Pacific Journal of Mathematics

RANDOM MÖBIUS GROUPS, I: RANDOM SUBGROUPS OF PSL(2, $\mathbb{R}$)

Gaven Martin and Graeme O'Brien

Volume 303 No. 2 December 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 303, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.303.669

RANDOM MÖBIUS GROUPS, I: RANDOM SUBGROUPS OF PSL(2, $\mathbb{R}$)

GAVEN MARTIN AND GRAEME O'BRIEN

We introduce a geometrically natural probability measure μ on the group PSL(2, $\mathbb{R}$), identified as the group of all Möbius transformations of the hyperbolic plane, which is mutually absolutely continuous with respect to the Haar measure. Our aim is to study topological generation and random subgroups, in particular random two-generator subgroups where the generators are selected randomly. This probability measure in effect establishes an isomorphism between random n-generator groups and collections of n random pairs of arcs on the circle. Our aim is to estimate the likelihood that such a random group topologically generates (or, conversely, is discrete). We also want to calculate the precise expectation of associated parameters, the geometry and topology, and to establish the effectiveness of tests for discreteness. We achieve an interesting mix of bounds and precise results. For instance, if $f, g \in_* \mathrm{PSL}(2,\mathbb{R})$ (that is, selected via μ), then $0.85 < \Pr\{\overline{\langle f, g\rangle} = \mathrm{PSL}(2,\mathbb{R})\} < 0.9$, thus the probability the group is discrete is at least $\frac{1}{10}$ (Theorem 8.3) and this increases to $\frac{2}{5}$ if we condition the selection to hyperbolic elements (Theorem 11.6). Further, if ζ is a primitive n-th root of unity, $n \geq 2$, and $f(z) = \zeta z$ is the elliptic of order n, and we choose $g \in_* \mathrm{PSL}(2,\mathbb{R})$ conditioned to be hyperbolic, then $\Pr\{\overline{\langle f, g\rangle} = \mathrm{PSL}(2,\mathbb{R})\} = 1 - 2/n^2$ (Theorem 12.5). We establish results such as the p.d.f. for the translation length τ_f of a random hyperbolic to be $H[\tau] = -4/\pi^2 \ \tanh\frac{\tau}{2}\ \log\tanh\frac{\tau}{4}$ (Theorem 4.9), along with related geometric invariants.

1. Introduction

This article is motivated in part by generalisations of a couple of specific problems and then explores the more general question of random subgroups of PSL(2, $\mathbb{R}$).

Research supported in part by grants from the N.Z. Marsden Fund. G. Martin was partly supported by the Hausdorff Institute during the completion of this project. Some of this work forms part of G. O'Brien's PhD thesis.
MSC2010: 22E40, 30F40.
Keywords: random discrete Möbius groups.

Here we will mean that generators are selected randomly from a probability distribution on $\mathrm{PSL}(2,\mathbb{R})$ and not the limiting random process as described in [Calegari and Walker 2015] and the references.

First, consider a well known and important result from [Kantor and Lubotzky 1990] (see also [Dixon 1969] and [Liebeck and Shalev 1995]) that shows that the probability that a pair of uniformly and randomly selected elements $u, v \in_u G$ of a classical finite group G generates tends to 1 as the order of the group tends to ∞. Here the notation $\in_u$ means randomly selected from the uniform distribution. So, for instance,

$$\mathbf{Pr}\{\langle u, v\rangle = \mathrm{PSL}(2,q) : u, v \in_u \mathrm{PSL}(2,q)\} \to 1 \quad \text{as } q \to \infty.$$

An earlier result of Auerbach [1934] shows that for a compact Lie group G, a generic pair (u, v) with respect to the product Haar measure on $G \times G$ topologically generates, that is

$$\mathbf{Pr}\{\overline{\langle u, v\rangle} = G : u, v \in_u G\} = 1.$$

There are very recent strengthenings of this result [Noskov 2018]. We ask if we can give meaning to, and answer, a similar question for a noncompact Lie group such as $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{C})$ where there can be no *invariant* probability measure.

For us, there are other questions as well. These are motivated by the increasing number of computer-supported searches of moduli spaces of discrete groups to solve problems in geometry and topology in recent times. These include the smallest volume hyperbolic manifold [Gabai et al. 2011], the noncompact manifold [Cao and Meyerhoff 2001], the orbifold (Siegel's problem) [Gehring and Martin 2009; Marshall and Martin 2012] and perhaps the biggest search of all in [Gabai et al. 2003] establishing topological rigidity. Many of these searches are based on tests for discreteness and related geometric estimates. Thus we ask how effective are elementary discreteness tests such as Jørgensen's inequality? This question can be phrased as follows: Suppose we somehow choose $u, v \in \mathrm{PSL}(2,\mathbb{C})$, what is the probability that $|\operatorname{tr}^2(u) - 4| + |\operatorname{tr}[u, v] - 2| \le 1$?

Another question is, given $\langle u, v\rangle$ discrete in $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{C})$, what is the distribution of the possible topologies of the quotient of the natural action on hyperbolic space. As an example, if we choose two "random" hyperbolic elements which generate a discrete group, then generically the quotient space is either the two-sphere with three holes, or a torus with one hole, with the latter occurring with probability $\frac{1}{3}$ and determined by whether or not the axes cross. We might also ask for the distribution of the dimension of the limit set, or shortest geodesic and so forth. We will answer some of these questions here and leave others to a sequel. For groups generated by two nilpotent elements (parabolic) of $\mathrm{PSL}(2,\mathbb{C})$, we give explicit answers in [Martin et al. 2019].

In each case there are a few problems to address: what generate means, how to measure effectiveness in a probabilistic sense, and what the "correct" probability density is. The first problem is straightforward.

Definition. Let G be a topological group. We say $g_1, g_2, \dots, g_n$ *topologically generate* if

$$\overline{\langle g_1, g_2, \dots, g_n \rangle} = G. \tag{1.1}$$

Notice that if G is a Lie group, then the left-hand side is a closed Lie subgroup, and so a Lie group itself. In $\mathrm{PSL}(2, \mathbb{R})$ these closed Lie subgroups can only be finite, discrete, $\mathbb{R}$ or $\mathbb{S}$. While in $\mathrm{PSL}(2, \mathbb{C})$ we have the same description if we add $\mathrm{PSL}(2, \mathbb{R})$. As an example, the n-torus $T^n = \overline{\langle a \rangle}$ for a generic $a \in T^n$ (with respect to the usual volume form). Advancing Auerbach's result, Noskov [2018] proved that for any compact simple Lie group G and any $g \in G \setminus \{I\}$ the subset of $\{h \in G : \overline{\langle g, h \rangle} = G\}$ is nonempty and Zariski open in G.

Now we must discuss probability measures. In the case of locally compact topological groups (which we will not stray from) there is always an invariant Haar measure. However, there can be no invariant probability measure unless the group is compact. Thus for $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$, our first significant problem is to define a geometrically natural probability measure on these spaces. Desirable properties should be that it is mutually absolutely continuous with respect to Haar measure, and invariant under the maximal compact subgroup. This latter property is useful from a computational point of view when using the Iwasawa decomposition. Another desirable property would be that the measure is "geometrically natural" and, finally, that we are able to be compute with it. Unfortunately this will also mean that parabolic elements and elements with a specific finite order occur with probability zero since this is the case for Haar measure. We deal with these cases by conditioning the selection.

In this paper we will focus on the case of $\mathrm{PSL}(2, \mathbb{R})$. This group acts as Möbius transformations (that is, linear fractional transformations or isometries) of hyperbolic 2-space. For us a random group will mean a finitely generated subgroup of $\mathrm{PSL}(2, \mathbb{R})$ where the generators are selected from our probability measure. Our ultimate aim is to study random subgroups of $\mathrm{PSL}(2, \mathbb{C})$ viewed as isometries of hyperbolic 3-space, but the two-dimensional case is quite distinct in many ways — for instance, since the trace is a continuous function to $\mathbb{R}$, the set of precompact cyclic subgroups (the elliptic elements) has nonempty interior, and therefore will have positive measure in any reasonable imposed measure (for our measure, the set of elliptics and the set of hyperbolics are both of measure equal to $\frac{1}{2}$). For $\mathrm{PSL}(2, \mathbb{C})$ this should not be the case.

However, the motivation for the probability measure we chose is similar in both cases. We seek something "geometrically natural" and with which we can compute. We should expect that almost surely (that is, with probability 1) a finitely generated subgroup of the Möbius group is free.

Let us give a couple of examples of the sorts of results we present. We write $f \in_* \mathrm{PSL}(2, \mathbb{R})$ to mean that f is a random variable in $\mathrm{PSL}(2, \mathbb{R})$ selected using the probability density described in Section 2A, although in what follows we chose a Möbius representation for $\mathrm{PSL}(2, \mathbb{R})$.

Theorem 1.2. (1) *Suppose $f, g \in_* \mathrm{PSL}(2, \mathbb{R})$. Then*

$$0.85 < \mathbf{Pr}\{\overline{\langle f, g\rangle} = \mathrm{PSL}(2, \mathbb{R})\} < 0.9.$$

(2) *Suppose $f, g \in_* \mathrm{PSL}(2, \mathbb{R})$ are hyperbolic. Then*

$$\tfrac{2}{5} < \mathbf{Pr}\{\overline{\langle f, g\rangle} = \mathrm{PSL}(2, \mathbb{R})\} < \tfrac{3}{5}.$$

(3) *For $f \in_* \mathrm{PSL}(2, \mathbb{R})$ hyperbolic, the p.d.f. for the translation length $\tau(f)$ is*

$$H[\tau] = -\frac{4}{\pi^2}\,\tanh\frac{\tau}{2}\,\log\tanh\frac{\tau}{4}.$$

We also consider such things as the probability distribution of the trace of f, the probability that the axes of randomly chosen hyperbolic generators cross and so forth. Finally we look at some specific cases where the calculations simplify a bit. For instance we prove the following theorem.

Theorem 1.3. *Let $f(z) = \zeta^n z$, n be an integer at least 2, and let $g \in_* \mathrm{PSL}(2, \mathbb{R})$ be hyperbolic. Then*

$$\mathbf{Pr}\bigl\{\overline{\langle f, g\rangle} = \mathrm{PSL}(2, \mathbb{R})\bigr\} = 1 - \frac{2}{n^2}.$$

To study these questions, our main idea is to set up a topological isomorphism between n pairs of random arcs on the circle and n-generator Möbius groups. We then determine the statistics of a random cyclic group completely and then consider pairs of generators. Unfortunately we are unable to determine the statistics of commutators of pairs of generators. This is an important challenge with topological consequences and which we only partially resolve.

2. Random Möbius groups

We introduce specific definitions in the context of Möbius groups of the hyperbolic plane, identified as the unit disk with the hyperbolic metric. These will naturally motivate more general definitions for the case of Möbius groups of hyperbolic 3-space in later work.

If $A \in \mathrm{PSL}(2,\mathbb{C})$ has the form

$$A = \pm \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix}, \quad |a|^2 - |c|^2 = 1, \tag{2.1}$$

then the associated linear fractional transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$f(z) = \frac{az+c}{\bar{c}z+\bar{a}} \tag{2.2}$$

preserves the unit circle since

$$\left|\frac{az+c}{\bar{c}z+\bar{a}}\right| = |\bar{z}| \left|\frac{az+c}{\bar{a}\bar{z}+\bar{c}|z|^2}\right|,$$

with the implication that $|z| = 1$ implies $|f(z)| = 1$.

The rotation subgroup $\boldsymbol{K}$ of the disk, $z \mapsto \zeta^2 z$, $|\zeta| = 1$, and the nilpotent or parabolic subgroup $\boldsymbol{P}$ (conjugate to the translations) have the respective representations

$$\begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix}, \quad |\zeta| = 1, \quad \begin{pmatrix} 1+it & t \\ t & 1-it \end{pmatrix}, \quad t \in \mathbb{R}.$$

The group of all matrices satisfying (2.1) will be denoted $\mathcal{F}$. It is not difficult to construct an algebraic isomorphism $\mathcal{F} \equiv \mathrm{PSL}(2,\mathbb{R}) \equiv \mathrm{Isom}^+(\mathbb{H}^2)$, the isometry group of two-dimensional hyperbolic space (see [Beardon 1983]) and we will often abuse notation and use A from (2.1) and the mapping f from (2.2) interchangeably. Despite some efforts to directly use $\mathrm{PSL}(2,\mathbb{R})$, we feel the approach we take is geometrically more natural by working in $\mathcal{F}$. In particular, our measures are obviously invariant under the action of the compact group $\boldsymbol{K}$. We also seek distributions from which we can make explicit calculations and which are geometrically natural (see, in particular, Lemma 4.2).

