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REGULARITY OF QUOTIENTS OF DRINFELD MODULAR SCHEMES

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Let A be the coordinate ring of a projective smooth curve over a finite field minus a closed point. For a nontrivial ideal $I \subset A$, Drinfeld defined the notion of structure of level I on a Drinfeld module.

We extend this to that of level N, where N is a finitely generated torsion A-module. The case where $N = (I^{-1}/A)^d$, where d is the rank of the Drinfeld module, coincides with the structure of level I. The moduli functor is representable by a regular affine scheme.

The automorphism group $\mathrm{Aut}_A(N)$ acts on the moduli space. Our theorem gives a class of subgroups for which the quotient of the moduli scheme is regular. Examples include generalizations of Γ_0 and of Γ_1. We also show that parabolic subgroups appearing in the definition of Hecke correspondences are such subgroups.

1. Introduction

1.1. ***Main Theorem and applications.*** We first recall the usual setup for Drinfeld modules. Let C be a smooth projective geometrically irreducible curve over the finite field $\mathbb{F}_q$ of q elements. Let F denote the function field of C. Fix a closed point ∞ of C. Let $A = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ be the coordinate ring of the affine $\mathbb{F}_q$-scheme $C \setminus \{\infty\}$.

In this article, we define the structure of level N on a Drinfeld module, generalizing the structure of level I of Drinfeld (also known as the full level I structure). This may also be regarded as a generalization of the Γ_1-structure.

Let us denote by $\mathcal{M}_N^d = \mathcal{M}_{N,A}^d$ the functor that associates an A-scheme S to the set of isomorphism classes of Drinfeld modules over S with structure of level N. (We will also mention $\mathcal{M}_{N,U}^d$ for an open subscheme U of Spec A.) The representability by an affine scheme and its regularity of the moduli functor when $|\mathrm{Supp}\, N| \geq 2$ can be proved in a manner similar to that of the full level case. See Proposition 4.2.1.

Note that the automorphism group $\mathrm{Aut}_A(N)$ of N as an A-module acts on N, hence on the set of level structures, and thereby on the moduli space. We define

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admissible subgroups of $\mathrm{Aut}_A(N)$ and denote by $\mathcal{S}_N$ the set of admissible subgroups. Our main theorem in a rough form is as follows. See Theorem 6.1.1 for a more general statement.

Theorem 1.1.1. *Let N be a torsion A-module, generated by at most d elements. Let $N = N_1 \oplus N_2$ be a direct sum decomposition such that $N_1 \neq 0$, $N_2 \neq 0$, and $\mathrm{Supp}\, N_1 \cap \mathrm{Supp}\, N_2 = \varnothing$. Let $H \in \mathcal{S}_{N_1}$ be an admissible subgroup, regarded as a subgroup of $\mathrm{Aut}_A(N)$. Then the quotient $\mathcal{M}^d_N/H$ is regular.*

Recall that moduli of elliptic curves with Γ_0 or Γ_1 structures, usually denoted $Y_0(N)$ or $Y_1(N)$ for some integer N, are quotients of the full level moduli $Y(N)$ by certain groups defined in terms of congruences modulo N. Our admissible subgroups include function field analogue of such groups.

The quotients of interest to us are those appearing in the definition of Hecke correspondences. Let $Y \xleftarrow{f} X \xrightarrow{g} Y$ be a diagram of schemes where Y is smooth over a base field and f is a finite surjective morphism. Recall that this gives rise to a finite correspondence; see [Mazza et al. 2006, p. 3]. A typical example of a Hecke correspondence is the finite correspondence corresponding to the diagram above where X is a quotient by some parabolic subgroup of a full level moduli and Y is a full level moduli. (See [Laumon 1996, p. 14] for Hecke correspondences in general over F; see Section 6.3 for examples.) A finite correspondence acts on higher Chow groups [Mazza et al. 2006, p. 142, Theorem 17.21] and the action corresponding to a Hecke correspondence is called Hecke operator.

Our theorem may make the computation of the Hecke operator easier in the following sense. Our regularity theorem implies that X is regular (smooth over the base field) and the maps f, g are finite flat. In this case, the action of the finite correspondence is given as the composition of the pullback by g and the pushforward by f.

In a future paper, we will construct certain elements in the higher Chow groups of Drinfeld modular schemes over A, extending, in a sense, our elements [Kondo and Yasuda 2012] in rational algebraic K-theory of Drinfeld modular schemes away from level. We show that they are (again) norm compatible (we say that they form an Euler system), that is, we express the pushforward of the elements in terms of Hecke actions relating to local L-factors. The computation uses Theorem 1.1.1 in a way described in the previous paragraph.

1.2. *Outline of proof.* An outline of the proof of regularity of quotients of moduli of elliptic curves is given by Katz and Mazur [1985, Theorem 7.5.1, p. 201]. We follow their outline except that we need one additional ingredient, namely, Dickson's theorem [1911] from modular invariant theory.

Let us briefly recall the outline. To prove the regularity of the quotient, one first proves the regularity at the points away from the level. The regularity in this case is the consequence of the fact that the quotient map becomes étale. Now, to prove the regularity at a point y on the bad fiber, we look at the point x lying above y. To the point x, there corresponds a Drinfeld module (with level N structure), and in turn a formal $\mathcal{O}$-module. It happens that the regularity at y depends only on the height of the formal $\mathcal{O}$-module. Since the singular locus is closed, and there are points of arbitrary height near the (most) supersingular point (we prove its existence in Section 7), we are reduced to the case where the formal $\mathcal{O}$-module has maximal height.

To show the regularity of the ring of invariants in the supersingular case, the two tools we use are Proposition 7.5.2 of [Katz and Mazur 1985] (Proposition 8.9.1 below) and Dickson's theorem (Theorem 8.12.1).

Note that a typical type of group encountered by Katz and Mazur are subgroups of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ for some integer N, and the essential ones are those contained in some Borel subgroup. In this case, successive applications of their Proposition 7.5.2 proves the regularity of the quotient. A typical type in our case is a parabolic subgroup of $\mathrm{GL}_d(A/I)$ for some ideal I with $d \geq 2$. In this case, using their Proposition 7.5.2 successively, we are reduced to the computation of invariants by the Levi subgroups, each of which is $\mathrm{GL}_{d'}(\kappa)$ for some finite field κ and some positive integer d'. This last case is the subject of modular invariant theory, where Dickson's theorem is the most basic theorem.

1.3. ***Organization.*** The paper is organized as follows. We start with Section 2 on formal $\mathcal{O}$-modules, much like in the original paper by Drinfeld [1974]. We introduce structure of level N for formal $\mathcal{O}$-modules and construct the universal deformation explicitly. This will be useful in the proof of theorem. Section 3 is on divisible $\mathcal{O}$-modules with structure of level N. In Section 4, we define Drinfeld modules with structure of level N and show, using the results of Sections 2 and 3, that the moduli are regular. In Section 5, we define a set of subgroups of $\mathrm{Aut}_A(N)$, which we call admissible. The definition is formulated so that the proof of our theorem can be given using Proposition 8.9.1 and Dickson's theorem. We give some examples. In Section 6, we give a statement of our main theorem (Theorem 6.1.1) and its applications. Hecke operators are discussed in this section. In Section 7, we prove the existence of a supersingular point on the moduli. This is logically independent of other sections. Section 8 is devoted to the proof of our main result.

2. Formal $\mathcal{O}$-modules with structure of level P

Let $\wp \subsetneq A$ be a nonzero prime ideal. Let $A_\wp$ be the ring of integers of the local field $F_\wp$ at $\wp$. We fix a uniformizer $\pi \in A_\wp$. Let $A_\wp^{\mathrm{ur}}$ denote the ring of integers of the maximal unramified extension and $\widehat{A}_\wp^{\mathrm{ur}}$ denote the completion. Let $\mathcal{O} = A_\wp$.

2.1. ***Definitions.*** Let $\mathcal{C}$ be the category of complete local $\widehat{A}_\wp^{\mathrm{ur}}$-algebras with residue field $\overline{\kappa(\wp)} := \widehat{A}_\wp^{\mathrm{ur}}/(\pi)$, where the morphisms are local $\widehat{A}_\wp^{\mathrm{ur}}$-algebra homomorphisms.

Let $R \in \mathcal{C}$ and let (F, f) be a formal $\mathcal{O}$-module (see [Drinfeld 1974, p. 563]) over R. Let P be a finitely generated torsion $\mathcal{O}$-module.

Definition 2.1.1. A structure of level P on (F, f) is a morphism of $\mathcal{O}$-modules

$$\psi : P \to \mathfrak{m}_R,$$

where $\mathfrak{m}_R \subset R$ is the maximal ideal endowed with the $\mathcal{O}$-module structure given by (F, f), such that the power series f_π is divisible by

$$\prod_{\alpha \in \operatorname{Ker} \pi : P \to P} (x - \psi(\alpha)).$$

Let (G, g) be a formal $\mathcal{O}$-module over $\overline{\kappa(\wp)}$ with (the unique) structure of level N.

Definition 2.1.2. A deformation of (G, g) is a formal $\mathcal{O}$-module (F, f) over R, for some $R \in \mathcal{C}$, with structure of level P such that $(F, f) \bmod \mathfrak{m}_R$ is isomorphic to (G, g).

Let $D_P = D_{d,P}$ denote the functor that associates $R \in \mathcal{C}$ with the set of isomorphism classes of deformations over R of (G, g).

2.2. *Universal deformation space of formal $\mathcal{O}$-modules over $\overline{\kappa(\wp)}$.* The following is due to Lubin and Tate, and Drinfeld, and the details are given in [Hopkins and Gross 1994].

Let $d \geq 1$. We define a formal $\mathcal{O}$-module $\widehat{F}_d$ over the ring $\mathcal{O}[[t_1, \dots, t_{d-1}]]$ of formal power series as follows. As a formal group, $\widehat{F}_d = \widehat{\mathbb{G}_a}$. The action of $a \in \kappa(\wp) \subset \mathcal{O}$ on $\widehat{F}_d$ is given by the power series $f_a(X) = aX$, and the action of π is given by the power series

$$f_\pi(X) = \pi X + t_1 X^{q_\wp} + \cdots + t_{d-1} X^{q_\wp^{d-1}} + X^{q_\wp^d}.$$

Set $F_d = \widehat{F}_d \otimes_{\mathcal{O}[[t_1,\dots,t_{d-1}]]} \kappa(\wp)$. Then F_d is a formal $\mathcal{O}$-module of height d over $\kappa(\wp)$. By [Drinfeld 1974, Proposition 1.6, p. 566], any formal $\mathcal{O}$-module over $\overline{\kappa(\wp)}$ of height d is isomorphic to $F_d \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$.

