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MORSE INEQUALITIES FOR FOURIER COMPONENTS OF KOHN–ROSSI COHOMOLOGY OF CR COVERING MANIFOLDS WITH S^1-ACTION

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Let X be a compact connected CR manifold of dimension $2n+1$, $n \geq 1$. Let $\widetilde{X}$ be a paracompact CR manifold with a transversal CR S^1-action, such that there is a discrete group Γ acting freely on $\widetilde{X}$ having $X = \widetilde{X}/\Gamma$. Based on an asymptotic formula for the Fourier components of the heat kernel with respect to the S^1-action, we establish the Morse inequalities for Fourier components of reduced L^2-Kohn–Rossi cohomology with values in a rigid CR vector bundle over $\widetilde{X}$. As a corollary, we obtain the Morse inequalities for Fourier components of Kohn–Rossi cohomology on X which were obtained by Hsiao and Li (2016) by using Szegő kernel method.

1. Introduction and statement of the results

Gromov, Henkin and Shubin [Gromov et al. 1998, Theorem 0.2] considered covering manifolds that are strongly pseudoconvex of complex manifolds and analyzed the holomorphic L^2-functions on the coverings. Todor, Chiose and Marinescu [Todor et al. 2001] generalized in a similar manner the Morse inequalities of Siu [1984] and Demailly [1985] on coverings of complex manifolds; they also considered coverings of weakly pseudoconvex domains in [Marinescu et al. 2002]. The study of problems on CR manifolds with S^1-action became active recently, see [Cheng et al. 2019; Hsiao 2018; Hsiao and Huang 2019; Hsiao and Li 2016; 2018; Hsiao and Shao 2019]. In particular, Hsiao and Li [2016] established the Morse inequalities for Fourier components of Kohn–Rossi cohomology on X by using the Szegő kernel method. In [Hsiao and Marinescu 2012] general Morse inequalities for CR bundles are proved, generalizing [Getzler 1989]. Inspired by the results of [Gromov et al. 1998; Hsiao and Li 2016; Todor et al. 2001; Siu 1984; Demailly 1985], we establish Morse inequalities for Fourier components of reduced L^2-Kohn–Rossi cohomology

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with values in a rigid CR vector bundle on a covering manifold over a compact connected CR manifold with S^1-action. This generalizes the results of [Hsiao and Li 2016] to CR covering manifolds with S^1-action. We present a proof by the heat kernel method, which is inspired by Bismut's proof [1987; Ma and Marinescu 2007] of the holomorphic Morse inequalities. The crucial estimate for Fourier components of the heat kernel of Kohn Laplacians was given in [Hsiao and Huang 2019].

Now we formulate the main results. We refer to other sections for notation and definitions (see Definition 2.1, 2.2, 2.3, 2.5 and (3-1), (3-25)) used here. Let X be a compact connected CR manifold of dimension $2n+1$, $n \geq 1$ with a transversal CR S^1-action $e^{i\theta}$ on X. For $x \in X$, we say that the period of x is $\frac{2\pi}{\ell}$, $\ell \in \mathbb{N}$, if $e^{i\theta} \circ x \neq x$, for every $0 < \theta < \frac{2\pi}{\ell}$, and $e^{i2\pi/\ell} \circ x = x$. For each $\ell \in \mathbb{N}$, put

$$X_\ell = \left\{ x \in X \mid \text{the period of } x \text{ is } \tfrac{2\pi}{\ell} \right\} \tag{1-1}$$

and let

$$p = \min\{\ell \in \mathbb{N} \mid X_\ell \neq \varnothing\}. \tag{1-2}$$

It is well-known that if X is connected, then X_p is an open and dense subset of X (see [Duistermaat and Heckman 1982, p. 265]). Assume $X = X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_k}$, $p =: p_1 < p_2 < \cdots < p_k$. Set $X_{\mathrm{reg}} := X_p$. We call $x \in X_{\mathrm{reg}}$ a regular point of the S^1 action. Let X_{sing} be the complement of X_{reg}.

Let $\widetilde{X}$ be a paracompact CR manifold, such that there is a discrete group Γ acting freely on $\widetilde{X}$ having $X = \widetilde{X}/\Gamma$. Let $\pi : \widetilde{X} \to X$ be the natural projection with the pull-back map $\pi^* : TX \to T\widetilde{X}$. Then $\widetilde{X}$ admits a pull-back CR structure $T^{1,0}\widetilde{X} := \pi^* T^{1,0} X$ and, hence, a CR manifold. We assume that $\widetilde{X}$ admits a transversal CR locally free S^1 action, denote by $e^{i\theta}$. We further assume that the map

$$\Gamma \times \widetilde{X} \to \widetilde{X}, \quad (\gamma, \tilde{x}) \mapsto \gamma \circ \tilde{x}, \quad \text{for all } \tilde{x} \in \widetilde{X},\ \gamma \in \Gamma.$$

is CR, see (2-6), and

$$e^{i\theta} \circ \gamma \circ \tilde{x} = \gamma \circ e^{i\theta} \circ \tilde{x}, \quad \text{for all } \tilde{x} \in \widetilde{X},\ \theta \in [0, 2\pi[,\ \gamma \in \Gamma.$$

Let $\widetilde{E} := \pi^* E$ be the pull-pack bundle of a rigid CR vector bundle E over X. Then $\widetilde{E}$ is a Γ-invariant rigid CR vector bundle over $\widetilde{X}$. We denote by $\widetilde{X}_{\mathrm{reg}}$ the set of regular points of the S^1-action on $\widetilde{X}$. Note that since Γ acts on $\widetilde{X}$ freely so that $\widetilde{X}/\Gamma = X$, hence, we have $\widetilde{X}_{\mathrm{reg}}/\Gamma = X_{\mathrm{reg}} = X_p$. We denote by $X(q)$ a subset of X such that

$$X(q) := \left\{ x \in X \mid \mathcal{L}_x \text{ has exactly } q \text{ negative eigenvalues and } n-q \text{ positive eigenvalues} \right\}.$$

We refer to Section 2 for more details. Our main theorem is:

Theorem 1.1. *With the above notation and assumptions, as $m \to \infty$, for $q = 0, 1, \dots, n$, the m-th Fourier components of reduced L^2-Kohn–Rossi cohomology* (see (3-25)) *satisfy the following strong Morse inequalities*:

$$\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) \leq \begin{cases} \dfrac{prm^n}{2\pi^{n+1}}\displaystyle\sum_{j=0}^{q}(-1)^{q-j}\int_{X(j)}|\det(\mathcal{L}_x)|\,dv_X(x)+o(m^n), & \text{for } p\,|\,m,\\ \displaystyle\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) = o(m^n), & \text{for } p\nmid m, \end{cases} \tag{1-3}$$

where r denotes the rank of $\widetilde{E}$, $\dim_\Gamma$ *denotes the von Neumann dimension* (see Section 2C, [Ma and Marinescu 2007, §3.6.1] or [Atiyah 1976, §3]) *and $\mathcal{L}_x$ is the Levi form at $x \in X$. When $p\,|\,m$, $q = n$, as $m \to \infty$, we have the asymptotic Riemann–Roch–Hirzebruch theorem*:

$$\sum_{j=0}^{n}(-1)^{j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) = \frac{prm^n}{2\pi^{n+1}}\sum_{j=0}^{n}(-1)^{j}\int_{X(j)}|\det(\mathcal{L}_x)|\,dv_X(x)+o(m^n). \tag{1-4}$$

In particular, we get the weak Morse inequalities,

$$\dim_\Gamma \overline{H}^q_{b,(2),m}(\widetilde{X},\widetilde{E}) \leq \frac{prm^n}{2\pi^{n+1}}\int_{X(q)}|\det(\mathcal{L}_x)|\,dv_X(x)+o(m^n). \tag{1-5}$$

By the standard argument in [Hsiao and Li 2016; Hsiao et al. 2017], we deduce easily the following Grauert–Riemenschneider criterion on coverings of CR manifolds.

Corollary 1.2. *With the notation and assumptions of Theorem 1.1, we also assume that X is weakly pseudoconvex and strongly pseudoconvex at a point. Then*

$$\dim_\Gamma \overline{H}^0_{b,(2),m}(\widetilde{X},\widetilde{E}) \approx m^n, \quad \text{for } p\,|\,m. \tag{1-6}$$

In particular, $\dim_\Gamma \overline{H}^0_{b,(2)}(\widetilde{X},\widetilde{E}) = \infty$.

When $\Gamma = \{e\}$, $p = 1$ and $\widetilde{E}$ is a trivial line bundle, we deduce the following Morse inequalities of Hsiao and Li [2016, Theorems 2.2 and 2.5].

Corollary 1.3. *With the above notation and assumptions, as $m \to \infty$, for $q = 0, 1, \dots, n$, the m-th Fourier components of Kohn–Rossi cohomology satisfy the* *following strong Morse inequalities*:

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j_{b,m}(X) \leq \frac{m^n}{2\pi^{n+1}}\sum_{j=0}^{q}(-1)^{q-j}\int_{X(j)}|\det(\mathcal{L}_x)|\,dv_X(x)+o(m^n), \tag{1-7}$$

where $\mathcal{L}_x$ is the Levi form at $x\in X$. In particular, *we get the weak Morse inequalities*,

$$\dim H^q_{b,m}(X)\leq \frac{m^n}{2\pi^{n+1}}\int_{X(q)}|\det(\mathcal{L}_x)|\,dv_X(x)+o(m^n). \tag{1-8}$$

Let X be a compact CR manifold of dimension $2n+1$, $n\geq 1$. A classical theorem due to Boutet de Monvel [1975] asserts that X can be globally CR embedded into $\mathbb{C}^N$, for some $N\in\mathbb{N}$, when X is strongly pseudoconvex with dimension $n\geq 5$. Epstein [1992] proved that if X is strongly pseudoconvex with dimension 3 and a global free transversal CR S^1-action, then X can be embedded into $\mathbb{C}^N$ by positive Fourier components of CR functions. Corollary 1.3 guarantees the abundance of positive Fourier components of CR functions to do embedding in general cases (e.g., the S^1-action can be only locally free). The proofs of Hsiao and Li [2016], include localization of analytic objects (eigenfunctions, Szegő kernels), Kohn L^2 estimates and scaling techniques. A more general version of Corollary 1.3 (with X being weakly pseudoconvex) is proved by Cheng, Hsiao and Tsai [Cheng et al. 2019, Proposition 1.20 and Corollary 1.21] in a different way. By using the Morse inequalities, (1-7) and (1-8), Hsiao and Li [2016, Theorem 2.6] proved that there are abundant CR functions on X when X is weakly pseudoconvex and strongly pseudoconvex at a point. Corollary 1.2 generalizes Theorem 2.6 of [Hsiao and Li 2016] to CR covering manifolds.

This paper is organized as follows. In Section 2 we introduce some basic notation, terminology and definitions. In Section 3 we study the asymptotic behavior of heat kernels of Kohn Laplacians. Section 4 is devoted to the heat kernel proof of the main theorem.

