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BOUNDS OF DOUBLE ZETA-FUNCTIONS AND THEIR APPLICATIONS

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In this paper, we shall improve the bounds on Euler–Zagier type double zeta-functions in the region $0 < \operatorname{Re} s_j < 1$ $(j = 1, 2)$ which provides a positive answer to a conjecture posed by Kiuchi and Tanigawa (2006). As an application, we improve the error term that appears in the asymptotic formula for the moment of the double zeta-function.

1. Introduction

The Euler–Zagier type multiple zeta-functions are defined by

$$\zeta_k(s_1, \dots, s_k) = \sum_{0<n_1<\cdots<n_k} n_1^{-s_1} \cdots n_k^{-s_k}, \tag{1-1}$$

where $s_j = \sigma_j + i t_j$ $(j = 1, \dots, k)$ are complex variables. The series in (1-1) is absolutely convergent for $\sigma_k > 1$ and $\sigma_{k-j} + \sigma_{k-j+1} + \cdots + \sigma_k > j + 1$ for all $j = 1, \dots, k-1$. Euler first studied values of the double zeta-function at positive integers in the eighteenth century. It came into more prominence after Atkinson [1949] investigated the analytic properties of $\zeta_2(s_1, s_2)$ in his study on the explicit representation of the error term in the asymptotic formula for the mean square of the Riemann zeta-function $\zeta(s)$ on the critical line.

Zhao [2000] and Akiyama, Egami and Tanigawa [Akiyama et al. 2001] proved the analytic continuation of $\zeta_k(s_1, \dots, s_k)$ to the whole $\mathbb{C}^k$. They determined the location of singularities. Many kinds of multiple zeta-functions are defined and their algebraic and analytic properties have been studied extensively. See, e.g., [Matsumoto and Tsumura 2006].

In the theory of zeta-functions, the order of a zeta-function on the vertical line in the critical strip plays an important role in various arithmetical problems. It is natural to study the order of the multiple zeta-functions. Kiuchi and Tanigawa [2006], applying the methods of double exponential sum of Titchmarsh, obtained upper bounds for $\zeta_2(s_1, s_2)$ for $0 \le \sigma_j < 1$ under the conditions $|t_1| \asymp |t_2|$ and

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$|t_1+t_2| \gg 1$ $(|t_1|, |t_2| \geq 2)$. Two special cases of their results are as follows [Kiuchi and Tanigawa 2006, (1.9)]:

$$\zeta_2(it_1, it_2) \ll |t_1| \log^2 |t_1|, \quad \zeta_2\left(\tfrac{1}{2}+it_1, \tfrac{1}{2}+it_2\right) \ll |t_1|^{\frac{1}{3}} \log^2 |t_1|.$$

In [loc. cit., p. 448, Remark 1.3], based on these bounds they conjectured that

$$\zeta_2(s_1, s_2) \ll |t_1|^{\mu(\sigma_1)+\varepsilon} |t_2|^{\mu(\sigma_2)+\varepsilon} \tag{1-2}$$

under the above conditions, where $\mu(\sigma)$ is the infimum of c such that $\zeta(\sigma+it) \ll |t|^c$, $\varepsilon > 0$ is any positive constant. In [Kiuchi et al. 2011], however, it is noticed that (1-2) does not hold in general if we do not assume such conditions like $|t_1| \asymp |t_2|$ and $|t_1+t_2| \gg 1$. In fact, it is shown that

$$\zeta_2\left(\tfrac{1}{2}+it_1, \tfrac{1}{2}+it_2\right) = \Omega(|t_1|^{\frac{1}{3}+\varepsilon}) \quad \text{for } |t_2| \ll |t_1|^{\frac{1}{6}-\varepsilon}. \tag{1-3}$$

The purpose of this paper is to derive more general upper bounds of $\zeta_2(s_1, s_2)$ and apply it to give sufficient conditions so that (1-2) holds with the function $p(\sigma)$ (defined in (2-7) below) as exponent instead of $\mu(\sigma)$.

We shall also apply our result for the mean square of double zeta-function which is a recent topic in the theory of multiple zeta-functions. In fact, Matsumoto and Tsumura [2015] studied the mean value of $\zeta_2(s_1, s_2)$ with respect to t_2 for various σ_2. Recently Kiuchi and Minamide [2016], using the expression of $\zeta_2(s_1, s_2)$ in [Kiuchi et al. 2011], obtained a good mean square estimate of $\zeta_2(s_1, s_2)$ in the cases $\sigma_1+\sigma_2 = 1$, and $\frac{1}{2} \leq \sigma_1+\sigma_2 < \frac{3}{2}$ and $\sigma_1+\sigma_2 \neq 1$. For instance, if $0 < \sigma_j < 1$, $1 < \sigma_1+\sigma_2 < \frac{3}{2}$ and $2 \leq t_2 \leq T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$, they showed that

$$\begin{aligned} &\int_2^T |\zeta_2(s_1, s_2)|^2 \, dt_1 \\ &\quad = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O(t_2^{-\frac{1}{2}} T^{\frac{5}{2}-\sigma_1-\sigma_2}). \end{aligned} \tag{1-4}$$

See [Kiuchi and Minamide 2016, Theorem 2]. By applying our result we shall obtain an improvement of (1-4).

2. Preliminaries

In this section we recall a formula for $\zeta_2(s_1, s_2)$ in a certain region obtained by Kiuchi, Tanigawa and Zhai [Kiuchi et al. 2011]. We also discuss the bounds on the zeta-function in the critical region which we use later to improve the bounds of the double zeta-function.

Theorem [Kiuchi et al. 2011, p. 17, Theorem 2]. Let $0 < \sigma_j < 1$ $(j = 1, 2)$ and $\varepsilon > 0$ any small constant. We have

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \tfrac{1}{2}\zeta(s_1 + s_2) + J(s_1, s_2), \tag{2-1}$$

where $J(s_1, s_2)$ is expressed by

$$J(s_1, s_2) = \chi(s_2) \sum_{n \le \frac{|t_2|}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O(|t_2|^{\delta+\varepsilon}) \tag{2-2}$$

and

$$\chi(s_2) = 2(2\pi)^{s_2-1} \sin(\pi s_2/2)\Gamma(1 - s_2), \quad \sigma_a(n) = \sum_{d|n} d^a,$$
$$\text{and} \quad \delta = \max(0, 1 - \sigma_1 - \sigma_2).$$

The function $\chi(s)$ is the function which appears in the functional equation of the Riemann zeta-function:

$$\zeta(s) = \chi(s)\zeta(1 - s). \tag{2-3}$$

It is well-known that [Ivić 1985, p. 9]

$$\chi(\sigma + it) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{i(t \pm \frac{\pi}{4})}\left(1 + O\left(\frac{1}{|t|}\right)\right) \quad (|t| \ge t_0 > 0) \tag{2-4}$$

or more precisely

$$\chi(\sigma + it) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{i(t \pm \frac{\pi}{4})}\left(1 - \frac{i(\sigma^2 - \sigma + \frac{1}{6})}{2t} + O\left(\frac{1}{|t|^2}\right)\right).$$

We always understand that $t \pm \frac{\pi}{4} = t + \operatorname{sgn}(t)\frac{\pi}{4}$.

In [Kiuchi et al. 2011], they estimated the sum in (2-2) trivially and derived

$$J(s_1, s_2) \ll |t_2|^{\frac{1}{2}+\delta+\varepsilon}, \tag{2-5}$$

[loc. cit., (1.6)], where δ is defined in the above Theorem.

In this paper we shall derive sharper upper bounds for $J(s_1, s_2)$. See Theorems 1 and 2 in Section 3. To evaluate $J(s_1, s_2)$, let us consider the sum

$$E(x; s_1, s_2) := \sum_{n \le x} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}}. \tag{2-6}$$

It is natural to apply Perron's formula to get an estimate for such a sum. We get an integral expression of $E(x; s_1, s_2)$ involving the Riemann zeta-function, and from which we can find a new kind of estimate by using the bounds of the Riemann zeta-function.

Let $p(\sigma)$ be the function defined by

$$\zeta(\sigma+it) \ll |t|^{p(\sigma)+\varepsilon}, \tag{2-7}$$

where ε is an arbitrary small positive number which needs not to be the same at each occurrence. We assume that $p(\sigma)$ satisfies the relation

$$\tfrac{1}{2}-\sigma+p(1-\sigma)=p(\sigma), \tag{2-8}$$

which is required by (2-3) and (2-4). We shall fix $p(\sigma)$ as $p(\sigma)=\frac{1}{2}-\sigma$ for $\sigma \le 0$ and $p(\sigma)=0$ for $\sigma \ge 1$. Hereafter, we take $p(\sigma)$ for $0 \le \sigma \le 1$ as

$$p(\sigma)=\begin{cases} \frac{1}{2}-\left(1-2p\left(\frac{1}{2}\right)\right)\sigma & \text{for } 0<\sigma\le\frac{1}{2}, \\ 2p\left(\frac{1}{2}\right)(1-\sigma) & \text{for } \frac{1}{2}<\sigma\le 1. \end{cases} \tag{2-9}$$

It is known that $p\left(\frac{1}{2}\right) \le \frac{1}{6}$ (Hardy and Littlewood), $p\left(\frac{1}{2}\right) \le \frac{32}{205}=0.156\ldots$ [Huxley 2005] and $p\left(\frac{1}{2}\right) \le \frac{13}{84}=0.1547\ldots$ [Bourgain 2017]. If the Lindelöf hypothesis is true we can take $p(\sigma)=\frac{1}{2}-\sigma$ for $0<\sigma\le\frac{1}{2}$ and $p(\sigma)=0$ for $\frac{1}{2}<\sigma\le 1$.

3. Statement of main results

In this section we will state our main results. The first two theorems deal with the bound on $J(s_1,s_2)$ defined in (2-2).

