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**MORSE INEQUALITIES FOR FOURIER COMPONENTS
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CR COVERING MANIFOLDS WITH S^1 -ACTION**

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Let X be a compact connected CR manifold of dimension $2n + 1$, $n \geq 1$. Let \tilde{X} be a paracompact CR manifold with a transversal CR S^1 -action, such that there is a discrete group Γ acting freely on \tilde{X} having $X = \tilde{X}/\Gamma$. Based on an asymptotic formula for the Fourier components of the heat kernel with respect to the S^1 -action, we establish the Morse inequalities for Fourier components of reduced L^2 -Kohn–Rossi cohomology with values in a rigid CR vector bundle over \tilde{X} . As a corollary, we obtain the Morse inequalities for Fourier components of Kohn–Rossi cohomology on X which were obtained by Hsiao and Li (2016) by using Szegő kernel method.

1. Introduction and statement of the results

Gromov, Henkin and Shubin [Gromov et al. 1998, Theorem 0.2] considered covering manifolds that are strongly pseudoconvex of complex manifolds and analyzed the holomorphic L^2 -functions on the coverings. Todor, Chiose and Marinescu [Todor et al. 2001] generalized in a similar manner the Morse inequalities of Siu [1984] and Demailly [1985] on coverings of complex manifolds; they also considered coverings of weakly pseudoconvex domains in [Marinescu et al. 2002]. The study of problems on CR manifolds with S^1 -action became active recently, see [Cheng et al. 2019; Hsiao 2018; Hsiao and Huang 2019; Hsiao and Li 2016; 2018; Hsiao and Shao 2019]. In particular, Hsiao and Li [2016] established the Morse inequalities for Fourier components of Kohn–Rossi cohomology on X by using the Szegő kernel method. In [Hsiao and Marinescu 2012] general Morse inequalities for CR bundles are proved, generalizing [Getzler 1989]. Inspired by the results of [Gromov et al. 1998; Hsiao and Li 2016; Todor et al. 2001; Siu 1984; Demailly 1985], we establish Morse inequalities for Fourier components of reduced L^2 -Kohn–Rossi cohomology

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with values in a rigid CR vector bundle on a covering manifold over a compact connected CR manifold with S^1 -action. This generalizes the results of [Hsiao and Li 2016] to CR covering manifolds with S^1 -action. We present a proof by the heat kernel method, which is inspired by Bismut’s proof [1987; Ma and Marinescu 2007] of the holomorphic Morse inequalities. The crucial estimate for Fourier components of the heat kernel of Kohn Laplacians was given in [Hsiao and Huang 2019].

Now we formulate the main results. We refer to other sections for notation and definitions (see Definition 2.1, 2.2, 2.3, 2.5 and (3-1), (3-25)) used here. Let X be a compact connected CR manifold of dimension $2n + 1$, $n \geq 1$ with a transversal CR S^1 -action $e^{i\theta}$ on X . For $x \in X$, we say that the period of x is $\frac{2\pi}{\ell}$, $\ell \in \mathbb{N}$, if $e^{i\theta} \circ x \neq x$, for every $0 < \theta < \frac{2\pi}{\ell}$, and $e^{i2\pi/\ell} \circ x = x$. For each $\ell \in \mathbb{N}$, put

$$(1-1) \quad X_\ell = \{x \in X \mid \text{the period of } x \text{ is } \frac{2\pi}{\ell}\}$$

and let

$$(1-2) \quad p = \min\{\ell \in \mathbb{N} \mid X_\ell \neq \emptyset\}.$$

It is well-known that if X is connected, then X_p is an open and dense subset of X (see [Duistermaat and Heckman 1982, p. 265]). Assume $X = X_{p_1} \cup X_{p_2} \cup \dots \cup X_{p_k}$, $p =: p_1 < p_2 < \dots < p_k$. Set $X_{\text{reg}} := X_p$. We call $x \in X_{\text{reg}}$ a regular point of the S^1 action. Let X_{sing} be the complement of X_{reg} .

Let \tilde{X} be a paracompact CR manifold, such that there is a discrete group Γ acting freely on \tilde{X} having $X = \tilde{X}/\Gamma$. Let $\pi : \tilde{X} \rightarrow X$ be the natural projection with the pull-back map $\pi^* : TX \rightarrow T\tilde{X}$. Then \tilde{X} admits a pull-back CR structure $T^{1,0}\tilde{X} := \pi^*T^{1,0}X$ and, hence, a CR manifold. We assume that \tilde{X} admits a transversal CR locally free S^1 action, denote by $e^{i\theta}$. We further assume that the map

$$\Gamma \times \tilde{X} \rightarrow \tilde{X}, \quad (\gamma, \tilde{x}) \mapsto \gamma \circ \tilde{x}, \quad \text{for all } \tilde{x} \in \tilde{X}, \gamma \in \Gamma.$$

is CR, see (2-6), and

$$e^{i\theta} \circ \gamma \circ \tilde{x} = \gamma \circ e^{i\theta} \circ \tilde{x}, \quad \text{for all } \tilde{x} \in \tilde{X}, \theta \in [0, 2\pi[, \gamma \in \Gamma.$$

Let $\tilde{E} := \pi^*E$ be the pull-pack bundle of a rigid CR vector bundle E over X . Then \tilde{E} is a Γ -invariant rigid CR vector bundle over \tilde{X} . We denote by \tilde{X}_{reg} the set of regular points of the S^1 -action on \tilde{X} . Note that since Γ acts on \tilde{X} freely so that $\tilde{X}/\Gamma = X$, hence, we have $\tilde{X}_{\text{reg}}/\Gamma = X_{\text{reg}} = X_p$. We denote by $X(q)$ a subset of X such that

$$X(q) := \{x \in X \mid \mathcal{L}_x \text{ has exactly } q \text{ negative eigenvalues} \\ \text{and } n - q \text{ positive eigenvalues}\}.$$

We refer to Section 2 for more details. Our main theorem is:

Theorem 1.1. *With the above notation and assumptions, as $m \rightarrow \infty$, for $q = 0, 1, \dots, n$, the m -th Fourier components of reduced L^2 -Kohn–Rossi cohomology (see (3-25)) satisfy the following strong Morse inequalities:*

$$(1-3) \quad \sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \leq \begin{cases} \frac{prm^n}{2\pi^{n+1}} \sum_{j=0}^q (-1)^{q-j} \int_{X^{(j)}} |\det(\mathcal{L}_x)| dv_X(x) + o(m^n), & \text{for } p \mid m, \\ \sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) = o(m^n), & \text{for } p \nmid m, \end{cases}$$

where r denotes the rank of \tilde{E} , \dim_{Γ} denotes the von Neumann dimension (see Section 2C, [Ma and Marinescu 2007, §3.6.1] or [Atiyah 1976, §3]) and \mathcal{L}_x is the Levi form at $x \in X$. When $p \mid m$, $q = n$, as $m \rightarrow \infty$, we have the asymptotic Riemann–Roch–Hirzebruch theorem:

$$(1-4) \quad \sum_{j=0}^n (-1)^j \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) = \frac{prm^n}{2\pi^{n+1}} \sum_{j=0}^n (-1)^j \int_{X^{(j)}} |\det(\mathcal{L}_x)| dv_X(x) + o(m^n).$$

In particular, we get the weak Morse inequalities,

$$(1-5) \quad \dim_{\Gamma} \bar{H}_{b,(2),m}^q(\tilde{X}, \tilde{E}) \leq \frac{prm^n}{2\pi^{n+1}} \int_{X^{(q)}} |\det(\mathcal{L}_x)| dv_X(x) + o(m^n).$$

By the standard argument in [Hsiao and Li 2016; Hsiao et al. 2017], we deduce easily the following Grauert–Riemenschneider criterion on coverings of CR manifolds.

Corollary 1.2. *With the notation and assumptions of Theorem 1.1, we also assume that X is weakly pseudoconvex and strongly pseudoconvex at a point. Then*

$$(1-6) \quad \dim_{\Gamma} \bar{H}_{b,(2),m}^0(\tilde{X}, \tilde{E}) \approx m^n, \quad \text{for } p \mid m.$$

In particular, $\dim_{\Gamma} \bar{H}_{b,(2),m}^0(\tilde{X}, \tilde{E}) = \infty$.

When $\Gamma = \{e\}$, $p = 1$ and \tilde{E} is a trivial line bundle, we deduce the following Morse inequalities of Hsiao and Li [2016, Theorems 2.2 and 2.5].

Corollary 1.3. *With the above notation and assumptions, as $m \rightarrow \infty$, for $q = 0, 1, \dots, n$, the m -th Fourier components of Kohn–Rossi cohomology satisfy the*

following strong Morse inequalities:

$$(1-7) \quad \sum_{j=0}^q (-1)^{q-j} \dim H_{b,m}^j(X) \leq \frac{m^n}{2\pi^{n+1}} \sum_{j=0}^q (-1)^{q-j} \int_{X^{(j)}} |\det(\mathcal{L}_x)| dv_X(x) + o(m^n),$$

where \mathcal{L}_x is the Levi form at $x \in X$. In particular, we get the weak Morse inequalities,

$$(1-8) \quad \dim H_{b,m}^q(X) \leq \frac{m^n}{2\pi^{n+1}} \int_{X^{(q)}} |\det(\mathcal{L}_x)| dv_X(x) + o(m^n).$$

Let X be a compact CR manifold of dimension $2n+1$, $n \geq 1$. A classical theorem due to Boutet de Monvel [1975] asserts that X can be globally CR embedded into \mathbb{C}^N , for some $N \in \mathbb{N}$, when X is strongly pseudoconvex with dimension $n \geq 1$. Epstein [1992] proved that if X is strongly pseudoconvex with dimension 3 and a global free transversal CR S^1 -action, then X can be embedded into \mathbb{C}^N by positive Fourier components of CR functions. Corollary 1.3 guarantees the abundance of positive Fourier components of CR functions to do embedding in general cases (e.g., the S^1 -action can be only locally free). The proofs of Hsiao and Li [2016], include localization of analytic objects (eigenfunctions, Szegő kernels), Kohn L^2 estimates and scaling techniques. A more general version of Corollary 1.3 (with X being weakly pseudoconvex) is proved by Cheng, Hsiao and Tsai [Cheng et al. 2019, Proposition 1.20 and Corollary 1.21] in a different way. By using the Morse inequalities, (1-7) and (1-8), Hsiao and Li [2016, Theorem 2.6] proved that there are abundant CR functions on X when X is weakly pseudoconvex and strongly pseudoconvex at a point. Corollary 1.2 generalizes Theorem 2.6 of [Hsiao and Li 2016] to CR covering manifolds.

