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We study the decomposability of Specht modules labeled by *bihooks*, bipartitions with a hook in each component, for the Iwahori–Hecke algebra of type *B*. In all characteristics, we determine a large family of decomposable Specht modules, and conjecture that these provide a complete list of decomposable Specht modules indexed by bihooks. We prove the conjecture for small *n*.

1. Introduction

Specht modules are of fundamental importance in the study of reflection groups and their deformations. We are particularly interested in the Iwahori–Hecke algebras of types A and B. In type B, these Hecke algebras have been studied from the point of view of their decomposition numbers [Fayers 2006; Ariki and Jacon 2010] and their Kazhdan–Lusztig theory [Geck et al. 2008; Jacon 2011] and via applications from higher representation theory [Brundan and Stroppel 2011]. Nowadays, the subject often takes on a more diagrammatic and categorical flavor, for example with the inception of Elias and Williamson's diagrammatic Hecke category [2016], Webster's diagrammatic Cherednik algebra [2017] and recent work of Elias and Losev [2017].

It is known by [Dipper and James 1991; Rouquier 2008b; Fayers and Speyer 2016] that the Specht modules are indecomposable if the *quantum characteristic e* is not 2, and under the further assumption in type B that the parameters κ_1 and κ_2 are distinct. Rouquier's work in fact gives us that the Hecke algebras admit faithful quasihereditary covers, whence indecomposability follows easily by considering the trivial endomorphism spaces of standard modules.

In type A, Murphy [1980] and Speyer [2014] completely determined the decomposability of Specht modules indexed by *hook* partitions. The general case is very difficult, owing to the complicated structure of the endomorphism rings of Specht modules. In the case of the symmetric group, Dodge and Fayers [2012] give the first new family of decomposable Specht modules in thirty years, which are

indexed by partitions of the form $(a, 3, 1^b)$. Parallel to this, the graded composition multiplicities for Specht modules indexed by hooks have been determined using Fock space machinery in [Chuang et al. 2004].

Here, we take the natural first step in extending this study of decomposable Specht modules to Iwahori–Hecke algebras of type B. We study Specht modules indexed by *bihooks*, that is bipartitions for which both components are hook partitions. In a certain subfamily of these, Sutton [2018; 2020] has determined graded decomposition numbers, drawing an analogue in type B with the aforementioned work of [Chuang et al. 2004]. As in [Speyer 2014], we study these Specht modules from the perspective of the cyclotomic Khovanov–Lauda–Rouquier algebras that were introduced by Khovanov and Lauda [2009] and Rouquier [2008a], an equivalent point of view by virtue of the isomorphism theorem of Brundan and Kleshchev [2009a]. In this framework, we investigate endomorphisms of Specht modules, and obtain nontrivial generalized eigenspace decompositions for several large families of Specht modules, which we conjecture are the only decomposable Specht modules indexed by bihooks if $e \neq 2$ and char $\mathbb{F} \neq 2$ (see Conjectures 4.2 and 5.5). In other words, our main results prove one direction of our conjectural classification in all of the cases where decomposable Specht modules may arise, and we prove our classification in full in a few cases. For small n or $e = 2$, we have some extra decomposable Specht modules — see Theorems 3.8 and 5.4, respectively. We summarize the majority of our decomposable Specht modules as follows.

Theorem 4.1. *Suppose that we take a Hecke algebra of type B with parameters $\kappa_1 = \kappa_2$. Let $\lambda = ((ke+a, 1^b), (je+a, 1^b))$ or $((b+1, 1^{je+a-1}), (b+1, 1^{ke+a-1}))$, for some $j, k \geqslant 1, 0 < a \leqslant e$, and $0 \leqslant b < e$ with $a + b \neq e$, or for $a = b = 0$.*

   (i) *For $j, k > 1$, if $j + k$ is even and char $\mathbb{F} \neq 2$, or if $j + k$ is odd, then S_λ is decomposable.*

   (ii) *If $j = 1$ or $k = 1$, then S_λ is decomposable if and only if char $\mathbb{F} \nmid j + k$.*

Theorem 5.3. *Let $e = 2$, and suppose that μ is a hook partition of n such that S_μ is a decomposable Specht module over the Hecke algebra of type A (see Theorems 5.1 and 5.2). Then, for any partition ν of m, the Specht modules $S_{(\mu,\nu)}$ and $S_{(\nu,\mu)}$ over the Hecke algebra of type B are decomposable.*

We now outline the layout of this paper. In Section 2, we will collect all necessary definitions and background from the literature, before studying the case of "small bihooks" (when $n \leqslant 2e$) in Section 3. In this case, we are able to completely determine the decomposability of Specht modules: we prove the above results in this special case, and furthermore show the converse, that all other bihooks index *indecomposable* Specht modules. Our method for this converse is a case-by-case analysis examining the tableaux that can appear in endomorphisms of these Specht

modules. We emphasize that this method will not readily extend to large n. Next, we conduct the majority of our study of Specht modules labeled by bihooks in Section 4, finding the aforementioned families of decomposable Specht modules. Our method here is to first use the divided power functors to reduce proving Theorem 4.1 to the case of bipartitions of the form $((ke), (je))$, and then determine certain endomorphisms for Specht modules in Theorem 4.13 indexed by these bipartitions. We compute three eigenvalues for these endomorphisms, yielding at least two distinct eigenvalues in any characteristic (with the exception of characteristic 2 when $j + k$ is even), resulting in a generalized eigenspace decomposition for the Specht modules having at least two nontrivial summands. Section 5 covers the $e = 2$ situation, which makes use of previous work of Speyer [2014] to yield quick results and prove Theorems 5.3 and 5.4. We leave some long technical calculations for Section 6, where the keen reader may find the grittier details of our work.

2. Background

In this section we give an overview of KLR algebras, Specht modules labeled by bihooks, and the associated combinatorics. Throughout, $\mathbb{F}$ will denote an arbitrary field.

Lie theoretic notation. Let $e \in \{2, 3, \dots\} \cup \{\infty\}$, which we call the *quantum characteristic*. If $e < \infty$, then we set $I := \mathbb{Z}/e\mathbb{Z}$, which we identify with the set $\{0, 1, \dots, e - 1\}$, whereas if $e = \infty$, we set $I := \mathbb{Z}$. We let Γ be the quiver with vertex set I and an arrow $i \to i - 1$ for each $i \in I$. If $e = \infty$, then Γ is the quiver of type A_∞; otherwise Γ is of type $A_{e-1}^{(1)}$.

Following Kac's book [1990], we recall standard notation for the Kac–Moody algebra associated to the generalized Cartan matrix $(a_{ij})_{i,j \in I}$. We have simple roots $\{\alpha_i \mid i \in I\}$, fundamental dominant weights $\{\Lambda_i \mid i \in I\}$, and the invariant symmetric bilinear form $(\cdot, \cdot)$ such that $(\alpha_i, \alpha_j) = a_{i,j}$ and $(\Lambda_i, \alpha_j) = \delta_{ij}$, for all $i, j \in I$. Let $Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geqslant 0} \alpha_i$ be the positive cone of the root lattice. If $\alpha = \sum_{i \in I} c_i \alpha_i \in Q_+$, then we define the *height of α* to be $\mathrm{ht}(\alpha) = \sum_{i \in I} c_i$.

An *e-bicharge* is an ordered pair $\kappa = (\kappa_1, \kappa_2) \in I^2$. We define its associated dominant weight Λ of level two to be $\Lambda = \Lambda_\kappa := \Lambda_{\kappa_1} + \Lambda_{\kappa_2}$.

The symmetric group. Let $\mathfrak{S}_n$ be the symmetric group on n letters. We denote by $s_1, \dots, s_{n-1}$ the standard Coxeter generators, where s_i is the simple transposition $(i, i + 1)$ for $1 \leqslant i < n$. We define a *reduced expression* for a permutation $w \in \mathfrak{S}_n$ to be an expression $s_{i_1} \dots s_{i_m}$ such that m is minimal, and call m the length of w, denoted $\ell(w)$.

We define the *Bruhat order* $\leqslant$ on $\mathfrak{S}_n$ as follows. If $x, w \in \mathfrak{S}_n$, then we write $x \leqslant w$ if there is a reduced expression for x which is a subexpression of a reduced expression for w.

For $1 \leqslant i \leqslant j \leqslant n-1$, we define $s_{\downarrow i}^{j} := s_j s_{j-1} \ldots s_i$ and $s_{\uparrow i}^{j} := s_i s_{i+1} \ldots s_j$.

Bipartitions. A *partition* λ of n is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $|\lambda| := \sum \lambda_i = n$. We write $\varnothing$ for the *empty partition* $(0, 0, \ldots)$. A *bipartition* λ of n is a pair $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ of partitions such that $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| = n$. We refer to $\lambda^{(1)}$ and $\lambda^{(2)}$ as the *first and second components*, respectively, of λ. We abuse notation and also write $\varnothing$ for the *empty bipartition* $(\varnothing, \varnothing)$. We denote the set of all bipartitions of n by $\mathscr{P}_n^2$.

For $\lambda, \mu \in \mathscr{P}_n^2$, we say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if for all $k \geqslant 1$,

$$\sum_{j=1}^{k} \lambda_j^{(1)} \geqslant \sum_{j=1}^{k} \mu_j^{(1)} \quad \text{and} \quad |\lambda^{(1)}| + \sum_{j=1}^{k} \lambda_j^{(1)} \geqslant |\mu^{(1)}| + \sum_{j=1}^{k} \mu_j^{(1)}.$$

The *Young diagram* of $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \mathscr{P}_n^2$ is defined to be

$$[\lambda] := \{(i, j, m) \in \mathbb{N} \times \mathbb{N} \times \{1, 2\} \mid 1 \leqslant j \leqslant \lambda_i^{(m)}\}.$$

We refer to elements of $[\lambda]$ as *nodes* of λ. We draw the Young diagram of a bipartition as a column vector of Young diagrams $[\lambda^{(1)}]$, $[\lambda^{(2)}]$. We say that a node $A \in [\lambda]$ is *removable* if $[\lambda] \setminus \{A\}$ is a Young diagram of a bipartition, while a node $A \notin [\lambda]$ is *addable* if $[\lambda] \cup \{A\}$ is a Young diagram of a bipartition.

If λ is a partition, the *conjugate partition*, denoted λ', is defined by

$$\lambda_i' = |\{j \geqslant 1 \mid \lambda_j \geqslant i\}|.$$

If $\lambda \in \mathscr{P}_n^2$, then we define the conjugate bipartition, also denoted λ', to be $\lambda' = (\lambda^{(2)'}, \lambda^{(1)'})$.

Tableaux. Let $\lambda \in \mathscr{P}_n^2$. Then a λ-*tableau* is a bijection $\mathtt{T} : [\lambda] \to \{1, \ldots, n\}$. We depict a λ-tableau $\mathtt{T}$ by inserting entries $1, \ldots, n$ into the Young diagram $[\lambda]$ with no repeats; we let $\mathtt{T}(i, j, m)$ denote the entry lying in node $(i, j, m) \in [\lambda]$. We say that $\mathtt{T}$ is *standard* if its entries increase down each column and along each row, within each component, and denote the set of all standard λ-tableaux by $\mathrm{Std}(\lambda)$.

