}(\mathcal{D})} + C\varepsilon^{b}\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho_2}(\mathcal{D})} \quad \textit{for all } \varepsilon \in (0, \varepsilon_0),$$

with C depending on c, $\{a_i, b_i, \gamma_i\}_{i=1}^{l}$, n, p, q, p_0, l and ε_0/ζ.

Remark 1.15. (1) If in addition $a, b > 0$, then (1-56) is equivalent to

$$\|f\|_{L^q_{\rho_1}(\mathcal{D})} \le C(\|f\|_{L^{p_0}_{\rho_0}(\mathcal{D})})^{a/(a+b)}\big(\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho_2}(\mathcal{D})} + \varepsilon_0^{-a-b}\|f\|_{L^{p_0}_{\rho_0}(\mathcal{D})}\big)^{b/(a+b)}; \tag{1-57}$$

see [Chua 2009, Remark 1.8(4)] for details.

(2) We may assume I^- in the above has more than one i. In [Chua 2009, Theorem 4.3; Caffarelli et al. 1984], the case with $l = 1$ in (1-54) and $z_1 = 0$ was considered, while we allow $l > 1$. Caffarelli et al. [1984] also showed that the conditions (i) and (ii) are necessary. The main difference (for $l > 1$) is that when $l = 1$ the measure induced is doubling ($|x|^\alpha$ is doubling on $\mathbb{R}^n$ if $\alpha > -n$) while it may not be doubling when $l > 1$ (see Example 1.3(iii)). This creates a problem for necessity of conditions. However, it is still possible to see that some of the conditions remain necessary. Indeed, condition (i) is necessary for the following weighted Poincaré inequality:

$$\|f - f_{\Omega,\rho_1}\|_{L^q_{\rho_1}(\Omega)} \le C\|\nabla^{\Omega}_{\alpha,p} f\|_{L^p_{\rho_2}(\Omega)} \quad \text{for all } f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \tag{1-58}$$

for any John domain Ω. To see this, just use the same Lipschitz function ϕ constructed in Remark 1.7(2) to see that

$$\rho_1(Q/2)^{1/q} \le Cr(Q)^{-\alpha}\rho_2(B)^{1/p}$$

for all δ-balls Q in $\Omega \setminus \{z_i\}_{i=1}^l$. For each fixed i, one could choose $r(Q) \sim \bar{d}_i(Q) = \sup_{x\in Q}|x - z_i|$ and let $r(Q) \to 0$. It is now clear that (i) holds. It will be more complicated if we only assume (1-56) holds. Condition (i) is still necessary for (1-55) provided the $L^q_{\rho_1}$ norm is not dominated by the $L^{p_0}_{\rho_0}$ norm. Indeed using ϕ as above again, for any δ-ball Q in $\mathcal{D} \setminus \{z_i\}_{i=1}^l$, (by translation and dilation) we may assume ϕ has support in $Q/2$ and vanishes outside Q, we have by using (1-56),

$$\rho_1(Q/2)^{1/q} \le C\varepsilon_0^{-a}\rho_0(Q)^{1/p_0} + C\varepsilon_0^b r(Q)^{-\alpha}\rho_2(Q)^{1/p}.$$

As $d_i(x) = |x - z_i|$ are essentially constant on δ-balls, we have

$$|Q|^{1/q}\prod \bar{d}_i(Q)^{a_i/q} \le C\varepsilon_0^{-a}|Q|^{1/p_0}\prod \bar{d}_i(Q)^{\gamma_i/p_0} + C\varepsilon_0^b|Q|^{-\alpha/n+1/p}\prod \bar{d}_i(Q)^{b_i/p}.$$

For each fixed i we could let $r(Q) \to 0$ with $\bar{d}_i(Q) \sim r(Q)$. So if $\frac{n+a_i}{q} < \frac{n+\gamma_i}{p_0}$, we must have $\frac{n+a_i}{q} \ge \frac{n+b_i}{p} - \alpha$.

Next, if we assume (1-56) holds for all $\mathcal{D} \in J(c, \varepsilon_0)$, then for any ball Q with $r(Q) \ge \varepsilon_0$ such that $2Q \subset \mathbb{R}^n \setminus \{z_i\}_{i=1}^l$, we may assume that Q is a connected component of some $\mathcal{D} \in J(c, \varepsilon_0)$. Taking $f = \chi_Q$, since (1-56) holds, we have

$$|Q|^{1/q}\prod \bar{d}_i(Q)^{a_i/q} \le C\varepsilon_0^{-a}|Q|^{1/p_0}\prod \bar{d}_i(Q)^{\gamma_i/p_0}.$$

It is then easy to see that $\sum \frac{a_i}{q} \le \sum \frac{\gamma_i}{p_0}$ as we could let $\bar{d}_i(Q) \to \infty$ (while fixing $r(Q)$).

2. Preliminaries

For easy reference, we collect in this section some definitions and terminology from [Chua 2009; Chua and Wheeden 2008; 2011].

Definition 2.1. A function d is called a (symmetric) quasimetric on a given set H if $d : H \times H \to [0, \infty)$ and there is a constant $\kappa \geq 1$ such that for all $x, y, z \in \Omega$,

$$\begin{aligned} & d(x, y) = d(y, x), \\ & d(x, y) = 0 \iff x = y, \quad \text{and} \\ & d(x, y) \leq \kappa[d(x, z) + d(z, y)]. \end{aligned} \tag{2-1}$$

If d is a quasimetric on H, we refer to the pair $\langle H, d\rangle$ as a quasimetric space. In this section, unless otherwise mentioned, H will always be a quasimetric space with quasimetric d and quasimetric constant κ. All measures on H will be defined on a fixed σ-algebra Σ that includes all balls. When $\kappa = 1$, H will be a metric space and we will just assume Σ to be the Borel algebra on H.

First, similar to [Chua and Wheeden 2008] for John domains, we see that δ-doubling is equivalent to doubling on weak John domains.

Proposition 2.2. *Let* $0 < \delta \leq \frac{1}{2}\kappa^2$. *If* $\Omega \subset H$, $\Omega \in J'(c)$ *with center* x', *then*

(i) $$d(x', \Omega^c) \geq c\, \mathrm{diam}(\Omega)/(2\kappa);$$

(ii) *for any* $x \in \Omega$, $0 < r_0 < \mathrm{diam}(\Omega)$, $B(x, r_0)$ *contains a* δ*-ball* Q *with* $r(Q) \geq Cr_0$, *where* C *depends only on* κ, δ *and* c, *hence*, *a measure* μ *is* δ*-doubling on* Ω *if and only if it is doubling on* Ω.

Proof. Given any $\varepsilon > 0$, there exist $z_1, z_2 \in \Omega$ with $d(z_1, z_2) > \mathrm{diam}(\Omega) - \varepsilon$. But

$$d(z_1, z_2) \leq \kappa(d(z_1, x') + d(z_2, x')).$$

Hence, without loss of generality, we may assume $d(z_1, x') \geq d(z_1, z_2)/(2\kappa)$. By the weak John condition (1-7), we have

$$d(x') = d(x', \Omega^c) \geq c\, d(z_1, x') \geq \frac{c(\mathrm{diam}(\Omega) - \varepsilon)}{2\kappa},$$

and (i) will then follow as $\varepsilon > 0$ is arbitrary.

For part (ii), recall from (i) that $d(x') \geq \frac{c}{2\kappa}\mathrm{diam}(\Omega)$. It suffices to show that if $x \in \Omega$ and $\delta d(x) \leq r \leq c\, \mathrm{diam}(\Omega)/(2\kappa)$, then $B_r(x)$ contains a δ-ball with radius comparable to r (with constant independent of r and x). The case when $x = x'$ in the above is easy. Now suppose $x \neq x'$. It is again clear if $x' \in B_r(x)$. So we may assume $d(x, x') \geq r$. Next, we need only a continuous path $\gamma : [0, l] \to \Omega$ connecting x to x' such that $d(\gamma(t), \Omega^c) \geq c\, d(\gamma(t), x)$. Since $d(\gamma(t), x)$ is a continuous function on $[0, l]$, there exists t_0 such that $d(\gamma(t_0), x) = r/(2\kappa)$ and hence $d(\gamma(t_0)) \geq cr/(2\kappa)$. We now observe that the δ-ball $B_{r'}(\gamma(t_0)) \subset B_r(x)$ with $r' = c\delta r/(2\kappa)$. This concludes the proof of part (ii). □

Remark 2.3. Doubling or δ-doubling will imply reverse doubling if Ω is assumed to have the "nonempty annuli property" (on symmetric quasimetric space; see [Chua and Wheeden 2008, Proposition 2.3]). Clearly, any weak John domain satisfies this "nonempty annuli property."

Now, let us state a theorem that is similar to [Chua and Wheeden 2011, Theorem 1.6].

Theorem 2.4. *Let $\Omega \subset H$, $\Omega \in J'(c)$ with central point x', let $0 < \delta \leq 1/(2\kappa^2)$, $1 \leq \tau \leq 1/(2\delta\kappa^2)$. Let $1 \leq p < q < \infty$. Suppose μ, σ and w are measures* (defined on a fixed σ-algebra that includes all balls and Ω) *where σ is δ-doubling on Ω and μ is absolutely continuous with respect to σ. Let $(f, g) \in L^1_{\sigma,\text{loc}}(\Omega) \times L^p_{w,\text{loc}}(\Omega)$ such that (1-4) holds. Suppose there exists a ball set function μ^* satisfying* Condition (R) *such that $\mu(B \cap \Omega) \leq \mu^*(B)$ for all balls B with center in Ω and (μ, μ^*) satisfies the Vitali-type property on Ω* ((1-10) in Theorem 1.1). *Suppose further that for any ball B with center in Ω and $r(B) \leq$* diam(Ω),

$$\mu^*(B)^{1/q} a(Q) \leq C_1 \tag{2-2}$$

for all δ-balls $Q \subset B$ such that $r(Q) \geq c\delta r(B)/(4\tau\kappa)$. Then

$$\mu\{x \in \Omega : |f(x) - f_{B',\sigma}| > t\} \leq C C_1^q V_\mu \|g\|^q_{L^p_w(\Omega)}/t^q \quad \text{for all } t > 0, \tag{2-3}$$

where $B' = B(x', \delta d(x'))$, and C depends on c, A_1, A_2, θ_1, θ_2, δ, τ, κ, p, q and the doubling constant D_σ of σ but is independent of C_1, V_μ and diam(Ω). *Moreover, if $\mathfrak{S}$ satisfies (1-4) with the truncation property, then the following strong-type inequality also holds*:

$$\|f - f_{B',\sigma}\|_{L^q_\mu(\Omega)} \leq C c_T C_1 V_\mu^{1/q} \|g\|_{L^p_w(\Omega)}, \tag{2-4}$$

where C depends on the parameters as above.

Remark 2.5. It follows from standard interpolation argument that (2-3) will imply

$$\|f - f_{B',\sigma}\|_{L^{\tilde{q}}_\mu(\Omega)} \leq C C_1 V_\mu^{1/q} \mu(\Omega)^{1/\tilde{q}-1/q} \|g\|_{L^p_w(\Omega)}$$

for any $1 \leq \tilde{q} < q$, where the constant C now also depends on $\tilde{q}$; see [Chua and Wheeden 2008, Remark 1.3].

In order to prove the above theorem, we will first extend a Whitney-type lemma similar to [Chua and Wheeden 2008, Proposition 2.6]. For simplicity, we will let $\lambda = \kappa + 2\kappa^2$.

Proposition 2.6. *Let* $0 < \delta \le 1/(2\kappa^2)$. *Suppose* $\Omega \subset H$ *such that* $d(x, \Omega^c) > 0$ *for any* $x \in \Omega$ (when $\Omega \ne H$) *and there is a* δ*-doubling measure* σ *on* Ω *with doubling constant* D_σ. *Then there exists a covering* $\widetilde{W} = \{\widetilde{B}_i\}$ *of* Ω *by* δ*-balls* $\widetilde{B}_i$ *such that*:

(a) $r(B_i) \le \delta d(x_{B_i}) \le \lambda^2 r(B_i)$, *where* x_{B_i} *is the center of* B_i *for all* $B_i \in W = \{2\kappa \widetilde{B}_i : \widetilde{B}_i \in \widetilde{W}\}$ *and given* $x \in \Omega$ *there exists* $\widetilde{B} \in \widetilde{W}$ *such that* ($\delta' = \delta/\lambda^3$)

$$B(x, \delta' d(x)) \subset \widetilde{B} \quad \text{and} \quad B(x, \lambda\delta' d(x)) \subset 2\kappa \widetilde{B} \subset B(x, \delta d(x)) \subset 2\kappa\lambda^2 \widetilde{B}. \tag{2-5}$$

(b) *For every* $\tau \ge 1$ *that satisfies* $\tau\delta \le 1/(2\kappa^2)$, *there is a constant* K *depending only on* τ, κ *and* D_σ *so that the balls* $\{\tau B_i : B_i \in W\}$ *have bounded intercepts with bound* K (*i.e.*, *each* τB_i *intersects at most* $K - 1$ *other* τB_j *in the family*); *in particular*, *the balls* $\{\tau B_i : B_i \in W\}$ *also have pointwise bounded overlaps with overlap constant* K. *Indeed*, *the existence of the* δ*-doubling measure* σ *guarantees that any collection of* $\{\tau B : B \in \mathfrak{F}\}$ *has bounded intercepts whenever* $\mathfrak{F}$ *consists of disjoint* δ*-balls.*

Now suppose further that $\Omega \in J'(c)$ *with center* x'. *Then*:

(c) *For any* $x \in \Omega$, $x \ne x'$, *there exists a finite chain of* δ*-balls* $\{B_i\}_{i=0}^L \subset W$, *depending on* x *and with* $L = L_x$, *such that* $x \in B_0$, $x' \in B_L$, B_L *is independent of* x *and satisfies* $\lambda^{-2} B(x', \delta d(x')) \subset B_L \subset B(x', \delta d(x'))$, $B_i \cap B_{i+1}$ *contains a* δ*-ball* B_i' *with* $B_i \cup B_{i+1} \subset \lambda^4 B_i'$ *for all* i, *and*

$$B_0 \subset \frac{4\lambda^4\kappa}{c\delta} B_i \quad \textit{for all } i. \tag{2-6}$$

Furthermore, *there is a finite chain of* δ*-Whitney balls* ($B(x, r)$ is said to be a δ-Whitney ball if $r = \delta d(x)$) $\{\mathcal{Q}_i\}_{i=0}^L$ *depending on* x *with bounded intercepts such that* $\mathcal{Q}_0 = B(x, \delta d(x))$, $\mathcal{Q}_L = B(x', \delta d(x'))$, $(1/\lambda^2)\mathcal{Q}_i \subset B_i \subset \mathcal{Q}_i$, *and* $\mathcal{Q}_i \cap \mathcal{Q}_{i+1}$ *contains a* δ*-ball* $\mathcal{Q}_i'$ *with* $\mathcal{Q}_i \cup \mathcal{Q}_{i+1} \subset \lambda^6 \mathcal{Q}_i'$.