This paper is organized as follows. In Section 2 we introduce some basic notation, terminology and definitions. In Section 3 we study the asymptotic behavior of heat kernels of Kohn Laplacians. Section 4 is devoted to the heat kernel proof of the main theorem.

2. Preliminaries

2A. Some standard notation. We use the following notation: $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{R} is the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$, $\bar{\mathbb{R}}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we set $|\alpha| = \alpha_1 + \dots + \alpha_n$. For $x = (x_1, \dots, x_n)$ we write

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Let $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, be coordinates of \mathbb{C}^n . We write

$$\begin{aligned} z^\alpha &= z_1^{\alpha_1} \cdots z_n^{\alpha_n}, & \bar{z}^\alpha &= \bar{z}_1^{\alpha_1} \cdots \bar{z}_n^{\alpha_n}, \\ \partial_{z_j} &= \frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right), & \partial_{\bar{z}_j} &= \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}} \right), \\ \partial_z^\alpha &= \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}, & \partial_{\bar{z}}^\alpha &= \partial_{\bar{z}_1}^{\alpha_1} \cdots \partial_{\bar{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha}. \end{aligned}$$

Let X be a C^∞ orientable paracompact manifold. We let TX and T^*X denote the tangent bundle of X and the cotangent bundle of X , respectively. The complexified tangent bundle of X and the complexified cotangent bundle of X will be denoted by $\mathbb{C}TX$ and $\mathbb{C}T^*X$, respectively. We write $\langle \cdot, \cdot \rangle$ to denote the pointwise duality between T^*X and TX . We extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathbb{C}T^*X \times \mathbb{C}TX$. For $u \in \mathbb{C}T^*X$, $v \in \mathbb{C}TX$, we also write $u(v) := \langle u, v \rangle$.

Let $Y \subset X$ be an open set. The spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y, E)$ and $D'(Y, E)$, respectively.

2B. CR manifolds with S^1 -action. Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of X . That is, $T^{1,0}X$ is a subbundle of rank n of the complexified tangent bundle $\mathbb{C}TX$, satisfying

$$T^{1,0}X \cap T^{0,1}X = \{0\},$$

where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$. We assume that X admits a S^1 action: $S^1 \times X \rightarrow X$. We write $e^{i\theta}$ to denote the S^1 action. Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the S^1 action given by $(Tu)(x) = \frac{\partial}{\partial \theta}(u(e^{i\theta} \circ x))|_{\theta=0}$, $u \in C^\infty(X)$.

Definition 2.1. We say that the S^1 action $e^{i\theta}$ is CR if $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and the S^1 action is transversal if for each $x \in X$,

$$\mathbb{C}T(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathbb{C}T_xX.$$

Moreover, we say that the S^1 action is locally free if $T \neq 0$ everywhere.

Note that if the S^1 action is transversal, then it is locally free. We assume throughout that $(X, T^{1,0}X)$ is a connected CR manifold with a transversal CR S^1 action $e^{i\theta}$ and we let T be the global vector field induced by the S^1 action. Let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form determined by $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0, T \rangle = -1$.

Definition 2.2. For $p \in X$, the Levi form \mathcal{L}_p is the Hermitian quadratic form on $T_p^{1,0}X$ given by $\mathcal{L}_p(U, \bar{V}) = -\frac{1}{2i} \langle d\omega_0(p), U \wedge \bar{V} \rangle$, $U, V \in T_p^{1,0}X$.

Definition 2.3. If the Levi form \mathcal{L}_p is positive definite, we say that X is strongly pseudoconvex at p . If the Levi form is positive definite at every point of X , we say that X is strongly pseudoconvex.

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. Define the vector bundle of $(0, q)$ forms by $T^{*0,q}X = \Lambda^q(T^{*0,1}X)$. Put $T^{*0,\bullet}X := \bigoplus_{j \in \{0,1,\dots,n\}} T^{*0,j}X$. Let $D \subset X$ be an open subset. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega_0^{0,q}(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D . Put

$$\Omega^{0,\bullet}(D) := \bigoplus_{j \in \{0,1,\dots,n\}} \Omega^{0,j}(D), \quad \Omega_0^{0,\bullet}(D) := \bigoplus_{j \in \{0,1,\dots,n\}} \Omega_0^{0,j}(D).$$

Similarly, if E is a vector bundle over D , then we let $\Omega^{0,q}(D, E)$ denote the space of smooth sections of $T^{*0,q}X \otimes E$ over D and let $\Omega_0^{0,q}(D, E)$ be the subspace of $\Omega^{0,q}(D, E)$ whose elements have compact support in D . Put

$$\Omega^{0,\bullet}(D, E) := \bigoplus_{j \in \{0,1,\dots,n\}} \Omega^{0,j}(D, E), \quad \Omega_0^{0,\bullet}(D, E) := \bigoplus_{j \in \{0,1,\dots,n\}} \Omega_0^{0,j}(D, E).$$

Fix $\theta_0 \in]-\pi, \pi[$, θ_0 small. Let

$$de^{i\theta_0} : \mathbb{C}T_x X \rightarrow \mathbb{C}T_{e^{i\theta_0}x} X$$

denote the differential map of $e^{i\theta_0} : X \rightarrow X$. By the CR property of the S^1 action, we can check that

$$(2-1) \quad \begin{aligned} de^{i\theta_0} : T_x^{1,0}X &\rightarrow T_{e^{i\theta_0}x}^{1,0}X, & de^{i\theta_0} : T_x^{0,1}X &\rightarrow T_{e^{i\theta_0}x}^{0,1}X, \\ de^{i\theta_0}(T(x)) &= T(e^{i\theta_0}x). \end{aligned}$$

Let

$$(e^{i\theta_0})^* : \Lambda^j(\mathbb{C}T^*X) \rightarrow \Lambda^j(\mathbb{C}T^*X)$$

be the pull-back map by $e^{i\theta_0}$, $j = 0, 1, \dots, 2n + 1$. From (2-1), it is easy to see that for every $q = 0, 1, \dots, n$,

$$(2-2) \quad (e^{i\theta_0})^* : T_{e^{i\theta_0}x}^{*0,q}X \rightarrow T_x^{*0,q}X.$$

Let $u \in \Omega^{0,q}(X)$. Define

$$(2-3) \quad Tu := \frac{\partial}{\partial \theta} ((e^{i\theta})^* u)|_{\theta=0} \in \Omega^{0,q}(X).$$

For every $\theta \in \mathbb{R}$ and every $u \in C^\infty(X, \Lambda^j(\mathbb{C}T^*X))$, we write $u(e^{i\theta} \circ x) := (e^{i\theta})^* u(x)$.

Let $\bar{\partial}_b : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X)$ be the tangential Cauchy–Riemann operator. From the CR property of the S^1 action, it is straightforward to see that

$$T\bar{\partial}_b = \bar{\partial}_b T \quad \text{on } \Omega^{0,\bullet}(X).$$

Definition 2.4. Let $D \subset U$ be an open set. We say that a function $u \in C^\infty(D)$ is rigid if $Tu = 0$. We say that a function $u \in C^\infty(X)$ is Cauchy–Riemann (CR for short) if $\bar{\partial}_b u = 0$. We call u a rigid CR function if $\bar{\partial}_b u = 0$ and $Tu = 0$.

Definition 2.5. Let F be a complex vector bundle over X . We say that F is rigid (CR) if X can be covered with open sets U_j with trivializing frames $\{f_j^1, f_j^2, \dots, f_j^r\}$, $j = 1, 2, \dots$, such that the corresponding transition matrices are rigid (CR). The frames $\{f_j^1, f_j^2, \dots, f_j^r\}$, $j = 1, 2, \dots$, are called rigid (CR) frames.

Definition 2.6. Let F be a complex rigid vector bundle over X and let $\langle \cdot | \cdot \rangle_F$ be a Hermitian metric on F . We say that $\langle \cdot | \cdot \rangle_F$ is a rigid Hermitian metric if for every rigid local frame f_1, \dots, f_r of F , we have $T\langle f_j | f_k \rangle_F = 0$, for every $j, k = 1, 2, \dots, r$.

It is known that there is a rigid Hermitian metric on any rigid vector bundle F (see [Cheng et al. 2019, Theorem 2.10; Hsiao 2018, Theorem 10.5]). Note that Baouendi, Rothschild and Treves [Baouendi et al. 1985] proved that $T^{1,0}X$ is a rigid complex vector bundle over X .

From now on, let E be a rigid CR vector bundle over X and we take a rigid Hermitian metric $\langle \cdot | \cdot \rangle_E$ on E and take a rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle T | T \rangle = 1$. The Hermitian metrics on $\mathbb{C}TX$ and on E induce Hermitian metrics $\langle \cdot | \cdot \rangle$ and $\langle \cdot | \cdot \rangle_E$ on $T^{*0,\bullet}X$ and $T^{*0,\bullet}X \otimes E$, respectively. We denote by $dv_X = dv_X(x)$ the volume form on X induced by the fixed Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$. Then we get natural global L^2 inner products $(\cdot | \cdot)_E$, $(\cdot | \cdot)$ on $\Omega^{0,\bullet}(X, E)$ and $\Omega^{0,\bullet}(X)$, respectively. We denote by $L^2(X, T^{*0,q}X \otimes E)$ and $L^2(X, T^{*0,q}X)$ the completions of $\Omega^{0,q}(X, E)$ and $\Omega^{0,q}(X)$ with respect to $(\cdot | \cdot)_E$ and $(\cdot | \cdot)$, respectively. Similarly, we denote by $L^2(X, T^{*0,\bullet}X \otimes E)$ and $L^2(X, T^{*0,\bullet}X)$ the completions of $\Omega^{0,\bullet}(X, E)$ and $\Omega^{0,\bullet}(X)$ with respect to $(\cdot | \cdot)_E$ and $(\cdot | \cdot)$, respectively. We extend $(\cdot | \cdot)_E$ and $(\cdot | \cdot)$ to $L^2(X, T^{*0,\bullet}X \otimes E)$ and $L^2(X, T^{*0,\bullet}X)$ in the standard way, respectively. For $f \in L^2(X, T^{*0,\bullet}X \otimes E)$, we denote $(f | f)_E$ by $\|f\|_E^2$. Similarly, for $f \in L^2(X, T^{*0,\bullet}X)$, we denote $(f | f)$ by $\|f\|^2$.

