Pacific Journal of Mathematics

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MIN LIU AND ZHONGWEI TANG

Volume 304 No. 1 January 2020

PACIFIC JOURNAL OF MATHEMATICS
Vol. 304, No. 1, 2020
dx.doi.org/10.2140/pjm.2020.304.103

PSEUDOINDEX THEORY AND NEHARI METHOD FOR A FRACTIONAL CHOQUARD EQUATION

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We study the following nonlinear fractional Choquard equation:

$$(\star)\quad \varepsilon^{2s}(-\Delta)^s w + V(x)w = \varepsilon^{-\theta} W(x)[I_\theta * (W|w|^p)]|w|^{p-2}w, \quad x \in \mathbb{R}^N,$$

where $\varepsilon > 0$, $s \in (0,1)$, $N > 2s$, I_θ is the Riesz potential with order $\theta \in (0,N)$, $p \in \big[2, \frac{N+\theta}{N-2s}\big)$, $\min_{\mathbb{R}^N} V > 0$ and $\inf_{\mathbb{R}^N} W > 0$. By specifying the ranges and interdependence of linear and nonlinear potentials, we achieve the existence, convergence, concentration, and decay estimate of positive groundstates for $(\star)$. The multiplicity of semiclassical solutions is established via pseudoindex theory. The existence of sign-changing solutions is constructed by minimizing the energy on Nehari nodal set.

1. Introduction and main results

This paper contributes to the multiplicity of semiclassical solutions and the convergence, concentration, decay estimate of positive groundstates for the nonlinear fractional Choquard equation,

$$\text{(1-1)}\quad \varepsilon^{2s}(-\Delta)^s w + V(x)w = \varepsilon^{-\theta} W(x)[I_\theta * (W|w|^p)]|w|^{p-2}w, \quad w \in H^s(\mathbb{R}^N),$$

where $\varepsilon > 0$, $s \in (0,1)$, $N > 2s$, $\theta \in (0,N)$, $p \in \big[2, \frac{N+\theta}{N-2s}\big)$, V and W are Hölder continuous bounded positive functions, and the Riesz potential I_θ is defined as follows:

$$\text{(1-2)}\qquad I_\theta(x) := \frac{A_\theta}{|x|^{N-\theta}}, \quad x \in \mathbb{R}^N \setminus \{0\}, \quad \text{where } A_\theta := \frac{\Gamma\big(\frac{N-\theta}{2}\big)}{2^\theta \pi^{N/2} \Gamma\big(\frac{\theta}{2}\big)}.$$

The Choquard equation first appeared in Fröhlich and Pekar's model of polaron [Pekar 1954] and was afterwards introduced by Ph. Choquard in the modeling of a one-component plasma [Lieb 1977]. It can be regarded as Schrödinger–Newton equation in models coupling the Schrödinger equation of quantum physics with

Tang is the corresponding author. He is supported by the National Natural Science Foundation of China (11571040, 11671331).
MSC2010: primary 35J20; secondary 35R11.
Keywords: Fractional Choquard equation, pseudoindex, sign-changing solution, concentration.

nonrelativistic Newtonian gravity and it also associates with the Einstein–Klein–Gordon and Einstein–Dirac system [Moroz and Van Schaftingen 2017].

Solitary wave solution with the type

$$\psi(x,t)=e^{-i\lambda t/\varepsilon}w(x)$$

for the time-dependent Hartree equation with $(t,x)\in(0,\infty)\times\mathbb{R}^N$,

$$i\varepsilon\frac{\partial\psi}{\partial t}=\varepsilon^{2s}(-\Delta)^s\psi+(V(x)+\lambda)\psi-\varepsilon^{-\theta}W(x)[I_\theta*(W|\psi|^p)]|\psi|^{p-2}\psi$$

corresponds to the solution of Equation (1-1). In this sense, Choquard equation is known as stationary Hartree equation.

Concerning the early works on Choquard equation, we can go back to E. H. Lieb [1977] and P. L. Lions [1980], where they both dealt with the following equation

$$-\Delta u+au=2\Big(\frac{1}{|x|}*|u|^2\Big)u,\quad x\in\mathbb{R}^3 \tag{1-3}$$

with $a\in\mathbb{R}$. Moreover, Lieb proved the existence and uniqueness of solutions by using symmetric decreasing rearrangement inequalities and Lions obtained the existence of infinitely many radially symmetric solutions.

Thereafter, the study about Choquard equation has grown steadily. Ma and Zhao [2010] settled the classification of all positive solutions to Equation (1-3) with $a=1$ and proved that all the positive solutions of this equation must be radially symmetric and monotone decreasing about some fixed point by moving plane method. Moroz and Schaftingen [2013] and Ghimenti and Schaftingen [2016] studied Equation (1-1) with $s=\varepsilon=V(x)=W(x)=1$ successively, the former proved the existence, regularity, symmetry and decay asymptotics of positive groundstates, and the latter constructed minimal energy odd solutions for $p\in\big(\frac{N+\theta}{N},\frac{N+\theta}{N-2}\big)$ and minimal energy nodal solutions for $p\in\big(2,\frac{N+\theta}{N-2}\big)$. Alves, Gao, Squassina and Yang [Alves et al. 2017] considered

$$-\varepsilon^2\Delta u+V(x)u=\varepsilon^{-\theta}W(x)\Big(\frac{1}{|x|^{3-\theta}}*(WG(u))\Big)g(u),\quad x\in\mathbb{R}^3,$$

where $\varepsilon>0$, $\theta\in(0,3)$, V, W are continuous real functions, G is the primitive of g with a critical growth due to the Hardy–Littlewood–Sobolev inequality, and they established the existence and multiplicity of semiclassical solutions and characterized the concentration behavior by variational methods.

Recently, the discussion about fractional Choquard equation has also appeared gradually. Lenzmann [2009] proved the uniqueness of groundstate for the pseudorelativistic Hartree equation

$$\sqrt{-\Delta+m^2}u+au=\Big(\frac{1}{|x|}*|u|^2\Big)u,\quad x\in\mathbb{R}^3$$

in the regime of u with sufficiently small L^2-mass, where $m > 0$, $a \in \mathbb{R}$. Cingolani and Secchi [2015] studied the semiclassical limit for the pseudorelativistic Hartree equation

$$\sqrt{-\varepsilon^2\Delta + m^2}u + V(x)u = (I_\theta * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $\varepsilon > 0$, $m > 0$, $p \in \left[2, \frac{2N}{N-1}\right)$, $\theta \in ((N-1)p - N, N)$ and V is a continuous and bounded function with $\inf_{\mathbb{R}^N} V > -m$. d'Avenia, Siciliano and Squassina [d'Avenia et al. 2015] investigated

$$(-\Delta)^s u + au = \left(\frac{1}{|x|^{N-\theta}} * |u|^p\right)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $s \in (0, 1)$, $a > 0$, $p > 1$, $N \geq 3$, $\theta \in (0, N)$, and they obtained the regularity, existence, nonexistence, symmetry as well as decay properties of groundstates. Bhattarai [2017] obtained the existence and stability results of solutions for the following fractional Choquard equation with combined power and Hartree type nonlinearities

$$(-\Delta)^s u + au = b|u|^{r-2}u + \lambda\left(\frac{1}{|x|^{N-\theta}} * |u|^p\right)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $s \in (0, 1)$, $N \geq 2$, $\theta \in (0, N)$, $r \in \left(2, 2 + \frac{4s}{N}\right)$, $p \in \left[2, 1 + \frac{2s+\theta}{N}\right)$, $a \in \mathbb{R}$ and b, λ are nonnegative constants satisfying $b + \lambda \neq 0$.

For more results about Choquard equation, see [Belchior et al. 2017; Moroz and Van Schaftingen 2017; Zhong and Tang 2018]. For classical local case, we mainly refer to [Ding and Wei 2017]. For fractional elliptic problems, we refer to [Ao et al. 2017; Brändle et al. 2013; Dávila et al. 2014; Dipierro et al. 2017; Fall et al. 2015; Felmer et al. 2012; Jin et al. 2014].

Ding and Wei [2017] considered the following Schrödinger equation:

$$-\varepsilon^2\Delta w + V(x)w = W(x)|w|^{p-2}w, \quad x \in \mathbb{R}^N,$$

where $\varepsilon > 0$, $p \in \left(2, \frac{2N}{N-2}\right)$ and V, W are continuous bounded positive functions. They studied the existence and concentration of positive groundstates and they also constructed the multiplicity of semiclassical solutions including at least a pair of sign-changing solutions by pseudoindex theory and Nehari method.

With regard to the multiplicity of solutions for fractional elliptic problems, Lyapunov–Schmidt reduction is also an important method. We can take [Ao et al. 2017; Dávila et al. 2014] as examples. Dávila, del Pino and Wei [Dávila et al. 2014] considered the following fractional nonlinear Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^N,$$

where $\varepsilon > 0$, $s \in (0,1)$, $p \in \left(1, \frac{N+2s}{N-2s}\right)$ and $V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\inf_{\mathbb{R}^N} V > 0$. For any positive integer k, the authors proved the existence of k-spike solution in $H^{2s}(\mathbb{R}^N)$ by applying Lyapunov–Schmidt variational reduction together with appropriate assumptions. Ao, Wei and Yang [Ao et al. 2017] later studied the above equation with $\varepsilon = 1$, $N = 2$ and constructed infinitely many nonradial high energy positive solutions for it by using an intermediate Lyapunov–Schmidt reduction and assuming

$$V(x) = V_\infty + \frac{a}{|x|^m} + O\left(\frac{1}{|x|^{m+\sigma}}\right), \quad \text{as } |x| \to \infty.$$

Motivated by the works cited above, we wish to study the multiplicity and concentration of solutions for the nonlinear fractional Choquard equation (1-1) by using pseudoindex theory and the Nehari method. Referring to [Ding and Wei 2017], we shall consider the ranges and interdependence of linear and nonlinear potentials to be the essential assumptions. According to [Felmer et al. 2012], we will suppose the two potentials to be Hölder continuous in order to lift the regularity of solutions and ensure the representation (2-2) holds. Compared with [Ding and Wei 2017], there appear two nonlocal terms in our equation, which are fractional Laplace operator and convolution type Hartree term, which bring us more difficulties. In order to overcome the nonlocality of the fractional Laplace operator, in some cases we will adopt the extension method proposed in [Caffarelli and Silvestre 2007]. On the Hartree term, the Hardy–Littlewood–Sobolev inequality and the Brézis–Lieb type lemma (see Lemma 3.5) play an important role. We finally obtain the existence of multiple solutions, groundstates and sign-changing solutions for Equation (1-1). In addition, we prove the positive groundstate concentrates in a special set related to the minimum of linear potential and the maximum of nonlinear potential.

Now we state our assumptions and main results.

(A0) $V, W \in C^{0,\lambda}(\mathbb{R}^N, \mathbb{R})$ are bounded with some $\lambda \in (0,1)$, V achieves a global minimum on $\mathbb{R}^N$ with $\min_{\mathbb{R}^N} V > 0$, and W achieves a global maximum on $\mathbb{R}^N$ with $\inf_{\mathbb{R}^N} W > 0$.

To describe our results and proofs, denote

$$\tau := \min_{\mathbb{R}^N} V, \quad \mathscr{V} := \{x \in \mathbb{R}^N : V(x) = \tau\}, \quad \tau_\infty := \liminf_{|x|\to\infty} V(x);$$
$$k := \max_{\mathbb{R}^N} W, \quad \mathscr{W} := \{x \in \mathbb{R}^N : W(x) = k\}, \quad k_\infty := \limsup_{|x|\to\infty} W(x);$$
$$x_v \in \mathscr{V} : \ k_v := W(x_v) = \max_{\mathscr{V}} W;$$
$$x_w \in \mathscr{W} : \ \tau_w := V(x_w) = \min_{\mathscr{W}} V.$$

(A1) Either (i) or (ii) holds, where

(i) $\tau < \tau_\infty$, and there exists $R_v > 0$ such that $W(x) \le k_v$, for all $|x| \ge R_v$;
(ii) $k > k_\infty$, and there exists $R_w > 0$ such that $V(x) \ge \tau_w$, for all $|x| \ge R_w$.

If (A1)(i) holds, define $\mathscr{A}_v := \{x \in \mathscr{V} : W(x) = k_v\} \cup \{x \notin \mathscr{V} : W(x) > k_v\}$.

If (A1)(ii) holds, define $\mathscr{A}_w := \{x \in \mathscr{W} : V(x) = \tau_w\} \cup \{x \notin \mathscr{W} : V(x) < \tau_w\}$.

In what follows, $\mathscr{A}$ stands for $\mathscr{A}_v$ in the case (A1)(i) and $\mathscr{A}_w$ in the case (A1)(ii). Clearly, $\mathscr{A}$ is bounded. Moreover, $\mathscr{A} = \mathscr{V} \cap \mathscr{W}$ if $\mathscr{V} \cap \mathscr{W} \neq \varnothing$.

Theorem 1.1. *Assume that* (A0) *holds and*

$$\tau < \tau_\infty, \quad k_v \ge k_\infty. \tag{1-4}$$

Then for the maximal integer $m \in \mathbb{N}$ with

$$m < \left(\frac{\tau_\infty}{\tau}\right)^{\frac{\theta+2s}{2s(p-1)} - \frac{N-2s}{2s}} \cdot \left(\frac{k_v}{k_\infty}\right)^{\frac{2}{p-1}}, \tag{1-5}$$

Equation (1-1) possesses at least m pairs of solutions for small $\varepsilon > 0$. Furthermore, when $m \ge 2$ and $p \in \left(2, \frac{N+\theta}{N-2s}\right)$, among the solutions, at least one is positive, one is negative and two change sign.

Theorem 1.2. *Assume that* (A0) *holds and*

$$\tau_w \le \tau_\infty, \quad k > k_\infty. \tag{1-6}$$

Then for the maximal integer $m \in \mathbb{N}$ with

$$m < \left(\frac{\tau_\infty}{\tau_w}\right)^{\frac{\theta+2s}{2s(p-1)} - \frac{N-2s}{2s}} \cdot \left(\frac{k}{k_\infty}\right)^{\frac{2}{p-1}}, \tag{1-7}$$

all the conclusions of Theorem 1.1 remain true.

Theorem 1.3. *Assume that* (A0)–(A1) *hold. Then for sufficiently small $\varepsilon > 0$, Equation (1-1) has a positive groundstate w_ε. If additionally $V, W \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\nabla V, \nabla W$ are bounded, then w_ε satisfies*:

(i) (concentration) *There exists a maximum point x_ε of w_ε with*

$$\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathscr{A}) = 0.$$

(ii) (decay estimate) *For $p \in \left(2, \frac{N+\theta}{N-2s}\right)$, there exist $C_2 > C_1 > 0$ and sufficiently large $R > 0$ such that*

$$\frac{C_1 \varepsilon^{N+2s}}{|x - x_\varepsilon|^{N+2s}} \le w_\varepsilon(x) \le \frac{C_2 \varepsilon^{N+2s}}{|x - x_\varepsilon|^{N+2s}}, \quad \textit{for all } |x| \ge R.$$

(iii) (convergence) *Setting $u_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$, for any sequence $x_\varepsilon \to x_0$ (as $\varepsilon \to 0$),*

$$u_\varepsilon \to u \quad \textit{in } H^s(\mathbb{R}^N),$$

where u is a groundstate of

$$(-\Delta)^s u + V(x_0)u = W^2(x_0)(I_\theta * u^p)u^{p-1}, \quad u > 0. \tag{1-8}$$

In particular, if $\mathscr{V} \cap \mathscr{W} \neq \varnothing$, *then* $\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathscr{V} \cap \mathscr{W}) = 0$ *and up to a sequence,* $u_\varepsilon \to u$ *in* $H^s(\mathbb{R}^N)$ *as* $\varepsilon \to 0$, *with u being a groundstate of*

$$(-\Delta)^s u + \tau u = k^2(I_\theta * u^p)u^{p-1}, \quad u > 0. \tag{1-9}$$

Our paper is organized as follows: Section 2 is a review of the fractional Sobolev space and the fractional Laplace operator. Section 3 offers some valuable information about the Riesz potential. Section 4 contains some preliminary results which are established by variational methods and play a key role in the proofs of main theorems. Sections 5 and 6 contribute to the proofs of main results. In Section 5, we prove the multiplicity of semiclassical solutions via Benci pseudoindex theory and show the existence of the groundstates and the sign-changing solutions. In Section 6, we discuss the convergence, concentration and decay estimate of the positive groundstate.

2. Fractional Sobolev space and fractional Laplace operator

In this section, we shall recall some important facts about the fractional Sobolev space and the fractional Laplace operator.

For any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{(N+2s)/2}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2},$$

where the term

$$[u]_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}$$

is the Gagliardo seminorm of u.

For any $u \in H^s(\mathbb{R}^N)$, the fractional Laplace operator $(-\Delta)^s$ is defined via Fourier transform by

$$\widehat{(-\Delta)^s u} = |\xi|^{2s} \hat{u} \in L^2(\mathbb{R}^N), \tag{2-1}$$

where $\widehat{\cdot}$ stands for the Fourier transform. When u is assumed additionally smooth, one can obtain the direct representation

$$(-\Delta)^s u(x) = -\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \, \mathrm{d}y, \tag{2-2}$$

where $C_{N,s} := \left(\int_{\mathbb{R}^N} (1-\cos\zeta_1)/|\zeta|^{N+2s} \, \mathrm{d}\zeta\right)^{-1}$ is the dimensional constant.

Lemma 2.1 [Di Nezza et al. 2012]. *Let $s \in (0, 1)$ and $u \in H^s(\mathbb{R}^N)$. Then*

$$[u]^2_{H^s(\mathbb{R}^N)} = 2C^{-1}_{N,s} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi = 2C^{-1}_{N,s} \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)}.$$

Thus one can see the fractional Sobolev space $H^s(\mathbb{R}^N)$ defined as above coincides with

$$\hat{H}^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1+|\xi|^{2s}) |\hat{u}(\xi)|^2 \, \mathrm{d}\xi < \infty \right\}.$$

Obviously, the following norms on $H^s(\mathbb{R}^N)$ are all equivalent:

$$u \mapsto \left(\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2},$$

$$u \mapsto \left(\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2},$$

$$u \mapsto \left(\int_{\mathbb{R}^N} u^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, \mathrm{d}x \right)^{1/2}.$$

The dual space $H^{-s}(\mathbb{R}^N)$ of $H^s(\mathbb{R}^N)$ is defined in the standard way.