2. Preliminaries

2A. ***Some standard notation.*** We use the following notation: $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+:=\{x\in\mathbb{R}\mid x>0\}$, $\overline{\mathbb{R}}_+:=\{x\in\mathbb{R}\mid x\geq 0\}$. For a multiindex $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}_0^n$ we set $|\alpha|=\alpha_1+\cdots+\alpha_n$. For $x=(x_1,\ldots,x_n)$ we write

$$x^\alpha=x_1^{\alpha_1}\ldots x_n^{\alpha_n},\quad \partial_{x_j}=\frac{\partial}{\partial x_j},\quad \partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}=\frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Let $z=(z_1,\dots,z_n)$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\dots,n$, be coordinates of $\mathbb{C}^n$. We write

$$z^\alpha=z_1^{\alpha_1}\cdots z_n^{\alpha_n},\quad \bar z^\alpha=\bar z_1^{\alpha_1}\cdots \bar z_n^{\alpha_n},$$

$$\partial_{z_j}=\frac{\partial}{\partial z_j}=\frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}}-i\frac{\partial}{\partial x_{2j}}\Big),\quad \partial_{\bar z_j}=\frac{\partial}{\partial \bar z_j}=\frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}}+i\frac{\partial}{\partial x_{2j}}\Big),$$

$$\partial_z^\alpha=\partial_{z_1}^{\alpha_1}\cdots\partial_{z_n}^{\alpha_n}=\frac{\partial^{|\alpha|}}{\partial z^\alpha},\quad \partial_{\bar z}^\alpha=\partial_{\bar z_1}^{\alpha_1}\cdots\partial_{\bar z_n}^{\alpha_n}=\frac{\partial^{|\alpha|}}{\partial \bar z^\alpha}.$$

Let X be a C^∞ orientable paracompact manifold. We let TX and T^*X denote the tangent bundle of X and the cotangent bundle of X, respectively. The complexified tangent bundle of X and the complexified cotangent bundle of X will be denoted by $\mathbb{C}TX$ and $\mathbb{C}T^*X$, respectively. We write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between T^*X and TX. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}T^*X\times\mathbb{C}TX$. For $u\in\mathbb{C}T^*X$, $v\in\mathbb{C}TX$, we also write $u(v):=\langle u,v\rangle$.

Let $Y\subset X$ be an open set. The spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y,E)$ and $D'(Y,E)$, respectively.

2B. *CR manifolds with S^1-action.* Let $(X,T^{1,0}X)$ be a compact CR manifold of dimension $2n+1$, $n\geq 1$, where $T^{1,0}X$ is a CR structure of X. That is, $T^{1,0}X$ is a subbundle of rank n of the complexified tangent bundle $\mathbb{C}TX$, satisfying

$$T^{1,0}X\cap T^{0,1}X=\{0\},$$

where $T^{0,1}X=\overline{T^{1,0}X}$, and $[\mathcal{V},\mathcal{V}]\subset\mathcal{V}$, where $\mathcal{V}=C^\infty(X,T^{1,0}X)$. We assume that X admits a S^1 action: $S^1\times X\to X$. We write $e^{i\theta}$ to denote the S^1 action. Let $T\in C^\infty(X,TX)$ be the global real vector field induced by the S^1 action given by $(Tu)(x)=\frac{\partial}{\partial\theta}(u(e^{i\theta}\circ x))|_{\theta=0}$, $u\in C^\infty(X)$.

Definition 2.1. We say that the S^1 action $e^{i\theta}$ is CR if $[T,C^\infty(X,T^{1,0}X)]\subset C^\infty(X,T^{1,0}X)$ and the S^1 action is transversal if for each $x\in X$,

$$\mathbb{C}T(x)\oplus T_x^{1,0}X\oplus T_x^{0,1}X=\mathbb{C}T_xX.$$

Moreover, we say that the S^1 action is locally free if $T\neq 0$ everywhere.

Note that if the S^1 action is transversal, then it is locally free. We assume throughout that $(X,T^{1,0}X)$ is a connected CR manifold with a transversal CR S^1 action $e^{i\theta}$ and we let T be the global vector field induced by the S^1 action. Let $\omega_0\in C^\infty(X,T^*X)$ be the global real one form determined by $\langle\omega_0,u\rangle=0$, for every $u\in T^{1,0}X\oplus T^{0,1}X$ and $\langle\omega_0,T\rangle=-1$.

Definition 2.2. For $p\in X$, the Levi form $\mathcal{L}_p$ is the Hermitian quadratic form on $T_p^{1,0}X$ given by $\mathcal{L}_p(U,\overline{V})=-\frac{1}{2i}\langle d\omega_0(p),U\wedge\overline{V}\rangle$, $U,V\in T_p^{1,0}X$.

Definition 2.3. If the Levi form $\mathcal{L}_p$ is positive definite, we say that X is strongly pseudoconvex at p. If the Levi form is positive definite at every point of X, we say that X is strongly pseudoconvex.

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. Define the vector bundle of $(0,q)$ forms by $T^{*0,q}X = \Lambda^q(T^{*0,1}X)$. Put $T^{*0,\bullet}X := \bigoplus_{j\in\{0,1,\dots,n\}} T^{*0,j}X$. Let $D \subset X$ be an open subset. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega_0^{0,q}(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D. Put

$$\Omega^{0,\bullet}(D) := \bigoplus_{j\in\{0,1,\dots,n\}} \Omega^{0,j}(D), \quad \Omega_0^{0,\bullet}(D) := \bigoplus_{j\in\{0,1,\dots,n\}} \Omega_0^{0,j}(D).$$

Similarly, if E is a vector bundle over D, then we let $\Omega^{0,q}(D,E)$ denote the space of smooth sections of $T^{*0,q}X \otimes E$ over D and let $\Omega_0^{0,q}(D,E)$ be the subspace of $\Omega^{0,q}(D,E)$ whose elements have compact support in D. Put

$$\Omega^{0,\bullet}(D,E) := \bigoplus_{j\in\{0,1,\dots,n\}} \Omega^{0,j}(D,E), \quad \Omega_0^{0,\bullet}(D,E) := \bigoplus_{j\in\{0,1,\dots,n\}} \Omega_0^{0,j}(D,E).$$

Fix $\theta_0 \in\,]-\pi,\pi[$, θ_0 small. Let

$$de^{i\theta_0} : \mathbb{C}T_xX \to \mathbb{C}T_{e^{i\theta_0}x}X$$

denote the differential map of $e^{i\theta_0} : X \to X$. By the CR property of the S^1 action, we can check that

$$\begin{gathered} de^{i\theta_0} : T_x^{1,0}X \to T_{e^{i\theta_0}x}^{1,0}X, \quad de^{i\theta_0} : T_x^{0,1}X \to T_{e^{i\theta_0}x}^{0,1}X, \\ de^{i\theta_0}(T(x)) = T(e^{i\theta_0}x). \end{gathered} \tag{2-1}$$

Let

$$(e^{i\theta_0})^* : \Lambda^j(\mathbb{C}T^*X) \to \Lambda^j(\mathbb{C}T^*X)$$

be the pull-back map by $e^{i\theta_0}$, $j = 0, 1, \dots, 2n+1$. From (2-1), it is easy to see that for every $q = 0, 1, \dots, n$,

$$(e^{i\theta_0})^* : T_{e^{i\theta_0}x}^{*0,q}X \to T_x^{*0,q}X. \tag{2-2}$$

Let $u \in \Omega^{0,q}(X)$. Define

$$Tu := \frac{\partial}{\partial\theta}((e^{i\theta})^*u)|_{\theta=0} \in \Omega^{0,q}(X). \tag{2-3}$$

For every $\theta \in \mathbb{R}$ and every $u \in C^\infty(X, \Lambda^j(\mathbb{C}T^*X))$, we write $u(e^{i\theta}\circ x) := (e^{i\theta})^*u(x)$.

Let $\bar\partial_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy–Riemann operator. From the CR property of the S^1 action, it is straightforward to see that

$$T\bar\partial_b = \bar\partial_b T \quad \text{on } \Omega^{0,\bullet}(X).$$

Definition 2.4. Let $D \subset U$ be an open set. We say that a function $u \in C^\infty(D)$ is rigid if $Tu = 0$. We say that a function $u \in C^\infty(X)$ is Cauchy–Riemann (CR for short) if $\bar{\partial}_b u = 0$. We call u a rigid CR function if $\bar{\partial}_b u = 0$ and $Tu = 0$.

Definition 2.5. Let F be a complex vector bundle over X. We say that F is rigid (CR) if X can be covered with open sets U_j with trivializing frames $\{f_j^1, f_j^2, \dots, f_j^r\}$, $j = 1, 2, \dots$, such that the corresponding transition matrices are rigid (CR). The frames $\{f_j^1, f_j^2, \dots, f_j^r\}$, $j = 1, 2, \dots$, are called rigid (CR) frames.

Definition 2.6. Let F be a complex rigid vector bundle over X and let $\langle \cdot \mid \cdot \rangle_F$ be a Hermitian metric on F. We say that $\langle \cdot \mid \cdot \rangle_F$ is a rigid Hermitian metric if for every rigid local frame $f_1, \dots, f_r$ of F, we have $T\langle f_j \mid f_k\rangle_F = 0$, for every $j, k = 1, 2, \dots, r$.

It is known that there is a rigid Hermitian metric on any rigid vector bundle F (see [Cheng et al. 2019, Theorem 2.10; Hsiao 2018, Theorem 10.5]). Note that Baouendi, Rothschild and Treves [Baouendi et al. 1985] proved that $T^{1,0}X$ is a rigid complex vector bundle over X.

From now on, let E be a rigid CR vector bundle over X and we take a rigid Hermitian metric $\langle \cdot \mid \cdot \rangle_E$ on E and take a rigid Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle T \mid T\rangle = 1$. The Hermitian metrics on $\mathbb{C}TX$ and on E induce Hermitian metrics $\langle \cdot \mid \cdot \rangle$ and $\langle \cdot \mid \cdot \rangle_E$ on $T^{*0,\bullet}X$ and $T^{*0,\bullet}X \otimes E$, respectively. We denote by $dv_X = dv_X(x)$ the volume form on X induced by the fixed Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TX$. Then we get natural global L^2 inner products $(\cdot \mid \cdot)_E$, $(\cdot \mid \cdot)$ on $\Omega^{0,\bullet}(X, E)$ and $\Omega^{0,\bullet}(X)$, respectively. We denote by $L^2(X, T^{*0,q}X \otimes E)$ and $L^2(X, T^{*0,q}X)$ the completions of $\Omega^{0,q}(X, E)$ and $\Omega^{0,q}(X)$ with respect to $(\cdot \mid \cdot)_E$ and $(\cdot \mid \cdot)$, respectively. Similarly, we denote by $L^2(X, T^{*0,\bullet}X \otimes E)$ and $L^2(X, T^{*0,\bullet}X)$ the completions of $\Omega^{0,\bullet}(X, E)$ and $\Omega^{0,\bullet}(X)$ with respect to $(\cdot \mid \cdot)_E$ and $(\cdot \mid \cdot)$, respectively. We extend $(\cdot \mid \cdot)_E$ and $(\cdot \mid \cdot)$ to $L^2(X, T^{*0,\bullet}X \otimes E)$ and $L^2(X, T^{*0,\bullet}X)$ in the standard way, respectively. For $f \in L^2(X, T^{*0,\bullet}X \otimes E)$, we denote $(f \mid f)_E$ by $\|f\|_E^2$. Similarly, for $f \in L^2(X, T^{*0,\bullet}X)$, we denote $(f \mid f)$ by $\|f\|^2$.