We also write $\bar{\partial}_b$ to denote the tangential Cauchy–Riemann operator acting on forms with values in E :

$$\bar{\partial}_b : \Omega^{0,\bullet}(X, E) \rightarrow \Omega^{0,\bullet}(X, E).$$

Since E is rigid, we can also define Tu for every $u \in \Omega^{0,q}(X, E)$ and we have

$$(2-4) \quad T\bar{\partial}_b = \bar{\partial}_b T \quad \text{on } \Omega^{0,\bullet}(X, E).$$

For every $m \in \mathbb{Z}$, let

$$(2-5) \quad \begin{aligned} \Omega_m^{0,q}(X, E) &:= \{u \in \Omega^{0,q}(X, E) \mid Tu = imu\}, \quad q = 0, 1, 2, \dots, n, \\ \Omega_m^{0,\bullet}(X, E) &:= \{u \in \Omega^{0,\bullet}(X, E) \mid Tu = imu\}. \end{aligned}$$

For each $m \in \mathbb{Z}$, we denote by $L_m^2(X, T^{*0,q}X \otimes E)$ and $L_m^2(X, T^{*0,q}X)$ the completions of $\Omega_m^{0,q}(X, E)$ and $\Omega_m^{0,q}(X)$ with respect to $(\cdot | \cdot)_E$ and $(\cdot | \cdot)$, respectively. Similarly, we denote by $L_m^2(X, T^{*0,\bullet}X \otimes E)$ and $L_m^2(X, T^{*0,\bullet}X)$ the completions of $\Omega_m^{0,\bullet}(X, E)$ and $\Omega_m^{0,\bullet}(X)$ with respect to $(\cdot | \cdot)_E$ and $(\cdot | \cdot)$, respectively.

2C. Covering manifolds, von Neumann dimension. Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n + 1$, $n \geq 1$. Let \tilde{X} be a paracompact CR manifold, such that there is a discrete group Γ acting freely on \tilde{X} having $X = \tilde{X}/\Gamma$. Let $\pi : \tilde{X} \rightarrow X$ be the natural projection with the pull-back map $\pi^* : TX \rightarrow T\tilde{X}$. Then \tilde{X} admits a pull-back CR structure $T^{1,0}\tilde{X} := \pi^*T^{1,0}X$ and, hence, a CR manifold. We assume that \tilde{X} admits a transversal CR locally free S^1 action, denoted by $e^{i\theta}$. We further assume that the map

$$\Gamma \times \tilde{X} \rightarrow \tilde{X}, \quad (\gamma, \tilde{x}) \mapsto \gamma \circ \tilde{x}, \quad \text{for all } \tilde{x} \in \tilde{X}, \gamma \in \Gamma.$$

is CR, i.e.,

$$(2-6) \quad \gamma_*(T_{\tilde{x}}^{1,0}\tilde{X}) \subseteq T_{\gamma \cdot \tilde{x}}^{1,0}\tilde{X},$$

and

$$e^{i\theta} \circ \gamma \circ \tilde{x} = \gamma \circ e^{i\theta} \circ \tilde{x}, \quad \text{for all } \tilde{x} \in \tilde{X}, \theta \in [0, 2\pi[, \gamma \in \Gamma.$$

It is easy to see that the S^1 -action $e^{i\theta}$ on \tilde{X} induces a transversal CR locally free S^1 action, also denoted by $e^{i\theta}$. We denote by $\tilde{T} := \pi^*T$ the pull-back one form on \tilde{X} , then T is the global real vector field induced by the S^1 -action on X . Let $\tilde{\omega}_0 := \pi^*\omega_0$ be the pull-back one form on \tilde{X} , where ω_0 is the global real one form on X as defined in Section 2B. Then, for $\tilde{p} \in \tilde{X}$, the Levi form $\tilde{\mathcal{L}}_{\tilde{p}}$ is the Hermitian quadratic form on $T_{\tilde{p}}^{1,0}\tilde{X}$ given by

$$(2-7) \quad \tilde{\mathcal{L}}_{\tilde{p}}(\tilde{U}, \tilde{V}) = -\frac{1}{2i} \langle d\tilde{\omega}_0(\tilde{p}), \tilde{U} \wedge \tilde{V} \rangle = -\frac{1}{2i} \langle d\omega_0(\pi(\tilde{p})), \pi_*\tilde{U} \wedge \pi_*\tilde{V} \rangle,$$

where $\tilde{U}, \tilde{V} \in T_{\tilde{p}}^{1,0}\tilde{X}$.

As usual, let $\Omega^{0,q}(\tilde{X})$ denote the space of smooth sections of $\wedge^q(T^{*0,1}\tilde{X})$. We also denote by $\bar{\partial}_b : \Omega^{0,q}(\tilde{X}) \rightarrow \Omega^{0,q+1}(\tilde{X})$ the tangential Cauchy–Riemann operator. Then $\tilde{T}\bar{\partial}_b = \bar{\partial}_b\tilde{T}$ on $\Omega^{0,\bullet}(\tilde{X})$. Let E be a rigid CR vector bundle over X , then $\tilde{E} := \pi^*E$ is a Γ -invariant rigid CR vector bundle over \tilde{X} . Again let $\Omega^{0,q}(\tilde{X}, \tilde{E})$ denote the space of smooth sections of $\wedge^q(T^{*0,1}\tilde{X}) \otimes \tilde{E}$. We again denote by $\bar{\partial}_b : \Omega^{0,q}(\tilde{X}, \tilde{E}) \rightarrow \Omega^{0,q+1}(\tilde{X}, \tilde{E})$ the tangential Cauchy–Riemann operator. Then again $\tilde{T}\bar{\partial}_b = \bar{\partial}_b\tilde{T}$ on $\Omega^{0,\bullet}(\tilde{X}, \tilde{E})$. We denote by $L^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})$ and $L^2(\tilde{X}, T^{*0,q}\tilde{X})$ the completions of $\Omega^{0,q}(\tilde{X}, \tilde{E})$ and $\Omega^{0,q}(\tilde{X})$ with respect to the corresponding pull-back metrics $(\cdot | \cdot)_{\tilde{E}}$ and $(\cdot | \cdot)$. Similarly, we denote by $L^2(\tilde{X}, T^{*0,\bullet}\tilde{X} \otimes \tilde{E})$ and $L^2(\tilde{X}, T^{*0,\bullet}\tilde{X})$ the completions of $\Omega^{0,\bullet}(\tilde{X}, \tilde{E})$ and $\Omega^{0,\bullet}(\tilde{X})$ with respect to the corresponding pull-back metrics $(\cdot | \cdot)_{\tilde{E}}$ and $(\cdot | \cdot)$.

As usual, for every $m \in \mathbb{Z}$, let

$$(2-8) \quad \begin{aligned} \Omega_m^{0,q}(\tilde{X}, \tilde{E}) &:= \{u \in \Omega^{0,q}(\tilde{X}, \tilde{E}) \mid \tilde{T}u = imu\}, \quad q = 0, 1, 2, \dots, n, \\ \Omega_m^{0,\bullet}(\tilde{X}, \tilde{E}) &:= \{u \in \Omega^{0,\bullet}(\tilde{X}, \tilde{E}) \mid \tilde{T}u = imu\}. \end{aligned}$$

For each $m \in \mathbb{Z}$, we denote by $L_m^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})$ and $L_m^2(\tilde{X}, T^{*0,q}\tilde{X})$ the completions of $\Omega_m^{0,q}(\tilde{X}, \tilde{E})$ and $\Omega_m^{0,q}(\tilde{X})$ with respect to the corresponding pull-back metrics $(\cdot | \cdot)_{\tilde{E}}$ and $(\cdot | \cdot)$. Similarly, we denote by $L_m^2(\tilde{X}, T^{*0,\bullet}\tilde{X} \otimes \tilde{E})$ and $L_m^2(\tilde{X}, T^{*0,\bullet}\tilde{X})$ the completions of $\Omega_m^{0,\bullet}(\tilde{X}, \tilde{E})$ and $\Omega_m^{0,\bullet}(\tilde{X})$ with respect to the corresponding pull-back metrics $(\cdot | \cdot)_{\tilde{E}}$ and $(\cdot | \cdot)$.

Recall that $U \subset \tilde{X}$ is called a fundamental domain of the action of Γ on \tilde{X} if the following conditions hold:

$$(2-9) \quad \begin{aligned} (1) \quad & \tilde{X} = \bigcup_{\gamma \in \Gamma} \gamma(\bar{U}), \\ (2) \quad & \gamma_1(U) \cap \gamma_2(U) = \emptyset \quad \text{for } \gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2, \\ (3) \quad & \bar{U} \setminus U \text{ is of measure 0.} \end{aligned}$$

We can take U to be S^1 -invariant and with the pull-back S^1 -action $e^{i\theta}$. We construct such a fundamental domain in the following: From the discussion in the proof of [Cheng et al. 2019, Theorem 2.11], we can find local trivializations W_1, \dots, W_N such that $X = \bigcup_{j=1}^N W_j$ and each W_j is S^1 -invariant. For each j , let $\tilde{W}_j \subset \tilde{X}$ be an S^1 -invariant open set such that $\pi : \tilde{W}_j \rightarrow W_j$ is a diffeomorphism and a CR map with inverse $\varphi_j : W_j \rightarrow \tilde{W}_j$. Define $U_j = W_j \setminus (\bigcup_{i < j} \bar{W}_i \cap W_j)$. Then $U := \bigcup_j \varphi_j(U_j)$ is the fundamental domain we want.

It is easy to see that

$$(2-10) \quad L^2(\tilde{X}, \tilde{E}) \simeq L^2\Gamma \otimes L^2(U, \tilde{E}) \simeq L^2\Gamma \otimes L^2(X, E).$$

We then have a unitary action of Γ by left translations on $L^2\Gamma$ by $t_\gamma \delta_\eta = \delta_{\gamma\eta}$, where $\{\delta_\eta \mid \eta \in \Gamma\}$ is the orthonormal basis of $L^2\Gamma$ formed by the delta functions. It induces a unitary action of Γ on $L^2(\tilde{X}, \tilde{E})$ by $\gamma \mapsto T_\gamma = t_\gamma \otimes \text{Id}$.