Proposition 2.2.1 (Lubin–Tate, Drinfeld, Gross–Hopkins). *The formal $\mathcal{O}$-module $\widehat{F}_d \widehat{\otimes}_{\mathcal{O}[[t_1,\dots,t_{d-1}]]} \widehat{A}_\wp^{\mathrm{ur}}[[t_1, \dots, t_{d-1}]]$ is the universal deformation of $F_d \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$.*

Proof. This is [Hopkins and Gross 1994, p. 45, Proposition 12.10]. □

2.3. *Universal deformation space of formal $\mathcal{O}$-module over $\overline{\kappa(\wp)}$ of level P.* We need the description of the universal deformation ring D_P of $F_d \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$ of level P. Set $D_0 = \mathcal{O}[[t_1, \dots, t_{d-1}]]$ by abuse of notation.

Proposition 2.3.1. *Let $\{e_i\}_{1 \leq i \leq r} \subset P$ be a minimal set of generators of P as $\mathcal{O}$-module. If $r > d$, then the functor D_P is representable by an empty scheme. Suppose that $r \leq d$. Then the functor D_P is representable by a regular D_0-algebra. We let,* *by abuse of notation*, D_P *denote the representing ring. Then the images of the* e_i*'s in* D_P *by the universal level structure and* t_i*'s for* $r \le i \le d-1$ *form a regular system of parameters.*

This statement follows essentially from the proof of the full level case treated in [Drinfeld 1974]. As we want to use the description of the ring D_P, we give some details below.

Proof. It suffices to treat the case $P = A/\wp^{n_1} \oplus \cdots \oplus A/\wp^{n_r}$ with $r \le d$, $n_1 \le \cdots \le n_r$ and $e_i = \bar{1} \in A/\wp^{n_i} \subset P$ for $1 \le i \le r$ with $1 \le n_1 \le \cdots \le n_r$. We construct the rings D_P by induction on the exponent (i.e., the n_r) of P.

Suppose the exponent is 1. Then $P = (A/\wp)^r$ for some $1 \le r \le d$. In this case, the claim holds true with $D_P = L_r$ of [Drinfeld 1974, Lemma on p. 572].

Suppose the claim holds true for $k < n_r$. Write

$$P = A/\wp^{n_1} \oplus \cdots \oplus A/\wp^{n_s} \oplus A/\wp^k \oplus \cdots \oplus A/\wp^k$$

with $1 \le n_1 \le \cdots \le n_s < k$. Set

$$P' = A/\wp^{n_1} \oplus \cdots \oplus A/\wp^{n_s} \oplus A/\wp^{k-1} \oplus \cdots \oplus A/\wp^{k-1}.$$

By inductive hypothesis, $D_{P'}$ is representable, and a regular system of parameters is given by the images of $e_1, \ldots, e_s, e_{s+1}, \ldots, e_r$ via the universal level structure and $t_r, \ldots, t_{d-1}$.

Now, set

$$D_P = D_{P'}[[\theta_{s+1}, \ldots, \theta_r]]/(f_\pi(\theta_{s+1}) - \psi(e_{s+1}), \ldots, f_\pi(\theta_r) - \psi(e_r)),$$

where ψ is the universal level structure of level P'. Then the claim holds true. □

3. Divisible $\mathcal{O}$-modules with structure of level P

3.1. Let $R \in \mathcal{C}$. We refer to [Drinfeld 1974, p. 574 C)] for the definition of a divisible $\mathcal{O}$-module over R. Let (F, f) be a divisible $\mathcal{O}$-module over R. The number j is defined by $F/F_{\text{loc}} \cong \operatorname{Spf} R \times (\mathcal{K}/\mathcal{O})^j$ where $\mathcal{K}$ is the field of fractions of $\mathcal{O}$. Let h denote the height (assumed to be finite) of the reduction of F_{loc}.

Definition 3.1.1. A structure of level P on a divisible $\mathcal{O}$-module (F, f) is an $\mathcal{O}$-module homomorphism

$$\phi : P \to \operatorname{Hom}_{\operatorname{Spf} R}(\operatorname{Spf} R, F)$$

such that there is a submodule $P_1 \subset P$, the restriction of ϕ to which is a structure of level P_1 on the formal $\mathcal{O}$-module F_{loc}, and such that the induced map $P/P_1 \to F/F_{\text{loc}}$ is an injection.

3.2. Let (G, g) be a divisible $\mathcal{O}$-module over $\overline{\kappa(\wp)}$ with structure of level P such that G_{loc} has height h and $G/G_{\mathrm{loc}} \cong (\mathcal{K}/\mathcal{O})^j$. The deformation of (G, g) is defined.

Proposition 3.2.1. *The functor that sends $R \in \mathcal{C}$ to the set of deformations of level P of the divisible module (G, g) over R is represented by the ring*

$$E_{(G,g)} \cong D_{P_1}[[d_1, \dots, d_j]],$$

where P_1 is the submodule that appears in the level P structure of (G, g).

Proof. The proof is almost identical to that of [Drinfeld 1974, Proposition 4.5, p. 574]; hence omitted. □

4. Regularity of moduli of level N

4.1. Let S be an A-scheme. Let $E \to S$ be a Drinfeld module over S. Let N be a torsion A-module.

Definition 4.1.1. A structure of level N on E is a homomorphism of A-modules $\psi : N \to \mathrm{Hom}_{S\text{-schemes}}(S, E)$ such that, for any element $a \in A$, the Cartier divisor

$$\sum_{x \in \mathrm{Ker}\, a : N \to N} \psi(x)$$

of E is a closed subscheme of $E[a] = \mathrm{Ker}(a : E \to E)$.

4.1.2. Let $I \subset A$ be a nonzero ideal and d be the rank of E. Then the case where $N = (I^{-1}/A)^d$ is the structure of level I as defined by Drinfeld [1974].

4.1.3. Let N be a nonzero finitely generated torsion A-module. For a nonzero prime ideal $\wp$ of A, let $N_\wp$ denote the $\wp$-primary part so that $N = \oplus_\wp N_\wp$ is the primary decomposition. Let $\psi : N \to \mathrm{Hom}(S, E)$ be a map and let $\psi_\wp : N_\wp \to \mathrm{Hom}(S, E)$ be the restriction for each nonzero prime ideal $\wp$. Then ψ is a structure of level N if and only if each $\psi_\wp$ is a structure of level $N_\wp$.

4.2. Let $U \subset \mathrm{Spec}\, A$ be an open subscheme. Let $\mathcal{M}^d_{N,U}$ denote the functor

$$(U\text{-scheme}) \to (\mathrm{Set})$$

that sends a U-scheme S to the set of isomorphism classes of Drinfeld modules of rank d over S with structure of level N.

Proposition 4.2.1. *Let N be a nonzero finitely generated torsion A-module.*

(1) *Suppose $|\mathrm{Supp}\, N| \geq 2$. Let $U \subset \mathrm{Spec}\, A$ be an open subscheme. Then the functor $\mathcal{M}^d_{N,U}$ is representable by a regular affine U-scheme.*

(2) *Let $Z \subset \mathrm{Supp}\, N$ be a nonempty subset. Let $U \subset \mathrm{Spec}\, A \setminus Z$ be an open subscheme. Then the functor $\mathcal{M}^d_{N,U}$ is representable by a regular affine U-scheme.*

Proof. The representability and regularity can be proved in a similar way to the case of structure of level I as in the proof of [Drinfeld 1974, p. 576, Proposition 5.3]. □

5. Admissible subgroups

The goal of this section is to define the set $\mathcal{S}_N$ of subgroups of $\mathrm{Aut}_A(N)$, which we call admissible.

5.1. Let N be a finitely generated torsion A-module, generated by at most d elements. Let $N = \bigoplus_{\wp \in \mathrm{Spec}\, A, \wp \neq (0)} N_\wp$ be the primary decomposition of N. We will define a set $\mathcal{S}_{N_\wp}$ of subgroups of $\mathrm{Aut}_A(N_\wp)$ (automorphisms as A-module) for each $\wp$. Set $\mathcal{S}_N = \prod_{\wp \in \mathrm{Spec}\, A, \wp \neq (0)} \mathcal{S}_{N_\wp}$. This $\mathcal{S}_N$ is regarded as a set of subgroups of $\mathrm{Aut}_A(N)$ in a natural manner.

From here on, we fix a nonzero prime ideal $\wp$ of A and assume that N is a $\wp$-primary torsion A-module generated by at most d elements.

5.1.1. Let us take an isomorphism $N \cong A_1 \oplus \cdots \oplus A_r$, where $1 \leq r \leq d$, $A_i = A/\wp^{n_i}$, $1 \leq n_i$, $1 \leq i \leq r$.

We use the following description of $\mathrm{Hom}_A(N, N)$ by matrices:

$$\mathrm{Hom}_A(N, N) = \{(\varphi_{i,j})_{1 \leq i,j \leq r} \mid \varphi_{i,j} \in \mathrm{Hom}_A(A_i, A_j)\}.$$

We have canonically $\mathrm{Hom}_A(A_i, A_j) = \wp^{n_{i,j}}/\wp^{n_j}$, where $n_{i,j} = \max\{0, n_j - n_i\}$. Let $m_{i,j} \in \mathbb{Z}$ with $n_{i,j} \leq m_{i,j} \leq n_j$ for $1 \leq i, j \leq r$. We set

$$H_{(m_{i,j})} = \{(\varphi_{i,j})_{1 \leq i,j \leq r} \mid \varphi_{i,j} \in (\delta_{i,j} + \wp^{m_{i,j}})/\wp^{n_j}\},$$

where $\delta_{i,j}$ is the Kronecker delta, and regard it as a subset of $\mathrm{Hom}_A(A_i, A_j)$.