(d) *If* $\mathcal{Q}_i \not\subset B(x, r)$, *then* $r(\mathcal{Q}_i) \ge c\delta r/(2\kappa)$ *where* x *and* $\mathcal{Q}_i$ *are given in* (c).

(e) *For all* $\varepsilon > 0$, *the number of disjoint* $\mathcal{Q}_i$ (in (c)) *having radius between* ε *and* 2ε *is at most* C (depending only on δ, κ, D_σ and c).

Proof. The proof of this proposition is just a simple modification of that of [Chua and Wheeden 2008, Proposition 2.6] even though the assumption on Ω is now weaker. For completeness, we provide this proof in Appendix B; see also [Chua and Wheeden 2015]. □

Proof of Theorem 2.4. The proof of this theorem is indeed similar but much simpler than that of [Chua and Wheeden 2011, Theorem 1.6]. However, as weak John domains are weaker than John domains, we will prove it using [Chua and Wheeden 2008, Theorem 1.2]. For easy reference, we have stated it as Theorem A.1 in Appendix A, where we have changed the notation slightly. First as in [Chua and Wheeden 2011, (1-6)], for each $x \in \Omega$, since μ^* satisfies Condition (R), let $B_j^x = B(x, r_j^x)$ as in (1-8), condition (2) of Theorem A.1 will then hold with $\wp = \mu(\Omega)/\mu(B')$. Moreover, Proposition 2.6(c) enable us to construct (see [Chua and Wheeden 2011, (1-6)]) a sequence $\{Q_i^x\}_{i=1}^{\infty}$ of δ-balls such that $Q_1^x = B(x', \delta d(x'))$ and $\{Q_i^x\}$ has the intersection property

$$Q_i^x \cap Q_{i+1}^x \text{ contains a } \delta\text{-ball } Q_i' \text{ with } Q_i^x \cup Q_{i+1}^x \subset N Q_i'$$

for some positive constant N independent of x and i. Equation (A-1) will then hold as σ is δ-doubling. Moreover, for large i, Q_i^x is centered at x; in fact, for balls $B_j^x = B(x, r_j^x)$, there exist $K_x, K_x' \in \mathbb{N}$ such that $\tau Q_{i+K_x}^x = B_{i+K_x'}^x$ for $i \geq 0$. B_j^x is a $\tau\delta$-ball if $j \geq K_x$, and Q_i^x is not centered at x if $i \leq K_x$ (indeed, such Q_i^x are δ-Whitney balls constructed in Proposition 2.6(c)). We associate with each ball $B_j^x = B(x, r_j^x)$, $j \geq 1$, the following special subcollection of $\{Q_i^x\}$ as in [Chua and Wheeden 2011, (1-6)]:

$$\mathcal{C}(B_j^x) = \{Q_i^x : \tau Q_i^x \subset B_j^x \text{ and } \tau Q_i^x \not\subset B_{j+1}^x\}. \tag{2-7}$$

In case $j \geq K_x$, then $\mathcal{C}(B_j^x)$ consists of just the single ball $\tau^{-1} B_j^x = Q_j^x$. By Proposition 2.6(d)–(e), we know that each $\mathcal{C}(B)$ has a bounded number (denoted by $L = C(\delta, \kappa, D_\sigma, c)$) of δ-balls Q and each δ-ball has radius $\geq c\delta r(B)/(4\tau\kappa)$. Hence if $I = \{B_\alpha\}$ is a countable collection of pairwise disjoint balls B_j^x in the above, then with the notation of condition (3) in Theorem A.1, we have by (2-2), taking $a_*(Q) = a(Q)\|g\|_{L_w^p(\tau Q)}$,

$$\sum_{B_\alpha \in I} (A(B_\alpha)^q \mu^*(B_\alpha))^{p/q} \leq L^{p/q} \sum_{B_\alpha \in I} C_1^p \|g\|_{L_w^p(B_\alpha)}^p \leq L^{p/q} C_1^p \|g\|_{L_w^p(\Omega)}^p.$$

Thus, (A-4) holds with $\theta = p/q$ and $(C_0^q \mu(\Omega))^{p/q} = C(\delta, \kappa, D_\sigma, c) C_1^p \|g\|_{L_w^p(\Omega)}^p$.

Finally, (A-2) holds with

$$p_0 = 1, \quad C(f, Q_j^x) = f_{Q_j^x, \sigma} \quad \text{and} \quad a_*(Q_j^x) = a(Q_j^x)\|g\|_{L_w^p(\tau Q_j^x)}$$

as $f_{Q_i^x, \sigma} \to f(x)$, σ-a.e. (and hence μ-a.e.) by the Lebesgue differentiation theorem as σ is δ-doubling and note that a Vitali-type property (1-10) holds with $\mu = \mu^* = \sigma$ (on metric spaces, see [Heinonen 2001]). Condition (4) holds because we have assumed the Vitali-type property (1-10) holds. (2-3) now follows from Theorem A.1.

Moreover, if the truncation property holds, the proof of the strong-type inequality (2-4) follows exactly the same argument as in [Chua and Wheeden 2008, proof of Theorem 1.10] (and has been used in many other papers listed there) and hence omitted here. We shall only note that our conclusion follows from [Chua and Wheeden 2011, Theorem 1.9]. □

Next, we prove a self improving Poincaré-type property for balls. Note that a metric ball will be a weak John domain if we assume certain geodesic path property. However, we will establish it without such an assumption.

Proposition 2.7. *Let $1 \le p < q$ and $\mathcal{D}$ be a measurable subset in H such that $d(x, \mathcal{D}^c) > 0$ for all $x \in \mathcal{D}$* (when $\mathcal{D} \neq H$). *Let*

$$0 < \delta \le 1/(2\kappa^2) \quad \textit{and} \quad 1 \le \tau \le 1/(2\delta\kappa^2).$$

Let σ, μ, w be measures on $\mathcal{D}$ such that σ is δ-doubling on $\mathcal{D}$ and μ is absolutely continuous with respect to σ. Suppose (1-4) holds for all $(f, g) \in \mathfrak{S} \subset L^1_{\sigma,\mathrm{loc}}(\mathcal{D}) \times L^p_{w,\mathrm{loc}}(\mathcal{D})$ and δ-balls B in $\mathcal{D}$. Suppose there exists a ball set function μ^* such that $\mu(B) \le \mu^*(B)$ for all δ-balls B in $\mathcal{D}$ and such that Condition (R) holds for any δ-ball $B_r(x_0)$ with $r_1^x = 2\kappa r$. Suppose further that

$$\mu^*(\widetilde{Q})^{1/q} a(Q) \le C_1 \quad \textit{for all } \delta\textit{-balls } Q, \widetilde{Q}, \quad Q \subset \widetilde{Q} \textit{ and } r(Q) \ge r(\widetilde{Q})/(2\kappa). \tag{2-8}$$

If (μ, μ^*) satisfies the Vitali-type property (1-10) on $\mathcal{D}$, then for any ball B such that $\lambda\tau B$ is a δ-ball, we have

$$\mu\{x \in B : |f(x) - f_{B,\sigma}| > t\} \le \frac{C C_1^q V_\mu}{t^q} \|g\|^q_{L^p_w(\lambda\tau B)} \quad \textit{for all } t > 0, (f, g) \in \mathfrak{S}, \tag{2-9}$$

where C depends on $A_1, A_2, \theta_1, \theta_2, \delta, \tau, \kappa, c, p, q$ and the doubling constant D_σ of σ. Furthermore, if $\mathfrak{S}$ satisfies (1-4) with the truncation property, then we also have the following strong-type inequality:

$$\|f - f_{B,\sigma}\|_{L^q_\mu(B)} \le C c_T C_1 V_\mu^{1/q} \|g\|_{L^p_w(\lambda\tau B)} \textit{ for all } (f, g) \in \mathfrak{S}. \tag{2-10}$$

Proof. This is again a consequence of Theorem A.1 [Chua and Wheeden 2008, Theorem 1.2]. For each ball $B_r(x_0)$ such that $B_{\lambda\tau r}(x_0)$ is a δ-ball, we will apply Theorem A.1 with $\Omega = B_r(x_0)$. For each $x \in B_r(x_0)$, we define $Q_1^x = B_r(x_0)$, $Q_2^x = B(x, 2\kappa r)$ and let $r_1^x = 2\kappa r$. By Condition (R), there exists a sequence $r_j^x \to 0$ such that $r_j^x/2 \le r_{j+1}^x < r_j^x$ and (1-8) holds. We now take $B_j^x = Q_{j+1}^x = B_{r_j}(x)$ for all $j \ge 1$ and define $\mathcal{C}(B_j^x) = \{B_j^x\}$ for $j > 1$ and $\mathcal{C}(B_1^x) = \{B_1^x, B_r(x_0)\}$. Note that B_j^x are δ-balls and (A-3) holds with $\wp = 1$ by Condition (R). Moreover, (A-1) holds since σ is δ-doubling. Also, let

$$a_*(Q_i^x) = a(Q_i^x) \|g\|_{L^p_w(\tau Q_i^x)}.$$

Similar to the proof of Theorem 2.4, (A-2) holds with $p_0 = 1$ by (1-4). Take $\theta = p/q$. We now observe that if I is a subcollection of pairwise disjoint balls B_j^x

defined above, we have

$$\begin{aligned}\sum_{B\in I}(a_*(B)^q\mu^*(B))^{p/q} &= \sum_{B\in I} a(B)^p\|g\|^p_{L^p_w(\tau B)}\mu^*(B)^{p/q}\\ &\le \sum_{B\in I} C_1^p\|g\|^p_{L^p_w(\cup\tau B)}\\ &\le CC_1^p\|g\|^p_{L^p_w(\lambda\tau B_r(x_0))}\end{aligned}$$

since $\{\tau B\}_{B\in I}$ has bounded overlap (see Proposition 2.6(b)) and $\tau B_i^x \subset \lambda\tau B_r(x_0)$ (see [Chua and Wheeden 2008, Observation 2.1(1)]). Equation (A-4) will then hold with

$$C_0^q\mu(\Omega) = 2^q CC_1^q\|g\|^q_{L^p_w(\lambda\tau B_r(x_0))}$$

since

$$a(B_r(x_0))^p\|g\|^p_{L^p_w(B_{\tau r}(x_0))}\mu^*(B_{2\kappa r}(x))^{p/q} \le C_1^p\|g\|^p_{L^p_w(B_{\tau r}(x_0))} \quad \text{for any } x\in B_r(x_0).$$

Again, note that $f_{B_r(x),\sigma} \to f(x)$ for σ-a.e. x as $r\to 0$. The first part of the proposition then follows from Theorem A.1 and once again the second part will follow from the standard truncation argument. □

Remark 2.8. (1) $\lambda = 3$ when H is a metric space as $\kappa = 1$. Moreover, checking through our proof, λ can be replaced by $(1+\varepsilon)$ (for any fixed $\varepsilon > 0$) provided for all $x\in B_r(x_0)$ such that $B_{(1+\varepsilon)r}(x_0)$ is a δ ball, we have

(i) $$\sigma(B_{\varepsilon r}(x)\cap B_r(x_0)) \ge C_\sigma\sigma(B_{\varepsilon r}(x)\cup B_r(x_0));$$

(ii) $$\mu^*(B_{\varepsilon r}(x))^{1/q}a(B_r(x_0)) \le C_1;$$

(iii) $$\mu(B_r(x_0)) \le \wp\mu^*(B_{\varepsilon r}(x)).$$

Indeed, we will then choose Q_2^x to be $B(x,\varepsilon r)$ instead of $B(x,2r)$. The rest of the proof is similar with the help of (i)–(iii).

(2) A similar inequality has been obtained in [Hajłasz and Koskela 2000, Theorem 5.1] on metric spaces with λ being replaced by 5 and $\mu = w$ being doubling and $a(Q) = Cr(Q)\mu(Q)^{-1/p}$ [Hajłasz and Koskela 2000, (22)].

(3) It is often true that metric balls are weak John domains; for example, when the ball satisfies the "geodesic path property." In that case, $\|g\|_{L^p_w(\lambda\tau B)}$ in (2-9) and (2-10) can be replaced by just $\|g\|_{L^p_w(B)}$ using Theorem 2.4; see also [Heinonen 2001, Theorem 9.5] when $\mu = w$ is doubling and the main results in [Franchi et al. 2003] for quasimetric balls. Indeed, in particular we obtain the main result of [Franchi et al. 2003] without the assumption of "geodesic path property" or "chain condition."

(4) Equation (2-10) will imply

$$\|f - f_{B,\mu}\|_{L^q_\mu(B)} \le CC_1V_\mu^{1/q}\|g\|_{L^p_w(\lambda\tau B)} \quad \text{for all } (f,g)\in\mathfrak{S}$$

and hence by Hölder's inequality,

(2-11) $$\|f - f_{B,\mu}\|_{L^p_\mu(B)} \le C C_1 V_\mu^{1/q} \mu(B)^{1/q-1/p} \|g\|_{L^p_w(\lambda\tau B)} \quad \text{for all } (f, g) \in \mathfrak{S}.$$

The idea of John domains has been extended to generalized John domains which include bounded and unbounded John domains in [Chua 2009]. It has been shown in [Chua 2009, Proposition 2.24] that a generalized John domain in a metric space that satisfies some "path property" is still a generalized John domain if a point is being removed. Indeed, by a simple modification of that proof, we can also show that a weak John domain in a metric space satisfying a "path property" (which is slightly weaker than that of [Chua 2009]) is still a weak John domain if a point has been removed. In particular, a John domain in $\mathbb{R}^n$ with finite number of points being removed will still be a John domain.

Proposition 2.9. *Let Ω be a subset of a metric space H and $\Omega \in J'(c)$. Suppose Ω satisfies the following path property*:

> *Given any two points $x, y \in B_r(z)$ with $B_{2r}(z) \subset \Omega$, there exists a continuous path $\eta : [0, 1] \to B_{\theta r_2}(z) \setminus B_{r_1/\theta}(z)$ such that $\eta(0) = x$ and $\eta(1) = y$ where $r_1 = \min\{d(x, z), d(y, z)\}$ and $r_2 = \max\{d(x, z), d(y, z)\}$ and θ is a fixed constant > 1.*

Then $\Omega \setminus \{z\} \in J'(C(c, \theta))$ is also a weak John domain.

Proof. As the proof is very similar to the proof of [Chua 2009, Proposition 2.24], we shall only provide it in Appendix B. □

Remark 2.10. (1) The above mentioned path property is weaker than the one used in [Chua 2009, Proposition 2.24]. Indeed, this property is a consequence of the "linearly connected property" defined in [Heinonen 2001, p. 64].

(2) Consequently, if $\Omega \subset \mathbb{R}^n$ is a weak John domain, then $\Omega \setminus \{z_i\}_{i=1}^l$, $l \in \mathbb{N}$ is also a weak John domain. Indeed, if $\Omega \in J'(c)$, then $\Omega \setminus \{z_i\}_{i=1}^l \in J'(\tilde{c})$ with $0 < \tilde{c} < c$ depending only on c, l and n.