Let us recall the definition of the von Neumann dimension or Γ -dimension of a Γ -module $V \subset L^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})$, see also [Ma and Marinescu 2007, Definition 3.6.1]. We shall denote by $\mathcal{L}(A)$ the space of bounded operators of the Hilbert space H . Let $\mathcal{A}_\Gamma \subset \mathcal{L}(L^2\Gamma)$ be the algebra of operators which commute with all left translations and denote the unit element of Γ by e . We define $\text{Tr}_\Gamma[A] := \langle A\delta_e, \delta_e \rangle$, $A \in \mathcal{A}_\Gamma$. Note that a Γ -module is a left Γ -invariant subspace $V \subset L^2\Gamma$. The orthogonal projection P_V on V is in \mathcal{A}_Γ for a Γ -module V . Set $\dim_\Gamma V := \text{Tr}_\Gamma[P_V]$. Now we replace $L^2\Gamma$ by $L^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})$. Then to any operator $A \in \mathcal{L}(L^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E}))$, we associate operators $a_{\gamma\eta} \in \mathcal{L}(L^2(U, T^{*0,q}\tilde{X} \otimes \tilde{E}))$ such that $a_{\gamma\eta}(f)$ is the projection of $A(\delta_\gamma \otimes f)$ on $\mathbb{C}\delta_\eta \otimes L^2(U, T^{*0,q}\tilde{X} \otimes \tilde{E})$. In

addition, if $A \in \mathcal{A}_\Gamma$ and A is positive, then $a_{\gamma\eta} = a_{e,\gamma^{-1}\eta}$ and

$$\mathrm{Tr}_\Gamma[A] := \mathrm{Tr}[a_{ee}] \geq 0,$$

is well-defined. The orthogonal projection P_V on $V \subset L^2(\widetilde{X}, T^{*0,q}\widetilde{X} \otimes \widetilde{E})$ is in \mathcal{A}_Γ for a Γ -module V .

Definition 2.7. The von Neumann dimension or Γ -dimension of a Γ -module V is defined by

$$\dim_\Gamma V := \mathrm{Tr}_\Gamma[P_V].$$

3. Asymptotic expansion of heat kernels of Kohn Laplacians

In this section, we recall the definition of heat kernels. Then we give a new version of asymptotic expansions of heat kernels of Kohn Laplacians.

3A. Asymptotics of heat kernels of Kohn Laplacians on a compact CR manifold.

Since $T\bar{\partial}_b = \bar{\partial}_b T$ and E is a rigid CR vector bundle with a rigid Hermitian metric, we have

$$\bar{\partial}_{b,m} := \bar{\partial}_b : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E), \quad \text{for all } m \in \mathbb{Z}.$$

The m -th Fourier component of Kohn–Rossi cohomology is given by

$$(3-1) \quad H_{b,m}^q(X, E) := \frac{\mathrm{Ker} \bar{\partial}_b : \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q+1}(X, E)}{\mathrm{Im} \bar{\partial}_b : \Omega_m^{0,q-1}(X, E) \rightarrow \Omega_m^{0,q}(X, E)}.$$

We also write

$$\bar{\partial}_b^* : \Omega^{0,\bullet}(X, E) \rightarrow \Omega^{0,\bullet}(X, E)$$

to denote the formal adjoint of $\bar{\partial}_b$ with respect to $(\cdot | \cdot)_E$. Since $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$ are rigid, we can check that

$$(3-2) \quad \begin{aligned} T\bar{\partial}_b^* &= \bar{\partial}_b^* T, && \text{on } \Omega^{0,\bullet}(X, E), \\ \bar{\partial}_{b,m}^* &:= \bar{\partial}_b^* : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E), && \text{for all } m \in \mathbb{Z}. \end{aligned}$$

Now, we fix $m \in \mathbb{Z}$. The m -th Fourier component of Kohn Laplacian is given by

$$(3-3) \quad \square_{b,m} := (\bar{\partial}_{b,m} + \bar{\partial}_{b,m}^*)^2 : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E).$$

We extend $\square_{b,m}$ to $L_m^2(X, T^{*0,\bullet}X \otimes E)$ by

$$(3-4) \quad \square_{b,m} : \mathrm{Dom} \square_{b,m} \subset L_m^2(X, T^{*0,\bullet}X \otimes E) \rightarrow L_m^2(X, T^{*0,\bullet}X \otimes E),$$

where $\mathrm{Dom} \square_{b,m} := \{u \in L_m^2(X, T^{*0,\bullet}X \otimes E) \mid \square_{b,m}u \in L_m^2(X, T^{*0,\bullet}X \otimes E)\}$, where for any $u \in L_m^2(X, T^{*0,\bullet}X \otimes E)$, $\square_{b,m}u$ is defined in the sense of distributions. We recall the following results (see Section 3 in [Cheng et al. 2019]).

Theorem 3.1. *The Kohn Laplacian $\square_{b,m}$ is self-adjoint, $\text{Spec } \square_{b,m}$ is a discrete subset of $[0, \infty[$ and for every $\nu \in \text{Spec } \square_{b,m}$, ν is an eigenvalue of $\square_{b,m}$ with finite multiplicity.*

For every $\nu \in \text{Spec } \square_{b,m}$, let $\{f_1^\nu, \dots, f_{d_\nu}^\nu\}$ be an orthonormal frame for the eigenspace of $\square_{b,m}$ with eigenvalue ν . The heat kernel $e^{-t\square_{b,m}}(x, y)$ is given by

$$(3-5) \quad e^{-t\square_{b,m}}(x, y) = \sum_{\nu \in \text{Spec } \square_{b,m}} \sum_{j=1}^{d_\nu} e^{-\nu t} f_j^\nu(x) \otimes (f_j^\nu(y))^\dagger,$$

where $f_j^\nu(x) \otimes (f_j^\nu(y))^\dagger$ denotes the linear map:

$$\begin{aligned} f_j^\nu(x) \otimes (f_j^\nu(y))^\dagger : T_y^{*0,\bullet} X \otimes E_y &\rightarrow T_x^{*0,\bullet} X \otimes E_x, \\ u(y) \in T_y^{*0,\bullet} X \otimes E_y &\rightarrow f_j^\nu(x) \langle u(y) | f_j^\nu(y) \rangle_E \in T_x^{*0,\bullet} X \otimes E_x. \end{aligned}$$

Let $e^{-t\square_{b,m}} : L^2(X, T^{*0,\bullet} X \otimes E) \rightarrow L_m^2(X, T^{*0,\bullet} X \otimes E)$ be the continuous operator with distribution kernel $e^{-t\square_{b,m}}(x, y)$.

We denote by $\dot{\mathcal{R}}$ the Hermitian matrix $\dot{\mathcal{R}} \in \text{End}(T^{1,0} X)$ such that for $V, W \in T^{1,0} X$,

$$(3-6) \quad id\omega_0(V, \bar{W}) = \langle \dot{\mathcal{R}}V | W \rangle.$$

Let $\{\omega_j\}_{j=1}^n$ be a local orthonormal frame of $T^{1,0} X$ with dual frame $\{\omega^j\}_{j=1}^n$. Set

$$(3-7) \quad \gamma_d = -i \sum_{l,j=1}^n d\omega_0(\omega_j, \bar{\omega}_l) \bar{\omega}^l \wedge \iota \bar{\omega}_j,$$

where $\iota_{\bar{\omega}_j}$ denotes the interior product of $\bar{\omega}_j$. Then $\gamma_d \in \text{End}(T^{*0,\bullet} X)$ and $-id\omega_0$ acts as the derivative γ_d on $T^{*0,\bullet} X$. If we choose $\{\omega_j\}_{j=1}^n$ to be an orthonormal basis of $T^{1,0} X$ such that

$$(3-8) \quad \dot{\mathcal{R}}(x) = \text{diag}(a_1(x), \dots, a_n(x)) \in \text{End}(T_x^{1,0} X),$$

then

$$(3-9) \quad \gamma_d(x) = - \sum_{j=1}^n a_j(x) \bar{\omega}^j \wedge \iota \bar{\omega}_j.$$

Define $\det \dot{\mathcal{R}}(x) := a_1(x) \cdots a_n(x)$.

Fix $x, y \in X$. Let $d(x, y)$ denote the standard Riemannian distance of x and y with respect to the given Hermitian metric. Take ζ

$$0 < \zeta < \inf \left\{ \frac{2\pi}{p_k}, \left| \frac{2\pi}{p_r} - \frac{2\pi}{p_{r+1}} \right|, r = 1, \dots, k-1 \right\}.$$

For $x \in X$, put

$$\hat{d}(x, X_{\text{sing}}) := \inf \left\{ d(x, e^{-i\theta} x) \mid \zeta \leq \theta \leq \frac{2\pi}{p} - \zeta \right\}.$$

The following result generalizes Theorem 3.1 in [Hsiao and Huang 2019].

Theorem 3.2. *With the above notation and assumptions, for every $\varepsilon > 0$, there are $m_0 > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that for all $m \geq m_0$, we have*

$$(3-10) \quad \left| e^{-(t/m)\square_{b,m}}(x, x) - \sum_{s=1}^p e^{(2\pi(s-1)/p)mi} (2\pi)^{-n-1} m^n \frac{\det(\dot{\mathcal{R}}) \exp(t\gamma_d)}{\det(1 - \exp(-t\dot{\mathcal{R}}))}(x) \otimes \text{Id}_{E_x} \right| \\ \leq \varepsilon m^n + C m^n t^{-n} e^{-\varepsilon_0 m \hat{d}(x, X_{\text{sing}})^2/t}, \quad \text{for all } (t, x) \in \mathbb{R}_+ \times X_{\text{reg}}.$$

Proof. We use the notation from Section 3 in [Hsiao and Huang 2019]. Recall that Γ_m is defined in [Hsiao and Huang 2019, (3.31)] (see also (3-29)). For $x \in X_{\text{reg}}$, we have

$$(3-11) \quad \Gamma_m(t, x, x) \\ = \frac{1}{2\pi} \sum_{j=1}^N \int_0^{2\pi} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du \\ = \frac{1}{2\pi} \sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \sum_{j=1}^N \int_0^{2\pi/p} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du \\ = \frac{1}{2\pi} \sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \sum_{j=1}^N \int_{u \in [\zeta, (2\pi/p) - \zeta]} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du \\ + \frac{1}{2\pi} \sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \sum_{j=1}^N \int_{- \zeta}^{\zeta} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du,$$

where $H_{j,m}$ is defined in [Hsiao and Huang 2019, (3.30)] (see also (3-29)). From [Hsiao and Huang 2019, (3.29), (3.34); Cheng et al. 2019, (6.4)], there are $\varepsilon_0 > 0$ and C_0 independent of j, x, m, t such that, for all $t \in \mathbb{R}_+$ and for all $m \in \mathbb{N}$, we have

$$(3-12) \quad \left| \frac{1}{2\pi} \int_{u \in [\zeta, \frac{2\pi}{p} - \zeta]} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du \right| \leq C_0 m^n t^{-n} e^{-\varepsilon_0 m \hat{d}(x, X_{\text{sing}})^2/t}.$$

Then the proof is completed by applying [Hsiao and Huang 2019, (3.32), (3.39)] and (3-12). \square

Remark 3.3. It is easy to check that

$$(3-13) \quad \sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} = \begin{cases} p & p \mid m \\ 0 & p \nmid m. \end{cases}$$

3B. BRT trivializations. To prove [Theorem 3.2](#), we need some preparation. We first need the following result due to Baouendi, Rothschild and Treves.