Theorem 1. *Let* $0<\sigma_j<1$ $(j=1,2)$ *and assume that* $\sigma_1+\sigma_2\ge 1$. *Then we have*

$$J(s_1,s_2) \ll |t_2|^{p(\sigma_1+\sigma_2-1)+\varepsilon}. \tag{3-1}$$

To state the next theorem we need to prepare two notations. Let

$$Q_{u_0}(s_1,s_2)=\begin{cases} \zeta(s_1)\zeta(s_2) & \text{if } u_0<0, \\ 0 & \text{otherwise} \end{cases} \tag{3-2}$$

and

$$Q_1(x)=\frac{\zeta(2-s_1-s_2)x^{1-s_1}}{1-s_1}. \tag{3-3}$$

Theorem 2. *Let* $0<\sigma_j<1$ $(j=1,2)$, $s_1+s_2\ne 1$, *and let* u_0 *be a real number such that* $u_0<1-\sigma_1$, $u_0\ne\sigma_2$ *and* $u_0\ne 0$.

(a) *Suppose that* $|t_1| \gg |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}}$. *Then we have*

$$J(s_1,s_2)=Q_{u_0}(s_1,s_2)+O\left(|t_2|^{p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_o+\sigma_1)+\varepsilon}+|t_2|^{\delta+\varepsilon}\right).$$

(b) *Suppose that* $|t_1| \ll |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}}$.

(i) *If* $\frac{1}{2} \le u_0+\sigma_1 < 1$, *we have*

$$J(s_1,s_2) = Q_{u_0}(s_1,s_2) + \chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) + O\left(|t_2|^{p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_0+\sigma_1)} + |t_2|^{\delta+\varepsilon}\right).$$

(ii) *If* $u_0+\sigma_1 < \frac{1}{2}$, *we have*

$$J(s_1,s_2) = Q_{u_0}(s_1,s_2) + \chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) + O(|t_2|^{\delta+\varepsilon}) + \begin{cases} O\left(|t_2|^{p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}\right) & \text{if } |t_1| \gg |t_2|^{\frac{(1/2-(u_0+\sigma_1))(\delta+1/2-p(\sigma_2-u_0))}{p(u_0+\sigma_1)(3/2-(u_0+\sigma_1))}}, \\ O\left(|t_2|^{\frac{(\delta+1/2)(1/2-(u_0+\sigma_1))+p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}+\varepsilon}\right) & \text{if } |t_1| \ll |t_2|^{\frac{(1/2-(u_0+\sigma_1))(\delta+1/2-p(\sigma_2-u_0))}{p(u_0+\sigma_1)(3/2-(u_0+\sigma_1))}}. \end{cases}$$

By (2-4), we observe that

$$\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) = \frac{\zeta(2-s_1-s_2)}{1-s_1}\left(\frac{|t_2|}{2\pi}\right)^{\frac{3}{2}-s_1-s_2} e^{i(t_2\pm\frac{\pi}{4})} + O\left(\frac{|\zeta(2-s_1-s_2)|}{|1-s_1|}|t_2|^{\frac{1}{2}-\sigma_1-\sigma_2}\right). \tag{3-4}$$

We shall prove Theorems 1 and 2 in Section 5. It seems to be useful to specialize the choice of u_0. We shall give several cases as Corollaries in Section 6.

As an application of Theorem 2, we shall give sufficient conditions for (1-2). More precisely, we shall prove

Theorem 3. *Let* $0<\sigma_j<1$ *and* $|t_1+t_2| \gg 1$.

(a) *If* $\sigma_1+\sigma_2 \ge 1$ *and* $|t_2|^{\frac{1/2-p(\sigma_2)}{1+p(\sigma_1)}} \ll |t_1| \ll |t_2|^{\frac{p(\sigma_2)+1}{p(\sigma_1+\sigma_2-1)-p(\sigma_1)}}$, *we have*

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

(b) *If* $\sigma_1+\sigma_2 \le 1$ *and*

$$\max\left\{|t_2|^{\frac{1-\sigma_1-\sigma_2-p(\sigma_2)}{p(\sigma_1)}}, |t_2|^{\frac{3/2-\sigma_1-\sigma_2-p(\sigma_2)}{1+p(\sigma_1)}}\right\} \ll |t_1| \ll |t_2|^{\frac{p(\sigma_2)+1}{3/2-\sigma_1-\sigma_2-p(\sigma_1)}},$$

we have

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

As complementary cases to Theorem 3(a) we can show the following theorem.

Theorem 4. *Let* $0<\sigma_j<1$ *and* $|t_1+t_2| \gg 1$.

(a) *If* $|t_2|^{\frac{p(\sigma_2)+1}{p(\sigma_1+\sigma_2-1)-p(\sigma_1)}} \ll |t_1|$, *then*

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1+\sigma_2-1)+\varepsilon}.$$

(b) *If* $\sigma_1+\sigma_2 \geq \frac{3}{2}$ *and* $|t_2|^{\frac{p(\sigma_1)}{p(\sigma_1)+1}} \ll |t_1| \ll |t_2|^{\frac{1/2-p(\sigma_2)}{1+p(\sigma_1)}}$, *then*

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

(c) *If* $1 \leq \sigma_1+\sigma_2 < \frac{3}{2}$, $\frac{1}{2} \leq \sigma_1 < 1$ *and* $|t_2|^{\frac{p(\sigma_1+\sigma_2-1)-p(\sigma_2)}{1+p(\sigma_1)}} \ll |t_1| \ll |t_2|^{\frac{1/2-p(\sigma_2)}{1+p(\sigma_1)}}$, *then*

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

(d) *If* $1 \leq \sigma_1+\sigma_2 < \frac{3}{2}$, $0 < \sigma_1 < \frac{1}{2}$ *and*

$$\max\left\{|t_2|^{\frac{(1/2-\sigma_1)(1/2-p(\sigma_2))}{p(\sigma_1)(3/2-\sigma_1)}}, |t_2|^{\frac{p(\sigma_1+\sigma_2-1)-p(\sigma_2)}{1+p(\sigma_1)}}\right\} \ll |t_1| \ll |t_2|^{\frac{1/2-p(\sigma_2)}{1+p(\sigma_1)}},$$

then we have

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

There are other cases.

Theorem 5. *Let* $0 < \sigma_j < 1$ *and* $|t_1+t_2| \gg 1$.

(a) *If* $\sigma_1+\sigma_2 \leq 1$, $\frac{1}{2} \leq \sigma_1 < 1$ *and* $1 \ll |t_1| \ll |t_2|^{\frac{3/2-\sigma_1-\sigma_2-p(\sigma_2)}{1+p(\sigma_1)}}$, *then we have*

$$\zeta_2(s_1,s_2) \ll |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2+\varepsilon}|t_1|^{-1}.$$

(b) *If* $\sigma_1+\sigma_2 \leq 1$, $0 < \sigma_1 < \frac{1}{2}$ *and*

$$1 \ll |t_1| \ll \min\left\{|t_2|^{\frac{3/2-\sigma_1-\sigma_2-p(\sigma_1-\sigma_2-1/2)}{1+p(1/2)}}, |t_2|^{\frac{3/2-\sigma_1-\sigma_2-p(\sigma_2)}{1+p(\sigma_1)}}\right\},$$

then we have

$$\zeta_2(s_1,s_2) \ll |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2+\varepsilon}|t_1|^{-1}.$$

Theorems 3, 4 and 5 will be proved in Section 7.

Finally we shall apply Theorem 1 to obtain the mean square of $\zeta_2(s_1,s_2)$ in case of $1 \leq \sigma_1+\sigma_2 < \frac{3}{2}$.

Theorem 6. *Let* $1 \leq \sigma_1+\sigma_2 < \frac{3}{2}$ *and* $t_2 > 2$. *Then we have*

$$\int_2^X |\zeta_2(s_1,s_2)|^2\,dt_1 = \frac{(2\pi)^{2\sigma_1+2\sigma_2-3}\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}X^{4-2\sigma_1-2\sigma_2} + O\left(X^{\frac{5}{2}-\sigma_1-\sigma_2+\varepsilon}t_2^{p(\sigma_1+\sigma_2-1)-1+\varepsilon}\right) \tag{3-5}$$

if $t_2 \ll X^{\frac{3/2-\sigma_1-\sigma_2}{1+p(\sigma_1+\sigma_2-1)}}$.

Theorem 6 will be proved in Section 8. Note that this (3-5) is an improvement of Kiuchi and Minamide's result (1-4).

After a slight modification, our method can be applied to the mean square of $\zeta_2(s_1,s_2)$ for the case $\sigma_1+\sigma_2 < 1$, which will be treated in our next paper.

4. Some Lemmas

To prove Theorems 1 and 2 we need to recall the following Lemmas. To express $E(x; s_1, s_2)$ as an integral of zeta-functions, we first require Perron's formula [Tenenbaum 2015, p. 220, Corollary 2.4].

Lemma 7. *Let $F(s) := \sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series with an abscissa of absolute convergence σ_a. Suppose there exists a real number $\alpha \geq 0$ such that*

$$\sum_{n\geq 1} |a_n| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\alpha} \quad (\sigma_a < \sigma < \sigma_a + 1),$$

and that B is a nondecreasing function satisfying

$$|a_n| \leq B(n) \quad (n \geq 1).$$

Then, for $x \geq 2$, $T \geq 2$, $\sigma \leq \sigma_a$, $\kappa = \sigma_a - \sigma + 1/\log x$, we have

$$\sum_{n\leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} F(s+w) \frac{x^w}{w}\, dw + O\biggl(x^{\sigma_a - \sigma} \frac{(\log x)^\alpha}{T} + \frac{B(2x)}{x^\sigma} \Bigl(1 + x \frac{\log T}{T} \Bigr) \biggr). \tag{4-1}$$

In order to prove Theorem 2 in Section 3 we need some results on the mean square of the Riemann zeta-function:

$$\int_1^T \bigl|\zeta\bigl(\tfrac{1}{2} + it\bigr)\bigr|^2\, dt = T \log \frac{T}{2\pi} + (2\gamma - 1)T + O(T^{\frac{1}{3}+\varepsilon}) \tag{4-2}$$

and

$$\int_1^T |\zeta(\sigma + it)|^2\, dt = \zeta(2\sigma)T + A(\sigma)T^{2-2\sigma} + O(T^{\frac{2}{3}(1-\sigma)} \log^{\frac{2}{9}} T) \tag{4-3}$$

for $\frac{1}{2} < \sigma < 1$. The formula (4-2) was shown by Balasubramanian [1978]. Later Heath-Brown and Huxley [1990] improved the error term to $O(T^{\frac{7}{22}})$.