Finally, let us discuss a density theorem that is an extension of [Hajłasz and Koskela 1998, Theorem 3] (see also [Hajłasz 1993]). For convenience, we say $C^\infty(\Omega)$ (or $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$) is dense in a norm space W if $C^\infty(\Omega) \cap W$ (or $\mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap W$) is dense in W.

Proposition 2.11. *Let $1 \le p_0, p < \infty$. Let Ω be a domain in $\mathbb{R}^n$, μ, w, ρ, ρ_0 be weights on Ω such that $\rho, \rho^{-1} \in L^\infty_{w,\mathrm{loc}}(\Omega)$* (locally bounded with respect to the measure dw) *and $\rho_0, \rho_0^{-1} \in L^\infty_{\mu,\mathrm{loc}}(\Omega)$. Suppose $C^\infty(\Omega)$* (or $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$) *is dense in $W^{1,p}_{w,\mathrm{loc}}(\Omega) \cap L^{p_0}_{\mu,\mathrm{loc}}(\Omega)$, i.e.,*

(A) *Given any $x \in \Omega$, there exists $B_{r_x}(x) \subset \Omega$ such that for all*

$$f \in W^{1,p}_{w,\mathrm{loc}}(\Omega) \cap L^{p_0}_{\mu,\mathrm{loc}}(\Omega),$$

and $\varepsilon > 0$, *there exists* $\phi \in C^\infty(\Omega)$ (or $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$) *such that*

$$\|f-\phi\|_{W^{1,p}_w(B_{r_x}(x))} < \varepsilon \quad \textit{and} \quad \|f-\phi\|_{L^{p_0}_\mu(B_{r_x}(x))} < \varepsilon. \tag{2-12}$$

Then $C^\infty(\Omega)$ (or $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$) *is dense in* $W^{1,p}_{w\rho}(\Omega) \cap L^{p_0}_{\mu\rho_0}(\Omega)$.

Proof. For each $x \in \Omega$, let $B_{r_x}(x) \subset \Omega$ such that (A) holds and $\overline{B_{r_x}(x)} \subset \Omega$. Since $\Omega = \bigcup_{x\in\Omega} B(x, r_x/2)$, there exists countable subfamily of bounded overlapping balls $\{B_i\}_{i=1}^\infty$ ($B_i = B(x, r_x/2)$ for some x) such that $\Omega \subset \bigcup_i B_i$. We will then choose a partition of unity. Indeed, for each B_i, we find $h_i \in C_0^\infty(\mathcal{D})$ with $\chi_{B_i} \le h_i \le \chi_{2B_i}$ and define $u_i = h_i / \sum_k h_k$ ($u_i = 0$ if $h_i = 0$). Next, for any $f \in W^{1,p}_{w\rho}(\Omega) \cap L^{p_0}_{\mu\rho_0}(\Omega)$, since $\rho^{-1} \in L^\infty_{w,\mathrm{loc}}(\Omega)$, $\rho_0^{-1} \in L^\infty_{\mu,\mathrm{loc}}(\Omega)$, it is clear that $f \in W^{1,p}_{w,\mathrm{loc}}(\Omega) \cap L^{p_0}_{\mu,\mathrm{loc}}(\Omega)$. Since $\rho \in L^\infty_{w,\mathrm{loc}}(\Omega)$ and $\rho_0 \in L^\infty_{\mu,\mathrm{loc}}(\Omega)$, for each B_i, there exists $A_i > 0$ such that $\rho \le A_i$ on $2B_i$ w-a.e. and $\rho_0 \le A_i$ on $2B_i$ μ-a.e. Now, by (A), given any $\varepsilon > 0$, there exists $g_i \in C^\infty(\Omega)$ such that

$$\|f-g_i\|_{L^p_w(2B_i)} + \|\nabla(f-g_i)\|_{L^p_w(2B_i)} \le \varepsilon / \bigl(2^i (A_i)^{1/p} \max\{\|\nabla u_i\|_{L^\infty(\Omega)}, 1\}\bigr)$$

and $\|f-g_i\|_{L^{p_0}_\mu(2B_i)} < \varepsilon/(2^i A_i^{1/p_0})$. Thus, by the triangle inequality and estimates on ρ,

$$\begin{aligned}
\|\nabla(fu_i - g_iu_i)\|_{L^p_{w\rho}(\Omega)} &\\
&= \|\nabla(fu_i - g_iu_i)\|_{L^p_{w\rho}(2B_i)} \\
&= \|u_i\nabla(f-g_i) + (f-g_i)\nabla u_i\|_{L^p_{w\rho}(2B_i)} \\
&\le \|\nabla(f-g_i)\|_{L^p_{w\rho}(2B_i)} + \|\nabla u_i\|_{L^\infty(\Omega)}\|f-g_i\|_{L^p_{w\rho}(2B_i)} \\
&\le A_i^{1/p}\|\nabla(f-g_i)\|_{L^p_w(2B_i)} + A_i^{1/p}\|\nabla u_i\|_{L^\infty(\Omega)}\|f-g_i\|_{L^p_w(2B_i)} \\
&\le 2\varepsilon/2^i.
\end{aligned}$$

Hence if $g = \sum g_iu_i$, then $g \in C^\infty(\Omega)$ (or $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ when $g_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$) and

$$\begin{aligned}
\|\nabla(f-g)\|_{L^p_{w\rho}(\Omega)} &= \bigl\|\nabla\bigl(f\textstyle\sum u_i - \sum g_iu_i\bigr)\bigr\|_{L^p_{w\rho}(\Omega)} \quad \text{since } \textstyle\sum_i u_i = 1 \\
&= \bigl\|\textstyle\sum_i \nabla(fu_i - g_iu_i)\bigr\|_{L^p_{w\rho}(\Omega)} \\
&\le \textstyle\sum_i \|\nabla(fu_i - g_iu_i)\|_{L^p_{w\rho}(\Omega)} \le 2\varepsilon.
\end{aligned}$$

Finally, it is easy to see that

$$\|f-g\|_{L^p_{w\rho}(\Omega)},\ \|f-g\|_{L^{p_0}_{\mu\rho_0}(\Omega)} < 2\varepsilon. \qquad \square$$

It is often true that the Poincaré inequality holds. The following observation is useful in applying the density theorem.

Proposition 2.12. *Let* $1 \le p, p_0 < \infty$, $\tau \ge 1$ *and* μ, w *be locally integrable weights on a domain* $\Omega \subset \mathbb{R}^n$. *Suppose for all balls* B *with* $2\tau B \subset \Omega$ *and* $f \in C^\infty(\Omega)$,

$$\|f - f_{B,\mu}\|_{L^{p_0}_\mu(B)} \le a(B)\|\nabla f\|_{L^p_w(\tau B)}, \tag{2-13}$$

where $a(B)$ is a finite ball set function that is independent of f. Suppose also $C^\infty(\Omega)$ is dense in $E^p_{w,\mathrm{loc}}(\Omega)$, i.e., given any $x \in \Omega$, there exists $B_{r_x}(x) \subset \Omega$ such that for any $\varepsilon > 0$, $f \in E^p_{w,\mathrm{loc}}(\Omega)$, there exists $\phi \in C^\infty(\Omega)$ such that

$$\|\nabla(f-\phi)\|_{L^p_w(B_{r_x}(x))} < \varepsilon. \tag{2-14}$$

Then $C^\infty(\Omega)$ is also dense in $E^p_{w,\mathrm{loc}}(\Omega) \cap L^{p_0}_{\mu,\mathrm{loc}}(\Omega)$.

Proof. The conclusion follows from (2-13) and (2-14). □

Remark 2.13. (1) One could generalize the above density theorem to domains in Riemannian manifolds where there are partitions of unity.

(2) Under the assumptions of Proposition 2.11, $C^\infty(\Omega)$ is also dense in

$$L^p_{w\rho_0,\mathrm{loc}}(\Omega) \cap W^{1,p}_{w\rho,\mathrm{loc}}(\Omega), \quad L^p_{w\rho_0}(\Omega) \cap E^p_{w\rho}(\Omega), \quad \ldots, \quad \text{etc.}$$

To see this, just check through the proof.

(3) If (2-13) holds and $w \in A_p$, then it follows from Proposition 2.12 that $C^\infty(\Omega)$ is dense in $L^{p_0}_\mu(\Omega) \cap E^p_w(\Omega)$ as $C^\infty(\Omega)$ is dense in $W^{1,p}_w(\Omega)$ [Turesson 2000].

(4) Condition (2-12) holds for example when $w \in A_p$ and $\mu \in A_{p_0}$. It is then easy to see that $C^\infty(\Omega)$ is dense in $L^{p_0}_{\mu\rho_0}(\Omega) \cap E^p_{w\rho}(\Omega)$. The case where $w = \mu = 1$, $p_0 = p$ and ρ, ρ_0 are positive continuous on Ω has been obtained in [Hajłasz and Koskela 1998, Theorem 3].

(5) The density theorem for weighted Sobolev spaces of different definitions has been studied in [Chiadò Piat and Serra Cassano 1994].

Finally, note that derivatives and the fractional derivatives satisfy the truncation property.

Proposition 2.14. *Let Ω be an open set in $\mathbb{R}^n$ and $1 \le p < \infty$ and $0 < \alpha \le 1$. Let w be any Borel measure on Ω. Then for any $f \in L^1_{\mathrm{loc}}(\Omega)$* (or $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ if $\alpha = 1$), *we have*

$$\sum_{k=1}^{\infty} \|\nabla^\Omega_{\alpha,p} f_b^{2^k\omega}\|^p_{L^p_w(\Omega)} \le C(p)\|\nabla^\Omega_{\alpha,p} f\|^p_{L^p_w(\Omega)} \quad \text{for any } \omega > 0 \text{ and } b \in \mathbb{R}. \tag{2-15}$$

Proof. The case $\alpha = 1$ is well-known and obvious as

$$\sum_{k=1}^{\infty} \|\nabla f_b^{2^k\omega}\|^p_{L^p_w(\Omega)} \le \|\nabla|f-b|\|^p_{L^p_w(\Omega)} \le \|\nabla f\|^p_{L^p_w(\Omega)}.$$

For $0 < \alpha < 1$, a result has been stated in [Dyda et al. 2016, Theorem 4.1]. Unfortunately, the statement is not quite the same as ours. So we will provide the details here. Fix any $\omega > 0$ and $b \in \mathbb{R}$, let

$$A_i = \{x \in \Omega : 2^{i-1}\omega < |f(x)-b| \le 2^i\omega\}.$$

Then

$$\begin{aligned}\sum_{k=1}^{\infty}&\|\nabla^{\Omega}_{\alpha,p} f_b^{2^k\omega}\|^p_{L^p_w(\Omega)}\\&=\sum_{k=1}^{\infty}\int_{\Omega}\int_{B(x,\rho_\Omega(x))}\frac{|f_b^{2^k\omega}(x)-f_b^{2^k\omega}(y)|^p}{|x-y|^{n+\alpha p}}\,dy\,dw(x)\\&=\Biggl(\sum_{k=1}^{\infty}\sum_{i\le k\le j}\int_{A_i}\int_{A_j\cap B(x,\rho_\Omega(x))}\\&\qquad\qquad+\sum_{k=1}^{\infty}\sum_{j\le k\le i}\int_{A_i}\int_{A_j\cap B(x,\rho_\Omega(x))}\Biggr)\frac{|f_b^{2^k\omega}(x)-f_b^{2^k\omega}(y)|^p}{|x-y|^{n+\alpha p}}\,dy\,dw(x).\end{aligned}$$

If $x\in A_i,\ y\in A_j,\ i<j-1$, then

$$\begin{aligned}&|f(x)-f(y)|\ge|f(y)-b|-|f(x)-b|\ge 2^{j-2}\omega,\\ \text{and}\quad&|f_b^{2^k\omega}(x)-f_b^{2^k\omega}(y)|\le 2^k\omega\le 4 2^{k-j}|f(x)-f(y)|.\end{aligned}$$

On the other hand $|f_b^{2^k\omega}(x)-f_b^{2^k\omega}(y)|\le|f(x)-f(y)|$ for all k. Hence,

$$|f_b^{2^k\omega}(x)-f_b^{2^k\omega}(y)|\le 4 2^{k-j}|f(x)-f(y)|\quad\text{for all } i\le k\le j. \tag{2-16}$$

Using the above (2-16), we have

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{i\le k\le j}&\int_{A_i}\int_{A_j\cap B(x,\rho_\Omega(x))}\frac{|f_b^{2^k\omega}(x)-f_b^{2^k\omega}(y)|^p}{|x-y|^{n+\alpha p}}\,dy\,dw(x)\\&\le 4^p\sum_{k=1}^{\infty}\sum_{i\le k\le j}2^{(k-j)p}\int_{A_i}\int_{A_j\cap B(x,\rho_\Omega(x))}\frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\,dy\,dw(x)\\&\le\frac{4^p}{1-2^{-p}}\int_{\Omega}\int_{B(x,\rho_\Omega(x))}\frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\,dy\,dw(x).\end{aligned}$$

A similar estimate can be done for the remaining term. □

3. Proof of the main theorem and related results

Proof of Theorem 1.1. First, on each Ω_j, by Theorem 2.4, for any $(f,g)\in\mathfrak{S}$,

$$\begin{aligned}\|f-f_{B'_j,\sigma}\|_{L^q_\mu(\Omega_j)}&\\&\le C\bigl(\theta_1,\theta_2,A_1,A_2,D_\sigma,\delta,\tau,c,q,p\bigr)c_T C_1 V_\mu^{1/q}\|g\|_{L^p_w(\Omega_j)}.\end{aligned}\tag{3-1}$$

Hence by the triangle inequality and Hölder's inequality,

$$\begin{aligned}\text{(3-2)}\qquad \|f\|_{L^q_\mu(\Omega_j)} &\le \|f_{B'_j,\sigma}\|_{L^q_\mu(\Omega_j)} + \|f - f_{B'_j,\sigma}\|_{L^q_\mu(\Omega_j)} \\ &\le \frac{\mu(\Omega_j)^{1/q}}{\sigma(B'_j)^{1/p_0}} \|f\|_{L^{p_0}_\sigma(B'_j)} + Cc_T C_1 V_\mu^{1/q} \|g\|_{L^p_w(\Omega_j)} \\ &\le C(c, D_\sigma) C_2 \|f\|_{L^{p_0}_\sigma(\Omega_j)} + Cc_T C_1 V_\mu^{1/q} \|g\|_{L^p_w(\Omega_j)}\end{aligned}$$

by (1-12) since σ is δ-doubling on Ω_j (with doubling constant D_σ) and $\Omega_j \subset C(c, \delta) B'_j$. Hence,

$$\begin{aligned}\Big(\sum \|f\|^q_{L^q_\mu(\Omega_j)}\Big)^{1/q} &\le CC_2\Big(\sum \|f\|^q_{L^{p_0}_\sigma(\Omega_j)}\Big)^{1/q} + Cc_T C_1 V_\mu^{1/q}\Big(\sum \|g\|^q_{L^p_w(\Omega_j)}\Big)^{1/q} \\ &\le CC_2\Big(\sum \|f\|^{p_0}_{L^{p_0}_\sigma(\Omega_j)}\Big)^{1/p_0} + Cc_T C_1 V_\mu^{1/q}\Big(\sum \|g\|^p_{L^p_w(\Omega_j)}\Big)^{1/p},\end{aligned}$$

since $1 \le p, p_0 \le q$. Thus since $\sum \chi_{\Omega_j} \le M$,

$$\text{(3-3)}\qquad \|f\|_{L^q_\mu(\Omega)} \le C\big[C_2 M^{1/p_0} \|f\|_{L^{p_0}_\sigma(\Omega)} + c_T C_1 V_\mu^{1/q} M^{1/p} \|g\|_{L^p_w(\Omega)}\big]$$

for all $(f, g) \in \mathfrak{S}$. We will now apply Theorem A.2 to prove Theorem 1.1(II)(b) and (II)(c).