Theorem 3.4 [[Baouendi et al. 1985](#)]. *For every point $x_0 \in X$, we can find local coordinates $x = (x_1, \dots, x_{2n+1}) = (z, \theta) = (z_1, \dots, z_n, \theta)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, $x_{2n+1} = \theta$, defined in some small neighborhood*

$$D = \{(z, \theta) : |z| < \delta, -\varepsilon_0 < \theta < \varepsilon_0\}$$

of x_0 , $\delta > 0$, $0 < \varepsilon_0 < \pi$, such that $(z(x_0), \theta(x_0)) = (0, 0)$ and

$$(3-14) \quad T = \frac{\partial}{\partial \theta}, \quad Z_j = \frac{\partial}{\partial z_j} + i \frac{\partial \varphi}{\partial z_j}(z) \frac{\partial}{\partial \theta}, \quad j = 1, \dots, n,$$

where $Z_j(x)$, $j = 1, \dots, n$, form a basis of $T_x^{1,0}X$, for each $x \in D$, and $\varphi(z) \in C^\infty(D, \mathbb{R})$ is independent of θ . We call $(D, (z, \theta), \varphi)$ a BRT trivialization.

By using a BRT trivialization, we get another way to define Tu , for all $u \in \Omega^{0,q}(X)$. Let $(D, (z, \theta), \varphi)$ be a BRT trivialization. It is clear that

$$\{d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q} \mid 1 \leq j_1 < \dots < j_q \leq n\}$$

is a basis for $T_x^{*0,q}X$, for every $x \in D$. Let $u \in \Omega^{0,q}(X)$. On D , we write

$$(3-15) \quad u = \sum_{1 \leq j_1 < \dots < j_q \leq n} u_{j_1 \dots j_q} d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}.$$

Then, on D , we can check that

$$(3-16) \quad Tu = \sum_{1 \leq j_1 < \dots < j_q \leq n} (Tu_{j_1 \dots j_q}) d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$$

and Tu is independent of the choice of BRT trivializations. Note that, on the BRT trivialization $(D, (z, \theta), \varphi)$, we have

$$(3-17) \quad \bar{\partial}_b = \sum_{j=1}^n d\bar{z}_j \wedge \left(\frac{\partial}{\partial \bar{z}_j} - i \frac{\partial \varphi}{\partial \bar{z}_j}(z) \frac{\partial}{\partial \theta} \right).$$

3C. Local heat kernels on BRT trivializations. Until further notice, we fix $m \in \mathbb{Z}$. Let $B := (D, (z, \theta), \varphi)$ be a BRT trivialization. We may assume that $D = U \times]-\varepsilon, \varepsilon[$, where $\varepsilon > 0$ and U is an open set of \mathbb{C}^n . Since E is rigid, we can consider E as a holomorphic vector bundle over U . We may assume that E is trivial on U . Consider a trivial line bundle $L \rightarrow U$ with nontrivial Hermitian fiber metric $|1|_{h^L}^2 = e^{-2\varphi}$. Let $(L^m, h^{L^m}) \rightarrow U$ be the m -th power of (L, h^L) . Let $\Omega^{0,q}(U, E \otimes L^m)$ and $\Omega^{0,q}(U, E)$ be the spaces of $(0, q)$ forms on U with values

in $E \otimes L^m$ and E , respectively, $q = 0, 1, 2, \dots, n$. Put

$$\begin{aligned}\Omega^{0,\bullet}(U, E \otimes L^m) &:= \bigoplus_{j \in \{0, 1, \dots, n\}} \Omega^{0,j}(U, E \otimes L^m), \\ \Omega^{0,\bullet}(U, E) &:= \bigoplus_{j \in \{0, 1, \dots, n\}} \Omega^{0,j}(U, E).\end{aligned}$$

Since L is trivial, from now on, we identify $\Omega^{0,\bullet}(U, E)$ with $\Omega^{0,\bullet}(U, E \otimes L^m)$. Since the Hermitian fiber metric $\langle \cdot | \cdot \rangle_E$ is rigid, we can consider $\langle \cdot | \cdot \rangle_E$ as a Hermitian fiber metric on the holomorphic vector bundle E over U . Let $\langle \cdot, \cdot \rangle$ be the Hermitian metric on $\mathbb{C}TU$ given by

$$\left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right\rangle = \left\langle \frac{\partial}{\partial z_j} + i \frac{\partial \varphi}{\partial z_j}(z) \frac{\partial}{\partial \theta} \mid \frac{\partial}{\partial z_k} + i \frac{\partial \varphi}{\partial z_k}(z) \frac{\partial}{\partial \theta} \right\rangle, \quad j, k = 1, 2, \dots, n.$$

$\langle \cdot, \cdot \rangle$ induces a Hermitian metric on $T^{*0,\bullet}U := \bigoplus_{j=0}^n T^{*0,j}U$, where $T^{*0,j}U$ is the bundle of $(0, j)$ forms on U , $j = 0, 1, \dots, n$. We shall also denote this induced Hermitian metric on $T^{*0,\bullet}U$ by $\langle \cdot, \cdot \rangle$. The Hermitian metrics on $T^{*0,\bullet}U$ and E induce a Hermitian metric on $T^{*0,\bullet}U \otimes E$. We shall also denote this induced metric by $\langle \cdot | \cdot \rangle_E$. Let (\cdot, \cdot) be the L^2 inner product on $\Omega^{0,\bullet}(U, E)$ induced by $\langle \cdot, \cdot \rangle$, $\langle \cdot | \cdot \rangle_E$. Similarly, let $(\cdot, \cdot)_m$ be the L^2 inner product on $\Omega^{0,\bullet}(U, E \otimes L^m)$ induced by $\langle \cdot, \cdot \rangle$, $\langle \cdot | \cdot \rangle_E$ and h^{L^m} .

The curvature of L induced by h^L is given by $R^L := 2\partial\bar{\partial}\varphi$. Let $\dot{R}^L \in \text{End}(T^{1,0}U)$ be the Hermitian matrix given by

$$R^L(W, \bar{Y}) = \langle \dot{R}^L W, Y \rangle, \quad W, Y \in T^{1,0}U.$$

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{1,0}U$ with dual frame $\{w^j\}_{j=1}^n$. Set

$$(3-18) \quad \omega_d = - \sum_{l,j} R^L(w_j, \bar{w}_l) \bar{w}^l \wedge \iota_{\bar{w}_j},$$

where $\iota_{\bar{w}_j}$ denotes the interior product of \bar{w}_j .

Let

$$\bar{\partial} : \Omega^{0,\bullet}(U, E \otimes L^m) \rightarrow \Omega^{0,\bullet}(U, E \otimes L^m)$$

be the Cauchy–Riemann operator and let

$$\bar{\partial}^{*,m} : \Omega^{0,\bullet}(U, E \otimes L^m) \rightarrow \Omega^{0,\bullet}(U, E \otimes L^m)$$

be the formal adjoint of $\bar{\partial}$ with respect to $(\cdot, \cdot)_m$. Put

$$(3-19) \quad \square_{B,m} := (\bar{\partial} + \bar{\partial}^{*,m})^2 : \Omega^{0,\bullet}(U, E \otimes L^m) \rightarrow \Omega^{0,\bullet}(U, E \otimes L^m).$$

We need the following result (see Lemma 5.1 in [Cheng et al. 2019])

Lemma 3.5. *Let $u \in \Omega_m^{0,\bullet}(X, E)$. On D , we write*

$$u(z, \theta) = e^{im\theta} \tilde{u}(z), \quad \tilde{u}(z) \in \Omega^{0,\bullet}(U, E).$$

Then,

$$(3-20) \quad e^{-m\varphi} \square_{B,m} (e^{m\varphi} \tilde{u}) = e^{-im\theta} \square_{b,m} (u).$$

Let $z, w \in U$ and let $T(z, w) \in (T_w^{*0,\bullet} U \otimes E_w) \boxtimes (T_z^{*0,\bullet} U \otimes E_z)$. We write $|T(z, w)|$ to denote the standard pointwise matrix norm of $T(z, w)$ induced by $\langle \cdot | \cdot \rangle_E$. Let $\Omega_0^{0,\bullet}(U, E)$ be the subspace of $\Omega^{0,\bullet}(U, E)$ whose elements have compact support in U . Let dv_U be the volume form on U induced by $\langle \cdot, \cdot \rangle$. Assume $T(z, w) \in C^\infty(U \times U, (T_w^{*0,\bullet} U \otimes E_w) \boxtimes (T_z^{*0,\bullet} U \otimes E_z))$. Let $u \in \Omega_0^{0,\bullet}(U, E)$. We define the integral $\int T(z, w)u(w) dv_U(w)$ in the standard way. Let $G(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T_w^{*0,\bullet} U \otimes E_w) \boxtimes (T_z^{*0,\bullet} U \otimes E_z))$. We write $G(t)$ to denote the continuous operator

$$G(t) : \Omega_0^{0,\bullet}(U, E) \rightarrow \Omega^{0,\bullet}(U, E), \quad u \mapsto \int G(t, z, w)u(w) dv_U(w)$$

and we write $G'(t)$ to denote the continuous operator

$$G'(t) : \Omega_0^{0,\bullet}(U, E) \rightarrow \Omega^{0,\bullet}(U, E), \quad u \mapsto \int \frac{\partial G(t, z, w)}{\partial t} u(w) dv_U(w).$$