We also write $\bar{\partial}_b$ to denote the tangential Cauchy–Riemann operator acting on forms with values in E:

$$\bar{\partial}_b : \Omega^{0,\bullet}(X, E) \to \Omega^{0,\bullet}(X, E).$$

Since E is rigid, we can also define Tu for every $u \in \Omega^{0,q}(X, E)$ and we have

$$T\bar{\partial}_b = \bar{\partial}_b T \quad \text{on } \Omega^{0,\bullet}(X, E). \tag{2-4}$$

For every $m \in \mathbb{Z}$, let

$$\begin{aligned} \Omega_m^{0,q}(X, E) &:= \{u \in \Omega^{0,q}(X, E) \mid Tu = imu\}, \quad q = 0, 1, 2, \dots, n, \\ \Omega_m^{0,\bullet}(X, E) &:= \{u \in \Omega^{0,\bullet}(X, E) \mid Tu = imu\}. \end{aligned} \tag{2-5}$$

For each $m \in \mathbb{Z}$, we denote by $L^2_m(X, T^{*0,q}X \otimes E)$ and $L^2_m(X, T^{*0,q}X)$ the completions of $\Omega^{0,q}_m(X, E)$ and $\Omega^{0,q}_m(X)$ with respect to $(\cdot\,|\,\cdot)_E$ and $(\cdot\,|\,\cdot)$, respectively. Similarly, we denote by $L^2_m(X, T^{*0,\bullet}X \otimes E)$ and $L^2_m(X, T^{*0,\bullet}X)$ the completions of $\Omega^{0,\bullet}_m(X, E)$ and $\Omega^{0,\bullet}_m(X)$ with respect to $(\cdot\,|\,\cdot)_E$ and $(\cdot\,|\,\cdot)$, respectively.

2C. ***Covering manifolds, von Neumann dimension.*** Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n+1$, $n \geq 1$. Let $\widetilde{X}$ be a paracompact CR manifold, such that there is a discrete group Γ acting freely on $\widetilde{X}$ having $X = \widetilde{X}/\Gamma$. Let $\pi : \widetilde{X} \to X$ be the natural projection with the pull-back map $\pi^* : TX \to T\widetilde{X}$. Then $\widetilde{X}$ admits a pull-back CR structure $T^{1,0}\widetilde{X} := \pi^* T^{1,0}X$ and, hence, a CR manifold. We assume that $\widetilde{X}$ admits a transversal CR locally free S^1 action, denoted by $e^{i\theta}$. We further assume that the map

$$\Gamma \times \widetilde{X} \to \widetilde{X}, \quad (\gamma, \tilde{x}) \mapsto \gamma \circ \tilde{x}, \quad \text{for all } \tilde{x} \in \widetilde{X},\ \gamma \in \Gamma.$$

is CR, i.e.,

$$\gamma_*(T^{1,0}_{\tilde{x}}\widetilde{X}) \subseteq T^{1,0}_{\gamma\cdot\tilde{x}}\widetilde{X}, \tag{2-6}$$

and

$$e^{i\theta} \circ \gamma \circ \tilde{x} = \gamma \circ e^{i\theta} \circ \tilde{x}, \quad \text{for all } \tilde{x} \in \widetilde{X},\ \theta \in [0, 2\pi[,\ \gamma \in \Gamma.$$

It is easy to see that the S^1-action $e^{i\theta}$ on $\widetilde{X}$ induces a transversal CR locally free S^1 action, also denoted by $e^{i\theta}$. We denote by $\widetilde{T} := \pi^* T$ the pull-back one form on $\widetilde{X}$, then T is the global real vector field induced by the S^1-action on X. Let $\widetilde{\omega}_0 := \pi^*\omega_0$ be the pull-back one form on $\widetilde{X}$, where ω_0 is the global real one form on X as defined in Section 2B. Then, for $\tilde{p} \in \widetilde{X}$, the Levi form $\widetilde{\mathcal{L}}_{\tilde{p}}$ is the Hermitian quadratic form on $T^{1,0}_{\tilde{p}}\widetilde{X}$ given by

$$\widetilde{\mathcal{L}}_{\tilde{p}}(\widetilde{U}, \overline{\widetilde{V}}) = -\frac{1}{2i}\langle d\widetilde{\omega}_0(\tilde{p}), \widetilde{U} \wedge \overline{\widetilde{V}}\rangle = -\frac{1}{2i}\langle d\omega_0(\pi(\tilde{p})), \pi_*\widetilde{U} \wedge \pi_*\overline{\widetilde{V}}\rangle, \tag{2-7}$$

where $\widetilde{U}, \widetilde{V} \in T^{1,0}_{\tilde{p}}\widetilde{X}$.

As usual, let $\Omega^{0,q}(\widetilde{X})$ denote the space of smooth sections of $\bigwedge^q(T^{*0,1}\widetilde{X})$. We also denote by $\bar{\partial}_b : \Omega^{0,q}(\widetilde{X}) \to \Omega^{0,q+1}(\widetilde{X})$ the tangential Cauchy–Riemann operator. Then $\widetilde{T}\bar{\partial}_b = \bar{\partial}_b\widetilde{T}$ on $\Omega^{0,\bullet}(\widetilde{X})$. Let E be a rigid CR vector bundle over X, then $\widetilde{E} := \pi^*E$ is a Γ-invariant rigid CR vector bundle over $\widetilde{X}$. Again let $\Omega^{0,q}(\widetilde{X}, \widetilde{E})$ denote the space of smooth sections of $\bigwedge^q(T^{*0,1}\widetilde{X}) \otimes \widetilde{E}$. We again denote by $\bar{\partial}_b : \Omega^{0,q}(\widetilde{X}, \widetilde{E}) \to \Omega^{0,q+1}(\widetilde{X}, \widetilde{E})$ the tangential Cauchy–Riemann operator. Then again $\widetilde{T}\bar{\partial}_b = \bar{\partial}_b\widetilde{T}$ on $\Omega^{0,\bullet}(\widetilde{X}, \widetilde{E})$. We denote by $L^2(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})$ and $L^2(\widetilde{X}, T^{*0,q}\widetilde{X})$ the completions of $\Omega^{0,q}(\widetilde{X}, \widetilde{E})$ and $\Omega^{0,q}(\widetilde{X})$ with respect to the corresponding pull-back metrics $(\cdot\,|\,\cdot)_{\widetilde{E}}$ and $(\cdot\,|\,\cdot)$. Similarly, we denote by $L^2(\widetilde{X}, T^{*0,\bullet}\widetilde{X} \otimes \widetilde{E})$ and $L^2(\widetilde{X}, T^{*0,\bullet}\widetilde{X})$ the completions of $\Omega^{0,\bullet}(\widetilde{X}, \widetilde{E})$ and $\Omega^{0,\bullet}(\widetilde{X})$ with respect to the corresponding pull-back metrics $(\cdot\,|\,\cdot)_{\widetilde{E}}$ and $(\cdot\,|\,\cdot)$.

As usual, for every $m \in \mathbb{Z}$, let

$$
\begin{aligned}
&\Omega^{0,q}_m(\widetilde{X}, \widetilde{E}) := \{u \in \Omega^{0,q}(\widetilde{X}, \widetilde{E}) \mid \widetilde{T}u = imu\}, \quad q = 0, 1, 2, \dots, n, \\
&\Omega^{0,\bullet}_m(\widetilde{X}, \widetilde{E}) := \{u \in \Omega^{0,\bullet}(\widetilde{X}, \widetilde{E}) \mid \widetilde{T}u = imu\}.
\end{aligned} \tag{2-8}
$$

For each $m \in \mathbb{Z}$, we denote by $L^2_m(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})$ and $L^2_m(\widetilde{X}, T^{*0,q}\widetilde{X})$ the completions of $\Omega^{0,q}_m(\widetilde{X}, \widetilde{E})$ and $\Omega^{0,q}_m(\widetilde{X})$ with respect to the corresponding pull-back metrics $(\cdot \mid \cdot)_{\widetilde{E}}$ and $(\cdot \mid \cdot)$. Similarly, we denote by $L^2_m(\widetilde{X}, T^{*0,\bullet}\widetilde{X} \otimes \widetilde{E})$ and $L^2_m(\widetilde{X}, T^{*0,\bullet}\widetilde{X})$ the completions of $\Omega^{0,\bullet}_m(\widetilde{X}, \widetilde{E})$ and $\Omega^{0,\bullet}_m(\widetilde{X})$ with respect to the corresponding pull-back metrics $(\cdot \mid \cdot)_{\widetilde{E}}$ and $(\cdot \mid \cdot)$.

Recall that $U \subset \widetilde{X}$ is called a fundamental domain of the action of Γ on $\widetilde{X}$ if the following conditions hold:

$$
\begin{aligned}
&(1) \quad \widetilde{X} = \textstyle\bigcup_{\gamma \in \Gamma} \gamma(\overline{U}), \\
&(2) \quad \gamma_1(U) \cap \gamma_2(U) = \varnothing \quad \text{for } \gamma_1, \gamma_2 \in \Gamma,\ \gamma_1 \neq \gamma_2, \\
&(3) \quad \overline{U} \setminus U \quad \text{is of measure 0.}
\end{aligned} \tag{2-9}
$$

We can take U to be S^1-invariant and with the pull-back S^1-action $e^{i\theta}$. We construct such a fundamental domain in the following: From the discussion in the proof of [Cheng et al. 2019, Theorem 2.11], we can find local trivializations $W_1, \dots, W_N$ such that $X = \bigcup_{j=1}^N W_j$ and each W_j is S^1-invariant. For each j, let $\widetilde{W}_j \subset \widetilde{X}$ be an S^1-invariant open set such that $\pi : \widetilde{W}_j \to W_j$ is a diffeomorphism and a CR map with inverse $\varphi_j : W_j \to \widetilde{W}_j$. Define $U_j = W_j \setminus \big(\bigcup_{i<j} \overline{W}_i \cap W_j\big)$. Then $U := \bigcup_j \varphi_j(U_j)$ is the fundamental domain we want.