Caffarelli and Silvestre [2007] introduced another valuable local representation of $(-\Delta)^s$, which is via the following boundary value problem

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla \tilde{u}) = 0 & \text{in } \mathbb{R}^{N+1}_+ := \{(x, y) : x \in \mathbb{R}^N, \, y > 0\}, \\ \tilde{u}(x, 0) = u(x) & \text{on } \mathbb{R}^N. \end{cases}$$

There, $\tilde{u}$ is called the s-harmonic extension of u, and $\tilde{u}$ belongs to

$$X^s(\mathbb{R}^{N+1}_+) := \overline{C^\infty_0(\overline{\mathbb{R}^{N+1}_+})}^{\|\cdot\|_{X^s(\mathbb{R}^{N+1}_+)}}$$

with the norm

$$\|\tilde{u}\|_{X^s(\mathbb{R}^{N+1}_+)} := \left(C_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \tilde{u}|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}$$

and the inner product

$$(\tilde{u}, \tilde{v})_{X^s(\mathbb{R}^{N+1}_+)} := C_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla \tilde{u} \cdot \nabla \tilde{v} \, \mathrm{d}x \, \mathrm{d}y,$$

where $C_s := \Gamma(s)/(2^{1-2s}\Gamma(1-s))$, is a normalization constant ensuring

$$\|\tilde{u}\|^2_{X^s(\mathbb{R}^{N+1}_+)} = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}(\xi)|^2 \,\mathrm{d}\xi = \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^N)}. \tag{2-3}$$

Moreover, $\tilde{u}$ can be explicitly expressed by

$$\tilde{u}(x, y) = \int_{\mathbb{R}^N} P_s(x-z, y)u(z)\,\mathrm{d}z, \tag{2-4}$$

where

$$P_s(x, y) = \frac{d_{N,s}\, y^{2s}}{(|x|^2 + y^2)^{(N+2s)/2}}$$

is the s-Poisson kernel. It has been proved in [Caffarelli and Silvestre 2007] that

$$(-\Delta)^s u(x) = -C_s \lim_{y\to 0^+} y^{1-2s}\partial_y \tilde{u}(x, y). \tag{2-5}$$

The characterizations (2-1), (2-2) and (2-5) are equivalent, for instance, in Schwartz's space of rapidly decaying C^∞ functions on $\mathbb{R}^N$. For more properties of the fractional Laplace operator, we refer readers to Section 2 of [Silvestre 2007]. In the following, we set $2^*_s := 2N/(N-2s)$.

Lemma 2.2 [Di Nezza et al. 2012]. *Let $s \in (0, 1)$. Then $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*_s]$ and compactly embedded into $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $q \in [2, 2^*_s)$.*

Lemma 2.3 [d'Avenia et al. 2015]. *Let $s \in (0, 1)$, $\varrho > 0$, $q \in [2, 2^*_s)$. If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N} \int_{B_\varrho(y)} |u_n|^q \,\mathrm{d}x \to 0, \quad \text{as } n \to \infty,$$

then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for any $r \in (2, 2^*_s)$.

Lemma 2.4 [Brändle et al. 2013]. *Let $s \in (0, 1)$ and $U \in X^s(\mathbb{R}^{N+1}_+)$. Then*

$$\left(\int_{\mathbb{R}^N} |u|^{2^*_s} \,\mathrm{d}x\right)^{2/2^*_s} \le k_{N,s} \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla U|^2 \,\mathrm{d}x\,\mathrm{d}y,$$

where

$$u(x) = \mathrm{tr}_{\mathbb{R}^N} U =: U(x, 0) \quad \text{and} \quad k_{N,s} = \frac{\Gamma(s)\Gamma\left(\frac{N-2s}{2}\right)}{2\pi^s\Gamma(1-s)\Gamma\left(\frac{N+2s}{2}\right)}\left[\frac{\Gamma(N)}{\Gamma(N/2)}\right]^{2s/N}.$$

Lemma 2.5 [Dipierro et al. 2017]. *Let $r > 0$, $B^+_r := \{(x, y) \in \mathbb{R}^{N+1}_+ : |(x, y)| < r\}$ and $\mathscr{T}$ be a subset of $X^s(\mathbb{R}^{N+1}_+)$ such that*

$$\sup_{U\in\mathscr{T}} \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla U|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Then $\mathscr{T}$ is precompact in $L^2(B_r^+, y^{1-2s})$. Namely, $X^s(\mathbb{R}_+^{N+1})$ is compactly embedded into $L^2_{\mathrm{loc}}(\mathbb{R}_+^{N+1}, y^{1-2s})$ with

$$\|U\|_{L^2(\mathbb{R}_+^{N+1}, y^{1-2s})} := \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} U^2(x, y) \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}.$$

Now we define a Hilbert space

$$E := \left\{ U \in X^s(\mathbb{R}_+^{N+1}) : \int_{\mathbb{R}^N} U^2(x, 0) \, \mathrm{d}x < \infty \right\}$$

with norm

$$\|U\|_E := \left(C_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla U|^2 \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^N} U^2(x, 0) \, \mathrm{d}x \right)^{1/2}$$

and inner product

$$(U_1, U_2)_E := C_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla U_1 \cdot \nabla U_2 \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^N} U_1(x, 0) U_2(x, 0) \, \mathrm{d}x.$$

Clearly, $\mathrm{tr}_{\mathbb{R}^N} E = H^s(\mathbb{R}^N)$. By Lemma 2.4, E is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*]$. Moreover, E is compactly embedded into $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$.

In the following, we drop the constant C_s for simplicity and without destruction. Moreover, we shall use different patterns of C to denote various positive constants, and $o(1)$ to denote the quantities that tend to 0 as $n \to \infty$ or $j \to \infty$. Meanwhile, set

$$\|u\|_s := \|u\|_{H^s(\mathbb{R}^N)}, \quad |u|_q := \|u\|_{L^q(\mathbb{R}^N)},$$
$$\|u\|_X := \|u\|_{X^s(\mathbb{R}_+^{N+1})}, \quad (u, v)_X := (u, v)_{X^s(\mathbb{R}_+^{N+1})},$$
$$u^+ := \max\{0, u\}, \quad u^- := \min\{0, u\}, \quad \mathbb{R}_+ := (0, \infty).$$

3. Riesz potential

In this section, we shall review and prove some useful results about the Riesz potential.

The Riesz potential with order $\theta \in (0, N)$ of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is defined by

$$(I_\theta * f)(x) := \int_{\mathbb{R}^N} \frac{A_\theta f(y)}{|x - y|^{N-\theta}} \, \mathrm{d}y, \tag{3-1}$$

where A_θ is the same as in (1-2). The integral in (3-1) converges in the classical Lebesgue sense for a.e. $x \in \mathbb{R}^N$ if and only if

$$f \in L^1(\mathbb{R}^N, (1 + |x|)^{\theta - N}). \tag{3-2}$$

Moreover, if (3-2) doesn't hold, then (3-1) diverges everywhere on $\mathbb{R}^N$. In addition, if $f \in C^{0,\nu}(\mathbb{R}^N)$ with $\nu \in (0,1)$, then either $I_\theta * f \in C^{0,\nu+\theta}(\mathbb{R}^N)$ for $\nu+\theta \in (0,1)$ or $I_\theta * f \in C^{0,1}(\mathbb{R}^N)$ for $\nu+\theta \in [1, N+1)$ (see Theorem 1 and its proof in [du Plessis 1955]).

The Riesz potential I_θ is well-defined as an operator in $L^q(\mathbb{R}^N)$ if and only if $q \in \left[1, \frac{N}{\theta}\right)$. Furthermore, if $q \in \left(1, \frac{N}{\theta}\right)$ and $r := \frac{Nq}{N-\theta q}$, then

$$I_\theta : L^q(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$$

is a bounded linear operator, which can be disclosed by the Hardy–Littlewood–Sobolev inequality.

Lemma 3.1 [Hardy et al. 1952]. *Let $\theta \in (0, N)$, $q \in \left(1, \frac{N}{\theta}\right)$. Then for any $f \in L^q(\mathbb{R}^N)$, $I_\theta * f \in L^{Nq/(N-\theta q)}(\mathbb{R}^N)$ and*

$$\left(\int_{\mathbb{R}^N} |I_\theta * f|^{Nq/(N-\theta q)}\,\mathrm{d}x\right)^{\frac{1}{q}-\frac{\theta}{N}} \le C_{N,\theta,q}\left(\int_{\mathbb{R}^N} |f|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Applying Lemma 3.1 to the function $f = |u|^p \in L^{2N/(N+\theta)}(\mathbb{R}^N)$, we get the following result.

Lemma 3.2 [Moroz and Van Schaftingen 2017]. *Let $\theta \in (0, N)$. Then for any $u \in L^{2Np/(N+\theta)}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (I_\theta * |u|^p)|u|^p\,\mathrm{d}x \le C_{N,\theta}\left(\int_{\mathbb{R}^N} |u|^{2Np/(N+\theta)}\,\mathrm{d}x\right)^{(N+\theta)/N}.$$

If, in particular, $s \in (0,1)$, $N > 2s$, $\theta \in (0, N)$, $p \in \left[\frac{N+\theta}{N}, \frac{N+\theta}{N-2s}\right]$ and $u \in H^s(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} (I_\theta * |u|^p)|u|^p\,\mathrm{d}x \le C_{N,\theta,p}\|u\|_s^{2p}.$$

As a matter of fact, $p \in \left[\frac{N+\theta}{N}, \frac{N+\theta}{N-2s}\right]$ if and only if $\frac{2Np}{N+\theta} \in [2, 2_s^*]$.

It's worthwhile to mention that the Brézis–Lieb type lemma holds for the Riesz potential. To prove this lemma, we need two essential results.

Lemma 3.3. *Let Ω be a domain on $\mathbb{R}^N$ and $q, r \in (1, \infty)$, $\frac{1}{q}+\frac{1}{r} = 1$. If $u_n \to u$ in $L^q(\Omega)$ and $v_n \rightharpoonup v$ in $L^r(\Omega)$ as $n \to \infty$, then*

$$\lim_{n\to\infty}\int_\Omega u_n v_n\,\mathrm{d}x = \int_\Omega uv\,\mathrm{d}x.$$

Lemma 3.4. *Let Ω be a domain on $\mathbb{R}^N$, $r \in [1, \infty)$ and $\{u_n\}$ be a bounded sequence in $L^r(\Omega)$. If $u_n \to u$ a.e. in Ω as $n \to \infty$, then for any $q \in [1, r]$,*

$$\lim_{n\to\infty}\int_\Omega \big||u_n|^q - |u_n-u|^q - |u|^q\big|^{r/q}\,\mathrm{d}x = 0,$$

$$\lim_{n\to\infty}\int_\Omega \big||u_n^\pm|^q - |(u_n-u)^\pm|^q - |u^\pm|^q\big|^{r/q}\,\mathrm{d}x = 0.$$

We would like to point out that the constraint $p \in \big[2, \frac{N+\theta}{N-2s}\big)$ is only needed for the second part of the following Brézis–Lieb type lemma, while the first part permits $p \in \big[\frac{N+\theta}{N}, \frac{N+\theta}{N-2s}\big]$.

Lemma 3.5. *Let $s \in (0,1)$, $N > 2s$, $\theta \in (0, N)$, $p \in \big[2, \frac{N+\theta}{N-2s}\big)$. If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$, then*

(i) *$\mathcal{D}(u_n) - \mathcal{D}(u_n - u) \to \mathcal{D}(u)$ as $n \to \infty$;*

(ii) *$\mathcal{D}'(u_n) - \mathcal{D}'(u_n - u) \to \mathcal{D}'(u)$ in $H^{-s}(\mathbb{R}^N)$ as $n \to \infty$,*

where

$$\mathcal{D}(u) := \int_{\mathbb{R}^N} (I_\theta * |u|^p)|u|^p \,\mathrm{d}x.$$

Proof. The proof is similar to that of Lemma 2.5 in [Liu and Tang 2019] with Lemmas 3.3 and 3.4 being used, and so it is omitted. □

In terms of Lemma 3.5, the following result is absolutely true.

Lemma 3.6. *Let $s \in (0,1)$, $N > 2s$, $\theta \in (0, N)$, $p \in \big[2, \frac{N+\theta}{N-2s}\big)$. If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$, then for any $v \in H^s(\mathbb{R}^N)$,*

$$\langle \mathcal{D}'(u_n), v\rangle \to \langle \mathcal{D}'(u), v\rangle \quad \text{as } n \to \infty,$$

where $\mathcal{D}(u)$ is defined as in Lemma 3.5.

4. Preliminary results

In this section, we establish some results which are necessary for the arguments of our main results. Let us first consider the following two equations for $s \in (0,1)$, $N > 2s$, $\theta \in (0, N)$, $p \in \big[2, \frac{N+\theta}{N-2s}\big)$,

$$(-\Delta)^s v + av = b^2 (I_\theta * |v|^p)|v|^{p-2} v, \quad v \in H^s(\mathbb{R}^N), \tag{4-1}$$

where $a > 0$, $b > 0$, and

$$(-\Delta)^s v + V_\varepsilon^a(x) v = W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b |v|^p)]|v|^{p-2} v, \quad v \in H^s(\mathbb{R}^N), \tag{4-2}$$

where $\tau \le a \le \tau_\infty$, $k_\infty \le b \le k$ and

$$V^a(x) := \max\{a, V(x)\}, \quad V_\varepsilon^a(x) := V^a(\varepsilon x),$$
$$W^b(x) := \min\{b, W(x)\}, \quad W_\varepsilon^b(x) := W^b(\varepsilon x).$$

Definition 4.1. (i) $v \in H^s(\mathbb{R}^N)$ is a weak solution of Equation (4-1) if for any $\varphi \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \big((-\Delta)^{s/2} v (-\Delta)^{s/2} \varphi + av\varphi\big) \,\mathrm{d}x = \int_{\mathbb{R}^N} b^2 (I_\theta * |v|^p)|v|^{p-2} v\varphi \,\mathrm{d}x.$$

(ii) $v_\varepsilon \in H^s(\mathbb{R}^N)$ is a weak solution of Equation (4-2) if for any $\varphi \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \big((-\Delta)^{s/2} v_\varepsilon (-\Delta)^{s/2}\varphi + V_\varepsilon^a(x) v_\varepsilon \varphi\big)\,\mathrm{d}x = \int_{\mathbb{R}^N} W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b |v_\varepsilon|^p)]|v_\varepsilon|^{p-2} v_\varepsilon \varphi\,\mathrm{d}x.$$

Associated with Equation (4-1) and Equation (4-2) respectively, we define the energy functionals for each $v \in H^s(\mathbb{R}^N)$,

$$J^{ab}(v) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} v|^2 + a v^2\big)\,\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N} b^2 (I_\theta * |v|^p)|v|^p\,\mathrm{d}x,$$
$$J_\varepsilon^{ab}(v) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} v|^2 + V_\varepsilon^a(x) v^2\big)\,\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N} W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b |v|^p)]|v|^p\,\mathrm{d}x;$$

the Nehari manifolds

$$\mathcal{N}^{ab} := \big\{v \in H^s(\mathbb{R}^N)\setminus\{0\} : \langle (J^{ab})'(v), v\rangle = 0\big\},$$
$$\mathcal{N}_\varepsilon^{ab} := \big\{v \in H^s(\mathbb{R}^N)\setminus\{0\} : \langle (J_\varepsilon^{ab})'(v), v\rangle = 0\big\};$$

the least energies

$$\vartheta^{ab} := \inf_{\mathcal{N}^{ab}} J^{ab}, \quad \vartheta_\varepsilon^{ab} := \inf_{\mathcal{N}_\varepsilon^{ab}} J_\varepsilon^{ab};$$

and the sets of least energy solutions

$$\mathscr{R}^{ab} := \big\{v \in H^s(\mathbb{R}^N) : J^{ab}(v) = \vartheta^{ab},\ (J^{ab})'(v) = 0\big\},$$
$$\mathscr{R}_\varepsilon^{ab} := \big\{v_\varepsilon \in H^s(\mathbb{R}^N) : J_\varepsilon^{ab}(v_\varepsilon) = \vartheta_\varepsilon^{ab},\ (J_\varepsilon^{ab})'(v_\varepsilon) = 0\big\}.$$

In particular, we set

$$J^\infty := J^{\tau_\infty k_\infty}, \quad \mathcal{N}^\infty := \mathcal{N}^{\tau_\infty k_\infty}, \quad \vartheta^\infty := \vartheta^{\tau_\infty k_\infty}, \quad V_\varepsilon^\infty := V_\varepsilon^{\tau_\infty},$$
$$J_\varepsilon^\infty := J_\varepsilon^{\tau_\infty k_\infty}, \quad \mathcal{N}_\varepsilon^\infty := \mathcal{N}_\varepsilon^{\tau_\infty k_\infty}, \quad \vartheta_\varepsilon^\infty := \vartheta_\varepsilon^{\tau_\infty k_\infty}, \quad W_\varepsilon^\infty := W_\varepsilon^{k_\infty}.$$

Moreover, J^{ab}, $J_\varepsilon^{ab} \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$. The critical points of J^{ab} and J_ε^{ab} correspond to the weak solutions of Equation (4-1) and Equation (4-2), respectively.

Additionally, Equations (4-1) and (4-2) can be reformulated as

$$\text{(4-3)}\qquad \begin{cases} \operatorname{div}(y^{1-2s}\nabla \tilde v) = 0 & \text{in } \mathbb{R}_+^{N+1},\\ \tilde v(x,0) = v(x) & \text{on } \mathbb{R}^N,\\ \lim_{y\to 0^+} y^{1-2s}\partial_y \tilde v = av - b^2(I_\theta * |v|^p)|v|^{p-2}v & \text{in } \mathbb{R}_+^{N+1},\end{cases}$$

and

$$\text{(4-4)}\qquad \begin{cases} \operatorname{div}(y^{1-2s}\nabla \tilde v) = 0 & \text{in } \mathbb{R}_+^{N+1},\\ \tilde v(x,0) = v(x) & \text{on } \mathbb{R}^N,\\ \lim_{y\to 0^+} y^{1-2s}\partial_y \tilde v = V_\varepsilon^a(x)v - W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b|v|^p)]|v|^{p-2}v & \text{in } \mathbb{R}_+^{N+1},\end{cases}$$

respectively, where $\tilde v \in E$ is the s-harmonic extension of $v \in H^s(\mathbb{R}^N)$.