The *column-initial tableau* $\mathtt{T}_\lambda$ is the λ-tableau where the entries $1, \ldots, n$ appear in order down consecutive columns, working from left to right, first in component 2, then component 1.

The symmetric group $\mathfrak{S}_n$ acts naturally on the left on the set of λ-tableaux. For $\mathtt{T}$ a λ-tableau, we define the permutation $w_\mathtt{T} \in \mathfrak{S}_n$ by $w_\mathtt{T} \mathtt{T}_\lambda = \mathtt{T}$.

Suppose $\lambda \in \mathscr{P}_n^2$. Let $\mathtt{S}$ and $\mathtt{T}$ be λ-tableaux with corresponding reduced expressions $w_\mathtt{S}$ and $w_\mathtt{T}$, respectively. Then we say that $\mathtt{T}$ *dominates* $\mathtt{S}$, written as $\mathtt{T} \trianglerighteq \mathtt{S}$, if and only if $w_\mathtt{T} \geqslant w_\mathtt{S}$.

Residues and degrees. Fix an e-bicharge $\kappa = (\kappa_1, \kappa_2)$. The *e-residue* of a node $A = (i, j, m) \in \mathbb{N} \times \mathbb{N} \times \{1, 2\}$ is defined to be

$$\text{res } A := \kappa_m + j - i \pmod{e}.$$

We call a node of residue r an *r-node*.

Let T be a λ-tableau. If $\text{T}(i, j, m) = r$, we set $\text{res}_\text{T}(r) = \text{res}(i, j, m)$. The *residue sequence* of T is defined to be

$$\boldsymbol{i}_\text{T} = (\text{res}_\text{T}(1), \ldots, \text{res}_\text{T}(n)).$$

We denote the residue sequence of the column-initial tableau T_λ by $\boldsymbol{i}_\lambda := \boldsymbol{i}_{\text{T}_\lambda}$.

We now define the degree of a standard tableau, which is the *codegree* as given in [Brundan et al. 2011, §3.5]. For $\lambda \in \mathscr{P}_n^2$ and an i-node A of λ, we define

$$d^A(\lambda) := \#\{\text{addable } i\text{-nodes of } \lambda \text{ strictly above } A\}$$
$$- \#\{\text{removable } i\text{-nodes of } \lambda \text{ strictly above } A\}.$$

Let $\text{T} \in \text{Std}(\lambda)$ with $\text{T}^{-1}(n) = A$. We define the *degree* of T, denoted $\deg(\text{T})$, recursively, by setting $\deg(\varnothing) := 0$, and

$$\deg(\text{T}) := d^A(\lambda) + \deg(\text{T}_{<n}),$$

where $\text{T}_{<n}$ is the standard tableau obtained from T by removing the node A.

Regular bipartitions. Let $\lambda \in \mathscr{P}_n^2$. We define the *i-signature of* λ by reading the Young diagram $[\lambda]$ from the top of the first component down to the bottom of the last component, writing a $+$ for each addable i-node and a $-$ for each removable i-node. We obtain the *reduced i-signature of* λ by successively deleting all adjacent pairs $+-$ from the i-signature of λ, always of the form $- \cdots - + \cdots +$.

The removable i-nodes corresponding to the $-$ signs in the reduced i-signature of λ are called the *normal i-nodes* of λ, while the addable i-nodes corresponding to the $+$ signs in the reduced i-signature of λ are called the *conormal i-nodes* of λ. The lowest normal i-node of $[\lambda]$, if there is one, is called the *good i-node* of λ, which corresponds to the last $-$ sign in the i-signature of λ. Analogously, the highest conormal i-node of $[\lambda]$, if there is one, is called the *cogood i-node* of λ, which corresponds to the first $+$ sign in the i-signature of λ.

We say that a bipartition $\lambda \in \mathscr{P}_n^2$ is *regular*, or *conjugate-Kleshchev*, if $[\lambda]$ can be obtained by successively adding cogood nodes to $\varnothing$. That is, we have a sequence $\varnothing = \lambda(0), \lambda(1), \ldots, \lambda(n) = \lambda$ such that $[\lambda(i)] \cup \{A\} = [\lambda(i + 1)]$, where A is a cogood node of $\lambda(i)$. Equivalently, λ is regular if and only if $\varnothing$ can be obtained by successively removing good nodes from $[\lambda]$. Observe in level one that the set of all regular partitions coincides with the set of all e-regular partitions.

Cyclotomic Khovanov–Lauda–Rouquier algebras. Suppose $\alpha \in Q^+$ has height n, and set

$$I^\alpha = \{i = (i_1, i_2, \ldots, i_n) \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \alpha\}.$$

Recalling that $\Lambda = \Lambda_\kappa$, we define $\mathscr{R}_\alpha^\Lambda$ to be the unital associative $\mathbb{F}$-algebra with generating set

$$\{e(i) \mid i \in I^\alpha\} \cup \{y_1, \ldots, y_n\} \cup \{\psi_1, \ldots, \psi_{n-1}\}$$

and relations

$$e(i)e(j) = \delta_{i,j}e(i),$$
$$\sum_{i \in I^\alpha} e(i) = 1,$$
$$y_r e(i) = e(i)y_r,$$
$$\psi_r e(i) = e(s_r i)\psi_r,$$
$$y_r y_s = y_s y_r,$$
$$\psi_r y_s = y_s \psi_r \qquad\qquad \text{if } s \neq r, r+1,$$
$$\psi_r \psi_s = \psi_s \psi_r \qquad\qquad \text{if } |r-s| > 1,$$
$$y_r \psi_r e(i) = (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}})e(i),$$
$$y_{r+1} \psi_r e(i) = (\psi_r y_r + \delta_{i_r, i_{r+1}})e(i),$$
$$\psi_r^2 e(i) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ e(i) & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r)e(i) & \text{if } i_r = i_{r+1} + 1, \\ (y_r - y_{r+1})e(i) & \text{if } i_r = i_{r+1} - 1, \end{cases}$$
$$\psi_r \psi_{r+1} \psi_r e(i) = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1)e(i) & \text{if } i_{r+2} = i_r = i_{r+1} + 1, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1)e(i) & \text{if } i_{r+2} = i_r = i_{r+1} - 1, \\ (\psi_{r+1} \psi_r \psi_{r+1})e(i) & \text{otherwise,} \end{cases}$$
$$y_1^{(\Lambda, \alpha_{i_1})} e(i) = 0$$

for all admissible r, s, i, j. When $e = 2$, we actually have slightly different "quadratic" and "braid" relations, which may be found, for example, in [Kleshchev et al. 2012, §3.1]. We omit them here, as we will not explicitly calculate with these relations when $e = 2$.

Lemma 2.1 [Brundan and Kleshchev 2009b, Corollary 1]. *There is a unique $\mathbb{Z}$-grading on $\mathscr{R}_\alpha^\Lambda$ such that, for all admissible r and i,*

$$\deg(e(i)) = 0, \qquad \deg(y_r) = 2, \qquad \deg \psi_r(e(i)) = -a_{i_r, r+1}.$$

The *cyclotomic Khovanov–Lauda–Rouquier* (*KLR*) *algebra* or *cyclotomic quiver Hecke algebra* $\mathscr{R}_n^\Lambda$ is defined to be the direct sum $\bigoplus_\alpha \mathscr{R}_\alpha^\Lambda$, where the sum is taken over all $\alpha \in Q^+$ of height n.

Here we sum over all $\alpha \in Q^+$ of height n, though in fact only finitely many of the summands will be nonzero, so (even when $e = \infty$) $\mathscr{R}_n^\Lambda$ is a unital algebra.

These $\mathbb{Z}$-graded algebras are connected to the Hecke algebras of type B via (a special case of) Brundan and Kleshchev's *graded isomorphism theorem*.

Theorem 2.2 [Brundan and Kleshchev 2009a, Main Theorem]. *If $e = \operatorname{char}(\mathbb{F})$ or* $\operatorname{char}(\mathbb{F}) \nmid e$, *then $\mathscr{R}_n^\Lambda$ is isomorphic to the integral Hecke algebra $\mathscr{H}_n(q, Q_1, Q_2)$ of type B with parameters $q \in \mathbb{F}$ a primitive e-th root of unity, $Q_1 = q^{\kappa_1}$, and $Q_2 = q^{\kappa_2}$. That is, $\mathscr{H}_n(q, Q_1, Q_2)$ has generators $T_0, \ldots, T_{n-1}$ satisfying type B Coxeter relations, with the quadratic relations replaced with*

$$(T_0 - q^{\kappa_1})(T_0 - q^{\kappa_2}) = 0 \quad and \quad (T_i - q)(T_i + 1) = 0 \quad for \ all \ 1 \leqslant i \leqslant n - 1.$$

Specht modules labeled by bihooks.

Definition 2.3. We call a bipartition λ a *bihook* if it is of the form $\lambda = ((a, 1^b), (c, 1^d))$ for some integers $a, c \geqslant 1$ and $b, d \geqslant 0$.

Definition 2.4. [Kleshchev et al. 2012, Definition 7.11] Let $\lambda = ((a, 1^b), (c, 1^d)) \in \mathscr{P}_n^2$. The (column) *Specht module* S_λ is the cyclic $\mathscr{R}_n^\Lambda$-module generated by z_λ of degree $\deg(z_\lambda) := \deg(T_\lambda)$ subject to the relations

- $e(\boldsymbol{i}_\lambda)z_\lambda = z_\lambda$,
- $y_r z_\lambda = 0$ for all $r \in \{1, \ldots, n\}$,
- $\psi_r z_\lambda = 0$ for all $r \in \{1, \ldots, n-1\} \setminus \{d+1, c+d, b+c+d+1\}$,
- $\psi_1 \psi_2 \ldots \psi_{d+1} z_\lambda = 0 = \psi_{c+d+1} \psi_{c+d+2} \ldots \psi_{b+c+d+1} z_\lambda$ (these are the Garnir relations arising from nodes $(1, 1, 2)$ and $(1, 1, 1)$, respectively).

For each $w \in \mathfrak{S}_n$, we fix a reduced expression $w = s_{i_1} \ldots s_{i_m}$ throughout. We define the associated element of $\mathscr{R}_n^\Lambda$ to be $\psi_w := \psi_{i_1} \ldots \psi_{i_m}$, which, in general, depends on the choice of reduced expression for w. For $\lambda \in \mathscr{P}_n^2$ and a λ-tableau T, we define $v_T := \psi_{w_T} z_\lambda$.

Whilst these vectors v_T of S_λ also depend on the choice of reduced expression in general, the following result does not.

Theorem 2.5 [Brundan et al. 2011, Corollary 4.6; Kleshchev et al. 2012, Proposition 7.14 and Corollary 7.20]. *For $\lambda \in \mathscr{P}_n^2$, the set of vectors $\{v_T \mid T \in \operatorname{Std}(\lambda)\}$ is a homogeneous $\mathbb{F}$-basis of S_λ, with $\deg(v_T) = \deg(T)$. Moreover, for any λ-tableau S, v_S is a linear combination of basis elements v_T such that $S \trianglerighteq T$.*

We record the following useful lemma that we will use frequently.