5.1.2. We consider the following condition for a subgroup $H \subset \mathrm{Aut}_A(N)$:

(a) $\quad H = H_{(m_{i,j})} \cap \mathrm{Aut}_A(N)$ for some $m_{i,j} \in \mathbb{Z}$ with $n_{i,j} \leq m_{i,j} \leq n_j$.

Definition 5.1.3. We say that a subgroup $H \subset \mathrm{Aut}_A(N)$ is an admissible subgroup if condition (a) is satisfied for some direct sum decomposition $N = A_1 \oplus \cdots \oplus A_r$, where $1 \leq r \leq d$, $A_i = A/\wp^{n_i}$, $1 \leq n_i$, $1 \leq i \leq r$. We denote by $\mathcal{S}_N$ the set of admissible subgroups of $\mathrm{Aut}_A(N)$.

5.1.4. We introduce some more notation to investigate properties satisfied by admissible subgroups. For an A-module B and an ideal $I \subset A$, we set

$$B[I] = \bigcap_{i \in I} \mathrm{Ker}[i : B \to B].$$

We have

$$\begin{aligned} N[\wp] &= \wp^{n_1 - 1}/\wp^{n_1} \oplus \cdots \oplus \wp^{n_r - 1}/\wp^{n_r} \\ &\cong \kappa(\wp) \oplus \cdots \oplus \kappa(\wp), \end{aligned}$$

where $\kappa(\wp) = A/\wp$.

Let $H = H_{(m_{i,j})} \cap \mathrm{Aut}_A(N)$ be an admissible subgroup for some direct sum decomposition $N = A_1 \oplus \cdots \oplus A_r$, where $1 \le r \le d$, $A_i = A/\wp^{n_i}$, $1 \le n_i$, $1 \le i \le r$. Set

$$K = \mathrm{Image}[H \subset \mathrm{Aut}_A(N) \to \mathrm{Aut}_A(N[\wp])],$$

where the arrow is the canonical map. Let S denote the set of pairs (i, j) of integers with $1 \le i, j \le r$ satisfying $m_{i,j} \ne n_j - n_i$. Since the composite

$$\wp^{m_{i,j}}/\wp^{n_j} \hookrightarrow \wp^{n_{i,j}}/\wp^{n_j} \cong \mathrm{Hom}_A(A_i, A_j) \to \mathrm{Hom}_A(A_i[\wp], A_j[\wp])$$

is the zero map if and only if $(i, j) \in S$, the group K is identified with the set of invertible $r \times r$ matrices B with coefficients in $A/\wp$ such that for any integers i, j with $1 \le i, j \le r$, the (i, j)-th entry of $B - 1_r$ is equal to zero when $(i, j) \in S$. Using that K is a group, one can check that S satisfies the following property: if $(i, j), (j, k) \in S$ then $(i, k) \in S$. For $i, j \in \{1, \dots, r\}$, let us write $i \sim j$ if either $i = j$ or $((i, j), (j, i) \in S)$. It is easy to see that $\sim$ gives an equivalence relation on the set $\{1, \dots, r\}$. For $i \in \{1, \dots, r\}$, the equivalence class of i will be denoted by $\bar{i}$. As is easily seen, this equivalence relation has the following property, which we will use later: if $i \in \{1, \dots, r\}$ satisfies $(i, i) \notin S$, then $\bar{i}$ is the singleton $\{i\}$. Let us consider the quotient set $\{1, \dots, r\}/\sim$ under this equivalence relation. For $i, j \in \{1, \dots, r\}$, we write $\bar{i} \le \bar{j}$ if either $i \sim j$ or $(i, j) \in S$ is satisfied. The property of S mentioned above implies that this condition depends only on the classes $\bar{i}$, $\bar{j}$ of i, j, and the relation $\le$ gives a partial order on the set $\{1, \dots, r\}/\sim$. Let us choose a total order on $\{1, \dots, r\}/\sim$ extending this partial order and write $\{1, \dots, r\}/\sim = \{R_1, \dots, R_u\}$, $R_1 < \cdots < R_u$. For $s = 1, \dots, u$, let d_s denote the cardinality of the subset $R_s \subset \{1, \dots, r\}$. By permuting the elements $1, \dots, r$ if necessary, we may assume that $R_1, \dots, R_u$ satisfy the following condition:

(b) For $s = 1, \dots, u$, the set R_s is equal to the set of integers i satisfying

$$d_1 + \cdots + d_{s-1} < i \le d_1 + \cdots + d_s.$$

For $i = 0, \dots, u$ set $F_s = \bigoplus_{i=1}^{d_1+\cdots+d_s} \wp^{n_i - 1}/\wp^{n_i} \subset N[\wp]$. This gives an increasing filtration of $N[\wp]$ as $\kappa(\wp)$-vector space:

$$\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_u = N[\wp].$$

Let

$$P_{F_\bullet} = \{g \in \mathrm{Aut}_{\kappa(\wp)}(N[\wp]) \mid g(F_i) = F_i \text{ for all } 1 \le i \le r.\}$$

Then K has the following property:

(c) $$K \subset P_{F_\bullet}.$$

For $1 \le i \le u$, let $L_i = \mathrm{Aut}_{\kappa(\wp)}(F_i/F_{i-1})$ and regard them as quotients of $P_{F_\bullet}$.

Let R denote the set of integers $s \in \{1, \dots, u\}$ such that any $i, j \in R_s$ satisfy $(i, j) \in S$. Then K satisfies the following property:

$$\text{(d)} \qquad \operatorname{Image}\left[K \subset P_{F_\bullet} \to \prod_{1 \le i \le u} L_i\right] = \prod_{i \in R} L_i,$$

where $\prod_{i \in R} L_i$ is the trivial group if $R = \varnothing$.

5.2. ***Examples:*** $\boldsymbol{\Gamma_0}$ ***and*** $\boldsymbol{\Gamma_1}$**.** Let $I \subsetneq A$ be a nonzero ideal. Let $d \geq 1$. Let $N = (A/I)^d$. We consider the subgroup Γ_0 (resp. Γ_1) of $\mathrm{GL}_d(A/I)$ consisting of elements $(a_{ij})_{1 \le i,j \le d}$ such that

$$(a_{d,1}, \dots, a_{d,d-1}) \equiv (0, \dots, 0) \bmod I$$
$$(\text{resp. } (a_{d,1}, \dots, a_{d,d}) \equiv (0, \dots, 0, 1) \bmod I).$$

Then Γ_0 and Γ_1 belong to $\mathcal{S}_N$.

5.3. ***Examples: parabolic subgroups.*** Let $N = (A/I)^d$. Let $\mathbf{d} = (d_1, \dots, d_r)$ be a partition of $d = d_1 + \cdots + d_r$. There is an associated parabolic subgroup $P_{\mathbf{d}} \subset \mathrm{GL}_d(A/I)$. Then $P_{\mathbf{d}}$ is admissible.

6. Main Theorem and its application

Let us state our main theorem and corollaries in this section. The proof will be given in Section 8.

6.1. Theorem 6.1.1. *Let $d \geq 1$. Let N be a torsion A-module generated by at most d elements. Suppose $N = N_1 \oplus N_2$ for some nonzero N_1 and N_2 such that* $\operatorname{Supp} N_1 \cap \operatorname{Supp} N_2 = \varnothing$. *Let $U \subset$* Spec A *be an open subscheme such that the pair (N_2, U) satisfies assumption* (1) *or* (2) *of Proposition 4.2.1. Let $H \subset \mathrm{Aut}_A(N_1)$ be a subgroup that belongs to $\mathcal{S}_{N_1}$, which is regarded as a subgroup of* $\mathrm{Aut}_A(N)$. *Then the quotient*

$$\mathcal{M}^d_{N,U}/H$$

is regular.

Remark 6.1.2. The following case is not covered by Theorem 6.1.1. Let N be a torsion A-module generated by at most d elements such that $\operatorname{Supp} N = \{\wp\}$ for some nonzero prime ideal $\wp$. Let $U \subset \operatorname{Spec} A \setminus \operatorname{Supp} N$ be an open subscheme. By Proposition 4.2.1, the moduli $\mathcal{M}^d_{N,U}$ is representable. Take an admissible subgroup $H \in \mathcal{S}_N$. Then Theorem 6.1.1 does not refer to the quotient $\mathcal{M}^d_{N,U}/H$. We think our proof will work if there exists a nonzero A-submodule $N_0 \subset N$ such that N_0 is H-invariant. This assumption gives us a sequence $\mathcal{M}^d_{N,U} \to \mathcal{M}^d_{N,U}/H \to \mathcal{M}_{N_0}$ of moduli schemes which is similar to the one that appears in Section 8.4, which may be a starting point. If there does not exist such an A-submodule, we do not know if the quotient is regular.

6.2. Let N_1', N_1'', N_2 be torsion A-modules. Suppose $N_1' \oplus N_2$ and $N_1'' \oplus N_2$ are generated by at most d elements. Assume that $\operatorname{Supp} N_1' \cap \operatorname{Supp} N_2 = \varnothing$ and $\operatorname{Supp} N_1'' \cap \operatorname{Supp} N_2 = \varnothing$. Let $U \subset \operatorname{Spec} A$ be an open subscheme such that the pair (N_2, U) satisfies assumption (1) or (2) of Proposition 4.2.1.

6.2.1. Suppose we are given a surjective morphism of A-modules $f : N_1' \to N_1''$. We write $f' : N_1' \oplus N_2 \to N_1'' \oplus N_2$ for the induced surjection. Consider the functor that sends a Drinfeld module over a scheme S with structure of level $N_1' \oplus N_2$,

$$E \to S, \quad \psi : N_1' \oplus N_2 \to \operatorname{Hom}(S, E),$$

to that with level $N_1'' \oplus N_2$,

$$E/\psi(N_1' \oplus N_2) \to S, \quad \psi' : N_1'' \oplus N_2 \to \operatorname{Hom}(S, E/\operatorname{Ker} f'),$$

where $\psi'(x) = \psi(y) \bmod \operatorname{Ker} f'$ for any lift y of x. We denote the induced morphism

$$m_f : \mathcal{M}^d_{N_1' \oplus N_2, U} \to \mathcal{M}^d_{N_1'' \oplus N_2, U}.$$

As the morphism is finite (cf. [Laumon 1996, p. 8, (1.4.2)]) and both target and source are regular, we deduce that m_f is flat using [Altman and Kleiman 1970, V, p. 95, 3.6].