Lemma 8 [Kiuchi and Minamide 2016, Lemma 3]. *Let $0 < \sigma < \frac{1}{2}$. For any sufficiently large positive number T, we have*

$$\int_2^T |\zeta(\sigma + it)|^2\, dt = (2\pi)^{2\sigma - 1} \frac{\zeta(2 - 2\sigma)}{2 - 2\sigma} T^{2-2\sigma} + \zeta(2\sigma)T + O(T^{1 - \frac{4}{3}\sigma}).$$

Using (4-2), (4-3) and Lemma 8 we obtain the following:

Lemma 9. *Let $s = \sigma + it$ be fixed and u a real number. Suppose that $|t| \leq T$, $u + \sigma < 1$ and $u \neq 0$. Let*

$$I = \int_{u-iT}^{u+iT} \left| \frac{\zeta(w+s)}{w} \right| |dw|.$$

Then we have

$$I \ll \begin{cases} |t|^{p(u+\sigma)+\varepsilon} + T^{\varepsilon} & \text{if } \frac{1}{2} \le u+\sigma < 1, \\ |t|^{p(u+\sigma)+\varepsilon} + T^{\frac{1}{2}-(u+\sigma)+\varepsilon} & \text{if } 0 \le u+\sigma \le \frac{1}{2}, \\ T^{\frac{1}{2}-(u+\sigma)+\varepsilon} & \text{if } u+\sigma \le 0, \end{cases}$$

where ε is any small positive constant.

Proof. It is easy to prove for $u+\sigma \le 0$. Suppose that $0 < u+\sigma < 1$. It is enough to consider the integral

$$K := \int_0^T \frac{|\zeta(u+\sigma+i(v+t))|}{1+v}\,dv$$

for $t > 0$. We divide the path of integration as

$$K = \int_0^t + \int_t^T =: K_1 + K_2,$$

say. It is trivial that

$$K_1 \ll |t|^{p(u+\sigma)+\varepsilon}.$$

Since $v \ge \frac{1}{2}(v+t)$ in K_2, we have

$$\begin{aligned} K_2 &\ll \int_t^T \frac{|\zeta(u+\sigma+i(v+t))|}{v+t}\,dv = \int_{2t}^{T+t} \frac{|\zeta(u+\sigma+iv)|}{v}\,dv \\ &\le \int_1^{2T} \frac{|\zeta(u+\sigma+iv)|}{v}\,dv \ll (\log T)^{\frac{1}{2}} \left(\int_1^{2T} \frac{|\zeta(u+\sigma+iv)|^2}{v}\,dv \right)^{\frac{1}{2}}. \end{aligned}$$

In the last step we have used Cauchy's inequality. Let

$$F_\sigma(T) = \int_1^T |\zeta(\sigma+it)|^2\,dt.$$

By integration by parts and (4-2), (4-3) and Lemma 8, we see that

$$\begin{aligned} \int_1^{2T} \frac{|\zeta(u+\sigma+iv)|^2}{v}\,dv &= \frac{F_{u+\sigma}(v)}{v}\Big|_1^{2T} + \int_1^{2T} \frac{F_{u+\sigma}(v)}{v^2}\,dv \\ &\ll \begin{cases} T^{1-2(u+\sigma)} & \text{if } u+\sigma < \frac{1}{2}, \\ \log^2 T & \text{if } u+\sigma = \frac{1}{2}, \\ \log T & \text{if } u+\sigma > \frac{1}{2}. \end{cases} \end{aligned}$$

Thus we get the assertion of Lemma 9. □

In Lemma 9 it is easy to improve T^{ε} to some power of $\log T$, but this is enough for our present purposes.

To complete the proof of Theorem 2, we require another lemma which is Lemma 2.4 of [Graham and Kolesnik 1991] (originally, Srinivasan [1962/63, p. 179, Lemma 3]).

Lemma 10. *Suppose that*

$$L(H)=\sum_{i=1}^{m} A_i H^{a_i}+\sum_{j=1}^{n} B_j H^{-b_j},$$

where A_i, B_j, a_i and b_j are positive. Assume that $H_1 \le H_2$. Then there is some H with $H_1 \le H \le H_2$ and

$$L(H)\ll \sum_{i=1}^{m}\sum_{j=1}^{n}(A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)}+\sum_{i=1}^{m} A_i H_1^{a_i}+\sum_{j=1}^{n} B_j H_2^{-b_j}.$$

5. Estimations of $J(s_1, s_2)$

In this section we give proofs of Theorem 1 and 2. To begin with we shall consider the Dirichlet series $F(w)$ defined by

$$F(w)=\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n^w}=\zeta(w)\zeta(w-(1-s_1-s_2)).$$

The abscissa of absolute convergence of $F(w)$ is given by

$$\sigma_a=\delta+1=\begin{cases}1 & \text{if } \sigma_1+\sigma_2\ge 1,\\ 2-\sigma_1-\sigma_2 & \text{if } \sigma_1+\sigma_2<1\end{cases}$$

and $B(n)\ll n^{\delta}d(n)$. We apply Lemma 7 to $F(w)$ with $s=1-s_2$. Here for the sake of simplicity we shall take $\kappa=\sigma_2+\delta+\varepsilon$, where ε is an arbitrary small positive number. We note that this choice gives a slightly weaker error term in (4-1) but it is of no harm for our present purposes. Let $E(x;s_1,s_2)$ denote the quantities defined by (2-6). Then it follows from Lemma 7 that

$$E(x;s_1,s_2)=\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT}\zeta(w+1-s_2)\zeta(w+s_1)\frac{x^w}{w}\,dw + O\left(x^{\sigma_2+\delta+\varepsilon}\left(\frac{\log T}{T}+\frac{1}{x}\right)\right). \tag{5-1}$$

Hereafter we shall assume that $T\ll |t_1|^A,\ |t_2|^A,\ x^A$, where $A>0$ is a certain positive constant.

In (5-1), w is in the domain of the absolute convergence of $\zeta(w+1-s_2)$ and $\zeta(w+s_1)$, so if we evaluate the above integral trivially we get

$$E(|t_2|/2\pi;s_1,s_2)\ll |t_2|^{\sigma_2+\delta+\varepsilon},$$

which gives the bound (2-5) by (2-2). To get sharper estimates, we move the line of integration to the left hand side of $\operatorname{Re} w = \kappa$.

5.1. ***Proof of Theorem 1.*** If $\sigma_1 + \sigma_2 = 1$ the assertion of Theorem 1 is the same as that of (2-5). Assume that $\sigma_1 + \sigma_2 > 1$. We move the line of integration to $\operatorname{Re} w = 1 - \sigma_1 + \varepsilon$ in (5-1). Since $\zeta(w + s_1)$ converges absolutely, we get

$$E(x; s_1, s_2) \ll |t_2|^{p(2-\sigma_1-\sigma_2)+\varepsilon} x^{1-\sigma_1+\varepsilon} \log T + x^{\sigma_2+\varepsilon}\left(\frac{\log T}{T} + \frac{1}{x}\right) + \frac{1}{T} \max_{1-\sigma_1+\varepsilon \le u \le \kappa} \left(|t_2|^{p(u+1-\sigma_2)+\varepsilon} x^u\right)$$

for $T \le |t_2| - 1$.

Let $x = |t_2|/2\pi$ and $T = |t_2| - 1$. By the relation

$$p(2 - \sigma_1 - \sigma_2) = p(\sigma_1 + \sigma_2 - 1) + \sigma_1 + \sigma_2 - \tfrac{3}{2},$$

the first term on the right-hand side is $\ll |t_2|^{p(\sigma_1+\sigma_2-1)+\sigma_2-\frac{1}{2}+\varepsilon}$. The last term on the right-hand side becomes

$$\begin{aligned}\max_{1-\sigma_1+\varepsilon \le u \le \kappa} \left(|t_2|^{p(u+1-\sigma_2)+u-1+\varepsilon}\right) &= \max_{1-\sigma_1+\varepsilon \le u \le \kappa} \left(|t_2|^{p(\sigma_2-u)+\sigma_2-\frac{3}{2}+\varepsilon}\right) \\ &\le |t_2|^{p(-\varepsilon)+\sigma_2-\frac{3}{2}+\varepsilon} \\ &\le |t_2|^{\sigma_2-1+\varepsilon},\end{aligned}$$

where in the last step we use the fact $p(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \le 0$. Hence we get

$$E(|t_2|/2\pi; s_1, s_2) \ll |t_2|^{p(\sigma_1+\sigma_2-1)+\sigma_2-\frac{1}{2}+\varepsilon},$$

and by (2-2) and (2-4), we get

$$J(s_1, s_2) \ll |t_2|^{p(\sigma_1+\sigma_2-1)+\varepsilon}. \qquad \square$$

We note that the right hand side of (3-1) does not depend on $|t_1|$ and improves the trivial estimate (2-5) for $\sigma_1 + \sigma_2 > 1$.

5.2. ***General formulation (Proof of Theorem 2).*** In order to prove Theorem 2, we move the line of integration of (5-1) to the left-hand side of that in Theorem 1. The residues of the integrands in (5-1) at $w = 1 - s_1, s_2$ and 0 are

$$\begin{aligned}Q_1(x) &= \zeta(2 - s_1 - s_2)\frac{x^{1-s_1}}{1 - s_1}, \\ Q_2(x) &= \zeta(s_1 + s_2)\frac{x^{s_2}}{s_2}, \\ Q_3 &= \zeta(1 - s_2)\zeta(s_1),\end{aligned}$$

respectively. The function $Q_1(x)$ is the same as that defined in (3-3).