Lemma 2.6 [Brundan et al. 2011, Lemma 4.4]. *Let $\lambda \in \mathscr{P}_n^2$, and $\mathrm{T} \in \mathrm{Std}(\lambda)$. Then* $e(\boldsymbol{i})v_{\mathrm{T}} = \delta_{\boldsymbol{i},\boldsymbol{i}_{\mathrm{T}}} v_{\mathrm{T}}$.

Of particular importance to the present paper is the following result on the decomposability of Specht modules, which is a special case of a result for higher level cyclotomic KLR algebras.

Proposition 2.7 [Fayers and Speyer 2016, Corollary 3.12]. *If $e \neq 2$ and $\kappa_1 \neq \kappa_2$, then the Specht modules S_λ are indecomposable for all $\lambda \in \mathscr{P}_n^2$.*

The following useful result is obtained from [Kleshchev et al. 2012, Theorems 7.25 and 8.5].

Theorem 2.8. S_λ *is decomposable if and only if* $\mathrm{S}_{\lambda'}$ *is.*

We know from [Brundan and Kleshchev 2009b, Theorem 5.10] that Specht modules S_λ indexed by regular bipartitions have simple heads, yielding the following:

Proposition 2.9. *If $\lambda \in \mathscr{P}_n^2$ is a regular bipartition, then the Specht module S_λ is indecomposable.*

Lemma 2.10 [Brundan et al. 2011, Lemma 4.9]. *Let $\lambda \in \mathscr{P}_n^2$, $1 \leqslant r < n$, and $\mathrm{T} \in \mathrm{Std}(\lambda)$. If r and $r + 1$ lie in the same row or in the same column of T, then*

$$\psi_r v_{\mathrm{T}} = \sum_{\substack{\mathrm{S} \in \mathrm{Std}(\lambda) \\ \boldsymbol{i}_{\mathrm{S}} = \boldsymbol{i}_{s_r \mathrm{T}} \\ \mathrm{S} \lhd \mathrm{T}}} a_{\mathrm{S}} v_{\mathrm{S}} \quad \text{for some } a_{\mathrm{S}} \in \mathbb{F}.$$

Definition 2.11. We define

$$\psi_{\substack{\downarrow \\ y}}^x = \psi_x \psi_{x-1} \ldots \psi_y \quad \text{and} \quad \psi_{\substack{\uparrow \\ y}}^x = \psi_y \psi_{y+1} \ldots \psi_x$$

if $x \geqslant y$ and set both equal to $1_{\mathbb{F}}$ if $x < y$. Furthermore, we use the shorthand

$$\psi_{\substack{\downarrow \\ y}}^{x} \psi_{\substack{\downarrow \\ y+1}}^{x+1} \ldots \psi_{\substack{\downarrow \\ y+c}}^{x+c} := \psi_{\substack{\downarrow \\ y}}^{x} \psi_{\substack{\downarrow \\ y+1}}^{x+1} \ldots \psi_{\substack{\downarrow \\ y+c}}^{x+c}, \qquad \psi_{\substack{\downarrow \\ y}}^{x} \psi_{\substack{\downarrow \\ y-1}}^{x-1} \ldots \psi_{\substack{\downarrow \\ y-c}}^{x-c} := \psi_{\substack{\downarrow \\ y}}^{x} \psi_{\substack{\downarrow \\ y-1}}^{x-1} \ldots \psi_{\substack{\downarrow \\ y-c}}^{x-c},$$

$$\psi_{\substack{\uparrow \\ x}}^{y} \psi_{\substack{\uparrow \\ x+1}}^{y+1} \ldots \psi_{\substack{\uparrow \\ x+c}}^{y+c} := \psi_{\substack{\uparrow \\ x}}^{y} \psi_{\substack{\uparrow \\ x+1}}^{y+1} \ldots \psi_{\substack{\uparrow \\ x+c}}^{y+c}, \qquad \psi_{\substack{\uparrow \\ x}}^{y} \psi_{\substack{\uparrow \\ x-1}}^{y-1} \ldots \psi_{\substack{\uparrow \\ x-c}}^{y-c} := \psi_{\substack{\uparrow \\ x}}^{y} \psi_{\substack{\uparrow \\ x-1}}^{y-1} \ldots \psi_{\substack{\uparrow \\ x-c}}^{y-c}.$$

3. Small bihooks

In light of Proposition 2.7, we suppose that $\kappa_1 = \kappa_2$ throughout Sections 3 and 4. In fact, we need only assume that $\kappa = (0, 0)$ since residue shifts do not change the isomorphism type of $\mathscr{R}_n^\Lambda$. Here, we begin our examination of Specht modules labeled by bihooks by completely determining which Specht modules are decomposable when $n \leqslant 2e$. We first make the following easy observation.

Lemma 3.1. *If $k < e$, then $\lambda = ((k), (k))$ is a regular bipartition. In particular, S_λ is indecomposable.*

Proof. Starting from $(\varnothing, \varnothing)$, we may add two cogood 0-nodes and then two cogood 1-nodes, and continue in this fashion until we have added two cogood $(k - 1)$-nodes. The resulting bipartition is λ. It follows from Proposition 2.9 that S_λ is indecomposable. □

Remark. In the above proof, we may go so far as adding two cogood $(e - 2)$-nodes to reach the regular bipartition $((e - 1), (e - 1))$, but can go no further. Adding a cogood e-node yields the bipartition $((e), (e - 1))$, but adding a second cogood e-node results in the bipartition $((e, 1), (e - 1))$, *not* $((e), (e))$. One can check that $((k), (k))$ is not regular for any $k \geqslant e$.

In fact, it is not difficult to see that we may generalize the previous lemma as follows.

Lemma 3.2. *Suppose $a \geqslant 1$ and $b \geqslant 0$ with $a + b < e$. Then $\lambda = ((a, 1^b), (a, 1^b))$ is a regular bipartition, so S_λ is indecomposable.*

Conversely, we will next show that if we instead have $a + b = e$, then the Specht modules $S_{((a, 1^b), (a, 1^b))}$ are all decomposable.

Lemma 3.3. *Suppose $a \geqslant 1$ and $b \geqslant 0$ with $a + b \leqslant e$, and let $\lambda = ((a, 1^b), (a, 1^b))$. Then there is a unique standard λ-tableau $T \neq T_\lambda$ with res $T = i_\lambda$.*

Proof. This is an easy consequence of the fact that $i_\lambda = (0, e - 1, \ldots, e - b, 1, 2, \ldots, a - 1, 0, e - 1, \ldots, e - b, 1, 2, \ldots, a - 1)$. If we write $T_\lambda = (T^{(1)}, T^{(2)})$, then $T = (T^{(2)}, T^{(1)})$. □

Lemma 3.4. *Suppose $a \geqslant 1$ and $b \geqslant 0$ with $a + b = e$, and let $\lambda = ((a, 1^b), (a, 1^b))$. There is an endomorphism φ of S_λ determined by $\varphi(z_\lambda) = v_T$, where T is the tableau in Lemma 3.3.*

Proof. The proof proceeds by checking that the annihilator of z_λ also annihilates v_T.

We already know by Lemmas 2.6 and 3.3 that $e(i)v_T = \delta_{i,i_\lambda} v_T$. It is easy to see that $\deg T = 1 = \deg T_\lambda$, and therefore that $\deg(y_r v_T) = 3$. However, $y_r v_T \in e(i_\lambda) S_\lambda = \langle v_{T_\lambda}, v_T \rangle_{\mathbb{F}}$, so $\deg(y_r v_T) = 1$. This contradiction gives us that $y_r v_T = 0$ for all $r \in \{1, \ldots, n\}$.

Finally, we check the relations involving ψ generators. We know by Lemma 2.10 that $\psi_w v_T = 0$ if there exists no standard λ-tableau with residue sequence $w i_\lambda$, for any $w \in \mathfrak{S}_n$.

Observe that for $1 \leqslant r \leqslant b$,

$$s_r i_\lambda = (0, e - 1, e - 2, \ldots, e - r + 2, e - r, e - r + 1, i_{r+2}, \ldots, i_n).$$

It is clear that there is no standard λ-tableau with residue sequence $s_r i_\lambda$, and hence $\psi_r v_T = 0$ for all $1 \leqslant r \leqslant b$. Similarly, $\psi_r v_T = 0$ for all $r \in \{b + 2, \ldots, e - 1\} \cup \{e + 1, \ldots, e + b\} \cup \{e + b + 2, \ldots, 2e - 1\}$.

If $a > 1$, we must also check the longer Garnir relations arising from the Garnir nodes $(1, 1, 1)$ and $(1, 1, 2)$. Applying $s_1 s_2 \ldots s_{b+1}$ to i_λ, we find that the residue sequence $s_1 s_2 \ldots s_{b+1} i_\lambda$ begins with 1, and thus $\psi_1 \psi_2 \ldots \psi_{b+1} z_\lambda = 0$. Similarly, if we apply $s_{e+1} s_{e+2} \ldots s_{e+b+1}$ to i_λ, we find that the residue sequence $s_{e+1} s_{e+2} \ldots s_{e+b+1} i_\lambda$ contains 1 in its e-th and $(e+1)$-th positions, and the 0 residues occur in the first and $(e+2)$-th positions. Clearly there is no standard λ-tableau with this residue sequence, and thus $\psi_{e+1} \psi_{e+2} \ldots \psi_{e+b+1} z_\lambda = 0$. □

Proposition 3.5. *If $\varphi(z_\lambda) = v_T$ as above, then*

$$\varphi(v_T) = \begin{cases} (-1)^{b+1} 2 v_T & \text{if } a > 1, \\ (-1)^{e-1} 2 v_T & \text{if } \lambda = ((1^e), (1^e)). \end{cases}$$

We prove Proposition 3.5 by a rather lengthy calculation, which we relegate to Section 6.

Theorem 3.6. *Suppose $a \geqslant 1$ and $b \geqslant 0$ with $a + b = e$, and let $\lambda = ((a, 1^b), (a, 1^b))$. Then S_λ is decomposable if and only if $\operatorname{char} \mathbb{F} \neq 2$.*

Proof. If $\operatorname{char} \mathbb{F} \neq 2$, then there is an endomorphism φ' of S_λ determined by $\varphi' = (-1)^{(b+1)} \frac{1}{2} \varphi$, which is an idempotent. If $\operatorname{char} \mathbb{F} = 2$, then it is easy to see from Lemma 3.3 and Proposition 3.5 that S_λ has no nontrivial idempotent endomorphisms, and the result follows. □

Next, we will generalize Lemma 3.2 and show that all Specht modules indexed by "small" bihooks not appearing in Theorem 3.6 are indecomposable. More precisely:

Theorem 3.7. *Suppose $e \neq 2$, and let $\lambda = ((a, 1^b), (c, 1^d)) \in \mathscr{P}_n^2$ such that either $a + b + c + d < 2e$ or $a + b + c + d = 2e$ with $a \neq c$ or $b \neq d$. Then S_λ is indecomposable.*

Proof. We determine all of the possible standard λ-tableaux with residue sequence

$$i_\lambda = (0, -1, \ldots, -d, 1, 2, \ldots, c-1, 0, -1, \ldots, -b, 1, 2, \ldots, a-1).$$

If no standard λ-tableau other than T_λ has residue sequence i_λ, then there exists no nontrivial endomorphism of S_λ and thus S_λ is indecomposable. For any other standard λ-tableau T distinct from T_λ such that $i_T = i_\lambda$, we show that there exists no nontrivial endomorphisms of S_λ in each of these cases.