Let $H \subset \operatorname{Aut}_A(N_1')$ be a subgroup. Suppose H is admissible, i.e., $H \in \mathcal{S}_{N_1'}$. We regard H as a subgroup of $\operatorname{Aut}_A(N_1' \oplus N_2)$ by letting it act trivially on N_2. Let H act on N_1'' trivially and assume that f' is a H-equivariant map. Then the morphism m_f factors as $\mathcal{M}^d_{N_1' \oplus N_2, U} \to \mathcal{M}^d_{N_1' \oplus N_2, U}/H \xrightarrow{h} \mathcal{M}^d_{N_1'' \oplus N_2, U}$.

Corollary 6.2.2. *The morphism h is finite and flat.*

Proof. The morphism m_f is finite. Now,

$$\mathcal{M}^d_{N_1' \oplus N_2, U}/H \to \mathcal{M}^d_{N_1'' \oplus N_2, U}$$

is finite since $\mathcal{M}^d_{N_2, U}$ is noetherian and m_f is finite. We know from Theorem 6.1.1 that $\mathcal{M}^d_{N_1' \oplus N_2, U}/H$ is regular. We use the fact [Altman and Kleiman 1970, V, p. 95, 3.6] that a finite morphism between regular schemes of the same dimension is flat to conclude. □

6.2.3. There is an analogous corollary for injections. Let N_1', N_1'', N_2, and U be as above. Suppose we are given an injective morphism $f : N_1'' \to N_1'$ of A-modules. Let $f' : N_1'' \oplus N_2 \to N_1' \oplus N_2$ be the induced map. Consider the functor that sends a Drinfeld module over a scheme S with structure of level $N_1' \oplus N_2$,

$$E \to S, \quad \psi : N_1' \oplus N_2 \to \operatorname{Hom}(S, E),$$

to that with level $N_1'' \oplus N_2$,

$$E \to S, \quad \psi' : N_1'' \oplus N_2 \to \operatorname{Hom}(S, E),$$

where ψ' is the restriction of ψ. We let

$$r_f : \mathcal{M}^d_{N_1'\oplus N_2,U} \to \mathcal{M}^d_{N_1''\oplus N_2,U}$$

denote the induced morphism of schemes.

Let $H \in \mathcal{S}_{N_1'}$ be an admissible subgroup. Let H act on N_2 and N_1'' trivially and assume $f' : N_1'' \oplus N_2 \to N_1' \oplus N_2$ is H-equivariant. Then the morphism r_f factors as $\mathcal{M}^d_{N_1'\oplus N_2,U} \to \mathcal{M}^d_{N_1'\oplus N_2,U}/H \xrightarrow{h} \mathcal{M}^d_{N_1''\oplus N_2,U}$.

Corollary 6.2.4. *The morphism h is finite and flat.*

Proof. The proof is analogous to the proof of Corollary 6.2.2; hence omitted. □

6.3. ***Application: Hecke operators on higher Chow groups.*** Using Corollaries 6.2.2 and 6.2.4 of the previous section, we obtain the following description of Hecke operators on higher Chow groups of Drinfeld modular schemes over A.

6.3.1. *Hecke operators as finite correspondences.* Let $d \geq 1$. Let $I \subsetneq A$ be a nonzero ideal. Let $\wp \subset A$ be a prime ideal which is prime to I. Take U to be an open subscheme of $\operatorname{Spec} A \setminus \operatorname{Spec}(A/I)$ if $|\operatorname{Supp} A/I| = 1$ and U to be any open subscheme of $\operatorname{Spec} A$ otherwise.

Set $N_{0,k} = (A/\wp)^k$, $N_2 = (A/I)^d$ and $N_{1,k} = N_2 \oplus N_{0,k}$ for $0 \leq k \leq d$. By Proposition 4.2.1, the functors $\mathcal{M}^d_{N_{1,k},U}$ and $\mathcal{M}^d_{N_2,U}$ are representable by regular schemes.

Let $G_k = \operatorname{Aut}_A(N_{0,k})$. Then $G_k \in \mathcal{S}_{N_{1,k}}$. Regard G_k as a subgroup of $\operatorname{Aut}_A(N_{1,k})$ with the identity on the direct factor N_2.

Let $f_k : N_2 \to N_{1,k}$ be the canonical injection into the direct summand for each k. As in Section 6.2.3, we obtain a morphism $r_{f_k} : \mathcal{M}^d_{N_{1,k},U} \to \mathcal{M}^d_{N_2,U}$ which factors as

$$\mathcal{M}^d_{N_{1,k},U} \to \mathcal{M}^d_{N_{1,k},U}/G_k \xrightarrow{\bar{r}_{f_k}} \mathcal{M}^d_{N_2,U}.$$

We denote by $\bar{r}_{f_k}$ the second morphism.

Let $g_k : N_{1,k} \to N_2$ denote the canonical surjection onto the direct summand for each k. As in Section 6.2.1, we obtain a morphism $m_{g_k} : \mathcal{M}^d_{N_{1,k},U} \to \mathcal{M}^d_{N_2,U}$, which factors as

$$\mathcal{M}^d_{N_{1,k},U} \to \mathcal{M}^d_{N_{1,k},U}/G_k \xrightarrow{\overline{m}_{g_k}} \mathcal{M}^d_{N_2,U}.$$

Let us denote by $\overline{m}_{g_k}$ the second morphism.

As we have seen in the proofs of Corollaries 6.2.4 and 6.2.2, without using the main results of this article, we know that the morphisms $\overline{m}_{g_k}$ and $\bar{r}_{f_k}$ are finite. Therefore the diagram $\mathcal{M}^d_{N_2,U} \xleftarrow{\overline{m}_{g_k}} \mathcal{M}^d_{N_{1,k},U}/G_k \xrightarrow{\bar{r}_{f_k}} \mathcal{M}^d_{N_2,U}$ defines a finite correspondence in the sense used in [Mazza et al. 2006, p. 142, Theorem 17.21]. The action of this finite correspondence on the higher Chow group $\mathrm{CH}^*(\mathcal{M}^d_{N_2,k}, *)$ is denoted by $T_{\wp,k}$ and we define this to be the k-th Hecke operator at $\wp$.

6.3.2. As an application of our theorem, we can express Hecke operators as composition of pullback and pushforward as follows. As seen in Corollaries 6.2.2 and 6.2.4, the morphisms $\bar{r}_{f_k}$ and $\overline{m}_{g_k}$ are finite and moreover flat. It follows that the graphs of $\bar{r}_{f_k}$ and $\overline{m}_{g_k}$ are finite correspondences, and one can check that the composition equals $T_{\wp,k}$. That is, we have

$$T_{\wp,k} = (\overline{m}_{g_k})_*(\bar{r}_{f_k})^* : \mathrm{CH}^*(\mathcal{M}^d_{N_2,U}, *) \to \mathrm{CH}^*(\mathcal{M}^d_{N_2,U}, *),$$

for each $0 \le k \le d$, where upper star is the pullback and lower star is the pushforward.

7. Existence of supersingular points

In the proof of our main result, we use the fact that there exists a supersingular point at any $\wp$. The aim of this section is to give a proof of this fact.

7.1. Let $I \subset A$ be a nonzero ideal such that $|\mathrm{Spec}\,(A/I)| \ge 2$. Let $\wp \subset A$ be a nonzero prime ideal and let $\kappa(\wp) = A/\wp$. Let $\mathcal{M}^d_{I,A}$ denote the moduli functor of full level I Drinfeld modules of rank d. The subscript A indicates that the moduli is a functor from the category of A-scheme and it is representable by a scheme because of the condition on I. (We view it as a scheme.)

Lemma 7.1.1. *We have $\mathcal{M}^1_{I,A}(\overline{\kappa(\wp)}) \neq \varnothing$.*

Proof. Let F_∞ be the completion at ∞ of F. By [Drinfeld 1974, Corollary, p. 570], there exists a Drinfeld module E of rank 1 over F^s_∞ where F^s_∞ is a separable closure of F_∞. As F^s_∞ is separably closed, the I-torsion points $E[I]$ of E is isomorphic over F^s_∞ to the constant A-module scheme A/I. A choice of an isomorphism gives a level I structure, thus we see that $\mathcal{M}^1_{I,A}(F^s_\infty) \neq 0$. Since $\mathcal{M}^1_{I,A} \times_A F$ is an F-scheme and $\overline{F} \subset \overline{F_\infty}$, it follows that $\mathcal{M}^1_{I,A}(\overline{F}) \neq \varnothing$. Take a finite extension L/F such that $\mathcal{M}^1_{I,A}(L) \neq \varnothing$. Take a place $\wp_L$ over $\wp$. Let $L_{\wp_L}$ denote the completion of L at $\wp_L$. We have $\mathcal{M}^1_{I,A}(L_{\wp_L}) \neq \varnothing$ using the canonical map $\mathrm{Spec}\, L_{\wp_L} \to \mathrm{Spec}\, L$. Then, by [Drinfeld 1974, Proposition 7.1, p. 584], there exists a finite extension R of $L_{\wp_L}$ such that $\mathcal{M}^1_{I,A}(\kappa) \neq \varnothing$, where κ is the residue field of R. This proves the claim. □

7.2. ***Construction of a cover.*** Let C, ∞, A, F, F_∞, C_∞ be as above. Let $\wp \subset A$ be a nonzero prime ideal and $\kappa(\wp) = A/\wp$.

We construct a covering C' of C of degree d as follows. Let $f \in F$ be a nonzero element such that f has a zero of order 1 at each $\wp$ and ∞. (The existence of such an f can be proved by, for example, using the Riemann–Roch theorem.)

Set $F' = F[y]/(y^d - f)$ and let C' be the smooth projective curve whose function field is F'. Let $h : C' \to C$ denote the canonical map corresponding to $F \subset F'$. Then, by construction, h is totally ramified at $\wp$ and ∞. It follows that $h^{-1}(\infty)$ and $h^{-1}(\wp)$ are singletons. Let ∞' and $\wp'$ denote the fibers of ∞ and $\wp$ respectively.