2A. ***The probability distribution.*** Our probability space is $(\mathcal{F}, \mu_{\mathcal{F}})$, the space of matrices with the following imposed distributions of the entries of an element of $\mathcal{F}$.

(i) $\zeta = a/|a|$ and $\eta = c/|c|$ are chosen uniformly in the circle $\mathbb{S}$, with arclength measure.

(ii) $t = |a| \geq 1$ is chosen so that

$$2\arcsin(1/t) \in [0, \pi]$$

is uniformly distributed.

Notice that the product $\zeta\eta$ is uniformly distributed on the circle as a simple consequence of the rotational invariance of arclength measure. Further, this measure is equivalent to the uniform probability measure $\arg(a) \in [0, 2\pi]$. It is thus clear that this selection process is invariant under the rotation subgroup of the circle.

Next, if θ is uniformly distributed in $[0,\pi]$, then the probability distribution function for $\sin\theta$ is $\frac{1}{\pi}(1/\sqrt{1-y^2})$ for $y\in[-1,1]$. Since $t\mapsto 1/t$, for $t>0$, is strictly decreasing, we can use the change of variables formula for distribution functions to deduce the p.d.f. for $|a|$.

Lemma 2.3. *The random variable* $|a|\in[1,\infty)$ *has the p.d.f.*

$$F_{|a|}(x)=\frac{2}{\pi}\,\frac{1}{x\sqrt{x^2-1}}.$$

Next notice that the equation $1+|c|^2=|a|^2$ tells us that $\arctan(1/|c|)$ is also uniformly distributed in $[0,\pi]$.

Thus we require that the matrix entries a and c have arguments $\arg(a)$ and $\arg(c)$ uniformly distributed on $\mathbb{R}$ mod 2π. We write this as $\arg(a)\in_u[0,2\pi]_{\mathbb{R}}$ and $\arg(c)\in_u[0,2\pi]_{\mathbb{R}}$. We illustrate with a lemma.

Lemma 2.4. *If* $\arg(a),\arg(b)\in_u[0,2\pi]_{\mathbb{R}}$, *then* $\arg(ab),\arg(a/b)\in_u[0,2\pi]_{\mathbb{R}}$. *Hence* $\arg(a^k)=k\arg(a)\in_u[0,2\pi]_{\mathbb{R}}$ *for* $k\in\mathbb{Z}$.

Proof. The usual method of calculating probability distributions for combinations of random variables via characteristic functions shows that if θ,η are selected from a uniformly distributed probability measure on $[0,2\pi]$, then the p.d.f. for $\theta+\eta\in[0,4\pi]$ is given by

$$g(\zeta)=\begin{cases}\zeta/(8\pi^2), & 0\le\zeta<2\pi,\\ (4\pi-\zeta)/(8\pi^2), & 2\pi\le\zeta\le4\pi.\end{cases}\tag{2.5}$$

We reduce mod 2π and observe

$$\frac{\zeta}{8\pi^2}+\frac{4\pi-\zeta}{8\pi^2}=\frac{1}{2\pi}$$

and this gives us once again the uniform probability density on $[0,2\pi]$. The remaining results are easy consequences. □

In what follows we will also need to consider variables supported in $[0,\pi]$ or smaller subintervals and as above we will write this as $a\in_u[0,\pi]_{\mathbb{R}}$ and so forth. Most often we will also drop the subscript $\mathbb{R}$.

In a moment we will calculate some distributions naturally associated with Möbius transformations such as traces and translation lengths. Every Möbius transformation of the unit disk $\mathbb{D}$ can be written in the form

$$z\mapsto\zeta^2\,\frac{z-w}{1-\bar{w}z},\qquad |\zeta|=1,\quad w\in\mathbb{D}.\tag{2.6}$$

Thus one could consider another approach by choosing distributions for $\zeta\in\mathbb{S}$ and $w\in\mathbb{D}$. It seems clear one would want ζ uniformly distributed in $\mathbb{S}$. The real question is by what probability measure should w be chosen on $\mathbb{D}$? If w is chosen rotationally invariant, then the choice boils down to probability measures on radii. The choices we have made turn out as follows. The matrix representation of (2.6) in the form (2.1) is

$$\zeta^2\,\frac{z-w}{1-\bar{w}z} \leftrightarrow \begin{pmatrix} \frac{\zeta}{\sqrt{1-|w|^2}} & -\frac{\zeta w}{\sqrt{1-|w|^2}} \\ -\frac{\bar{\zeta}\bar{w}}{\sqrt{1-|w|^2}} & \frac{\bar{\zeta}}{\sqrt{1-|w|^2}} \end{pmatrix}.$$

Hence ζ and $w/|w|$ will be uniformly distributed in $\mathbb{S}$. Then, $|w|<1$ necessarily and

$$\arccos(|w|) = \arcsin(\sqrt{1-|w|^2}) \in [0,\pi/2]$$

is uniformly distributed and we find $|w| = |f(0)|$ has the p.d.f. $2/(\pi\sqrt{1-y^2})$, $y\in[0,1]$).

Corollary 2.7. *Let $f\in\mathcal{F}$ be a random Möbius transformation. Then the p.d.f. for $y=|f(0)|$ is $2/(\pi\sqrt{1-y^2})$. The expected value of $|f(0)|$ is*

$$E[\|f(0)|] = \frac{2}{\pi}\int_0^1 \frac{y}{\sqrt{1-y^2}}\,dy = \frac{2}{\pi} = 0.63662\ldots$$

The hyperbolic distance here between 0 and $E[|f(0)|]$ is

$$\log\frac{1+|f(0)|}{1-|f(0)|} = \log\frac{\pi+2}{\pi-2} = 1.50494\ldots.$$

3. Fixed points

The fixed points of a random $f\in\mathcal{F}$ are solutions to the same quadratic equation and one should therefore expect some correlation. From (2.2) we see the fixed points are the solutions to $az+c = z(\bar{c}z+\bar{a})$. That is

$$z_\pm = \frac{1}{\bar{c}}\big(i\Im m(a) \pm \sqrt{\Re e(a)^2-1}\big), \quad |a|^2 = 1+|c|^2. \tag{3.1}$$

We consider two cases. Further we will soon establish that $\Pr\{|\Re e(a)|\le 1\} = \frac{1}{2}$, so each case occurs with equal probability.

Case 1: (f elliptic or parabolic). $|\Re e(a)|\le 1$ and so $\arg(z_\pm) = \frac{\pi}{2}+\arg(c)$. Thus the argument of both fixed points is the same and that angle is uniformly distributed in $[0,\pi]$.

Case 2: (f hyperbolic). $\Re e(a)>1$ and $|z_\pm|=1$. We calculate the derivative

$$|f'(z_\pm)| = \frac{1}{|\bar{c}z_\pm+\bar{a}|^2} = \frac{1}{|\Re e(a)\pm\sqrt{\Re e(a)^2-1}|^2}.$$

Hence $|f'(z_+)|<1$ and z_+ is an *attracting* fixed point, with z_- being *repelling*.

Figure 1. The p.d.f. H_Y for the angle $\phi/2$ between fixed points of a random hyperbolic $f \in \mathcal{F}$ and the convolution $H_Y * H_Y$.

We have chosen $\arg(c)$ to be uniformly distributed and so the argument of either fixed point, say z_+, is uniformly distributed. The interesting question is the distribution of the angle (at 0) between the fixed points. That is the argument of $z_+\bar{z}_-$. This will reflect the correlation we are looking for. This angle is easily seen to be the angle $\phi \in [0, \pi]$ where $\cos(\phi/2) = \Im m(a)/|c|$. Then

$$\cos(\phi/2) = \Im m(a)/|c| = \frac{|a| \sin\theta}{\sqrt{|a|^2 - 1}} = \frac{\sin\theta}{\cos\alpha},$$

where we are able to assume that both θ and α are uniformly distributed in $[0, \pi/2]$ and we are conditioned by $\sin\theta \leq \cos\alpha$.

We will calculate the distribution of $\sin\theta/\cos\alpha$ carefully when we come to the calculation of the parameters determining a Möbius group. We report the p.d.f. here as follows.

Theorem 3.2. *The distribution of the random variable* $X = \sin(\theta)/\cos(\alpha)$, *for* θ *and* α *uniformly distributed in* $[0, \pi/2]$ *is given by the formula*

$$h_X(x) = \frac{4}{\pi^2 x} \log\frac{1+x}{1-x}, \quad 0 \leq x < 1. \tag{3.3}$$

We can now use the change of variables formula to compute the p.d.f. for $\phi/2$. That is, we want the distribution for $Y = \cos^{-1}(h_X(x))$, given $h_X(x) \leq 1$. We can compute this distribution to be

$$h_Y(y) = \frac{4}{\pi^2} \tan(y) \log\frac{1+\cos(y)}{1-\cos(y)}.$$

Theorem 3.4. *Let* $\phi \in [0, \pi]$ *be the angle subtended at* 0 *by the fixed points of a random hyperbolic element in* $\mathcal{F}$. *Then the p.d.f. for* $\eta = \phi/2$, *as seen in Figure 1, is given by*

$$H_Y(\eta) = \frac{4}{\pi^2} \tan(\eta) \log\frac{1+\cos(\eta)}{1-\cos(\eta)}. \tag{3.5}$$

Some hyperbolic trigonometry reveals the hyperbolic line between a pair of points $z_\pm \in \mathbb{S}$ meets the closed disk of hyperbolic radius r (denoted $\mathbb{D}_\rho(r)$) when the angle ϕ formed at 0 satisfies $\cosh(r) \geq 1/\sin(\phi/2)$. If $z_\pm$ are the fixed points of a hyperbolic element f, then this hyperbolic line joining them is called the axis of f, denoted $\operatorname{axis}(f)$. We can therefore compute the probability that the axis of a random hyperbolic element meets $\mathbb{D}_\rho(r)$ by setting $\delta = \sin^{-1}(1/\cosh(r))$ and computing

$$\begin{aligned}\mathcal{P}(\operatorname{axis}(f) \cap \mathbb{D}_\rho(r) \neq \varnothing) &= \frac{4}{\pi^2} \int_0^\delta \tan(\eta)\, \log \frac{1+\cos(\eta)}{1-\cos(\eta)}\, d\eta \\ &= \frac{4}{\pi^2} \int_0^{\tanh(r)} \frac{1}{x}\, \log \frac{1+x}{1-x}\, dx \\ &= \frac{4}{\pi^2}[\mathrm{Li}_2(\tanh(r)) - \mathrm{Li}_2(-\tanh(r))].\end{aligned}$$

Here $\mathrm{Li}_2(s) = \sum_1^\infty n^{-2}s^n$ is a polylog function. Thus, for instance, this probability exceeds $\frac{1}{2}$ as soon as $r > 0.678\ldots$ and exceeds 0.95 as soon as $r > 2.24419$.

Now, the bisector ζ_f of the smaller circular arc between the fixed points of a random hyperbolic element of f is uniformly distributed on the circle. Then, given f and g random hyperbolic elements of $\mathcal{F}$ and angles ϕ_f and ϕ_g between their fixed points, the p.d.f. for $\phi_f/2 + \phi_g/2$ is the convolution $H_Y * H_Y$. We note that $e^{i\theta} = \xi = \zeta_f \overline{\zeta}_g$ is uniformly distributed as well. Given ξ, the fixed points of f and of g intertwine (so that the axes cross) if both $\phi_f + \phi_g \geq 2\theta$ and $|\phi_f - \phi_g| < 2\theta$. We can use the distributions above to calculate these probabilities, but it is quite complicated and we will find another route to this probability a bit later.

4. Isometric circles and traces

The isometric circles of the Möbius transformation f defined at (2.2) are defined to be the two circles

$$C_+ = \left\{ |z + \frac{\bar{a}}{\bar{c}}| = \frac{1}{|c|} \right\}, \quad C_- = \left\{ z : |z - \frac{a}{\bar{c}}| = \frac{1}{|c|} \right\},$$

which are paired by the action of f and f^{-1}, with $f^{\pm 1}(C_\pm) = C_\mp$. The *isometric disks* are the finite regions bounded by these two circles. Since $|a|^2 = 1 + |c|^2 \geq 1$, both these circles meet the unit circle in an arc of angle $\theta \in [0, \pi]$. Some elementary trigonometry reveals that

$$\sin(\theta/2) = 1/|a|. \tag{4.1}$$

Thus by our choice of distribution for $|a|$ we obtain the following key result.

Lemma 4.2. *The arcs determined by the intersections of the finite disks bounded by the isometric circles of f, where f is chosen according to the distributions* (i) *and* (ii), *are centred on uniformly distributed points of* $\mathbb{S}$ *and have arc length uniformly distributed in* $[0, \pi]$.