Definition 4.2. (i) $\tilde{v} \in E$ is a weak solution of problem (4-3) if for all $\tilde{\varphi} \in E$

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla \tilde{v} \cdot \nabla \tilde{\varphi} \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^N} a v \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} b^2 (I_\theta * |v|^p) |v|^{p-2} v \varphi \, \mathrm{d}x,$$

(ii) $\tilde{v}_\varepsilon \in E$ is a weak solution of problem (4-4) if for all $\tilde{\varphi} \in E$

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla \tilde{v}_\varepsilon \cdot \nabla \tilde{\varphi} \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon^a(x) v_\varepsilon \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} W_\varepsilon^b(x) [I_\theta * (W_\varepsilon^b |v|^p)] |v_\varepsilon|^{p-2} v_\varepsilon \varphi \, \mathrm{d}x,$$

where $v = \mathrm{tr}_{\mathbb{R}^N} \tilde{v}$, $v_\varepsilon = \mathrm{tr}_{\mathbb{R}^N} \tilde{v}_\varepsilon$, $\varphi = \mathrm{tr}_{\mathbb{R}^N} \tilde{\varphi}$.

Associated with (4-3) and (4-4) respectively, we define the energy functionals for each $\tilde{v} \in E$,

$$\tilde{J}^{ab}(\tilde{v}) := \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla \tilde{v}|^2 \, \mathrm{d}x \, \mathrm{d}y + \frac{1}{2} \int_{\mathbb{R}^N} a v^2 \, \mathrm{d}x - \frac{1}{2p} \int_{\mathbb{R}^N} b^2 (I_\theta * |v|^p) |v|^p \, \mathrm{d}x,$$

$$\tilde{J}_\varepsilon^{ab}(\tilde{v}) := \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla \tilde{v}|^2 \, \mathrm{d}x \, \mathrm{d}y + \frac{1}{2} \int_{\mathbb{R}^N} V_\varepsilon^a(x) v^2 \, \mathrm{d}x - \frac{1}{2p} \int_{\mathbb{R}^N} W_\varepsilon^b(x) [I_\theta * (W_\varepsilon^b |v|^p)] |v|^p \, \mathrm{d}x;$$

and we denote the Nehari manifolds by $\tilde{\mathscr{N}}^{ab}$, $\tilde{\mathscr{N}}_\varepsilon^{ab}$; the least energies by $\tilde{\vartheta}^{ab}$, $\tilde{\vartheta}_\varepsilon^{ab}$; and the sets of least energy solutions by $\tilde{\mathscr{R}}^{ab}$, $\tilde{\mathscr{R}}_\varepsilon^{ab}$.

In addition, $\tilde{J}^{ab}$, $\tilde{J}_\varepsilon^{ab} \in C^1(E, \mathbb{R})$. The critical points of $\tilde{J}^{ab}$ and $\tilde{J}_\varepsilon^{ab}$ correspond to the weak solutions of (4-3) and (4-4), respectively.

In terms of (2-3), we find that

$$\tilde{J}^{ab}(\tilde{v}) = J^{ab}(v), \quad \tilde{J}_\varepsilon^{ab}(\tilde{v}) = J_\varepsilon^{ab}(v)$$

for any $\tilde{v} \in E$ as the s-harmonic extension of $v \in H^s(\mathbb{R}^N)$. And $v \in \mathscr{N}^{ab}$ if and only if $\tilde{v} \in \tilde{\mathscr{N}}^{ab}$, $v \in \mathscr{N}_\varepsilon^{ab}$ if and only if $\tilde{v} \in \tilde{\mathscr{N}}_\varepsilon^{ab}$. Hence

$$\tilde{\vartheta}^{ab} = \vartheta^{ab}, \quad \tilde{\vartheta}_\varepsilon^{ab} = \vartheta_\varepsilon^{ab}.$$

Indeed, $\tilde{v} \in E$ is a solution of (4-3) if and only if $v = \mathrm{tr}_{\mathbb{R}^N} \tilde{v}$ is a solution of (4-1), and $\tilde{v}_\varepsilon \in E$ is a solution of (4-4) if and only if $v_\varepsilon = \mathrm{tr}_{\mathbb{R}^N} \tilde{v}_\varepsilon$ is a solution of (4-2). Since (4-1), (4-2) and (4-3), (4-4) are equivalent, respectively, we shall take some advantages in different cases.

***Equation* (4-1).** In this subsection, we will derive some results for Equation (4-1), and the corresponding results are also true for (4-3).

Lemma 4.3. *There exist $\rho > 0$ and $\sigma > 0$ such that*

(i) $J^{ab}(v) > \sigma$, *for all* $\|v\|_s = \rho$; (ii) $\lim_{t \to +\infty} J^{ab}(tv) = -\infty$ *if* $v \neq 0$.

Lemma 4.4. $\vartheta^{ab} = \inf_{v\in H^s(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} J^{ab}(tv) = \inf_{\gamma\in\Gamma^{ab}} \max_{t\in[0,1]} J^{ab}(\gamma(t)) > 0,$ *where*

$$\Gamma^{ab} := \big\{\gamma \in C\big([0,1], H^s(\mathbb{R}^N)\big) : \gamma(0) = 0,\ J^{ab}(\gamma(1)) < 0\big\}.$$

Lemma 4.5. *Let* $v \in H^s(\mathbb{R}^N)\setminus\{0\}$. *Then*

$$\max_{t\geq 0} J^{ab}(tv) = \frac{p-1}{2p}(S^{ab}(v))^{p/(p-1)},$$

where

$$S^{ab}(v) := \frac{\int_{\mathbb{R}^N}\big(|(-\Delta)^{s/2}v|^2 + av^2\big)\,\mathrm{d}x}{\big(\int_{\mathbb{R}^N} b^2(I_\theta * |v|^p)|v|^p\,\mathrm{d}x\big)^{1/p}}.$$

From the proof of Theorem 1 in [Moroz and Van Schaftingen 2013], the following result is true.

Lemma 4.6. *Both* ς^{ab} *and* ϑ^{ab} *are attained in* $H^s(\mathbb{R}^N)$, *where*

$$\begin{aligned}\varsigma^{ab} &:= \inf\big\{S^{ab}(v) : v \in H^s(\mathbb{R}^N)\setminus\{0\}\big\}\\ &= \inf\Big\{S^{ab}(v) : \int_{\mathbb{R}^N} b^2(I_\theta * |v|^p)|v|^p\,\mathrm{d}x = 1,\ v \in H^s(\mathbb{R}^N)\Big\}\end{aligned}$$

with $S^{ab}(v)$ *defined as in Lemma 4.5.*

In view of Theorem 1.1 in [d'Avenia et al. 2015], we have the following result.

Lemma 4.7. *There exists a groundstate* $v \in H^s(\mathbb{R}^N)$ *for Equation (4-1), where* v *is positive, radially symmetric and decreasing. Moreover, if* $s \in \big(0, \frac{1}{2}\big]$, *then* $v \in L^1(\mathbb{R}^N)\cap C^{0,\mu}(\mathbb{R}^N)$ *for* $\mu \in (0, 2s)$; *if* $s \in \big(\frac{1}{2}, 1\big)$, *then* $v \in L^1(\mathbb{R}^N)\cap C^{1,\mu}(\mathbb{R}^N)$ *for* $\mu \in (0, 2s-1)$.

Lemma 4.8. *Let* $a_i > 0$ *and* $b_i > 0$ *for* $i = 1, 2$.

(i) *If* $\min\{a_2 - a_1, b_1 - b_2\} \geq 0$, *then* $\vartheta^{a_1b_1} \leq \vartheta^{a_2b_2}$.

(ii) *If* $\min\{a_2 - a_1, b_1 - b_2\} \geq 0$ *and* $\max\{a_2 - a_1, b_1 - b_2\} > 0$, *then* $\vartheta^{a_1b_1} < \vartheta^{a_2b_2}$.

Lemma 4.9. $$\vartheta^{ab} = \left(\frac{a}{\tau_\infty}\right)^{\frac{\theta+2s}{2s(p-1)} - \frac{N-2s}{2s}} \cdot \left(\frac{k_\infty}{b}\right)^{\frac{2}{p-1}} \vartheta^\infty.$$

Moreover, (1-5) *holds if and only if* $m\vartheta^{\tau k_v} < \vartheta^\infty$; (1-7) *holds if and only if* $m\vartheta^{\tau_w k} < \vartheta^\infty$.

Proof. Setting

$$u(x) := \left(\left(\frac{\tau_\infty}{a}\right)^{\frac{\theta+2s}{4s}} \frac{b}{k_\infty}\right)^{\frac{1}{p-1}} v\left(\left(\frac{\tau_\infty}{a}\right)^{\frac{1}{2s}} x\right),$$

then Equation (4-1) is equivalent to

$$(-\Delta)^s u + \tau_\infty u = k_\infty^2 (I_\theta * |u|^p)|u|^{p-2}u, \quad u \in H^s(\mathbb{R}^N).$$

One can check that $v \in \mathcal{N}^{ab}$ if and only if $u \in \mathcal{N}^\infty$, and for any $v \in \mathcal{N}^{ab}$,

$$J^{ab}(v) = \left(\frac{a}{\tau_\infty}\right)^{\frac{\theta+2s}{2s(p-1)} - \frac{N-2s}{2s}} \cdot \left(\frac{k_\infty}{b}\right)^{\frac{2}{p-1}} J^\infty(u). \qquad \square$$

Lemma 4.10. *For any given $r \geq 2$ and $a > 0$, there exists a continuous function $v \in H^s(\mathbb{R}^N)$ satisfying*

$$(-\Delta)^s v + av = 0, \quad |x| \geq r$$

and

$$\frac{C_1}{|x|^{N+2s}} \leq v(x) \leq \frac{C_2}{|x|^{N+2s}}, \quad \textit{for all } |x| \geq r,$$

where $C_2 > C_1 > 0$ dependent on s, a, r, N.

Proof. Consider the function $v := \mathcal{K}_a * \chi_{B_{r/2}}$, where $\mathcal{K}_a$ is the fundamental solution of $(-\Delta)^s + a$ with $\widehat{\mathcal{K}_a(x)} = 1/(|\xi|^{2s} + a)$, and $\chi_{B_{r/2}}$ is the characteristic function of the ball $B_{r/2}$. For any $|x| \geq r$,

$$v(x) = \int_{|z| \leq r/2} \mathcal{K}_a(x-z)\,\mathrm{d}z \quad \text{and} \quad (-\Delta)^s v + av = 0.$$

According to Felmer, Quaas and Tan [Felmer et al. 2012], $\mathcal{K}_a$ is positive, radially symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$, moreover,

$$\frac{\bar{C}_1}{|x|^{N+2s}} \leq \mathcal{K}_a(x) \leq \frac{\bar{C}_2}{|x|^{N+2s}}, \quad \text{for all } |x| \geq 1$$

with $\bar{C}_2 > \bar{C}_1 > 0$ dependent on s, c, N. Thus for all $|x| \geq r$,

$$\int_{|z| \leq r/2} \frac{\bar{C}_1}{|x-z|^{N+2s}}\,\mathrm{d}z \leq \int_{|z| \leq r/2} \mathcal{K}_a(x-z)\,\mathrm{d}z \leq \int_{|z| \leq r/2} \frac{\bar{C}_2}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

where

$$\begin{aligned}\int_{|z| \leq r/2} \frac{\bar{C}_1}{|x-z|^{N+2s}}\,\mathrm{d}z &\geq \int_{|z| \leq r/2} \frac{\bar{C}_1}{(|x|+|z|)^{N+2s}}\,\mathrm{d}z \\ &\geq \int_{|z| \leq r/2} \frac{\bar{C}_1}{(|x|+|x|/2)^{N+2s}}\,\mathrm{d}z = \frac{C_1}{|x|^{N+2s}}\end{aligned}$$

and

$$\begin{aligned}\int_{|z| \leq r/2} \frac{\bar{C}_2}{|x-z|^{N+2s}}\,\mathrm{d}z &\leq \int_{|z| \leq r/2} \frac{\bar{C}_2}{(|x|-|z|)^{N+2s}}\,\mathrm{d}z \\ &\leq \int_{|z| \leq r/2} \frac{\bar{C}_2}{(|x|-|x|/2)^{N+2s}}\,\mathrm{d}z = \frac{C_2}{|x|^{N+2s}}.\end{aligned} \qquad \square$$

***Equation* (4-2).** In this subsection, we will establish some results for Equation (4-2), and the corresponding results are also true for (4-4).

Lemma 4.11. *There exist $\rho > 0$ and $\sigma > 0$ both independent of ε, a, b and just dependent on $N,\ \theta,\ p,\ \tau,\ k$, such that*

$$\text{(i) } J_\varepsilon^{ab}(v) > \sigma, \quad \textit{for all } \|v\|_s = \rho; \qquad \text{(ii) } \lim_{t\to+\infty} J_\varepsilon^{ab}(tv) = -\infty \quad \textit{if } v \neq 0.$$

Lemma 4.12. $$\vartheta_\varepsilon^{ab} = \inf_{v\in H^s(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} J_\varepsilon^{ab}(tv) = \inf_{\gamma\in\Gamma_\varepsilon^{ab}} \max_{t\in[0,1]} J_\varepsilon^{ab}(\gamma(t)) > 0,$$

where

$$\Gamma_\varepsilon^{ab} := \big\{\gamma \in C\big([0,1], H^s(\mathbb{R}^N)\big) : \gamma(0) = 0,\ J_\varepsilon^{ab}(\gamma(1)) < 0\big\}.$$

Lemma 4.13. *If J_ε^∞ possesses a $(\mathrm{PS})_c$ sequence, then either $c = 0$ or $c \geq \vartheta_\varepsilon^\infty$. Moreover, $\vartheta_\varepsilon^\infty \geq \vartheta^\infty$.*

Proof. Let $\{v_n\} \subset H^s(\mathbb{R}^N)$ and $J_\varepsilon^\infty(v_n) \to c$, $(J_\varepsilon^\infty)'(v_n) \to 0$ in $H^{-s}(\mathbb{R}^N)$ as $n \to \infty$. Assume $c \neq 0$, we will prove $c \geq \vartheta_\varepsilon^\infty$.

Since $\{v_n\}$ is bounded in $H^s(\mathbb{R}^N)$, we may assume $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$ along a subsequence. Set $z_n := v_n - v$. By the Brézis–Lieb lemma, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} v_n|^2 + V_\varepsilon^\infty(x) v_n^2\big)\,\mathrm{d}x \\ &\qquad = \int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} v|^2 + V_\varepsilon^\infty(x) v^2\big)\,\mathrm{d}x \\ &\qquad\qquad + \int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} z_n|^2 + V_\varepsilon^\infty(x) z_n^2\big)\,\mathrm{d}x + o(1). \end{aligned} \tag{4-5}$$

By slight amendment in the proof of Lemma 3.5, we get that

$$\begin{aligned} &\int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |v_n|^p)]|v_n|^p\,\mathrm{d}x \\ &\qquad = \int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |v|^p)]|v|^p\,\mathrm{d}x \\ &\qquad\qquad + \int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |z_n|^p)]|z_n|^p\,\mathrm{d}x + o(1), \end{aligned} \tag{4-6}$$

and for any $\varphi \in H^s(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |v_n|^p)]|v_n|^{p-2} v_n \varphi\,\mathrm{d}x \\ &\qquad = \int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |v|^p)]|v|^{p-2} v \varphi\,\mathrm{d}x \\ &\qquad\qquad + \int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |z_n|^p)]|z_n|^{p-2} z_n \varphi\,\mathrm{d}x + o(1)\|\varphi\|_s. \end{aligned} \tag{4-7}$$

As the proof of Lemma 3.6, we get that for any $\varphi \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |v_n|^p)]|v_n|^{p-2} v_n \varphi \,\mathrm{d}x \to \int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |v|^p)]|v|^{p-2} v \varphi \,\mathrm{d}x \quad \text{as } n \to \infty,$$

which ensures that $(J_\varepsilon^\infty)'(v) = 0$. In virtue of (4-5), (4-6) and (4-7), we obtain that

$$J_\varepsilon^\infty(v_n) = J_\varepsilon^\infty(v) + J_\varepsilon^\infty(z_n) + o(1),$$
$$(J_\varepsilon^\infty)'(v_n) = (J_\varepsilon^\infty)'(v) + (J_\varepsilon^\infty)'(z_n) + o(1),$$

which imply that

$$J_\varepsilon^\infty(z_n) \to c - J_\varepsilon^\infty(v), \qquad \text{as } n \to \infty, \tag{4-8}$$

$$(J_\varepsilon^\infty)'(z_n) \to 0 \quad \text{in } H^{-s}(\mathbb{R}^N), \quad \text{as } n \to \infty. \tag{4-9}$$

If there exists $z_{n_k} \equiv 0$, that is $v_{n_k} \equiv v$, then $J_\varepsilon^\infty(v) = c \neq 0$ and $v \in \mathcal{N}_\varepsilon^\infty$. Thus $c \geq \vartheta_\varepsilon^\infty$.