We fill $[\lambda]$ with $1, \ldots, n$ to find all standard λ-tableaux with residue sequence i_λ.

(i) Suppose $b < d < e$. Since $n \leqslant 2e$, we must have that $b \leqslant e - 2$. First, we shall assume that $b < e - 2$. Since $b \neq e - 1, e - 2$, it follows that $-b - 1 \not\equiv 0$ or 1 (mod e), so we cannot put $1, \ldots, b+1$ down the first column of the first component, as $b+2$ would have nowhere to go. Thus, we must have $1, \ldots, b+2$ down the first column of the second component, and in fact the only way to fill in the remaining entries yields T_λ, so there are no nontrivial endomorphisms of S_λ.

If instead we assume $b = e - 2$, then our conditions give $\lambda = ((a, 1^{e-2}), (c, 1^{e-1}))$, for which there are only 3 possible bihooks: $((1^{e-1}), (1^e))$, $((1^{e-1}), (2, 1^{e-1}))$, and $((2, 1^{e-2}), (1^e))$. It is easy to check that the first has no standard tableaux of residue i_λ besides T_λ, the second also has the standard tableau $T = s_e T_\lambda$, and the third has the standard tableau $S = s \underset{1}{\overset{e}{\downarrow}} s \underset{2}{\overset{e+1}{\downarrow}} \ldots s \underset{e}{\overset{2e-1}{\downarrow}} T_\lambda$. Since $\psi_{e-1} v_T = v_{s_{e-1}T}$, and $\psi_{e-1} v_S = v_{s_{e-1}S}$ (plus possibly some lower order terms), while $\psi_{e-1} z_\lambda = 0$, these cases yield no nontrivial homomorphisms.

(ii) Suppose $b < d$ and $d \geqslant e$. If $b = e - 2$, then we must have $\lambda = ((1^{e-1}), (1^{e+1}))$, for which we may easily see that only T_λ and $s_{e+1}T_\lambda$ have the correct residue sequence, and that $\psi_e v_{s_{e+1}T_\lambda} = v_{s_e s_{e+1}T_\lambda}$, and therefore there are no nontrivial homomorphisms. So we may assume that $b < e - 2$. As in part (i), we must place $1, \ldots, b + 2$ down the first column of the second component, and in fact must place $1, \ldots, e - 1$ in there. If we place the entry e in node $(e, 1, 2)$, we can either place $e + 1$ in node $(e + 1, 1, 2)$ or node $(1, 1, 1)$. One can check that in the former case, we may only obtain T_λ. In the latter, we must then place $e + 2, \ldots, d + 1$ down the first column of the first component, which is only possible if $b \geqslant d - e$. If so, we may continue, placing $d + 2, \ldots, d + c$ in the first row of the second component, then $d + c + 1, \ldots, 2d + c - e + 1$ down the remaining nodes in the second component, and $2d + c - e + 2, \ldots, n$ as in T_λ. Call this standard tableau R.

If $a \geqslant c \geqslant 2$, we could have also placed $d + 2, \ldots, d + c$ in the first row of the first component, whence we are forced to place $d + c + 1, \ldots, 2d + c - e + 1$ down the first column of the second component, then $2d + c - e + 2, \ldots, b + c + d + 1$ down the first column of the first component, then $b + c + d + 2, \ldots, b + 2c + d$ in the first row of the second component, and then $b + 2c + d + 1, \ldots, n$ as in T_λ. Call this standard tableau S.

Finally, suppose that $c \geqslant 2$, and we instead placed the entry e in the node $(1, 2, 2)$. Then we must put the entries $e + 1, \ldots, d + 1$ down the first column of the first component, which is only possible if $b \geqslant d - e$. We must then place $d + 2$ in node $(e, 1, 2)$ and place $d + 3, \ldots, d + c$ along the first row of the second component and then $d + c + 1, \ldots, 2d + c - e + 1$ down the first column of the second component, and then $2d + c - e + 2, \ldots, n$ are filled as in T_λ. Call this standard tableau T.

Then $\psi_e z_\lambda = 0$, while $\psi_e v_R = v_{s_e R}$, $\psi_e v_S = v_{s_e S}$, and $\psi_e v_T = v_{s_e T}$ (plus possibly some lower order terms in this final case). It follows that there is no nontrivial endomorphism of S_λ in any of these cases.

(iii) Suppose $b \geqslant d$. If $c < a$, the result follows from parts (i) and (ii) by applying Theorem 2.8. So we may assume that $c \geqslant a$. The only bihook for fitting these conditions with $d \geqslant e - 1$ is $((1^e), (1^e))$, so we may further assume that $d \leqslant e - 2$. If we place $1, \ldots, d + 1$ down the first column of the second component, then $d + 2, \ldots, d + c$ must also be in the second component, if $c \leqslant e$. If $b \neq e - 1$, then

the only option at this point is to fill in $d + c + 1, \ldots, n$ as in T_λ. If $b = e - 1$ and $a \geqslant 2$, we may also obtain the standard tableau $R = s_{b+c+d+1} T_\lambda$. If $c > e$ (in which case we also have $b < e - 1$ and $c > a$), then we must fill $d + 2, \ldots, d + e$ in the second component, but can place $d + e + 1, \ldots, d + c$ along the first row of the first component, if $c \leqslant a + e$. In this case, we may obtain a standard tableau U by filling in $d + c + 1$ in node $(1, e + 1, 2)$, then $d + c + 2, \ldots, d + c + b + 1$ down the first column of the first component, and then $d + c + b + 2, \ldots, d + 2c + b - e$ along the first row of the second component, with $d + 2c + b - e + 1, \ldots, n$ being in the first component, as in T_λ. Since $\psi_{d+e} v_U = v_{s_{d+e} U}$ (plus possibly lower order terms), there is no homomorphism mapping z_λ to v_U.

If instead we place $1, \ldots, d + 1$ down the first column of the first component, then $d + 2, \ldots, d + a$ must be placed in the first row of the first component. If $c > a$, then we have nowhere left to put $d + a + 1$, so there is no such standard tableau. If $c = a$, then we may place $d + a + 1, \ldots, 2d + a + 1$ down the first column of the second component, then $2d + a + 2, \ldots d + a + b + 1$ down the first column of the first component, and the remaining entries along the first row of the second component, obtaining a standard tableau we shall call S. If we further have that $b = e - 1$, we also have the standard tableau $T = s_{d+a+b+1} S$. There are no other standard tableaux of the correct residue sequence.

We now show that there exists no homomorphism φ such that $\varphi(z_\lambda) = \alpha v_R + \beta v_S + \gamma v_T$ for some $\alpha, \beta, \gamma \in \mathbb{F}$. We observe that $\psi_{c+2d+1} v_R = \psi_{c+2d+1} \psi_{c+d+e} z_\lambda = \psi_{c+d+e} \psi_{c+2d+1} z_\lambda = 0$, since $d + 1 \leqslant e - 1$, and we cannot have that $d = e - 2$, $b = e - 1$, and $c \geqslant a \geqslant 2$, while $\psi_{c+2d+1} v_S = v_{s_{c+2d+1} S}$ (plus lower order terms), and $\psi_{c+2d+1} v_T = v_{s_{c+2d+1} T}$ (plus lower order terms). Thus, $\psi_{c+2d+1}(v_R + v_S + v_T) = v_{s_{c+2d+1} S} + v_{s_{c+2d+1} T}$ (plus lower order terms), so that there exists no homomorphism mapping z_λ to a linear combination of v_S and v_T. Furthermore, $\psi_{c+d+e-1} v_R = v_{s_{c+d+e-1} R}$, whilst $\psi_{c+d+e-1} z_\lambda = \psi_{c+2d+1} z_\lambda = 0$. Hence, there is no nontrivial endomorphism of S_λ. $\qquad \square$

We summarize the results of this section in the following theorem.

Theorem 3.8. *Let $n \leqslant 2e$ and $\lambda \in \mathscr{P}_n^2$ be a bihook. If $e \neq 2$, then S_λ is decomposable if and only if $\mathrm{char}\, \mathbb{F} \neq 2$, $n = 2e$, and $\lambda = ((a, 1^b), (a, 1^b))$ for some $a \geqslant 1$ and $b \geqslant 0$. If $e = 2$, then S_λ is decomposable if and only if $\mathrm{char}\, \mathbb{F} \neq 2$ and $\lambda = ((2), (2))$, $((1^2), (1^2))$, $((2), (1^2))$, or $((1^2), (2))$.*

Remark. If $e = 2$, the four decomposable Specht modules are pairwise isomorphic. It is easy to check that no other small bihooks label decomposable Specht modules.

4. General bihooks

This section comprises our main body of work, where we determine several large families of decomposable Specht modules in any characteristic when e is finite.

Throughout this section, we will assume that $\kappa = (0, 0)$. We begin by applying induction and restriction functors to reduce our proof to the case of Specht modules $S_{((ke),(je))}$. We then show that these are decomposable by exploiting certain endomorphisms, for which we determine at least two distinct eigenvalues, so that the generalized eigenspace decompositions contain at least two summands. The amalgamation of these results can be stated in our main theorem of this section as follows.

Theorem 4.1. *Let* $\lambda = ((ke+a, 1^b), (je+a, 1^b))$ *or* $((b+1, 1^{je+a-1}), (b+1, 1^{ke+a-1}))$, *for some* $j, k \geqslant 1$, $0 < a \leqslant e$, *and* $0 \leqslant b < e$ *with* $a + b \neq e$, *or for* $a = b = 0$.

(i) *For* $j, k > 1$, *if* $j + k$ *is even and* $\mathrm{char}\,\mathbb{F} \neq 2$, *or if* $j + k$ *is odd, then* S_λ *is decomposable.*

(ii) *If* $j = 1$ *or* $k = 1$, *then* S_λ *is decomposable if and only if* $\mathrm{char}\,\mathbb{F} \nmid j + k$.

Conjecture 4.2. *When* $e \neq 2$ *and* $\mathrm{char}\,\mathbb{F} \neq 2$, *Theorems 4.1 and 3.8 provide a complete list of decomposable Specht modules indexed by bihooks.*

Remark. We have checked Conjecture 4.2 in GAP for all $e \in \{3, 4, 5\}$, $\mathrm{char}\,\mathbb{F} \in \{0, 3, 5\}$, and $n \leqslant 22$. In characteristic 2, we have found an extra handful of decomposable Specht modules which our theorem and our methods do not detect — namely those indexed by $((4e), (2e))$, $((2e), (4e))$, $((8e), (2e))$, $((2e), (8e))$, $((5e), (3e))$, $((3e), (5e))$, their conjugates, and those bihooks obtained from these by our induction functor arguments. It is tempting to speculate that the extra decomposable Specht modules in characteristic 2 correspond to λ as in Theorem 4.1 with $j \neq k$ and $j - k \equiv 2 \pmod 4$.