Let u_0 be a real number such that $u_0 < 1-\sigma_1,\ u_0 \neq \sigma_2$ and $u_0 \neq 0$. In (5-1), we move the line of integration to $\operatorname{Re} w = u_0$. By using Cauchy's theorem we get

$$E(x,s_1,s_2) = G(T,x)+V(T,x)+H(T,x)+O\Bigl(x^{\sigma_2+\delta+\varepsilon}\Bigl(\frac{\log T}{T}+\frac{1}{x}\Bigr)\Bigr), \tag{5-2}$$
$$=: G(T,x)+R(T,x) \quad \text{(say)},$$

where

$$V(T,x) = \frac{1}{2\pi i}\int_{u_0-iT}^{u_0+iT} \zeta(w+1-s_2)\zeta(w+s_1)\frac{x^w}{w}\,dw,$$
$$H(T,x) = \frac{1}{2\pi i}\biggl\{\int_{\kappa-iT}^{u_0-iT}+\int_{u_0+iT}^{\kappa+iT}\biggr\}\zeta(w+1-s_2)\zeta(w+s_1)\frac{x^w}{w}\,dw,$$
$$G(T,x) = Q_1(T,x)+Q_2(T,x)+Q_3(u_0)$$

with

$$Q_1(T,x) = \begin{cases} Q_1(x) & \text{if } T>|t_1|,\\ 0 & \text{otherwise,}\end{cases}$$
$$Q_2(T,x) = \begin{cases} Q_2(x) & \text{if } T>|t_2| \text{ and } u_0<\sigma_2,\\ 0 & \text{otherwise,}\end{cases}$$
$$Q_3(u_0) = \begin{cases} Q_3 & \text{if } u_0<0,\\ 0 & \text{otherwise.}\end{cases}$$

For the integral over the horizontal line we have

$$H(T,x) \ll \frac{1}{T}\max_{u_0\le u\le\kappa}\bigl((T+|t_2|)^{p(u+1-\sigma_2)+\varepsilon}(T+|t_1|)^{p(u+\sigma_1)+\varepsilon}x^u\bigr).$$

To derive the estimate of $V(T,x)$ we divide into two cases.

Case 1. First we choose

$$T \ll \min(|t_1|,|t_2|)-1.$$

In this case we have

$$G(T,x) = Q_3(u_0).$$

As for $V(T,x)$ we have

$$V(T,x) \ll |t_2|^{p(u_0+1-\sigma_2)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}x^{u_0}\log T.$$

Therefore we have

$$\begin{aligned}R(T,x) &\ll |t_2|^{p(u_0+1-\sigma_2)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}x^{u_0}\\ &\qquad + \frac{1}{T}\max_{u_0\le u\le\kappa}\bigl(|t_2|^{p(u+1-\sigma_2)+\varepsilon}|t_1|^{p(u+\sigma_1)+\varepsilon}x^{u}\bigr)\\ &\qquad\qquad + x^{\sigma_2+\delta+\varepsilon}\left(\frac{\log T}{T}+\frac{1}{x}\right)\\ &\ll |t_2|^{p(u_0+1-\sigma_2)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}x^{u_0}\\ &\qquad + \frac{1}{T}\Bigl\{|t_2|^{p(u_0+1-\sigma_2)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}x^{u_0}\\ &\qquad\qquad + |t_2|^{p(1+\delta)+\varepsilon}|t_1|^{p(\sigma_1+\sigma_2+\delta)+\varepsilon}x^{\sigma_2+\delta+\varepsilon}\Bigr\}\\ &\qquad\qquad + x^{\sigma_2+\delta+\varepsilon}\left(\frac{\log T}{T}+\frac{1}{x}\right)\\ &\ll |t_2|^{p(u_0+1-\sigma_2)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}x^{u_0} + x^{\sigma_2+\delta+\varepsilon}\left(\frac{1}{T}+\frac{1}{x}\right).\end{aligned}$$

In the right-hand side of the above formula, there are no terms with positive powers of T, so we can choose T as large as possible.

Put $x_0 = |t_2|/2\pi$ for short. By (2-8) we have

$$\begin{aligned}|t_2|^{p(u_0+1-\sigma_2)}x_0^{u_0} &\ll |t_2|^{p(u_0+1-\sigma_2)+u_0}\\ &\ll |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)}.\end{aligned} \tag{5-3}$$

Hereafter when we take $x = x_0$, we change the term $|t_2|^{p(u_0+1-\sigma_2)}x_0^{u_0}$ as in the right-hand side of (5-3) without referring (2-8).

Suppose that $|t_2| \le |t_1|$ and take $T = |t_2|-1$. Then we have

$$\begin{aligned}R(|t_2|-1, x_0) &\ll |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon} + |t_2|^{\sigma_2+\delta-1+\varepsilon}\\ &\ll |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}.\end{aligned} \tag{5-4}$$

The second inequality is verified as follows. If $\delta = 0$, it is trivial. So assume that $\delta = 1-\sigma_1-\sigma_2$. Since $|t_2| \le |t_1|$, it is enough to show that

$$\sigma_2 - \tfrac{1}{2} + p(\sigma_2-u_0) + p(u_0+\sigma_1) + \sigma_1 \ge 0.$$

Since $\sigma + p(\sigma) \ge \frac{1}{2}$ for any σ, we can see easily that the left-hand side of the above is greater than $\frac{1}{2}$.

Suppose next that $|t_1| \le |t_2|$. Taking $T = |t_1| - 1$, we have

(5-5) $$R(|t_1|-1, x_0) \ll |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon} + |t_2|^{\sigma_2+\delta+\varepsilon}|t_1|^{-1} \ll \begin{cases} |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_o+\sigma_1)+\varepsilon} & \text{if } |t_2| \ge |t_1| \gg |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}}, \\ |t_2|^{\sigma_2+\delta+\varepsilon}|t_1|^{-1} & \text{if } |t_1| \ll |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}}. \end{cases}$$

From (5-4) and (5-5) we obtain

(5-6)
$$R(T, x_0) \ll \begin{cases} |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_o+\sigma_1)+\varepsilon} & \text{if } |t_1| \gg |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}}, \\ |t_2|^{\sigma_2+\delta+\varepsilon}|t_1|^{-1} & \text{if } |t_1| \ll |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}} \end{cases}$$

for $T \le \min(|t_1|, |t_2|) - 1$.

Case 2. To improve the latter estimate of (5-6) we assume that

$$|t_1| \ll |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}}$$

and take

$$|t_1| + 1 \le T \le |t_2| - 1.$$

In this case

$$G(T, x) = Q_1(x) + Q_3(u_0).$$

As for $H(T, x)$ and $V(T, x)$ we have

$$H(T, x) \ll |t_2|^{p(u_0+1-\sigma_2)+\varepsilon} T^{p(u_0+\sigma_1)-1+\varepsilon} x^{u_0} + x^{\sigma_2+\delta+\varepsilon} T^{-1}$$

and by using Lemma 9, we get

(5-7)
$$V(T, x) \ll |t_2|^{p(u_0+1-\sigma_2)+\varepsilon} x^{u_0} \int_{u_0-iT}^{u_0+iT} \left| \frac{\zeta(w+s_1)}{w} \right| |dw| \ll \begin{cases} |t_2|^{p(u_0+1-\sigma_2)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon} x^{u_0} & \text{for } \frac{1}{2} \le u_0+\sigma_1 < 1, \\ |t_2|^{p(u_0+1-\sigma_2)+\varepsilon}\left(|t_1|^{p(u_0+\sigma_1)+\varepsilon} + T^{\frac{1}{2}-(u_0+\sigma_1)}\right) x^{u_0} & \text{for } 0 < u_0+\sigma_1 < \frac{1}{2}, \\ |t_2|^{p(u_0+1-\sigma_2)+\varepsilon} T^{\frac{1}{2}-(u_0+\sigma_1)+\varepsilon} x^{u_0} & \text{for } u_0+\sigma_1 \le 0. \end{cases}$$

We find that $H(T, x)$ is absorbed in $V(T, x) + O(x^{\sigma_2+\delta+\varepsilon}(1/T + 1/x))$ in each case.

Now we will treat each case in (5-7) separately.

(i) The Case $\frac{1}{2} \le u_0 + \sigma_1 < 1$. We take $T = |t_2| - 1$ as before. Then we have

$$R(|t_2|-1, x_0) \ll |t_2|^{\sigma_2 - \frac{1}{2} + p(\sigma_2 - u_0) + \varepsilon} |t_1|^{p(u_0+\sigma_1)+\varepsilon} + |t_2|^{\sigma_2+\delta-1+\varepsilon}.$$

We shall show that the first term is greater than the second term. If $\sigma_1 + \sigma_2 \ge 1$, it is trivial since $\delta = 0$. So we assume that $\sigma_1 + \sigma_2 < 1$ and $\delta = 1 - \sigma_1 - \sigma_2$. If $\frac{1}{2} \le \sigma_1 + \sigma_2 < 1$, then it is again trivial. Assume that $0 < \sigma_1 + \sigma_2 < \frac{1}{2}$. Then from $\frac{1}{2} \le u_0 + \sigma_1 < 1$, we have $-1 < \sigma_2 - u_0 < 0$, hence $p(\sigma_2 - u_0) > \frac{1}{2}$. This shows that $\sigma_2 - \frac{1}{2} + p(\sigma_2 - u_0) > -\sigma_1$. Therefore we have

$$R(|t_2|-1, x_0) \ll |t_2|^{\sigma_2 - \frac{1}{2} + p(\sigma_2 - u_0) + \varepsilon} |t_1|^{p(u_0+\sigma_1)+\varepsilon}. \tag{5-8}$$

(ii) The Case $0 < u_0 + \sigma_1 < \frac{1}{2}$. In this case we have

$$\begin{aligned} &R(T, x_0) \\ &\quad \ll |t_2|^{\sigma_2 - \frac{1}{2} + p(\sigma_2 - u_0) + \varepsilon} \left(|t_1|^{p(u_0+\sigma_1)+\varepsilon} + T^{\frac{1}{2}-(u_0+\sigma_1)+\varepsilon}\right) + |t_2|^{\sigma_2+\delta+\varepsilon} T^{-1}. \end{aligned}$$

By Lemma 10, there exists $T_0 \in [|t_1|+1, |t_2|-1]$ such that

$$\begin{aligned} R(T_0, x_0) \ll\ & |t_2|^{\sigma_2 - \frac{1}{2} + p(\sigma_2 - u_0) + \varepsilon} |t_1|^{p(u_0+\sigma_1)+\varepsilon} \\ & + |t_2|^{\sigma_2 - \frac{1}{2} + p(\sigma_2 - u_0) + \varepsilon} |t_1|^{\frac{1}{2}-(u_0+\sigma_1)+\varepsilon} + |t_2|^{\sigma_2+\delta-1+\varepsilon} \\ & + |t_2|^{\sigma_2+\delta-\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}+\varepsilon} \\ =:\ & R_1 + R_2 + R_3 + R_4, \end{aligned}$$

say. It is clearly seen that $R_2 \ll R_1$. In order to show that $R_3 \ll R_4$, it is enough to check that

$$1 - (u_0 + \sigma_1) - \delta + p(\sigma_2 - u_0) > 0.$$

If $\delta = 0$ it is trivial. If $\delta = 1 - \sigma_1 - \sigma_2$, the left-hand side of the above is equal to $\sigma_2 - u_0 + p(\sigma_2 - u_0)$, which is positive. Therefore we have $R_3 \ll R_4$ and

$$\begin{aligned} R(T_0, x_0) \ll\ & |t_2|^{\sigma_2 - \frac{1}{2} + p(\sigma_2 - u_0) + \varepsilon} |t_1|^{p(u_0+\sigma_1)+\varepsilon} \\ & + |t_2|^{\sigma_2+\delta-\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}+\varepsilon}. \end{aligned} \tag{5-9}$$