It is easy to see that

$$
L^2(\widetilde{X}, \widetilde{E}) \simeq L^2\Gamma \otimes L^2(U, \widetilde{E}) \simeq L^2\Gamma \otimes L^2(X, E). \tag{2-10}
$$

We then have a unitary action of Γ by left translations on $L^2\Gamma$ by $t_\gamma \delta_\eta = \delta_{\gamma\eta}$, where $\{\delta_\eta \mid \eta \in \Gamma\}$ is the orthonormal basis of $L^2\Gamma$ formed by the delta functions. It induces a unitary action of Γ on $L^2(\widetilde{X}, \widetilde{E})$ by $\gamma \mapsto T_\gamma = t_\gamma \otimes \mathrm{Id}$.

Let us recall the definition of the von Neumann dimension or Γ-dimension of a Γ-module $V \subset L^2(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})$, see also [Ma and Marinescu 2007, Definition 3.6.1]. We shall denote by $\mathscr{L}(A)$ the space of bounded operators of the Hilbert space H. Let $\mathscr{A}_\Gamma \subset \mathscr{L}(L^2\Gamma)$ be the algebra of operators which commute with all left translations and denote the unit element of Γ by e. We define $\mathrm{Tr}_\Gamma[A] := \langle A\delta_e, \delta_e \rangle$, $A \in \mathscr{A}_\Gamma$. Note that a Γ-module is a left Γ-invariant subspace $V \subset L^2\Gamma$. The orthogonal projection P_V on V is in $\mathscr{A}_\Gamma$ for a Γ-module V. Set $\dim_\Gamma V := \mathrm{Tr}_\Gamma[P_V]$. Now we replace $L^2\Gamma$ by $L^2(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})$. Then to any operator $A \in \mathscr{L}(L^2(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E}))$, we associate operators $a_{\gamma\eta} \in \mathscr{L}(L^2(U, T^{*0,q}\widetilde{X} \otimes \widetilde{E}))$ such that $a_{\gamma\eta}(f)$ is the projection of $A(\delta_\gamma \otimes f)$ on $\mathbb{C}\delta_\eta \otimes L^2(U, T^{*0,q}\widetilde{X} \otimes \widetilde{E})$. In addition, if $A \in \mathscr{A}_\Gamma$ and A is positive, then $a_{\gamma\eta} = a_{e,\gamma^{-1}\eta}$ and

$$\mathrm{Tr}_\Gamma[A] := \mathrm{Tr}[a_{ee}] \geq 0,$$

is well-defined. The orthogonal projection P_V on $V \subset L^2(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})$ is in $\mathscr{A}_\Gamma$ for a Γ-module V.

Definition 2.7. The von Neumann dimension or Γ-dimension of a Γ-module V is defined by

$$\dim_\Gamma V := \mathrm{Tr}_\Gamma[P_V].$$

3. Asymptotic expansion of heat kernels of Kohn Laplacians

In this section, we recall the definition of heat kernels. Then we give a new version of asymptotic expansions of heat kernels of Kohn Laplacians.

3A. ***Asymptotics of heat kernels of Kohn Laplacians on a compact CR manifold.*** Since $T\bar\partial_b = \bar\partial_b T$ and E is a rigid CR vector bundle with a rigid Hermitian metric, we have

$$\bar\partial_{b,m} := \bar\partial_b : \Omega^{0,\bullet}_m(X, E) \to \Omega^{0,\bullet}_m(X, E), \quad \text{for all } m \in \mathbb{Z}.$$

The m-th Fourier component of Kohn–Rossi cohomology is given by

$$H^q_{b,m}(X, E) := \frac{\mathrm{Ker}\,\bar\partial_b : \Omega^{0,q}_m(X, E) \to \Omega^{0,q+1}_m(X, E)}{\mathrm{Im}\,\bar\partial_b : \Omega^{0,q-1}_m(X, E) \to \Omega^{0,q}_m(X, E)}. \tag{3-1}$$

We also write

$$\bar\partial^*_b : \Omega^{0,\bullet}(X, E) \to \Omega^{0,\bullet}(X, E)$$

to denote the formal adjoint of $\bar\partial_b$ with respect to $(\,\cdot\,|\,\cdot\,)_E$. Since $\langle\,\cdot\,|\,\cdot\,\rangle_E$ and $\langle\,\cdot\,|\,\cdot\,\rangle$ are rigid, we can check that

$$\begin{aligned} T\bar\partial^*_b &= \bar\partial^*_b T, && \text{on } \Omega^{0,\bullet}(X, E), \\ \bar\partial^*_{b,m} &:= \bar\partial^*_b : \Omega^{0,\bullet}_m(X, E) \to \Omega^{0,\bullet}_m(X, E), && \text{for all } m \in \mathbb{Z}. \end{aligned} \tag{3-2}$$

Now, we fix $m \in \mathbb{Z}$. The m-th Fourier component of Kohn Laplacian is given by

$$\Box_{b,m} := (\bar\partial_{b,m} + \bar\partial^*_{b,m})^2 : \Omega^{0,\bullet}_m(X, E) \to \Omega^{0,\bullet}_m(X, E). \tag{3-3}$$

We extend $\Box_{b,m}$ to $L^2_m(X, T^{*0,\bullet}X \otimes E)$ by

$$\Box_{b,m} : \mathrm{Dom}\,\Box_{b,m} \subset L^2_m(X, T^{*0,\bullet}X \otimes E) \to L^2_m(X, T^{*0,\bullet}X \otimes E), \tag{3-4}$$

where $\mathrm{Dom}\,\Box_{b,m} := \{u \in L^2_m(X, T^{*0,\bullet}X \otimes E) \mid \Box_{b,m}u \in L^2_m(X, T^{*0,\bullet}X \otimes E)\}$, where for any $u \in L^2_m(X, T^{*0,\bullet}X \otimes E)$, $\Box_{b,m}u$ is defined in the sense of distributions. We recall the following results (see Section 3 in [Cheng et al. 2019]).

Theorem 3.1. *The Kohn Laplacian* $\Box_{b,m}$ *is self-adjoint*, $\operatorname{Spec}\Box_{b,m}$ *is a discrete subset of* $[0,\infty[$ *and for every* $\nu\in\operatorname{Spec}\Box_{b,m}$, ν *is an eigenvalue of* $\Box_{b,m}$ *with finite multiplicity.*

For every $\nu\in\operatorname{Spec}\Box_{b,m}$, let $\{f_1^\nu,\dots,f_{d_\nu}^\nu\}$ be an orthonormal frame for the eigenspace of $\Box_{b,m}$ with eigenvalue ν. The heat kernel $e^{-t\Box_{b,m}}(x,y)$ is given by

$$e^{-t\Box_{b,m}}(x,y)=\sum_{\nu\in\operatorname{Spec}\Box_{b,m}}\sum_{j=1}^{d_\nu}e^{-\nu t}f_j^\nu(x)\otimes(f_j^\nu(y))^\dagger, \tag{3-5}$$

where $f_j^\nu(x)\otimes(f_j^\nu(y))^\dagger$ denotes the linear map:

$$\begin{aligned}f_j^\nu(x)\otimes(f_j^\nu(y))^\dagger : T_y^{*0,\bullet}X\otimes E_y&\to T_x^{*0,\bullet}X\otimes E_x,\\ u(y)\in T_y^{*0,\bullet}X\otimes E_y&\to f_j^\nu(x)\langle u(y)\mid f_j^\nu(y)\rangle_E\in T_x^{*0,\bullet}X\otimes E_x.\end{aligned}$$

Let $e^{-t\Box_{b,m}}:L^2(X,T^{*0,\bullet}X\otimes E)\to L^2_m(X,T^{*0,\bullet}X\otimes E)$ be the continuous operator with distribution kernel $e^{-t\Box_{b,m}}(x,y)$.

We denote by $\dot{\mathcal{R}}$ the Hermitian matrix $\dot{\mathcal{R}}\in\operatorname{End}(T^{1,0}X)$ such that for $V,W\in T^{1,0}X$,

$$i\,d\omega_0(V,\overline{W})=\langle\dot{\mathcal{R}}V\mid W\rangle. \tag{3-6}$$

Let $\{\omega_j\}_{j=1}^n$ be a local orthonormal frame of $T^{1,0}X$ with dual frame $\{\omega^j\}_{j=1}^n$. Set

$$\gamma_d=-i\sum_{l,j=1}^n d\omega_0(\omega_j,\bar\omega_l)\bar\omega^l\wedge\iota_{\bar\omega_j}, \tag{3-7}$$

where $\iota_{\bar\omega_j}$ denotes the interior product of $\bar\omega_j$. Then $\gamma_d\in\operatorname{End}(T^{*0,\bullet}X)$ and $-i\,d\omega_0$ acts as the derivative γ_d on $T^{*0,\bullet}X$. If we choose $\{\omega_j\}_{j=1}^n$ to be an orthonormal basis of $T^{1,0}X$ such that

$$\dot{\mathcal{R}}(x)=\operatorname{diag}(a_1(x),\dots,a_n(x))\in\operatorname{End}(T_x^{1,0}X), \tag{3-8}$$

then

$$\gamma_d(x)=-\sum_{j=1}^n a_j(x)\bar\omega^j\wedge\iota_{\bar\omega_j}. \tag{3-9}$$

Define $\det\dot{\mathcal{R}}(x):=a_1(x)\cdots a_n(x)$.

Fix $x,y\in X$. Let $d(x,y)$ denote the standard Riemannian distance of x and y with respect to the given Hermitian metric. Take ζ

$$o<\zeta<\inf\left\{\frac{2\pi}{p_k},\ \left|\frac{2\pi}{p_r}-\frac{2\pi}{p_{r+1}}\right|,\ r=1,\dots,k-1\right\}.$$

For $x\in X$, put

$$\hat d(x,X_{\mathrm{sing}}):=\inf\left\{d(x,e^{-i\theta}x)\mid \zeta\le\theta\le\frac{2\pi}{p}-\zeta\right\}.$$

The following result generalizes Theorem 3.1 in [Hsiao and Huang 2019].