We begin with a reduction result, greatly simplifying the work we must do to prove Theorem 4.1.

Proposition 4.3. *Let* $k, j \geqslant 1$, *and* $0 \leqslant a < e$. *The Specht module* $S_{((ke),(je))}$ *is decomposable if and only if* $S_{((ke+a),(je+a))}$ *is.*

Proof. Our argument is similar to that in [Speyer 2014, Theorem 3.2], using the graded "cyclotomic divided power functors" of [Brundan and Kleshchev 2009b, §4.6], which we denote here by $e_i^{(r)}$ and $f_i^{(r)}$. Let $\lambda = ((ke), (je))$ and $\mu = ((ke+a), (je+a))$. Then $e_0^{(2)} e_1^{(2)} \ldots e_{a-1}^{(2)} S_\mu = S_\lambda$ and $f_{a-1}^{(2)} f_{a-2}^{(2)} \ldots f_0^{(2)} S_\lambda = S_\mu$. It follows that S_λ and S_μ have the same composition length, and that $e_0^{(2)} e_1^{(2)} \ldots e_{a-1}^{(2)} D \neq 0$ for any composition factor D of S_μ. Hence, by exactness, $e_0^{(2)} e_1^{(2)} \ldots e_{a-1}^{(2)} M \neq 0$ for any submodule $M \subseteq S_\mu$. It follows that if S_μ is decomposable, then so is S_λ. Repeating the argument the other way around completes the proof.   □

In fact, we may extend the above proposition as follows.

Proposition 4.4. *Let* $k, j \geqslant 1$, $0 < a \leqslant e$, *and* $0 \leqslant b < e$ *with* $a + b \neq e$. *The Specht module* $S_{((ke),(je))}$ *is decomposable if and only if* $S_{((ke+a, 1^b),(je+a, 1^b))}$ *is.*

Proof. The argument is similar to the proof of Proposition 4.3. By Proposition 4.3, we know that $S_{((ke),(je))}$ is decomposable if and only if $S_{((ke+1),(je+1))}$ is.

Let $\lambda = ((ke+1),(je+1))$ and $\mu = ((ke+a,1^b),(je+a,1^b))$. If $a+b < e$, we find that $e_1^{(2)} e_2^{(2)} \ldots e_{a-1}^{(2)} \cdot e_{e-1}^{(2)} e_{e-2}^{(2)} \ldots e_{e-b}^{(2)} S_\mu = S_\lambda$ and $f_{e-b}^{(2)} f_{e-b+1}^{(2)} \cdots f_{e-1}^{(2)} \cdot f_{a-1}^{(2)} f_{a-2}^{(2)} \cdots f_1^{(2)} S_\lambda = S_\mu$.

If $a+b > e$, we find that

$$e_1^{(2)} e_2^{(2)} \ldots e_{a-2}^{(2)} \cdot e_{e-1}^{(2)} e_{e-2}^{(2)} \ldots e_a^{(2)} \cdot e_{a-1}^{(4)} \cdot e_{a-2}^{(2)} e_{a-3}^{(2)} \ldots e_{e-b}^{(2)} S_\mu = S_\lambda$$

and

$$f_{e-b}^{(2)} \cdots f_{a-3}^{(2)} f_{a-2}^{(2)} \cdot f_{a-1}^{(4)} \cdot f_a^{(2)} f_{a+1}^{(2)} \cdots f_{e-1}^{(2)} \cdot f_{a-2}^{(2)} f_{a-3}^{(2)} \ldots f_1^{(2)} S_\lambda = S_\mu$$

where we adopt the convention that $f_a^{(2)} f_{a+1}^{(2)} \ldots f_{e-1}^{(2)} = \mathrm{id} = e_{e-1}^{(2)} e_{e-2}^{(2)} \ldots e_a^{(2)}$ if $a = e$.

We may now complete the proof identically to Proposition 4.3. □

The following result handles some indecomposable Specht modules when n is reasonably small, essentially extending Theorem 3.7.

Proposition 4.5. *Let* $1 \leqslant a \leqslant e$, *and* $0 \leqslant b < e$ *with* $a + b \neq e$, *and let* $\lambda = ((a,1^b),(a,1^b))$. *Then* S_λ *is indecomposable.*

Proof. It is easy to see that λ is regular, from which the result also follows: starting with $\varnothing$, we may add two cogood 0-nodes, followed by two cogood 1-nodes, and so on up to adding two cogood $(a-1)$-nodes, then two cogood (-1)-nodes, and so on up to adding two cogood $(-b)$-nodes. If $n > 2e$, then after adding the first $2e - 2$ cogood nodes in the above, we must add four cogood nodes of the next residue before reverting to adding two at a time. □

In view of the above results, we may assume that $\lambda = ((ke),(je))$ as we prove our main result. We fix this choice of λ for the remainder of the section, and compute an endomorphism of S_λ which we will use to prove our result.

We now introduce notation for the basis vectors v_T of S_λ, analogous to [Sutton 2018, §5.2]. Observe that a standard λ-tableau T is determined by the entries $a_r := T(1,r,2)$ lying in its second component, for all $r \in \{1, \ldots, je\}$. We can thus write $T = w_T T_\lambda$, where

$$w_T = s_{\underset{1}{\overset{a_1-1}{\downarrow}}} s_{\underset{2}{\overset{a_2-1}{\downarrow}}} \ldots s_{\underset{je}{\overset{a_{je}-1}{\downarrow}}} \in \mathfrak{S}_n.$$

It follows that $v_T = \psi_T z_\lambda$ where

$$\psi_T = \psi_{\underset{1}{\overset{a_1-1}{\downarrow}}} \psi_{\underset{2}{\overset{a_2-1}{\downarrow}}} \ldots \psi_{\underset{je}{\overset{a_{je}-1}{\downarrow}}} \in \mathscr{R}_n^\Lambda.$$

In order to distinguish our standard tableaux compactly, we will often write $v(a_1, a_2, \ldots, a_{je})$ for the standard λ-tableau with entries $a_1, a_2, \ldots, a_{je}$ in the second component.

Definition 4.6. Let $T \in \text{Std}(\lambda)$. An *e-brick* is a sequence of e adjacent nodes containing entries $je+1$, $je+2$, ..., $(j+1)e$ for $j \geqslant 0$. We say that T is an *e-brick tableau* if all entries of T lie in *e-bricks*. We denote the set of all standard *e-brick* λ-tableaux by $\mathcal{T}_e$.

Example. If $e = 3$ and $\lambda = ((6), (6))$, then $\mathcal{T}_e$ consists of the following six tableaux, obtained by permuting the four *e-bricks*:

$$T_1 = \begin{array}{|c|c|c|c|c|c|} \hline 7 & 8 & 9 & 10 & 11 & 12 \\ \hline \end{array} \quad T_2 = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 5 & 6 & 10 & 11 & 12 \\ \hline \end{array} \quad T_3 = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 7 & 8 & 9 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 10 & 11 & 12 \\ \hline \end{array}$$

$$T_4 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 10 & 11 & 12 \\ \hline \end{array} \quad T_5 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 7 & 8 & 9 \\ \hline \end{array} \quad T_6 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|c|} \hline 4 & 5 & 6 & 10 & 11 & 12 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|c|} \hline 7 & 8 & 9 & 10 & 11 & 12 \\ \hline \end{array}$$

These tableaux correspond to $v_{T_1} = v(1, 2, 3, 4, 5, 6)$, $v_{T_2} = v(1, 2, 3, 7, 8, 9)$, $v_{T_3} = v(1, 2, 3, 10, 11, 12)$, $v_{T_4} = v(4, 5, 6, 7, 8, 9)$, $v_{T_5} = v(4, 5, 6, 10, 11, 12)$, and $v_{T_6} = v(7, 8, 9, 10, 11, 12)$ in $S_{((6),(6))}$.

The following easy lemma is our motivation for introducing this definition.

Lemma 4.7. *For any* $T \in \text{Std}(\lambda)$, $v_T \in e(i_\lambda) S_\lambda$ *if and only if* $T \in \mathcal{T}_e$.

In particular, this lemma tells us that for any endomorphism φ of S_λ,

$$\varphi(z_\lambda) = \sum_{T \in \mathcal{T}_e} a_T v_T \quad \text{for some } a_T \in \mathbb{F}.$$

Lemma 4.8. *For all* $T \in \mathcal{T}_e$, $\deg T = j$.

Proof. Any nodes in the first component of T cannot contribute to the degree, since there can't be any nodes above them. In the second component, each *e-brick* contributes $+1$ to the degree. If the first component is empty when adding such a brick, this comes from only having an addable 0-node higher up in the diagram. Otherwise, there is an addable 0-node, as well as one addable $(e-1)$-node and one removable $(e-1)$-node. $\qquad\square$

Lemma 4.9. *For all* $T \in \mathcal{T}_e$ *and* $1 \leqslant r \leqslant n$, $y_r v_T = 0$.

Proof. Since v_T has degree j, $\deg(y_r v_T) = j+2$. But $y_r v_T \in e(i_\lambda) S_\lambda$, so $\deg(y_r v_T) = j$. This contradiction yields the result. $\qquad\square$

Similarly, the following result computes the actions of many ψ_r generators on $v_T \in e(i_\lambda) S_\lambda$.

Lemma 4.10. *For all* $T \in \mathcal{T}_e$ *and* $1 \leqslant r < n$ *with* $r \not\equiv 0 \pmod{e}$, $\psi_r v_T = 0$.

Proof. We know that

$$\psi_r v_{\mathrm{T}} = \sum_{\substack{\mathrm{T}\in\mathrm{Std}(\lambda) \\ i_{\mathrm{S}}=i_{s_r\mathrm{T}}}} a_{\mathrm{S}} v_{\mathrm{S}} \quad \text{for some } a_{\mathrm{S}} \in \mathbb{F},$$

and the result follows since no standard λ-tableau can have residue sequence $i_{s_r\mathrm{T}}$. $\square$

In order to calculate an endomorphism of S_λ, it remains to understand the action of the generators ψ_{re} on basis vectors v_{T}. In general, ψ_{re} does not annihilate v_{T}. We will find an endomorphism of S_λ which maps z_λ to a *linear combination* of elements v_{T} which we will show is annihilated by ψ_{re} if $r \neq j$. First, we will introduce some necessary notation for working with tableaux in $\mathcal{T}_e$.

For any $\mathrm{T} \in \mathcal{T}_e$, we number the e-bricks in the order of their entries, i.e., T comprises bricks $1, 2, \ldots, j+k$. Then we have brick transpositions and their corresponding ψ expressions, which we will denote by Ψ_r. In particular, the brick transposition which transposes the r-th and $(r+1)$-th bricks corresponds to

$$\Psi_r = \psi \underset{(r-1)e+1}{\overset{re}{\downarrow}} \underset{(r-1)e+2}{\overset{re+1}{\downarrow}} \cdots \underset{re}{\overset{(r+1)e-1}{\downarrow}}.$$

As with our ψ generators, we introduce the shorthand $\Psi^x_{\underset{y}{\downarrow}} = \Psi_x \Psi_{x-1} \ldots \Psi_y$ and $\Psi^x_{\underset{y}{\uparrow}} = \Psi_y \Psi_{y+1} \ldots \Psi_x$.