From this expression we conclude that

$$R(T_0, x_0) \ll \begin{cases} |t_2|^{\sigma_2 - \frac{1}{2} + p(\sigma_2 - u_0) + \varepsilon} |t_1|^{p(u_0+\sigma_1)+\varepsilon} \\ \qquad \text{if } |t_1| \gg |t_2|^{\frac{(1/2-(u_0+\sigma_1))(\delta+1/2-p(\sigma_2-u_0))}{p(u_0+\sigma_1)(3/2-(u_0+\sigma_1))}}, \\ |t_2|^{\sigma_2+\delta-\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}+\varepsilon} \\ \qquad \text{if } |t_1| \ll |t_2|^{\frac{(1/2-(u_0+\sigma_1))(\delta+1/2-p(\sigma_2-u_0))}{p(u_0+\sigma_1)(3/2-(u_0+\sigma_1))}}. \end{cases} \tag{5-10}$$

(iii) The Case $u_0+\sigma_1 \le 0$. Similarly to the case $0 < u_0+\sigma_1 < \frac{1}{2}$, there exists $T_1 \in [|t_1|+1, |t_2|-1]$ such that

$$\begin{aligned} R(T_1, x_0) &\ll |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{\frac{1}{2}-(u_0+\sigma_1)+\varepsilon} + |t_2|^{\sigma_2+\delta-1+\varepsilon} \\ &\qquad\qquad + |t_2|^{\sigma_2+\delta-\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}+\varepsilon} \\ &\ll |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{\frac{1}{2}-(u_0+\sigma_1)+\varepsilon} + |t_2|^{\sigma_2+\delta-\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}+\varepsilon}. \end{aligned}$$

Therefore we have

$$R(T_1, x_0) \ll \begin{cases} |t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{\frac{1}{2}-(u_0+\sigma_1)+\varepsilon} & \text{if } |t_1| \gg |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}}, \\ |t_2|^{\sigma_2+\delta-\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}+\varepsilon} & \text{if } |t_1| \ll |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}}. \end{cases} \tag{5-11}$$

However, since $p(u_0+\sigma_1) = \frac{1}{2}-(u_0+\sigma_1)$ for $u_0+\sigma_1 < 0$, (5-11) has the same form as the estimate (5-10) in the previous case.

Now from (5-2), (5-6), (5-8), (5-10), and the remark after (5-11), we can summarize the above calculations as follows.

(1) Suppose that $|t_1| \gg |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}}$. Then we have

$$E\left(\frac{|t_2|}{2\pi}; s_1, s_2\right) = Q_3(u_0) + O\big(|t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_o+\sigma_1)+\varepsilon}\big).$$

(2) Suppose that $|t_1| \ll |t_2|^{\frac{\delta+1/2-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)}}$.

(i) If $\frac{1}{2} \le u_0+\sigma_1 < 1$, we have

$$\begin{aligned} E\left(\frac{|t_2|}{2\pi}; s_1, s_2\right) &= Q_1(|t_2|/2\pi) + Q_3(u_0) \\ &\qquad + O\big(|t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}\big). \end{aligned}$$

(ii) If $u_0+\sigma_1 < \frac{1}{2}$, we have

$$\begin{aligned} E\left(\frac{|t_2|}{2\pi}; s_1, s_2\right) &= Q_1(|t_2|/2\pi) + Q_3(u_0) \\ &+ \begin{cases} O\big(|t_2|^{\sigma_2-\frac{1}{2}+p(\sigma_2-u_0)+\varepsilon}|t_1|^{p(u_0+\sigma_1)+\varepsilon}\big) \\ \qquad\qquad \text{if } |t_1| \gg |t_2|^{\frac{(1/2-(u_0+\sigma_1))(\delta+1/2-p(\sigma_2-u_0))}{p(u_0+\sigma_1)(3/2-(u_0+\sigma_1))}}, \\ O\big(|t_2|^{\sigma_2+\delta-\frac{\delta+1/2-p(\sigma_2-u_0)}{3/2-(u_0+\sigma_1)}+\varepsilon}\big) \\ \qquad\qquad \text{if } |t_1| \ll |t_2|^{\frac{(1/2-(u_0+\sigma_1))(\delta+1/2-p(\sigma_2-u_0))}{p(u_0+\sigma_1)(3/2-(u_0+\sigma_1))}}. \end{cases} \end{aligned}$$

In the case (ii) of (2), it is not difficult to see that

$$\frac{\left(\frac{1}{2}-(u_0+\sigma_1)\right)\left(\delta+\frac{1}{2}-p(\sigma_2-u_0)\right)}{p(u_0+\sigma_1)\left(\frac{3}{2}-(u_0+\sigma_1)\right)}<\frac{\delta+\frac{1}{2}-p(\sigma_2-u_0)}{1+p(u_0+\sigma_1)},$$

hence the range of $|t_1|$ is not empty.

Recalling

$$J(s_1,s_2)=\chi(s_2)E\left(\frac{|t_2|}{2\pi};s_1,s_2\right)+O(|t_2|^{\delta+\varepsilon})$$

and noting

$$\chi(s_2)Q_3(u_0)=Q_{u_0}(s_1,s_2)$$

by (2-3), we get Theorem 2.

6. Some special cases

In this section, we shall give several concrete evaluations of $J(s_1,s_2)$ by taking special u_0, which may be useful when we consider applications. Since it is obtained from Theorem 2, we only give simple remarks.

Corollary 11. *Let* $0<\sigma_j<1$ *for* $j=1,2$ *and* $s_1+s_2\neq 1$.

(a) *If* $|t_1|\gg|t_2|^{\frac{\delta+1/2-p(\sigma_2)}{1+p(\sigma_1)}}$, *we have*

$$J(s_1,s_2)\ll|t_2|^{p(\sigma_2)+\varepsilon}|t_1|^{p(\sigma_1)+\varepsilon}+|t_2|^{\delta+\varepsilon}.$$

(b) *If* $\frac{1}{2}\leq\sigma_1<1$ *and* $|t_1|\ll|t_2|^{\frac{\delta+1/2-p(\sigma_2)}{1+p(\sigma_1)}}$, *we have*

$$J(s_1,s_2)=\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)+O\left(|t_2|^{p(\sigma_2)+\varepsilon}|t_1|^{p(\sigma_1)+\varepsilon}+|t_2|^{\delta+\varepsilon}\right).$$

(c) *If* $0<\sigma_1<\frac{1}{2}$ *and* $|t_2|^{\frac{(1/2-\sigma_1)(1/2-p(\sigma_2)+\delta)}{p(\sigma_1)(3/2-\sigma_1)}}\ll|t_1|\ll|t_2|^{\frac{\delta+1/2-p(\sigma_2)}{1+p(\sigma_1)}}$, *we have*

$$J(s_1,s_2)=\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)+O\left(|t_2|^{p(\sigma_2)+\varepsilon}|t_1|^{p(\sigma_1)+\varepsilon}+|t_2|^{\delta+\varepsilon}\right).$$

(d) *If* $0<\sigma_1<\frac{1}{2}$ *and* $|t_1|\ll|t_2|^{\frac{(1/2-\sigma_1)(\delta+1/2-p(\sigma_2))}{p(\sigma_1)(3/2-\sigma_1)}}$, *we have*

$$J(s_1,s_2)=\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)+O\left(|t_2|^{\frac{p(\sigma_2)+(1/2+\delta)(1/2-\sigma_1)}{3/2-\sigma_1}+\varepsilon}+|t_2|^{\delta+\varepsilon}\right).$$

Proof. We take $u_0=\varepsilon$ such that

$$0<\varepsilon<\min(\sigma_2,1-\sigma_1).$$

In this choice we have $Q_3(\varepsilon)=0$. In Theorem 2, the exponents of $|t_2|$ in the assumption on $|t_1|$ and $|t_2|$ have factors like $p(\sigma_2-\varepsilon)$ and $p(\sigma_1+\varepsilon)$, but we see easily that we can replace these factors by $p(\sigma_2)$ and $p(\sigma_1)$, respectively. We always make such a simplification. Then the assertions are derived by Theorem 2 immediately. □

Corollary 12. *Let* $0<\sigma_j<1$ *for* $j=1,2$ *and* $s_1+s_2\neq 1$.