We consider the heat operator of $\square_{B,m}$. By using the standard Dirichlet heat kernel construction (see [Grigoryan and Saloff-Coste 2002]) and the proofs of Theorem 1.6.1 and Theorem 5.5.9 in [Ma and Marinescu 2007], we deduce the following:

Theorem 3.6. *There is*

$$A_{B,m}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T_w^{*0,\bullet} U \otimes E_w) \boxtimes (T_z^{*0,\bullet} U \otimes E_z))$$

such that

$$(3-21) \quad \lim_{t \rightarrow 0^+} A_{B,m}(t) = I, \quad \text{in } D'(U, T^{*0,\bullet} U \otimes E),$$

$$A'_{B,m}(t)u + \frac{1}{m} A_{B,m}(t)(\square_{B,m}u) = 0, \quad \text{for all } u \in \Omega_0^{0,\bullet}(U, E), \quad t > 0,$$

and $A_{B,m}(t, z, w)$ satisfies the following:

(I) *For every compact set $K \Subset U$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0^n$, there are constants $C_{\alpha_1, \alpha_2, \beta_1, \beta_2, K} > 0$ and $\varepsilon_0 > 0$ independent of t and m such that*

$$(3-22) \quad \left| \partial_z^{\alpha_1} \partial_{\bar{z}}^{\alpha_2} \partial_w^{\beta_1} \partial_{\bar{w}}^{\beta_2} (A_{B,m}(t, z, w) e^{m(\varphi(w) - \varphi(z))}) \right|$$

$$\leq C_{\alpha_1, \alpha_2, \beta_1, \beta_2, K} \left(\frac{m}{t} \right)^{n + |\alpha_1| + |\alpha_2| + |\beta_1| + |\beta_2|} e^{-m\varepsilon_0|z-w|^2/t},$$

for all $(t, z, w) \in \mathbb{R}_+ \times K \times K$.

(II) $A_{B,m}(t, z, z)$ admits an asymptotic expansion:

$$(3-23) \quad A_{B,m}(t, z, z) = (2\pi)^{-n} m^n \frac{\det(\dot{R}^L) \exp(t\omega_d)}{\det(1 - \exp(-t\dot{R}^L))} (z) \otimes \text{Id}_{E_z} + o(m^n)$$

in $C^\ell(U, \text{End}(T^{*0,\bullet}U) \otimes E)$ locally uniformly on $\mathbb{R}_+ \times U$, for every $\ell \in \mathbb{N}$. Here we use the convention that if an eigenvalue $a_j(z)$ of $\dot{R}^L(z)$ is zero, then its contribution for $(\det(\dot{R}^L)/\det(1 - \exp(-t\dot{R}^L)))(z)$ is $\frac{1}{t}$.

3D. L^2 Kohn–Rossi cohomology on a covering manifold. Let

$$\tilde{\square}_b : \text{Dom } \tilde{\square}_b \subset L^2(\tilde{X}, T^{*0,\bullet}\tilde{X}) \rightarrow L^2(\tilde{X}, T^{*0,\bullet}\tilde{X})$$

be the Gaffney extension of the pull-back Kohn Laplacian on \tilde{X} . By a result of Gaffney, $\tilde{\square}_b$ is a positive self-adjoint operator (see Proposition 3.1.2 in [Ma and Marinescu 2007]). That is, $\tilde{\square}_b$ is self-adjoint and the spectrum of $\tilde{\square}_b$ is contained in \mathbb{R}_+ . Now, we fix $m \in \mathbb{Z}$. As in (3-3), we introduce the m -th Fourier component of the Kohn Laplacian $\tilde{\square}_{b,m}$ on $\Omega_m^{0,\bullet}(\tilde{X}, \tilde{E})$. We can easily see that $\tilde{\square}_{b,m}$ is also self-adjoint. By the second isomorphism of (2-10), we can see that, for any $\gamma \in \Gamma$,

$$(3-24) \quad T_\gamma(\text{Dom}(\tilde{\square}_{b,m})) \subset \text{Dom}(\tilde{\square}_{b,m}), \quad T_\gamma \tilde{\square}_{b,m} = \tilde{\square}_{b,m} T_\gamma \quad \text{on } \text{Dom}(\tilde{\square}_{b,m}).$$

Consider the spectral resolution $E_\lambda^q(\tilde{\square}_{b,m})$ of $\tilde{\square}_{b,m}$ acting on $L_m^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})$. (See [Ma and Marinescu 2007, Appendix C.2]). The proof of the following lemma is similar to Lemma 3.6.3 in [Ma and Marinescu 2007].

Lemma 3.7. *For any $q = 0, 1, \dots, n$ and $\lambda \in \mathbb{R}$, then $E_\lambda^q(\tilde{\square}_{b,m})$ commutes with Γ , its Schwartz kernel is smooth and*

$$\dim_\Gamma E_\lambda^q(\tilde{\square}_{b,m}) < +\infty.$$

Proof. By (2-10) and (3-24), we can see that, for any $\lambda \in \mathbb{R}$, $E_\lambda^q(\tilde{\square}_{b,m})$ commutes with Γ . We claim that $\tilde{\square}_{b,m} - \tilde{T}^2 \equiv \Delta$ is a second order elliptic operator, so is $\Delta - m^2$. Its principal symbol is locally written as

$$\sigma_\Delta(\tilde{x}, \tilde{\xi}) = \sigma_{\tilde{\square}_{b,m}}(\tilde{x}, \tilde{\xi}) - \sigma_{\tilde{T}^2}(\tilde{x}, \tilde{\xi}) = \sum_{j=1}^n |\sigma_{L_j}(\tilde{x}, \tilde{\xi})|^2 - \sigma_{\tilde{T}}(\tilde{x}, \tilde{\xi})^2,$$

where $\tilde{\xi} = (\xi_1, \dots, \xi_{2n}, \xi_{2n+1})$ and $\{L_j\}$ is an orthonormal basis of $T_x^{0,1}\tilde{X}$. It is well-known that the characteristic manifold of $\tilde{\square}_b$ is

$$\Sigma = \{(\tilde{x}, c\tilde{\omega}_0(\tilde{x})) \in T^*\tilde{X} \mid c \neq 0\}.$$

It means that $\sigma_{\tilde{\square}_{b,m}}(\tilde{x}, \tilde{\xi}) > 0$ if and only if $(\xi_1, \dots, \xi_{2n}) \neq 0$. Meanwhile, in a local BRT coordinate [Baouendi et al. 1985], we have $\tilde{T} = \frac{\partial}{\partial \theta}$, then $\sigma_{\tilde{T}} = i\xi_{2n+1}$. That is, $\sigma_{\tilde{T}^2} = -\xi_{2n+1}^2$. Then the claim is proved. By the spectral theorem, see [Ma and Marinescu 2007, Theorem C.2.1], we have $\text{Im}(E_\lambda(\Delta - m^2)) \subset \text{Dom}((\Delta - m^2)^k)$

for $k \in \mathbb{N}$. Using the uniform Sobolev spaces [Shubin 1995, pp. 511–512], it is easy to see that $\text{Im}(E_\lambda(\Delta - m^2)) \subset \Omega^\bullet(\tilde{X}, \tilde{E})$, so that

$$E_\lambda(\Delta - m^2) : L^2(\tilde{X}, T^{*0,\bullet}\tilde{X} \otimes \tilde{E}) \rightarrow \Omega^\bullet(\tilde{X}, \tilde{E})$$

is linear continuous. Hence,

$$\begin{aligned} \text{Im } E_\lambda(\tilde{\square}_{b,m}) &= \text{Im}(E_\lambda(\Delta - m^2)) \cap L_m^2(\tilde{X}, T^{*0,\bullet}\tilde{X} \otimes \tilde{E}) \\ &\subset \Omega^\bullet(\tilde{X}, \tilde{E}) \cap L_m^2(\tilde{X}, T^{*0,\bullet}\tilde{X} \otimes \tilde{E}) = \Omega_m^\bullet(\tilde{X}, \tilde{E}) \end{aligned}$$

and $E_\lambda(\tilde{\square}_{b,m}) : L_m^2(\tilde{X}, T^{*0,\bullet}\tilde{X} \otimes \tilde{E}) \rightarrow \Omega_m^\bullet(\tilde{X}, \tilde{E})$ is also linear continuous. By the Schwartz kernel theorem, the kernel $E_\lambda(\tilde{\square}_{b,m})(\tilde{x}, \tilde{x})$ of $E_\lambda(\tilde{\square}_{b,m})$ with respect to $dv_{\tilde{X}}(\tilde{x})$ is smooth. By [Ma and Marinescu 2007, (3.6.12)],

$$\dim_\Gamma E_\lambda(\tilde{\square}_{b,m}) = \int_U \text{Tr}[E_\lambda(\tilde{\square}_{b,m})(\tilde{x}, \tilde{x})] dv_{\tilde{X}}(\tilde{x}) < +\infty. \quad \square$$

Definition 3.8. (a) The m -th Fourier component of the space of harmonic forms $\mathcal{H}^\bullet(\tilde{X}, \tilde{E})$ is defined by

$$\mathcal{H}_{b,m}^\bullet(\tilde{X}, \tilde{E}) := \text{Ker}(\tilde{\square}_{b,m}) = \{s \in \text{Dom } \tilde{\square}_{b,m} \mid \tilde{\square}_{b,m}s = 0\}.$$

(b) The m -th Fourier component of the q -th reduced L^2 Kohn–Rossi cohomology is given by

$$(3-25) \quad \bar{H}_{b,(2),m}^q(\tilde{X}, \tilde{E}) := \frac{\text{Ker } \bar{\partial}_b \cap L_m^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})}{[\text{Im } \bar{\partial}_b \cap L_m^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})]},$$

where $[V]$ denotes the closure of the space V .