Theorem 3.2. *With the above notation and assumptions, for every* $\varepsilon > 0$, *there are* $m_0 > 0$, $\varepsilon_0 > 0$ *and* $C > 0$ *such that for all* $m \geq m_0$, *we have*

$$\left| e^{-(t/m)\Box_{b,m}}(x,x) - \sum_{s=1}^{p} e^{(2\pi(s-1)/p)mi} (2\pi)^{-n-1} m^n \frac{\det(\dot{\mathcal{R}})\exp(t\gamma_d)}{\det(1-\exp(-t\dot{\mathcal{R}}))}(x) \otimes \mathrm{Id}_{E_x} \right| \leq \varepsilon m^n + C m^n t^{-n} e^{-\varepsilon_0 m \hat{d}(x, X_{\mathrm{sing}})^2/t}, \quad \text{for all } (t,x) \in \mathbb{R}_+ \times X_{\mathrm{reg}}. \tag{3-10}$$

Proof. We use the notation from Section 3 in [Hsiao and Huang 2019]. Recall that Γ_m is defined in [Hsiao and Huang 2019, (3.31)] (see also (3-29)). For $x \in X_{\mathrm{reg}}$, we have

$$\begin{aligned} \Gamma_m(t,x,x) &= \frac{1}{2\pi}\sum_{j=1}^{N}\int_0^{2\pi} H_{j,m}(t,x,e^{iu}\circ x)e^{imu}\,du \\ &= \frac{1}{2\pi}\sum_{s=1}^{p} e^{\frac{2\pi(s-1)}{p}mi}\sum_{j=1}^{N}\int_0^{2\pi/p} H_{j,m}(t,x,e^{iu}\circ x)e^{imu}\,du \\ &= \frac{1}{2\pi}\sum_{s=1}^{p} e^{\frac{2\pi(s-1)}{p}mi}\sum_{j=1}^{N}\int_{u\in[\zeta,(2\pi/p)-\zeta]} H_{j,m}(t,x,e^{iu}\circ x)e^{imu}\,du \\ &\qquad + \frac{1}{2\pi}\sum_{s=1}^{p} e^{\frac{2\pi(s-1)}{p}mi}\sum_{j=1}^{N}\int_{-\zeta}^{\zeta} H_{j,m}(t,x,e^{iu}\circ x)e^{imu}\,du, \end{aligned} \tag{3-11}$$

where $H_{j,m}$ is defined in [Hsiao and Huang 2019, (3.30)] (see also (3-29)). From [Hsiao and Huang 2019, (3.29), (3.34); Cheng et al. 2019, (6.4)], there are $\varepsilon_0 > 0$ and C_0 independent of j, x, m, t such that, for all $t \in \mathbb{R}_+$ and for all $m \in \mathbb{N}$, we have

$$\left| \frac{1}{2\pi}\int_{u\in[\zeta,\frac{2\pi}{p}-\zeta]} H_{j,m}(t,x,e^{iu}\circ x)e^{imu}\,du \right| \leq C_0 m^n t^{-n} e^{-\varepsilon_0 m \hat{d}(x,X_{\mathrm{sing}})^2/t}. \tag{3-12}$$

Then the proof is completed by applying [Hsiao and Huang 2019, (3.32), (3.39)] and (3-12). □

Remark 3.3. It is easy to check that

$$\sum_{s=1}^{p} e^{\frac{2\pi(s-1)}{p}mi} = \begin{cases} p & p \mid m \\ 0 & p \nmid m. \end{cases} \tag{3-13}$$

3B. ***BRT trivializations.*** To prove Theorem 3.2, we need some preparation. We first need the following result due to Baouendi, Rothschild and Treves.

Theorem 3.4 [Baouendi et al. 1985]. *For every point $x_0 \in X$, we can find local coordinates $x = (x_1, \dots, x_{2n+1}) = (z, \theta) = (z_1, \dots, z_n, \theta)$, $z_j = x_{2j-1} + i x_{2j}$, $j = 1, \dots, n$, $x_{2n+1} = \theta$, defined in some small neighborhood*

$$D = \{(z, \theta) : |z| < \delta, -\varepsilon_0 < \theta < \varepsilon_0\}$$

of x_0, $\delta > 0$, $0 < \varepsilon_0 < \pi$, such that $(z(x_0), \theta(x_0)) = (0, 0)$ and

$$T = \frac{\partial}{\partial \theta}, \quad Z_j = \frac{\partial}{\partial z_j} + i \frac{\partial \varphi}{\partial z_j}(z) \frac{\partial}{\partial \theta}, \quad j = 1, \dots, n, \tag{3-14}$$

where $Z_j(x)$, $j = 1, \dots, n$, form a basis of $T_x^{1,0}X$, for each $x \in D$, and $\varphi(z) \in C^\infty(D, \mathbb{R})$ is independent of θ. We call $(D, (z, \theta), \varphi)$ a BRT trivialization.

By using a BRT trivialization, we get another way to define Tu, for all $u \in \Omega^{0,q}(X)$. Let $(D, (z, \theta), \varphi)$ be a BRT trivialization. It is clear that

$$\{d\overline{z_{j_1}} \wedge \dots \wedge d\overline{z_{j_q}} \mid 1 \le j_1 < \dots < j_q \le n\}$$

is a basis for $T_x^{*0,q}X$, for every $x \in D$. Let $u \in \Omega^{0,q}(X)$. On D, we write

$$u = \sum_{1 \le j_1 < \dots < j_q \le n} u_{j_1 \cdots j_q} d\overline{z_{j_1}} \wedge \dots \wedge d\overline{z_{j_q}}. \tag{3-15}$$

Then, on D, we can check that

$$Tu = \sum_{1 \le j_1 < \dots < j_q \le n} (Tu_{j_1 \cdots j_q}) d\overline{z_{j_1}} \wedge \dots \wedge d\overline{z_{j_q}} \tag{3-16}$$

and Tu is independent of the choice of BRT trivializations. Note that, on the BRT trivialization $(D, (z, \theta), \varphi)$, we have

$$\bar\partial_b = \sum_{j=1}^n d\bar z_j \wedge \left(\frac{\partial}{\partial \bar z_j} - i \frac{\partial \varphi}{\partial \bar z_j}(z) \frac{\partial}{\partial \theta} \right). \tag{3-17}$$

3C. ***Local heat kernels on BRT trivializations.*** Until further notice, we fix $m \in \mathbb{Z}$. Let $B := (D, (z, \theta), \varphi)$ be a BRT trivialization. We may assume that $D = U \times]-\varepsilon, \varepsilon[$, where $\varepsilon > 0$ and U is an open set of $\mathbb{C}^n$. Since E is rigid, we can consider E as a holomorphic vector bundle over U. We may assume that E is trivial on U. Consider a trivial line bundle $L \to U$ with nontrivial Hermitian fiber metric $|1|^2_{h^L} = e^{-2\varphi}$. Let $(L^m, h^{L^m}) \to U$ be the m-th power of (L, h^L). Let $\Omega^{0,q}(U, E \otimes L^m)$ and $\Omega^{0,q}(U, E)$ be the spaces of $(0, q)$ forms on U with values in $E \otimes L^m$ and E, respectively, $q = 0, 1, 2, \ldots, n$. Put

$$\Omega^{0,\bullet}(U, E \otimes L^m) := \bigoplus_{j \in \{0,1,\ldots,n\}} \Omega^{0,j}(U, E \otimes L^m),$$
$$\Omega^{0,\bullet}(U, E) := \bigoplus_{j \in \{0,1,\ldots,n\}} \Omega^{0,j}(U, E).$$

Since L is trivial, from now on, we identify $\Omega^{0,\bullet}(U, E)$ with $\Omega^{0,\bullet}(U, E \otimes L^m)$. Since the Hermitian fiber metric $\langle \cdot \,|\, \cdot \rangle_E$ is rigid, we can consider $\langle \cdot \,|\, \cdot \rangle_E$ as a Hermitian fiber metric on the holomorphic vector bundle E over U. Let $\langle \cdot\,, \cdot \rangle$ be the Hermitian metric on $\mathbb{C}TU$ given by

$$\left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right\rangle = \left\langle \frac{\partial}{\partial z_j} + i\frac{\partial \varphi}{\partial z_j}(z)\frac{\partial}{\partial \theta} \,\middle|\, \frac{\partial}{\partial z_k} + i\frac{\partial \varphi}{\partial z_k}(z)\frac{\partial}{\partial \theta} \right\rangle, \quad j, k = 1, 2, \ldots, n.$$

$\langle \cdot\,, \cdot \rangle$ induces a Hermitian metric on $T^{*0,\bullet}U := \bigoplus_{j=0}^n T^{*0,j}U$, where $T^{*0,j}U$ is the bundle of $(0, j)$ forms on U, $j = 0, 1, \ldots, n$. We shall also denote this induced Hermitian metric on $T^{*0,\bullet}U$ by $\langle \cdot\,, \cdot \rangle$. The Hermitian metrics on $T^{*0,\bullet}U$ and E induce a Hermitian metric on $T^{*0,\bullet}U \otimes E$. We shall also denote this induced metric by $\langle \cdot \,|\, \cdot \rangle_E$. Let $(\cdot\,, \cdot)$ be the L^2 inner product on $\Omega^{0,\bullet}(U, E)$ induced by $\langle \cdot\,, \cdot \rangle$, $\langle \cdot \,|\, \cdot \rangle_E$. Similarly, let $(\cdot\,, \cdot)_m$ be the L^2 inner product on $\Omega^{0,\bullet}(U, E \otimes L^m)$ induced by $\langle \cdot\,, \cdot \rangle$, $\langle \cdot \,|\, \cdot \rangle_E$ and h^{L^m}.

The curvature of L induced by h^L is given by $R^L := 2\partial\bar{\partial}\varphi$. Let $\dot{R}^L \in \mathrm{End}(T^{1,0}U)$ be the Hermitian matrix given by

$$R^L(W, \overline{Y}) = \langle \dot{R}^L W, Y \rangle, \quad W, Y \in T^{1,0}U.$$

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{1,0}U$ with dual frame $\{w^j\}_{j=1}^n$. Set

$$\omega_d = -\sum_{l,j} R^L(w_j, \overline{w}_l)\overline{w}^l \wedge \iota_{\overline{w}_j}, \tag{3-18}$$

where $\iota_{\overline{w}_j}$ denotes the interior product of $\overline{w}_j$.

Let

$$\bar{\partial} : \Omega^{0,\bullet}(U, E \otimes L^m) \to \Omega^{0,\bullet}(U, E \otimes L^m)$$

be the Cauchy–Riemann operator and let

$$\bar{\partial}^{*,m} : \Omega^{0,\bullet}(U, E \otimes L^m) \to \Omega^{0,\bullet}(U, E \otimes L^m)$$

be the formal adjoint of $\bar{\partial}$ with respect to $(\cdot\,, \cdot)_m$. Put

$$\Box_{B,m} := (\bar{\partial} + \bar{\partial}^{*,m})^2 : \Omega^{0,\bullet}(U, E \otimes L^m) \to \Omega^{0,\bullet}(U, E \otimes L^m). \tag{3-19}$$

We need the following result (see Lemma 5.1 in [Cheng et al. 2019])

Lemma 3.5. *Let $u \in \Omega^{0,\bullet}_m(X, E)$. On D, we write*

$$u(z,\theta) = e^{im\theta}\widetilde{u}(z), \quad \widetilde{u}(z) \in \Omega^{0,\bullet}(U, E).$$

Then,

$$e^{-m\varphi}\Box_{B,m}(e^{m\varphi}\widetilde{u}) = e^{-im\theta}\Box_{b,m}(u). \tag{3-20}$$