(a) *If* $|t_1|\gg|t_2|^{\frac{\delta+1/2}{p(\sigma_1+\sigma_2-1)+1}}$, *we have*

$$J(s_1,s_2)=\zeta(s_1)\zeta(s_2)+O\big(|t_1|^{p(\sigma_1+\sigma_2-1)+\varepsilon}+|t_2|^{\delta+\varepsilon}\big).$$

(b) *If* $\sigma_1+\sigma_2\geq\frac{3}{2}$ *and* $|t_1|\ll|t_2|^{\frac{1/2}{p(\sigma_1+\sigma_2-1)+1}}$, *we have*

$$J(s_1,s_2)=\zeta(s_1)\zeta(s_2)+\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)+O\big(|t_1|^{p(\sigma_1+\sigma_2-1)+\varepsilon}+|t_2|^{\varepsilon}\big).$$

(c) *If* $\sigma_1+\sigma_2<\frac{3}{2}$ *and* $|t_2|^{\frac{(1/2+\delta)(3/2-\sigma_1-\sigma_2)}{p(\sigma_1+\sigma_2-1)(5/2-(\sigma_1+\sigma_2))}}\ll|t_1|\ll|t_2|^{\frac{\delta+1/2}{p(\sigma_1+\sigma_2-1)+1}}$, *we have*

$$J(s_1,s_2)=\zeta(s_1)\zeta(s_2)+\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)+O\big(|t_1|^{p(\sigma_1+\sigma_2-1)+\varepsilon}+|t_2|^{\delta+\varepsilon}\big).$$

(d) *If* $\sigma_1+\sigma_2<\frac{3}{2}$ *and* $|t_1|\ll|t_2|^{\frac{(1/2+\delta)(3/2-\sigma_1-\sigma_2)}{p(\sigma_1+\sigma_2-1)(5/2-(\sigma_1+\sigma_2))}}$, *we have*

$$J(s_1,s_2)=\zeta(s_1)\zeta(s_2)+\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)+O\big(|t_2|^{\frac{(\delta+1/2)(3/2-(\sigma_1+\sigma_2))}{5/2-(\sigma_1+\sigma_2)}+\varepsilon}+|t_2|^{\delta+\varepsilon}\big).$$

Proof. We take $u_0=\sigma_2-1$ in Theorem 2. Since $u_0<0$, we have $Q_{u_0}(s_1,s_2)=\zeta(s_1)\zeta(s_2)$. In the case (b) which corresponds to the case (b)(i) in Theorem 2, we have $\delta=0$ since $\sigma_1+\sigma_2\geq\frac{3}{2}$. □

Corollary 13. *Let* $0<\sigma_j<1$ *for* $j=1,2$ *and* $s_1+s_2\neq 1$.

(a) *If* $|t_1|\gg|t_2|^{\frac{2(\delta+1/2-p(\sigma_1+\sigma_2))}{3}}$, *we have*

$$J(s_1,s_2)=\zeta(s_1)\zeta(s_2)+O\big(|t_2|^{p(\sigma_1+\sigma_2)+\varepsilon}|t_1|^{\frac{1}{2}+\varepsilon}+|t_2|^{\delta+\varepsilon}\big).$$

(b) *If* $|t_1|\ll|t_2|^{\frac{2(\delta+1/2-p(\sigma_1+\sigma_2))}{3}}$, *we have*

$$J(s_1,s_2)=\zeta(s_1)\zeta(s_2)+\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)+O\big(|t_2|^{\frac{(\delta+1/2)+2p(\sigma_1+\sigma_2)}{3}+\varepsilon}+|t_2|^{\delta+\varepsilon}\big).$$

Proof. We take $u_0=-\sigma_1$ in Theorem 2. Since $u_0<0$, $Q_{u_0}(s_1,s_2)=\zeta(s_1)\zeta(s_2)$. The case of (i) and the upper case of (ii) in (b) of Theorem 2 do not occur. □

Corollary 14. *Let $0<\sigma_j<1$ for $j=1,2$ and $s_1+s_2\neq 1$. Suppose that $\sigma_1+\sigma_2<\frac{3}{2}$ and $\sigma_2\neq\frac{1}{2}$.*

(a) *If $|t_1|\gg|t_2|^{\frac{\delta+1/2-p(1/2)}{1+p(\sigma_1+\sigma_2-1/2)}}$, we have*

$$J(s_1,s_2)=Q_{\sigma_2-\frac{1}{2}}(s_1,s_2)+O\big(|t_2|^{p(\frac{1}{2})+\varepsilon}|t_1|^{p(\sigma_1+\sigma_2-\frac{1}{2})+\varepsilon}+|t_2|^{\delta+\varepsilon}\big).$$

(b) *If $1\leq\sigma_1+\sigma_2<\frac{3}{2}$ and $|t_1|\ll|t_2|^{\frac{1/2-p(1/2)}{1+p(\sigma_1+\sigma_2-1/2)}}$, we have*

$$J(s_1,s_2)=Q_{\sigma_2-\frac{1}{2}}(s_1,s_2)+\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)$$
$$+O\big(|t_2|^{p(\frac{1}{2})+\varepsilon}|t_1|^{p(\sigma_1+\sigma_2-\frac{1}{2})+\varepsilon}+|t_2|^{\varepsilon}\big).$$

(c) *If $\sigma_1+\sigma_2<1$ and $|t_2|^{\frac{(1-\sigma_1-\sigma_2)(3/2+\sigma_1-\sigma_2-p(1/2))}{p(\sigma_1+\sigma_2-1/2)(2-\sigma_1-\sigma_2)}}\ll|t_1|\ll|t_2|^{\frac{3/2-\sigma_1-\sigma_2-p(1/2)}{1+p(\sigma_1+\sigma_2-1/2)}}$, we have*

$$J(s_1,s_2)=Q_{\sigma_2-\frac{1}{2}}(s_1,s_2)+\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)$$
$$+O\big(|t_2|^{p(\frac{1}{2})+\varepsilon}|t_1|^{p(\sigma_1+\sigma_2-\frac{1}{2})+\varepsilon}+|t_2|^{1-\sigma_1-\sigma_2+\varepsilon}\big).$$

(d) *If $\sigma_1+\sigma_2<1$ and $|t_1|\ll|t_2|^{\frac{(1-\sigma_1-\sigma_2)(3/2-\sigma_1-\sigma_2-p(1/2))}{p(\sigma_1+\sigma_2-1/2)(2-\sigma_1-\sigma_2)}}$, we have*

$$J(s_1,s_2)=Q_{\sigma_2-\frac{1}{2}}(s_1,s_2)+\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)$$
$$+O\big(|t_2|^{\frac{(3/2-\sigma_1-\sigma_2+1/2)(1-\sigma_1-\sigma_2)+p(1/2)}{2-\sigma_1-\sigma_2}+\varepsilon}+|t_2|^{1-\sigma_1-\sigma_2+\varepsilon}\big).$$

Proof. We take $u_0=\sigma_2-\frac{1}{2}$ in Theorem 2. In (b), we have $\delta=0$ and in (c) and (d) we have $\delta=1-\sigma_1-\sigma_2$. □

Corollary 15. *Let $0<\sigma_j<1$ for $j=1,2$ and $s_1+s_2\neq 1$. Let $Q_{u_0}(s_1,s_2)$ be the function defined by (3-2). Suppose that $\sigma_1+\sigma_2\neq\frac{1}{2}$ and $\sigma_1\neq\frac{1}{2}$. Then we have*

(a) *If $|t_1|\gg|t_2|^{\frac{\delta+1/2-p(\sigma_1+\sigma_2-1/2)}{p(1/2)+1}}$ we have*

$$J(s_1,s_2)=Q_{\sigma_1-\frac{1}{2}}(s_1,s_2)+O\big(|t_2|^{p(\sigma_1+\sigma_2-\frac{1}{2})+\varepsilon}|t_1|^{p(\frac{1}{2})+\varepsilon}\big)+O(|t_2|^{\delta+\varepsilon}).$$

(b) *If $|t_1|\ll|t_2|^{\frac{\delta+1/2-p(\sigma_1+\sigma_2-1/2)}{p(1/2)+1}}$ we have*

$$J(s_1,s_2)=Q_{\sigma_1-\frac{1}{2}}(s_1,s_2)+\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right)$$
$$+O\big(|t_2|^{p(\sigma_1+\sigma_2-\frac{1}{2})+\varepsilon}|t_1|^{p(\frac{1}{2})+\varepsilon}+|t_2|^{\delta+\varepsilon}\big).$$

Proof. We take $u_0=\frac{1}{2}-\sigma_1$ in Theorem 2. As $u_0=\frac{1}{2}-\sigma_1$, only possible subcase is (i) of (b) in Theorem 2. □

In the above corollaries we recall that $\chi(s_2)Q_1(|t_2|/2\pi)$ is given by

$$\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) = \frac{\zeta(2-s_1-s_2)}{1-s_1}\left(\frac{|t_2|}{2\pi}\right)^{\frac{3}{2}-s_1-s_2} e^{i(t_2 \pm \frac{\pi}{4})} + O\big(|\zeta(2-s_1-s_2)||t_2|^{\frac{1}{2}-\sigma_1-\sigma_2}|t_1|^{-1}\big) \quad \text{(see (3-4))}.$$

7. Proof of Theorems 3, 4 and 5

We shall give sufficient conditions for

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}. \tag{7-1}$$

In view of (2-1) and (2-2), we need at least the following three conditions for (7-1).

$$|t_1+t_2|^{p(\sigma_1+\sigma_2-1)}|t_2|^{-1} \ll |t_1|^{p(\sigma_1)}|t_2|^{p(\sigma_2)}, \tag{7-2}$$

$$|t_1+t_2|^{p(\sigma_1+\sigma_2)} \ll |t_1|^{p(\sigma_1)}|t_2|^{p(\sigma_2)}, \tag{7-3}$$

$$|t_2|^{\delta} \ll |t_1|^{p(\sigma_1)}|t_2|^{p(\sigma_2)}. \tag{7-4}$$

Since $p(\sigma)$ is monotonically decreasing, (7-3) is automatically satisfied. For (7-2) and (7-4), we divide into two cases.

The case $\sigma_1+\sigma_2 \geq 1$. Since $\delta = 0$ in this case, (7-4) is trivial. We have to consider (7-2) only.

(i) We assume that $|t_2| \ll |t_1|$. Then (7-2) follows from the fact

$$|t_1| \ll |t_2|^{\frac{p(\sigma_2)+1}{p(\sigma_1+\sigma_2-1)-p(\sigma_1)}}.$$

(ii) Next we assume that $|t_1| \ll |t_2|$. Then the left-hand side of (7-2) is bounded by $|t_2|^{p(\sigma_1+\sigma_2-1)-1}$, whose exponent is negative, hence (7-2) holds trivially.

The case $\sigma_1+\sigma_2 \leq 1$. In this case we have $\delta = 1-\sigma_1-\sigma_2$ and $p(\sigma_1+\sigma_1-1) = \frac{3}{2}-\sigma_1-\sigma_2$.