Set $A' = H^0(C' \setminus \{\infty'\}, \mathcal{O}_{C'})$. Recall that (e.g., as in [Deligne and Husemoller 1987, p. 33, Remark 2.1]) we normalize the absolute values so that $|a|_\infty = |A/(a)|$ and $|a'|_{\infty'} = |A'/(a')|$ for $a \in A$ and $a' \in A'$ respectively, where $|\cdot|$ denotes the cardinality. In particular, we have $|h_A(a)|_{\infty'} = |a|_\infty^d$ where $h_A : A \to A'$ is the map induced by h.

7.3. *A rank 1 Drinfeld module for* (C', ∞') *and its formal module.*

7.3.1. Let $I' \subset A'$ be an (auxiliary) nonzero ideal such that $|\mathrm{Supp}\, A'/I'| \geq 2$. Take a nonzero prime ideal $\wp' \subset A'$. By Lemma 7.1.1 for $d = 1$, we have $M^1_{I',A'}(\overline{\kappa(\wp')}) \neq \varnothing$. Take $x' \in M^1_{I',A'}(\overline{\kappa(\wp')})$ and let $E'_{x'}$ denote the corresponding Drinfeld module over $\overline{\kappa(\wp')}$. We write $\varphi_{x'} : A' \to \mathrm{End}_{A'-\mathrm{gpsch}}(\mathbb{G}_{a,\overline{\kappa(\wp)}})$ for the corresponding ring homomorphism.

7.3.2. The universal deformation of $E'_{x'}$ is computed in [Drinfeld 1974, p. 576, Section 5C]. Let $\widetilde{E}'_{x'}$ denote the associated divisible $\widehat{A'_{\wp'}}^{\mathrm{ur}}$-module, and $\widetilde{E}'^{\mathrm{loc}}_{x'}$ denote the connected component containing zero (which is a formal $\widehat{A'_{\wp'}}^{\mathrm{ur}}$-module). This is isomorphic to the additive formal group with $f_{\pi'}(x) = \pi' x + x^{|\pi'|_{\infty'}}$ where π' is a uniformizer in $A'_{\wp'}$. Hence the formal $\widehat{A'_{\wp'}}^{\mathrm{ur}}$-module associated with $E'_{x'}$ is isomorphic to the additive formal group with $f_{\pi'}(x) = x^{|\pi'|_{\infty'}}$.

7.4. Using the ring homomorphism $\varphi_{x'}$, we construct a Drinfeld module (E, φ) for (C, ∞) as follows. Using the map $h_A : A \to A'$, we identify $\overline{\kappa(\wp)} = \overline{\kappa(\wp')}$. We define a ring homomorphism φ as the composite

$$A \xrightarrow{h_A} A' \xrightarrow{\varphi_{x'}} \mathrm{End}_{A'-\mathrm{gpsch}}(\mathbb{G}_{a,\overline{\kappa(\wp')}}) \to \mathrm{End}_{A-\mathrm{gpsch}}(\mathbb{G}_{a,\overline{\kappa(\wp)}}).$$

It can be checked that this defines a Drinfeld module (E, φ) for (C, ∞) over $\overline{\kappa(\wp)}$. The rank is d since $\deg(\varphi(a)) = |\varphi(a)|_{\infty'} = |a|_\infty^d$ for all nonzero a.

Proposition 7.4.1. *There exists a supersingular Drinfeld module (for (C, ∞)) of rank d over $\overline{\kappa(\wp)}$.*

Proof. A candidate (E, φ) was constructed above. It remains to show that E is supersingular. Let us take for our uniformizer π' the generator y of F' over F. Then $\pi = (\pi')^d = y^d$ is a uniformizer in $A^{\mathrm{ur}}_\wp$. We have

$$E[\wp]^{\mathrm{loc}} \cong \hat{E}[\wp] = \hat{E}[\pi] \cong \hat{E}'_{x'}[\pi'^d].$$

Here, the superscript ˆ denotes the associated formal module. The isogeny that has the last term as the kernel is of degree $|\pi|^d_\infty$. Since the isogeny that has $E[\wp]$ as the kernel is of degree $|\pi|^d_\infty$, it follows that $E[\wp]^{\mathrm{loc}} = E[\wp]$. This implies that E is supersingular. □

8. Proof of Main Theorem

8.1. Let the notation be as in Theorem 6.1.1.

8.2. ***One prime at a time.*** Let the setup be as in Theorem 6.1.1. Let $N_1 = N_{1,\wp_1} \oplus \cdots \oplus N_{1,\wp_r}$ be the primary decomposition of N_1. Let $H = H_{\wp_1} \times \cdots \times H_{\wp_r}$ be the decomposition given by the definition of admissible subgroup. (In particular, $H_{\wp_i} \in \mathcal{S}_{N_{1,\wp_i}}$ for $1 \le i \le r$.)

Now, the quotient is expressed as the fiber product

$$\mathcal{M}^d_{N,U}/H = \mathcal{M}^d_{N_{1,\wp_1}\oplus N_2,U}/H_{\wp_1} \times_{\mathcal{M}^d_{N_2,U}} \cdots \times_{\mathcal{M}^d_{N_2,U}} \mathcal{M}^d_{N_{1,\wp_r}\oplus N_2,U}/H_{\wp_r}.$$

Lemma 8.2.1. *Assume that Theorem 6.1.1 holds for each factor. Then Theorem 6.1.1 holds for the product.*

Proof. Because the singular locus is closed, it suffices to show that $\mathcal{M}^d_{N,U}/H$ is regular at every closed point.

Observe that the i-th map $\mathcal{M}^d_{N_{1,\wp_i}\oplus N_2,U}/H_{\wp_i} \to \mathcal{M}^d_{N_2,U}$ in the construction of the product is étale when restricted to $U \setminus \{\wp_i\}$ for $1 \le i \le r$.

Let x be a closed point of (the A-scheme) $\mathcal{M}^d_{N,U}/H$. If the image of x in Spec A is not contained in $\{\wp_1, \dots, \wp_r\}$, then by the observation above, there exists an open neighborhood of x such that $\mathcal{M}^d_{N,U}/H \to \mathcal{M}^d_{N_2,U}$ is étale. By Proposition 4.2.1, $\mathcal{M}^d_{N_2,U}$ is regular. Hence $\mathcal{M}^d_{N,U}/H$ is regular in the neighborhood of x.

If the image of x in Spec A is $\{\wp_i\}$ for some $1 \le i \le r$, it follows from the observation above that there exists an open neighborhood of x such that the i-th projection map $\mathcal{M}^d_{N,U}/H \to \mathcal{M}^d_{N_{1,\wp_i}\oplus N_2,U}/H_{\wp_i}$ is étale. By assumption, the target is regular. Hence $\mathcal{M}^d_{N,U}/H$ is regular at x. □

Hence it suffices to treat the case where N_1 is $\wp$-torsion for some nonzero prime ideal $\wp$. We assume from now that $N_1 = N_{1,\wp}$ and $H = H_{\wp_1} \in S_{N_1}$.

8.3. ***Away from the prime $\wp$.*** Let x be a closed point of $\mathcal{M}^d_{N,U}/H$. Take a closed point y of $\mathcal{M}^d_{N,U}$ that is sent to x via the canonical quotient map $\mathcal{M}^d_{N,U} \to \mathcal{M}^d_{N,U}/H$. Let $U_{N_1} = U \cap (\operatorname{Spec} A \setminus \operatorname{Supp} N_1) = U \setminus \{\wp\}$. Note that the restriction to U_{N_1} (the base change from U to U_{N_1}) $\mathcal{M}^d_{N,U_{N_1}} \to \mathcal{M}^d_{N,U_{N_1}}/H$ of the canonical quotient map is étale. Therefore the regularity at x follows from the regularity at y, which in turn follows from Proposition 4.2.1.

8.4. ***At the prime $\wp$; dependence on the height.*** We follow the outline given in [Katz and Mazur 1985].

We use Section 6.2.1 with $N'_1 = N_1$, $N''_1 = 0$, and $f : N'_1 \to N''_1$ the zero map. Then we obtain a morphism $m_f : \mathcal{M}^d_{N_1\oplus N_2,U} \to \mathcal{M}^d_{N_2,U}$. This morphism factors as

$$\mathcal{M}^d_{N_1\oplus N_2,U} \to \mathcal{M}^d_{N_1\oplus N_2,U}/H \to \mathcal{M}^d_{N_2,U}.$$

(Recall $N = N_1 \oplus N_2$.) Let $x \in \mathcal{M}^d_{N,U}/H(\overline{\kappa(\wp)})$, $y \in \mathcal{M}^d_{N,U}(\overline{\kappa(\wp)})$ be a preimage of x, and $z \in \mathcal{M}^d_{N_2,U}(\overline{\kappa(\wp)})$ be the image of x.

Let $\mathcal{O}_z$ be the local ring of $\mathcal{M}^d_{N_2,U} \times_U \operatorname{Spec} A^{\mathrm{ur}}_{\wp}$ at z and U_z be the completion of $\mathcal{O}_z$. By [Drinfeld 1974, p. 576 C], the ring U_z is isomorphic to the deformation ring of the formal $\mathcal{O}$-module with level N_2 structure associated with the Drinfeld module corresponding to the point z (see Proposition 3.2.1 for a description of the corresponding divisible $\mathcal{O}$-module and hence of the formal $\mathcal{O}$-module). We note that U_z depends only on the height of the associated formal $\mathcal{O}$-module.

Let us consider the following commutative diagram, where each of the squares is cartesian:

$$\begin{array}{ccccc}
\mathcal{M}^d_{N,U} & \longleftarrow & \mathcal{M}^d_{N,U} \times_{\mathcal{M}^d_{N_2,U}} \operatorname{Spec} \mathcal{O}_z & \longleftarrow & \mathcal{M}^d_{N,U} \times_{\mathcal{M}^d_{N_2,U}} \operatorname{Spec} U_z \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}^d_{N,U}/H & \longleftarrow & \mathcal{M}^d_{N,U}/H \times_{\mathcal{M}^d_{N_2,U}} \operatorname{Spec} \mathcal{O}_z & \longleftarrow & \mathcal{M}^d_{N,U}/H \times_{\mathcal{M}^d_{N_2,U}} \operatorname{Spec} U_z \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}^d_{N_2,U} & \longleftarrow & \operatorname{Spec} \mathcal{O}_z & \longleftarrow & \operatorname{Spec} U_z.
\end{array}$$

We note that the bottom horizontal arrows are flat, hence all horizontal arrows are flat.