If $z_n \neq 0$, for all $n \in \mathbb{N}$, then there exists $t_n > 0$ such that $t_n z_n \in \mathcal{N}_\varepsilon^\infty$. Hence

$$J_\varepsilon^\infty(t_n z_n) \geq \vartheta_\varepsilon^\infty. \tag{4-10}$$

It follows from $\langle (J_\varepsilon^\infty)'(t_n z_n), t_n z_n \rangle = 0$ and $\langle (J_\varepsilon^\infty)'(z_n), z_n \rangle = o(1)$ that

$$(1 - t_n^{2p-2}) \int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |z_n|^p)]|z_n|^p \,\mathrm{d}x = o(1). \tag{4-11}$$

Additionally,

$$\begin{aligned} \|z_n\|_s^2 &\leq C \int_{\mathbb{R}^N} \big(|(-\Delta)^{\frac{s}{2}} z_n|^2 + V_\varepsilon^\infty(x) z_n^2 \big) \,\mathrm{d}x \\ &= C \int_{\mathbb{R}^N} W_\varepsilon^\infty(x)[I_\theta * (W_\varepsilon^\infty |z_n|^p)]|z_n|^p \,\mathrm{d}x + o(1). \end{aligned}$$

If $\int_{\mathbb{R}^N} (I_\theta * |z_n|^p)|z_n|^p \,\mathrm{d}x \to 0$ as $n \to \infty$, then $\|z_n\|_s \to 0$ as $n \to \infty$. Thus $v_n \to v$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$ and $c = J_\varepsilon^\infty(v) \geq \vartheta_\varepsilon^\infty$. If $\int_{\mathbb{R}^N} (I_\theta * |z_n|^p)|z_n|^p \,\mathrm{d}x \geq \delta > 0$, then $t_n \to 1$ as $n \to \infty$ by (4-11). Hence $J_\varepsilon^\infty(t_n z_n) \to c - J_\varepsilon^\infty(v)$ as $n \to \infty$ by (4-8), which implies $c \geq J_\varepsilon^\infty(v) + \vartheta_\varepsilon^\infty \geq \vartheta_\varepsilon^\infty$ by (4-10).

Finally, it follows from $V_\varepsilon^\infty(x) \geq \tau_\infty$ and $W_\varepsilon^\infty(x) \leq k_\infty$ for any $x \in \mathbb{R}^N$ that

$$J_\varepsilon^\infty(u) \geq J^\infty(u), \quad \text{for all } u \in H^s(\mathbb{R}^N).$$

In virtue of Lemmas 4.4 and 4.12, $\vartheta_\varepsilon^\infty$ and ϑ^∞ are mountain pass levels of J_ε^∞ and J^∞, respectively. Therefore, $\vartheta_\varepsilon^\infty \geq \vartheta^\infty$. □

Remark 4.14. Similarly, if J_ε^{ab} has a $(\mathrm{PS})_c$ sequence, then either $c = 0$ or $c \geq \vartheta_\varepsilon^{ab}$.

Lemma 4.15. *J_ε^{ab} satisfies the $(\mathrm{PS})_c$ condition for all $c < \vartheta_\varepsilon^\infty$.*

Proof. Let $\{v_n\} \subset H^s(\mathbb{R}^N)$ and $J_\varepsilon^{ab}(v_n) \to c$, $(J_\varepsilon^{ab})'(v_n) \to 0$ in $H^{-s}(\mathbb{R}^N)$ as $n \to \infty$.

Since $\{v_n\}$ is bounded in $H^s(\mathbb{R}^N)$, we may assume $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$ along a subsequence. Then $(J_\varepsilon^{ab})'(v) = 0$ by Lemma 3.6.

Set $z_n := v_n - v$. Then $z_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$ and

$$z_n \to 0 \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \quad \text{as } n \to \infty, \quad \text{for } q \in [2, 2_s^*). \tag{4-12}$$

Due to the classical Brézis–Lieb lemma and Lemma 3.5, we have

$$J_\varepsilon^{ab}(z_n) \to c - J_\varepsilon^{ab}(v), \quad (J_\varepsilon^{ab})'(z_n) \to 0 \quad \text{in } H^{-s}(\mathbb{R}^N) \quad \text{as } n \to \infty. \tag{4-13}$$

Now we prove

$$J_\varepsilon^{\infty}(z_n) \to c - J_\varepsilon^{ab}(v), \quad (J_\varepsilon^{\infty})'(z_n) \to 0 \quad \text{in } H^{-s}(\mathbb{R}^N) \quad \text{as } n \to \infty.$$

By definition, we can get that

$$\lim_{|x|\to\infty} \big(V_\varepsilon^{\infty}(x) - V_\varepsilon^{a}(x)\big) = 0, \qquad \lim_{|x|\to\infty} \big(W_\varepsilon^{\infty}(x) - W_\varepsilon^{b}(x)\big) = 0,$$

which imply that for all $\delta > 0$, there exists $R > 0$ such that

$$\big|V_\varepsilon^{\infty}(x) - V_\varepsilon^{a}(x)\big| \le \delta, \quad \big|W_\varepsilon^{\infty}(x) - W_\varepsilon^{b}(x)\big| \le \delta, \quad \text{for all } |x| > R.$$

Therefore, by Lemma 3.2 and the Hölder inequality, we get that

$$\big|J_\varepsilon^{\infty}(z_n) - J_\varepsilon^{ab}(z_n)\big| \le \Big(\frac{1}{2}|z_n|_2^2 + \frac{k}{p}|z_n|^{2p}_{2Np/(N+\theta)}\Big)\delta + C\big(|z_n|^2_{L^2(B_R)} + |z_n|^p_{L^{2Np/(N+\theta)}(B_R)}\big),$$

which jointly with (4-12) and (4-13), implies that

$$J_\varepsilon^{\infty}(z_n) \to c - J_\varepsilon^{ab}(v), \quad \text{as } n \to \infty. \tag{4-14}$$

For any $\varphi \in H^s(\mathbb{R}^N)$, by the Hölder inequality and Lemma 3.1, we have

$$\begin{aligned}
&\big|\big\langle (J_\varepsilon^{\infty})'(z_n) - (J_\varepsilon^{ab})'(z_n), \varphi\big\rangle\big| \\
&\le \int_{\mathbb{R}^N} \big|V_\varepsilon^{\infty}(x) - V_\varepsilon^{a}(x)\big| |z_n||\varphi| \,\mathrm{d}x \\
&\qquad + k\int_{\mathbb{R}^N} \big|W_\varepsilon^{\infty}(x) - W_\varepsilon^{b}(x)\big| (I_\theta * |z_n|^p)|z_n|^{p-1}|\varphi| \,\mathrm{d}x \\
&\qquad\qquad + k\int_{\mathbb{R}^N} \big|W_\varepsilon^{\infty}(x) - W_\varepsilon^{b}(x)\big| [I_\theta * (|z_n|^{p-1}|\varphi|)]|z_n|^p \,\mathrm{d}x \\
&\le C_1\delta\big(|z_n|_2 + |z_n|^{2p-1}_{2Np/(N+\theta)}\big)\|\varphi\|_s + C_2\big(|z_n|_{L^2(B_R)} + |z_n|^{p-1}_{L^{2Np/(N+\theta)}(B_R)}\big)\|\varphi\|_s,
\end{aligned}$$

which together with (4-12) and (4-13), implies that

$$(J_\varepsilon^{\infty})'(z_n) \to 0 \quad \text{in } H^{-s}(\mathbb{R}^N), \quad \text{as } n \to \infty. \tag{4-15}$$

It follows from (4-14) and (4-15) that $\{z_n\}$ is a $(\mathrm{PS})_{c-J_\varepsilon^{ab}(v)}$ sequence of J_ε^∞. By Lemma 4.13, either $c = J_\varepsilon^{ab}(v)$ or $c \geq J_\varepsilon^{ab}(v) + \vartheta_\varepsilon^\infty$. But the latter contradicts the assumption $c < \vartheta_\varepsilon^\infty$ in this lemma. Thus $c = J_\varepsilon^{ab}(v)$ and

$$J_\varepsilon^{ab}(v_n) \to J_\varepsilon^{ab}(v), \quad \text{as } n \to \infty. \tag{4-16}$$

Next, since $\langle (J_\varepsilon^{ab})'(v_n), v_n\rangle = o(1)$ and $\langle (J_\varepsilon^{ab})'(v), v\rangle = 0$, we have

$$J_\varepsilon^{ab}(v_n) = \frac{p-1}{2p}\int_{\mathbb{R}^N}\big(|(-\Delta)^{s/2}v_n|^2 + V_\varepsilon^a(x)v_n^2\big)\,\mathrm{d}x + o(1),$$
$$J_\varepsilon^{ab}(v) = \frac{p-1}{2p}\int_{\mathbb{R}^N}\big(|(-\Delta)^{s/2}v|^2 + V_\varepsilon^a(x)v^2\big)\,\mathrm{d}x,$$

which together with (4-16) imply that $\|v_n\|_s \to \|v\|_s$ as $n \to \infty$. Therefore,

$$v_n \to v \quad \text{in } H^s(\mathbb{R}^N), \quad \text{as } n \to \infty. \qquad \square$$

Lemma 4.16. $\displaystyle\limsup_{\varepsilon\to 0} \vartheta_\varepsilon^{ab} \leq \vartheta^{\alpha\beta}$, *where* $\alpha = V^a(0)$, $\beta = W^b(0)$.

Proof. Let $\overline{V}_\varepsilon(x) := V_\varepsilon^a(x) - \alpha$ and $\overline{W}_\varepsilon(x) := \beta - W_\varepsilon^b(x)$. Then

$$\overline{V}_\varepsilon(x) \to 0, \quad \overline{W}_\varepsilon(x) \to 0, \quad \text{a.e. on } \mathbb{R}^N, \quad \text{as } \varepsilon \to 0. \tag{4-17}$$

Meanwhile,

$$J_\varepsilon^{ab}(v) = J^{\alpha\beta}(v) + \frac{1}{2}\int_{\mathbb{R}^N}\overline{V}_\varepsilon(x)v^2\,\mathrm{d}x + \frac{\beta}{p}\int_{\mathbb{R}^N}\overline{W}_\varepsilon(x)(I_\theta * |v|^p)|v|^p\,\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N}\overline{W}_\varepsilon(x)[I_\theta * (\overline{W}_\varepsilon|v|^p)]|v|^p\,\mathrm{d}x. \tag{4-18}$$

By Lemma 4.6, there exists $e \in \mathscr{R}^{\alpha\beta}$. Let $t_\varepsilon > 0$ satisfy $t_\varepsilon e \in \mathscr{N}_\varepsilon^{ab}$, then

$$\max_{t\geq 0} J_\varepsilon^{ab}(te) = J_\varepsilon^{ab}(t_\varepsilon e) \geq \vartheta_\varepsilon^{ab}. \tag{4-19}$$

Since $J_\varepsilon^{ab}(te) \to -\infty$ as $t \to +\infty$, there exists $T_0 > 0$ such that

$$J_\varepsilon^{ab}(te) < 0, \quad \text{for all } t > T_0. \tag{4-20}$$

By (4-19) and (4-20), we get $t_\varepsilon \leq T_0$. Without loss of generality, we can assume $t_\varepsilon \to t_0$ as $\varepsilon \to 0$. Noting (4-17), (4-18) and (4-19), it follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}\vartheta_\varepsilon^{ab} \leq J_\varepsilon^{ab}(t_\varepsilon e) = J^{\alpha\beta}(t_\varepsilon e) &+ \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}\overline{V}_\varepsilon(x)e^2\,\mathrm{d}x + \frac{\beta t_\varepsilon^{2p}}{p}\int_{\mathbb{R}^N}\overline{W}_\varepsilon(x)(I_\theta * |e|^p)|e|^p\,\mathrm{d}x \\ &- \frac{t_\varepsilon^{2p}}{2p}\int_{\mathbb{R}^N}\overline{W}_\varepsilon(x)[I_\theta * (\overline{W}_\varepsilon|e|^p)]|e|^p\,\mathrm{d}x \\ &\to J^{\alpha\beta}(t_0 e) \leq J^{\alpha\beta}(e) = \vartheta^{\alpha\beta} \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Thus $\limsup_{\varepsilon\to 0}\vartheta_\varepsilon^{ab}\le\vartheta^{\alpha\beta}$. □

Lemma 4.17. *Let $\tau\le a\le\tau_\infty$, $k_\infty\le b\le k$. If*

$$\left(\frac{V^a(0)}{\tau_\infty}\right)^{\frac{\theta+2s}{4s}-\frac{(N-2s)(p-1)}{4s}}<\frac{W^b(0)}{k_\infty}, \tag{4-21}$$

then there exists $\varepsilon^{ab}>0$ such that $\vartheta_\varepsilon^{ab}$ is attained at $v_\varepsilon^{ab}>0$ for all $\varepsilon\le\varepsilon^{ab}$.

Proof. Noting Lemma 4.9 and (4-21), we have $\vartheta^{\alpha\beta}<\vartheta^\infty$, where $\alpha=V^a(0)$ and $\beta=W^b(0)$. By Lemmas 4.16 and 4.13, there exists $\varepsilon^{ab}>0$ such that

$$\vartheta_\varepsilon^{ab}<\vartheta^\infty\le\vartheta_\varepsilon^\infty,\quad \text{for all }\ \varepsilon\le\varepsilon^{ab}.$$

By Lemma 4.15, J_ε^{ab} satisfies the $(\mathrm{PS})_{\vartheta_\varepsilon^{ab}}$ condition for all $\varepsilon\le\varepsilon^{ab}$, which together with Lemmas 4.11 and 4.12 implies that $\vartheta_\varepsilon^{ab}$ is attained at $v_\varepsilon^{ab}\in H^s(\mathbb{R}^N)$.

Next we shall prove $v_\varepsilon^{ab}>0$. Let $\tilde v_\varepsilon^{ab}\in E$ be the s-harmonic extension of v_ε^{ab}, then $\tilde v_\varepsilon^{ab}$ is a groundstate of problem (4-4). Since $\tilde J_\varepsilon^{ab}(\tilde v_\varepsilon^{ab})=\tilde J_\varepsilon^{ab}(|\tilde v_\varepsilon^{ab}|)$, we may assume $\tilde v_\varepsilon^{ab}\ge 0$. Then $v_\varepsilon^{ab}(x)=\tilde v_\varepsilon^{ab}(x,0)\ge 0$. As in Theorem 3.2 in [d'Avenia et al. 2015], we have $v_\varepsilon^{ab}\in L^1(\mathbb{R}^N)\cap C^{0,\mu}(\mathbb{R}^N)$ with $\mu\in(0,1)$, which implies $|v_\varepsilon^{ab}(x)|\to 0$ as $|x|\to\infty$. Hence $v_\varepsilon^{ab}\in L^\infty(\mathbb{R}^N)$. Then $I_\theta*|v_\varepsilon^{ab}|^p\in C^{0,\mu+\theta}(\mathbb{R}^N)$ for $\mu+\theta\in(0,1)$ or $I_\theta*|v_\varepsilon^{ab}|^p\in C^{0,1}(\mathbb{R}^N)$ for $\mu+\theta\in[1,N+1)$. Noting that $V,W\in C^{0,\lambda}(\mathbb{R}^N)$, we can use a similar proof as Theorem 1.4 in [Felmer et al. 2012] to obtain that $v_\varepsilon^{ab}\in C^{0,2s+\nu}(\mathbb{R}^N)$ for $2s+\nu\le 1$ and $v_\varepsilon^{ab}\in C^{1,2s+\nu-1}(\mathbb{R}^N)$ for $2s+\nu>1$, where $\nu=\min\{\lambda,\mu\}$. This regularity makes sure (2-2) holds. If $v_\varepsilon^{ab}(x_0)=0$ for some $x_0\in\mathbb{R}^N$, then (4-2) implies that $((-\Delta)^s v_\varepsilon^{ab})(x_0)=0$. Due to (2-2), we get $v_\varepsilon^{ab}\equiv 0$, which is impossible. Hence $v_\varepsilon^{ab}>0$. □

In order to prove the existence of sign-changing solutions for Equation (1-1), we turn to problem (4-4) and define the Nehari nodal set

$$\tilde{\mathscr{M}}_\varepsilon^{ab}:=\left\{\tilde v\in E:\tilde v^\pm\ne 0,\ \langle(\tilde J_\varepsilon^{ab})'(\tilde v),\tilde v^\pm\rangle=0\right\}$$

and the least energy nodal value

$$\tilde\zeta_\varepsilon^{ab}:=\inf_{\tilde{\mathscr{M}}_\varepsilon^{ab}}\tilde J_\varepsilon^{ab}.$$

Indeed, $\tilde{\mathscr{N}}_\varepsilon^{ab}\supset\tilde{\mathscr{M}}_\varepsilon^{ab}\ne\varnothing$ and $\tilde\zeta_\varepsilon^{ab}\ge\tilde\vartheta_\varepsilon^{ab}>0$.