(i) Firstly we assume that $|t_2| \ll |t_1|$. Then (7-2) and (7-4) follow from

$$|t_1|^{\frac{3}{2}-\sigma_1-\sigma_2}|t_2|^{-1} \ll |t_1|^{p(\sigma_1)}|t_2|^{p(\sigma_2)}$$

and

$$|t_2|^{1-\sigma_1-\sigma_2} \ll |t_1|^{p(\sigma_1)}|t_2|^{p(\sigma_2)},$$

respectively. Since $1-\sigma_1-\sigma_2 \leq p(\sigma_1)+p(\sigma_2)$ and $|t_2| \ll |t_1|$, the latter condition is automatically satisfied. The former one becomes

$$|t_2| \ll |t_1| \ll |t_2|^{\frac{p(\sigma_2)+1}{3/2-\sigma_1-\sigma_2-p(\sigma_1)}}. \tag{7-5}$$

(ii) Secondly we assume that $|t_1| \ll |t_2|$. Then (7-2) and (7-4) follow from

$$|t_2|^{\frac{1}{2}-\sigma_1-\sigma_2} \ll |t_1|^{p(\sigma_1)}|t_2|^{p(\sigma_2)}$$

and

$$|t_2|^{1-\sigma_1-\sigma_2} \ll |t_1|^{p(\sigma_1)}|t_2|^{p(\sigma_2)},$$

respectively. Clearly the latter is stronger than the former, hence they are unified as

$$|t_2|^{\frac{1-\sigma_1-\sigma_2-p(\sigma_2)}{p(\sigma_1)}} \ll |t_1| \ll |t_2|.$$

The above consideration leads to the following lemma.

Lemma 16. (a) *If* $\sigma_1+\sigma_2 \geq 1$ *and* $|t_1| \ll |t_2|^{\frac{p(\sigma_2)+1}{p(\sigma_1+\sigma_2-1)-p(\sigma_1)}}$, *then* (7-2) *and* (7-4) *hold.*

(b) *If* $\sigma_1+\sigma_2 \leq 1$ *and* $|t_2|^{\frac{1-\sigma_1-\sigma_2-p(\sigma_2)}{p(\sigma_1)}} \ll |t_1| \ll |t_2|^{\frac{p(\sigma_2)+1}{3/2-\sigma_1-\sigma_2-p(\sigma_1)}}$, *then* (7-2) *and* (7-4) *hold.*

We remark again that (7-3) is automatically satisfied.

We note that in (b) above we have

$$\frac{1-\sigma_1-\sigma_2-p(\sigma_2)}{p(\sigma_1)} \leq 1 \leq \frac{p(\sigma_2)+1}{\frac{3}{2}-\sigma_1-\sigma_2-p(\sigma_1)},$$

since we always have $\sigma+p(\sigma)=\frac{1}{2}+p(1-\sigma)\geq\frac{1}{2}$. Furthermore if

$$1-\sigma_1-\sigma_2-p(\sigma_2) \geq 0,$$

σ_1 must satisfy $\sigma_1 \leq \frac{1}{2}$.

Proof of Theorem 3. We shall combine Lemma 16 and Corollary 11(a). The additional condition is $|t_1| \gg |t_2|^{\frac{\delta+1/2-p(\sigma_2)}{1+p(\sigma_1)}}$. Thus we get

(a) If $\sigma_1+\sigma_2 \geq 1$ and $|t_2|^{\frac{1/2-p(\sigma_2)}{1+p(\sigma_1)}} \ll |t_1| \ll |t_2|^{\frac{p(\sigma_2)+1}{p(\sigma_1+\sigma_2-1)-p(\sigma_1)}}$, we have

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

(b) If $\sigma_1+\sigma_2 \leq 1$ and

$$\max\left\{|t_2|^{\frac{1-\sigma_1-\sigma_2-p(\sigma_2)}{p(\sigma_1)}}, |t_2|^{\frac{3/2-\sigma_1-\sigma_2-p(\sigma_2)}{1+p(\sigma_1)}}\right\} \ll |t_1| \ll |t_2|^{\frac{p(\sigma_2)+1}{3/2-\sigma_1-\sigma_2-p(\sigma_1)}},$$

we have

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

This proves Theorem 3. ∎

Proof of Theorem 4. (a) We shall use Corollary 12(a) where the condition is $|t_2|^{\frac{\delta+1/2}{1+p(\sigma_1+\sigma_2-1)}} \ll |t_1|$. From the assumption on $|t_1|$ and $|t_2|$ and the inequality $p(\sigma_2) \geq \delta - \frac{1}{2}$, the above condition is always satisfied. Hence we get

$$J(s_1,s_2) = \zeta(s_1)\zeta(s_2) + O\big(|t_1|^{p(\sigma_1+\sigma_2-1)+\varepsilon} + |t_2|^{\varepsilon}\big)$$
$$\ll |t_1|^{p(\sigma_1+\sigma_2-1)+\varepsilon}.$$

By (2-1) we also find that

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1+\sigma_2-1)+\varepsilon},$$

which proves (a) of Theorem 4.

(b) For the proof, we shall use Corollary 12(b) where the additional condition to Lemma 16 is $|t_1| \ll |t_2|^{\frac{1/2}{1+p(\sigma_1+\sigma_2-1)}}$ ($\ll |t_2|$). Since $\sigma_1+\sigma_2 \geq \frac{3}{2}$ by assumption, we have $\frac{1}{2} \leq \sigma_j < 1$ $(j=1,2)$ and $\sigma_1+\sigma_2-1 \geq \frac{1}{2}$, from which $p(\sigma_1+\sigma_2-1) = p(\sigma_1)+p(\sigma_2)$ follows. This implies that

$$\frac{\frac{1}{2}-p(\sigma_2)}{1+p(\sigma_1)} \leq \frac{\frac{1}{2}}{1+p(\sigma_1+\sigma_2-1)},$$

therefore the condition of Corollary 12(b) is satisfied by the assumption. Furthermore

$$|t_1|^{p(\sigma_1+\sigma_2-1)} \ll |t_1|^{p(\sigma_1)}|t_2|^{p(\sigma_2)}$$

holds true. Hence we get

$$J(s_1,s_2) = \chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) + O(|t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}).$$

By (3-4) and the relation

$$\tfrac{3}{2}-\sigma_1-\sigma_2+p(2-\sigma_1-\sigma_2) = p(\sigma_1+\sigma_2-1) = p(\sigma_1)+p(\sigma_2),$$

where we used (2-9) in the latter equality, we find that

$$\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon},$$

if $|t_2|^{\frac{p(\sigma_1)}{p(\sigma_1)+1}} \ll |t_1|$. Also conditions (7-2), (7-3) and (7-4) are automatically satisfied, thus we get

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

(c) We assume that $1 \leq \sigma_1+\sigma_2 < \frac{3}{2}$, $\frac{1}{2} \leq \sigma_1 < 1$ and $|t_1| \ll |t_2|^{\frac{1/2-p(\sigma_2)}{1+p(\sigma_1)}}$. Corollary 11(b) gives

$$J(s_1,s_2) = \chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) + O\big(|t_2|^{p(\sigma_2)+\varepsilon}|t_1|^{p(\sigma_1)+\varepsilon}\big).$$

By (3-4) and $t_1| \ll |t_2|$, we see that

$$\chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) \ll |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2+p(2-\sigma_1-\sigma_2)}|t_1|^{-1}$$
$$= |t_2|^{p(\sigma_1+\sigma_2-1)}|t_1|^{-1}.$$

Hence if we assume $|t_2|^{\frac{p(\sigma_1+\sigma_2-1)-p(\sigma_2)}{1+p(\sigma_1)}} \ll |t_1|$, we get

$$J(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

By $|t_1| \ll |t_2|$ and $p(\sigma_1+\sigma_2-1) < 1$, the condition (7-2) is satisfied. Conditions (7-3) and (7-4) are also trivially satisfied, hence we get

$$\zeta_2(s_1,s_2) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}.$$

(d) We assume that $1 \le \sigma_1+\sigma_2 < \frac{3}{2}$, $0 < \sigma_1 < \frac{1}{2}$ and $|t_1| \ll |t_2|^{\frac{1/2-p(\sigma_2)}{1+p(\sigma_1)}}$. If $|t_2|^{\frac{(1/2-\sigma_1)(1/2-p(\sigma_2))}{p(\sigma_1)(3/2-\sigma_1)}} \ll |t_1|$, we have

$$J(s_1,s_2) = \chi(s_2)Q_1\left(\frac{|t_2|}{2\pi}\right) + O\big(|t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon}\big)$$

by Corollary 11(c). As in the case (c), if we combine the condition

$$\chi(s_2)Q_1(|t_2|/2\pi) \ll |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon},$$

we get the assertion (d) of Theorem 4. □

Proof of Theorem 5. (a) Since $\delta = 1-\sigma_1-\sigma_2$ in this case, we can use Corollary 11(b). By (3-4) and $p(2-\sigma_1-\sigma_2) = 0$, we get

$$J(s_1,s_2) \ll |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2+\varepsilon}|t_1|^{-1} + |t_1|^{p(\sigma_1)+\varepsilon}|t_2|^{p(\sigma_2)+\varepsilon} + |t_2|^{1-\sigma_1-\sigma_2+\varepsilon}.$$

By assumption, the middle term on the right-hand side is smaller than the first term. We also observe that $\frac{3/2-\sigma_1-\sigma_2-p(\sigma_2)}{1+p(\sigma_1)} \le \frac{1}{2}$ from $\frac{1}{2} \le \sigma_1 < 1$. Therefore

$$J(s_1,s_2) \ll |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2+\varepsilon}|t_1|^{-1}.$$

Furthermore we can see easily that

$$\zeta(s_1+s_2-1)/(s_2-1) \ll |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2+\varepsilon}|t_1|^{-1},$$
$$\zeta(s_1+s_2) \ll |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2+\varepsilon}|t_1|^{-1}.$$

This proves (a) of Theorem 5.

(b) Similarly to the above, this can be proved by using Corollary 15(b), and we omit the details. □

Examples. We assume that $|t_1+t_2| \gg 1$ in this example.

(1) The case $\sigma_1 = \sigma_2 = \frac{1}{2}$.
By Theorem 3(a) we get the following estimates.