Note that for any $\mathrm{T} \in \mathcal{T}_e$, w_{T} is fully commutative since the reading word is 321-avoiding. We can write v_{T} in terms of the e-bricks lying in the second component of T; i.e., we may write $v_{\mathrm{T}} = v(B_{i_1}, B_{i_2}, \ldots, B_{i_j})$, for $1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant j+k$. Then

$$v_{\mathrm{T}} = \Psi^{i_1-1}_{\underset{1}{\downarrow}} \Psi^{i_2-1}_{\underset{2}{\downarrow}} \ldots \Psi^{i_j-1}_{\underset{j}{\downarrow}} z_\lambda.$$

Analogously, we write $\underline{v}(B_{i_1}, B_{i_2}, \ldots, B_{i_k})$ for the standard basis vector of S_λ indexed by the standard λ-tableau that has the e-bricks $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ lying in its *first* component.

Example. As in the previous example, let $e = 3$ and $\lambda = ((6), (6))$. Then, for example,

$$\Psi_1 = \psi^3_{\underset{1}{\downarrow}} \psi^4_{\underset{2}{\downarrow}} \psi^5_{\underset{3}{\downarrow}} = \psi_3 \psi_2 \psi_1 \psi_4 \psi_3 \psi_2 \psi_5 \psi_4 \psi_3.$$

The six tableaux in $\mathcal{T}_e$, given in the previous example, are determined by which two bricks (from the available bricks $1, 2, 3, 4$) are in the second component, and correspond to the basis elements

$$z_\lambda, \qquad \Psi_2 z_\lambda, \qquad \Psi^3_{\underset{2}{\downarrow}} z_\lambda = \Psi_3 \Psi_2 z_\lambda,$$

$$\Psi_1 \Psi_2 z_\lambda, \qquad \Psi_1 \Psi^3_{\underset{2}{\downarrow}} z_\lambda = \Psi_1 \Psi_3 \Psi_2 z_\lambda, \qquad \Psi^2_{\underset{1}{\downarrow}} \Psi^3_{\underset{2}{\downarrow}} z_\lambda = \Psi_2 \Psi_1 \Psi_3 \Psi_2 z_\lambda.$$

The following lemma is easy to see from the relations in the KLR algebras and their Specht modules, and we will use it frequently without reference.

Lemma 4.11. (i) *If* $|r - s| > 1$, *then* $\Psi_r \Psi_s = \Psi_s \Psi_r$.

(ii) *If* $\lambda = ((ke), (je))$, *for some* $j, k \geqslant 1$ *and* $r \neq j$, *then* $\Psi_r z_\lambda = 0$.

The following proposition will be crucial in our computations, particularly when $e > 2$. For $e = 2$, it follows from the proof of [Speyer 2014, Lemma 5.5], where the result for hook partitions in level one is proved, assuming that the residue sequence $\boldsymbol{i}_\lambda$ alternates between 0 and 1. Its proof for $e \geqslant 3$ is long and technical, requiring many preliminary lemmas, and we relegate this to Section 6.

Proposition 4.12. *Suppose that* $e \in \{2, 3, \dots\}$, *and let* $\lambda = ((ke), (je))$, *for some* $j, k \geqslant 1$. *If* $v \in e(\boldsymbol{i}_\lambda) S_\lambda$ *and* $1 \leqslant r \leqslant j + k - 1$, *then*

(i) $\psi_{re} \Psi_r v = -2\psi_{re} v$,

(ii) *for* $r < j + k - 1$, $\psi_{re} \Psi_{r+1} \Psi_r v = \psi_{re} v$,

(iii) *for* $r > 1$, $\psi_{re} \Psi_{r-1} \Psi_r v = \psi_{re} v$.

Theorem 4.13. *Suppose that* $k \geqslant j \geqslant 1$.

(i) *Let* $\lambda = ((ke), (je))$. *Then there is an endomorphism of* S_λ *defined by*

$$\varphi(z_\lambda) = \sum_{\substack{0 \leqslant i \leqslant k-1 \\ 0 \leqslant l \leqslant j-1}} (j - l)(k - i) \Psi_{\underset{j-l}{\uparrow}}^{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda.$$

(ii) *Let* $\lambda = ((je), (ke))$. *Then there is an endomorphism of* S_λ *defined by*

$$\varphi(z_\lambda) = \sum_{\substack{0 \leqslant i \leqslant j-1 \\ 0 \leqslant l \leqslant k-1}} (k - l)(j - i) \Psi_{\underset{k-l}{\uparrow}}^{k-1} \Psi_{\underset{k}{\downarrow}}^{k+i} z_\lambda.$$

Proof. (i) By Lemmas 4.7, 4.9, and 4.10, we just need to show that $\psi_{re} \varphi(z_\lambda) = 0$, for all $r \in \{1, \dots, j-1\} \cup \{j+1, \dots, k+j-1\}$. So we shall fix r and look at the action of ψ_{re} on each summand of $\varphi(z_\lambda)$. We will use Proposition 4.12 many times in this proof, without further reference.

We first suppose that $r \in \{1, \dots, j-1\}$. If $j - l < r - 1$, we observe that

$$\psi_{re} \Psi_{\underset{j-l}{\uparrow}}^{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda = \psi_{re} \Psi_{j-l} \Psi_{j-l+1} \dots \Psi_{r-2} \underbrace{\Psi_{r-1} \Psi_r}_{} \Psi_{r+1} \Psi_{r+2} \dots \Psi_{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda = 0.$$

Similarly, if $j - l > r + 1$, we have

$$\psi_{re} \Psi_{\underset{j-l}{\uparrow}}^{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda = \Psi_{\underset{j-l}{\uparrow}}^{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} \psi_{re} z_\lambda = 0.$$

If $j - l = r - 1$, we obtain the reduced expression

$$\psi_{re} \Psi_{r-1} \Psi_r \Psi_{r+1} \Psi_{r+2} \dots \Psi_{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda = \psi_{re} \Psi_{r+1} \Psi_{r+2} \dots \Psi_{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda.$$

If $j - l = r$, we obtain the reduced expression

$$\psi_{re} \Psi_r \Psi_{r+1} \Psi_{r+2} \dots \Psi_{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda = -2\psi_{re} \Psi_{r+1} \Psi_{r+2} \dots \Psi_{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda.$$

If $j - l = r + 1$, we immediately obtain the reduced expression

$$\psi_{re}\Psi_{r+1}\Psi_{r+2}\ldots\Psi_{j-1}\Psi\overset{j+i}{\underset{j}{\downarrow}} z_\lambda.$$

Thus, the only summands of $\varphi(z_\lambda)$ which are not killed by ψ_{re} are those corresponding to $j - l \in \{r - 1, r, r + 1\}$, and for a fixed i all three yield the same basis vector, so we must check the coefficients to show that they cancel. If $r > 1$, then the corresponding coefficients are $(r - 1)(k - i)$, $-2r(k - i)$, and $(r + 1)(k - i)$, respectively, so they cancel. If $r = 1$, we do not have a term corresponding to $j - l = r - 1$, so we only have the latter two terms, which clearly cancel.

We now suppose that $r \in \{j + 1, \ldots, j + k - 1\}$. If $j + i < r - 1$, we have

$$\psi_{re}\Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi_{j+i}\Psi_{j+i-1}\ldots\Psi_j z_\lambda = \Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi_{j+i}\Psi_{j+i-1}\ldots\Psi_j\psi_{re}z_\lambda = 0.$$

Similarly, if $j + i > r + 1$, we have

$$\psi_{re}\Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi\overset{j+i}{\underset{r+2}{\downarrow}}\Psi_{r+1}\Psi_r\Psi\overset{r-1}{\underset{j}{\downarrow}} z_\lambda = \psi_{re}\Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi\overset{j+i}{\underset{r+3}{\downarrow}}\Psi\overset{r-1}{\underset{j}{\downarrow}}\Psi_{r+2}z_\lambda = 0.$$

If $j + i = r - 1$, we see that the following expression is reduced:

$$\psi_{re}\Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi_{r-1}\Psi_{r-2}\ldots\Psi_j z_\lambda.$$

If $j + i = r$, we obtain the reduced expression

$$\psi_{re}\Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi_r\Psi_{r-1}\ldots\Psi_j z_\lambda = -2\psi_{re}\Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi_{r-1}\ldots\Psi_j z_\lambda.$$

If $j + i = r + 1$, we obtain the reduced expression

$$\psi_{re}\Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi_{r+1}\Psi_r\Psi_{r-1}\Psi_{r-2}\ldots\Psi_j z_\lambda = \psi_{re}\Psi\overset{j-1}{\underset{j-l}{\uparrow}}\Psi_{r-1}\Psi_{r-2}\ldots\Psi_j z_\lambda.$$

As in the previous case, it is easy to verify that the coefficients $(j-l)(k-r+j+1)$, $-2(j-l)(k-r+j)$, and $(j-l)(k-r+j-1)$ cancel. We note that here $r = j+k-1$ is the exceptional case, for which there is no term corresponding to $j+i = r+1$, but the $j+i = r-1$ and $j+i = r$ terms contribute $2(j-l)$ and $-2(j-l)$, respectively, so that the terms still cancel.