First, by Proposition 2.7 and Remark 2.8(4) and Hölder's inequality, we have

$$\text{(3-4)}\qquad \|f - f_{B,\mu}\|_{L^p_\mu(B)} \le C\mu(B)^{1/p-1/q} \|g\|_{L^p_w(3\tau B)}$$

for all $(f, g) \in \mathfrak{S}$ and any δ-ball B of any Ω_j. Now suppose $\{(f_n, g_n)\} \subset \mathfrak{S}$ such that $\{f_n\}$ and $\{g_n\}$ are bounded in $L^{p_0}_\sigma(\Omega)$ and $L^p_w(\Omega)$ respectively. Then $\{f_n\}$ is also bounded in $L^q_\mu(\Omega)$ by (3-3). Since $\mu(\Omega) < \infty$, given any $\varepsilon > 0$, there exists $L \in \mathbb{N}$ such that $\mu\big(\Omega \setminus \bigcup_{j=1}^L \Omega_j\big) < \varepsilon/2$. Next, for each $1 \le j \le L$, let $\{Q_{i,j}\}_{i=1}^\infty$ be a collection of bounded intersecting δ-balls of Ω_j such that $\Omega_j = \bigcup Q_{i,j}$ guaranteed by Proposition 2.6(a). Then there exists $k_j \in \mathbb{N}$ such that

$$\mu\big(\Omega_j \setminus \textstyle\bigcup_{i=1}^{k_j} Q_{i,j}\big) < \varepsilon/(2L).$$

For each

$$\text{(3-5)}\qquad 0 < r < \delta \min\big\{d\big(\Omega_j^c, \textstyle\bigcup_{i=1}^{k_j} Q_{i,j}\big) : 1 \le j \le L\big\}\big/(6\tau),$$

we choose any maximum family of pairwise disjoint balls $\{B(x_m, r/3)\}_{m=1}^K$ contained in $\bigcup_{1 \le j \le L} \bigcup_{i=1}^{k_j} Q_{i,j}$. Then it is easy to see that each $B_r(x_m)$ is a δ-ball in some Ω_j and

$$\textstyle\bigcup_{m=1}^K B_r(x_m) \supset \bigcup_{1 \le j \le L} \bigcup_{i=1}^{k_j} Q_{i,j}.$$

Note that the family $\{B(x_m, 3\tau r)\}_{m=1}^K$ has bounded overlaps. It is now clear by (3-4) that

$$\text{(3-6)}\quad \sum_{m=1}^{K} \|f_n - (f_n)_{B_r(x_m),\mu}\|^p_{L^p_\mu(B_r(x_m))} \le \sum_{m=1}^{K} C\mu(B_r(x_m))^{1-p/q}\|g_n\|^p_{L^p_w(B(x_m,3\tau r))} \le C\sup_m \mu(B_r(x_m))^{1-p/q}\|g_n\|^p_{L^p_w(\Omega)}.$$

We now choose r smaller if necessary such that the right hand side of (3-6) is less than ε^p, which is possible by Remark 1.2(8) and the fact that $\{g_n\}$ is bounded in $L^p_w(\Omega)$.

Taking $\{E_\ell\}$ as $\{B_r(x_m)\}_{m=1}^K$, by Theorem A.2, we conclude the proof of (II)(b). Part (c) of (II) is similar but easier; see Remark 1.2(10).

Finally, note that if we only assume (1-4) without the truncation property, then instead of (3-3), we will only have (if $p_0, p \le \tilde{q} < q$)

$$\text{(3-7)}\quad \|f\|_{L^{\tilde{q}}_\mu(\Omega)} \le C\sup_j \mu(\Omega_j)^{1/\tilde{q}-1/q}\big[C_2 M^{1/p_0}\|f\|_{L^{p_0}_\sigma(\Omega)} + C_1 V_\mu^{1/q} M^{1/p}\|g\|_{L^p_w(\Omega)}\big]$$

for all $(f, g) \in \mathfrak{S}$ while (3-4) remains valid. Remark 1.2(1) is now clear.

Proof of Theorem 1.4. First, clearly,

$$\rho_1\mu(B\cap\Omega_j) \le \mu^*(B)\prod_{i=1}^{l}\Psi_i(\bar{\eta}_i(B))^{a_i} \quad \text{for any } \Omega_j.$$

We now see that $\mu^*(B)\prod_{i=1}^{l}\Psi_i(\bar{\eta}_i(B))^{a_i}$ satisfies Condition (R). Indeed since Ψ_i's are monotone increasing, $a_i > 0$ and $\Psi_i(2t) \le C_{\Psi_i}\Psi_i(t)$ for all $t > 0$ and all i, we have

$$\Psi_i(\bar{\eta}_i(B))^{a_i} \le \Psi_i(\bar{\eta}_i(2B))^{a_i} \le C_{\rho_1}\Psi_i(\bar{\eta}_i(B))^{a_i} \quad \text{for all balls } B \text{ with center in } \Omega_j,$$

where $C_{\rho_1} = C\big(\{C_{\Psi_i}, a_i\}_{i=1}^l\big)$. Given any $x \in \Omega_j$, suppose $B_i = B(x, r_i^x)$ is the sequence given in Condition (R) for μ^*. The sequence will then work for $\mu^*(B)\prod_{i=1}^{l}\Psi_i(\bar{\eta}_i(B))^{a_i}$. That is, it satisfies Condition (R) (but with smaller constant A_1 and θ_1 on the left). Next, for any δ-ball Q of Ω_j, we have by (1-4),

$$\text{(3-8)}\quad \|f - f_{Q,\sigma}\|_{L^1_\sigma(Q)} \le a(Q)\|g\|_{L^p_w(\tau Q)} \le C(\{C_{\Psi_i}, b_i\}, p, \tau\delta)a(Q)\prod_{i=1}^{l}\Psi_i(\bar{\eta}_i(Q))^{-b_i/p}\|g\|_{L^p_{\rho_2 w}(\tau Q)}$$

as τQ is a $\tau\delta$-ball and ρ_2 is essentially constant on τQ. Moreover, if $Q \subset B$, where Q is a δ-ball, B is a ball with center in Ω_j such that $r(B) \le \operatorname{diam}(\Omega_j)$ and $r(Q) \ge c\delta r(B)/4$, then since $r(Q) \le \bar{\eta}_i(Q) \le \bar{\eta}_i(B) \le C(\delta, c)\bar{\eta}_i(Q)$ and

$\Psi_i(2t) \le C_{\Psi_i}\Psi_i(t),\ a_i > 0$ for all i, we have by (1-25),

$$\begin{aligned}\mu^*(B)^{1/q}a(Q)\prod_{i=1}^{l}\Psi_i(\bar\eta_i(B))^{a_i/q}\prod_{i=1}^{l}\Psi_i(\bar\eta_i(Q))^{-b_i/p} \\ &\le C(\delta, q, c, \{a_i, b_i, C_{\Psi_i}\})\mu^*(B)^{1/q}a(Q)\prod_{i=1}^{l}\Psi_i(\bar\eta_i(Q))^{a_i/q-b_i/p} \\ &\le C(\delta, q, c, \{a_i, b_i, C_{\Psi_i}\})C_1.\end{aligned}$$

Hence, we have by Theorem 2.4 that (1-26) holds. Next, by the triangle inequality and Hölder's inequality,

$$\|f\|_{L^p_{\rho_1\mu}(\Omega_j)} \le \frac{\rho_1\mu(\Omega_j)^{1/q}}{\sigma(B'_j)}\|f\|_{L^{p_0}_\sigma(B'_j)} + \|f - f_{B'_j,\sigma}\|_{L^p_{\rho_1\mu}(\Omega_j)} = I + II.$$

Using (1-27), noting that σ is δ-doubling on Ω_j on each j with doubling constant D_σ and $\Omega_j \subset C(c,\delta)B'_j$, we have

$$\begin{aligned}I &\le C\big(\delta, \tau, c, \{C_{\Psi_i}, a_i\}, q, p_0\big)\frac{\mu(\Omega_j)^{1/q}\prod_{i=1}^{l}\Psi_i(\bar\eta_i(B'_j))^{a_i/q}}{\sigma(B'_j)^{1/p_0}}\|f\|_{L^{p_0}_\sigma(B'_j)} \\ &\le C\big(c, \delta, \tau, \{C_{\Psi_i}, a_i, \gamma_i\}, p_0, q\big)\frac{\prod_{i=1}^{l}\Psi_i(\bar\eta_i(B'_j))^{a_i/q-\gamma_i/p_0}\mu(\Omega_j)^{1/q}}{\sigma(B'_j)^{1/p_0}}\|f\|_{L^{p_0}_{\rho_0\sigma}(B'_j)} \\ &\le C\big(c, \delta, \tau, D_\sigma, \{C_{\Psi_i}, a_i, \gamma_i\}, p_0, q\big)C_2\|f\|_{L^{p_0}_{\rho_0\sigma}(\Omega_j)},\end{aligned}$$

Combining with (1-26), we have

$$\begin{aligned}\|f\|_{L^p_{\rho_1\mu}(\Omega_j)} &\le C\big(c, \delta, \tau, D_\sigma, \{C_{\Psi_i}, a_i, b_i, \gamma_i\}, p, p_0, q\big) \\ &\quad\times\big(C_2\|f\|_{L^{p_0}_{\rho_0\sigma}(\Omega_j)} + C_1c_TV_\mu^{1/q}\|g\|_{L^p_{\rho_2 w}(\Omega_j)}\big).\end{aligned}$$

Finally, we can conclude Theorem 1.4 by an argument as in the proof of Theorem 1.1.

Proof of Theorem 1.8. We will only prove the second part where $\mathcal{D} = \bigcup\Omega_j$. We will use Theorem 1.4 with $d\sigma = dx$ the Lebesgue measure, $\Psi = 1$, $\eta = \rho$, $\delta = \frac{1}{2}$ and $\mu^*(B) = C_\mu r(B)^N$. Since $\Omega_j \subset \mathbb{R}^n$, (μ_a, μ_a^*) (where $\mu_a^*(B) = \bar\rho(B)^a\mu^*(B)$) satisfies the Vitali-type property (1-10) with parameter depending only on n. Let $\mathfrak{S}$ be as in the proof of Corollary 1.6. Then the Poincaré inequality (1-4) holds for $\mathfrak{S}$ with $\sigma = 1$ and $g = \nabla^{\Omega_j}_{\alpha,p}f$ by (1-2). Note that $\nabla^{\Omega_j}_{\alpha,p}f \le \nabla^{\mathcal{D}}_{\alpha,p}f$. Again, $\mathfrak{S}$ satisfies (1-4) with the truncation property by Proposition 2.14.

Next, if B is a ball with center in Ω_j, $r(B) \le \operatorname{diam}(\Omega_j)$ and Q is a δ-ball in B such that $r(Q) \ge c\,r(B)/8$, by the fact that $\bar\rho(Q) \ge Cr(Q)$ and (1-46),

$$\begin{aligned}\mu^*(B)^{1/q}a(Q)\bar\rho(B)^{a/q-b/p} &\le CC_*C_\mu^{1/q}r(Q)^{N/q+\beta_1}\bar\rho(Q)^{\beta_2+a/q-b/p} \\ &\le C\big(M_2, a, b, p, q, N, \beta_1, \beta_2\big)C_*C_\mu^{1/q}\end{aligned}$$

since $\beta_1+\frac{N}{q}+\min\{0,\beta_2+\frac{a}{q}-\frac{b}{p}\}\ge 0$ (by (i)) and $\beta_2+\frac{a}{q}-\frac{b}{p}\le 0$ when ρ is unbounded (by (ii)). Hence (1-25) holds. We now check that (1-27) holds. As $\Omega_j\subset C(c)B'_j$, by (1-45),

$$\begin{aligned}(3\text{-}9)\quad &\mu(\Omega_j)^{1/q}|\Omega_j|^{-1/p_0}\bar\rho(\Omega_j)^{a/q-\gamma/p_0}\\ &\le CC_\mu^{1/q}\min\{r(B'_j)^N, r(B'_j)^{N_1}\bar\rho(B'_j)^{N_2}\}^{1/q}r(B'_j)^{-n/p_0}\bar\rho(\Omega_j)^{a/q-\gamma/p_0}\\ &\le CC_\mu^{1/q}\min\Big\{r(B'_j)^{N/q-n/p_0}\bar\rho(B'_j)^{a/q-\gamma/p_0},\\ &\qquad\qquad r(B'_j)^{N_1/q-n/p_0}\bar\rho(B'_j)^{(N_2+a)/q-\gamma/p_0}\Big\},\end{aligned}$$

which is bounded by $C(M_1,M_2,a,b,p_0,q,N,N_2,N_1,\gamma)C_\mu^{1/q}$ using (i), (ii) and the fact that $r(B'_j)\ge C(c,M_1)$. Equation (1-47) will then hold for all $f\in\mathfrak{S}_\alpha(\mathcal{D})$ by Theorem 1.4.

For the part of compact embedding, as we now allow $\mu_a(\mathcal{D})=\infty$, we cannot use Theorem 1.4 directly, we will use Theorem A.3 instead of Theorem A.2. Now, suppose we have strict inequalities in conditions (i) and (ii). Then we can find $\tilde q>q$ and $\tilde a>a$ such that conditions (i) and (ii) hold with a and q being replaced by $\tilde a$ and $\tilde q$ respectively. We can then apply the first part of the theorem to conclude that (1-47) holds with either a being replaced by $\tilde a$ or q being replaced by $\tilde q$.