Lemma 4.18. *For any $\tilde v\in E$ with $\tilde v^\pm\ne 0$, there exists a unique pair of $(l_{\tilde v},t_{\tilde v})\in\mathbb{R}_+\times\mathbb{R}_+$ such that*

$$l_{\tilde v}\tilde v^+ + t_{\tilde v}\tilde v^-\in\tilde{\mathscr{M}}_\varepsilon^{ab},\quad \tilde J_\varepsilon^{ab}(l_{\tilde v}\tilde v^+ + t_{\tilde v}\tilde v^-)=\max_{l,t\ge 0}\tilde J_\varepsilon^{ab}(l\tilde v^+ + t\tilde v^-).$$

Proof. For any $\tilde v\in E$ with $\tilde v^\pm\ne 0$, consider the mapping

$$F(l,t):=\left(\left\langle(\tilde J_\varepsilon^{ab})'(l\tilde v^+ + t\tilde v^-),l\tilde v^+\right\rangle,\left\langle(\tilde J_\varepsilon^{ab})'(l\tilde v^+ + t\tilde v^-),t\tilde v^-\right\rangle\right)$$

for $(l, t) \in \mathbb{R}_+ \times \mathbb{R}_+$. Since

$$\begin{aligned}\big\langle (\tilde{J}_\varepsilon^{ab})'(l\tilde{v}^+ + t\tilde{v}^-), l\tilde{v}^+\big\rangle = l^2\Big(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla \tilde{v}^+|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon^a(x)|v^+|^2\,\mathrm{d}x\Big)\\ - l^p t^p \int_{\mathbb{R}^N} W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b|v^-|^p)]|v^+|^p\,\mathrm{d}x\\ - l^{2p}\int_{\mathbb{R}^N} W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b|v^+|^p)]|v^+|^p\,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}\big\langle (\tilde{J}_\varepsilon^{ab})'(l\tilde{v}^+ + t\tilde{v}^-), t\tilde{v}^-\big\rangle = t^2\Big(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla \tilde{v}^-|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon^a(x)|v^-|^2\,\mathrm{d}x\Big)\\ - t^p l^p \int_{\mathbb{R}^N} W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b|v^+|^p)]|v^-|^p\,\mathrm{d}x\\ - t^{2p}\int_{\mathbb{R}^N} W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b|v^-|^p)]|v^-|^p\,\mathrm{d}x,\end{aligned}$$

where $v^\pm = \mathrm{tr}_{\mathbb{R}^N}\,\tilde{v}^\pm = \tilde{v}^\pm(x,0)$, there exist $R > r > 0$ such that for any $l, t \in [r, R]$,

$$\begin{aligned}&\big\langle (\tilde{J}_\varepsilon^{ab})'(r\tilde{v}^+ + t\tilde{v}^-), r\tilde{v}^+\big\rangle > 0, \quad \big\langle (\tilde{J}_\varepsilon^{ab})'(R\tilde{v}^+ + t\tilde{v}^-), R\tilde{v}^+\big\rangle < 0,\\ &\big\langle (\tilde{J}_\varepsilon^{ab})'(l\tilde{v}^+ + r\tilde{v}^-), r\tilde{v}^-\big\rangle > 0, \quad \big\langle (\tilde{J}_\varepsilon^{ab})'(l\tilde{v}^+ + R\tilde{v}^-), R\tilde{v}^-\big\rangle < 0.\end{aligned}$$

By Miranda's theorem [1940], there exists $(l_{\tilde{v}}, t_{\tilde{v}}) \in (r, R) \times (r, R)$ such that

$$\big\langle (\tilde{J}_\varepsilon^{ab})'(l_{\tilde{v}}\tilde{v}^+ + t_{\tilde{v}}\tilde{v}^-), l_{\tilde{v}}\tilde{v}^+\big\rangle = 0 = \big\langle (\tilde{J}_\varepsilon^{ab})'(l_{\tilde{v}}\tilde{v}^+ + t_{\tilde{v}}\tilde{v}^-), t_{\tilde{v}}\tilde{v}^-\big\rangle.$$

That is $l_{\tilde{v}}\tilde{v}^+ + t_{\tilde{v}}\tilde{v}^- \in \tilde{\mathscr{M}}_\varepsilon^{ab}$.

Noting $p \geq 2$ and $\tilde{J}_\varepsilon^{ab}(l^{1/p}\tilde{v}^+ + t^{1/p}\tilde{v}^-)$ is strictly concave for $(l, t) \in \mathbb{R}_+ \times \mathbb{R}_+$, it follows from

$$\begin{aligned}\frac{\partial \tilde{J}_\varepsilon^{ab}(l^{1/p}\tilde{v}^+ + t^{1/p}\tilde{v}^-)}{\partial l}\Big|_{(l_{\tilde{v}}^p, t_{\tilde{v}}^p)} &= \frac{1}{p l_{\tilde{v}}^p}\big\langle (\tilde{J}_\varepsilon^{ab})'(l_{\tilde{v}}\tilde{v}^+ + t_{\tilde{v}}\tilde{v}^-), l_{\tilde{v}}\tilde{v}^+\big\rangle = 0,\\ \frac{\partial \tilde{J}_\varepsilon^{ab}(l^{1/p}\tilde{v}^+ + t^{1/p}\tilde{v}^-)}{\partial t}\Big|_{(l_{\tilde{v}}^p, t_{\tilde{v}}^p)} &= \frac{1}{p t_{\tilde{v}}^p}\big\langle (\tilde{J}_\varepsilon^{ab})'(l_{\tilde{v}}\tilde{v}^+ + t_{\tilde{v}}\tilde{v}^-), t_{\tilde{v}}\tilde{v}^-\big\rangle = 0\end{aligned}$$

that $(l_{\tilde{v}}^p, t_{\tilde{v}}^p)$ is the unique maximum point of $\tilde{J}_\varepsilon^{ab}(l^{1/p}\tilde{v}^+ + t^{1/p}\tilde{v}^-)$ for $(l, t) \in \mathbb{R}_+ \times \mathbb{R}_+$. Hence

$$\tilde{J}_\varepsilon^{ab}(l_{\tilde{v}}\tilde{v}^+ + t_{\tilde{v}}\tilde{v}^-) = \max_{l,t\geq 0} \tilde{J}_\varepsilon^{ab}(l^{1/p}\tilde{v}^+ + t^{1/p}\tilde{v}^-) = \max_{l,t\geq 0} \tilde{J}_\varepsilon^{ab}(l\tilde{v}^+ + t\tilde{v}^-). \qquad \square$$

Now we define for any $\tilde{u}, \tilde{v} \in E$ and $u = \mathrm{tr}_{\mathbb{R}^N}\,\tilde{u},\ v = \mathrm{tr}_{\mathbb{R}^N}\,\tilde{v}$,

$$f_\varepsilon^{ab}(\tilde{u}, \tilde{v}) := \begin{cases} \dfrac{\int_{\mathbb{R}^N} W_\varepsilon^b(x)[I_\theta * (W_\varepsilon^b(|u|^p + |v|^p))]|u|^p\,\mathrm{d}x}{\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla \tilde{u}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon^a(x)u^2\,\mathrm{d}x} & \text{if } \tilde{u} \neq 0,\\ 0 & \text{if } \tilde{u} = 0.\end{cases}$$

Obviously, $\tilde{v} \in \tilde{\mathscr{M}}_\varepsilon^{ab}$ if and only if $f_\varepsilon^{ab}(\tilde{v}^+, \tilde{v}^-) = f_\varepsilon^{ab}(\tilde{v}^-, \tilde{v}^+) = 1$.

Set

$$\tilde{U}_\varepsilon^{ab} := \left\{\tilde{v} \in E : |f_\varepsilon^{ab}(\tilde{v}^+, \tilde{v}^-) - 1| < \tfrac{1}{2},\ |f_\varepsilon^{ab}(\tilde{v}^-, \tilde{v}^+) - 1| < \tfrac{1}{2}\right\},$$

then $\tilde{\mathscr{M}}_\varepsilon^{ab} \subset \tilde{U}_\varepsilon^{ab} \neq \varnothing$. Moreover, one can verify that for $p \in \left(2, \frac{N+\theta}{N-2s}\right)$,

$$\tilde{v}_n \in \tilde{U}_\varepsilon^{ab} \text{ and } \|\tilde{v}_n\|_E \le C \Longrightarrow \|\tilde{v}_n^\pm\|_E \ge \bar{C} > 0. \tag{4-22}$$

We continue to denote $P := \{\tilde{v} \in E : \tilde{v} \ge 0\}$ and let Q be the set of mappings q such that

(i) $\quad q \in C(D, E), \quad$ where $D := [0, 1] \times [0, 1]$,

and for all $t \in [0, 1]$,

(ii) $\quad q(t, 0) = 0,$

(iii) $\quad q(0, t) \in P,$

(iv) $\quad q(1, t) \in -P,$

(v) $\quad (\tilde{J}_\varepsilon^{ab} \circ q)(t, 1) \le 0, \quad f_\varepsilon^{ab}(q(t, 1), 0) \ge 2.$

Then $Q \neq \varnothing$. By a standard proof (see Lemma 3.2 in [Cerami et al. 1986]), we have the following result.

Lemma 4.19. $$\inf_{q\in Q} \sup_{\tilde{v}\in q(D)} \tilde{J}_\varepsilon^{ab}(\tilde{v}) = \inf_{\tilde{v}\in \tilde{\mathscr{M}}_\varepsilon^{ab}} \tilde{J}_\varepsilon^{ab}(\tilde{v}).$$

5. Proofs of multiplicity of semiclassical solutions

Setting $v(x) := w(\varepsilon x)$, the Equation (1-1) is equivalent to

$$(-\Delta)^s v + V(\varepsilon x)v = W(\varepsilon x)[I_\theta * (W(\varepsilon y)|v(y)|^p)]|v|^{p-2}v, \qquad v \in H^s(\mathbb{R}^N). \tag{5-1}$$

If $v_\varepsilon(x)$ is a solution of Equation (5-1), then $w_\varepsilon(x) = v_\varepsilon(x/\varepsilon)$ is a solution of Equation (1-1).

By extension, we can reformulate Equation (5-1) as

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla\tilde{v}) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \tilde{v}(x, 0) = v(x) & \text{on } \mathbb{R}^N, \\ \lim_{y\to 0^+} y^{1-2s}\partial_y\tilde{v} = V(\varepsilon x)v - W(\varepsilon x)[I_\theta * (W(\varepsilon y)|v(y)|^p)]|v|^{p-2}v & \text{in } \mathbb{R}_+^{N+1}. \end{cases} \tag{5-2}$$

Noting $V(\varepsilon x) = V_\varepsilon^\tau(x)$, $W(\varepsilon x) = W_\varepsilon^k(x)$, we find that Equation (5-1) and problem (5-2) are particular forms of Equation (4-2) and problem (4-4), respectively. For simplicity, we denote

$$\begin{aligned} &J_\varepsilon := J_\varepsilon^{\tau k}, \quad \mathscr{N}_\varepsilon := \mathscr{N}_\varepsilon^{\tau k}, \quad \vartheta_\varepsilon := \vartheta_\varepsilon^{\tau k}, \quad \mathscr{R}_\varepsilon := \mathscr{R}_\varepsilon^{\tau k}, \\ &\tilde{J}_\varepsilon := \tilde{J}_\varepsilon^{\tau k}, \quad \tilde{\mathscr{N}}_\varepsilon := \tilde{\mathscr{N}}_\varepsilon^{\tau k}, \quad \tilde{\vartheta}_\varepsilon := \tilde{\vartheta}_\varepsilon^{\tau k}, \quad \tilde{\mathscr{R}}_\varepsilon := \tilde{\mathscr{R}}_\varepsilon^{\tau k}, \quad V_\varepsilon := V_\varepsilon^\tau, \\ &f_\varepsilon := f_\varepsilon^{\tau k}, \quad \tilde{\mathscr{M}}_\varepsilon := \tilde{\mathscr{M}}_\varepsilon^{\tau k}, \quad \tilde{\zeta}_\varepsilon := \tilde{\zeta}_\varepsilon^{\tau k}, \quad \tilde{U}_\varepsilon := \tilde{U}_\varepsilon^{\tau k}, \quad W_\varepsilon := W_\varepsilon^k. \end{aligned}$$

Proof of Theorem 1.1. Without loss of generality, we may assume $x_v = 0$. Then $V(0) = \tau,\ W(0) = k_v$. By (1-4) and (1-5), we get $m \geq 1$.

Step 1. We shall construct an m-dimensional subspace E_{rm} of E such that

$$\sup_{\tilde{v}\in E_{rm}} \tilde{J}_\varepsilon(\tilde{v}) < \vartheta^\infty, \quad \text{for all } r \geq r_m,\ \varepsilon \leq \varepsilon_m,$$

where r_m and ε_m are existing constants depending on m.

Choose $a = \tau,\ b = k_v$ in Equation (4-1), by Lemmas 4.6 and 4.7, there exist $v \in \mathscr{R}^{\tau k_v}$ and $v(x) = v(|x|) > 0$. Let $\tilde{v} \in E$ be the s-harmonic extension of v, then $\tilde{v} \in \widetilde{\mathscr{R}}^{\tau k_v}$ and $\tilde{v}(x, 0) = v(x)$. Let $r > 0,\ \chi_r \in C_0^\infty(\mathbb{R}_+)$ satisfy $\chi_r(t) = 1$ for $t \leq r$ and $\chi_r(t) = 0$ for $t \geq r + 1$ with $|\chi_r'(t)| \leq 2$. Set

$$\tilde{v}_r(x, y) := \chi_r(|(x, y)|)\tilde{v}(x, y), \quad (x, y) \in \mathbb{R}_+^{N+1}.$$

Taking into account Lemmas 2.5, 2.2, 3.1 and 3.3, it follows from

$$\begin{aligned}
\|\tilde{v}_r - \tilde{v}\|_E^2 &= \int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla(\tilde{v}_r - \tilde{v})|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} |v_r - v|^2\,\mathrm{d}x \\
&\leq C\biggl(\int_{\mathbb{R}_+^{N+1}\setminus B_r^+(0)} y^{1-2s}|\nabla\tilde{v}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{B_{r+1}^+(0)\setminus B_r^+(0)} y^{1-2s}\tilde{v}^2\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\qquad + \int_{|x|>r} v^2\,\mathrm{d}x\biggr) \\
&\leq \overline{C}\biggl(\int_{\mathbb{R}_+^{N+1}\setminus B_r^+(0)} y^{1-2s}|\nabla\tilde{v}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{|x|>r} v^2\,\mathrm{d}x\biggr) \to 0, \quad \text{as } r \to \infty
\end{aligned}$$

that

$$\begin{aligned}
\tilde{v}_r &\to \tilde{v} \quad \text{in } E, && \text{as } r \to \infty, \\
v_r &\to v \quad \text{in } H^s(\mathbb{R}^N), && \text{as } r \to \infty, \\
v_r &\to v \quad \text{in } L^{2Np/(N+\theta)}(\mathbb{R}^N), && \text{as } r \to \infty, \\
\int_{\mathbb{R}^N} (I_\theta * v_r^p)v_r^p\,\mathrm{d}x &\to \int_{\mathbb{R}^N} (I_\theta * v^p)v^p\,\mathrm{d}x, && \text{as } r \to \infty,
\end{aligned}$$

where $v_r := \mathrm{tr}_{\mathbb{R}^N}\,\tilde{v}_r$. Therefore, we obtain that as $r \to \infty$,

$$\begin{aligned}
\max_{t\geq 0} \tilde{J}^{\tau k_v}(t\tilde{v}_r) &= \frac{p-1}{2p}\left(\frac{\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\tilde{v}_r|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \tau v_r^2\,\mathrm{d}x}{\left(\int_{\mathbb{R}^N} k_v^2(I_\theta * v_r^p)v_r^p\,\mathrm{d}x\right)^{1/p}}\right)^{\frac{p}{p-1}} \\
&\to \frac{p-1}{2p}\left(\frac{\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\tilde{v}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \tau v^2\,\mathrm{d}x}{\left(\int_{\mathbb{R}^N} k_v^2(I_\theta * v^p)v^p\,\mathrm{d}x\right)^{1/p}}\right)^{\frac{p}{p-1}} \\
&= \max_{t\geq 0} \tilde{J}^{\tau k_v}(t\tilde{v}) = \tilde{J}^{\tau k_v}(\tilde{v}) = \tilde{\vartheta}^{\tau k_v} = \vartheta^{\tau k_v}.
\end{aligned} \tag{5-3}$$

Additionally,

$$V_\varepsilon(x) \to V(0) = \tau, \quad W_\varepsilon(x) \to W(0) = k_v \tag{5-4}$$

as $\varepsilon \to 0$ uniformly on any bounded set of x. Thus (5-3) and (5-4) imply that

$$\begin{aligned} \max_{t\ge 0} \tilde{J}_\varepsilon(t\tilde{v}_r) &= \frac{p-1}{2p}\left(\frac{\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{v}_r|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon(x) v_r^2 \,\mathrm{d}x}{\left(\int_{\mathbb{R}^N} W_\varepsilon(x)[I_\theta * (W_\varepsilon v_r^p)] v_r^p \,\mathrm{d}x\right)^{1/p}}\right)^{\frac{p}{p-1}} \\ &\to \frac{p-1}{2p}\left(\frac{\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{v}_r|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \tau v_r^2 \,\mathrm{d}x}{\left(\int_{\mathbb{R}^N} k_v^2 (I_\theta * v_r^p) v_r^p \,\mathrm{d}x\right)^{1/p}}\right)^{\frac{p}{p-1}} \\ &= \max_{t\ge 0} \tilde{J}^{\tau k_v}(t\tilde{v}_r) \to \vartheta^{\tau k_v} \quad \text{as } \varepsilon \to 0,\ r \to \infty,\ \text{respectively}. \end{aligned} \tag{5-5}$$

Define

$$\tilde{\phi}_{rj}(x, y) := \tilde{v}_r\big(x_1 - 2j(r+1), x_2, \dots, x_N, y\big), \quad \text{for } j = 0, 1, \dots, m-1$$

and set

$$E_{rm} := \operatorname{span}\big\{\tilde{\phi}_{rj}(x, y) : j = 0, 1, \dots, m-1\big\}.$$

One can check that $(\tilde{\phi}_{ri}, \tilde{\phi}_{rj})_E = 0$ if $i \ne j$. Thus $\dim E_{rm} = m$. Similarly as (5-5), for all $j = 1, 2, \dots, m-1$ and $\phi_{rj} = \mathrm{tr}_{\mathbb{R}^N} \tilde{\phi}_{rj}$,

$$\begin{aligned} \max_{t\ge 0} \tilde{J}_\varepsilon(t\tilde{\phi}_{rj}) &= \frac{p-1}{2p}\left(\frac{\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{\phi}_{rj}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon(x) \phi_{rj}^2 \,\mathrm{d}x}{\left(\int_{\mathbb{R}^N} W_\varepsilon(x)[I_\theta * (W_\varepsilon \phi_{rj}^p)] \phi_{rj}^p \,\mathrm{d}x\right)^{1/p}}\right)^{\frac{p}{p-1}} \\ &\to \frac{p-1}{2p}\left(\frac{\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{\phi}_{rj}|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \tau \phi_{rj}^2 \,\mathrm{d}x}{\left(\int_{\mathbb{R}^N} k_v^2 (I_\theta * \phi_{rj}^p) \phi_{rj}^p \,\mathrm{d}x\right)^{1/p}}\right)^{\frac{p}{p-1}} \\ &= \max_{t\ge 0} \tilde{J}^{\tau k_v}(t\tilde{\phi}_{rj}) = \max_{t\ge 0} \tilde{J}^{\tau k_v}(t\tilde{v}_r) \\ &\to \vartheta^{\tau k_v} \quad \text{as } \varepsilon \to 0,\ r \to \infty,\ \text{respectively}. \end{aligned}$$

Consequently, for all $\delta > 0$, there exist $r_\delta > 0$, $\varepsilon_\delta > 0$ such that

$$\max_{t\ge 0} \tilde{J}_\varepsilon(t\tilde{\phi}_{rj}) \le \vartheta^{\tau k_v} + \delta, \quad \text{for all } r \ge r_\delta,\ \varepsilon \le \varepsilon_\delta,\ j = 0, 1, \dots, m-1. \tag{5-6}$$

For any $\tilde{v} \in E_{rm}$, we may assume $\tilde{v} = t_0\tilde{\phi}_{r0} + t_1\tilde{\phi}_{r1} + \cdots + t_{m-1}\tilde{\phi}_{r(m-1)}$, where $t_j \in \mathbb{R}$ for $j = 0, 1, \dots, m-1$. Then

$$[I_\theta * (W_\varepsilon |\tilde{v}|^p)]|\tilde{v}|^p \ge \sum_{j=0}^{m-1} [I_\theta * (W_\varepsilon |t_j\tilde{\phi}_{rj}|^p)]|t_j\tilde{\phi}_{rj}|^p.$$

Hence by (5-6),

$$\begin{aligned}\tilde{J}_\varepsilon(\tilde{v}) &\le \tilde{J}_\varepsilon(t_0\tilde{\phi}_{r0}) + \tilde{J}_\varepsilon(t_1\tilde{\phi}_{r1}) + \cdots + \tilde{J}_\varepsilon(t_{m-1}\tilde{\phi}_{r(m-1)}) \\ &\le m(\vartheta^{\tau k_v} + \delta), \quad \text{for all } r \ge r_\delta,\ \varepsilon \le \varepsilon_\delta,\end{aligned}$$

which implies that

$$\sup_{\tilde{v}\in E_{rm}} \tilde{J}_\varepsilon(\tilde{v}) \le m(\vartheta^{\tau k_v} + \delta), \quad \text{for all } r \ge r_\delta,\ \varepsilon \le \varepsilon_\delta.$$

Noting Lemma 4.9, we can choose $0 < \delta < \frac{\vartheta^\infty}{m} - \vartheta^{\tau k_v}$, then there exist $r_m > 0$, $\varepsilon_m > 0$ such that

$$\sup_{\tilde{v}\in E_{rm}} \tilde{J}_\varepsilon(\tilde{v}) < \vartheta^\infty, \quad \text{for all } r \ge r_m,\ \varepsilon \le \varepsilon_m. \tag{5-7}$$

<u>Step 2</u>. We will define constants $c_1, c_2, \ldots, c_m$ and verify that they are critical values of $\tilde{J}_\varepsilon$.