Let $z, w \in U$ and let $T(z, w) \in (T^{*0,\bullet}_w U \otimes E_w) \boxtimes (T^{*0,\bullet}_z U \otimes E_z)$. We write $|T(z, w)|$ to denote the standard pointwise matrix norm of $T(z, w)$ induced by $\langle \cdot \,|\, \cdot \rangle_E$. Let $\Omega^{0,\bullet}_0(U, E)$ be the subspace of $\Omega^{0,\bullet}(U, E)$ whose elements have compact support in U. Let dv_U be the volume form on U induced by $\langle \cdot\,, \cdot \rangle$. Assume $T(z, w) \in C^\infty\big(U \times U, (T^{*0,\bullet}_w U \otimes E_w) \boxtimes (T^{*0,\bullet}_z U \otimes E_z)\big)$. Let $u \in \Omega^{0,\bullet}_0(U, E)$. We define the integral $\int T(z, w)u(w)\, dv_U(w)$ in the standard way. Let $G(t, z, w) \in C^\infty\big(\mathbb{R}_+ \times U \times U, (T^{*0,\bullet}_w U \otimes E_w) \boxtimes (T^{*0,\bullet}_z U \otimes E_z)\big)$. We write $G(t)$ to denote the continuous operator

$$G(t) : \Omega^{0,\bullet}_0(U, E) \to \Omega^{0,\bullet}(U, E), \quad u \mapsto \int G(t, z, w)u(w)\, dv_U(w)$$

and we write $G'(t)$ to denote the continuous operator

$$G'(t) : \Omega^{0,\bullet}_0(U, E) \to \Omega^{0,\bullet}(U, E), \quad u \mapsto \int \frac{\partial G(t, z, w)}{\partial t} u(w)\, dv_U(w).$$

We consider the heat operator of $\Box_{B,m}$. By using the standard Dirichlet heat kernel construction (see [Grigoryan and Saloff-Coste 2002]) and the proofs of Theorem 1.6.1 and Theorem 5.5.9 in [Ma and Marinescu 2007], we deduce the following:

Theorem 3.6. *There is*

$$A_{B,m}(t, z, w) \in C^\infty\big(\mathbb{R}_+ \times U \times U, (T^{*0,\bullet}_w U \otimes E_w) \boxtimes (T^{*0,\bullet}_z U \otimes E_z)\big)$$

such that

$$\begin{aligned} &\lim_{t\to 0+} A_{B,m}(t) = I, \quad \textit{in } D'(U, T^{*0,\bullet}U \otimes E), \\ &A'_{B,m}(t)u + \frac{1}{m} A_{B,m}(t)(\Box_{B,m}u) = 0, \quad \textit{for all } u \in \Omega^{0,\bullet}_0(U, E),\ t > 0, \end{aligned} \tag{3-21}$$

and $A_{B,m}(t, z, w)$ satisfies the following:

(I) *For every compact set $K \Subset U$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}^n_0$, there are constants $C_{\alpha_1,\alpha_2,\beta_1,\beta_2,K} > 0$ and $\varepsilon_0 > 0$ independent of t and m such that*

$$\begin{aligned} &\big|\partial^{\alpha_1}_z \partial^{\alpha_2}_{\bar z} \partial^{\beta_1}_w \partial^{\beta_2}_{\bar w} \big(A_{B,m}(t, z, w) e^{m(\varphi(w)-\varphi(z))}\big)\big| \\ &\qquad \le C_{\alpha_1,\alpha_2,\beta_1,\beta_2,K} \Big(\frac{m}{t}\Big)^{n+|\alpha_1|+|\alpha_2|+|\beta_1|+|\beta_2|} e^{-m\varepsilon_0|z-w|^2/t}, \\ &\qquad\qquad \textit{for all } (t, z, w) \in \mathbb{R}_+ \times K \times K. \end{aligned} \tag{3-22}$$

(II) $A_{B,m}(t,z,z)$ *admits an asymptotic expansion*:

$$A_{B,m}(t,z,z) = (2\pi)^{-n} m^n \frac{\det(\dot{R}^L)\exp(t\omega_d)}{\det(1-\exp(-t\dot{R}^L))}(z) \otimes \mathrm{Id}_{E_z} + o(m^n) \tag{3-23}$$

in $C^\ell\big(U, \mathrm{End}(T^{*0,\bullet}U) \otimes E\big)$ *locally uniformly on* $\mathbb{R}_+ \times U$, *for every* $\ell \in \mathbb{N}$. *Here we use the convention that if an eigenvalue* $a_j(z)$ *of* $\dot{R}^L(z)$ *is zero, then its contribution for* $\big(\det(\dot{R}^L)/\det(1-\exp(-t\dot{R}^L))\big)(z)$ *is* $\frac{1}{t}$.

3D. L^2 *Kohn–Rossi cohomology on a covering manifold.* Let

$$\widetilde{\Box}_b : \mathrm{Dom}\,\widetilde{\Box}_b \subset L^2(\widetilde{X}, T^{*0,\bullet}\widetilde{X}) \to L^2(\widetilde{X}, T^{*0,\bullet}\widetilde{X})$$

be the Gaffney extension of the pull-back Kohn Laplacian on $\widetilde{X}$. By a result of Gaffney, $\widetilde{\Box}_b$ is a positive self-adjoint operator (see Proposition 3.1.2 in [Ma and Marinescu 2007]). That is, $\widetilde{\Box}_b$ is self-adjoint and the spectrum of $\widetilde{\Box}_b$ is contained in $\overline{\mathbb{R}}_+$. Now, we fix $m \in \mathbb{Z}$. As in (3-3), we introduce the m-th Fourier component of the Kohn Laplacian $\widetilde{\Box}_{b,m}$ on $\Omega^{0,\bullet}_m(\widetilde{X}, \widetilde{E})$. We can easily see that $\widetilde{\Box}_{b,m}$ is also self-adjoint. By the second isomorphism of (2-10), we can see that, for any $\gamma \in \Gamma$,

$$T_\gamma(\mathrm{Dom}(\widetilde{\Box}_{b,m})) \subset \mathrm{Dom}(\widetilde{\Box}_{b,m}), \quad T_\gamma \widetilde{\Box}_{b,m} = \widetilde{\Box}_{b,m} T_\gamma \quad \text{on } \mathrm{Dom}(\widetilde{\Box}_{b,m}). \tag{3-24}$$

Consider the spectral resolution $E^q_\lambda(\widetilde{\Box}_{b,m})$ of $\widetilde{\Box}_{b,m}$ acting on $L^2_m(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})$. (See [Ma and Marinescu 2007, Appendix C.2]). The proof of the following lemma is similar to Lemma 3.6.3 in [Ma and Marinescu 2007].

Lemma 3.7. *For any* $q = 0, 1, \dots, n$ *and* $\lambda \in \mathbb{R}$, *then* $E^q_\lambda(\widetilde{\Box}_{b,m})$ *commutes with* Γ, *its Schwartz kernel is smooth and*

$$\dim_\Gamma E^q_\lambda(\widetilde{\Box}_{b,m}) < +\infty.$$

Proof. By (2-10) and (3-24), we can see that, for any $\lambda \in \mathbb{R}$, $E^q_\lambda(\widetilde{\Box}_{b,m})$ commutes with Γ. We claim that $\widetilde{\Box}_{b,m} - \widetilde{T}^2 \equiv \Delta$ is a second order elliptic operator, so is $\Delta - m^2$. Its principal symbol is locally written as

$$\sigma_\Delta(\tilde{x}, \xi) = \sigma_{\widetilde{\Box}_{b,m}}(\tilde{x}, \xi) - \sigma_{\widetilde{T}^2}(\tilde{x}, \xi) = \sum_{j=1}^n |\sigma_{L_j}(\tilde{x}, \xi)|^2 - \sigma_{\widetilde{T}}(\tilde{x}, \xi)^2,$$

where $\xi = (\xi_1, \dots, \xi_{2n}, \xi_{2n+1})$ and $\{L_j\}$ is an orthonormal basis of $T^{0,1}_x\widetilde{X}$. It is well-known that the characteristic manifold of $\widetilde{\Box}_b$ is

$$\Sigma = \{(\tilde{x}, c\widetilde{\omega}_0(\tilde{x})) \in T^*\widetilde{X} \mid c \neq 0\}.$$

It means that $\sigma_{\widetilde{\Box}_{b,m}}(\tilde{x}, \xi) > 0$ if and only if $(\xi_1, \dots, \xi_{2n}) \neq 0$. Meanwhile, in a local BRT coordinate [Baouendi et al. 1985], we have $\widetilde{T} = \frac{\partial}{\partial\theta}$, then $\sigma_{\widetilde{T}} = i\xi_{2n+1}$. That is, $\sigma_{\widetilde{T}^2} = -\xi^2_{2n+1}$. Then the claim is proved. By the spectral theorem, see [Ma and Marinescu 2007, Theorem C.2.1], we have $\mathrm{Im}(E_\lambda(\Delta - m^2)) \subset \mathrm{Dom}((\Delta - m^2)^k)$

for $k \in \mathbb{N}$. Using the uniform Sobolev spaces [Shubin 1995, pp. 511–512], it is easy to see that $\operatorname{Im}(E_\lambda(\Delta - m^2)) \subset \Omega^\bullet(\widetilde{X}, \widetilde{E})$, so that

$$E_\lambda(\Delta - m^2) : L^2(\widetilde{X}, T^{*0,\bullet}\widetilde{X} \otimes \widetilde{E}) \to \Omega^\bullet(\widetilde{X}, \widetilde{E})$$

is linear continuous. Hence,

$$\begin{aligned}\operatorname{Im} E_\lambda(\widetilde{\Box}_{b,m}) &= \operatorname{Im}(E_\lambda(\Delta - m^2)) \cap L^2_m(\widetilde{X}, T^{*0,\bullet}\widetilde{X} \otimes \widetilde{E}) \\ &\subset \Omega^\bullet(\widetilde{X}, \widetilde{E}) \cap L^2_m(\widetilde{X}, T^{*0,\bullet}\widetilde{X} \otimes \widetilde{E}) = \Omega^\bullet_m(\widetilde{X}, \widetilde{E})\end{aligned}$$

and $E_\lambda(\widetilde{\Box}_{b,m}) : L^2_m(\widetilde{X}, T^{*0,\bullet}\widetilde{X} \otimes \widetilde{E}) \to \Omega^\bullet_m(\widetilde{X}, \widetilde{E})$ is also linear continuous. By the Schwartz kernel theorem, the kernel $E_\lambda(\widetilde{\Box}_{b,m})(\tilde{x}, \tilde{x})$ of $E_\lambda(\widetilde{\Box}_{b,m})$ with respect to $dv_{\widetilde{X}}(\tilde{x})$ is smooth. By [Ma and Marinescu 2007, (3.6.12)],

$$\dim_\Gamma E_\lambda(\widetilde{\Box}_{b,m}) = \int_U \operatorname{Tr}[E_\lambda(\widetilde{\Box}_{b,m})(\tilde{x}, \tilde{x})]\, dv_{\widetilde{X}}(\tilde{x}) < +\infty. \qquad \Box$$

Definition 3.8. (a) The m-th Fourier component of the space of harmonic forms $\mathcal{H}^\bullet(\widetilde{X}, \widetilde{E})$ is defined by

$$\mathcal{H}^\bullet_{b,m}(\widetilde{X}, \widetilde{E}) := \operatorname{Ker}(\widetilde{\Box}_{b,m}) = \{s \in \operatorname{Dom} \widetilde{\Box}_{b,m} \mid \widetilde{\Box}_{b,m} s = 0\}.$$

(b) The m-th Fourier component of the q-th reduced L^2 Kohn–Rossi cohomology is given by

$$\overline{H}^q_{b,(2),m}(\widetilde{X}, \widetilde{E}) := \frac{\operatorname{Ker} \bar{\partial}_b \cap L^2_m(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})}{\left[\operatorname{Im} \bar{\partial}_b \cap L^2_m(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})\right]}, \tag{3-25}$$

where $[V]$ denotes the closure of the space V.