It is this lemma which supports our claim that the p.d.f. on $\mathcal{F}$ is natural and suggests the way forward for an analysis of random subgroups of $\mathrm{PSL}(2, \mathbb{C})$.

The isometric circles of f are disjoint if

$$\left|\frac{a}{\bar{c}} + \frac{\bar{a}}{\bar{c}}\right| \geq \frac{2}{|c|}.$$

This occurs if $|\mathrm{tr}(f)| = |a + \bar{a}| = 2|\Re e(a)| \geq 2$. Since the disjointness of isometric circles has important geometric consequences we will need to find the p.d.f. for the random variable $t = |\mathrm{tr}(f)|$. As $|\Re e(a)| = |a|\,|\cos(\theta)|$, for a fixed $\theta \in [0, \pi/2]$, the probability

$$\Pr[\{|a| \geq 1/\cos\theta\}] = 1 - \frac{2}{\pi}\int_1^{1/\cos\theta} \frac{dx}{x\sqrt{x^2-1}} = 1 - \frac{2}{\pi}\theta. \tag{4.3}$$

As $a/|a|$ is uniformly distributed on the circle, we have $\theta\,|\,[0, \pi/2]$ uniformly distributed in $[0, \pi/2]$. Therefore using the obvious symmetries we may calculate

$$\Pr[\{|a + \bar{a}| \geq 2\}] = \frac{2}{\pi}\int_0^{\pi/2} 1 - \frac{2}{\pi}\theta \; d\theta = \frac{1}{2}.$$

Corollary 4.4. *Let* $f \in \mathcal{F}$ *be a Möbius transformation chosen randomly from the distribution described in* (i) *and* (ii). *Then the probability that the isometric circles of* f *are disjoint is equal to* $\frac{1}{2}$.

Therefore we have the following simple consequence concerning random cyclic groups.

Corollary 4.5. *Let* $f \in \mathcal{F}$ *be a Möbius transformation chosen randomly from the distribution described in* (i) *and* (ii). *Then the probability that the cyclic group* $\langle f \rangle$ *is discrete is equal to* $\frac{1}{2}$.

Proof. The matrix $A \in \mathrm{SL}(2, \mathbb{C})$ represents an elliptic or parabolic Möbius transformation f if and only if $-2 \leq \mathrm{tr}\, A \leq 2$. This occurs with probability $\frac{1}{2}$. The matrix A represents an elliptic transformation of finite order, or a parabolic transformation if and only if $\mathrm{tr}(A) = \pm 2\cos(p\pi/q),\ \ p, q \in \mathbb{Z}$, and this set is countable and therefore has measure zero. The result follows. □

We now note the following trivial consequence.

Corollary 4.6. *Let* $f, g \in \mathcal{F}$ *be Möbius transformations chosen randomly from the distribution described in* (i) *and* (ii). *Then the probability that the group* $\langle f, g \rangle$ *is discrete is no more than* $\frac{1}{4}$.

Actually we can use (4.3) to determine the p.d.f. for $|\mathrm{tr}(A)|$. We will do this two ways. First, for $s \geq 2$,

$$\begin{aligned}\Pr[\{|\,\mathrm{tr}(A)| \geq s\}] = \Pr[\{2|a|\cos\theta \geq s\}] &= \Pr[\{|a| \geq s/(2\cos\theta)\}] \\ &= 1 - \frac{4}{\pi^2}\int_0^{\pi/2}\int_1^{s/2\cos\theta} \frac{dx}{x\sqrt{x^2-1}}\,d\theta \\ &= 1 - \frac{4}{\pi^2}\int_0^{\pi/2} \cos^{-1}\left(\frac{2\cos\theta}{s}\right)d\theta.\end{aligned}$$

We can now differentiate this function of s under the integral, integrate with respect to θ (using the symmetry to reduce it to being over $[0, \pi/2]$) to obtain the probability density function for $|\mathrm{tr}(A)|$ (for $|\mathrm{tr}(A)| \geq 2$),

$$F[s] = \frac{4}{\pi^2 s}\cosh^{-1}\left[\frac{s}{\sqrt{s^2-4}}\right],$$

with $s \geq 2$. This gives the distribution for $\mathrm{tr}^2 A$ as

$$G[t] = \frac{2}{\pi^2 t}\cosh^{-1}\left(\frac{\sqrt{t}}{\sqrt{t-4}}\right) = \frac{2}{\pi^2 t}\log\frac{\sqrt{t}+2}{\sqrt{t-4}}, \quad t \geq 4.$$

Then the random variable $\beta = \mathrm{tr}^2 A - 4 \geq 0$ has distribution

$$G[\beta] = \frac{1}{\pi^2(\beta+4)}\log\left(1 + \frac{8+4\sqrt{\beta+4}}{\beta}\right), \quad \beta \geq 0. \tag{4.7}$$

We could now follow through a similar, but more difficult, calculation to determine the distribution for β in the interval $-4 \leq \beta \leq 0$. It turns out to be

$$G[\beta] = \frac{1}{\pi^2(\beta+4)}\log\left(\frac{2+\sqrt{\beta+4}}{2-\sqrt{\beta+4}}\right), \quad \beta \in [-4, 0]. \tag{4.8}$$

We will return to this in a moment through a different approach as we can immediately use (4.7) to find the distribution of the translation length of hyperbolic elements.

As we have seen, every element $f \in \mathcal{F}$ which is not elliptic (conjugate to a rotation, equivalently $\beta(f) \in [-4, 0)$) or parabolic (conjugate to a translation, equivalently $\beta(f) = 0$) fixes two points on the circle and the hyperbolic line $\mathrm{axis}(f)$ with those points as endpoints. The transformation acts as a translation by constant hyperbolic distance $\tau(f)$ along its axis. This number $\tau(f)$ is called the *translation length* and is related to the trace via the formula $\beta(f) = 4\sinh^2 \tau/2$ [Gehring and Martin 1994]. We obtain the distribution for $\tau = \tau(f)$ from the change of variables

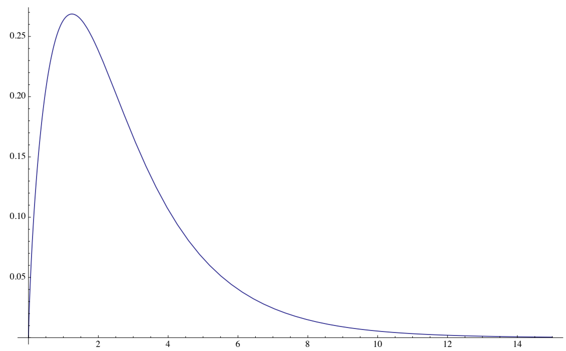

Figure 2. The p.d.f. for the translation length τ of a random hyperbolic element of $\mathcal{F}$.

formula for p.d.f. using (4.7)

$$H[\tau]=\frac{2}{\pi^2}\tanh\frac{\tau}{2}\log\left(\frac{\cosh\frac{\tau}{2}+1}{\cosh\frac{\tau}{2}-1}\right)=-\frac{4}{\pi^2}\tanh\frac{\tau}{2}\log\tanh\frac{\tau}{4}.$$

Unlike our earlier distribution G, the p.d.f. for τ has all moments. In particular once we observe

$$\int_0^\infty t\tanh\frac{t}{2}\log\left[\tanh\frac{t}{4}\right]dt=-\pi^2\log 2,$$

we have the following theorem.

Theorem 4.9. *For randomly selected hyperbolic $f\in_*\mathcal{F}$ the p.d.f. for the translation length $\tau=\tau(f)$, as seen in Figure 2, is*

$$H[\tau]=-\frac{4}{\pi^2}\tanh\frac{\tau}{2}\log\tanh\frac{\tau}{4} \tag{4.10}$$

and the expected value of the translation length is $E[\tau]=4\log 2\approx 2.77259\ldots$

However there is another way to see these results and which is more useful in what is to follow in that it more clearly relates to the geometry.

5. The parameter $\beta=\mathrm{tr}^2(A)-4$

Theorem 5.1. *If a Möbius transformation f is randomly chosen in $\mathcal{F}$, then*

$$\beta(f)=4\left(\frac{\cos^2(\theta)}{\sin^2(\alpha)}-1\right),\quad \theta\in_u[0,2\pi],\quad \alpha\in_u\left[0,\frac{\pi}{2}\right], \tag{5.2}$$

where 2α is the arc length intersection of the isometric circles of f with $\mathbb{S}$ and θ is the argument of the leading entry of A, the matrix representative for f.

Proof. Let $A = \left(\begin{smallmatrix} a & c \\ \bar{c} & \bar{a} \end{smallmatrix}\right)$. Then $\beta = \operatorname{tr}^2 A - 4 = [2\Re e(a)]^2 - 4 = 4|a|^2 \cos^2(\theta) - 4$ and the result follows by (4.1) and Lemma 4.2. □

Theorem 5.3. *The distribution of the random variable*

$$w = \frac{\cos^2(\theta)}{\sin^2(\alpha)} \quad \textit{for } \theta \in_u [0, 2\pi] \textit{ and } \alpha \in_u \left[0, \frac{\pi}{2}\right]$$

is given by the formula

$$h(w) = \frac{1}{\pi^2 w} \log\left|\frac{\sqrt{w}+1}{\sqrt{w}-1}\right|, \quad w \geq 0. \tag{5.4}$$

Proof. The probability distribution functions of $x = \cos^2(\theta)$ and $y = \sin^2(\alpha)$ are independent and identically distributed $F(x)$ and $F(y)$,

$$F(x) = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{5.5}$$

F is monotonic for $x, y \in \left[0, \frac{1}{2}\right)$ and also for $x, y \in \left(\frac{1}{2}, 1\right]$ and antisymmetric about $\frac{1}{2}$. Therefore we can use the change of variables formula and the Mellin convolution to compute the p.d.f. Write $x = \cos^2(\theta)$, $y = \sin^2(\alpha)$ and $w = \cos^2(\theta)/\sin^2(\alpha)$. We use the Mellin convolution for quotients as in [Springer 1979]. For $x, y \in (0, 1)$ the upper integration limits for the convolution integral are $y < 1 \times \frac{1}{w}$ whenever $w > 1$ and $y < 1$ otherwise; accordingly the Mellin convolution for the quotient of the probability distribution functions over $(0, \infty)$ is calculated as follows, where we have ensured the piecewise differentiability of the integrand.

$$h(w) = \int_0^1 y\, f(x) f(y) dy \ \text{ for } w < 1 \quad \text{and} \quad \int_0^{\frac{1}{w}} y\, f(x) f(y) dy \ \text{ for } w \geq 1 \tag{5.6}$$

and the indefinite integral embedded in both components of (5.6) is given as

$$\begin{aligned} \int y\, f(yw) f(y)\, dy &= \int y \frac{1}{\pi\sqrt{yw(1-yw))}} \frac{1}{\pi\sqrt{y(1-y)}}\, dy \\ &= \frac{2}{\pi^2 w} \log\left(w\sqrt{(y-1)} + \sqrt{w(yw-1)}\right). \end{aligned} \tag{5.7}$$

Simplification of the log term in (5.7) yields

$$\log\left(w(w(2y-1) - 1 + 2\sqrt{w(y-1)(yw-1)})\right)$$

$$= \begin{cases} e_0 = \log(-w(w+1-2\sqrt{w})) & \text{at } y = 0, \\ e_1 = \log(w(w-1)) & \text{at } y = 1, \\ e_{\frac{1}{w}} = \log(-w(w-1)) & \text{at } y = \frac{1}{w}, \end{cases}$$

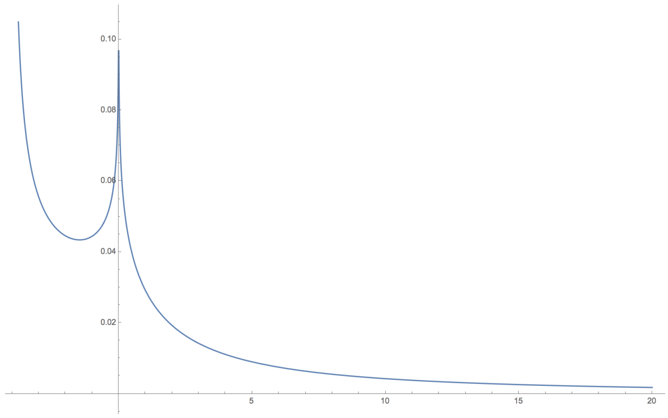

Figure 3. The p.d.f. for the parameter $\beta(f)$ for a random element $f \in \mathcal{F}$.

and accordingly the definite integrals in (5.6) evaluate to

$$\int_0^1 y\ f(yw)f(y)dy = \frac{1}{\pi^2 w}(e_1 - e_0)$$

and

$$\int_0^{\frac{1}{w}} y\ f(yw)f(y)dy = \frac{1}{\pi^2 w}(e_{1/w} - e_0).$$

If we now let $v = \sqrt{w}$, then

$$e_1 - e_0 = \log(w(w-1)) - \log(-w(w+1-2\sqrt{w})) = \log\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)$$

and

$$e_{1/w} - e_0 = \log(-w(w-1)) - \log(-w(w+1-2\sqrt{w})) = \log\left(\frac{\sqrt{w}+1}{\sqrt{w}-1}\right)$$

We deduce that the distribution of $w = \cos^2(\theta)/\sin^2(\alpha)$ is given by (5.4). $\square$

From this, and a little obvious manipulation to see these formulas actually agree with those obtained earlier, we obtain the result we were looking for.