Consider the symmetric group $\mathbb{Z}_2 = \{\text{id}, -\text{id}\}$ and denote

$$\Sigma := \{A \subset E : A \text{ is closed and } A = -A\}.$$

For any $A \in \Sigma$, the Krasnoselskii genus of A is defined by

$$\text{gen}(A) := \inf\big\{n : \text{there exists } g \in C(A, \mathbb{R}^n \setminus \{0\}) \text{ and } g \text{ is odd}\big\}.$$

Set

$$\mathcal{H} := \{h \in C(E, E) : h \text{ is an odd homeomorphism}\}$$

and for any $A \in \Sigma$, define

$$i(A) := \min_{h\in\mathcal{H}} \text{gen}(h(A) \cap \partial B_\rho),$$

where $\rho > 0$ is a constant defined in Lemma 4.11. Thus $i(A)$ is a version of Benci pseudoindex of A. Let

$$c_j := \inf_{i(A)\ge j} \sup_{\tilde{v}\in A} \tilde{J}_\varepsilon(\tilde{v}), \quad j = 1, 2, \ldots, m.$$

Clearly, $c_1 \le c_2 \le \cdots \le c_m$. Next we will show $c_1 \ge \sigma$ and $c_m \le \sup_{\tilde{v}\in E_{rm}} \tilde{J}_\varepsilon(\tilde{v})$, where $\sigma > 0$ was defined in Lemma 4.11.

For any $A \in \Sigma$ and $i(A) \ge 1$, we have $\text{gen}(A \cap \partial B_\rho) \ge 1$, which implies $A \cap \partial B_\rho \ne \varnothing$. It follows from Lemma 4.11(i) that

$$\tilde{J}_\varepsilon(\tilde{v}) = J_\varepsilon(v) > \sigma, \quad \text{for all } \|\tilde{v}\|_E = \|v\|_s = \rho,$$

where $v = \text{tr}_{\mathbb{R}^N} \tilde{v}$. Hence $\sup_{\tilde{v}\in A} \tilde{J}_\varepsilon(\tilde{v}) > \sigma$ and $c_1 \ge \sigma$.

Taking into account that the Krasnoselskii genus satisfies the dimension property (see [Benci 1982]), we get

$$\mathrm{gen}(h(E_{rm}) \cap \partial B_\rho) = \dim E_{rm} = m, \quad \text{for all } h \in \mathcal{H},$$

which implies $i(E_{rm}) = m$. Thus $c_m \le \sup_{\tilde{v} \in E_{rm}} \tilde{J}_\varepsilon(\tilde{v})$.

Noting (5-7) and Lemma 4.13, we get that for any $r \ge r_m$, $\varepsilon \le \varepsilon_m$,

(5-8) $$\sigma \le c_1 \le c_2 \le \cdots \le c_m \le \sup_{\tilde{v} \in E_{rm}} \tilde{J}_\varepsilon(\tilde{v}) < \vartheta^\infty \le \vartheta_\varepsilon^\infty.$$

Now we prove $c_1, c_2, \dots, c_m$ are critical values of $\tilde{J}_\varepsilon$ by applying the Theorem 1.4 in [Benci 1982]. Set

$$c_0 := \sigma, \quad c_\infty := \sup_{\tilde{v} \in E_{rm}} \tilde{J}_\varepsilon(\tilde{v}),$$

$$(\tilde{J}_\varepsilon)^c := \{\tilde{v} \in E : \tilde{J}_\varepsilon(\tilde{v}) \le c\}, \quad K_c := \{\tilde{v} \in E : \tilde{J}_\varepsilon(\tilde{v}) = c,\ (\tilde{J}_\varepsilon)'(\tilde{v}) = 0\}.$$

Since $\tilde{J}_\varepsilon$ is an even functional,

(5-9) $$(\tilde{J}_\varepsilon)^c \in \Sigma, \quad K_c \in \Sigma, \quad \text{for all } c \in [c_0, c_\infty].$$

By (5-8) and Lemma 4.15, $\tilde{J}_\varepsilon$ satisfies the $(\mathrm{PS})_c$ condition for any $c \in [c_0, c_\infty]$, which implies that

(5-10) $$K_c \text{ is compact in } E, \quad \text{for all } c \in [c_0, c_\infty].$$

For any $c \in [c_0, c_\infty]$, $d > 0$ and $(K_c)_d := \{\tilde{v} \in E : \mathrm{dist}(\tilde{v}, K_c) < d\}$, choose $\delta = \frac{d}{4}$, then by the contradiction method we can get that there exists $\tilde{\varepsilon} > 0$ such that

(5-11) $$\|(\tilde{J}_\varepsilon)'(\tilde{v})\| \ge \frac{8\tilde{\varepsilon}}{\delta}, \quad \text{for all } \tilde{v} \in \tilde{J}_\varepsilon^{-1}([c - 2\tilde{\varepsilon}, c + 2\tilde{\varepsilon}]) \setminus \overline{(K_c)_{d/2}}.$$

Choose $S := E \setminus (K_c)_d$ and by the deformation lemma (see Lemma 2.3 in [Willem 1996]), there exists $\tilde{\eta} \in C([0, 1] \times E, E)$ such that $\tilde{\eta}(t, \cdot)$ is an odd homeomorphism of E for any $t \in [0, 1]$ and $\tilde{\eta}(1, (\tilde{J}_\varepsilon)^{c+\tilde{\varepsilon}} \cap S) \subset (\tilde{J}_\varepsilon)^{c-\tilde{\varepsilon}}$. Set $\eta(\cdot) := \tilde{\eta}(1, \cdot)$; then η is an odd homeomorphism of E and

(5-12) $$\eta((\tilde{J}_\varepsilon)^{c+\tilde{\varepsilon}} \setminus (K_c)_d) \subset (\tilde{J}_\varepsilon)^{c-\tilde{\varepsilon}}.$$

For any $A \in \Sigma$ and $A \subset (\tilde{J}_\varepsilon)^{c_0}$, it follows from Lemma 4.11(i) that $A \cap \partial B_\rho = \varnothing$. Thus $\mathrm{gen}(A \cap \partial B_\rho) = 0$ and

(5-13) $$i(A) = \min_{h \in \mathcal{H}} \mathrm{gen}(h(A) \cap \partial B_\rho) = 0.$$

Choose $\tilde{A} = E_{rm}$, then

(5-14) $$\tilde{A} \subset (\tilde{J}_\varepsilon)^{c_\infty} \quad \text{and} \quad i(\tilde{A}) = i(E_{rm}) = m \ge 1.$$

Uniting (5-9), (5-10), (5-12), (5-13) and (5-14), we obtain that $c_1, c_2, \dots, c_m$ are critical values of $\tilde{J}_\varepsilon$, and $\mathrm{gen}(K_c) \geq r+1$ if $c := c_k = c_{k+1} = \cdots = c_{k+r}$ with $k \geq 1$ and $k+r \leq m$. Since $\tilde{J}_\varepsilon$ is even, we conclude that $\tilde{J}_\varepsilon$ has at least m pairs of critical points which are also solutions of Equation (5-2). Consequently, J_ε has at least m pairs of critical points being solutions of Equation (5-1).

Step 3. We will prove Equation (5-1) has at least one positive and one negative groundstate.

Noting (1-5) and $m \geq 1$, we get that

$$\left(\frac{\tau}{\tau_\infty}\right)^{\frac{\theta+2s}{4s}-\frac{(N-2s)(p-1)}{4s}} < \frac{k_v}{k_\infty},$$

which together with $V(0) = \tau,\ W(0) = k_v$ implies that

$$\left(\frac{V^\tau(0)}{\tau_\infty}\right)^{\frac{\theta+2s}{4s}-\frac{(N-2s)(p-1)}{4s}} < \frac{W^k(0)}{k_\infty}.$$

By Lemma 4.17, there exists $\varepsilon^{\tau k} > 0$ such that ϑ_ε is attained at $v_\varepsilon > 0$ for all $\varepsilon \leq \varepsilon^{\tau k}$. Therefore, v_ε and $-v_\varepsilon$ are positive and negative groundstates of Equation (5-1).

Step 4. We will prove Equation (5-1) has at least one pair of sign-changing solutions when $m \geq 2$ and $2 < p < \frac{N+\theta}{N-2s}$.

Firstly, let $e \in \mathscr{R}^{\tau k_v}$ with $e > 0$ and $\tilde{e} \in E$ be the s-harmonic extension of e. Then $\tilde{e} \in \tilde{\mathscr{R}}^{\tau k_v}$ with $\tilde{e} > 0$. Let $\chi_r \in C_0^\infty(\mathbb{R}_+)$ satisfy $\chi_r(t) = 1$ for $t \leq r$ and $\chi_r(t) = 0$ for $t \geq r+1$ with $|\chi_r'(t)| \leq 2$. Choose $r > 0$ and $X_r := (x_r, y_r) \in \mathbb{R}_+^{N+1}$ satisfying $|X_r|$ large and

$$\mathrm{dist}\big(B_{r+1}^+(0), B_{r+1}^+(X_r)\big) > 0.$$

Define for $(x, y) \in \mathbb{R}_+^{N+1}$,

$$\begin{aligned} \tilde{e}_r^+(x, y) &:= \chi_r(|(x, y)|)\tilde{e}(x, y) \geq 0, \\ \tilde{e}_r^-(x, y) &:= -\chi_r(|(x - x_r, y - y_r)|)\tilde{e}(x, y) \leq 0. \end{aligned}$$

Then $\mathrm{supp}\, \tilde{e}_r^+ \cap \mathrm{supp}\, \tilde{e}_r^- = \varnothing$ and

$$\tilde{e}_r^+ \to e, \quad \tilde{e}_r^- \to -e \quad \text{in } E, \quad \text{as } r \to \infty. \tag{5-15}$$

By Lemma 4.18, there exists $(l_r, t_r) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$l_r \tilde{e}_r^+ + t_r \tilde{e}_r^- \in \tilde{\mathscr{M}}_\varepsilon. \tag{5-16}$$

Secondly, since $\langle (\tilde{J}_\varepsilon)'(l_r\tilde{e}_r^+ + t_r\tilde{e}_r^-), l_r\tilde{e}_r^+\rangle = \langle (\tilde{J}_\varepsilon)'(l_r\tilde{e}_r^+ + t_r\tilde{e}_r^-), t_r\tilde{e}_r^-\rangle = 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\tilde{e}_r^+|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon(x)|e_r^+|^2\,\mathrm{d}x &= l_r^{p-2}t_r^p\int_{\mathbb{R}^N} W_\varepsilon(x)[I_\theta*(W_\varepsilon|e_r^-|^p)]|e_r^+|^p\,\mathrm{d}x \\ &\quad + l_r^{2p-2}\int_{\mathbb{R}^N} W_\varepsilon(x)[I_\theta*(W_\varepsilon|e_r^+|^p)]|e_r^+|^p\,\mathrm{d}x\end{aligned}\tag{5-17}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\tilde{e}_r^-|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon(x)|e_r^-|^2\,\mathrm{d}x &= t_r^{p-2}l_r^p\int_{\mathbb{R}^N} W_\varepsilon(x)[I_\theta*(W_\varepsilon|e_r^+|^p)]|e_r^-|^p\,\mathrm{d}x \\ &\quad + t_r^{2p-2}\int_{\mathbb{R}^N} W_\varepsilon(x)[I_\theta*(W_\varepsilon|e_r^-|^p)]|e_r^-|^p\,\mathrm{d}x,\end{aligned}\tag{5-18}$$

where $e_r^\pm = \mathrm{tr}_{\mathbb{R}^N}\,\tilde{e}_r^\pm$, which imply that l_r and t_r are bounded for $r \in (0,\infty)$. Without loss of generality, we may assume that

$$l_r \to l_0, \quad t_r \to t_0, \quad \text{as } r \to \infty.$$

Letting $\varepsilon \to 0$ in (5-17) and (5-18), we get that

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\tilde{e}_r^+|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \tau|e_r^+|^2\,\mathrm{d}x \\ = l_r^{p-2}t_r^p\int_{\mathbb{R}^N} k_v^2(I_\theta*|e_r^-|^p)|e_r^+|^p\,\mathrm{d}x + l_r^{2p-2}\int_{\mathbb{R}^N} k_v^2(I_\theta*|e_r^+|^p)|e_r^+|^p\,\mathrm{d}x\end{aligned}\tag{5-19}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\tilde{e}_r^-|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \tau|e_r^-|^2\,\mathrm{d}x \\ = t_r^{p-2}l_r^p\int_{\mathbb{R}^N} k_v^2(I_\theta*|e_r^+|^p)|e_r^-|^p\,\mathrm{d}x + t_r^{2p-2}\int_{\mathbb{R}^N} k_v^2(I_\theta*|e_r^-|^p)|e_r^-|^p\,\mathrm{d}x.\end{aligned}\tag{5-20}$$

Letting $r \to \infty$ in (5-19) and (5-20), by (5-15) we obtain

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\tilde{e}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N}\tau e^2\,\mathrm{d}x = \left(l_0^{p-2}t_0^p + l_0^{2p-2}\right)\int_{\mathbb{R}^N} k_v^2(I_\theta*e^p)e^p\,\mathrm{d}x,$$
$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\tilde{e}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N}\tau e^2\,\mathrm{d}x = \left(t_0^{p-2}l_0^p + t_0^{2p-2}\right)\int_{\mathbb{R}^N} k_v^2(I_\theta*e^p)e^p\,\mathrm{d}x.$$

Since $\tilde{e} \in \tilde{\mathcal{N}}^{\tau k_v}$, we have $l_0 = t_0 = 2^{1/(2-2p)}$.

Thirdly, by (5-16) we have

$$
\begin{aligned}
\tilde{\zeta}_\varepsilon &\le \tilde{J}_\varepsilon(l_r\tilde{e}_r^+ + t_r\tilde{e}_r^-) \\
&= \frac{p-1}{2p}\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla(l_r\tilde{e}_r^+ + t_r\tilde{e}_r^-)|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon(x)|l_r e_r^+ + t_r e_r^-|^2\,\mathrm{d}x\right) \\
&\to \frac{p-1}{2p}\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla(l_r\tilde{e}_r^+ + t_r\tilde{e}_r^-)|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \tau|l_r e_r^+ + t_r e_r^-|^2\,\mathrm{d}x\right) \\
&\to \frac{p-1}{2p}(l_0^2+t_0^2)\left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\tilde{e}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \tau e^2\,\mathrm{d}x\right) \\
&= (l_0^2+t_0^2)\tilde{\vartheta}^{\tau k_v} < 2\vartheta^{\tau k_v}, \quad \text{as } \varepsilon\to 0 \quad \text{and} \quad r\to\infty.
\end{aligned}
\tag{5-21}
$$

Noting $m \ge 2$, due to Lemma 4.9 and Lemma 4.13, we get

$$
2\vartheta^{\tau k_v} < \vartheta^\infty \le \vartheta_\varepsilon^\infty. \tag{5-22}
$$

In view of (5-21) and (5-22), we conclude that $\tilde{\zeta}_\varepsilon < \vartheta_\varepsilon^\infty$ for ε small enough, which implies that $\tilde{J}_\varepsilon$ satisfies the $(\mathrm{PS})_{\tilde{\zeta}_\varepsilon}$ condition for ε small enough.