(ii) The proof is similar to that of the first part. $\qquad\square$

Example. Let $e = 3$, $\kappa = (0, 0)$, and $\lambda = ((9), (9))$. Then we have the endomorphism of S_λ

$$\varphi(z_\lambda) = 9\Psi_3 z_\lambda + 6\Psi_4\Psi_3 z_\lambda + 3\Psi_5\Psi_4\Psi_3 z_\lambda + 6\Psi_2\Psi_3 z_\lambda + 4\Psi_2\Psi_4\Psi_3 z_\lambda$$
$$+ 2\Psi_2\Psi_5\Psi_4\Psi_3 z_\lambda + 3\Psi_1\Psi_2\Psi_3 z_\lambda + 2\Psi_1\Psi_2\Psi_4\Psi_3 z_\lambda + \Psi_1\Psi_2\Psi_5\Psi_4\Psi_3 z_\lambda,$$

where the summands correspond to the tableaux

9
7	8	9	13	14	15	16	17	18
1	2	3	4	5	6	10	11	12

6
7	8	9	10	11	12	16	17	18
1	2	3	4	5	6	13	14	15

3
7	8	9	10	11	12	13	14	15
1	2	3	4	5	6	16	17	18

6
4	5	6	13	14	15	16	17	18
1	2	3	7	8	9	10	11	12

4
4	5	6	10	11	12	16	17	18
1	2	3	7	8	9	13	14	15

2
4	5	6	10	11	12	13	14	15
1	2	3	7	8	9	16	17	18

3
1	2	3	13	14	15	16	17	18
4	5	6	7	8	9	10	11	12

2
1	2	3	10	11	12	16	17	18
4	5	6	7	8	9	13	14	15

1
1	2	3	10	11	12	13	14	15
4	5	6	7	8	9	16	17	18

Proposition 4.14. *Suppose that $\lambda = ((e), (ke))$ or $\lambda = ((ke), (e))$ with $k > 1$. Then S_λ is decomposable if and only if $\mathrm{char}\,\mathbb{F} \nmid k + 1$.*

Proof. We first let $\lambda = ((e), (ke))$. For any nontrivial endomorphism φ of S_λ, we have

$$\varphi(z_\lambda) = \sum_{j=1}^{k} a_j \Psi^{k}_{\uparrow\, j} z_\lambda \quad \text{for some } a_j \in \mathbb{F}.$$

Since $\psi_{re}\varphi(z_\lambda) = 0$ for all $r \in \{1, 2, \ldots, k-1\}$, and by Proposition 4.12 we have

$$\psi_{re}\Psi^{k}_{\uparrow\, j} z_\lambda = \begin{cases} \psi_{re}\Psi^{k}_{\uparrow\, j} z_\lambda & \text{if } j = r-1 \text{ or } r+1, \\ -2\psi_{re}\Psi^{k}_{\uparrow\, j} z_\lambda & \text{if } j = r, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that $a_1 = \frac{1}{2}a_2$ and $2a_j = a_{j-1} + a_{j+1}$ for $j \in \{2, 3, \ldots, k-1\}$. One can check that, up to scalar multiplication, the only nontrivial endomorphism of S_λ is thus the one given in Theorem 4.13, which simplifies to

$$\varphi(z_\lambda) = \sum_{j=1}^{k} j \Psi^{k}_{\uparrow\, j} z_\lambda.$$

Using Proposition 4.12, we find that

$$\varphi^2(z_\lambda) = \left(\sum_{j=1}^{k} j \Psi^{k}_{\uparrow\, j} \right) \left(\sum_{j=1}^{k} j \Psi^{k}_{\uparrow\, j} \right) z_\lambda$$

$$= \left(\sum_{j=1}^{k} j \Psi^{k}_{\uparrow\, j} \right) ((k-1) \Psi^{k}_{\uparrow\, k-1} + k \Psi_k) z_\lambda$$

$$= (k-1)\varphi(z_\lambda) - (2k)\varphi(z_\lambda),$$

and so $-\frac{1}{k+1}\varphi$ is an idempotent when char $\mathbb{F} \nmid k+1$, or equivalently, when $(k+1, \text{char } \mathbb{F}) = 1$. Moreover, it is clear that $\varphi^2 = 0$ when char $\mathbb{F} \mid k+1$, so that there are no nontrivial idempotent endomorphisms.

We now let $\lambda = ((ke), (e))$. We similarly find that there exists only a single nontrivial endomorphism of S_λ, up to scalar multiplication, which is defined by

$$\varphi(z_\lambda) = \sum_{j=1}^{k}(k+1-j)\Psi_{\downarrow 1}^{j}z_\lambda.$$

Analogously, $-\frac{1}{k+1}\varphi$ is an idempotent when char $\mathbb{F} \nmid k+1$ and $\varphi^2 = 0$ when char $\mathbb{F} \mid k+1$. $\qquad\square$

Corollary 4.15. *Let* $1 \leqslant a < e$ *and* $0 \leqslant b < e$ *such that* $a + b < e$. *Then* $S_{((e+a,1^b),(ke+a,1^b))}$ *and* $S_{((ke+a,1^b),(e+a,1^b))}$ *are decomposable if and only if* $(k+1, \text{char } \mathbb{F}) = 1$ *with* $k > 1$.

Proof. The result follows from [Proposition 4.14](#) by applying [Proposition 4.4](#). $\qquad\square$

Theorem 4.16. *Suppose that* $k \geqslant j > 1$. *Let* $\lambda = ((ke), (je))$, *and let* φ *be the endomorphism of* S_λ *from [Theorem 4.13](#)(i). Then*

(i) φ *has an eigenvalue* $-j(k+1)$ *with corresponding eigenvector*

$$v(B_{k+1}, B_{k+2}, B_{k+3}, \ldots, B_{k+j}),$$

(ii) φ *has an eigenvalue* $-(j-1)(k+2)$ *with corresponding eigenvector*
   - $v(B_2, B_3, B_{k+2}, B_{k+3}, \ldots, B_{2k-1}) - v(B_1, B_{k+2}, B_{k+3}, \ldots, B_{2k})$ *if* $k = j$,
   - $\sum_{i=1}^{k-j+2} i\, v(B_2, B_4, B_6, \ldots, B_{2j-2}, B_{k+j-i+1})$ *if* $k > j$,

(iii) *if* $k > j > 2$, *then* φ *has an eigenvalue* $-(j-2)(k+3)$ *with corresponding eigenvector*

$$\sum_{i=1}^{k-j+3} \tfrac{1}{2}i(i+1)v(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-i}, B_{k+j-i+1})$$
$$+ \sum_{i=1}^{k-j+2} i(i+1)v(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-i-1}, B_{k+j-i+1})$$
$$+ \sum_{i=1}^{k-j+1}\sum_{l=1}^{i} l(i+2)v(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-i-2}, B_{k+j-l+1}),$$

(iv) *if* $k > j = 2$, *then* φ *has an eigenvalue* 0 *with corresponding eigenvector*

$$\sum_{i=0}^{k} \tfrac{1}{2}(i+1)(i+2)v(B_{k-i+1}, B_{k-i+2}) + \sum_{l=0}^{k-1}\sum_{i=1}^{k-l-1}(i+1)(k-l+1)v(B_{l+1}, B_{k-i+2}).$$

Analogously, we also have the following theorem.

Theorem 4.17. *Suppose that* $k \geqslant j > 1$. *Let* $\lambda = ((je), (ke))$, *and let* φ *be the endomorphism of* S_λ *from Theorem 4.13(ii). Then*

(i) φ *has an eigenvalue* $-j(k+1)$ *with corresponding eigenvector*

$$\underline{v}(B_{k+1}, B_{k+2}, B_{k+3}, \ldots, B_{k+j}),$$

(ii) φ *has an eigenvalue* $-(j-1)(k+2)$ *with corresponding eigenvector*
- $\underline{v}(B_2, B_3, B_{k+2}, B_{k+3}, \ldots, B_{2k-1}) - \underline{v}(B_1, B_{k+2}, B_{k+3}, \ldots, B_{2k})$ *if* $k = j$,
- $\sum_{i=1}^{k-j+2} i \, \underline{v}(B_2, B_4, B_6, \ldots, B_{2j-2}, B_{k+j-i+1})$ *if* $k > j$,

(iii) *if* $k > j > 2$, *then* φ *has an eigenvalue* $-(j-2)(k+3)$ *with corresponding eigenvector*

$$\sum_{i=1}^{k-j+3} \tfrac{1}{2} i(i+1) \underline{v}(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-i}, B_{k+j-i+1})$$
$$+ \sum_{i=1}^{k-j+2} i(i+1) \underline{v}(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-i-1}, B_{k+j-i+1})$$
$$+ \sum_{i=1}^{k-j+1} \sum_{l=1}^{i} l(i+2) \underline{v}(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-i-2}, B_{k+j-l+1}),$$

(iv) *if* $k > j = 2$, *then* φ *has an eigenvalue* 0 *with corresponding eigenvector*

$$\sum_{i=0}^{k} \tfrac{1}{2}(i+1)(i+2) \underline{v}(B_{k-i+1}, B_{k-i+2}) + \sum_{l=0}^{k-1} \sum_{i=1}^{k-l-1} (i+1)(k-l+1) \underline{v}(B_{l+1}, B_{k-i+2}).$$

Below, we will prove Theorem 4.16(iii). This is the most difficult part of Theorem 4.16 to prove, and the others are proved analogously. Likewise, similar calculations prove Theorem 4.17.

Proof of Theorem 4.16(iii). Let $\eta = \Psi_1 \Psi_{\underset{2}{\downarrow}}^{3} \Psi_{\underset{3}{\downarrow}}^{5} \ldots \Psi_{\underset{j-3}{\downarrow}}^{2j-7} \Psi_{\underset{j-2}{\downarrow}}^{2j-5}$, and

$$\psi_{w_{(1,s)}} := \eta \cdot \Psi_{\underset{j-1}{\downarrow}}^{k+j-s-1} \Psi_{\underset{j}{\downarrow}}^{k+j-s} \quad (1 \leqslant s \leqslant k-j+3),$$
$$\psi_{w_{(2,s)}} := \eta \cdot \Psi_{\underset{j-1}{\downarrow}}^{k+j-s-2} \Psi_{\underset{j}{\downarrow}}^{k+j-s} \quad (1 \leqslant s \leqslant k-j+2),$$
$$\psi_{w_{(r,s)}} := \eta \cdot \Psi_{\underset{j-1}{\downarrow}}^{k+j-s-3} \Psi_{\underset{j}{\downarrow}}^{k+j-r+2} \quad (1 \leqslant s \leqslant k-j+1, \ 3 \leqslant r \leqslant s+2),$$

and define $v_{r_s} = \psi_{w_{(r,s)}} z_\lambda$ in all cases above. We can now write

$$\sum_{s=1}^{k-j+3} \tfrac{1}{2}s(s+1)v(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-s}, B_{k+j-s+1})$$

$$+ \sum_{s=1}^{k-j+2} s(s+1)v(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-s-1}, B_{k+j-s+1})$$

$$+ \sum_{s=1}^{k-j+1} \sum_{r=3}^{s+2} (r-2)(s+2)v(B_2, B_4, B_6, \ldots, B_{2j-4}, B_{k+j-s-2}, B_{k+j-r+3})$$

$$= \sum_{s=1}^{k-j+3} \tfrac{1}{2}s(s+1)v_{1_s} + \sum_{s=1}^{k-j+2} s(s+1)v_{2_s} + \sum_{s=1}^{k-j+1} \sum_{r=3}^{s+2} (r-2)(s+2)v_{r_s}.$$

We want to show that

$$\left(\sum_{s=1}^{k-j+3} \tfrac{1}{2}s(s+1)\psi_{w_{(1,s)}} + \sum_{s=1}^{k-j+2} s(s+1)\psi_{w_{(2,s)}} + \sum_{s=1}^{k-j+1} \sum_{r=3}^{s+2} (r-2)(s+2)\psi_{w_{(r,s)}} \right) \cdot \varphi(z_\lambda)$$

$$= -(j-2)(k+3)\left(\sum_{s=1}^{k-j+3} \tfrac{1}{2}s(s+1)v_{1_s} + \sum_{s=1}^{k-j+2} s(s+1)v_{2_s} + \sum_{s=1}^{k-j+1} \sum_{r=3}^{s+2} (r-2)(s+2)v_{r_s} \right).$$

First suppose we are in one of the following cases, where l is the index in the summation form for $\varphi(z_\lambda)$ from Theorem 4.13:

- $s \leqslant k - j$ for all r and l,
- $s = k - j + 1$ and $r = 1$ or $r = 2$ or $(r \geqslant 3$ and $l \leqslant j - 2)$,
- $s = k - j + 2$ and $l \leqslant j - 2$,
- $s = k - j + 3$ and $l \leqslant j - 3$.