Now suppose $\{u_i\}_{i=1}^\infty\subset\mathfrak{S}_\alpha(\mathcal{D})$ such that both $\|u_i\|_{L^{p_0}_{\rho^\gamma}(\mathcal{D})}$ and $\|\nabla^{\mathcal{D}}_{\alpha,p}u_i\|_{L^p_{w_b}(\mathcal{D})}$ are bounded. Then $\{u_i\}$ is a bounded sequence in both $L^q_{\mu_{\tilde a}}(\mathcal{D})$ and $L^{\tilde q}_{\mu_a}(\mathcal{D})$. Hence it has a weakly convergent subsequence (in both $L^q_{\mu_{\tilde a}}(\mathcal{D})$ and $L^{\tilde q}_{\mu_a}(\mathcal{D})$) and for convenience, we will still denote the subsequence by $\{u_i\}$ and we may also assume that $\|u_i\|_{L^q_{\mu_{\tilde a}}(\mathcal{D})}\le A$ for all i. Now, given any $\eta>0$, let $\mathcal{D}_\eta=\{x\in\mathcal{D}:\rho(x)<\eta^{q/(a-\tilde a)}\}$ and $\mathcal{D}'_\eta=\mathcal{D}\setminus\mathcal{D}_\eta$. Then,

$$\begin{aligned}(3\text{-}10)\quad \|u_i-u_j\|^q_{L^q_{\mu_a}(\mathcal{D}'_\eta)}&=\int_{\mathcal{D}'_\eta}|u_i-u_j|^q\rho(x)^{a-\tilde a}d\mu_{\tilde a}\le\eta^q\int_{\mathcal{D}'_\eta}|u_i-u_j|^q d\mu_{\tilde a}\\ &\le(2A)^q\eta^q.\end{aligned}$$

Again, by Proposition 2.7, for all i, we know inequality (2-11) holds with $f=u_i$, $g=\nabla^{\mathcal{D}}_{\alpha,p}u_i$, $\mu=\mu_a$, $w=w_b$, $\lambda=3$ and $\tau=1$. Next, given any $\varepsilon>0$, as $q>p$, by Remark 1.2(8), we see that there exists $\delta_\varepsilon>0$ such that (A-7) holds with $f=u_i$ if $r(B)<\delta_\varepsilon$ and $6B\subset\mathcal{D}$. Further, since $\mu_a(\mathcal{D}_\eta)<\infty$ by assumption, as $\{u_i\}$ and $\{\nabla^{\mathcal{D}}_{\alpha,p}u_i\}$ are bounded in $L^{\tilde q}_{\mu_a}(\mathcal{D}_\eta)$ and $L^p_{w_b}(\mathcal{D}_\eta)$ respectively, using Theorem A.3, $\{u_i\}$ has a subsequence (still denoted by $\{u_i\}$) converging in $L^{\tilde q}_{\mu_a}(\mathcal{D}_\eta)$ and hence Cauchy. Thus, there exists N_ε such that

$$(3\text{-}11)\qquad \|u_i-u_j\|_{L^q_{\mu_a}(\mathcal{D}_\eta)}\le\varepsilon\quad\text{if } i,j\ge N_\varepsilon.$$

It is now clear that $\{u_i\}$ is a Cauchy sequence in $L^q_{\mu_a}(\mathcal{D})$.

Proof of Corollary 1.10. Let $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{D}) \cap L^{p_0}_{\rho^\gamma}(\mathbb{R}^n)$ if $\alpha = 1$ (or, if $\alpha < 1$, $f \in \widehat{W}^{\alpha,p}_{\rho^b}(\mathcal{D}) \cap L^{p_0}_{\rho^\gamma}(\mathbb{R}^n)$). It is known that $\mathbb{R}^n \setminus \mathrm{G} \in J(c,\infty)$ [Chua 1995, Proposition 2.7; 2009, Proposition 2.21]. Moreover, $\mathcal{D} = \mathbb{R}^n \setminus (\mathrm{G} \cup F) \in J(\tilde{c},\infty)$ [Chua 2009, Proposition 2.24]. Thus, given any $K > 0$, there exists $\{\Omega^K_j\} \subset J'(\tilde{c})$ such that $\mathrm{diam}(\Omega^K_j) \sim K$, center ball B^K_j of Ω^K_j with $r(B^K_j) \sim K$, $\bigcup \Omega^K_j = \mathcal{D}$ and $\sum \chi_{\Omega^K_j} \le C(n)$.

By (1-44), in Theorem 1.8(I), taking $dw = dx$, $\beta = \alpha - n/p$ and we have

$$\begin{aligned}\|f - f_{B^K_j}\|_{L^q_{\mu_a}(\Omega^K_j)} &\le C\bar{\rho}(R)^{\alpha+N+a/q-n+b/p}\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho^b}(\Omega^K_j)}\\ &= C\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho^b}(\Omega^K_j)}.\end{aligned}$$

Hence, by the triangle inequality, Hölder's inequality, the fact that $\bar{\rho}(B^K_j) \ge C\,\mathrm{diam}(\Omega^K_j)$ and ρ is essentially constant on B^K_j, we have

$$\begin{aligned}\text{(3-12)}\quad \|f\|_{L^q_{\mu_a}(B^K_j)} &\le C\mu_a(B^K_j)^{1/q}|B^K_j|^{-1/p_0}\bar{\rho}(B^K_j)^{-\gamma/p_0}\|f\|_{L^{p_0}_{\rho^\gamma}(B^K_j)} + C\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho^b}(\Omega^K_j)}\\ &\le C|\Omega^K_j|^{N/nq-1/p_0}\bar{\rho}(B^K_j)^{a/q-\gamma/p_0}\|f\|_{L^{p_0}_{\rho^\gamma}(B^K_j)} + C\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho^b}(\Omega^K_j)}\\ &\le C\,\mathrm{diam}(\Omega^K_j)^{(N+a)/q-(n+\gamma)/p_0}\|f\|_{L^{p_0}_{\rho^\gamma}(\Omega^K_j)} + C\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho^b}(\Omega^K_j)}\end{aligned}$$

since $a/q \le \gamma/p_0$. Finally, as $q \ge p, p_0$, we have by summing over Ω^K_j,

$$\text{(3-13)}\qquad \|f\|_{L^q_{\mu_a}(\mathbb{R}^n)} \le CK^{(N+a)/q-(n+\gamma)/p_0}\|f\|_{L^{p_0}_{\rho^\gamma}(\mathcal{D})} + C\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho^b}(\mathbb{R}^n)}.$$

Taking $K \to \infty$, as $(N+a)/q < (n+\gamma)/p_0$, we obtain (1-49) for $\alpha < 1$. For $\alpha = 1$, recall that by Remark 2.13(4), we know $\mathrm{Lip}_{\mathrm{loc}}(\mathcal{D})$ is dense in $L^{p_0}_{\rho^\gamma}(\mathcal{D}) \cap E^p_{\rho^b}(\mathcal{D})$ and this concludes the proof of the first part. Finally, the last part of the corollary follows from Remark 1.2(10) and is similar to the proof of the second part of Corollary 1.6. Indeed, instead of (3-12), we will have

$$\|f\|_{L^q_{\mu_a}(\Omega^k_j)} \le C\mu_a(\Omega^k_j)^{1/q-1/p_0}\|f\|_{L^{p_0}_{\mu_a}(\Omega^k_j)} + C\|\nabla^{\mathcal{D}}_{\alpha,p} f\|_{L^p_{\rho^b}(\Omega^k_j)}.$$

Summing up as before, by (1-51), again letting $K \to \infty$, we now see that (1-49) holds if $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{D}) \cap L^{p_0}_{\mu_a}(\mathbb{R}^n)$ or $f \in L^{p_0}_{\mu_a}(\mathbb{R}^n) \cap \widehat{W}^{\alpha,p}_{\rho^b}(\mathcal{D})$. Obviously, the condition involving γ is now redundant. Furthermore for $\alpha = 1$, (1-49) holds for all $f \in L^{p_0}_{\mu_a}(\mathbb{R}^n) \cap E^p_{\rho^b}(\mathcal{D})$ by Propositions 2.11 and 2.12.

Proof of Corollary 1.12. Since $\mathcal{D}$ is bounded, there exists a finite collection of dyadic cubes $\{R_j\}$ of the same size such that $\mathcal{D} \subset \bigcup \overline{R_j}$ and the center ball B'_j of each $R_j \setminus F$ does not intersect $\mathcal{D}$. In particular $f_{B'_j} = 0$ for each j if $f \in C^\infty_0(\mathcal{D})$. Hence, for each $R_j \setminus F$, by Theorem 1.8(I), we have (note that $R_j \setminus F \in J'(c)$ with c independent of j by Proposition 2.9 since F is finite),

$$\|f\|_{L^q_{\mu_a}(R_j)} = \|f\|_{L^q_{\mu_a}(R_j \setminus F)} \le CC^{1/q}_\mu\bar{\rho}(R_j)^{1+(N+a)/q-(n+b)/p}\|\nabla f\|_{L^p_{\rho^b}(R_j)}.$$

And hence

$$\|f\|_{L^q_{\mu_a}(\mathcal{D})} \le C\|\nabla f\|_{L^p_{\rho^b}(\mathcal{D})} \tag{3-14}$$

since $q \ge p$. Now if the inequality in (1-52) is strict, we can find $\tilde{q} > q$ such that the above inequality (3-14) holds with q being replaced by $\tilde{q}$. We can then apply Theorem A.3 to conclude that the embedding of $C_0^\infty(\mathcal{D})$ (and hence also the closure of $C_0^\infty(\mathcal{D})$ in $E^p_{\rho^b}(\mathcal{D})$) to $L^q_{\mu_a}(\mathcal{D})$ is compact. This completes the proof of Corollary 1.12.

Remark 3.1. If $\mathcal{D}$ is unbounded but there exists a collection of countable dyadic cubes $\{R_j\}$ of the same size such that $\mathcal{D} \subset \bigcup R_j$, $|R_j| \ge |\mathcal{D} \cap R_j|/2$ for all j and ρ is bounded on $\bigcup R_j$, then by taking the "parents" of those R_j (for convenience, we will still denote them by $\{R_j\}$), we may assume that the center ball B'_j of $R_j \setminus F$ does not intersect $\mathcal{D}$. We could then derive (3-14). Compact embedding of $C_0^\infty(\mathcal{D})$ (and hence also the closure of $C_0^\infty(\mathcal{D})$ in $E^p_{\rho^b}(\mathcal{D})$) to $L^q_{\mu_a}(\mathcal{D})$ can again be established under similar assumptions if $\mu_a\{x \in \mathcal{D} : \rho(x) < r\} < \infty$ for any $r > 0$.

Proof of Theorem 1.14. Instead of applying Theorem 1.8, we will apply techniques similar to those of [Chua 2009, Theorem 4.1, 4.3]. Moreover, we will also need either [Chua and Wheeden 2008, Theorem 2.9] or [Chua 2009, Theorem 2.11]. Note that a weak John domain is a Boman domain (see Proposition 2.6(c)). Next, by Proposition 2.9, there exists $\tilde{c}$ depending only on l, c and n such that for each j, $\widetilde{\Omega}_j = \Omega_j \backslash F \in J'(\tilde{c})$, where $F = \{z_i\}_{i=1}^l$. For convenience, we will let

$$\bar{d}_i(B) = \sup_{x \in B} |x - z_i| \quad \text{for each } i. \tag{3-15}$$

If Q is a δ-ball $\big(\delta = \frac{1}{5}\big)$ of $\widetilde{\Omega}_j$ for any j, then by Remark 1.2(5), (1-2) will hold with $a(Q) = C(n)|Q|^{\alpha/n - 1/p}$ and $w = 1$. Hence for all $f \in \mathfrak{S}_\alpha(\widetilde{\mathcal{D}})$, where $\widetilde{\mathcal{D}} = \mathcal{D} \setminus F$, since ρ_1, ρ_2 are both essentially constant on δ-balls with constant depending only on $\{a_i\}$ and $\{b_i\}$ respectively,

$$\begin{aligned}&\|f - f_Q\|_{L^q_{\rho_1}(Q)} \\ &\qquad \le C\big(n, p, q, \{a_i, b_i\}_{i=1}^l\big)|Q|^{\alpha/n - 1/p + 1/q} \bar{\rho}_1(Q)^{1/q} \bar{\rho}_2(Q)^{-1/p} \|\nabla^{\widetilde{\mathcal{D}}}_{\alpha,p} f\|_{L^p_{\rho_2}(Q)},\end{aligned}$$

where $\bar{\rho}_i(Q) = \sup_{x \in Q} \rho_i(x)$. Thus,

$$\begin{aligned}&\|f - f_Q\|_{L^q_{\rho_1}(Q)} \\ &\quad \le C\big(n, p, q, \{a_i, b_i\}_{i=1}^l\big) r(Q)^{\alpha - n/p + n/q} \prod_{i=1}^l \bar{d}_i(Q)^{a_i/q - b_i/p} \|\nabla^{\widetilde{\mathcal{D}}}_{\alpha,p} f\|_{L^p_{\rho_2}(Q)} \\ &\quad \le C\big(\tilde{c}, n, p, q, \{a_i, b_i\}_{i=1}^l, c_0\big) \varepsilon_0^b \|\nabla_{\alpha,p} f\|_{L^p_{\rho_2}(Q)}\end{aligned} \tag{3-16}$$

since

$$r(Q)^{\alpha-n/p+n/q}\prod_{i=1}^{l}\bar{d}_i(Q)^{a_i/q-b_i/p}\le C\big(\tilde{c},n,p,q,\{a_i,b_i\}_{i=1}^{l}\big)r(Q)^b$$

as $\bar{d}_i(Q)\ge r(Q)$ and assumptions (i) and (ii).

By Proposition A.4, ρ_1 is δ-doubling on $\widetilde{\Omega}_j$ with doubling constant

$$C\big(\{a_i\}_{i=1}^{l},c_0\varepsilon_0/\zeta,n\big).$$

we can conclude (by either [Chua 2009, Theorem 2.11] or [Chua and Wheeden 2008, Theorem 2.9] as John domains are Boman domains [Buckley et al. 1996]) that

$$\|f-f_{B'_j}\|_{L^q_{\rho_1}(\widetilde{\Omega}_j)}\le C\big(\tilde{c},n,p,q,\{a_i,b_i\}_{i=1}^{l},c_0\varepsilon_0/\zeta,c_0\big)\varepsilon_0^b\|\nabla^{\widetilde{\mathcal{D}}}_{\alpha,p}f\|_{L^p_{\rho_2}(\widetilde{\Omega}_j)}. \tag{3-17}$$

Using the triangle inequality and Hölder's inequality, we have

$$\|f\|_{L^q_{\rho_1}(\widetilde{\Omega}_j)}\le\rho_1(\widetilde{\Omega}_j)^{1/q}|B'_j|^{-1/p_0}\|f\|_{L^{p_0}(B'_j)}+C\varepsilon_0^b\|\nabla^{\widetilde{\mathcal{D}}}_{\alpha,p}f\|_{L^p_{\rho_2}(\widetilde{\Omega}_j)}=I+II.$$