Fourthly, consider a minimizing sequence $\{\tilde{v}_n\} \subset \tilde{\mathscr{M}}_\varepsilon$ of $\tilde{\zeta}_\varepsilon$ and denote by $\{q_n\} \subset Q$ a mapping sequence satisfying

$$
q_n(D) \subset \{\alpha\tilde{v}_n^+ + \beta\tilde{v}_n^- : \alpha, \beta \in \mathbb{R}_+ \cup \{0\}\}.
$$

Then

$$
\lim_{n\to\infty} \max_{\tilde{v}\in q_n(D)} \tilde{J}_\varepsilon(\tilde{v}) = \lim_{n\to\infty} \tilde{J}_\varepsilon(\tilde{v}_n) = \tilde{\zeta}_\varepsilon. \tag{5-23}
$$

By Lemma 4.19, (5-23) and a similar statement in [Cerami et al. 1986], there exists $\{\tilde{u}_n\} \subset E$ such that

$$
\lim_{n\to\infty} \operatorname{dist}(\tilde{u}_n, q_n(D)) = 0, \qquad \lim_{n\to\infty} \tilde{J}_\varepsilon(\tilde{u}_n) = \tilde{\zeta}_\varepsilon, \qquad \lim_{n\to\infty} (\tilde{J}_\varepsilon)'(\tilde{u}_n) = 0. \tag{5-24}
$$

Next we will show $\tilde{u}_n \in \tilde{U}_\varepsilon$ for n large enough. Indeed we only need to prove $\tilde{u}_n^\pm \ne 0$ for n large enough, which jointly with $\lim_{n\to\infty}\langle(\tilde{J}_\varepsilon)'(\tilde{u}_n), \tilde{u}_n^\pm\rangle = 0$ ensures that

$$
\lim_{n\to\infty} f_\varepsilon(\tilde{u}_n^+, \tilde{u}_n^-) = \lim_{n\to\infty} f_\varepsilon(\tilde{u}_n^-, \tilde{u}_n^+) = 1.
$$

By (5-24), there exists $\tilde{z}_n = \alpha_n\tilde{v}_n^+ + \beta_n\tilde{v}_n^- \in q_n(D)$ such that

$$
\lim_{n\to\infty} \|\tilde{u}_n - \tilde{z}_n\|_E = 0.
$$

Thus it suffices to show $\alpha_n\tilde{v}_n^+ \ne 0$ and $\beta_n\tilde{v}_n^- \ne 0$ for n large enough. It follows from $\tilde{v}_n \in \tilde{\mathscr{M}}_\varepsilon$ and (5-24) that $0 < C_1 \le \|\tilde{v}_n^\pm\|_E \le C_2$. We turn to show $\alpha_n \not\to 0$ and $\beta_n \not\to 0$ as $n\to\infty$. For a contradiction, assume that $\lim_{n\to\infty}\alpha_n = 0$, then

$$
0 < \tilde{\zeta}_\varepsilon = \lim_{n\to\infty} \tilde{J}_\varepsilon(\tilde{v}_n) = \lim_{n\to\infty} \tilde{J}_\varepsilon(\beta_n\tilde{v}_n^-).
$$

Hence $\{\beta_n\}$ is bounded and $\lim_{n\to\infty}\beta_n \neq 0$. Noting $p>2$, we have

$$\begin{aligned}
\tilde{\zeta}_\varepsilon &= \lim_{n\to\infty} \tilde{J}_\varepsilon(\tilde{v}_n) = \lim_{n\to\infty} \max_{\alpha,\beta\geq 0} \tilde{J}_\varepsilon(\alpha\tilde{v}_n^+ + \beta\tilde{v}_n^-) \geq \lim_{n\to\infty} \max_{\alpha\geq 0} \tilde{J}_\varepsilon(\alpha\tilde{v}_n^+ + \beta_n\tilde{v}_n^-) \\
&= \lim_{n\to\infty} \max_{\alpha\geq 0}\biggl[\frac{\alpha^2}{2}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\tilde{v}_n^+|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{\alpha^2}{2}\int_{\mathbb{R}^N} V_\varepsilon(x)|v_n^+|^2\,\mathrm{d}x \\
&\qquad - \frac{\alpha^{2p}}{2p}\int_{\mathbb{R}^N} W_\varepsilon(x)[I_\theta * (W_\varepsilon|v_n^+|^p)]|v_n^+|^p\,\mathrm{d}x \\
&\qquad - \frac{\alpha^p\beta_n^p}{p}\int_{\mathbb{R}^N} W_\varepsilon(x)[I_\theta * (W_\varepsilon|v_n^+|^p)]|v_n^-|^p\,\mathrm{d}x\biggr] + \lim_{n\to\infty}\tilde{J}_\varepsilon(\beta_n\tilde{v}_n^-) \\
&\geq \max_{\alpha\geq 0}(C_3\alpha^2 - C_4\alpha^p - C_5\alpha^{2p}) + \tilde{\zeta}_\varepsilon > \tilde{\zeta}_\varepsilon,
\end{aligned}$$

which is impossible.

Finally, noting that both of the above sequences $\{\tilde{v}_n\}$ and $\{\tilde{u}_n\}$ are related to ε, going if necessary to a subsequence, for ε small enough we can assume

$$\tilde{u}_n \to \tilde{u}_\varepsilon \quad \text{in } E, \quad \text{as } n\to\infty,$$

which together with (4-22) and (5-24) implies that

$$\tilde{u}_\varepsilon \in \tilde{\mathscr{M}}_\varepsilon, \quad \tilde{J}_\varepsilon(\tilde{u}_\varepsilon) = \tilde{\zeta}_\varepsilon, \quad (\tilde{J}_\varepsilon)'(\tilde{u}_\varepsilon) = 0.$$

That is $\pm\tilde{u}_\varepsilon$ are a pair of sign-changing solutions for problem (5-2). Hence we conclude that $\pm u_\varepsilon = \pm\,\mathrm{tr}_{\mathbb{R}^N}\,\tilde{u}_\varepsilon$ are a pair of sign-changing solutions for Equation (5-1).

This completes the proof. □

Proof of Theorem 1.2. We can assume without loss of generality that $x_w = 0$. Then $V(0) = \tau_w$, $W(0) = k$. By (1-6) and (1-7), we get $m \geq 1$. Taking $a = \tau_w$, $b = k$ in Equation (4-1), there exists $v \in \mathscr{R}^{\tau_w k}$. The following arguments are similar to the proof of Theorem 1.1, thus the details are omitted. □

6. Proofs of convergence, concentration and decay estimate of groundstates

In this section, we shall prove the convergence, concentration and decay estimate of the positive groundstates for Equation (1-1). Namely we offer the proof of Theorem 1.3.

Proof of Theorem 1.3. We shall deal with the case (A1)(i), the other case can be handled similarly. Without loss of generality, we assume $x_v = 0$. Then $V(0) = \tau$, $W(0) = k_v$. Obviously, (A1)(i) implies that (1-4) holds. It follows from Theorem 1.1 that Equation (1-1) has a positive groundstate $w_\varepsilon(x)$. Moreover, Equation (5-1) has a positive groundstate $v_\varepsilon(x) = w_\varepsilon(\varepsilon x)$, and problem (5-2) has a positive groundstate $\tilde{v}_\varepsilon(x, y)$ which is also the s-harmonic extension of v_ε.

Step 1. We shall prove the convergence of v_ε as $\varepsilon \to 0$ up to a sequence after translations.

Let $\varepsilon_j \to 0$ (as $j \to \infty$), $v_j := v_{\varepsilon_j} \in \mathscr{R}_{\varepsilon_j}$ with $v_j > 0$. Since

$$\vartheta_{\varepsilon_j} = J_{\varepsilon_j}(v_j) = \frac{p-1}{2p} \int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} v_j|^2 + V_{\varepsilon_j}(x) v_j^2\big)\,\mathrm{d}x \geq C\|v_j\|_s^2,$$

it follows from Lemma 4.16 that $\{v_j\}$ is bounded in $H^s(\mathbb{R}^N)$. By $v_j \in \mathscr{N}_{\varepsilon_j}$ and Lemma 3.2, we have

$$\|v_j\|_s^2 \leq C \int_{\mathbb{R}^N} (I_\theta * v_j^p) v_j^p\,\mathrm{d}x \leq \bar{C}\|v_j\|_s^{2p}. \tag{6-1}$$

Suppose $\lim_{j\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} v_j^2\,\mathrm{d}x = 0$, then by Lemmas 2.3, 3.1 and 3.3, we have

$$v_j \to 0 \quad \text{in } L^{2Np/(N+\theta)}(\mathbb{R}^N), \quad \text{as } j \to \infty,$$
$$\int_{\mathbb{R}^N} (I_\theta * v_j^p) v_j^p\,\mathrm{d}x \to 0, \quad \text{as } j \to \infty,$$

which contradicts (6-1). Therefore, there exist $\delta > 0$ and $y_j' \in \mathbb{R}^N$ such that

$$\int_{B_1(y_j')} v_j^2\,\mathrm{d}x \geq \delta. \tag{6-2}$$

Set $u_j(x) := v_j(x + y_j')$, $\widehat{V}_{\varepsilon_j}(x) := V_{\varepsilon_j}(x + y_j')$, $\widehat{W}_{\varepsilon_j}(x) := W_{\varepsilon_j}(x + y_j')$. Then u_j solves

$$(-\Delta)^s u_j + \widehat{V}_{\varepsilon_j}(x) u_j = \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j} u_j^p)] u_j^{p-1}, \quad u_j > 0 \tag{6-3}$$

with least energy

$$\begin{aligned}\hat{\vartheta}_{\varepsilon_j} = \hat{J}_{\varepsilon_j}(u_j) &:= \frac{1}{2}\int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} u_j|^2 + \widehat{V}_{\varepsilon_j}(x) u_j^2\big)\,\mathrm{d}x \\ &\qquad - \frac{1}{2p}\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j} u_j^p)] u_j^p\,\mathrm{d}x \\ &= \frac{p-1}{2p}\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j} u_j^p)] u_j^p\,\mathrm{d}x.\end{aligned} \tag{6-4}$$

Moreover, $[I_\theta * (W_{\varepsilon_j} v_j^p)](x + y_j') = [I_\theta * (\widehat{W}_{\varepsilon_j} u_j^p)](x)$ for any $x \in \mathbb{R}^N$, which ensures that

$$\hat{\vartheta}_{\varepsilon_j} = \hat{J}_{\varepsilon_j}(u_j) = J_{\varepsilon_j}(v_j) = \vartheta_{\varepsilon_j}. \tag{6-5}$$

In view of the boundedness of $\{u_j\}$, we can assume without loss of generality that

$$u_j \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^N), \qquad \text{as } j \to \infty, \tag{6-6}$$

$$u_j \to u \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^N), \quad \text{as } j \to \infty, \quad \text{for } q \in [2, 2^*_s), \tag{6-7}$$

which together with (6-2) implies $u \neq 0$.

Since V and W are bounded, going if necessary to a subsequence, we assume

$$V_{\varepsilon_j}(y'_j) \to V_0 \quad \text{and} \quad W_{\varepsilon_j}(y'_j) \to W_0, \quad \text{as } j \to \infty. \tag{6-8}$$

By the boundedness of $\nabla V : |\nabla V(x)| \le M$, for all $x \in \mathbb{R}^N$, we get that for any given $r > 0$,

$$\left|\widehat{V}_{\varepsilon_j}(x) - V_{\varepsilon_j}(y'_j)\right| = \left|\int_0^1 \nabla V(\varepsilon_j y'_j + t\varepsilon_j x)\varepsilon_j x \,\mathrm{d}t\right| \le \varepsilon_j M r, \quad \text{for all } x \in B_r(0).$$

Thus $\widehat{V}_{\varepsilon_j}(x) \to V_0$ as $j \to \infty$ uniformly on any bounded set of x. Similarly, $\widehat{W}_{\varepsilon_j}(x) \to W_0$ as $j \to \infty$ uniformly on any bounded set of x. As in the proof of Lemma 4.16, we have

$$\limsup_{j\to\infty} \hat{\vartheta}_{\varepsilon_j} \le \vartheta^{V_0 W_0}. \tag{6-9}$$

By (6-3), (6-6), (6-7), and Lemmas 3.1 and 3.3, we obtain that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int_{\mathbb{R}^N} \big[(-\Delta)^{s/2} u_j (-\Delta)^{s/2}\varphi + \widehat{V}_{\varepsilon_j}(x) u_j \varphi - \widehat{W}_{\varepsilon_j}(x)(I_\theta * (\widehat{W}_{\varepsilon_j} u_j^p)) u_j^{p-1}\varphi\big]\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \big[(-\Delta)^{s/2} u (-\Delta)^{s/2}\varphi + V_0 u\varphi - W_0 (I_\theta * (W_0 u^p)) u^{p-1}\varphi\big]\,\mathrm{d}x, \end{aligned}$$

which implies that u solves

$$(-\Delta)^s u + V_0 u = W_0^2 (I_\theta * u^p) u^{p-1}, \quad u > 0 \tag{6-10}$$

with energy

$$\begin{aligned} J^{V_0 W_0}(u) &:= \frac{1}{2}\int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} u|^2 + V_0 u^2\big)\,\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N} W_0^2 (I_\theta * u^p) u^p \,\mathrm{d}x \\ &= \frac{p-1}{2p}\int_{\mathbb{R}^N} W_0^2 (I_\theta * u^p) u^p \,\mathrm{d}x \ge \vartheta^{V_0 W_0}. \end{aligned} \tag{6-11}$$

According to Fatou's Lemma,

$$\int_{\mathbb{R}^N} W_0^2 (I_\theta * u^p) u^p \,\mathrm{d}x \le \liminf_{j\to\infty} \int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j} u_j^p)] u_j^p \,\mathrm{d}x. \tag{6-12}$$

Uniting (6-4), (6-9), (6-11) and (6-12), we obtain

$$\vartheta^{V_0 W_0} \le J^{V_0 W_0}(u) \le \liminf_{j\to\infty} \hat{J}_{\varepsilon_j}(u_j) \le \limsup_{j\to\infty} \hat{\vartheta}_{\varepsilon_j} \le \vartheta^{V_0 W_0}.$$

Therefore,

$$\lim_{j\to\infty} \hat{\vartheta}_{\varepsilon_j} = \vartheta^{V_0W_0} = J^{V_0W_0}(u). \tag{6-13}$$

Let $\tilde{u}_j$ and $\tilde{u}$ be the s-harmonic extension of u_j and u, respectively. Then as $j \to \infty$,

$$\tilde{u}_j \rightharpoonup \tilde{u}, \quad \text{in } E \text{ and in } X^s(\mathbb{R}^{N+1}_+).$$

Let $\eta \in C_0^\infty(\mathbb{R}_+)$ satisfy $\eta(t) = 1$ for $t \le 1$ and $\eta(t) = 0$ for $t \ge 2$ with $|\eta'(t)| \le 2$. Define

$$\tilde{\mu}_j(x, y) := \eta\left(\frac{|(x, y)|}{j}\right)\tilde{u}(x, y) \quad \text{and} \quad \tilde{z}_j(x, y) := \tilde{u}_j(x, y) - \tilde{\mu}_j(x, y)$$

for $(x, y) \in \mathbb{R}^{N+1}_+$ and denote $\mu_j := \operatorname{tr}_{\mathbb{R}^N} \tilde{\mu}_j,\ z_j := \operatorname{tr}_{\mathbb{R}^N} \tilde{z}_j$. Then as $j \to \infty$,

$$\begin{aligned} &\tilde{\mu}_j \to \tilde{u} && \text{in } E \text{ and in } X^s(\mathbb{R}^{N+1}_+),\\ &\mu_j \to u && \text{in } H^s(\mathbb{R}^N),\\ &\mu_j \to u && \text{in } L^q(\mathbb{R}^N), \quad \text{for } q \in [2, 2_s^*],\\ &\mu_j \to u && \text{a.e. on } \mathbb{R}^N, \end{aligned}$$

and

$$\begin{aligned} &\tilde{z}_j \rightharpoonup 0 && \text{in } E,\\ &z_j \rightharpoonup 0 && \text{in } H^s(\mathbb{R}^N),\\ &z_j \to 0 && \text{in } L^q_{\text{loc}}(\mathbb{R}^N), \quad \text{for } q \in [2, 2_s^*),\\ &z_j \to 0 && \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

Next we will show $\tilde{\hat{J}}_{\varepsilon_j}(\tilde{z}_j) \to 0$ and $\langle (\tilde{\hat{J}}_{\varepsilon_j})'(\tilde{z}_j), \tilde{z}_j \rangle \to 0$ as $j \to \infty$, where

$$\begin{aligned} \tilde{\hat{J}}_{\varepsilon_j}(\tilde{z}_j) := \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla \tilde{z}_j|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^N} \widehat{V}_{\varepsilon_j}(x)|z_j|^2 \,\mathrm{d}x \\ - \frac{1}{2p}\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j}|z_j|^p)]|z_j|^p \,\mathrm{d}x. \end{aligned}$$

Firstly, we claim

$$\|\tilde{z}_j\|_X^2 = \|\tilde{u}_j\|_X^2 - \|\tilde{\mu}_j\|_X^2 + o(1). \tag{6-14}$$

Indeed, $\|\tilde{u}_j\|_X^2 - \|\tilde{\mu}_j\|_X^2 - \|\tilde{z}_j\|_X^2 = 2(\tilde{u}_j, \tilde{\mu}_j)_X - 2\|\tilde{\mu}_j\|_X^2$, where $\|\tilde{\mu}_j\|_X \to \|\tilde{u}\|_X$ and

$$|(\tilde{u}_j, \tilde{\mu}_j)_X - (\tilde{u}, \tilde{u})_X| \le |(\tilde{u}_j, \tilde{\mu}_j - \tilde{u})_X| + |(\tilde{u}_j - \tilde{u}, \tilde{u})_X| \to 0, \quad \text{as } j \to \infty.$$

Secondly, we claim

$$\int_{\mathbb{R}^N} \widehat{V}_{\varepsilon_j}(x)|z_j|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} \widehat{V}_{\varepsilon_j}(x)|u_j|^2\,\mathrm{d}x - \int_{\mathbb{R}^N} \widehat{V}_{\varepsilon_j}(x)|\mu_j|^2\,\mathrm{d}x + o(1). \tag{6-15}$$

Indeed, $\{z_j\}$ is bounded in $H^s(\mathbb{R}^N)$ and in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^*]$. Moreover, for all $\delta > 0$, there exists $c(\delta) > 0$ such that

$$||u_j|^q - |z_j|^q| = ||z_j + \mu_j|^q - |z_j|^q| \le \delta|z_j|^q + c(\delta)|\mu_j|^q$$

and

$$g_j^\delta := \big(\big||u_j|^q - |z_j|^q - |\mu_j|^q\big| - \delta|z_j|^q\big)^+ \le (1 + c(\delta))|\mu_j|^q \le (1 + c(\delta))|u|^q.$$

By the Lebesgue dominated convergence theorem, $\int_{\mathbb{R}^N} g_j^\delta\,\mathrm{d}x \to 0$ as $j \to \infty$. Thus

$$\int_{\mathbb{R}^N} \big||u_j|^q - |z_j|^q - |\mu_j|^q\big|\,\mathrm{d}x \le \int_{\mathbb{R}^N} g_j^\delta\,\mathrm{d}x + \delta\int_{\mathbb{R}^N} |z_j|^q\,\mathrm{d}x = o(1). \tag{6-16}$$

Choose $q = 2$ in (6-16), we find (6-15) holds.