We can easily obtain the following weak Hodge decomposition

$$(3-26) \quad L_m^2(\tilde{X}, T^{*0,\bullet}\tilde{X} \otimes \tilde{E}) = \mathcal{H}^\bullet(\tilde{X}, \tilde{E}) \oplus [\text{Im}(\bar{\partial}_{b,m})] \oplus [\text{Im}(\bar{\partial}_{b,m}^*)]$$

By (3-26), we get the isomorphism

$$(3-27) \quad \bar{H}_{b,(2),m}^\bullet(\tilde{X}, \tilde{E}) \cong \mathcal{H}_{b,m}^\bullet(\tilde{X}, \tilde{E}).$$

3E. Asymptotics of heat kernels of Kohn Laplacians on a covering manifold. Assume that $X = D_1 \cup D_2 \cup \dots \cup D_N$, where $B_j := (D_j, (z, \theta), \varphi_j)$ is a BRT trivialization, for each j . We may assume that, for each j , $D_j = U_j \times]-2\delta_j, 2\tilde{\delta}_j[\subset \mathbb{C}^n \times \mathbb{R}$, $\delta_j > 0$, $\tilde{\delta}_j > 0$, $U_j = \{z \in \mathbb{C}^n \mid |z| < l_j\}$. For each j , put $\hat{D}_j = \hat{U}_j \times]-\delta_j/2, \tilde{\delta}_j/2[$, where $\hat{U}_j = \{z \in \mathbb{C}^n \mid |z| < l_j/2\}$. We may suppose that $X = \hat{D}_1 \cup \hat{D}_2 \cup \dots \cup \hat{D}_N$.

Let $\{\psi_j\}$ be a partition of unity subordinate to $\{\hat{D}_j\}$. Then $\{\tilde{\psi}_{\gamma,j} := \psi_j \circ \pi\}$ is a partition of unity subordinate to $\{\tilde{D}_{\gamma,j}\}$, where $\pi^{-1}(\hat{D}_j) = \bigcup_{\gamma \in \Gamma} \tilde{D}_{\gamma,j}$ and $\tilde{D}_{\gamma_1,j}$ and $\tilde{D}_{\gamma_2,j}$ are disjoint for $\gamma_1 \neq \gamma_2$. For each $\gamma \in \Gamma$ and each j , we have

$\tilde{D}_{\gamma,j} = \tilde{U}_{\gamma,j} \times]-\delta_{\gamma,j}/2, \tilde{\delta}_{\gamma,j}/2[$, where $\tilde{U}_{\gamma,j} = \{z \in \mathbb{C}^n \mid |z| < l_{\gamma,j}/2\}$. Then $\tilde{X} = \bigcup_{\gamma \in \Gamma} \bigcup_{j=1}^N \tilde{D}_{\gamma,j}$.

Fix $\gamma \in \Gamma$ and $j = 1, 2, \dots, N$. Put

$$K_{\gamma,j} = \{z \in \tilde{U}_{\gamma,j} \mid \text{there is a } \theta \in]-\delta_{\gamma,j}/2, \tilde{\delta}_{\gamma,j}/2[\text{ such that } \tilde{\psi}_{\gamma,j}(z, \theta) \neq 0\}.$$

Let $\tau_{\gamma,j}(z) \in C_0^\infty(\tilde{U}_{\gamma,j})$ with $\tau_{\gamma,j} \equiv 1$ on some neighborhood $W_{\gamma,j}$ of $K_{\gamma,j}$. Let $\sigma_{\gamma,j} \in C_0^\infty(]-\delta_{\gamma,j}/2, \tilde{\delta}_{\gamma,j}/2[)$ with $\int \sigma_{\gamma,j}(\theta) d\theta = 1$. Let

$$\tilde{A}_{B_{\gamma,j},m}(t, z, w) \in C^\infty(\mathbb{R}_+ \times \tilde{U}_{\gamma,j} \times \tilde{U}_{\gamma,j}, (T_w^{*0} \cdot \tilde{U}_{\gamma,j} \otimes \tilde{E}_w) \boxtimes (T_z^{*0} \cdot \tilde{U}_{\gamma,j} \otimes \tilde{E}_z))$$

be as in [Theorem 3.6](#). Put

$$(3-28) \quad \tilde{H}_{\gamma,j,m}(t, \tilde{x}, \tilde{y}) = \tilde{\psi}_{\gamma,j}(\tilde{x}) e^{-m\varphi_j(z)+im\theta} \tilde{A}_{B_{\gamma,j},m}(t, z, w) e^{m\varphi_{\gamma,j}(w)-im\eta} \tau_{\gamma,j}(w) \sigma_{\gamma,j}(\eta),$$

where $\tilde{x} = (z, \theta)$, $\tilde{y} = (w, \eta) \in \mathbb{C}^n \times \mathbb{R}$. Let

$$(3-29) \quad \tilde{\Gamma}_m(t, \tilde{x}, \tilde{y}) := \frac{1}{2\pi} \sum_{\gamma \in \Gamma} \sum_{j=1}^N \int_{-\pi}^\pi \tilde{H}_{\gamma,j,m}(t, \tilde{x}, e^{iu} \circ \tilde{y}) e^{imu} du.$$

Note that when $\Gamma = \{e\}$, $\tilde{\Gamma}_m(t, \tilde{x}, \tilde{y}) = \Gamma_m(t, \pi(\tilde{x}), \pi(\tilde{y}))$ is defined in [\[Hsiao and Huang 2019, \(3.31\)\]](#).

From [Lemma 3.5](#), with off-diagonal estimates of $\tilde{A}_{B_{\gamma,j},m}(t, \tilde{x}, \tilde{y})$ (see [\(3-22\)](#)), we can repeat the proof of [Theorem 5.14](#) in [\[Cheng et al. 2019\]](#) with minor change and deduce that

Theorem 3.9. *For every $\ell \in \mathbb{N}$, $\ell \geq 2$, and every $M > 0$, there are $\varepsilon_0 > 0$ and $m_0 > 0$ independent of t and m such that for every $m \geq m_0$, we have*

$$(3-30) \quad \|e^{-\frac{t}{m} \tilde{\square}_{b,m}}(\tilde{x}, \tilde{y}) - \tilde{\Gamma}_m(t, \tilde{x}, \tilde{y})\|_{C^\ell(\tilde{X} \times \tilde{X})} \leq e^{-\frac{m}{t} \varepsilon_0}, \quad \text{for all } t \in (0, M).$$

From [\[Ma and Marinescu 2007, Theorem 3.6.4\]](#), we have

Proposition 3.10. *For any $t_0 > 0$, $\varepsilon > 0$ and any $\gamma \in \Gamma$, $j = 1, 2, \dots, N$, there exists $C > 0$ such that for any $z \in \tilde{U}_{\gamma,j}$, $m \in \mathbb{N}$, $t > t_0$,*

$$\|\tilde{A}_{B_{\gamma,j},m}(t, z, z) - A_{B_{\gamma,j},m}(t, \pi(z), \pi(z))\|_{C^1(\tilde{U}_{\gamma,j} \times \tilde{U}_{\gamma,j})} \leq C \exp\left(-\frac{m}{32t} \varepsilon\right).$$

From [\(3-11\)](#) (see [\[Hsiao and Huang 2019, \(3.31\)\]](#)), [\(3-28\)](#), [\(3-29\)](#), [Proposition 3.10](#) and the fact that $\tilde{\psi}_{\gamma,j} = \psi_j \circ \pi$, we can easily deduce:

Lemma 3.11. *With the notation and assumptions as in [Theorem 3.9](#), we have*

$$\|\tilde{\Gamma}_m(t, \tilde{x}, \tilde{x}) - \Gamma_m(t, \pi(\tilde{x}), \pi(\tilde{x}))\|_{C^1(\tilde{X} \times \tilde{X})} \leq C \exp\left(-\frac{m}{t} \varepsilon_0\right).$$

From [Theorem 3.9](#), [Lemma 3.11](#) and [Theorem 3.5](#) of [\[Hsiao and Huang 2019\]](#), we have:

Theorem 3.12. *For every $\ell \in \mathbb{N}$, $\ell \geq 2$, and every $M > 0$, there are $\varepsilon_0 > 0$ and $m_0 > 0$ independent of t and m such that for any $\tilde{x} \in \tilde{X}$ and $m \geq m_0$, we have*

$$\begin{aligned} \left\| e^{-\frac{t}{m} \tilde{\square}_{b,m}}(\tilde{x}, \tilde{x}) - e^{-\frac{t}{m} \square_{b,m}}(\pi(\tilde{x}), \pi(\tilde{x})) \right\|_{C^\ell(\tilde{X} \times \tilde{X})} \\ \leq C \exp\left(-\frac{m}{t} \varepsilon_0\right), \quad \text{for all } t \in (0, M). \end{aligned}$$

By Theorems 3.2 and 3.12, we have:

Theorem 3.13. *With the above notations and assumptions, for every $\varepsilon > 0$, there are $m_0 > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that for all $m \geq m_0$, we have*

$$\begin{aligned} (3-31) \quad \left| e^{-\frac{t}{m} \tilde{\square}_{b,m}}(\tilde{x}, \tilde{x}) - \sum_{s=1}^p e^{\frac{2\pi(s-1)}{p} m i} (2\pi)^{-n-1} m^n \frac{\det(\dot{\mathcal{R}}) \exp(t\gamma_d)}{\det(1 - \exp(-t\dot{\mathcal{R}}))} (\pi(\tilde{x})) \otimes \text{Id}_{E_{\pi(\tilde{x})}} \right| \\ \leq \varepsilon m^n + C m^n t^{-n} e^{-\varepsilon_0 m \hat{d}(\pi(\tilde{x}), X_{\text{sing}})^2/t}, \quad \text{for all } (t, \tilde{x}) \in \mathbb{R}_+ \times \tilde{X}_{\text{reg}}. \end{aligned}$$

Recall that since Γ acts on \tilde{X} freely so that $\tilde{X}/\Gamma = X$, hence, $\tilde{X}_{\text{reg}}/\Gamma = X_{\text{reg}}$.

4. Heat kernel proof

In this section, we will present the heat kernel proof of the main theorem.

We denote by $\text{Tr}_{\Gamma,q}$ the Γ -trace of operators acting on $L_m^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})$, see Section 2C or [Ma and Marinescu 2007, Subsection 3.6.1].

Lemma 4.1. *For any $t > 0$, $m \in \mathbb{N}$, $0 \leq q \leq n$, we have*

$$(4-1) \quad \sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \leq \sum_{j=0}^q (-1)^{q-j} \text{Tr}_{\Gamma,j} \left[\exp\left(-\frac{t}{m} \tilde{\square}_{b,m}\right) \right],$$

with equality for $q = n$.