Using the fact that ρ_0 is essentially constant on B'_j, we have

$$\begin{aligned}I&\le C\big(p_0,\{\gamma_i\}_{i=1}^{l},\tilde{c}\big)\rho_1(\widetilde{\Omega}_j)^{1/q}|B'_j|^{-1/p_0}\prod_{i=1}^{l}\bar{d}_i(B'_j)^{-\gamma_i/p_0}\|f\|_{L^{p_0}(B'_j)}\\&\le C\big(p_0,q,\tilde{c},\{a_i,\gamma_i\}_{i=1}^{l}\big)|B'_j|^{1/q-1/p_0}\prod_{i=1}^{l}\bar{d}_i(B'_j)^{a_i/q-\gamma_i/p_0}\|f\|_{L^{p_0}_{\rho_0}(B'_j)}\\&\le C\big(\tilde{c},p_0,q,\{a_i,\gamma_i\}_{i=1}^{l},c_0\big)\varepsilon_0^{1/q-1/p_0+\sum(a_i/q-\gamma_i/p_0)}\|f\|_{L^{p_0}_{\rho_0}(B'_j)}\\&=C\big(\tilde{c},p_0,q,\{a_i,\gamma_i\}_{i=1}^{l},c_0\big)\varepsilon_0^{a}\|f\|_{L^{p_0}_{\rho_0}(B'_j)}\end{aligned}$$

as $\operatorname{diam}(\widetilde{\Omega}_j)\sim\varepsilon_0$ (with constant c_0) and $\sum a_i/q\le\sum\gamma_i/p_0$. Since $1\le p,p_0\le q$, we have

$$\begin{aligned}\|f\|_{L^q_{\rho_1}(\mathcal{D})}&\le\Big(\sum_j\|f\|^q_{L^q_{\rho_1}(\widetilde{\Omega}_j)}\Big)^{1/q}\\&\le C\varepsilon_0^a\Big(\sum_j\|f\|^q_{L^{p_0}_{\rho_0}(B'_j)}\Big)^{1/q}+C\varepsilon_0^b\Big(\sum_j\|\nabla^{\widetilde{\mathcal{D}}}_{\alpha,p}f\|^q_{L^p_{\rho_2}(\Omega_j)}\Big)^{1/q}\\&\le C\varepsilon_0^a\Big(\sum_j\|f\|^{p_0}_{L^{p_0}_{\rho_0}(B'_j)}\Big)^{1/p_0}+C\varepsilon_0^b\Big(\sum_j\|\nabla^{\widetilde{\mathcal{D}}}_{\alpha,p}f\|^p_{L^p_{\rho_2}(\Omega_j)}\Big)^{1/p}\\&\le C\varepsilon_0^aM^{1/p_0}\|f\|_{L^{p_0}_{\rho_0}(\mathcal{D})}+C\varepsilon_0^bM^{1/p}\|\nabla^{\widetilde{\mathcal{D}}}_{\alpha,p}f\|_{L^p_{\rho_2}(\mathcal{D})}.\end{aligned}$$

Hence we have (1-55) when $0 < \alpha < 1$. If $\alpha = 1$, we use density of $\mathrm{Lip}_{\mathrm{loc}}(\widetilde{\mathcal{D}})$ in $L^{p_0}_{\rho_0}(\widetilde{\mathcal{D}}) \cap E^p_{\rho_2}(\widetilde{\mathcal{D}})$ which contains $L^{p_0}_{\rho_0}(\mathcal{D}) \cap E^p_{\rho_2}(\mathcal{D})$; see Remark 2.13(4).

Next, if we have strict inequalities in both conditions (i) and (ii), we can find $\tilde{q} > q$ and $\alpha_1 > a_1$ such that both conditions (i) and (ii) hold with q being replaced by $\tilde{q}$ and a_1 being replaced by α_1. We define

$$\rho_1(x) = \prod_{i=1}^{l} |x - z_i|^{\alpha_i},$$

where $\alpha_i = a_i$ for $i = 2, \ldots, l$ and α_1 is chosen above. Then (1-55) holds with either q being replaced by $\tilde{q}$ or ρ_1 being replaced by ρ_1. In case $\rho_1(\mathcal{D}) < \infty$, the fact about compact embedding will follow from Theorem A.3. Next, in case $\rho_1(\mathcal{D}) = \infty$, clearly $\mathcal{D}$ is unbounded. Suppose $\{f_i\}$ is a bounded sequence of functions in $L^{p_0}_{\rho_0}(\mathcal{D})$ and $E^p_{\rho_2}(\mathcal{D})$ (or $\widehat{W}^{\alpha,p}_{\rho_2}(\mathcal{D})$). We will show that it has a subsequence that is Cauchy in $L^q_{\rho_1}(\mathcal{D})$. First, as (1-55) holds with either q being replaced by $\tilde{q}$ or ρ_1 being replaced by ρ_1, we know the sequence is also bounded in $L^q_{\rho_1}(\mathcal{D})$ and $L^{\tilde{q}}_{\rho_1}(\mathcal{D})$. Thus it has a weakly convergent subsequence (still denoted by $\{f_i\}$) in $L^q_{\rho_1}(\mathcal{D})$ and $\|f_i\|_{L^q_{\rho_1}(\mathcal{D})} \leq A$. Similar to the proof of Theorem 1.8, given any $\eta > 0$, we define

$$\mathcal{D}_\eta = \left\{x \in \mathcal{D} : |x - z_1| < \eta^{\tilde{q}/(a_1 - \alpha_1)}\right\}$$

and $\mathcal{D}'_\eta = \mathcal{D} \setminus \mathcal{D}_\eta$. As the rest of the proof is almost identical to that of Theorem 1.8, we will not repeat it here.

Finally, if in particular $\mathcal{D} \in J(c, \varepsilon_0)$ (see [Chua 2009]), we have (1-55) for all $0 < \varepsilon < \varepsilon_0$ with ε being replaced by ε.

Remark 3.2. Suppose μ, w are Borel measures such that μ is δ-doubling on $\mathcal{D}$ with $\mathcal{D}$ given in Theorem 1.14. Checking through the proof above, we see that

$$\|f\|_{L^q_{\mu\rho_1}(\mathcal{D})} \leq C\big(M^{1/p_0}\varepsilon_0^{-a}\|f\|_{L^{p_0}_{\mu\rho_0}(\mathcal{D})} + M^{1/p}\varepsilon_0^{b}\|g\|_{L^p_{w\rho_2}(\mathcal{D})}\big) \tag{3-18}$$

provided f and g are measurable functions on $\mathcal{D}$ such that

$$\|f - f_{Q,\mu}\|_{L^q_\mu(Q)} \leq Cr(Q)^\beta \|g\|_{L^p_w(Q)} \quad \text{for all } \delta\text{-balls } Q \text{ of } \mathcal{D} \tag{3-19}$$

and condition (i) in Theorem 1.14 holds with α being replaced by β ($\beta \geq 0$). See also Theorem 1.8.

Appendix A.

For convenience, we state [Chua and Wheeden 2008, Theorem 1.2] here for easy reference.

Theorem A.1. *Let σ and μ be measures on a σ-algebra Σ of subsets of X. Let Ω be a measurable subset of X and f a fixed measurable function which satisfies* *the following assumptions for some constants* $0 < p_0, q < \infty,\ 0 < \theta < 1,\ C_\sigma \ge 1,$ $0 < \theta_1 < \theta_2 < 1,\ 0 < A_1, A_2 < \infty$ *and* $\wp \ge 1$:

(1) *For each* $x \in \Omega$, *there is a sequence of measurable sets* $\{Q_i^x\}_{i=1}^\infty$, *depending on* x, *and a fixed set* $B' \subset X$ *such that* $Q_1^x = B'$,

$$0 < \sigma(Q_i^x \cup Q_{i+1}^x) \le C_\sigma \sigma(Q_i^x \cap Q_{i+1}^x) < \infty, \quad i = 1, 2, \dots, \tag{A-1}$$

and

$$\left(\frac{1}{\sigma(Q_i^x)} \int_{Q_i^x} |f - C(f, Q_i^x)|^{p_0} \, d\sigma \right)^{1/p_0} \le a_*(Q_i^x), \tag{A-2}$$

where $\{C(f, Q_i^x)\}$ *is a sequence of constants that converges to* $f(x)$ *and* $\{a_*(Q_i^x)\}$ *is a sequence of nonnegative numbers.*

(2) *For each* $x \in \Omega$, *there is a sequence* $\{B_j^x\}_{j=1}^\infty$ *of measurable sets and a sequence* $\{\mu^*(B_j^x)\}$ *of positive numbers such that*

$$\mu(\Omega) \le \wp \mu^*(B_1^x) \quad \text{and} \quad A_1 \theta_1^k \le \frac{\mu^*(B_{j+k}^x)}{\mu^*(B_j^x)} \le A_2 \theta_2^k, \quad j, k \in \mathbb{N}. \tag{A-3}$$

(3) *Let* $\mathfrak{F} = \{B_j^x\}_{x \in \Omega, j \in \mathbb{N}}$. *Assume for any* $B_j^x \in \mathfrak{F}$, *there is* $\mathcal{C}(B_j^x) \subset \{Q_l^x\}_{l \in \mathbb{N}}$ *such that for each* $x \in \Omega$, $\bigcup_{j \in \mathbb{N}} \mathcal{C}(B_j^x) = \{Q_l^x\}_{l \in \mathbb{N}}$ *and* $\mathcal{C}(B_i^x) \cap \mathcal{C}(B_j^x) = \varnothing$ *when* $i \ne j$. *Further, for any countable subcollection* I *of pairwise disjoint sets* $\{B_\alpha\}$ *in* $\mathfrak{F}$, *let*

$$A(B_\alpha) = \sum_{Q \in \mathcal{C}(B_\alpha)} a_*(Q)$$

and assume that

$$\sum_{B_\alpha \in I} \big(A(B_\alpha)^q \mu^*(B_\alpha) \big)^\theta \le (C_0^q \mu(\Omega))^\theta. \tag{A-4}$$

(4) *Suppose the collection* $\mathfrak{F}$ *is a cover of Vitali-type of subsets of* Ω *with respect to* (μ, μ^*), *i.e., given any measurable set* $E \subset \Omega$ *and a collection* $\mathcal{B}_E = \{B_{i(x)}^x : x \in E\}$, *there is a countable pairwise disjoint collection* $\mathcal{B}'_E \subset \mathcal{B}_E$ *such that*

$$\mu(E) \le V_\mu \sum_{B_\alpha \in \mathcal{B}'_E} \mu^*(B_\alpha), \quad V_\mu \ge 1.$$

Then

$$\sup_{t > 0} t \mu\{x \in \Omega : |f(x) - f_{B'}| > t\}^{1/q} \le C C_0 [\wp V_\mu \, \mu(\Omega)]^{1/q}, \tag{A-5}$$

where C *depends on* $C_\sigma, p_0, q, A_1, A_2, \theta, \theta_1$ *and* θ_2.

We will now state a general theorem that gives a necessary condition for precompact subsets of L^p spaces.

Theorem A.2 [Chua et al. 2013, Theorem 1.2]. *Let w be a finite measure on a σ-algebra Σ of subsets of a set Ω, with $\Omega \in \Sigma$. Let $1 \le p < \infty$, $1 < N \le \infty$ and $\mathcal{P}$ be a bounded subset of $L^N_\mu(\Omega)$. Suppose there is a positive constant C so that for every $\varepsilon > 0$, there are a finite number of sets $E_\ell \in \Sigma$ with*

(i) $$\mu\big(\Omega \setminus \textstyle\bigcup_\ell E_\ell\big) < \varepsilon \quad \textit{and} \quad \mu(E_\ell) > 0;$$

(ii) *for every $f \in \mathcal{P}$,*

$$\sum_\ell \|f - f_{E_\ell,\mu}\|^p_{L^p_\mu(E_\ell)} \le C\varepsilon^p \quad \textit{where } f_{E_\ell,\mu} = \int_{E_\ell} f \, d\mu/\mu(E_\ell). \tag{A-6}$$

Then for every sequence $\{f_k\} \subset \mathcal{P}$, there is a single subsequence $\{f_{k_i}\}$ and a function $f \in L^N_\mu(\Omega)$ such that $f_{k_i} \to f$ pointwise μ-a.e. in Ω and in $L^{\tilde q}_\mu(\Omega)$ norm for $1 \le \tilde q < N$.

Next, we state a useful special case of the above on Euclidean spaces. It is an extension of [Chua et al. 2013, Theorem 2.1]. Here we include the case of fractional derivatives. As almost the same proof as in [Chua et al. 2013] will give us the theorem, we shall not prove it.

Theorem A.3 [Chua et al. 2013, Theorem 2.1]. *Let $\widetilde\Omega \subset \Omega$ be both open sets in $\mathbb{R}^n$. Let μ, w be Borel measures on Ω with $\mu(\widetilde\Omega) = \mu(\Omega) < \infty$. Let $1 \le p < \infty$, $0 < \alpha \le 1$, $1 < N \le \infty$, $\tau_0 \ge 1$ and $\mathfrak{S} \subset L^N_\mu(\Omega) \cap E^p_w(\Omega)$ or $L^N_\mu(\Omega) \cap \widehat{W}^{\alpha,p}_w(\Omega)$, and suppose that for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that*

$$\|f - f_{B,\mu}\|_{L^p_\mu(B)} \le \varepsilon \|\nabla_{\alpha,p} f\|_{L^p_w(\tau_0 B)} \quad \textit{for all } f \in \mathfrak{S} \tag{A-7}$$

and all Euclidean balls B with $r(B) < \delta_\varepsilon$ and $2\tau_0 B \subset \widetilde\Omega$. Then for any sequence $\{f_k\} \subset \mathfrak{S}$ that is bounded in $L^N_\mu(\Omega) \cap E^p_w(\Omega)$ or $L^N_\mu(\Omega) \cap \widehat{W}^{\alpha,p}_w(\Omega)$, there is a subsequence $\{f_{k_i}\}$ and a function $f \in L^N_\mu(\Omega)$ such that $f_{k_i} \to f$ pointwise μ-a.e. in Ω and in $L^{\tilde q}_\mu(\Omega)$ norm for $1 \le \tilde q < N$.

Finally, note that ρ_1 in Theorem 1.14 is δ-doubling with doubling constant independent of $\widetilde\Omega_j$. Indeed, we have the following more general result.

Proposition A.4. *Let $\{S_i\}_{i=1}^l$, $l \in \mathbb{N}$ be such that each S_i is a set of finite points in $\mathbb{R}^n$ and $S_i \cap S_j = \varnothing$ if $i \ne j$. Suppose Ω is an open set with* $\operatorname{diam}(\Omega) \le d$, *$a_i > -n$ for all i and $z \in \Omega^c$ for all $z \in \bigcup_{i=1}^l S_i$. Then the weight $\prod_{i=1}^l d(x, S_i)^{a_i}$ is δ-doubling on Ω with doubling constant depends only on n, $\{a_i\}_{i=1}^l$, and d/ζ, where $\zeta = \min\big\{|z-y| : z \ne y, z, y \in \bigcup_{i\in I^-} S_i = S'\big\}$, $I^- = \{i : a_i < 0\}$* (independent of d when S' has ≤ 1 point).