The regularity of $\mathcal{M}^d_{N,U}/H$ at the image of the morphism from Spec $(\overline{\kappa(\wp)})$ corresponding to the $\overline{\kappa(\wp)}$-valued point x is equivalent to the regularity of

$$\mathcal{M}^d_{N,U}/H \times_{\mathcal{M}^d_{N_2,U}} \operatorname{Spec} U_z$$

at the unique point over x because the morphism $\operatorname{Spec} U_z \to \mathcal{M}^d_{N_2,U}$ is regular.

It follows from [Katz and Mazur 1985, p. 217, Proposition A7.1.3 (1)] that

$$(\mathcal{M}^d_{N,U}/H \times_{\mathcal{M}^d_{N_2,U}} \operatorname{Spec} U_z) \cong (\mathcal{M}^d_{N,U} \times_{\mathcal{M}^d_{N_2,U}} \operatorname{Spec} U_z)^H.$$

Notice that this scheme on the right depends only on the height of the associated formal module corresponding to z. Thus the regularity at x depends only on the height.

Remark 8.4.1. We remark that the isomorphism above is an analogue of [Katz and Mazur 1985, p. 194, Remark 7.1.4]. The idea for the argument above is also taken from [loc. cit.]. The main difference is that we do not use relative moduli problems.

8.5. ***Reduction to supersingular case.*** The height h of the formal $\mathcal{O}$-module corresponding to the point y ranges from 1 to d. (Note that this height is the same as that corresponding to the point z.) We call the point y supersingular if the height equals d. Note that in terms of f_π (see Section 2.2), h is the number such that $t_1 = \cdots = t_{h-1} = 0$ and $t_h \neq 0$.

Now since the supersingular locus is nonempty by Proposition 7.4.1 and closed, and since there exists a point of arbitrary height near a supersingular point, it suffices to treat the case $h = d$.

8.6. ***Reduction to the standard case.*** We now consider the case where the point y is supersingular. The completion of the local ring at y is isomorphic to D_{N_1}; see [Drinfeld 1974, p. 576]. The task is to show that the ring of H-invariants $(D_{N_1})^H$ is regular. The aim of this subsection is to reduce to the case where $N_1 = (A/\wp^n)^d$ for some n.

8.6.1. Let $P = (\wp^{-n}/A)^d$ and $Q = \wp^{-n_1}/A \oplus \cdots \oplus \wp^{-n_r}/A$ for some $1 \le n_1, \ldots, n_r \le n$, $1 \le r \le d$. There is an inclusion $Q \subset P$ induced by the inclusions $\wp^{-n_i}/A \subset \wp^{-n}/A \subset P$ where the first is the canonical one and the second is into the i-th summand for $1 \le i \le r$. Let us take an A-module L, whose $\wp$-primary component is zero, and an open subscheme $U \subset \operatorname{Spec} A$, so that the pairs $(Q \oplus L, U)$ and $(P \oplus L, U)$ satisfy one of the assumptions in Proposition 4.2.1.

Let us write $Q' = Q \oplus L$ and $P' = P \oplus L$. We then obtain a morphism $Q' \subset P'$ induced by the inclusion $Q \subset P$ and the identity on L, and hence a morphism $r : \mathcal{M}^d_{P',U} \to \mathcal{M}^d_{Q',U}$ using Section 6.2.3.

Set $G_{P',Q'} = \{g \in \operatorname{Aut}_A(P') \mid gQ' = Q', g|_{Q'} = \mathrm{id}_{Q'}\}$. Then the morphism r above factors as

$$\mathcal{M}^d_{P',U} \to \mathcal{M}^d_{P',U}/G_{P',Q'} \xrightarrow{h} \mathcal{M}^d_{Q',U}.$$

Lemma 8.6.2. *The morphism h is an isomorphism.*

Proof. One can check directly that h is an isomorphism away from $\wp$ (that is, over $U \setminus \{\wp\}$).

Now, $\mathcal{M}^d_{P',U}/G_{P',Q'}$ is normal, being a quotient of an affine regular scheme by a finite group action. We also have that $\mathcal{M}^d_{Q',U}$ is regular; hence normal. As they agree away from $\wp$ and h is finite, being the normalizations, they agree over U. □

8.6.3. Recall that N_1 is a $\wp$-torsion A-module which is generated by at most d elements. Write $N_1 = A_1 \oplus \cdots \oplus A_r$, where $A_i = A/\wp^{n_i}$ for $1 \le i \le r$ with some integer $n_i \ge 1$. Let us choose an integer $n \ge \max_i n_i$. Take an injection $A_i \hookrightarrow A/\wp^n$ of A-modules for $i = 1, \ldots, r$. Set $\widetilde{N}_1 = (A/\wp^n)^d$. These injections give an injection $N \hookrightarrow (A/\wp^n)^r \hookrightarrow (A/\wp^n)^d = \widetilde{N}_1$ where the second arrow is the injection into the first r factors. For a subgroup $H \subset \operatorname{Aut}_A(N_1)$, let $\widetilde{H}$ denote the subgroup $\{g \in \operatorname{Aut}_A(\widetilde{N}_1) \mid g(N_1) = N_1, g|_{N_1} \in H\}$ of $\operatorname{Aut}_A(\widetilde{N}_1)$. As is easily seen, H is an admissible subgroup of $\operatorname{Aut}_A(N_1)$ with respect to the direct sum decomposition as above if and only if $\widetilde{H}$ is an admissible subgroup of $\operatorname{Aut}_A(\widetilde{N}_1)$.

From this and Lemma 8.6.2, we see that it suffices to treat the case $N_1 = (A/\wp^n)^d$.

8.7. ***Some subgroups of H.*** We now consider the case $N_1 = (A/\wp^n)^d$ for some $n \ge 1$ in more detail. We will need certain subgroups of an admissible subgroup H for the proof of Theorem 6.1.1. We label them below.

8.7.1. To introduce subgroups of an admissible subgroup $H \subset \mathrm{Aut}_A(N_1)$, we introduce some more notation. Let $e_i = \bar{1} \in A/\wp^n \subset (A/\wp^n)^d = N_1$ where the inclusion is into the i-th factor. We regard $N_1 = (A/\wp^n)^d$ as the set of row vectors, on which $\mathrm{Aut}_A(N_1) = \mathrm{GL}_d(A/\wp^n)^{\mathrm{op}}$ acts as the multiplication from the right.

Let $R_{i,j} \subset A/\wp^n$ be subsets for $1 \le i, j \le d$. We use the notation $\{(R_{i,j})\}$ to denote the subset

$$\{(r_{i,j}) \in M_d(A/\wp^n) \mid r_{i,j} \in \delta_{i,j} + R_{i,j} \text{ for } 1 \le i, j \le d\}$$

of the set $M_d(A/\wp^n)$ of d-by-d matrices, where $\delta_{i,j}$ is the Kronecker delta.

8.7.2. Let H be an admissible subgroup with respect to the standard direct sum decomposition of N_1. We assume that the subsets $R_1, \dots, R_u$ of $\{1, \dots, d\}$ introduced in Section 5.1.4 satisfy condition (b)enumi. Then, by condition (a), there exists $(m_{i,j}) \in M_d(\mathbb{Z})$ with $0 \le m_{i,j} \le n$ such that

$$H = \{(\bar{\wp}^{m_{i,j}})\},$$

where $\bar{\wp} = \wp/\wp^n$ is the maximal ideal of $A/\wp^n$. Let K and L_i $(i = 1, \dots, u)$ be as in Section 5.1 and set

$$J = \mathrm{Ker}[H \twoheadrightarrow K \to \prod_{1 \le i \le u} L_i].$$

Let

$$m'_{i,j} = \begin{cases} 1 & \text{if } m_{i,j} = m_{j,i} = 0, \\ m_{i,j} & \text{otherwise.} \end{cases}$$

Then we have $J = \{(\bar{\wp}^{m'_{i,j}})\}$.

Write $J_{i,j} = \bar{\wp}^{m'_{i,j}}$. Set $J^k_{i,j} = J_{i,j} \cap \bar{\wp}^k$ for $0 \le k \le n$, and $J^k = \{(J^k_{i,j})\}$. Set

$$J^{k,\ell}_{i,j} = \begin{cases} J^k_{i,j} & \text{if } i \le l, \\ J^{k+1}_{i,j} & \text{if } i > l, \end{cases}$$

for $0 \le l \le d$ and $J^{k,\ell} = \{(J^{k,\ell}_{i,j})\}$. We have

$$\begin{aligned} J = J^0 = J^{0,d} \supset \cdots \supset J^{0,0} = J^1 = J^{1,d} \supset \cdots \supset J^{1,0} = J^2 \\ = J^{2,d} \supset \cdots \supset J^{2,0} = J^3 = \cdots . \end{aligned}$$

Note that

$$J^k = \mathrm{Ker}[J \subset \mathrm{GL}_d(A/\wp^n) \to \mathrm{GL}_d(A/\wp^k)],$$

where the arrow is the canonical map, and

$$J^{k,\ell} = \{h \in J^k \mid e_m h \equiv e_m \bmod \bar{\wp}^{k+1} \text{ for } \ell < m \le d\}.$$

8.7.3. It follows from this description that J^{k+1} is a normal subgroup of J^k for $k \geq 0$, and the quotient J^k/J^{k+1} is abelian for $k \geq 1$. Hence $J^{k,\ell-1}$ is a normal subgroup of $J^{k,\ell}$ for $k \geq 1$, $1 \leq l \leq d$. The situation is different for $k = 0$. However a similar statement also holds for $k = 0$.