We can easily obtain the following weak Hodge decomposition

$$L^2_m(\widetilde{X}, T^{*0,\bullet}\widetilde{X} \otimes \widetilde{E}) = \mathcal{H}^\bullet(\widetilde{X}, \widetilde{E}) \oplus [\operatorname{Im}(\bar{\partial}_{b,m})] \oplus [\operatorname{Im}(\bar{\partial}^*_{b,m})] \tag{3-26}$$

By (3-26), we get the isomorphism

$$\overline{H}^\bullet_{b,(2),m}(\widetilde{X}, \widetilde{E}) \cong \mathcal{H}^\bullet_b(\widetilde{X}, \widetilde{E}). \tag{3-27}$$

3E. ***Asymptotics of heat kernels of Kohn Laplacians on a covering manifold.*** Assume that $X = D_1 \bigcup D_2 \bigcup \cdots \bigcup D_N$, where $B_j := (D_j, (z, \theta), \varphi_j)$ is a BRT trivialization, for each j. We may assume that, for each j, $D_j = U_j \times]-2\delta_j, 2\tilde{\delta}_j[\subset \mathbb{C}^n \times \mathbb{R}$, $\delta_j > 0$, $\tilde{\delta}_j > 0$, $U_j = \{z \in \mathbb{C}^n \mid |z| < l_j\}$. For each j, put $\hat{D}_j = \hat{U}_j \times]-\delta_j/2, \tilde{\delta}_j/2[$, where $\hat{U}_j = \{z \in \mathbb{C}^n \mid |z| < l_j/2\}$. We may suppose that $X = \hat{D}_1 \bigcup \hat{D}_2 \bigcup \cdots \bigcup \hat{D}_N$.

Let $\{\psi_j\}$ be a partition of unity subordinate to $\{\hat{D}_j\}$. Then $\{\widetilde{\psi}_{\gamma,j} := \psi_i \circ \pi\}$ is a partition of unity subordinate to $\{\widetilde{D}_{\gamma,j}\}$, where $\pi^{-1}(\hat{D}_j) = \bigcup_{\gamma \in \Gamma} \widetilde{D}_{\gamma,j}$ and $\widetilde{D}_{\gamma_1,j}$ and $\widetilde{D}_{\gamma_2,j}$ are disjoint for $\gamma_1 \neq \gamma_2$. For each $\gamma \in \Gamma$ and each j, we have

$\widetilde{D}_{\gamma,j}=\widetilde{U}_{\gamma,j}\times\,]-\delta_{\gamma,j}/2,\ \tilde{\delta}_{\gamma,j}/2[$, where $\widetilde{U}_{\gamma,j}=\{z\in\mathbb{C}^n\mid |z|<l_{\gamma,j}/2\}$. Then $\widetilde{X}=\bigcup_{\gamma\in\Gamma}\bigcup_{j=1}^{N}\widetilde{D}_{\gamma,j}$.

Fix $\gamma\in\Gamma$ and $j=1,2,\dots,N$. Put

$$K_{\gamma,j}=\big\{z\in\widetilde{U}_{\gamma,j}\mid \text{there is a }\theta\in\,]-\delta_{\gamma,j}/2,\ \tilde{\delta}_{\gamma,j}/2[\text{ such that }\widetilde{\psi}_{\gamma,j}(z,\theta)\neq 0\big\}.$$

Let $\tau_{\gamma,j}(z)\in C_0^\infty(\widetilde{U}_{\gamma,j})$ with $\tau_{\gamma,j}\equiv 1$ on some neighborhood $W_{\gamma,j}$ of $K_{\gamma,j}$. Let $\sigma_{\gamma,j}\in C_0^\infty\big(]-\delta_{\gamma,j}/2,\ \tilde{\delta}_{\gamma,j}/2[\big)$ with $\int\sigma_{\gamma,j}(\theta)\,d\theta=1$. Let

$$\widetilde{A}_{B_{\gamma,j},m}(t,z,w)\in C^\infty\big(\mathbb{R}_+\times\widetilde{U}_{\gamma,j}\times\widetilde{U}_{\gamma,j},\ (T_w^{*0,\bullet}\widetilde{U}_{\gamma,j}\otimes\widetilde{E}_w)\boxtimes(T_z^{*0,\bullet}\widetilde{U}_{\gamma,j}\otimes\widetilde{E}_z)\big)$$

be as in Theorem 3.6. Put

$$\begin{aligned}\text{(3-28)}\quad &\widetilde{H}_{\gamma,j,m}(t,\tilde{x},\tilde{y})\\ &\qquad=\widetilde{\psi}_{\gamma,j}(\tilde{x})e^{-m\varphi_j(z)+im\theta}\widetilde{A}_{B_{\gamma,j},m}(t,z,w)e^{m\varphi_{\gamma,j}(w)-im\eta}\tau_{\gamma,j}(w)\sigma_{\gamma,j}(\eta),\end{aligned}$$

where $\tilde{x}=(z,\theta),\ \tilde{y}=(w,\eta)\in\mathbb{C}^n\times\mathbb{R}$. Let

$$\text{(3-29)}\qquad \widetilde{\Gamma}_m(t,\tilde{x},\tilde{y}):=\frac{1}{2\pi}\sum_{\gamma\in\Gamma}\sum_{j=1}^{N}\int_{-\pi}^{\pi}\widetilde{H}_{\gamma,j,m}(t,\tilde{x},e^{iu}\circ\tilde{y})e^{imu}\,du.$$

Note that when $\Gamma=\{e\}$, $\widetilde{\Gamma}_m(t,\tilde{x},\tilde{y})=\Gamma_m(t,\pi(\tilde{x}),\pi(\tilde{y}))$ is defined in [Hsiao and Huang 2019, (3.31)].

From Lemma 3.5, with off-diagonal estimates of $\widetilde{A}_{B_j,m}(t,\tilde{x},\tilde{y})$ (see (3-22)), we can repeat the proof of Theorem 5.14 in [Cheng et al. 2019] with minor change and deduce that

Theorem 3.9. *For every $\ell\in\mathbb{N},\ \ell\geq 2$, and every $M>0$, there are $\varepsilon_0>0$ and $m_0>0$ independent of t and m such that for every $m\geq m_0$, we have*

$$\text{(3-30)}\qquad \big\|e^{-\frac{t}{m}\widetilde{\Box}_{b,m}}(\tilde{x},\tilde{y})-\widetilde{\Gamma}_m(t,\tilde{x},\tilde{y})\big\|_{C^l(\widetilde{X}\times\widetilde{X})}\leq e^{-\frac{m}{t}\varepsilon_0},\quad \textit{for all } t\in(0,M).$$

From [Ma and Marinescu 2007, Theorem 3.6.4], we have

Proposition 3.10. *For any $t_0>0$, $\varepsilon>0$ and any $\gamma\in\Gamma$, $j=1,2,\dots,N$, there exists $C>0$ such that for any $z\in\widetilde{U}_{\gamma,j}$, $m\in\mathbb{N}$, $t>t_0$,*

$$\big\|\widetilde{A}_{B_{\gamma,j},m}(t,z,z)-A_{B_j,m}(t,\pi(z),\pi(z))\big\|_{C^l(\widetilde{U}_{\gamma,j}\times\widetilde{U}_{\gamma,j})}\leq C\exp\Big(-\frac{m}{32t}\varepsilon\Big).$$

From (3-11) (see [Hsiao and Huang 2019, (3.31)]), (3-28), (3-29), Proposition 3.10 and the fact that $\widetilde{\psi}_{\gamma,j}=\psi_j\circ\pi$, we can easily deduce:

Lemma 3.11. *With the notation and assumptions as in Theorem 3.9, we have*

$$\big\|\widetilde{\Gamma}_m(t,\tilde{x},\tilde{x})-\Gamma_m(t,\pi(\tilde{x}),\pi(\tilde{x}))\big\|_{C^l(\widetilde{X}\times\widetilde{X})}\leq C\exp\Big(-\frac{m}{t}\varepsilon_0\Big).$$

From Theorem 3.9, Lemma 3.11 and Theorem 3.5 of [Hsiao and Huang 2019], we have:

Theorem 3.12. *For every $\ell \in \mathbb{N}$, $\ell \geq 2$, and every $M > 0$, there are $\varepsilon_0 > 0$ and $m_0 > 0$ independent of t and m such that for any $\tilde{x} \in \widetilde{X}$ and $m \geq m_0$, we have*

$$\left\| e^{-\frac{t}{m}\widetilde{\Box}_{b,m}}(\tilde{x},\tilde{x}) - e^{-\frac{t}{m}\Box_{b,m}}(\pi(\tilde{x}),\pi(\tilde{x})) \right\|_{C^l(\widetilde{X}\times\widetilde{X})} \leq C\exp\Big(-\frac{m}{t}\varepsilon_0\Big), \quad \text{for all } t \in (0, M).$$

By Theorems 3.2 and 3.12, we have:

Theorem 3.13. *With the above notations and assumptions, for every $\varepsilon > 0$, there are $m_0 > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that for all $m \geq m_0$, we have*

$$\left| e^{-\frac{t}{m}\widetilde{\Box}_{b,m}}(\tilde{x},\tilde{x}) - \sum_{s=1}^{p} e^{\frac{2\pi(s-1)}{p}mi}(2\pi)^{-n-1}m^n \frac{\det(\dot{\mathcal{R}})\exp(t\gamma_d)}{\det(1-\exp(-t\dot{\mathcal{R}}))}(\pi(\tilde{x}))\otimes \mathrm{Id}_{E_{\pi(\tilde{x})}} \right| \leq \varepsilon m^n + Cm^n t^{-n} e^{-\varepsilon_0 m \hat{d}(\pi(\tilde{x}),X_{\mathrm{sing}})^2/t}, \quad \text{for all } (t,\tilde{x}) \in \mathbb{R}_+ \times \widetilde{X}_{\mathrm{reg}}. \tag{3-31}$$

Recall that since Γ acts on $\widetilde{X}$ freely so that $\widetilde{X}/\Gamma = X$, hence, $\widetilde{X}_{\mathrm{reg}}/\Gamma = X_{\mathrm{reg}}$.

4. Heat kernel proof

In this section, we will present the heat kernel proof of the main theorem.

We denote by $\mathrm{Tr}_{\Gamma,q}$ the Γ-trace of operators acting on $L^2_m(\widetilde{X}, T^{*0,q}\widetilde{X}\otimes\widetilde{E})$, see Section 2C or [Ma and Marinescu 2007, Subsection 3.6.1].