Thirdly, we claim

$$\begin{aligned}\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j}|z_j|^p)]|z_j|^p\,\mathrm{d}x &= \int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j}|u_j|^p)]|u_j|^p\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j}|\mu_j|^p)]|\mu_j|^p\,\mathrm{d}x + o(1).\end{aligned} \tag{6-17}$$

Indeed, taking into account that $\{z_j\}$ is bounded in $L^{2Np/(N+\theta)}(\mathbb{R}^N)$, it follows from (6-16) with $q = p$ that

$$\begin{aligned}|u_j|^p - |\mu_j|^p - |z_j|^p &\to 0 \quad \text{in } L^{2N/(N+\theta)}(\mathbb{R}^N), \quad \text{as } j \to \infty,\\ I_\theta * (|u_j|^p - |\mu_j|^p - |z_j|^p) &\to 0 \quad \text{in } L^{2N/(N-\theta)}(\mathbb{R}^N), \quad \text{as } j \to \infty,\end{aligned}$$

which together with

$$\begin{aligned}|z_j|^p &\rightharpoonup 0 && \text{in } L^{2N/(N+\theta)}(\mathbb{R}^N), \quad \text{as } j \to \infty,\\ \mu_j^p &\to u^p && \text{in } L^{2N/(N+\theta)}(\mathbb{R}^N), \quad \text{as } j \to \infty,\\ I_\theta * \mu_j^p &\to I_\theta * u^p && \text{in } L^{2N/(N-\theta)}(\mathbb{R}^N), \quad \text{as } j \to \infty\end{aligned}$$

and Lemma 3.3 imply that

$$\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)\Big[(I_\theta * (\widehat{W}_{\varepsilon_j}|u_j|^p))|u_j|^p - (I_\theta * (\widehat{W}_{\varepsilon_j}|\mu_j|^p))|\mu_j|^p - (I_\theta * (\widehat{W}_{\varepsilon_j}|z_j|^p))|z_j|^p\Big]\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)\big[I_\theta * \big(\widehat{W}_{\varepsilon_j}(|u_j|^p - |\mu_j|^p - |z_j|^p)\big)\big](|u_j|^p - |\mu_j|^p - |z_j|^p)\,\mathrm{d}x + 2\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)\big[I_\theta * \big(\widehat{W}_{\varepsilon_j}(|u_j|^p - |\mu_j|^p - |z_j|^p)\big)\big]|\mu_j|^p\,\mathrm{d}x + 2\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)\big[I_\theta * \big(\widehat{W}_{\varepsilon_j}(|u_j|^p - |\mu_j|^p - |z_j|^p)\big)\big]|z_j|^p\,\mathrm{d}x + 2\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j}|\mu_j|^p)]|z_j|^p\,\mathrm{d}x$$

$$\to 0 \quad \text{as } j \to \infty.$$

Fourthly, it follows from the Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}^N} \widehat{V}_{\varepsilon_j}(x)\mu_j^2\,\mathrm{d}x = \int_{\mathbb{R}^N} V_0 u^2\,\mathrm{d}x + o(1), \tag{6-18}$$

$$\int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j}\mu_j^p)]\mu_j^p\,\mathrm{d}x = \int_{\mathbb{R}^N} W_0^2(I_\theta * u^p)u^p\,\mathrm{d}x + o(1). \tag{6-19}$$

Moreover,

$$|(-\Delta)^{s/2}\mu_j|_2^2 = |(-\Delta)^{s/2}u|_2^2 + o(1). \tag{6-20}$$

Uniting (6-14), (6-15), (6-17), (6-18), (6-19), (6-20), (6-13), (6-3) and (6-10), we obtain

$$\tilde{\hat{J}}_{\varepsilon_j}(\tilde{z}_j) = \tilde{\hat{J}}_{\varepsilon_j}(\tilde{u}_j) - \tilde{\hat{J}}_{\varepsilon_j}(\tilde{\mu}_j) + o(1) = \hat{\vartheta}_{\varepsilon_j} - J^{V_0W_0}(u) + o(1) = o(1) \tag{6-21}$$

and

$$\begin{aligned} \langle(\tilde{\hat{J}}_{\varepsilon_j})'(\tilde{z}_j), \tilde{z}_j\rangle &= \langle(\tilde{\hat{J}}_{\varepsilon_j})'(\tilde{u}_j), \tilde{u}_j\rangle - \langle(\tilde{\hat{J}}_{\varepsilon_j})'(\tilde{\mu}_j), \tilde{\mu}_j\rangle + o(1) \\ &= \langle(\hat{J}_{\varepsilon_j})'(u_j), u_j\rangle - \langle(J^{V_0W_0})'(u), u\rangle + o(1) = o(1). \end{aligned} \tag{6-22}$$

Finally, by (6-21) and (6-22), we obtain that

$$\begin{aligned} o(1) &= \tilde{\hat{J}}_{\varepsilon_j}(\tilde{z}_j) - \frac{1}{2p}\langle(\tilde{\hat{J}}_{\varepsilon_j})'(\tilde{z}_j), \tilde{z}_j\rangle \\ &= \frac{p-1}{2p}\bigg(\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\tilde{z}_j|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \hat{V}_{\varepsilon_j}(x)|z_j|^2\,\mathrm{d}x\bigg) \ge C\|\tilde{z}_j\|_E^2, \end{aligned}$$

which implies $\tilde{z}_j \to 0$ in E and $z_j \to 0$ in $H^s(\mathbb{R}^N)$ as $j \to \infty$. Thus

$$\|u_j - u\|_s \le \|z_j\|_s + \|\mu_j - u\|_s \to 0, \quad \text{as } j \to \infty.$$

That is

$$u_j \to u \quad \text{in } H^s(\mathbb{R}^N), \quad \text{as } j \to \infty.$$

Step 2. We claim $u_j(x) \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$.

We can use the contradiction method to obtain that

$$\sup_j \int_{|x|\geq r} u_j^2 \,\mathrm{d}x \to 0, \quad \text{as } r \to \infty. \tag{6-23}$$

Moreover, noting that $(-\Delta)^s u_j - \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j} u_j^p)]u_j^{p-1} \leq 0$ on $\mathbb{R}^N$ and

$$[I_\theta * (\widehat{W}_{\varepsilon_j} u_j^p)]u_j^{p-2} \in L^q_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for some } q > \frac{N}{2s},$$

we deduce from Proposition 2.6 in [Jin et al. 2014] that for any compact set K,

$$\max_K u_j(x) \leq C\left(\int_K u_j^2 \,\mathrm{d}x\right)^{1/2},$$

which together with (6-23) ensures the claim true.

Step 3. We claim $\{\xi_j y_j'\}_j$ is bounded on $\mathbb{R}^N$.

Otherwise, there exists $|\xi_j y_j'| \to \infty$ as $j \to \infty$ along a subsequence. Hence $V_0 \geq \tau_\infty > \tau$ and $W_0 \leq k_\infty \leq k_v$, which together with Lemma 4.8, imply that $\vartheta^{V_0 W_0} > \vartheta^{\tau k_v}$. However, by (6-5), (6-13) and Lemma 4.16,

$$\vartheta^{V_0 W_0} = \lim_{j\to\infty} \hat{\vartheta}_{\varepsilon_j} = \lim_{j\to\infty} \vartheta_{\varepsilon_j} \leq \limsup_{j\to\infty} \vartheta_{\varepsilon_j} \leq \vartheta^{\tau k_v}.$$

That is a contradiction.

Therefore, without loss of generality we may assume

$$\varepsilon_j y_j' \to x_0, \quad \text{as } j \to \infty. \tag{6-24}$$

Noting (6-8), we have

$$V_0 = V(x_0) \quad \text{and} \quad W_0 = W(x_0). \tag{6-25}$$

By (6-10), we find that u is a groundstate of Equation (1-8).

Step 4. We claim $\{\varepsilon y_\varepsilon\}_\varepsilon$ is bounded, where $y_\varepsilon \in \mathbb{R}^N$ is a maximum point of v_ε.

For a contradiction, assume that there is $\varepsilon_j \to 0$ with $|\varepsilon_j y_j| \to \infty$, where $y_j := y_{\varepsilon_j}$ is a maximum point of $v_j := v_{\varepsilon_j}$. Repeating Steps 1, 2, 3, one can get that there exists $y_j' \in \mathbb{R}^N$ such that

$$\begin{gathered} u_j = v_j(\cdot + y_j') \to u \neq 0 \quad \text{in } H^s(\mathbb{R}^N), \quad \text{as } j \to \infty, \\ u_j(x) \to 0, \quad \text{as } |x| \to \infty, \quad \text{uniformly in } j \in \mathbb{N}, \\ \{\varepsilon_j y_j'\}_j \text{ is bounded on } \mathbb{R}^N. \end{gathered} \tag{6-26}$$

Thus $|\varepsilon_j y_j - \varepsilon_j y'_j| \geq |\varepsilon_j y_j| - |\varepsilon_j y'_j| \to \infty$, as $j \to \infty$, which implies that $|y_j - y'_j| \to \infty$ as $j \to \infty$. Then $\max_{\mathbb{R}^N} v_j = v_j(y_j) = u_j(y_j - y'_j) \to 0$ as $j \to \infty$. Hence $\max_{\mathbb{R}^N} u_j \to 0$ as $j \to \infty$. Noting $u_j > 0$, one has $u_j(x) \to 0$ as $j \to \infty$ uniformly in $x \in \mathbb{R}^N$, which contradicts (6-26).

Step 5. We claim $\lim_{\varepsilon \to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{A}_v) = 0$.

By Step 4, there exists $\varepsilon_j \to 0$ with

$$\varepsilon_j y_j \to y_0, \quad \text{as } j \to \infty, \tag{6-27}$$

where $y_j := y_{\varepsilon_j}$ is a maximum point of $v_j := v_{\varepsilon_j}$. It is sufficient to verify that $y_0 \in \mathscr{A}_v$.

By Step 1 and Step 3, there exists $y'_j \in \mathbb{R}^N$ satisfying $u_j(x) = v_j(x + y'_j)$ and (6-24). By Step 2, we can assume $u_j(x'_j) = \max_{\mathbb{R}^N} u_j$ and $\{x'_j\}_j$ is bounded on $\mathbb{R}^N$. Thus $y_j = x'_j + y'_j$ and $\varepsilon_j y_j - \varepsilon_j y'_j = \varepsilon_j x'_j \to 0$ as $j \to \infty$, which together with (6-24), (6-25) and (6-27), imply that

$$y_0 = x_0, \quad V(y_0) = V_0, \quad W(y_0) = W_0. \tag{6-28}$$

For a contradiction, assume that $y_0 \notin \mathscr{A}_v$. Then either $V(y_0) = \tau,\ W(y_0) < k_v$ or $V(y_0) > \tau,\ W(y_0) \leq k_v$. By Lemma 4.8,

$$\vartheta^{V(y_0)W(y_0)} > \vartheta^{\tau k_v}. \tag{6-29}$$

Uniting (6-5), (6-13), (6-28), (6-29) and Lemma 4.16, we obtain

$$\lim_{j\to\infty} \vartheta_{\varepsilon_j} = \lim_{j\to\infty} \hat{\vartheta}_{\varepsilon_j} = \vartheta^{V_0 W_0} = \vartheta^{V(y_0)W(y_0)} > \vartheta^{\tau k_v} \geq \limsup_{j\to\infty} \vartheta_{\varepsilon_j}.$$

That is impossible.

In particular, if $\mathscr{V} \cap \mathscr{W} \neq \varnothing$, then $x_0 \in \mathscr{A}_v = \mathscr{V} \cap \mathscr{W}$. Hence

$$\lim_{\varepsilon \to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{V} \cap \mathscr{W}) = 0 \quad \text{and} \quad V(x_0) = \tau,\ W(x_0) = k,$$

which jointly with Equation (1-8) imply that u is a groundstate of Equation (1-9).

Step 6. For $p \in \left(2, \frac{N+\theta}{N-2s}\right)$, we claim there exist $C_2 > C_1 > 0$ and $r > 2$ large enough such that for all small $\varepsilon > 0$,

$$\frac{C_1}{|x|^{N+2s}} \leq u_\varepsilon(x) \leq \frac{C_2}{|x|^{N+2s}}, \quad \text{for all } |x| \geq r.$$

We verify its correctness for any sequence.

Now choose $a = \sup_{\mathbb{R}^N} \widehat{V}_{\varepsilon_j}(x)$ and $r = 2$ in Lemma 4.10, then there is a continuous function $v_1 \in H^s(\mathbb{R}^N)$ such that

$$(-\Delta)^s v_1 + a v_1 = 0 \quad \text{and} \quad v_1(x) \geq \frac{\bar{C}_1}{|x|^{N+2s}}, \quad \text{for all } |x| \geq 2, \tag{6-30}$$

where $\overline{C}_1$ depends on a, N, s. By the continuity of u_{ε_j} and v_1, there is $M_1 > 0$ such that

$$u_{\varepsilon_j} - M_1 v_1 \geq 0, \quad \text{for all } |x| \leq 2.$$

Additionally, by (6-3) and (6-30), we have

$$((-\Delta)^s + a)(u_{\varepsilon_j} - M_1 v_1) \geq \int_{\mathbb{R}^N} \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j} u_{\varepsilon_j}^p)] u_{\varepsilon_j}^{p-1}\, \mathrm{d}x \geq 0, \quad \text{for all } |x| \geq 2.$$

Hence $u_{\varepsilon_j} - M_1 v_1 \geq 0$ for any $|x| \geq 2$, which together with (6-30) implies that

$$u_{\varepsilon_j}(x) \geq \frac{C_1}{|x|^{N+2s}}, \quad \text{for all } |x| \geq 2, \quad \text{where } C_1 = M_1 \overline{C}_1.$$

Next, by Step 2 and (6-3), choose $0 < a < \tau$ and $r > 2$ sufficiently large such that

$$\begin{aligned} (-\Delta)^s u_{\varepsilon_j} + a u_{\varepsilon_j} \\ = \widehat{W}_{\varepsilon_j}(x)[I_\theta * (\widehat{W}_{\varepsilon_j} u_{\varepsilon_j}^p)] u_{\varepsilon_j}^{p-1} + (a - \widehat{V}_{\varepsilon_j}(x)) u_{\varepsilon_j} \leq 0, \quad \text{for all } |x| \geq r. \end{aligned} \tag{6-31}$$

By Lemma 4.10, there is a continuous function $v_2 \in H^s(\mathbb{R}^N)$ such that

$$(-\Delta)^s v_2 + a v_2 = 0 \quad \text{and} \quad v_2(x) \leq \frac{\overline{C}_2}{|x|^{N+2s}}, \quad \text{for all } |x| \geq r, \tag{6-32}$$

where $\overline{C}_2$ depends on a, N, s, r. Moreover, there is $M_2 > 0$ such that

$$u_{\varepsilon_j} - M_2 v_2 \leq 0, \quad \text{for all } |x| \leq r.$$

By (6-31) and (6-32), we have

$$((-\Delta)^s + a)(u_{\varepsilon_j} - M_2 v_2) = (-\Delta)^s u_{\varepsilon_j} + a u_{\varepsilon_j} \leq 0, \quad \text{for all } |x| \geq r.$$

Thus $u_{\varepsilon_j} - M_2 v_2 \leq 0$ for any $|x| \geq r$, which together with (6-32) implies that

$$u_{\varepsilon_j}(x) \leq \frac{C_2}{|x|^{N+2s}}, \quad \text{for all } |x| \geq r, \quad \text{where } C_2 = M_2 \overline{C}_2.$$

Step 7. Set $x_\varepsilon = \varepsilon y_\varepsilon$. Then $w_\varepsilon(x_\varepsilon) = v_\varepsilon(y_\varepsilon)$. By Step 4, x_ε is a maximum point of w_ε and $\{x_\varepsilon\}_\varepsilon$ is bounded on $\mathbb{R}^N$. By Step 5, $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathscr{A}_v) = 0$. By Step 1 and Step 2, $u_\varepsilon(x) = v_\varepsilon(x + y'_\varepsilon) = w_\varepsilon(\varepsilon x + x_\varepsilon - \varepsilon x'_\varepsilon)$, where $x'_\varepsilon = y_\varepsilon - y'_\varepsilon$ is a maximum point of u_ε with $\varepsilon x'_\varepsilon \to 0$ as $\varepsilon \to 0$. By Step 6,

$$\frac{C_1 \varepsilon^{N+2s}}{|x - x_\varepsilon|^{N+2s}} \leq w_\varepsilon(x) \leq \frac{C_2 \varepsilon^{N+2s}}{|x - x_\varepsilon|^{N+2s}}, \quad \text{for all } |x| \geq R,$$

where $R = 2r + \sup_\varepsilon |x_\varepsilon|$ and C_1, C_2 depend on R, N, s, τ, $\sup_{\mathbb{R}^N} V(x)$.

The proof is completed. □

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Received October 9, 2018. Revised August 11, 2019.

MIN LIU
SCHOOL OF MATHEMATICAL SCIENCES
BEIJING NORMAL UNIVERSITY
BEIJING
CHINA
minliu@mail.bnu.edu.cn

ZHONGWEI TANG
SCHOOL OF MATHEMATICAL SCIENCES
BEIJING NORMAL UNIVERSITY
LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS
MINISTRY OF EDUCATION
BEIJING
CHINA
tangzw@bnu.edu.cn

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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Volume 304 No. 1 January 2020

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