Proof. The result is easy when S' has ≤ 1 point; recall that $|x|^\alpha$ is doubling on $\mathbb{R}^n$ for $\alpha > -n$ and see Example 1.3(ii). Moreover, again by Example 1.3(ii), we only need to show that $\prod_{i\in I^-} d(x, S_i)^{a_i}$ is δ-doubling on Ω. Thus, we will only show that $\rho(x) = \prod_{i \in I^-} d(x, S_i)^{a_i}$ induces a δ-doubling measure on Ω. For convenience, we will let

$$\bar{d}_i(B) = \sup_{x \in B} d(x, S_i).$$

Let $\beta = \sum_{i \in I^-} a_i$. Given any ball B with $2B \subset \Omega$, clearly (as $a_i < 0$ for all i)

$$\rho(B) \geq C(n, \{a_i\}) \Big(\prod_{i \in I^-} \bar{d}_i(B)^{a_i} \Big) r(B)^n \geq C(n, \{a_i\}) d^\beta r(B)^n. \tag{A-8}$$

Now, let $\widetilde{B}$ be a ball with the same center as B and $r(\widetilde{B}) \geq 2r(B)$. Since

$$\widetilde{B} \subset \Big(\bigcup_{z \in S'} (B(z, \zeta/2) \cap \widetilde{B}) \Big) \cup \{x \in \widetilde{B} : |x - z| \geq \zeta/2 \ \forall z \in S'\}.$$

For the first term note that the number of such balls $B(z, \zeta/2)$ that intersect $\widetilde{B}$ is less than $C(n) \max\{1, (4r(\widetilde{B})/\zeta)^n)\}$. Now suppose $B(z, \zeta/2)$ intersects $\widetilde{B}$, $z \in S_1$. We see that as $-n < a_i < 0$,

$$\begin{aligned} \rho(B(z, \zeta/2) \cap \widetilde{B}) &\leq C(n, \{a_i\}) (\zeta/2)^{\beta - a_1} \int_{B(z, \zeta/2) \cap \widetilde{B}} |x - z|^{a_1} \, dx \\ &\leq C(n, \{a_i\}) \zeta^{\beta - a_1} \min\{r(\widetilde{B})^{n + a_1}, (\zeta/2)^{n + a_1}\}. \end{aligned}$$

Hence if $r(\widetilde{B}) \geq \zeta/4$, we have

$$\begin{aligned} \rho(\widetilde{B}) &\leq \sum_{z \in S'} \rho(B(z, \zeta/2) \cap \widetilde{B}) + \rho\{x \in \widetilde{B} : |x - z| \geq \zeta/2 \ \forall z \in S'\} \\ &\leq C(n, \{a_i\}) \zeta^\beta (4r(\widetilde{B})/\zeta)^n) \zeta^n + C(n, \{a_i\}) \zeta^\beta r(\widetilde{B})^n \\ &\leq C(n, \{a_i\}) \zeta^\beta r(\widetilde{B})^n. \end{aligned}$$

On the other hand, if $r(\widetilde{B}) \leq \zeta/4$, then there is at most one $z_1 \in S'$ with $d(z_1, \widetilde{B}) < \zeta/4$. For simplicity, let us assume $z_1 \in S_1$. We have

$$\begin{aligned} \rho(\widetilde{B}) &\leq \rho(B(z_1, \zeta/2) \cap \widetilde{B}) + \rho\{x \in \widetilde{B} : |x - z| \geq \zeta/2 \ \forall z \in S'\} \\ &\leq C(n, \{a_i\}) \zeta^{\beta - a_1} r(\widetilde{B})^{n + a_1} + C(n, \{a_i\}) \zeta^\beta r(\widetilde{B})^n \\ &\leq C(n, \{a_i\}) \zeta^{\beta - a_1} r(\widetilde{B})^{n + a_1}. \end{aligned}$$

Moreover, if $d(z, \widetilde{B}) \geq \zeta/4$ for all $z \in S'$, we have $\rho(\widetilde{B}) \leq C(n, \{a_i\}) \zeta^\beta r(\widetilde{B})^n$. It is now easy to see that

$$\rho(\widetilde{B})/\rho(B) \leq C(\zeta, n, \{a_i\}_{i \in I'}) \max_i \{(r(\widetilde{B})/r(B))^{n + a_i}, (\zeta/d)^{\beta - a_i}\}.$$

In the above, we have assumed that the total number of points in $\bigcup S_i$ is more than 1. If there is only one point z, it is well-known that the weight $|x - z|^a$ induces a measure that is doubling on $\mathbb{R}^n$ if $a > -n$. □

Appendix B.

Proof of Proposition 2.6. In this proof, we will only assume the following condition:

There is a fixed "center" $x' \in \Omega$ such that for any $x \neq x'$ in Ω, there exists $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = x,\ \gamma(1) = x'$ and γ is "continuous," i.e.,

(B-1) for all $\varepsilon > 0$ and $t_0 \in [0, l]$, there exists $\delta > 0$ such that
$$d(\gamma(t), \gamma(t_0)) < \varepsilon \quad \text{when } |t - t_0| < \delta,\ t \in [0, l];$$

and γ satisfies the weak John condition

(B-2) $$d(\gamma(t), \Omega^c) = \inf\{d(\gamma(t), y) : y \notin \Omega\} \geq c\, d(\gamma(t), x) \quad \text{for all } t.$$

Note that while (B-2) remains the same, the main paper assumes γ is Lipschitz continuous instead of (B-1).

Even though we have allowed $\tau\delta \leq 1/(2\kappa^2)$ here instead of $\tau\delta < 1/(2\kappa^2)$ in [Chua and Wheeden 2008, Proposition 2.6], the proof of part (a) and (b) are essentially the same. For (2.5), just see [Chua and Wheeden 2008, (2.6)]. However, the assumptions in (c)–(e) are more different from those of [Chua and Wheeden 2008, Proposition 2.6]; we will provide a proof here. We will now prove (c). Fix a point $x \in \Omega$, and let $\gamma(t),\ t \in [0, l]$, be a curve connecting x and x' satisfying conditions guaranteed by the weak John property (B-2). With $\delta' = \delta/\lambda^3$, we begin by constructing a special sequence of δ'-Whitney balls centered along γ. For $t \in [0, l]$, let
$$\mathcal{R}_{\gamma(t)} = B\big(\gamma(t), \delta' d(\gamma(t))\big).$$
Use (2-5) to pick $\widetilde{B}_0 \in \widetilde{W}$ containing $\mathcal{R}_{\gamma(0)}$, and let
$$t_1 = \sup\{t \in [0, 1] : \gamma(t) \in \widetilde{B}_0\}.$$
Note that $t_1 > 0$ by continuity of γ. Moreover, $\mathcal{R}_{\gamma(t_1)}$ intersects $\widetilde{B}_0$ by definition of t_1 and continuity of γ. We then use (2-5) again to choose a ball $\widetilde{B}_1 \in \widetilde{W}$ containing $\mathcal{R}_{\gamma(t_1)}$. Then clearly $\widetilde{B}_0$ intersects $\widetilde{B}_1$. If $t_1 = l$, we stop the construction process. If $t_1 < l$, we define
$$t_2 = \sup\{t \in [t_1, l] : \gamma(t) \in \widetilde{B}_1\}$$
and choose $\widetilde{B}_2 \in \widetilde{W}$ containing $\mathcal{R}_{\gamma(t_2)}$. Again, $t_1 < t_2 \leq l$ and $\widetilde{B}_1 \cap \widetilde{B}_2 \neq \varnothing$. In general, if $0 = t_0 < t_1 < \cdots < t_k$ and $\widetilde{B}_0, \widetilde{B}_1, \ldots, \widetilde{B}_k$ with $\widetilde{B}_i \cap \widetilde{B}_{i+1} \neq \varnothing$ have been constructed and if $t_k < l$, we continue by defining

(B-3) $$t_{k+1} = \sup\{t \in [t_k, l] : \gamma(t) \in \widetilde{B}_k\}$$

and using (2-5) to pick $\widetilde{B}_{k+1} \in \widetilde{W}$ containing $\mathcal{R}_{\gamma(t_{k+1})}$. As before, we have $t_k < t_{k+1} \le l$ and $\widetilde{B}_k \cap \widetilde{B}_{k+1} \ne \varnothing$. We stop the construction if $t_{k+1} = l$.

Let us show that the process must end after a finite number of steps, i.e., that there is a positive integer $L = L_x$ such that $t_L = l$. To see this, note that since γ is continuous, taking $\varepsilon = \min\{c\delta' d(\gamma(t_1), x), \delta' d(x)\}$, we can find $\eta > 0$ such that

$$d(\gamma(s_1), \gamma(s_2)) < \varepsilon \quad \text{if } |s_1 - s_2| < \eta \text{ and } s_1, s_2 \in [0, l]. \tag{B-4}$$

Claim: $\quad |t_k - t_{k+1}| \ge \eta \quad$ for all $k \ge 1$ such that $t_{k+1} < l$.

Note that we are done if $t_{k+1} = l$. Suppose $|t_{k+1} - t_k| < \eta$, then there exists $l \ge t' > t_{k+1}$ and $|t' - t_k| < \eta$. But by (B-4), we have $d(\gamma(t'), \gamma(t_k)) < \varepsilon$. On the other hand by (B-2)

$$\delta' d(\gamma(t_k)) \ge c\delta' d(\gamma(t_k), x) \ge c\delta' d(\gamma(t_1), x) \ge \varepsilon,$$

as $\gamma(t_k) \notin \mathcal{R}_{\gamma(0)}$ if $k > 1$ and hence $\gamma(t') \in \mathcal{R}_{\gamma(t_k)} \subset \widetilde{B}_k$ while $t' > t_{k+1}$ and it is a contradiction to (B-3). This proves the claim. It is now easy to see that $L - 1 \le l/\eta$.

For each $\widetilde{B}_i$ constructed above, let $B_i = 2\kappa \widetilde{B}_i$ just as in the proof of [Chua and Wheeden 2008, Proposition 2.6(c)], we see that except for (2-6), the first part of (c) is proved.

Let us now prove (2-6). The case when $B_0 \cap B_i \ne \varnothing$ is easy since then $B_0 \subset \lambda^4 B_i$ (see [Chua and Wheeden 2008, p. 2996]) and hence (2-6) is obvious.

Next, suppose that $B_0 \cap B_i = \varnothing$. The following is just a simple modification of [Chua and Wheeden 2008, p. 2996]. Due to the construction of B_i, there is a point $\xi \in \widetilde{B}_i \cap \gamma[0, l]$. Since $\xi \notin B_0$ and $x \in \widetilde{B}_0 = B_0/(2\kappa)$, the quasitriangle inequality gives $d(\xi, x) \ge r(B_0)/(2\kappa)$. Similarly, since $x \notin B_i$ and $\xi \in \widetilde{B}_i$, we have $d(\xi, x) \ge r(B_i)/(2\kappa)$. Hence,

$$d(\xi, x) \ge \frac{1}{2\kappa} \max\{r(B_0), r(B_i)\}.$$

We can use this to show that

$$B_0 \subset \frac{\lambda^2 d(\xi, x)}{r(B_i)} B_i.$$

In fact, if $z \in B_0$ then

$$\begin{aligned} d(z, x_{B_i}) &\le \kappa[d(z, x) + d(x, x_{B_i})] \\ &\le \kappa\big[\kappa\{d(z, x_{B_0}) + d(x, x_{B_0})\} + \kappa\{d(x_{B_i}, \xi) + d(\xi, x)\}\big] \\ &< \kappa\big[2\kappa r(B_0) + \kappa r(\widetilde{B}_i) + \kappa d(\xi, x)\big], \end{aligned}$$

and thus by the previous estimate for $d(\xi, x)$, we have

$$d(z, x_{B_i}) < (4\kappa^3 + 2\kappa^2)\, d(\xi, x) < \lambda^2 d(\xi, x)$$

as desired. To complete the proof of (2-6), we now recall from (B-2) that $d(\xi) \geq c\, d(x, \xi)$. But since $\xi \in B_i$ and B_i is a δ-ball ($\delta \leq 1/(2\kappa^2)$), triangle inequality and (a) give

$$d(\xi) \leq 2\kappa d(x_{B_i}) \leq 2\kappa \frac{\lambda^2}{\delta} r(B_i).$$

Combining estimates, we obtain $d(\xi, x) \leq (2\kappa\lambda^2/(c\delta)) r(B_i)$, so that

$$B_0 \subset \frac{\lambda^2 d(\xi, x)}{r(B_i)} B_i \subset \frac{2\kappa\lambda^4}{c\delta} B_i,$$

which proves (2-6) in all cases.

To prove the last statement in (c), we return to the δ'-Whitney balls $\{\mathcal{R}_{\gamma(t_i)}\}_{i=0}^{L}$ centered on the weak John curve γ from x to x', and define balls $\mathcal{Q}_i$ by

$$\mathcal{Q}_i = \lambda^3 \mathcal{R}_{\gamma(t_i)}.$$

Then $\mathcal{Q}_i$ has center on γ and is a δ-Whitney ball since $r(\mathcal{Q}_i) = \lambda^3 \delta' d(\gamma(t_i)) = \delta d(\gamma(t_i))$. The same argument as in the proof of [Chua and Wheeden 2008, Proposition 2.6(c)] then establishes the second part of (c).

To verify part (d), note that the hypothesis $\mathcal{Q}_i \not\subset B(x, r)$ implies there exists $z \in \mathcal{Q}_i$ such that $d(z, x) \geq r$. Let $x_i = \gamma(t_i)$ be the center of $\mathcal{Q}_i$ and $r_i = r(\mathcal{Q}_i)$. Then by the triangle inequality and the fact that $d(x_i, x) = d(\gamma(t_i), x) \leq d(\gamma(t_i))/c = r_i/(c\delta)$, we have

$$r \leq d(z, x) \leq \kappa(d(z, x_i) + d(x_i, x)) \leq \kappa \frac{c\delta + 1}{c\delta} r_i < \frac{2\kappa r_i}{c\delta}.$$

This completes the proof of (d).

To prove part (e), we will again use the estimate

$$r(\mathcal{Q}_i) = \delta d(\gamma(t_i)) \geq c\delta d(\gamma(t_i), x),$$

which follows from the weak John condition (B-2). Thus if $r(\mathcal{Q}_i) \leq 2\varepsilon$, then

$$2\varepsilon \geq c\delta d(\gamma(t_i), x) \quad \text{and hence} \quad \mathcal{Q} \subset B(x, 4\kappa\varepsilon/(c\delta)).$$

However, as there is a δ-doubling measure σ on Ω, the number of disjoint $\mathcal{Q}$ of radius between ε and 2ε is bounded with bound depending only on D_σ, κ, δ and c. This completes the proof of Proposition 2.6.

Proof of Proposition 2.9. For this proof, we will be again assuming only (B-1) instead of Lipschitz continuity.