Theorem 5.8. *The distribution of $\beta(f)$ for f randomly chosen from $\mathcal{F}$, as in Figure 3, is given by*

$$G[\beta] = \frac{1}{2\pi^2(\beta+4)} \log\left|\frac{\sqrt{\beta+4}+2}{\sqrt{\beta+4}-2}\right|, \quad \beta \geq -4. \tag{5.9}$$

This is quite a slowly converging integral, $G[x] \approx 2/(\pi^2 x^{3/2})$ for $x \gg 1$. In order to discuss the effectiveness of Jørgensen's inequality [1976] we will want the

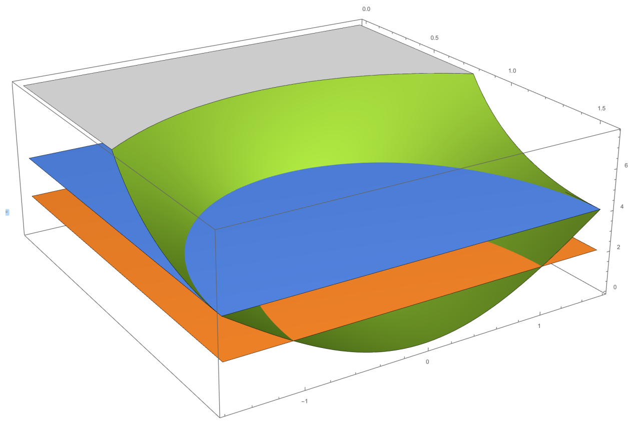

Figure 4. The graph of $\|A - I\|^2$ together with the planes $t^2 = 2$ and $t^2 = 4$.

cumulative distribution for $|\beta|$. We put down what we need in the following. The proof simply consists of calculating the integral.

Theorem 5.10. *For f randomly chosen from $\mathcal{F}$, $s \geq 0$, and $\beta = \beta(f)$,*

$$\mathbf{Pr}\{-s \leq \beta \leq 0\} = \frac{1}{2} + \frac{1}{\pi^2}\left(\mathrm{Li}_2(1 - s/4) - 4\,\mathrm{Li}_2\left(\sqrt{1 - s/4}\right)\right)$$

$$\begin{aligned}\mathbf{Pr}\{0 \leq \beta \leq s\} = \frac{2}{3} + \frac{2}{\pi^2}\Bigg[\mathrm{Li}_2\left(-\frac{\sqrt{s+4}+2}{\sqrt{s+4}-2}\right) + \mathrm{Li}_2\left(\frac{-4}{\sqrt{s+4}-2}\right) \\ + \log\left(\frac{\sqrt{s+4}+2}{\sqrt{s+4}-2}\right)\log\left(\frac{2\sqrt{s+4}}{\sqrt{s+4}-2}\right)\Bigg].\end{aligned}$$

This result gives us an indication of how likely it is that Jørgensen's inequality will have useful content since if we choose a random hyperbolic element in $\mathcal{F}$, then $\mathbf{Pr}\{0 < \beta < 1\} \approx 0.175745$.

5A. ***The metric in* PSL(2, ℝ).** Here we would like to identify the p.d.f. for the distance between an element of PSL(2, ℝ) and the identity when that element $A \in_* \mathrm{PSL}(2, \mathbb{R})$. Again we will see an elliptic/hyperbolic dichotomy in the singularities of the metric. We choose a representative A with positive trace,

$$\begin{pmatrix} e^{i\phi}\csc(\theta) & e^{i\alpha}\cot(\theta) \\ e^{-i\alpha}\cot(\theta) & e^{-i\phi}\csc(\theta) \end{pmatrix}, \quad \theta \in_u \left[0, \frac{\pi}{2}\right], \quad \phi \in_u \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

and calculate

$$\|A - I\|^2 = 2|e^{i\phi}\csc(\theta) - 1|^2 + 2\cot^2(\theta) = 4\csc^2(\theta) - 4\csc(\theta)\cos(\phi),$$

as illustrated in Figure 4.

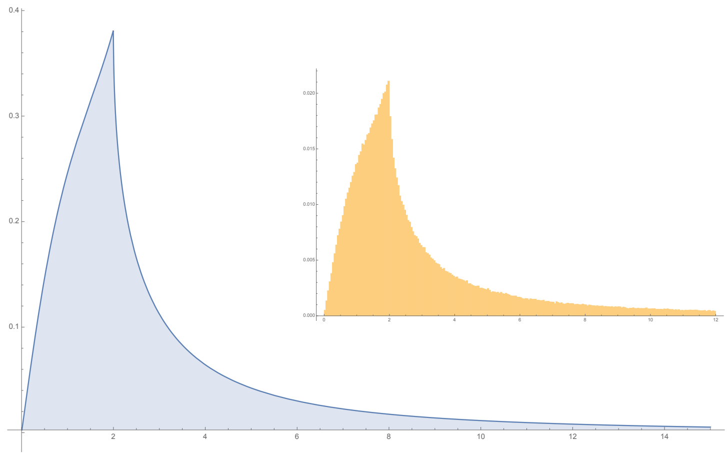

Figure 5. The p.d.f. of $\|A-I\|$. Inset a run on 10^7 random trials.

Then $\|A-I\|^2 \geq t^2$ implies $t^2\sin^2(\theta)+4\sin(\theta)\cos(\phi)-4 \leq 0$ and hence, since $\sin\theta \geq 0$,

$$\sin\theta \leq \frac{2}{t^2}\big(\sqrt{\cos^2\phi+t^2}-\cos\phi\big). \tag{5.11}$$

Notice that the right-hand side of (5.11) is ≤ 1. We want to find the measure of the subset of $\left[0,\frac{\pi}{2}\right]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ where (5.11) holds to find the p.d.f. Fix ϕ, then the length of the θ-interval where (5.11) holds is $\frac{\pi}{2}-\arcsin(2/(t^2)(\sqrt{\cos^2\phi+t^2}-\cos\phi))$, until $2/(t^2)(\sqrt{\cos^2\phi+t^2}-\cos\phi)=1$ whereafter the length stays 0. This latter condition is $1=t^2/4+\cos\phi$ which places no restriction on $\phi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ as soon as $t^2\geq 4$. Otherwise we only add up the lengths while $|\phi|\leq\arccos\big(1-\frac{t^2}{4}\big)$. We now have the following theorem after making the change of variables $x=\cos(\phi)$ for $t\geq 4$, and related changes for $0\leq t\leq 4$ to simplify their form so as to be able to differentiate under the integral sign to obtain the p.d.f.

Theorem 5.12. *The cumulative distribution for $\|A-I\|$, as shown in Figure 5, is*

$$\frac{4}{\pi^2}\int_{1-\frac{t^2}{4}}^{1}\frac{\pi}{2}-\sin^{-1}\left(\frac{2}{t^2}\big(\sqrt{x^2+t^2}-x\big)\right)\frac{dx}{\sqrt{1-x^2}}$$
$$=\frac{2}{\pi}\cos^{-1}\left(1-\frac{t^2}{4}\right)+\frac{4}{\pi^2}\int_0^1 t\sin^{-1}\left[2z+\frac{2\sqrt{t^4z^2+8t^2(2-z)+16}-8}{t^2}\right]\frac{dz}{\sqrt{z(8-t^2z)}}$$

for $0\leq t\leq 4$, and

$$1-\frac{4}{\pi^2}\int_0^1\sin^{-1}\left(\frac{2}{t^2}\big(\sqrt{x^2+t^2}-x\big)\right)\frac{dx}{\sqrt{1-x^2}}$$

for $4\leq t$.

6. The topology of the quotient space

Topologically there are two surfaces whose fundamental group is isomorphic to F_2, the free group on two generators. These are the 2-sphere with three holes $\mathbb{S}^2_3$, and the torus with one hole T^2_1. Thus a group $\Gamma = \langle f, g\rangle$ generated by two random hyperbolic elements of $\mathcal{F}$ if discrete, has quotient space $\mathbb{D}^2/\Gamma \in \{\mathbb{S}^2_3, T^2_1\}$. We would like to understand the likelihood of one of these topologies over the other. The topology is determined by whether the axes of f and g cross (giving T^2_1) or not (giving $\mathbb{S}^2_3$). This is the same thing as asking if the hyperbolic lines between the fixed points of f and the fixed points of g cross or not, and this in turn is determined by a suitable cross ratio of the fixed points. In fact, the geometry of the commutator $\gamma(f, g) = \mathrm{tr}[f, g] - 2$ determines not only the topology of the quotient, but also the hyperbolic length of the shortest geodesic — it is represented by either f, g, or $[f, g] = fgf^{-1}g^{-1}$ and their Nielsen equivalents. In fact the three numbers $\beta(f)$, $\beta(g)$ and $\gamma(f, g)$ determine the group $\langle f, g\rangle$ uniquely up to conjugacy. Since we have already determined the natural probability densities for $\beta(f)$ and $\beta(g)$ we need only identify the p.d.f. for $\gamma = \gamma(f, g)$ to find a conjugacy invariant way to identify random discrete groups. Unfortunately this is not so straightforward and we do not know this distribution. However important aspects of this distribution can be determined.

6A. ***Commutators and cross ratios.*** We follow [Beardon 1983] and define the cross ratio of four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ to be

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}. \tag{6.1}$$

In order to address the distribution of $\gamma(f, g) = \mathrm{tr}[f, g] - 2$, we need to understand the cross ratio distribution. This is because of the following result from [Beardon 1983, §7.23 and §7.24] together with a little manipulation.

Theorem 6.2. *Let ℓ_1, with endpoints z_1 and z_2, and ℓ_2, with endpoints w_1 and w_2, be hyperbolic lines in the unit disk model of hyperbolic space. So z_1, z_2, w_1, w_2 are in $\mathbb{S}$, the circle at infinity. Let δ be the hyperbolic distance between ℓ_1 and ℓ_2, and should they cross, let $\theta \in [0, \pi/2]$ be the angle at the intersection. Then*

$$\sinh^2\left[\tfrac{1}{2}(\delta + i\theta)\right] \times [z_1, w_1, z_2, w_2] = -1. \tag{6.3}$$

The number $\delta + i\theta$ is called the *complex distance* between the lines ℓ_1 and ℓ_2 where we put $\theta = 0$ if the lines do not meet. The proof of this theorem is simply to use Möbius invariance of the cross ratio and the two different models of the hyperbolic plane. If the two lines do not intersect, we choose the Möbius transformation which sends the disk to the upper half-plane and $\{z_1, z_2\}$ to $\{-1, +1\}$ and $\{w_1, w_2\}$

to $\{-s, s\}$ for some $s > 1$. Then $\delta = \log s$ and

$$[-1, -s, 1, s] = \frac{-4s}{(1-s)^2} = \frac{-4}{(e^{\delta/2} - e^{-\delta/2})^2} = -\frac{1}{\sinh^2(\delta/2)},$$

while if the axes meet at a finite point, we choose a Möbius transformation of the disk so the line endpoints are ± 1 and $e^{\pm i\theta}$ and the result follows similarly.

Next, Lemma 4.2 of [Gehring and Martin 1994] relates the parameters and cross ratios.

Theorem 6.4. *Let f and g be Möbius transformations and let $\delta + i\theta$ be the complex distance between their axes. Then*

$$4\gamma(f, g) = \beta(f)\,\beta(g)\, \sinh^2(\delta + i\theta). \tag{6.5}$$

We note from (6.3) that $\sinh^2(\delta + i\theta) = (1 - 2/[z_1, w_1, z_2, w_2])^2 - 1$. For a pair of hyperbolics f and g we have $\beta(f), \beta(g) \geq 0$ with $\delta = 0$ if the axes meet. Thus the axes cross if and only if $\gamma < 0$, or equivalently,

$$[z_1, w_1, z_2, w_2] > 1. \tag{6.6}$$

Actually to see the latter point, we choose the Möbius transformation which sends $z_1 \mapsto 0$, $z_2 \mapsto \infty$, $w_1 \mapsto 1$. Then $z_2 \mapsto z$, say, and

$$[z_1, w_1, z_2, w_2] = \frac{(0-1)(\infty - z)}{(0-\infty)(1-z)} = \frac{1}{1-z}.$$

The image of the axes (and therefore the axes themselves) cross when $z < 0$, equivalently when (6.6) holds.