Proof. Let $E_{\lambda}^{j,m}$ be the spectral resolution of $\tilde{\square}_{b,m}$ acting on $L_m^2(\tilde{X}, T^{*0,q}\tilde{X} \otimes \tilde{E})$. We consider the projectors $E_{\lambda_2}^{j,m}([\lambda_1, \lambda_2]) = E_{\lambda_2}^{j,m} - E_{\lambda_1}^{j,m}$, where $\lambda_2 > \lambda_1 \geq 0$. Then, by the Hodge decomposition (3-26), $\sum_{j=0}^q (-1)^{q-j} E_{\lambda_1, \lambda_2}^{j,m}$ is the projection on the range of $\bar{\partial}_{b,m} E^{q,m}([\lambda_1, \lambda_2])$ and thus a positive operator. Hence the Γ -invariant measure $\sum_{j=0}^q (-1)^{q-j} dE_{\lambda}^{j,m}$ is positive on $\{\lambda > 0\}$. It follows that

$$(4-2) \quad R := \int_{\lambda>0} e^{-\frac{t}{m}\lambda} \sum_{j=0}^q (-1)^{q-j} dE_{\lambda}^{j,m} \geq 0,$$

and R commutes with Γ . On the other hand,

$$(4-3) \quad \text{Tr}_{\Gamma,j} \left[\exp\left(-\frac{t}{m} \tilde{\square}_{b,m}\right) \right] = \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) + \text{Tr}_{\Gamma} \int_{\lambda>0} e^{-\frac{t}{m}\lambda} dE_{\lambda}^{j,m}.$$

By (4-2) and (4-3), we obtain the result. □

Let $\text{Tr}_q[\exp(-\frac{t}{m}\square_{b,m})]$ be the trace of the operator $\exp(-\frac{t}{m}\square_{b,m})$ acting on $\Omega_m^{0,q}(X, E)$. It is well-known (see Theorem 8.10 in [Roe 1998]) that

$$(4-4) \quad \text{Tr}_q\left[\exp\left(-\frac{t}{m}\square_{b,m}\right)\right] = \int_X \text{Tr}_q\left[\exp\left(-\frac{t}{m}\square_{b,m}\right)(x, x)\right] dv_X(x).$$

By [Ma and Marinescu 2007, (3.6.7) and (3.6.8)], as in (4-4), we obtain:

Proposition 4.2. *We have*

$$(4-5) \quad \text{Tr}_{\Gamma,q}\left[\exp\left(-\frac{t}{m}\tilde{\square}_{b,m}\right)\right] = \int_U \text{Tr}_q\left[e^{-\frac{t}{m}\tilde{\square}_{b,m}}(\tilde{x}, \tilde{x})\right] dv_{\tilde{X}}(\tilde{x}).$$

Now we are in a position to give the heat kernel proof of the Morse inequalities for the Fourier components of reduced L^2 Kohn–Rossi cohomology.

Proof of Theorem 1.1. Denote by $\text{Tr}_{\Lambda^{0,q}}$ the trace on $T^{*0,q}X$. The basis for $T^{*0,q}X$ is

$$(4-6) \quad \{\bar{\omega}^{j_1} \wedge \dots \wedge \bar{\omega}^{j_q} \mid j_1 < \dots < j_q\}.$$

We write for the index $(1, \dots, q)$

$$(4-7) \quad \begin{aligned} \exp(t\gamma_d)(\bar{\omega}^1 \wedge \dots \wedge \bar{\omega}^q) &= \prod_{j=1}^q (1 + (e^{-ta_j} - 1)\bar{\omega}^j \wedge t_{\bar{\omega}^j})(\bar{\omega}^1 \wedge \dots \wedge \bar{\omega}^q) \\ &= \sum_{k_1 < \dots < k_q} c_{k_1 \dots k_q}(x) \bar{\omega}^{k_1} \wedge \dots \wedge \bar{\omega}^{k_q}. \end{aligned}$$

From direct calculations, we see that

$$(4-8) \quad c_{1 \dots q}(x) = \exp\left(-t \sum_{j=1}^q a_j(x)\right).$$

Then we have

$$(4-9) \quad \text{Tr}_{\Lambda^{0,q}}[\exp(t\gamma_d)] = \sum_{j_1 < \dots < j_q} \exp\left(-t \sum_{i=1}^q a_{j_i}(x)\right).$$

Hence

$$(4-10) \quad \begin{aligned} \lim_{t \rightarrow \infty} \frac{\text{Tr}_{\Lambda^{0,q}}[\exp(t\gamma_d)]}{\det(1 - \exp(-t\hat{\mathcal{R}}))} &= \lim_{t \rightarrow \infty} \frac{\sum_{j_1 < \dots < j_q} \exp(-t \sum_{i=1}^q a_{j_i}(x))}{\prod_{j=1}^n (1 - \exp(-ta_j(x)))} \\ &= (-1)^q 1_{X(q)}, \end{aligned}$$

where the function $X(q)$ is defined by 1 on $X(q)$, 0 otherwise. As usual, for $\tilde{x} \in \tilde{X}$, $\pi(\tilde{x}) = x \in X$. It follows from [Theorem 3.13](#), (4-5) and [Lemma 4.1](#) that

$$\begin{aligned}
 (4-11) \quad & \frac{1}{m^n} \sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \\
 & \leq \frac{1}{m^n} \sum_{j=0}^q (-1)^{q-j} \text{Tr}_{\Gamma,q} \left[\exp\left(-\frac{t}{m} \tilde{\square}_{b,m}\right) \right] \\
 & = \frac{1}{m^n} \sum_{j=0}^q (-1)^{q-j} \int_U \text{Tr}_{\Gamma,q} \left[\exp\left(-\frac{t}{m} \tilde{\square}_{b,m}(\tilde{x}, \tilde{x})\right) \right] dv_{\tilde{X}}(\tilde{x}) \\
 & \leq (2\pi)^{-n-1} \sum_{s=1}^p e^{\frac{2\pi(s-1)}{p} mi} \sum_{j=0}^q (-1)^{q-j} \\
 & \quad \times \int_X \frac{\det(\dot{\mathcal{R}}) \text{Tr}_{\Lambda^{0,q}}[\exp(t\gamma_d) \otimes \text{Id}_{E_x}]}{\det(1 - \exp(-t\dot{\mathcal{R}}))} dv_X(x) \\
 & \quad + \varepsilon \sum_{j=0}^q (-1)^{q-j} \text{Vol}(X) + C \sum_{j=0}^q (-1)^{q-j} \int_X t^{-n} e^{-\varepsilon_0 m \hat{d}(x_0, X_{\text{sing}})^2/t} dv_X(x).
 \end{aligned}$$

Note that ε is arbitrarily small. By the dominant convergence theorem with $t \rightarrow \infty$, we have

$$\begin{aligned}
 (4-12) \quad & \limsup_{m \rightarrow \infty} \frac{1}{m^n} \sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \\
 & \leq \frac{pr}{(2\pi)^{n+1}} \sum_{j=0}^q (-1)^{q-j} \int_{X(j)} |\det(\dot{\mathcal{R}})| dv_X(x), \quad \text{for } p \mid m, \\
 & \limsup_{m \rightarrow \infty} \frac{1}{m^n} \sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) = 0, \quad \text{for } p \nmid m.
 \end{aligned}$$

From [Definition 2.2](#), (3-6) and (4-12), we finally get

$$\begin{aligned}
 (4-13) \quad & \sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \\
 & \leq \frac{prm^n}{2\pi^{n+1}} \sum_{j=0}^q (-1)^{q-j} \int_{X(j)} |\det(\mathcal{L}_x)| dv_X(x) + o(m^n), \quad \text{for } p \mid m, \\
 & \sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) = o(m^n), \quad \text{for } p \nmid m.
 \end{aligned}$$

Let $q = n$ in (4-1), by applying Theorem 3.13, we obtain for $p \mid m$,

$$\begin{aligned}
 (4-14) \quad & \frac{1}{m^n} \sum_{j=0}^n (-1)^{n-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \\
 & \geq \frac{1}{m^n} \sum_{j=0}^n (-1)^{n-j} \int_U \text{Tr}_{\Gamma,j} \left[\exp\left(-\frac{t}{m} \tilde{\square}_{b,m}(\tilde{x}, \tilde{x})\right) \right] dv_{\tilde{X}}(\tilde{x}) \\
 & \geq (2\pi)^{-n-1} p \sum_{j=0}^n (-1)^{n-j} \int_X \frac{\det(\dot{\mathcal{R}}) \text{Tr}_{\Lambda^{0,j}}[\exp(t\gamma_d) \otimes \text{Id}_{E_x}]}{\det(1 - \exp(-t\dot{\mathcal{R}}))} dv_X(x) \\
 & \quad - \varepsilon n \text{Vol}(X) - Cn \int_X t^{-n} e^{-\varepsilon_0 m \hat{d}(x_0, X_{\text{sing}})^2/t} dv_X(x).
 \end{aligned}$$

Note that ε is arbitrarily small. By the dominant convergence theorem with $t \rightarrow \infty$, we have

$$\begin{aligned}
 (4-15) \quad & \liminf_{m \rightarrow \infty, p \mid m} \frac{1}{m^n} \sum_{j=0}^n (-1)^{n-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \\
 & \geq \frac{pr}{(2\pi)^{n+1}} \sum_{j=0}^n (-1)^{n-j} \int_{X(j)} |\det(\dot{\mathcal{R}})| dv_X(x).
 \end{aligned}$$

Then

$$\begin{aligned}
 (4-16) \quad & \liminf_{m \rightarrow \infty, p \mid m} \frac{1}{m^n} \sum_{j=0}^n (-1)^{n-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \\
 & = \frac{pr}{(2\pi)^{n+1}} \sum_{j=0}^n (-1)^{n-j} \int_{X(j)} |\det(\dot{\mathcal{R}})| dv_X(x).
 \end{aligned}$$

We finally get

$$\begin{aligned}
 (4-17) \quad & \sum_{j=0}^n (-1)^{n-j} \dim_{\Gamma} \bar{H}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \\
 & = \frac{prm^n}{2\pi^{n+1}} \sum_{j=0}^n (-1)^{n-j} \int_{X(j)} |\det(\mathcal{L}_x)| dv_X(x) + o(m^n) \quad \text{for } p \mid m.
 \end{aligned}$$

Then the proof is completed. □

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