Let x' be the central point of Ω and let $d(z, \Omega^c) = (\theta + 2)\varepsilon$. We will consider two cases:

Case (i): $x' \in B_\varepsilon(z)$. We will assume $B_\varepsilon(z) \neq \Omega$ as the case $B_\varepsilon(z) = \Omega$ follows immediately from the path property. Using the path property, we know that there exists $x'' \notin B_\varepsilon(z)$ such that $d(x'', z) = \varepsilon$. Moreover, note that

$$d(x', \Omega^c) \le d(x', z) + d(z, \Omega^c) \le (\theta+3)\varepsilon.$$

For any $x \in \Omega\backslash\{z\}$, $x \neq x''$, we will now construct a continuous path connecting x to x''. First suppose $x \in B_\varepsilon(z)$. By assumption, there exists continuous

$$\eta : [0, 1] \to B_{\theta\varepsilon}(z) \setminus B_{d(x,z)/\theta}(z).$$

Clearly $d(\eta(t), \Omega^c) > 2\varepsilon > \frac{2}{\theta} d(\eta(t), z)$ and $d(\eta(t), z) > \frac{1}{1+\theta} d(\eta(t), x)$ since

$$d(\eta(t), x) \le d(\eta(t), z) + d(x, z) \le (1+\theta) d(\eta(t), z).$$

Next, suppose $x \notin B_\varepsilon(z)$. Since $\Omega \in J'(c)$, there exists continuous $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = x$, $\gamma(l) = x'$ and

$$d(\gamma(t), \Omega^c) \ge c\, d(\gamma(t), x) \quad \text{for all } t \in [0, l]. \tag{B-5}$$

Since $\gamma(l) \in B(z, \varepsilon)$, we now define

$$t' = \inf\{t \in [0, l] : \gamma(t) \in B_\varepsilon(z)\}.$$

Note that by continuity, we know $d(\gamma(t'), z) = \varepsilon$. Since $t' < l$, by the path property, there exists a continuous $\eta : [t', l] \to B_{\theta\varepsilon}(z) \setminus B_{\varepsilon/\theta}(z)$ such that $\eta(t') = \gamma(t')$ and $\eta(l) = x''$. Note that $d(\eta(t), \Omega^c), d(\eta(t), z) \ge \varepsilon/\theta$. Since

$$\begin{aligned} d(\eta(t), x) &\le d(\eta(t), z) + d(z, x') + d(x, x') < \theta\varepsilon + \varepsilon + \frac{1}{c} d(x', \Omega^c) \\ &< \frac{2\theta+3}{c}\varepsilon, \end{aligned} \tag{B-6}$$

it is now clear that

$$d(\eta(t), \Omega^c \cup z) \ge c_0 d(\eta(t), x) \quad \text{with } c_0 = \frac{c}{(2\theta+3)\theta}.$$

Combining γ with η, we obtain a continuous curve satisfying (B-2) connecting x to x''.

Case (ii): $x' \notin B_\varepsilon(z)$. Again, there exists a continuous $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(l) = x'$ satisfies (B-5). We now consider two subcases.

Subcase (a): $\gamma[0, l] \cap B_\varepsilon(z) = \varnothing$. Then $d(\gamma(t), z) \ge \varepsilon$ for all t. Moreover,

$$d(\gamma(t), \Omega^c) \le d(\gamma(t), z) + d(z, \Omega^c) \le (\theta+3)\, d(\gamma(t), z). \tag{B-7}$$

Hence

$$d(\gamma(t), \Omega^c \cup \{z\}) \ge \frac{1}{\theta+3} d(\gamma(t), \Omega^c) \ge \frac{c}{\theta+3} d(\gamma(t), x).$$

Subcase (b): $\gamma[0, l] \cap B_\varepsilon(z) \neq \varnothing$. Similar to case (i), we will let

$$t' = \inf\{t : \gamma(t) \in B_\varepsilon(z)\}.$$

Moreover, we also let

$$t'' = \sup\{t : \gamma(t) \in B_\varepsilon(z)\}.$$

Again, there exists $\eta : [t', t''] \to B_{\theta\varepsilon}(z) \setminus B_{\varepsilon/\theta}(z)$ with $\eta(t') = \gamma(t'),\ \eta(t'') = \gamma(t'')$.

We now define

$$\tilde{\gamma} = \begin{cases} \gamma(t) & \text{for } t \in [0, t'] \cup [t'', l], \\ \eta(t) & \text{for } t \in [t', t'']. \end{cases}$$

The case $t \in [0, t'] \cup [t'', l]$ follows from (B-7) and the case $t \in [t', t'']$ follows from (B-6).

References

[Bourgain et al. 2002] J. Bourgain, H. Brezis, and P. Mironescu, "Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications", *J. Anal. Math.* **87** (2002), 77–101. MR Zbl

[Brezis and Nirenberg 1983] H. Brezis and L. Nirenberg, "Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents", *Comm. Pure Appl. Math.* **36**:4 (1983), 437–477. MR Zbl

[Brown and Hinton 1988] R. C. Brown and D. B. Hinton, "Weighted interpolation inequalities of sum and product form in $\mathbb{R}^n$", *Proc. London Math. Soc.* (3) **56**:2 (1988), 261–280. MR

[Buckley et al. 1996] S. Buckley, P. Koskela, and G. Lu, "Boman equals John", pp. 91–99 in *16th Rolf Nevanlinna Colloquium* (Joensuu, Finland, 1995), edited by I. Laine and O. Martio, de Gruyter, Berlin, 1996. MR Zbl

[Caffarelli et al. 1984] L. Caffarelli, R. Kohn, and L. Nirenberg, "First order interpolation inequalities with weights", *Compositio Math.* **53**:3 (1984), 259–275. MR Zbl

[Chanillo and Wheeden 1985] S. Chanillo and R. L. Wheeden, "Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions", *Amer. J. Math.* **107**:5 (1985), 1191–1226. MR Zbl

[Chanillo and Wheeden 1992] S. Chanillo and R. L. Wheeden, "Poincaré inequalities for a class of non-A_p weights", *Indiana Univ. Math. J.* **41**:3 (1992), 605–623. MR Zbl

[Cheeger 1999] J. Cheeger, "Differentiability of Lipschitz functions on metric measure spaces", *Geom. Funct. Anal.* **9**:3 (1999), 428–517. MR Zbl

[Chiadò Piat and Serra Cassano 1994] V. Chiadò Piat and F. Serra Cassano, "Some remarks about the density of smooth functions in weighted Sobolev spaces", *J. Convex Anal.* **1**:2 (1994), 135–142. MR Zbl

[Chua 1995] S.-K. Chua, "Weighted Sobolev interpolation inequalities on certain domains", *J. London Math. Soc.* (2) **51**:3 (1995), 532–544. MR Zbl

[Chua 2005] S.-K. Chua, "Sharp conditions for weighted Sobolev interpolation inequalities", *Forum Math.* **17**:3 (2005), 461–478. MR Zbl

[Chua 2009] S.-K. Chua, "Sobolev interpolation inequalities on generalized John domains", *Pacific J. Math.* **242**:2 (2009), 215–258. MR Zbl

[Chua 2016] S.-K. Chua, "A Variation of maximal functions on nonsymmetric quasimetric spaces and its application on fractional Poincaré and Poincaré inequalities", preprint, 2016.

[Chua 2017a] S.-K. Chua, "Boundedness and regularity of solutions of degenerate elliptic partial differential equations", *J. Differential Equations* **263**:7 (2017), 3714–3736. MR Zbl

[Chua 2017b] S.-K. Chua, "Weighted inequalities of Poincaré type on chain domains", pp. 27–48 in *Harmonic analysis, partial differential equations and applications*, edited by S. Chanillo et al., Springer, 2017. MR Zbl

[Chua and Wheeden 2008] S.-K. Chua and R. L. Wheeden, "Self-improving properties of inequalities of Poincaré type on measure spaces and applications", *J. Funct. Anal.* **255**:11 (2008), 2977–3007. MR Zbl

[Chua and Wheeden 2011] S.-K. Chua and R. L. Wheeden, "Self-improving properties of inequalities of Poincaré type on s-John domains", *Pacific J. Math.* **250**:1 (2011), 67–108. MR Zbl

[Chua and Wheeden 2015] S.-K. Chua and R. L. Wheeden, "Global Subrepresentation Formulas in Chain Domains with Irregular Boundaries", preprint, 2015.

[Chua and Wheeden 2017] S.-K. Chua and R. L. Wheeden, "Existence of weak solutions to degenerate p-Laplacian equations and integral formulas", *J. Differential Equations* **263**:12 (2017), 8186–8228. MR Zbl

[Chua et al. 2013] S.-K. Chua, S. Rodney, and R. Wheeden, "A compact embedding theorem for generalized Sobolev spaces", *Pacific J. Math.* **265**:1 (2013), 17–57. MR Zbl

[Drelichman and Durán 2008] I. Drelichman and R. G. Durán, "Improved Poincaré inequalities with weights", *J. Math. Anal. Appl.* **347**:1 (2008), 286–293. MR Zbl

[Dyda et al. 2016] B. Dyda, L. Ihnatsyeva, and A. V. Vähäkangas, "On improved fractional Sobolev–Poincaré inequalities", *Ark. Mat.* **54**:2 (2016), 437–454. MR Zbl

[Franchi et al. 1995] B. Franchi, G. Lu, and R. L. Wheeden, "Representation formulas and weighted Poincaré inequalities for Hörmander vector fields", *Ann. Inst. Fourier* (*Grenoble*) **45**:2 (1995), 577–604. MR Zbl

[Franchi et al. 2003] B. Franchi, C. Pérez, and R. L. Wheeden, "A sum operator with applications to self-improving properties of Poincaré inequalities in metric spaces", *J. Fourier Anal. Appl.* **9**:5 (2003), 511–540. MR Zbl

[Gatto and Wheeden 1989] A. E. Gatto and R. L. Wheeden, "Sobolev inequalities for products of powers", *Trans. Amer. Math. Soc.* **314**:2 (1989), 727–743. MR Zbl

[Goldshtein and Ukhlov 2009] V. Goldshtein and A. Ukhlov, "Weighted Sobolev spaces and embedding theorems", *Trans. Amer. Math. Soc.* **361**:7 (2009), 3829–3850. MR Zbl

[Gurka and Opic 1988] P. Gurka and B. Opic, "Continuous and compact imbeddings of weighted Sobolev spaces, I", *Czechoslovak Math. J.* **38**:4 (1988), 730–744. MR Zbl

[Gurka and Opic 1989] P. Gurka and B. Opic, "Continuous and compact imbeddings of weighted Sobolev spaces, II", *Czechoslovak Math. J.* **39**:1 (1989), 78–94. MR Zbl

[Gurka and Opic 1991] P. Gurka and B. Opic, "Continuous and compact imbeddings of weighted Sobolev spaces, III", *Czechoslovak Math. J.* **41**:2 (1991), 317–341. MR Zbl

[Hajłasz 1993] P. Hajłasz, "Note on Meyers–Serrin's theorem", *Exposition. Math.* **11**:4 (1993), 377–379. MR Zbl

[Hajłasz 1996] P. Hajłasz, "Sobolev spaces on an arbitrary metric space", *Potential Anal.* **5**:4 (1996), 403–415. MR Zbl

[Hajłasz and Koskela 1998] P. Hajłasz and P. Koskela, "Isoperimetric inequalities and imbedding theorems in irregular domains", *J. London Math. Soc.* (2) **58**:2 (1998), 425–450. MR Zbl

[Hajłasz and Koskela 2000] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **688**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Heinonen 2001] J. Heinonen, *Lectures on analysis on metric spaces*, Springer, 2001. MR Zbl

[Heinonen et al. 1993] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, New York, 1993. MR Zbl

[Hurri-Syrjänen 2004] R. Hurri-Syrjänen, "The Poincaré inequality and reverse doubling weights", *Canad. Math. Bull.* **47**:2 (2004), 206–214. MR Zbl

[Jerison 1986] D. Jerison, "The Poincaré inequality for vector fields satisfying Hörmander's condition", *Duke Math. J.* **53**:2 (1986), 503–523. MR Zbl

[Keith 2004] S. Keith, "A differentiable structure for metric measure spaces", *Adv. Math.* **183**:2 (2004), 271–315. MR Zbl

[Keith and Zhong 2008] S. Keith and X. Zhong, "The Poincaré inequality is an open ended condition", *Ann. of Math.* (2) **167**:2 (2008), 575–599. MR Zbl

[Kufner 1985] A. Kufner, *Weighted Sobolev spaces*, Wiley, New York, 1985. Translated from the Czech. MR Zbl

[Kufner and Opic 1984] A. Kufner and B. Opic, "How to define reasonably weighted Sobolev spaces", *Comment. Math. Univ. Carolin.* **25**:3 (1984), 537–554. MR Zbl

[Lu 1992a] G. Lu, "Existence and size estimates for the Green's functions of differential operators constructed from degenerate vector fields", *Comm. Partial Differential Equations* **17**:7-8 (1992), 1213–1251. MR Zbl

[Lu 1992b] G. Lu, "Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications", *Rev. Mat. Iberoamericana* **8**:3 (1992), 367–439. MR Zbl

[Lu 1994] G. Lu, "The sharp Poincaré inequality for free vector fields: an endpoint result", *Rev. Mat. Iberoamericana* **10**:2 (1994), 453–466. MR Zbl

[Lu 1996] G. Lu, "A note on a Poincaré type inequality for solutions to subelliptic equations", *Comm. Partial Differential Equations* **21**:1-2 (1996), 235–254. MR Zbl

[Martio and Sarvas 1979] O. Martio and J. Sarvas, "Injectivity theorems in plane and space", *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4**:2 (1979), 383–401. MR Zbl

[Mazya 2011] V. Mazya, *Sobolev spaces with applications to elliptic partial differential equations*, 2nd ed., Grundlehren der Math. Wissenschaften **342**, Springer, Heidelberg, 2011. MR Zbl

[Mazya and Shaposhnikova 2002] V. Mazya and T. Shaposhnikova, "On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces", *J. Funct. Anal.* **195**:2 (2002), 230–238. Erratum in *J. Funct. Anal.* **201**:1 (2003), 298–300. MR Zbl

[Monticelli et al. 2012] D. D. Monticelli, S. Rodney, and R. L. Wheeden, "Boundedness of weak solutions of degenerate quasilinear equations with rough coefficients", *Differential Integral Equations* **25**:1-2 (2012), 143–200. MR Zbl

[Monticelli et al. 2015] D. D. Monticelli, S. Rodney, and R. L. Wheeden, "Harnack's inequality and Hölder continuity for weak solutions of degenerate quasilinear equations with rough coefficients", *Nonlinear Anal.* **126** (2015), 69–114. MR Zbl

[Rodney 2010] S. Rodney, "A degenerate Sobolev inequality for a large open set in a homogeneous space", *Trans. Amer. Math. Soc.* **362**:2 (2010), 673–685. MR Zbl

[Saloff-Coste 2002] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series **289**, Cambridge University Press, 2002. MR Zbl

[Sawyer and Wheeden 2006] E. T. Sawyer and R. L. Wheeden, *Hölder continuity of weak solutions to subelliptic equations with rough coefficients*, Mem. Amer. Math. Soc. **847**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Sawyer and Wheeden 2010] E. T. Sawyer and R. L. Wheeden, "Degenerate Sobolev spaces and regularity of subelliptic equations", *Trans. Amer. Math. Soc.* **362**:4 (2010), 1869–1906. MR Zbl

[Turesson 2000] B. O. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, Lecture Notes in Mathematics **1736**, Springer, 2000. MR Zbl

[Väisälä 1988] J. Väisälä, "Uniform domains", *Tohoku Math. J.* (2) **40**:1 (1988), 101–118. MR Zbl

[Väisälä 1989] J. Väisälä, "Quasiconformal maps of cylindrical domains", *Acta Math.* **162**:3-4 (1989), 201–225. MR Zbl

[Xuan 2005] B. Xuan, "The solvability of quasilinear Brezis–Nirenberg-type problems with singular weights", *Nonlinear Anal.* **62**:4 (2005), 703–725. MR Zbl

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