6B. ***Cross ratio of fixed points.*** Supposing that f and g are randomly chosen hyperbolic elements, we want to discuss the probability of their axes crossing, if f has fixed points z_1, z_2 and g has fixed points w_1, w_2. We identified the formula for the fixed points above at (3.1) and if we notate the random variables (matrix entries) a, c for f and α, β for g we have

$$z_1, z_2 = \frac{1}{\bar{c}}(i\Im m(a) \pm \sqrt{\Re e(a)^2 - 1}), \quad |a|^2 = 1 + |c|^2,$$
$$w_1, w_2 = \frac{1}{\bar{\beta}}(i\Im m(\alpha) \pm \sqrt{\Re e(\alpha)^2 - 1}), \quad |\alpha|^2 = 1 + |\beta|^2,$$

and as both elements are hyperbolic we have $\Re e(a) \geq 1$ and $\Re e(\alpha) \geq 1$. We put $U = i\Im m(a) + \sqrt{\Re e(a)^2 - 1}$ and $V = i\Im m(\alpha) + \sqrt{\Re e(\alpha)^2 - 1}$. Then

$$[z_1, w_1, z_2, w_2] = \frac{4\sqrt{\Re e(a)^2 - 1}\sqrt{\Re e(\alpha)^2 - 1}}{\bar{c}\,\bar{\beta}\,\big(\frac{U}{\bar{c}} - \frac{V}{\bar{\beta}}\big)\big(\frac{-\bar{U}}{\bar{c}} - \frac{-\bar{V}}{\bar{\beta}}\big)} = \frac{2\sqrt{\Re e(a)^2 - 1}\sqrt{\Re e(\alpha)^2 - 1}}{\Re e[U\bar{V}] - \Re e[c\bar{\beta}]}$$

since, as we recall, $1 = |z_i| = |U|/|c|$, and similarly $|V|/|\beta| = 1$. Thus we want to understand the statistics of the cross ratio, and in particular to determine when

$$[z_1, w_1, z_2, w_2] = \frac{2\sqrt{\Re e(a)^2 - 1}\sqrt{\Re e(\alpha)^2 - 1}}{\Re e[U\bar{V}] - \Re e[c\bar{\beta}]} \geq 1. \tag{6.7}$$

We have

$$\begin{aligned} a &= \frac{1}{\sin\theta}e^{i\phi}, \quad \theta \in_u [0, \pi/2], \quad \phi \in_u [0, 2\pi], \quad c = \cot\theta e^{i\delta}, \quad \delta \in_u [0, 2\pi], \\ \alpha &= \frac{1}{\sin\eta}e^{i\psi}, \quad \eta \in_u [0, \pi/2], \quad \psi \in_u [0, 2\pi] \quad \beta = \cot\eta e^{i\zeta}, \quad \zeta \in_u [0, 2\pi]. \end{aligned}$$

Then $\sqrt{\Re e(a)^2 - 1} = \sqrt{\cos^2\phi/\sin^2\theta} - 1$, $\sqrt{\Re e(\alpha)^2 - 1} = \sqrt{\cos^2\psi/\sin^2\eta - 1}$, $\Phi = \arg c\bar{\beta}$ is uniformly distributed in $[0, 2\pi]$ and

$$\Re e[U\bar{V}] - \Re e[c\bar{\beta}] = \frac{\sin\phi}{\sin\theta}\,\frac{\sin\psi}{\sin\eta} + \sqrt{\frac{\cos^2\phi}{\sin^2\theta} - 1}\,\sqrt{\frac{\cos^2\psi}{\sin^2\eta} - 1} - \cot\eta\cot\theta\cos\Phi.$$

This gives

$$\begin{aligned} &\frac{2\sqrt{\Re e(a)^2 - 1}\,\sqrt{\Re e(\alpha)^2 - 1}}{\Re e[U\bar{V}] - \Re e[c\bar{\beta}]} \\ &\qquad = \frac{2\sqrt{\cos^2\phi - \sin^2\theta}\,\sqrt{\cos^2\psi - \sin^2\eta}}{\sin\phi\,\sin\psi + \sqrt{\cos^2\phi - \sin^2\theta}\,\sqrt{\cos^2\psi - \sin^2\eta} - \cos\eta\cos\theta\cos\Phi} \\ &\qquad = \frac{2\sqrt{1 - X^2}\,\sqrt{1 - Y^2}}{XY + \sqrt{1 - X^2}\,\sqrt{1 - Y^2} - \cos\Phi} = Z, \end{aligned}$$

where we define the random variables $X = \sin\phi/\cos\theta$, and $Y = \sin\psi/\cos\eta$. To have $Z \geq 1$, we need $|X| \leq 1$, $|Y| \leq 1$ and

$$\sqrt{1 - X^2}\,\sqrt{1 - Y^2} \geq \cos\Phi - XY. \tag{6.8}$$

If this last condition holds, then $[z_1, w_1, z_2, w_2] \geq 1$ requires

$$\sqrt{1 - X^2}\sqrt{1 - Y^2} \geq XY - \cos\Phi. \tag{6.9}$$

Notice that X, Y and $\Phi \in_u [0, 2\pi]$ are independent, with X and Y identically distributed. Unfortunately $\sqrt{1 - X^2}\sqrt{1 - Y^2} \pm XY$ is difficult to find directly as $\sqrt{1 - X^2}\sqrt{1 - Y^2}$ and XY are not independent. We therefore write

$$X = \sin S, \quad S \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad Y = \sin T, \quad T \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

so that $\sqrt{1-X^2}\sqrt{1-Y^2} \pm XY = \cos(S \mp T)$ and we have the two requirements

$$\cos(S \mp T) \geq \pm\cos(\Phi). \tag{6.10}$$

Following the arguments of Section 5, we have the probability distribution functions X and S, with probability distribution functions, respectively,

$$F_X(x) = \frac{2}{\pi^2 x}\log\left|\frac{1+x}{1-x}\right|, \qquad -1 \leq x \leq 1,$$

$$F_S(\theta) = \frac{2}{\pi^2}\cot(\theta)\log\left|\frac{1+\sin(\theta)}{1-\sin(\theta)}\right|, \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

We can remove various symmetries and redundancies for the situation to simplify. For instance we may assume $S \geq 0$ and reduce to ranges where cos is either increasing or decreasing so we can remove it. We quickly come to the following conditions equivalent to (6.10) with S and T identically distributed as above and $\Phi \in_u [0, \pi/2]$:

$$0 \leq S, \quad -\Phi \leq S - T \leq \Phi, \quad \text{and} \quad S + T + \Phi \leq \pi. \tag{6.11}$$

This now sets up an integral which we implemented on Mathematica numerically and which returned the value 0.429 . . . . In the next section we correlate this with independent experiments to determine when $\gamma \leq 0$. This agrees with the results of integration as above at (6.11). We record this in the following.

Theorem 6.12. *Let f, g be randomly chosen hyperbolic elements of $\mathcal{F}$. Then the probability that the axes of f and g cross is* ≈ 0.429.

We should point out here the following well-known observation.

Lemma 6.13. *Let $f, g \in \mathcal{F}$. If $\gamma(f, g) < 0$, then both f and g are hyperbolic.*

Proof. As the result is conjugacy invariant we may first suppose f is hyperbolic and represented by $f = \left(\begin{smallmatrix}\lambda & 0\\ 0 & 1/\lambda\end{smallmatrix}\right)$, and $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, with $ad - bc = 1$. We calculate $\gamma = -(\lambda - 1/\lambda)^2 bc < 0$. Thus f hyperbolic gives $bc > 0$, $ad = 1 + bc > 1$ and $(a+d)^2 > 4$ showing g is hyperbolic. If f is parabolic we put $f = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, and calculate $\gamma(f, g) = c^2 \geq 0$. Finally, we write $f = \left(\begin{smallmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{smallmatrix}\right)$ if f is elliptic, and compute that

$$\gamma(f, g) = (a^2 + b^2 + c^2 + d^2 - 2)\sin^2\alpha \geq 0. \qquad \square$$

In contrast to Theorem 6.12, we have the following result.

Theorem 6.14. *Let ζ_1, ζ_1 and η_1, η_2 be two pairs of points, each randomly and uniformly chosen on the circle. Let α be the hyperbolic line between ζ_1 and ζ_2 and let β be the hyperbolic line between η_1 and η_2. Then the probability that α and β cross is* $\frac{1}{3}$.

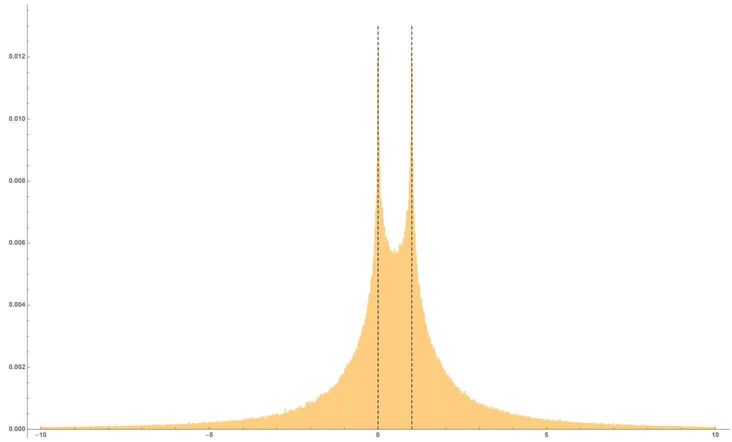

Figure 6. Histogram of the cross ratio of the fixed points of a randomly chosen pair of hyperbolic elements.

Proof. We can forget the points come in pairs and label them z_i, $i = 1, 2, 3, 4$, in order around the circle. There are three different cases, all with the same probability.

- z_1 connects to z_2, hence z_3 to z_4 and the lines are disjoint.
- z_1 connects to z_3, hence z_2 to z_4 and the lines intersect.
- z_1 connects to z_4, hence z_2 to z_3 and the lines are disjoint. □

Together these theorems quantify the degree to which the fixed points are correlated on the circle. We also include the following example.

In the histogram in Figure 6, the singularities are at 0 and 1. We make the observation that it seems quite likely that $\mathbf{Pr}\{[z1, w1, z2, w2] > 1\} = \frac{1}{5}$. It is somewhat of a chore to calculate the cross ratio distribution X_{cr} of four randomly selected points on the circle. This is done in [Martin 2019] and the distribution is very similar to that above, with singularities at 0 and 1. However for that distribution the probabilities are $\mathbf{Pr}\{X_{cr} < 0\} = \mathbf{Pr}\{0 < X_{cr} < 1\} = \mathbf{Pr}\{X_{cr} > 1\} = \frac{1}{3}$ (as can be seen from the action of the group $S4$ on the cross ratio [Beardon 1983]). This shows the distributions are definitely different.

7. The effectiveness of Jørgensen's inequality

In order to computationally explore the moduli spaces of discrete groups we need effective tests for discreteness in groups of Möbius transformations. In practice, it is very difficult to discern if a group is discrete, especially if we know a priori that the group is free on its generators. Discreteness is typically established by constructing a fundamental domain using the Poincaré polyhedral theorem, or using arithmetic information [Gehring et al. 1997], or algorithmically [Gilman 1995]. We would like to discern, with high confidence, that a group is discrete. The most common test is the following from [Jørgensen 1976] (see also [Gehring and Martin 1991a]).

Theorem 7.1 (Jørgensen's inequality). *Let $A, B \in \mathrm{SL}(2, \mathbb{C})$. Suppose that $\langle A, B\rangle$ is discrete and not virtually abelian. Set $\beta = \mathrm{tr}^2(A) - 4$ and $\gamma = \mathrm{tr}[A, B] - 2$. Then $|\gamma| + |\beta| \geq 1$, and if $\gamma \neq \beta$, then $|\beta - \gamma| + |\beta| \geq 1$ also.*

Another common test used can be found in [Gehring and Martin 1991b; Cao 1995].