Lemma 8.7.4. *Let $1 \leq \ell \leq d$. Then $J^{0,\ell-1}$ is a normal subgroup of $J^{0,\ell}$ and the quotient $J^{0,\ell}/J^{0,\ell-1}$ is abelian.*

Proof. It suffices to prove that $J^{0,\ell-1}/J^1$ is a normal subgroup of J/J^1 and the quotient $(J^{0,\ell}/J^1)/(J^{0,\ell-1}/J^1)$ is abelian.

By definition we have $J^1 = \mathrm{Ker}[H \twoheadrightarrow K]$. Hence the surjection $H \twoheadrightarrow K$ induces an isomorphism $J/J^1 \cong \mathrm{Ker}[K \to \prod_{1 \leq i \leq u} L_i]$. Via this isomorphism we regard J/J^1 as a subgroup of $K \subset \mathrm{Aut}_A(N_1[\wp]) = \mathrm{GL}_d(\kappa(\wp))^{\mathrm{op}}$. In particular each $g \in J/J^1$ is an element of $\mathrm{GL}_d(\kappa(\wp))$ and acts on $N_1[\wp] = (\wp^{n-1}/\wp^n)^d$, whose elements we regard as row vectors, as the multiplication by g from the right. Note that the submodule

$$N_1[\wp]_{\geq \ell} = \bigoplus_{\ell \leq i \leq d} \wp^{n-1}/\wp^n$$

of $N_1[\wp]$ is stable under the action of J/J^1. It follows from the definition of $J^{0,\ell-1}$ that $J^{0,\ell-1}/J^1$ is equal to the kernel of $J/J^1 \to \mathrm{Aut}_A(N_1[\wp]_{\geq \ell})$. This in particular shows that $J^{0,\ell-1}/J^1$ is a normal subgroup of J/J^1. Let us fix an element $e \in N_1[\wp]_{\geq \ell} \setminus N_1[\wp]_{\geq \ell+1}$. Let us consider the map $f : J^{0,\ell}/J^1 \to N_1[\wp]_{\geq \ell}$ that sends $g \in J^{0,\ell}/J^1$ to $eg - e \in N_1[\wp]_{\geq \ell}$. It is then easy to check that f is a homomorphism of groups and the kernel of f is equal to $J^{0,\ell-1}/J^1$. This shows that the quotient $(J^{0,\ell}/J^1)/(J^{0,\ell-1}/J^1)$ is abelian and the proof is complete. □

8.7.5. Set $Q^{k,\ell} = J^{k,\ell}/J^{k,\ell-1}$ for $k \geq 0$, $1 \leq l \leq d$. We take the set of representatives of $Q^{k,\ell}$ as follows. Let us choose a uniformizer $\pi \in A_\wp$ and set

$$Q^{k,\ell}_{i,j} = \begin{cases} \{0\} & \text{if } J^{k,\ell}_{i,j} = J^{k,\ell-1}_{i,j}, \\ \{a\pi^k \mid a \in \kappa(\wp)\} & \text{if } J^{k,\ell}_{i,j} \neq J^{k,\ell-1}_{i,j}. \end{cases}$$

Then the set $\{(Q^{k,\ell}_{i,j})\}$ is a subset of $J^{k,\ell}$ and is a complete set of representatives for $Q^{k,\ell}$. Moreover, the image of $\{(Q^{k,\ell}_{i,j})\}$ under the homomorphism $J^{k,\ell} \to J^{k,\ell}/J^{k+1}$ is a subgroup of $J^{k,\ell+1}/J^{k+1}$.

8.8. *Some additive polynomials.*

8.8.1. We regard the $\wp$-torsion A-module N_1 as an $\mathcal{O}$-module. A-submodules are regarded as $\mathcal{O}$-submodules and vice versa.

Recall (see Section 2) that D_{N_1} is the universal deformation ring of the formal $\mathcal{O}$-module $F_d \widehat{\otimes}_{\kappa(\wp)} \overline{\kappa(\wp)}$ (see Definition 2.1.2 and Proposition 2.2.1) with maximal height equipped with a structure of level N_1. The formal $\mathcal{O}$-module is isomorphic as a formal group to $\widehat{\mathbb{G}}_a$, and we use f as in Section 2.2 to denote the $\mathcal{O}$-structure. The explicit description of D_{N_1} is found in (the proof of) Proposition 2.3.1. We regard elements of N_1 as elements of $\mathfrak{m}_{D_{N_1}} \subset D_{N_1}$ via the universal level structure

$$\psi : N_1 \to \mathfrak{m}_{D_{N_1}},$$

where $\mathfrak{m}_{D_{N_1}}$ is the maximal ideal of D_{N_1}. (That is, we write n_1 to mean $\psi(n_1)$.)

We regard elements of N_1 as row vectors with coordinates in $A/\wp^n$. Then the group $\mathrm{GL}_d(A/\wp^n)$ acts on N_1 from the right, where the action of $g \in \mathrm{GL}_d(A/\wp^n)$ is given by the multiplication $- \cdot g : N_1 \to N_1$ by g from the right. Let $\mathcal{C}$ be as in Section 2. Let $g \in \mathrm{GL}_d(A/\wp^n)$. For a deformation (F, f) over some $R \in \mathcal{C}$ with structure ψ_F of level N_1, let $g\psi_F$ denote the composite $\psi_F \circ (- \cdot g)$. Then $g\psi_F$ is another structure of level N_1. By sending (F, f, ψ_F) to $(F, f, g\psi_F)$ for each (F, f, ψ_F), we obtain an automorphism of the universal deformation ring D_{N_1}. We denote this automorphism also by $- \cdot g$. It is then straightforward to check that the equality $\psi(x) \cdot g = \psi(xg)$ holds for any $x \in N_1$.

8.8.2. Let us introduce some additive polynomials with coefficients in D_{N_1} and list some of the properties. Let $M \subset N_1 = (A/\wp^n)^d$ be an A-submodule. We set

$$f^M(x) = \prod_{\alpha \in M} (x - \alpha) \in D_{N_1}[x].$$

As M is an abelian group, $f^M(x)$ is an additive polynomial, that is $f^M(x+y) = f^M(x) + f^M(y)$ holds where x and y are indeterminates.

Since $\kappa(\wp) \subset A_\wp = \mathcal{O}$, any M as above is a $\kappa(\wp)$-vector space. Hence we have $f^M(sx) = sf^M(x)$ for any $s \in \kappa(\wp)$. It follows from the construction of $\widehat{F}_d$ given in Section 2.2 that the action of $s \in \kappa(\wp) \subset \mathcal{O}$ on $\mathfrak{m}_{D_{N_1}}$ as a formal $\mathcal{O}$-module is equal to the multiplication by s in the $\overline{\kappa(\wp)}$-algebra D_{N_1}. Hence we have $f^M(sn) = sf^M(n)$ for any $s \in \kappa(\wp)$ and for any $n \in N_1$.

Let $y \in D_{N_1}$ and $g \in \mathrm{GL}_d(A/\wp^n)$. Then we have

$$(f^M(y)) \cdot g = f^{Mg}(yg) = \prod_{\alpha \in Mg} (yg - \alpha).$$

8.8.3. Now we look at the $\mathcal{O}$-module structure f of the universal deformation (D_{N_1}, f) (see Section 2.2).

For $z \in \kappa(\wp)$, we have $f_z(x) = zx$.

Let $n \geq m \geq 0$. Then the power series (actually a polynomial in the case at hand)

$$f_{\pi^m}(x) \in D_{N_1}[[x]]$$

giving the multiplication-by-π^m has as the set of roots the set of π^m-torsion points. Thus we have

$$f_{\pi^m}(x) = f^M(x),$$

where $M = N_1[\pi^m] \subset (A/\wp^n)^d$ is the set of π^m-torsion points.

As the universal level structure ψ is an $\mathcal{O}$-module homomorphism, we have $f_a(n_1) = an_1$ for $a \in \mathcal{O}$ and $n_1 \in N_1$ (by abuse of notation, we write n_1 for $\psi(n_1)$). Using the additivity, we have $f_{c\pi^k}(n_1+n_1') = f_{c\pi^k}(n_1)+f_{c\pi^k}(n_1') = c\pi^k n_1 + c\pi^k n_1'$ for $0 \leq k \leq n$, $n_1, n_1' \in N_1$, and $c \in \kappa(\wp)$.

8.9. ***A proposition of Katz and Mazur.*** We recall the following proposition, using the notation in [Katz and Mazur 1985].

Proposition 8.9.1 [Katz and Mazur 1985, Proposition 7.5.2, p. 205]. *Let A be a complete noetherian local ring which is regular of dimension n and whose residue field is perfect. Let G be a finite subgroup of* $\mathrm{Aut}(A)$, *such that every $g \in G$ acts trivially on the residue field of A. Let $(x_1, \dots, x_{n-1}, y)$ be a regular system of parameters in A. Assume that for each $g \in G$ we have*

1. $g(x_i) = x_i$ *for* $i = 1, \dots, n-1$,
2. $g(y) \equiv uy \bmod (x_1, \dots, x_{n-1})$ *for some unit* $u \in A^\times$.

Then

(i) *A is free over A^G with basis* $1, y, y^2, \dots, (y)^{\#G-1}$.

(ii) *A^G is a regular local ring of dimension n.*

(iii) *A regular system of parameters for A^G is* $(x_1, \dots, x_{n-1}, N(y))$, *where $N(y)$ is the norm $\prod_{g\in G} g(y)$.* □

Lemma 8.9.2. *In the setting of Proposition 8.9.1, assumption* 2 *follows from assumption* 1.