(i) Take $p\left(\frac{1}{2}\right) = \frac{1}{6}$ (Hardy and Littlewood). Then if $|t_2|^{\frac{2}{7}} \ll |t_1| \ll |t_2|^{\frac{7}{2}}$, we have

$$\zeta_2\left(\tfrac{1}{2} + it_1, \tfrac{1}{2} + it_2\right) \ll |t_1|^{\frac{1}{6}+\varepsilon}|t_2|^{\frac{1}{6}+\varepsilon}.$$

(ii) Take $p\left(\frac{1}{2}\right) = \frac{1}{4}$ (Phragmén–Lindelöf bound). Then if $|t_2|^{\frac{1}{5}} \ll |t_1| \ll |t_2|^5$, we have

$$\zeta_2\left(\tfrac{1}{2} + it_1, \tfrac{1}{2} + it_2\right) \ll |t_1|^{\frac{1}{4}+\varepsilon}|t_2|^{\frac{1}{4}+\varepsilon}.$$

(iii) Assuming the Lindelöf hypothesis for $\zeta(s)$, we take $p\left(\frac{1}{2}\right) = 0$. Then if $|t_2|^{\frac{1}{2}} \ll |t_1| \ll |t_2|^2$, we have

$$\zeta_2\left(\tfrac{1}{2} + it_1, \tfrac{1}{2} + it_2\right) \ll |t_1|^{\varepsilon}|t_2|^{\varepsilon}.$$

(2) The case $\sigma_1 = \frac{1}{2}$, $\sigma_2 = \frac{3}{4}$ and $p\left(\frac{1}{2}\right) = \frac{1}{6}$.
The condition of Theorem 3(a) is $|t_2|^{\frac{5}{14}} \ll |t_1| \ll |t_2|^{\frac{13}{2}}$, while the condition of Theorem 4(c) is $|t_2|^{\frac{3}{14}} \ll |t_1| \ll |t_2|^{\frac{5}{14}}$. Therefore if $|t_2|^{\frac{3}{14}} \ll |t_1| \ll |t_2|^{\frac{13}{2}}$, we have

$$\zeta_2\left(\tfrac{1}{2} + it_1, \tfrac{3}{4} + it_2\right) \ll |t_1|^{\frac{1}{6}+\varepsilon}|t_2|^{\frac{1}{12}+\varepsilon}.$$

(3) The case $\sigma_1 = \frac{3}{4}$, $\sigma_2 = \frac{1}{2}$ and $p\left(\frac{1}{2}\right) = \frac{1}{6}$.
The condition of Theorem 3(a) is $|t_2|^{\frac{4}{13}} \ll |t_1| \ll |t_2|^{\frac{14}{3}}$, while the condition of Theorem 4(c) is $|t_2|^{\frac{2}{13}} \ll |t_1| \ll |t_2|^{\frac{4}{13}}$. Therefore if $|t_2|^{\frac{2}{13}} \ll |t_1| \ll |t_2|^{\frac{14}{3}}$, we have

$$\zeta_2\left(\tfrac{3}{4} + it_1, \tfrac{1}{2} + it_2\right) \ll |t_1|^{\frac{1}{12}+\varepsilon}|t_2|^{\frac{1}{6}+\varepsilon}.$$

(4) The case $\sigma_1 = \sigma_2 = \frac{1}{4}$ and $p\left(\frac{1}{2}\right) = \frac{1}{6}$.
By Theorem 3(b), if $|t_2|^{\frac{1}{2}} \ll |t_1| \ll |t_2|^2$, we have

$$\zeta_2\left(\tfrac{1}{4} + it_1, \tfrac{1}{4} + it_2\right) \ll |t_1|^{\frac{1}{3}+\varepsilon}|t_2|^{\frac{1}{3}+\varepsilon}.$$

In Examples 5 and 6, ε_0 is a fixed small positive constant.

(5) The case $\sigma_1 = 1 - \varepsilon_0$, $\sigma_2 = \frac{1}{2}$ and $p\left(\frac{1}{2}\right) = \frac{1}{6}$.
The condition of Theorem 3(a) is $|t_2|^{1/(3+\varepsilon_0)} \ll |t_1| \ll |t_2|^{7/(1+2\varepsilon_0)}$, while the condition of Theorem 4(c) is $|t_2|^{2\varepsilon_0/(3+\varepsilon_0)} \ll |t_1| \ll |t_2|^{1/(3+\varepsilon_0)}$. Therefore if $|t_2|^{2\varepsilon_0/(3+\varepsilon_0)} \ll |t_1| \ll |t_2|^{7/(1+2\varepsilon_0)}$, we have

$$\zeta_2\left(1 - \varepsilon_0 + it_1, \tfrac{1}{2} + it_2\right) \ll |t_1|^{\frac{\varepsilon_0}{3}+\varepsilon}|t_2|^{\frac{1}{6}+\varepsilon}.$$

(6) The case $\sigma_1 = \frac{1}{2}$, $\sigma_2 = 1 - \varepsilon_0$ and $p\left(\frac{1}{2}\right) = \frac{1}{6}$.
The condition of Theorem 3(a) is $|t_2|^{(3-2\varepsilon_0)/7} \ll |t_1| \ll |t_2|^{\frac{3}{2}\varepsilon_0+\frac{1}{2}}$, while the condition of Theorem 4(c) is $|t_2|^{(1+2\varepsilon_0)/7} \ll |t_1| \ll |t_2|^{(3-2\varepsilon_0)/7}$. Therefore if

$|t_2|^{(1+2\varepsilon_0)/7} \ll |t_1| \ll |t_2|^{\frac{3}{2}\varepsilon_0+\frac{1}{2}}$, we have

$$\zeta_2\big(\tfrac{1}{2}+it_1, 1-\varepsilon_0+it_2\big) \ll |t_1|^{\frac{1}{6}+\varepsilon}|t_2|^{\frac{\varepsilon_0}{3}+\varepsilon}.$$

(7) Theorem 4(a) with $\sigma_1=\sigma_2=\frac{1}{2}$ and $p\big(\frac{1}{2}\big)=0$ gives

$$\zeta_2\big(\tfrac{1}{2}+it_1, \tfrac{1}{2}+it_2\big) \ll |t_1|^{\frac{1}{2}+\varepsilon} \quad \text{for } |t_2|^2 \ll |t_1|.$$

This does not conflict with (1-3).

8. The mean square of the double zeta-function

In this section we shall prove Theorem 6 as an application of Theorem 1. First we recall (2-1). Let

$$I_1 = \frac{1}{|s_2-1|^2}\int_2^X |\zeta(s_1+s_2-1)|^2\,dt_1,$$
$$I_2 = \frac{1}{4}\int_2^X |\zeta(s_1+s_2)|^2\,dt_1,$$
$$I_3 = \int_2^X |J(s_1,s_2)|^2\,dt_1.$$

From (2-1) we have

$$\int_2^X |\zeta_2(s_1,s_2)|^2 dt_1 = I_1+I_2+I_3+O\big(\sqrt{I_1I_2}+\sqrt{I_1I_3}+\sqrt{I_2I_3}\big). \tag{8-1}$$

As for I_1 we have

$$\begin{aligned} I_1 &= \frac{1}{|s_2-1|^2}\int_2^X |\zeta(\sigma_1+\sigma_2-1+i(t_2+t_1))|^2\,dt_1 \\ &= \frac{1}{|s_2-1|^2}\bigg(\int_2^{X+t_2}|\zeta(\sigma_1+\sigma_2-1+iv)|^2\,dv - \int_2^{2+t_2}|\zeta(\sigma_1+\sigma_2-1+iv)|^2\,dv\bigg) \\ &=: \frac{1}{|s_2-1|^2}(I_{11}-I_{12}). \end{aligned}$$

By Lemma 8, we find that

$$\begin{aligned} I_{11} = \frac{(2\pi)^{2\sigma_1+2\sigma_2-3}\zeta(4-2\sigma_1-2\sigma_2)}{4-2\sigma_1-2\sigma_2}X^{4-2\sigma_1-2\sigma_2} + \zeta(2\sigma_1+2\sigma_2-2)X \\ + O(X^{3-2\sigma_1-2\sigma_2}t_2) + O(t_2) + O(X^{\frac{7}{3}-\frac{4}{3}(\sigma_1+\sigma_2)}), \end{aligned}$$

and

$$I_{12} = O(t_2^{4-2\sigma_1-2\sigma_2}).$$

Hence we get

$$I_1 = \frac{(2\pi)^{2\sigma_1+2\sigma_2-3}\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|1-s_2|^2} X^{4-2\sigma_1-2\sigma_2} + \frac{\zeta(2\sigma_1+2\sigma_2-2)}{|1-s_2|^2} X + O(X^{3-2\sigma_1-2\sigma_2} t_2^{-1}) + O(t_2^{-1}) + O\big(X^{\frac{7}{3}-\frac{4}{3}(\sigma_1+\sigma_2)} t_2^{-2}\big) + O(t_2^{2-2\sigma_1-2\sigma_2}). \tag{8-2}$$

Similarly we have

$$I_2 = \tfrac{1}{4}\zeta(2\sigma_1+2\sigma_2)X + O(t_2). \tag{8-3}$$

For I_3 we apply Theorem 1 and get

$$I_3 \ll |t_2|^{2p(\sigma_1+\sigma_2-1)+\varepsilon} X. \tag{8-4}$$

By (8-2), (8-3) and (8-4) we have

$$\sqrt{I_1 I_2} \ll t_2^{-1} X^{\frac{5}{2}-\sigma_1-\sigma_2}, \tag{8-5}$$

$$\sqrt{I_1 I_3} \ll t_2^{p(\sigma_1+\sigma_2-1)-1+\varepsilon} X^{\frac{5}{2}-\sigma_1-\sigma_2}, \tag{8-6}$$

$$\sqrt{I_2 I_3} \ll t^{p(\sigma_1+\sigma_2-1)+\varepsilon} X. \tag{8-7}$$

By (8-1)–(8-7), we obtain

$$\int_2^X |\zeta_2(s_1,s_2)|^2\,dt_1 = \frac{(2\pi)^{2\sigma_1+2\sigma_2-3}\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|1-s_2|^2} X^{4-2\sigma_1-2\sigma_2} + O\big(t_2^{2p(\sigma_1+\sigma_2-1)+\varepsilon} X\big) + O\big(t_2^{p(\sigma_1+\sigma_2-1)-1+\varepsilon} X^{\frac{5}{2}-\sigma_1-\sigma_2}\big). \tag{8-8}$$

In (8-8), the second O-term is bigger than the first O-term if and only if

$$t_2 \ll X^{\frac{3/2-\sigma_1-\sigma_2}{1+p(\sigma_1+\sigma_2-1)}}.$$

Theorem 6 is now proved.

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