Theorem 7.2. *Let $A, B \in \mathrm{SL}(2, \mathbb{C})$. Suppose that $\langle A, B\rangle$ is discrete and not virtually abelian. Then*

$$|\gamma(\gamma - \beta)| \geq 2 - 2\cos(\pi/7) = 0.198\ldots$$

All these tests are sharp: there are nonelementary discrete examples where equality holds (for example, lattices). We have already seen that a randomly selected hyperbolic element in $\mathcal{F}$ has $|\beta| < 1$ with probability about 0.175745. Thus for a group generated by random hyperbolics $f_1, f_2, \ldots, f_n$, the probability that one has $|\beta_i| < 1$, so we can even consider the inequality, is quite high:

$$\mathbf{Pr}\{\beta_i < 1 \text{ for some } i = 1, 2\ldots, n\} \geq 1 - \left(\tfrac{33}{40}\right)^n.$$

Further, we are at liberty to consider other generators. For instance in the case of two generators, as in Figure 7, we note that $\langle f, g\rangle = \langle f, gf\rangle = \langle f, gf^{-1}\rangle$ (but do note that if f and g are randomly selected, then fg etc. are not). The unfortunate thing here is that all these pairs of generators have the same commutator,

$$\gamma(f, g) = \gamma(f, gf^{\pm 1}) = \gamma(g, fg^{\pm 1}).$$

In fact the commutator value γ is an invariant of the Nielsen class of generators, and since a random group is free with probability 1, all generating pairs are equivalent. That is, any generating pair has the same value for γ. Thus the principal obstruction to the effectiveness of a discreteness test such as those at Theorems 7.1 and 7.2 is the value of the trace of the commutator. We now explore this.

If we select two random Möbius transformations, say,

$$f = \begin{pmatrix} e^{i\phi_1}\csc(\theta_1) & e^{i(\alpha_1)}\cot(\theta_1) \\ e^{-i\alpha_1}\cot(\theta_1) & e^{-i\phi_1}\csc(\theta_1)\end{pmatrix}, \quad g = \begin{pmatrix} e^{i\phi_2}\csc(\theta_2) & e^{i\alpha_2}\cot(\theta_2) \\ e^{-i\alpha_2}\cot(\theta_2) & e^{-i\phi_2}\csc(\theta_2)\end{pmatrix},$$

and then compute and simplify (making some variable substitutions etc.) $\gamma = \gamma(f, g) = \mathrm{tr}[f, g] - 2$, we find

$$\begin{aligned}(7.3)\quad \gamma = 4\csc^2\theta_1\csc^2\theta_2\big[&2\cos^2\theta_1(\sin^2\phi_2 - \sin^2\alpha\cos^2\theta_2)\\ &+ \cos^2\theta_2\sin^2\phi_1 - 2\cos\alpha\cos\theta_1\cos\theta_2\sin\phi_1\sin\phi_2\big].\end{aligned}$$

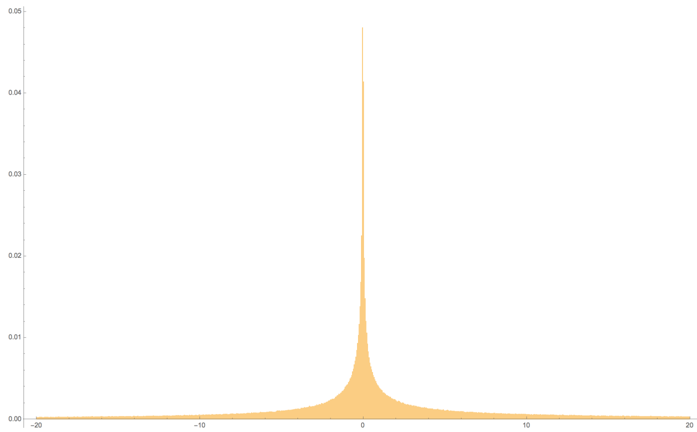

Figure 7. Histogram of γ values conditioned by f and g in $\mathrm{PSL}(2,\mathbb{R})$ hyperbolic.

Here $\theta_1, \theta_2 \in_u [0, \pi/2]$, $\alpha, \phi_1, \phi_2 \in_u [0, \pi]$. There seems to be no easy way to compute this p.d.f.

We made several independent runs through about 10^7 random matrix pairs of hyperbolic elements to generate the histogram in Figure 7. We found the probability that $\gamma < 0$ to be about 0.429601, in alignment with Theorem 6.12. Notice that if $\mathrm{tr}[f, g] \in [-2, 2]$, then almost surely $\langle f, g\rangle$ is not discrete since $[f, g]$ would be elliptic.

We also found

$$\mathbf{Pr}\{-4 \le \gamma \le 0\} \approx 0.162\ldots \tag{7.4}$$

$$\mathbf{Pr}\{|\gamma| + |\beta(f)| \le 1 \ \text{ or } \ |\gamma| + |\beta(g)| \le 1\} \approx 0.113\ldots \tag{7.5}$$

$$\mathbf{Pr}\{|\gamma(\gamma - \beta(f))| \le 0.198 \ \text{ or } \ |\gamma(\gamma - \beta(g))| \le 0.198\} \approx 0.119\ldots \tag{7.6}$$

$$\mathbf{Pr}\{\, fg \text{ or } fg^{-1} \text{ is elliptic } \} \approx 0.203\ldots \tag{7.7}$$

Of course these tests for nondiscreteness are not independent. If we put them all together and use additional inequalities found by replacing f by fg or fg^{-1} we found the following.

Conjecture 7.8. *Let $\langle f, g\rangle \in \mathcal{F}$ be randomly chosen hyperbolic elements. Then*

$$\mathbf{Pr}\Big\{\overline{\langle f, g\rangle} = \mathcal{F}\Big\} > 0.414986\ldots$$

That is to say we found that with probability $0.414986\ldots$ one of the discreteness tests is violated. This probability was not supported by rigorous calculations. However, it is not difficult to establish the lower bound 0.4 rigorously. We discuss this in the next section in a different setting but the ideas are the same.

8. Discreteness

An easy lower bound for the probability a group generated by two random elements of $\mathcal{F}$ is discrete based on the following Klein combination theorem (or "ping pong" lemma).

Lemma 8.1. *Let f_i, $i = 1, 2, \dots, n$, be hyperbolic transformations of the disk whose isometric disks are all disjoint. Then the group generated by these hyperbolic transformations $\langle f_1, f_2, \dots, f_n\rangle$ is discrete and isomorphic to the free group F_n.*

We have already seen that the probability that the isometric disks of a randomly chosen $f \in \mathcal{F}$ are disjoint is $\frac{1}{2}$.

Lemma 8.2. *Let α and β be arcs on $\mathbb{S}^1$ with uniformly randomly chosen midpoints ζ_α and ζ_β and subtending angles θ_α and θ_β uniformly chosen from $[0, \pi]$. The probability that α and β meet is $\frac{1}{2}$.*

Proof. The smaller arc subtended between ζ_α and ζ_β has length $\Theta = \arg(\zeta_\alpha\overline{\zeta}_\beta)$ and is uniformly distributed in $[0, \pi]$. Then α and β are disjoint if $\Theta - \theta_\alpha/2 - \theta_\beta/2 \geq 0$. Since $2\Theta - \theta_\alpha - \theta_\beta$ is uniformly distributed in $[-2\pi, 2\pi]$, the probability this number is positive is $\frac{1}{2}$. □

Using Lemma 8.1 this quickly gives us the obvious bound that if $f, g \in \mathcal{F}$ are randomly chosen, then the probability that $\langle f, g\rangle$ is discrete is at least $\frac{1}{64}$. For n generator groups this number is at least $2^{-(2n-1)!}$. However we are going to have to build a bit more theory to prove the following substantial improvements of these estimates.

Theorem 8.3. *The probability that randomly chosen $f, g \in_* \mathcal{F}$ generate a discrete group $\langle f, g\rangle$ is at least $\frac{1}{10}$.*

Theorem 8.4. *The probability that two randomly chosen hyperbolic transformation $f, g \in_* \mathcal{F}$ have disjoint isometric circles, and hence generate a discrete group $\langle f, g\rangle$, is at least $\frac{1}{5}$.*

Theorem 8.3 follows from Theorem 8.4 and Theorem 11.1 (another discreteness test) and the fact that the probability we choose two hyperbolic elements is independent and of probability equal to $\frac{1}{4}$. We now give a proof for Theorem 8.4. It is an immediate consequence of Lemmas 8.1 and 9.4 below.

9. Random arcs on a circle

Let α be an arc on the circle $\mathbb{S}$. We denote its midpoint by $m_\alpha \in \mathbb{S}$ and its arclength by $\ell_\alpha \in [0, 2\pi]$. Conversely, given $m_\alpha \in \mathbb{S}$ and $\ell_\alpha \in [0, 2\pi]$ we determine a unique arc $\alpha = \alpha(m_\alpha, \ell_\alpha)$ with this data.

A random arc α is the arc uniquely determined when we choose $m_\alpha \in \mathbb{S}$ uniformly (equivalently, $\arg(m_\alpha) \in_u [0, 2\pi]$) and $\ell_\alpha \in_u [0, 2\pi]$. We will abuse notation and also refer to random arcs when we restrict to $\ell_\alpha \in_u [0, \pi]$ as for the case of isometric disk intersections. We will make the distinction clear in context.

A simple consequence of our earlier result is the following corollary.

Corollary 9.1. *If* $m_\alpha, m_\beta \in_u \mathbb{S}$ *and* $\ell_\alpha, \ell_\beta \in_u [0, \pi]$, *then* $\Pr\{\alpha \cap \beta = \varnothing\} = \frac{1}{2}$.

We need to observe the following lemma.

Lemma 9.2. *If* $m_\alpha, m_\beta \in_u \mathbb{S}$ *and* $\ell_\alpha, \ell_\beta \in_u [0, 2\pi]$, *then* $\Pr\{\alpha \cap \beta = \varnothing\} = \frac{1}{6}$.

Proof. We need to calculate the probability that the argument of $\zeta = m_\alpha \overline{m}_\beta$ is greater than $(\ell_\alpha + \ell_\beta)/2$. Now $\theta = \arg(\zeta)$ is uniformly distributed in $[0, \pi]$. The joint distribution is uniform, and so we calculate

$$\Pr\{\theta \geq \ell_\alpha + \ell_\beta\} = \frac{1}{\pi^3} \iiint_{\{\theta \geq \alpha+\beta\}} d\theta\, d\alpha\, d\beta = \frac{1}{\pi^3} \int_0^\pi \int_0^\theta \int_0^{\theta-\alpha} d\beta\, d\alpha\, d\theta = \tfrac{1}{6}. \quad \square$$

Next we consider the probability of disjoint pairs of arcs.

Lemma 9.3. *Let* $m_{\alpha_1}, m_{\alpha_2}, m_{\beta_1}, m_{\beta_2} \in_u \mathbb{S}$ *and* $\ell_\alpha, \ell_\beta \in_u [0, \pi]$. *Set*

$$\alpha_i = \alpha(m_{\alpha_i}, \ell_{\alpha_i}), \quad \beta_i = \alpha(m_{\beta_i}, \ell_{\beta_i}).$$

Then the probability that all the arcs $\alpha_i, \beta_i,\ i = 1, 2$, *are disjoint is* $\frac{1}{20}$,

$$\Pr\{(\alpha_1 \cap \alpha_2) \cup (\beta_1 \cap \beta_2) \cup (\alpha_1 \cap \beta_1) \cup (\alpha_1 \cap \beta_2) \cup (\alpha_2 \cap \beta_1) \cup (\alpha_2 \cap \beta_2) = \varnothing\} = \tfrac{1}{20}.$$

Proof. We first observe that the events

$$(\alpha_1 \cap \beta_1) = \varnothing, \ (\alpha_1 \cap \beta_2) = \varnothing, \ (\alpha_2 \cap \beta_1) = \varnothing, \ (\alpha_2 \cap \beta_2) = \varnothing$$

are not independent since (among other reasons) α_1 and α_2, and similarly β_1 and β_2, may overlap. The probability that $(\alpha_1 \cap \beta_1) = \varnothing$ and $(\alpha_2 \cap \beta_2) = \varnothing$ we have already determined to be equal to $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$. The result now follows from Lemma 9.4. $\square$

Lemma 9.4. *Let* $m_{\alpha_1}, m_{\alpha_2}, m_{\beta_1}, m_{\beta_2} \in_u \mathbb{S}$ *and* $\ell_\alpha, \ell_\beta \in_u [0, \pi]$. *Set*

$$\alpha_i = \alpha(m_{\alpha_i}, \ell_{\alpha_i}), \quad \beta_i = \alpha(m_{\beta_i}, \ell_{\beta_i}),$$

and suppose we are given that $(\alpha_1 \cap \alpha_2) = (\beta_1 \cap \beta_2) = \varnothing$. *Then the probability that all the arcs* α_i *are disjoint from the arcs* $\beta_j,\ i, j = 1, 2$, *is* $\frac{1}{5}$,

$$\Pr\{(\alpha_1 \cap \beta_1) \cup (\alpha_1 \cap \beta_2) \cup (\alpha_2 \cap \beta_1) \cup (\alpha_2 \cap \beta_2) = \varnothing\} = \tfrac{1}{5}.$$