Proof. Assumption 1 implies that the action of G on A induces the action of G on the quotient ring $\bar{A} = A/(x_1, \dots, x_{n-1})$. Since $x_1, \dots, x_{n-1}, y$ form a regular system of parameters, $\bar{A}$ is a discrete valuation ring and the image $\bar{y}$ of y in $\bar{A}$ is a uniformizer of $\bar{A}$. Hence $g(\bar{y})$ is also a uniformizer and we have $g(\bar{y}) = u\bar{y}$ for some unit u of $\bar{A}$. Since any lift $\tilde{u} \in A$ of u is a unit in A, it follows that assumption 2 is satisfied. □

8.10. For $k \geq 0$ and $1 \leq i \leq d$, we let J_i^k denote the A-submodule

$$J_i^k = J_{i,1}^k \oplus \cdots \oplus J_{i,d}^k$$

of N_1. For $0 \leq \ell \leq d$, we set

$$J_i^{k,\ell} = J_{i,1}^{k,\ell} \oplus \cdots \oplus J_{i,d}^{k,\ell}.$$

Proposition 8.10.1. *Let $k \geq 0$ and $0 \leq \ell \leq d$.*

1. *The ring of invariants $(D_{N_1})^{J^{k,\ell}}$ is regular.*
2. *$f^{J_i^{k,\ell}}(e_i)$ for $1 \leq i \leq d$ form a regular system of parameters in $(D_{N_1})^{J^{k,\ell}}$.*

Proof. We prove this inductively. The groups $J^{k,\ell}$ are ordered by inclusion (both k and ℓ run). For k large, $J^{k,\ell} = \{1\}$, so the claim holds true by Proposition 2.3.1.

Let $1 \le \ell \le d$. (The case $\ell = 0$ appears as the case $\ell = d$.) Suppose the claim holds true for $J^{k,\ell-1}$. We prove the claim for $J^{k,\ell}$.

We use Proposition 8.9.1 in the following manner. Recall that $J^{k,\ell-1}$ is a normal subgroup of $J^{k,\ell}$. We set $G = J^{k,\ell}/J^{k,\ell-1}$ and $A = (D_{N_1})^{J^{k,\ell-1}}$ so that $A^G = (D_{N_1})^{J^{k,\ell}}$. We let $y = f^{J_\ell^{k,\ell-1}}$, n in Proposition 8.9.1 to be d, and

$$(x_1, \dots, x_{d-1}) = (f^{J_1^{k,\ell-1}}(e_1), \overset{\ell}{\dots}, f^{J_d^{k,\ell-1}}(e_d))$$

(equality not as ideals but as tuples) where the ℓ means omission of the ℓ-th component.

Lemma 8.10.2. *With notation as above, assumptions* 1, 2 *of Proposition 8.9.1 hold.*

Proof. By Lemma 8.9.2, it suffices to prove that assumption 1 holds.

By the inductive hypothesis, the elements $f^{J_i^{k,\ell-1}}(e_i) \in (D_{N_1})^{J^{k,\ell-1}}$ for $1 \le i \le d$ form a regular system of parameters. Let i be an integer with $1 \le i \le d$, $i \ne \ell$. We show that $f^{J_i^{k,\ell-1}}(e_i)$ is fixed under the action of $J^{k,\ell}$.

Take $g \in Q^{k,\ell} = J^{k,\ell}/J^{k,\ell-1}$. Take the representative $\widetilde{g}$ of g in $\{(Q_{i,j}^{k,\ell})\}$. Since $i \ne \ell$, we have $e_i\widetilde{g} = e_i$ and $J_i^{k,\ell-1} = J_i^{k,\ell}$. Hence we have $f^{J_i^{k,\ell-1}}(e_i) = f^{J_i^{k,\ell}}(e_i)$ and $f^{J_i^{k,\ell-1}}(e_i)g = f^{J_i^{k,\ell}\widetilde{g}}(e_i)$. Hence it suffices to show $J_i^{k,\ell}\widetilde{g} = J_i^{k,\ell}$.

Let $x \in J_i^{k,\ell}$. Let $A_i(x)$ denote the d-by-d matrix whose i-th row is x and the rest is zero. Then we have $I_d + A_i(x) \in J^{k,\ell}$, where I_d denotes the d-by-d identity matrix. Observe that the i-th row of the product $(I_d + A_i(x))\widetilde{g} \in J^{k,\ell}$ is equal to $e_i + x\widetilde{g}$. This shows $x\widetilde{g} \in J_i^{k,\ell}$. Since $x \in J_i^{k,\ell}$ is arbitrary, we have $J_i^{k,\ell}\widetilde{g} \subset J_i^{k,\ell}$. Since $J_i^{k,\ell}$ is finite, we obtain the equality $J_i^{k,\ell}\widetilde{g} = J_i^{k,\ell}$, as desired. □

With Lemma 8.10.2 above, we apply Proposition 8.9.1 to our situation. This completes the proof of Proposition 8.10.1. □

8.11. Using Proposition 8.10.1 with $k = 0$ and $\ell = d$, we see that the ring of invariants $(D_{N_1})^J$ is regular and that $f^{J_i^{0,d}}(e_i)$ for $1 \le i \le d$ form a regular system of parameters. As $H/J \cong \prod_{j \in R} L_j$, it remains to take the invariants under the action of $\prod_{j \in R} L_j$.

Take $g = (g_{c,d}) \in \prod_{j \in R} L_j$. We have

$$\begin{aligned} f^{J_b^{0,d}}(e_b) \cdot g = f^{J_b^{0,d}g}(e_b g) &= f^{J_b^{0,d}}(e_b g) = f^{J_b^{0,d}}(g_{b,1}e_1 + \dots + g_{b,d}e_d) \\ &= f^{J_b^{0,d}}(g_{b,1}e_1) + \dots + f^{J_b^{0,d}}(g_{b,d}e_d) \\ &= g_{b,1} f^{J_b^{0,d}}(e_1) + \dots + g_{b,d} f^{J_b^{0,d}}(e_d) \end{aligned}$$

for $1 \le b \le d$. This follows from the properties of additive polynomials collected in Section 8.8.

Let $d_1, \ldots, d_u$ be as in Section 5.1.4. For $s \in \{1, \ldots, u\}$, set $I_s = \{d_1 + \cdots + d_{s-1} + 1, \ldots, d_1 + \cdots + d_s\}$.

Lemma 8.11.1. *For any $i, i' \in I_s$, we have $J_i^{0,d} = J_{i'}^{0,d}$.*

Proof. We use the equivalence relation $\sim$ on the set $\{1, \ldots, d\}$ that was introduced in Section 5.1.4 (we are taking r there to be d here).

Since H is a group, it therefore follows that for $j, j', j'' \in \{1, \ldots, d\}$, we have $m_{j,j'} + m_{j',j''} \geq m_{j,j''}$. This implies that, if $j \sim j'$, then we have $m_{j,j''} = m_{j',j''}$. Note that $i \sim i'$. Hence $m_{i,j} = m_{i',j}$ for any $j \in \{1, \ldots, d\}$. This proves the claim. □

Let us write $x_i = f^{J_i^{0,d}}(e_i)$ for $1 \leq i \leq d$. We identify $D_{N_1}^J$ with $\overline{\kappa(\wp)}[[x_1, \ldots, x_d]]$. Consider the subring $B_s = \overline{\kappa(\wp)}[x_i]_{i \in I_s}$ for $s \in R$.

Lemma 8.11.2. *The subring B_s is stable under the action of L_s and the action is $\kappa(\wp)$-linear.*

Proof. This follows from the previous lemma and the computation of the action of $\prod_{j \in R} L_j$. □

By Dickson's theorem (Theorem 8.12.1), we have that $B_s^{L_s}$ is a polynomial ring $\overline{\kappa(\wp)}[f_i]_{i \in R_s}$ for some homogeneous polynomials f_i in the x_i's. Hence we obtain $\overline{\kappa(\wp)}[x_1, \ldots, x_d]^{\prod_{j \in R} L_j} = \overline{\kappa(\wp)}[f_1, \ldots, f_d]$ for some homogeneous polynomials f_i.

Lemma 8.11.3. *We have $\overline{\kappa(\wp)}[[x_1, \ldots, x_d]]^{\prod_{j \in R} L_j} = \overline{\kappa(\wp)}[[f_1, \ldots, f_d]]$.*

Proof. The inclusion of the right-hand side into the left-hand side is obvious.

Let $y \in \overline{\kappa(\wp)}[[x_1, \ldots, x_d]]^{\prod_{j \in R} L_j}$ and write $y = \sum_{i \geq 0} y_i$, where each y_i is a homogeneous polynomial of degree i in the x's. By definition, $\sigma \in \prod_{j \in R} L_j$ sends a homogeneous polynomial to a homogeneous polynomial of the same degree. Therefore, each y_i belongs to $\overline{\kappa(\wp)}[x_1, \ldots, x_d]^{\prod_{j \in R} L_j}$, hence to $\overline{\kappa(\wp)}[f_1, \ldots, f_d]$.

Let $g = c \prod_{i=1}^d f_i^{a_i} \in \overline{\kappa(\wp)}[f_1, \ldots, f_d]$ be a monomial, where $c \in \overline{\kappa(\wp)}$ and $a_i \geq 0$ for each $1 \leq i \leq d$. Then g is homogeneous of degree $\sum_{i=1}^d a_i \deg f_i$ in the x's. Hence g appears as a summand of y_m only if $m = \sum_{i=1}^d a_i \deg f_i$. In particular, g appears only finitely many times. Therefore $\sum_{i \geq 0} y_i$ defines an element of $\overline{\kappa(\wp)}[[f_1, \ldots, f_d]]$. This gives the other inclusion. □

Proof of Theorem 6.1.1. The theorem follows from Lemma 8.11.3. □

8.12. We recall Dickson's theorem. Let V be a d-dimensional $\kappa(\wp)$-vector space. Taking a basis $\alpha_1, \ldots, \alpha_d$, we identify the symmetric algebra $\mathrm{Sym}_{\kappa(\wp)} V$ with the polynomial ring $\kappa(\wp)[\alpha_1, \ldots, \alpha_d]$. Let $G = \mathrm{GL}(V)$. Let $\mathbf{F}$ be a field containing $\kappa(\wp)$. Then G acts on $\mathbf{F}[\alpha_1, \ldots, \alpha_d] = \mathrm{Sym}_{\kappa(\wp)}(V) \otimes_{\kappa(\wp)} \mathbf{F}$.

Theorem 8.12.1 [Dickson 1911]. (see [Smith 1995, p. 239, Theorem 8.1.5]) *The ring of invariants $\mathbf{F}[\alpha_1, \ldots, \alpha_d]^G$ is a polynomial ring $\mathbf{F}[\beta_1, \ldots, \beta_d]$ for some explicitly given homogeneous polynomials $\beta_1, \ldots, \beta_d$.* □

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