Lemma 4.1. *For any $t > 0$, $m \in \mathbb{N}$, $0 \leq q \leq n$, we have*

$$\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) \leq \sum_{j=0}^{q}(-1)^{q-j}\mathrm{Tr}_{\Gamma,j}\Big[\exp\Big(-\frac{t}{m}\widetilde{\Box}_{b,m}\Big)\Big], \tag{4-1}$$

with equality for $q = n$.

Proof. Let $E^{j,m}_\lambda$ be the spectral resolution of $\widetilde{\Box}_{b,m}$ acting on $L^2_m(\widetilde{X}, T^{*0,q}\widetilde{X}\otimes\widetilde{E})$. We consider the projectors $E^{j,m}(]\lambda_1,\lambda_2]) = E^{j,m}_{\lambda_2} - E^{j,m}_{\lambda_1}$, where $\lambda_2 > \lambda_1 \geq 0$. Then, by the Hodge decomposition (3-26), $\sum_{j=0}^{q}(-1)^{q-j}E^{j,m}(]\lambda_1,\lambda_2])$ is the projection on the range of $\bar{\partial}_{b,m}E^{q,m}(]\lambda_1,\lambda_2])$ and thus a positive operator. Hence the Γ-invariant measure $\sum_{j=0}^{q}(-1)^{q-j}dE^{j,m}_\lambda$ is positive on $\{\lambda > 0\}$. It follows that

$$R := \int_{\lambda>0} e^{-\frac{t}{m}\lambda}\sum_{j=0}^{q}(-1)^{q-j}dE^{j,m}_\lambda \geq 0, \tag{4-2}$$

and R commutes with Γ. On the other hand,

$$\mathrm{Tr}_{\Gamma,j}\Big[\exp\Big(-\frac{t}{m}\widetilde{\Box}_{b,m}\Big)\Big] = \dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) + \mathrm{Tr}_\Gamma\int_{\lambda>0} e^{-\frac{t}{m}\lambda}\,dE^{j,m}_\lambda. \tag{4-3}$$

By (4-2) and (4-3), we obtain the result. ∎

Let $\mathrm{Tr}_q\big[\exp\big(-\frac{t}{m}\Box_{b,m}\big)\big]$ be the trace of the operator $\exp\big(-\frac{t}{m}\Box_{b,m}\big)$ acting on $\Omega_m^{0,q}(X, E)$. It is well-known (see Theorem 8.10 in [Roe 1998]) that

$$\mathrm{Tr}_q\Big[\exp\Big(-\frac{t}{m}\Box_{b,m}\Big)\Big] = \int_X \mathrm{Tr}_q\Big[\exp\Big(-\frac{t}{m}\Box_{b,m}\Big)(x, x)\Big]\, dv_X(x). \tag{4-4}$$

By [Ma and Marinescu 2007, (3.6.7) and (3.6.8)], as in (4-4), we obtain:

Proposition 4.2. *We have*

$$\mathrm{Tr}_{\Gamma,q}\Big[\exp\Big(-\frac{t}{m}\widetilde{\Box}_{b,m}\Big)\Big] = \int_U \mathrm{Tr}_q\big[e^{-\frac{t}{m}\widetilde{\Box}_{b,m}}(\tilde{x}, \tilde{x})\big]\, dv_{\widetilde{X}}(\tilde{x}). \tag{4-5}$$

Now we are in a position to give the heat kernel proof of the Morse inequalities for the Fourier components of reduced L^2 Kohn–Rossi cohomology.

Proof of Theorem 1.1. Denote by $\mathrm{Tr}_{\Lambda^{0,q}}$ the trace on $T^{*0,q}X$. The basis for $T^{*0,q}X$ is

$$\big\{\bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q} \mid j_1 < \cdots < j_q\big\}. \tag{4-6}$$

We write for the index $(1, \ldots, q)$

$$\begin{aligned}\exp(t\gamma_d)(\bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^q) &= \prod_{j=1}^{q}(1 + (e^{-ta_j} - 1)\bar{\omega}^j \wedge \iota_{\bar{\omega}^j})(\bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^q) \\ &= \sum_{k_1<\cdots<k_q} c_{k_1\cdots k_q}(x)\bar{\omega}^{k_1} \wedge \cdots \wedge \bar{\omega}^{k_q}.\end{aligned} \tag{4-7}$$

From direct calculations, we see that

$$c_{1\cdots q}(x) = \exp\Big(-t\sum_{j=1}^{q} a_j(x)\Big). \tag{4-8}$$

Then we have

$$\mathrm{Tr}_{\Lambda^{0,q}}[\exp(t\gamma_d)] = \sum_{j_1<\cdots<j_q} \exp\Big(-t\sum_{i=1}^{q} a_{j_i}(x)\Big). \tag{4-9}$$

Hence

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathrm{Tr}_{\Lambda^{0,q}}[\exp(t\gamma_d)]}{\det(1 - \exp(-t\dot{\mathcal{R}}))} &= \lim_{t\to\infty} \frac{\sum_{j_1<\cdots<j_q} \exp\big(-t\sum_{i=1}^{q} a_{j_i}(x)\big)}{\prod_{j=1}^{n}(1 - \exp(-ta_j(x)))} \\ &= (-1)^q 1_{X(q)},\end{aligned} \tag{4-10}$$

where the function $X(q)$ is defined by 1 on $X(q)$, 0 otherwise. As usual, for $\tilde{x} \in \widetilde{X}$, $\pi(\tilde{x}) = x \in X$. It follows from Theorem 3.13, (4-5) and Lemma 4.1 that

$$\begin{aligned}
&\frac{1}{m^n}\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) \\
&\le \frac{1}{m^n}\sum_{j=0}^{q}(-1)^{q-j}\operatorname{Tr}_{\Gamma,q}\Big[\exp\Big(-\frac{t}{m}\widetilde{\Box}_{b,m}\Big)\Big] \\
&= \frac{1}{m^n}\sum_{j=0}^{q}(-1)^{q-j}\int_U \operatorname{Tr}_{\Gamma,q}\Big[\exp\Big(-\frac{t}{m}\widetilde{\Box}_{b,m}(\tilde{x},\tilde{x})\Big)\Big]\,dv_{\widetilde{X}}(\tilde{x}) \\
&\le (2\pi)^{-n-1}\sum_{s=1}^{p} e^{\frac{2\pi(s-1)}{p}mi}\sum_{j=0}^{q}(-1)^{q-j} \\
&\qquad\times\int_X \frac{\det(\dot{\mathcal{R}})\operatorname{Tr}_{\Lambda^{0,q}}[\exp(t\gamma_d)\otimes \mathrm{Id}_{E_x}]}{\det(1-\exp(-t\dot{\mathcal{R}}))}\,dv_X(x) \\
&\quad+\varepsilon\sum_{j=0}^{q}(-1)^{q-j}\operatorname{Vol}(X)+C\sum_{j=0}^{q}(-1)^{q-j}\int_X t^{-n}e^{-\varepsilon_0 m\hat{d}(x_0,X_{\mathrm{sing}})^2/t}\,dv_X(x).
\end{aligned} \tag{4-11}$$

Note that ε is arbitrarily small. By the dominant convergence theorem with $t \to \infty$, we have

$$\begin{aligned}
&\limsup_{m\to\infty}\frac{1}{m^n}\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) \\
&\qquad\le \frac{pr}{(2\pi)^{n+1}}\sum_{j=0}^{q}(-1)^{q-j}\int_{X(j)}|\det(\dot{\mathcal{R}})|\,dv_X(x), \quad \text{for } p\mid m, \\
&\limsup_{m\to\infty}\frac{1}{m^n}\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) = 0, \qquad \text{for } p\nmid m.
\end{aligned} \tag{4-12}$$

From Definition 2.2, (3-6) and (4-12), we finally get

$$\begin{aligned}
&\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) \\
&\qquad\le \frac{prm^n}{2\pi^{n+1}}\sum_{j=0}^{q}(-1)^{q-j}\int_{X(j)}|\det(\mathcal{L}_x)|\,dv_X(x)+o(m^n), \quad \text{for } p\mid m, \\
&\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E}) = o(m^n), \qquad \text{for } p\nmid m.
\end{aligned} \tag{4-13}$$

Let $q = n$ in (4-1), by applying Theorem 3.13, we obtain for $p \mid m$,

$$\begin{aligned}(4\text{-}14)\quad &\frac{1}{m^n}\sum_{j=0}^{n}(-1)^{n-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E})\\ &\geq \frac{1}{m^n}\sum_{j=0}^{n}(-1)^{n-j}\int_U \mathrm{Tr}_{\Gamma,j}\Big[\exp\Big(-\frac{t}{m}\widetilde{\Box}_{b,m}(\tilde{x},\tilde{x})\Big)\Big]\,dv_{\widetilde{X}}(\tilde{x})\\ &\geq (2\pi)^{-n-1}p\sum_{j=0}^{n}(-1)^{n-j}\int_X \frac{\det(\dot{\mathcal{R}})\,\mathrm{Tr}_{\Lambda^{0,j}}[\exp(t\gamma_d)\otimes \mathrm{Id}_{E_x}]}{\det(1-\exp(-t\dot{\mathcal{R}}))}\,dv_X(x)\\ &\qquad -\varepsilon n\,\mathrm{Vol}(X) - Cn\int_X t^{-n}e^{-\varepsilon_0 m\hat{d}(x_0,X_{\mathrm{sing}})^2/t}\,dv_X(x).\end{aligned}$$

Note that ε is arbitrarily small. By the dominant convergence theorem with $t \to \infty$, we have

$$\begin{aligned}(4\text{-}15)\quad &\liminf_{m\to\infty,\, p\mid m}\frac{1}{m^n}\sum_{j=0}^{n}(-1)^{n-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E})\\ &\qquad\geq \frac{pr}{(2\pi)^{n+1}}\sum_{j=0}^{n}(-1)^{n-j}\int_{X(j)}|\det(\dot{\mathcal{R}})|\,dv_X(x).\end{aligned}$$

Then

$$\begin{aligned}(4\text{-}16)\quad &\liminf_{m\to\infty,\, p\mid m}\frac{1}{m^n}\sum_{j=0}^{n}(-1)^{n-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E})\\ &\qquad= \frac{pr}{(2\pi)^{n+1}}\sum_{j=0}^{n}(-1)^{n-j}\int_{X(j)}|\det(\dot{\mathcal{R}})|\,dv_X(x).\end{aligned}$$

We finally get

$$\begin{aligned}(4\text{-}17)\quad &\sum_{j=0}^{n}(-1)^{n-j}\dim_\Gamma \overline{H}^j_{b,(2),m}(\widetilde{X},\widetilde{E})\\ &\qquad= \frac{prm^n}{2\pi^{n+1}}\sum_{j=0}^{n}(-1)^{n-j}\int_{X(j)}|\det(\mathcal{L}_x)|\,dv_X(x)+o(m^n)\quad \text{for } p\mid m.\end{aligned}$$

Then the proof is completed. □

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