Proof. Conditioned by the assumption that α_1 and α_2 are disjoint, and that β_1 and β_2 are disjoint, we note the events

$$(\alpha_1 \cap \beta_1) = \varnothing, \quad (\alpha_1 \cap \beta_2) = \varnothing, \quad (\alpha_2 \cap \beta_1) = \varnothing, \quad (\alpha_2 \cap \beta_2) = \varnothing$$

are independent. A little trigonometry reveals that

$$\alpha_i \cap \beta_j = \varnothing \leftrightarrow \frac{\ell_\alpha + \ell_\beta}{2} \le 2 \arcsin \frac{|m_{\alpha_i} - m_{\beta_j}|}{2} = \arg(m_{\alpha_i}\overline{m}_{\beta_j}).$$

Now the four variables $\theta_{i,j} = \arg(m_{\alpha_i}\overline{m}_{\beta_j})$, $i, j = 1, 2$, are uniformly distributed in $[0, \pi]$ and independent. We require $\min_{i,j} \theta_{i,j} \ge (\ell_\alpha + \ell_\beta)/2$. Now $\frac{1}{2}(\ell_\alpha + \ell_\beta) = \psi$ is uniformly distributed in $[0, \pi]$ and

$$\Pr\{\min_{i,j} \theta_{i,j} \ge \psi\} = \Big(1 - \frac{\psi}{\pi}\Big)^4. \tag{9.5}$$

Since $\frac{1}{\pi} \int_0^\pi \big(1 - \frac{\psi}{\pi}\big)^4 = \frac{1}{5}$, the result claimed follows. □

In passing we further note that (9.5) gives us a density function $\rho(\psi) = 4\big(1 - \frac{\psi}{\pi}\big)^3$ and hence an expected value of

$$\frac{4}{\pi^2} \int_0^\pi \psi \Big(1 - \frac{\psi}{\pi}\Big)^3 \, d\psi = 4 \int_0^1 (1-t)t^3 \, dt = \frac{1}{5}.$$

Generalising this result for a greater number of disjoint pairs of arcs quickly gets quite complicated. We state without proof the following, which we will not use.

Lemma 9.6. *Let $m_{\alpha_1}, m_{\alpha_2}, m_{\beta_1}, m_{\beta_2}, m_{\gamma_1}, m_{\gamma_2} \in_u \mathbb{S}$ and $\ell_\alpha, \ell_\beta, \ell_\gamma \in_u [0, \pi]$. Set*

$$\alpha_i = \alpha(m_{\alpha_i}, \ell_{\alpha_i}), \quad \beta_i = \alpha(m_{\beta_i}, \ell_{\beta_i}), \quad \gamma_i = \alpha(m_{\gamma_i}, \ell_{\gamma_i}).$$

Then the probability that all the arcs $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2$, are all disjoint is $\frac{3}{1000}$.

One can get results if there is additional symmetry; for instance, if the lengths of all the arcs are the same.

Theorem 9.7. *Let $m_{i_1}, m_{i_2} \in_u \mathbb{S}^1$, $i = 2, \dots, n$, and $\ell_\alpha \in_u [0, \pi]$. Then the probability that the arcs $\alpha_{ij} = \alpha(m_{i_j}, \ell_\alpha)$ are disjoint is*

$$\frac{1}{(2n)n!} \int_0^1 \sum_{k=0}^{[2-x]} (-1)^k \binom{n}{k} (2 - x - k)^n \, dx. \tag{9.8}$$

Proof. We cyclically order the set $\{m_{i_i} : i = 2, \dots, n, \ j = 1, 2\}$ and let θ_k be the angle between the k-th and $(k+1)$-st point (mod k). Then $\sum_{k=1}^{2n} \theta_k = 2\pi$. The arcs are disjoint if $\theta_k \ge \ell_\alpha$. We have $2n - 1$ independent random variables $\{\theta_k\}_{k=1}^{2n-1}$ which, firstly, must have a minimum which exceeds α, and secondly, must satisfy $2\pi - \sum_{k=1}^{2n-1} \theta_k \ge \ell_\alpha$. The first of these requirements gives us a factor $\frac{1}{2n}$, and from the second we note that the sum of m uniformly distributed random variables in $[0, 1]$ has the Irwin–Hall distribution

$$F_n(x) = \frac{1}{(m-1)!} \sum_{k=0}^{[x]} (-1)^k \binom{m}{k} (x - k)^{m-1}. \tag{9.9}$$

Thus

$$\Pr\left\{2-\frac{\ell_\alpha}{\pi} \geq \sum_{k=1}^{2n-1} \frac{\theta_k}{\pi}\right\} = \int_0^{2-t} F_{2n-1}(t)\, dt.$$

The result follows. □

As an example, for two pairs of equilength arcs we have

$$F_3(x) = \begin{cases} x^2/2, & 0 \leq x \leq 1, \\ (-2x^2+6x-3)/2, & 1 \leq x \leq 2, \\ (x^2-6x+9)/2, & 2 \leq x \leq 3. \end{cases}$$

We see that

$$\int_0^{2-t} F_3(x)\, dx = \int_0^1 F_3(x)\, dx + \int_1^{2-t} F_3(x)\, dx = \frac{1}{6}+\frac{2}{3}-\frac{t}{2}-\frac{t^2}{2}+\frac{t^3}{3}$$

$$\int_0^1 \int_0^{2-t} F_3(x)\, dx dt = \frac{1}{6} + \int_0^1 \frac{2}{3}-\frac{t}{2}-\frac{t^2}{2}+\frac{t^3}{3}\, dt = \frac{1}{6}+\frac{1}{3} = \frac{1}{2}$$

and the probability that two pairs of random equiarclength arcs with arclength uniformly distributed in $[0, \pi]$ are disjoint is $\frac{1}{8}$. Similarly for three pairs the probability is $\frac{9}{200}$.

10. Random arcs to Möbius groups

Given data $m_{\alpha_1}, m_{\alpha_2} \in \mathbb{S}$ with arclength $\ell_\alpha \in [0, \pi]$ we see, just as above, that the arcs centred on the m_{α_i} and of length ℓ_α determine a matrix which can be calculated by examination of the isometric circles. We have

$$A = \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix}, \quad c = i\sqrt{m_{\alpha_1}\, m_{\alpha_2}}\, \cot\frac{\ell_\alpha}{2}, \quad a = i\sqrt{\overline{m}_{\alpha_1}\, m_{\alpha_2}}\, \operatorname{cosec}\frac{\ell_\alpha}{2}, \tag{10.1}$$

where we make a consistent choice of sign by ensuring $c/a = m_{\alpha_1}\cos(\ell_\alpha/2)$. Of course, interchanging m_{α_1} and m_{α_2} sends a to $-\bar{a}$, and so the data actually uniquely determines the cyclic group $\langle f \rangle$ generated by the associated Möbius transformation

$$f(z) = -m_{\alpha_2}\, \frac{z + m_{\alpha_1}\cos\frac{\ell_\alpha}{2}}{z\, \cos\frac{\ell_\alpha}{2} + m_{\alpha_1}}$$

and not necessarily f itself.

As a consequence we have the following theorem.

Theorem 10.2. *There is a one-to-one correspondence between collections of n pairs of random arcs and n-generator Fuchsian groups preserving the associated probability distributions.*

A randomly chosen $\langle f\rangle \subset \mathcal{F}$ according to the distribution defined in Section 2A, corresponds uniquely to $m_{\alpha_1}, m_{\alpha_2} \in_u \mathbb{S}^1$ and $\ell_\alpha \in_u [0,\pi]$ with the distribution defined in Section 9.

Notice also that if we recognise the association of cyclic groups with the data and say two cyclic groups are close if they have close generators, then this association is continuous.

We have already seen that, for a pair of hyperbolic elements, if all the isometric disks are disjoint then the "ping pong" lemma implies discreteness of the groups in question. Then the association between Fuchsian groups and random arcs quickly establishes Theorems 8.3 and 8.4 via Lemma 9.4.

If f is a parabolic element of $\mathcal{F}$, then the isometric circles are adjacent and meet at the fixed point. Conversely, if two random arcs of arclength ℓ_α are adjacent we have $\arg(m_{\alpha_1}\overline{m}_{\alpha_2}) = \ell_\alpha$, and from (10.1)

$$a = i\left(\cos\frac{\ell_\alpha}{2} + i\sin\frac{\ell_\alpha}{2}\right)\operatorname{cosec}\frac{\ell_\alpha}{2} = -1 + i\cot\frac{\ell_\alpha}{2}$$

and $\operatorname{tr}^2(A) - 4 = 0$ so that A represents a parabolic transformation. Similarly, if the arcs overlap, then $\operatorname{tr}^2(A) \le 2$ and A represents an elliptic transformation.

Theorem 10.3. *Let f, g be randomly chosen parabolic elements in $\mathcal{F}$, by which we mean the isometric circles have diameter chosen as in Section 2A but are conditioned to be tangent. Then the probability $\langle f, g\rangle$ is discrete is at least $\frac{1}{6}$.*

Proof. It is not difficult to see that in fact the point of tangency is uniformly distributed in the circle and from our discussion above we see that this is the same as considering pairs of adjacent arcs. So as f and g are parabolic, their isometric disks are tangent and the point of intersection lies in a random arc of arclength uniformly distributed in $[0, 2\pi]$. Discreteness follows from the "ping pong" lemma and Lemma 9.2. □

11. Another discreteness criterion

Theorem 8.4 gives a discreteness criterion based on the disjointness of isomeric circles. Another criterion can be found as the first condition in [Rosenberger 1986, Theorem 3].

Theorem 11.1. *Let f and g be hyperbolic. If $\gamma(f,g) \le -4$, then $\langle f, g\rangle$ is discrete.*

With f and g selected randomly we recall from (7.3) that the condition $\gamma \le -4$ reads as

$$\begin{aligned}(11.2)\quad -\sin^2\theta_1\sin^2\theta_2 \ge 2\cos^2\theta_1(\sin^2\phi_2 - \sin^2\alpha\cos^2\theta_2)&\\ +\cos^2\theta_2\sin^2\phi_1 - 2\cos\alpha\cos\theta_1\cos\theta_2\sin\phi_1\sin\phi_2.&\end{aligned}$$

In our computational investigations we were drawn to the following remarkable observation.

Conjecture 11.3. *Let $f, g \in_* \mathcal{F}$ be hyperbolic and suppose the pairs of isometric circles are not disjoint. Then*

$$\mathbf{Pr}\{\gamma(f, g) \leq -4\} = \tfrac{1}{5}. \tag{11.4}$$

The expression in (11.2) provides good gradient bounds for a test and we were able to search this space to verify the conjecture to two decimal places. To gain just a little bit more accuracy without a great deal more care (and time) in our searches, we added in the additional result adapted from [Rosenberger 1986, Theorem 2].

Theorem 11.5. *Let f and g be hyperbolic and $\gamma(f, g) > 0$. Then $\langle f, g\rangle$ is discrete if there are representatives A and B in* PSL(2, ℝ), *for f and g, respectively, such that*

(1) $0 \leq \operatorname{tr}(A) \leq \operatorname{tr}(B) \leq |\operatorname{tr}(AB)|$,

(2) $\operatorname{tr}(AB) \leq -2$.

Of course the theorem applies to Nielsen equivalent pairs of generators. If $f, g \in_* \mathcal{F}$, we can compute

$$\begin{aligned}
\operatorname{tr}(f) &= 2\csc\theta_1\cos\phi_1,\\
\operatorname{tr}(g) &= 2\csc\theta_2\cos\phi_2,\\
\operatorname{tr}(fg) &= 2\csc\theta_1\csc\theta_2(\cos\theta_1\cos\theta_2\cos\alpha+\cos\phi_1+\phi_2),\\
\operatorname{tr}(fg^{-1}) &= 2\csc\theta_1\csc\theta_2(\cos(\phi_1-\phi_2)-2\cos\theta_1\cos\theta_2\cos\alpha).
\end{aligned}$$

Rearranging to avoid singularities for our gradient estimates (because of our normalisations we use f and g^{-1}), the tests we therefore derive from Theorem 11.5 are

(1) $\sin(\theta_2)\cos(\phi_1) \leq \sin(\theta_1)\cos(\phi_2)$,

(2) $\sin(\theta_1)\cos(\phi_2) \leq |\cos(\phi_1-\phi_2) - 2\cos(\theta_1)\cos(\theta_2)\cos(\alpha)|$,

(3) $\cos(\phi_1-\phi_2) + \sin(\theta_1)\sin(\theta_2) \leq \cos(\theta_1)\cos(\theta_2)\cos(\alpha)$.

A few simple experiments show that if f and g are hyperbolic with intersection isometric circles, then the test above as well as that obtained by the interchange of θ_1 and θ_2 (and the immaterial interchanging of ϕ_1 and ϕ_2) occurs about 5% of the time. It is easy to prove that it happens at least 2% of the time, giving us an easy error bound for our computational verification of Conjecture 11.3. Putting these together, with the bound from isometric circle disjointness yields the following theorem.

Theorem 11.6. *Let $f, g \in_* \mathcal{F}$ be hyperbolic. Then*

$$\mathbf{Pr}\{\langle f, g\rangle \textit{is discrete}\} \geq \tfrac{2}{5}. \tag{11.7}$$

12. Representations of $\mathbb{Z}_n * \mathbb{Z}$ in PSL(2, $\mathbb{R}$)

The discreteness criteria we have used above are not particularly sophisticated, but only minor improvements are known in the generality in which we use them. These amount to looking at deeper level configurations of isometric circles and quickly become extremely complicated. However there is one case where rather more precise results are known in general and that is the case where one generator has order 2. For Fuchsian groups we know more when a generator has finite order. In [Gehring et al. 2001], precise results are given to determine when $G = \langle f, g\rangle$ is discrete, where $\beta(g) = -4$, $\beta(f) \in \mathbb{R}$ and $\gamma(f, g) \in \mathbb{R}$.

It is important to note that this case is not so special, as evidenced by Theorem 12.1.

Theorem 12.1 [Gehring and Martin 1994]. *Let $\langle f, g\rangle$ be a discrete subgroup of* PSL(2, $\mathbb{C}$). *Then there is an elliptic Φ of order 2 such that $\langle f, \Phi\rangle$ is discrete, and*

$$\gamma(f, g) = \operatorname{tr}[f, g] - 2 = \operatorname{tr}[f, \Phi] - 2 = \gamma(f, \Phi)$$

This theorem explains in part why we would like to identify the p.d.f. for $\gamma(f, g)$. Care must be taken in using this result in our setting since although f, g might be randomly selected, it is not the case that Φ is.

If we choose a random matrix B conditioned by the assumptions $\operatorname{tr}(B) = 0$, and another random matrix A, then we have the forms

$$A = \begin{pmatrix} e^{i\phi}\csc(\eta) & e^{i\alpha}\cot(\eta) \\ e^{-i\alpha}\cot(\eta) & e^{-i\phi}\csc(\eta) \end{pmatrix}, \quad B = \begin{pmatrix} i\csc(\theta) & e^{i\psi}\cot(\theta) \\ e^{-i\psi}\cot(\theta) & -i\csc(\theta) \end{pmatrix},$$

where all the angles are chosen uniformly in $[0, \pi]$ and we have simplified the variables, replacing η and θ with $\eta/2$ and $\theta/2$ as above:

$$\beta = 4\csc^2(\eta)\cos^2(\phi) - 4,$$

$$\gamma = 4\cot^2(\eta)(\csc^2(\theta) - \cot^2(\theta)\sin^2(\alpha)) + 4\csc^2(\eta)\cot^2(\theta)\sin^2(\phi) \\ -8\cot(\eta)\csc(\eta)\cot(\theta)\csc(\theta)\sin(\phi)\cos(\alpha),$$

where we have assumed $\beta \geq 0$ to simplify the last equation and written α for $\alpha - \psi$ since both are uniformly distributed.

Now by [Gehring et al. 2001, Theorem 3.1], the group $\langle A, B | A^2 = 1\rangle$ projects to a faithful discrete nonelementary subgroup of PSL(2, $\mathbb{R}$) if and only if $4 \leq \beta + 4 \leq \gamma$. After some manipulation, we need to decide when

$$\sin^2(\alpha)\sin^2(\eta)\sin^2(\theta) \leq (\cos(\alpha)\cos(\eta) - \cos(\theta)\cos(\phi))^2.$$

Some parity and symmetry considerations reduce the problem to finding $8/\pi^4$ times the measure of the set

$$\{(\eta, \theta, \alpha, \phi) \in [0, \pi/2]^2 \times [0, \pi]^2 : \sin\alpha\sin\eta\sin\theta \leq \cos\alpha\cos\eta - \cos\theta\cos\phi\}.$$

We could not find a closed form for this number, but used numerical techniques to obtain the following theorem.

Theorem 12.2. *Let $f, g \in_* \mathcal{F}$ be conditioned by $f^2 = 1$. Then*

$$\mathbf{Pr}\{\overline{\langle f, g\rangle}\} = \mathcal{F}\} = 0.706 \pm 0.001.$$

However there is one family of special cases to which we can give a precise theorem. A slight generalisation of [Gehring et al. 2001] yields the following:

Lemma 12.3. *Let $\Gamma = \langle f, g\rangle$ be a Möbius group with $f^n = 1$ and g hyperbolic. Then Γ is discrete and free on its generators if and only if*

$$\gamma \geq \left(\sqrt{\beta+4} + 2\cos\frac{\pi}{n}\right)^2,$$

where $\beta = \operatorname{tr}^2(g) - 4$.

The "boundary groups" are the groups with presentation $\langle a, b | a^n = b^\infty = 1\rangle$.

Thus if $A \sim f : z \to \zeta z$ with $\zeta^n = 1$ and $B \sim g \in_* \mathcal{F}$ is hyperbolic and randomly chosen, then we compute

$$\beta = \operatorname{tr}^2(B) - 4 = 4\csc^2\eta\cos^2\phi - 4, \gamma = \operatorname{tr}[A, B] - 2 = 4\sin^2\frac{\pi}{n}\cot^2\eta,$$

and our test for discreteness is $\gamma \geq \left(\sqrt{\beta+4} + 2\cos\frac{\pi}{n}\right)^2$. That is,

$$4\sin^2\frac{\pi}{n}\cot^2\eta \geq \left(2\csc\eta\cos(\phi) + 2\cos\frac{\pi}{n}\right)^2.$$

We want to take the square roots here. Since $\eta \in_u \left[0, \frac{\pi}{2}\right]$, the left-hand side is positive. Similarly, since $\phi \in_u [0, \pi]$ it makes no difference to assume $\phi \in_u \left[0, \frac{\pi}{2}\right]$. We therefore should determine when

$$\sin\left(\frac{\pi}{n} - \eta\right) \geq \cos(\phi), \quad \eta, \phi \in_u \left[0, \frac{\pi}{2}\right]. \tag{12.4}$$

It is immediate that this probability is no more than $\frac{1}{n}$ as the right-hand side is positive. In fact, this shows that for the angles distributed as above,

$$\Pr\left\{\sin\left(\frac{\pi}{n} - \eta\right) \geq \cos(\phi)\right\} = \frac{1}{n}\Pr\{\sin(\theta) \geq \cos(\phi)\}.$$

This probability is then

$$\frac{4}{\pi^2}\int_0^{\pi/n}\int_0^{\sin(\theta)} \frac{dx\,d\theta}{\sqrt{1-x^2}} = \frac{2}{n^2}.$$

In terms of our original question about topological generation this reads as:

Theorem 12.5. *Let $f(z) = \zeta z$ and $\zeta^n = 1$. Let $g \in_* \mathcal{F}$ be a randomly chosen hyperbolic. Then*

$$\mathbf{Pr}\left\{\overline{\langle f, g\rangle} = \mathcal{F}\right\} = 1 - \frac{2}{n^2}.$$

References

[Auerbach 1934] H. Auerbach, "Sur les groupes linéaires bornes, III", *Stud. Math.* **5** (1934), 43–49. Zbl

[Beardon 1983] A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics **91**, Springer, 1983. MR Zbl

[Calegari and Walker 2015] D. Calegari and A. Walker, "Random groups contain surface subgroups", *J. Amer. Math. Soc.* **28**:2 (2015), 383–419. MR Zbl

[Cao 1995] C. Cao, "Some trace inequalities for discrete groups of Möbius transformations", *Proc. Amer. Math. Soc.* **123**:12 (1995), 3807–3815. MR Zbl

[Cao and Meyerhoff 2001] C. Cao and G. R. Meyerhoff, "The orientable cusped hyperbolic 3-manifolds of minimum volume", *Invent. Math.* **146**:3 (2001), 451–478. MR Zbl

[Dixon 1969] J. D. Dixon, "The probability of generating the symmetric group", *Math. Z.* **110** (1969), 199–205. MR Zbl

[Gabai et al. 2003] D. Gabai, G. R. Meyerhoff, and N. Thurston, "Homotopy hyperbolic 3-manifolds are hyperbolic", *Ann. of Math.* (2) **157**:2 (2003), 335–431. MR Zbl

[Gabai et al. 2011] D. Gabai, R. Meyerhoff, and P. Milley, "Mom technology and volumes of hyperbolic 3-manifolds", *Comment. Math. Helv.* **86**:1 (2011), 145–188. MR Zbl

[Gehring and Martin 1991a] F. W. Gehring and G. J. Martin, "Inequalities for Möbius transformations and discrete groups", *J. Reine Angew. Math.* **418** (1991), 31–76. MR Zbl

[Gehring and Martin 1991b] F. W. Gehring and G. J. Martin, "Some universal constraints for discrete Möbius groups", pp. 205–220 in *Paul Halmos*, edited by J. H. Ewing and F. W. Gehring, Springer, 1991. MR Zbl

[Gehring and Martin 1994] F. W. Gehring and G. J. Martin, "Commutators, collars and the geometry of Möbius groups", *J. Anal. Math.* **63** (1994), 175–219. MR Zbl

[Gehring and Martin 2009] F. W. Gehring and G. J. Martin, "Minimal co-volume hyperbolic lattices, I: The spherical points of a Kleinian group", *Ann. of Math.* (2) **170**:1 (2009), 123–161. MR Zbl

[Gehring et al. 1997] F. W. Gehring, C. Maclachlan, G. J. Martin, and A. W. Reid, "Arithmeticity, discreteness and volume", *Trans. Amer. Math. Soc.* **349**:9 (1997), 3611–3643. MR Zbl

[Gehring et al. 2001] F. W. Gehring, J. P. Gilman, and G. J. Martin, "Kleinian groups with real parameters", *Commun. Contemp. Math.* **3**:2 (2001), 163–186. MR Zbl

[Gilman 1995] J. Gilman, *Two-generator discrete subgroups of* $\mathrm{PSL}(2,\mathbb{R})$, Mem. Amer. Math. Soc. **561**, American Mathematical Society, Providence, RI, 1995. MR Zbl

[Jørgensen 1976] T. Jørgensen, "On discrete groups of Möbius transformations", *Amer. J. Math.* **98**:3 (1976), 739–749. MR Zbl

[Kantor and Lubotzky 1990] W. M. Kantor and A. Lubotzky, "The probability of generating a finite classical group", *Geom. Dedicata* **36**:1 (1990), 67–87. MR Zbl

[Liebeck and Shalev 1995] M. W. Liebeck and A. Shalev, "The probability of generating a finite simple group", *Geom. Dedicata* **56**:1 (1995), 103–113. MR Zbl

[Marshall and Martin 2012] T. H. Marshall and G. J. Martin, "Minimal co-volume hyperbolic lattices, II: Simple torsion in a Kleinian group", *Ann. of Math.* (2) **176**:1 (2012), 261–301. MR Zbl

[Martin 2019] G. J. Martin, "Random ideal hyperbolic quadrilaterals, the cross ratio distribution and punctured tori", *J. London Math. Soc.* (online publication June 2019).

[Martin et al. 2019] G. J. Martin, G. O'Brien, and Y. Yamashita, "Random Kleinian groups, II: Two parabolic generators", *Exp. Math.* (online publication February 2019).

[Noskov 2018] G. A. Noskov, "Efficient topological generation in compact Lie groups", *J. Phys. Conf. Ser.* **1050** (2018), art. id. 012059.

[Rosenberger 1986] G. Rosenberger, "All generating pairs of all two-generator Fuchsian groups", *Arch. Math.* (*Basel*) **46**:3 (1986), 198–204. MR Zbl

[Springer 1979] M. D. Springer, *The algebra of random variables*, Wiley, New York, 1979. MR Zbl

Received November 4, 2018. Revised July 11, 2019.

GAVEN MARTIN
INSTITUTE FOR ADVANCED STUDY
MASSEY UNIVERSITY
ALBANY
AUCKLAND
NEW ZEALAND

g.j.martin@massey.ac.nz

GRAEME O'BRIEN
INSTITUTE FOR ADVANCED STUDY
MASSEY UNIVERSITY
AUCKLAND
NEW ZEALAND

qegraeme@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2019 is US $490/year for the electronic version, and $665/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 303 No. 2 December 2019

0030-8730(201912)303:2;1-H