In these cases, we will show that

$$\psi_{w_{(r,s)}} \cdot \Psi_{\underset{j-l}{\uparrow}}^{j-1} \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda = \begin{cases} -2v_{r_s} & \text{if } l = i, \\ v_{r_s} & \text{if } l = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $l = 0$. If $i = 0$ or 1, we have

$$\psi_{w_{(r,s)}} \cdot \Psi_j z_\lambda = -2v_{r_s} \quad \text{or} \quad \psi_{w_{(r,s)}} \cdot \Psi_{j+1}\Psi_j z_\lambda = v_{r_s}, \quad \text{respectively.}$$

If $i > 1$, we have

$$\psi_{w_{(r,s)}} \cdot \Psi_{\underset{j}{\downarrow}}^{j+i} z_\lambda = \eta \cdot \Psi_{\underset{j-1}{\downarrow}}^{ij-1} \Psi_{\underset{j}{\downarrow}}^{ij-1} \Psi_{\underset{j+2}{\downarrow}}^{j+i} \Psi_{j+1}\Psi_j z_\lambda = 0.$$

Now suppose $l = 1$. If $i = 0$, we have

$$\psi_{w_{(r,s)}} \cdot \Psi_{j-1}\Psi_j z_\lambda = v_{r_s}.$$

If $i = 1$, we have

$$\psi_{w_{(r,s)}} \cdot \Psi_{j-1}\Psi_{j+1}\Psi_j z_\lambda = \eta \cdot \Psi^{ij-1}_{\underset{j-1}{\downarrow}}{}^{-1} \Psi^{ij-1}_{\underset{j+1}{\downarrow}} \underline{\Psi_j \Psi_{j+1}}\Psi_{j-1}\Psi_j z_\lambda$$

$$= -2\eta \cdot \Psi^{ij-1}_{\underset{j-1}{\downarrow}}{}^{-1} \Psi^{ij-1}_{\underset{j}{\downarrow}} z_\lambda = -2v_{r_s}.$$

If $i = 2$, we have

$$\psi_{w_{(r,s)}} \cdot \Psi_{j-1}\Psi_{j+2}\Psi_{j+1}\Psi_j z_\lambda = \eta \cdot \Psi^{ij-1}_{\underset{j-1}{\downarrow}}{}^{-1} \Psi^{ij-1}_{\underset{j+2}{\downarrow}} \underline{\Psi_{j+1}\Psi_{j+2}}\Psi_j \Psi_{j-1}\Psi_{j+1}\Psi_j z_\lambda$$

$$= \eta \cdot \Psi^{ij-1}_{\underset{j-1}{\downarrow}}{}^{-1} \underline{\Psi_j \Psi_{j-1}}\Psi^{ij-1}_{\underset{j}{\downarrow}} z_\lambda$$

$$= v_{r_s}.$$

If $i > 2$, we have

$$\psi_{w_{(r,s)}}\Psi_{j-1}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda = \eta \cdot \Psi^{ij-1}_{\underset{j-1}{\downarrow}}{}^{-1} \Psi^{ij-1}_{\underset{j+i}{\downarrow}} \underline{\Psi_{j+i-1}\Psi_{j+i}}\Psi^{j+i-2}_{\underset{j-1}{\downarrow}}\Psi^{j+i-1}_{\underset{j}{\downarrow}} z_\lambda$$

$$= \eta \cdot \Psi^{ij-1}_{\underset{j}{\downarrow}}{}^{-1} \Psi^{j+i-2}_{\underset{j+1}{\downarrow}} \Psi_{j-1}\underline{\Psi_j \Psi_{j-1}}\Psi^{ij-1}_{\underset{j}{\downarrow}} z_\lambda$$

$$= \eta \cdot \Psi^{ij-1}_{\underset{j-1}{\downarrow}}{}^{-1} \Psi^{j+i-2}_{\underset{j+2}{\downarrow}} \Psi^{ij-1}_{\underset{j+3}{\downarrow}}\Psi_{j+1}\underline{\Psi_{j+2}\Psi_{j+1}}\Psi_j z_\lambda$$

$$= 0.$$

Now assuming that $l > 1$, we have

$$\psi_{w(r,s)} \cdot \Psi^{j-1}_{\underset{j-l}{\uparrow}} \Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda$$

$$= \Psi_1 \Psi^3_{\underset{2}{\downarrow}}\Psi^5_{\underset{3}{\downarrow}}\ldots\Psi^{2j-2l-1}_{\underset{j-l}{\downarrow}}\Psi^{2j-2l+1}_{\underset{j-l}{\downarrow}}\Psi^{2j-2l+3}_{\underset{j-l+1}{\downarrow}}\ldots\Psi^{2j-5}_{\underset{j-3}{\downarrow}}\Psi^{ij-1}_{\underset{j-2}{\downarrow}}{}^{-1}\Psi^{ij-1}_{\underset{j-1}{\downarrow}}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda$$

$$= \Psi_1 \Psi^3_{\underset{2}{\downarrow}}\Psi^5_{\underset{3}{\downarrow}}$$

$$\ldots\Psi^{2j-2l-1}_{\underset{j-l+1}{\downarrow}}\Psi^{2j-2l+1}_{\underset{j-l+2}{\downarrow}}\Psi_{j-l}\underline{\Psi_{j-l+1}\Psi_{j-l}}\Psi^{2j-2l+3}_{\underset{j-l+1}{\downarrow}}\ldots\Psi^{2j-5}_{\underset{j-3}{\downarrow}}\Psi^{ij-1}_{\underset{j-2}{\downarrow}}{}^{-1}\Psi^{ij-1}_{\underset{j-1}{\downarrow}}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda$$

$$= \Psi_1 \Psi^3_{\underset{2}{\downarrow}}\Psi^5_{\underset{3}{\downarrow}}\ldots\Psi^{2j-2l-1}_{\underset{j-l}{\downarrow}}\Psi^{2j-2l+1}_{\underset{j-l+3}{\downarrow}}\Psi^{2j-2l+3}_{\underset{j-l+4}{\downarrow}}\Psi_{j-l+2}\underline{\Psi_{j-l+3}\Psi_{j-l+2}}\Psi_{j-l+1}\Psi^{2j-2l+5}_{\underset{j-l+2}{\downarrow}}$$

$$\ldots\Psi^{2j-5}_{\underset{j-3}{\downarrow}}\Psi^{ij-1}_{\underset{j-2}{\downarrow}}{}^{-1}\Psi^{ij-1}_{\underset{j-1}{\downarrow}}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda$$

$$= \Psi_1 \Psi^3_{\underset{2}{\downarrow}}\Psi^5_{\underset{3}{\downarrow}}\ldots\Psi^{2j-2l+1}_{\underset{j-l+1}{\downarrow}}\Psi^{2j-2l+3}_{\underset{j-l+5}{\downarrow}}\Psi^{2j-2l+5}_{\underset{j-l+6}{\downarrow}}\Psi_{j-l+4}\underline{\Psi_{j-l+5}\Psi_{j-l+4}}\Psi^{j-l+3}_{\underset{j-l+2}{\downarrow}}\Psi^{2j-2l+7}_{\underset{j-l+3}{\downarrow}}$$

$$\ldots\Psi^{2j-5}_{\underset{j-3}{\downarrow}}\Psi^{ij-1}_{\underset{j-2}{\downarrow}}{}^{-1}\Psi^{ij-1}_{\underset{j-1}{\downarrow}}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda$$

$$\vdots$$

$$= \Psi_1 \Psi^3_{\underset{2}{\downarrow}}\Psi^5_{\underset{3}{\downarrow}}\ldots\Psi^{2j-7}_{\underset{j-3}{\downarrow}}\Psi^{2j-5}_{\underset{j+l-3}{\downarrow}}\Psi^{ij-1}_{\underset{j+l-2}{\downarrow}}{}^{-1}\Psi_{j+l-4}\underline{\Psi_{j+l-3}\Psi_{j+l-4}}\Psi^{j+l-5}_{\underset{j-2}{\downarrow}}\Psi^{ij-1}_{\underset{j-1}{\downarrow}}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda$$

$$= \Psi_1 \Psi^3_{\underset{2}{\downarrow}}\Psi^5_{\underset{3}{\downarrow}}\ldots\Psi^{2j-7}_{\underset{j-3}{\downarrow}}\Psi^{2j-5}_{\underset{j-2}{\downarrow}}\Psi^{ij-1}_{\underset{j+l-1}{\downarrow}}{}^{-1}\Psi^{ij-1}_{\underset{j+l}{\downarrow}}\Psi_{j+l-2}\underline{\Psi_{j+l-1}\Psi_{j+l-2}}\Psi^{j+l-3}_{\underset{j-1}{\downarrow}}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda$$

$$= \Psi_1 \Psi^3_{\underset{2}{\downarrow}}\Psi^5_{\underset{3}{\downarrow}}\ldots\Psi^{2j-7}_{\underset{j-3}{\downarrow}}\Psi^{2j-5}_{\underset{j-2}{\downarrow}}\Psi^{ij-1}_{\underset{j-1}{\downarrow}}{}^{-1}\Psi^{ij-1}_{\underset{j+l}{\downarrow}}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda.$$

If $l = i$, observe

$$\Psi^{ij-1}_{\underset{j+i}{\downarrow}}{}^{-1}\Psi^{j+i}_{\underset{j}{\downarrow}} z_\lambda = -2\Psi^{ij-1}_{\underset{j}{\downarrow}}{}^{-1} z_\lambda.$$

If $l = i - 1$, observe

$$\Psi^{i}{}_{\underset{j+i-1}{\downarrow}}{}^{j-1}\underline{\Psi_{j+i}\Psi_{j+i-1}}\Psi^{j+i-2}{}_{\underset{j}{\downarrow}}z_\lambda = \Psi^{i}{}_{\underset{j}{\downarrow}}{}^{j-1}z_\lambda.$$

If $l = i + 1$, observe

$$\Psi^{i}{}_{\underset{j+i+1}{\downarrow}}{}^{j-1}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda = \Psi^{i}{}_{\underset{j}{\downarrow}}{}^{j-1}z_\lambda.$$

If $l > i + 1$, observe

$$\Psi^{i}{}_{\underset{j+l}{\downarrow}}{}^{j-1}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda = \Psi^{i}{}_{\underset{j+l+1}{\downarrow}}{}^{j-1}\Psi^{j+i}{}_{\underset{j}{\downarrow}}\Psi_{j+l}z_\lambda = 0.$$

If $l < i - 1$, observe

$$\Psi^{i}{}_{\underset{j+l}{\downarrow}}{}^{j-1}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda = \Psi^{i}{}_{\underset{j+l}{\downarrow}}{}^{j-1}\Psi^{j+i}{}_{\underset{j+l+2}{\downarrow}}\underline{\Psi_{j+l+1}\Psi_{j+l}}\Psi^{j+l-1}{}_{\underset{j}{\downarrow}}z_\lambda = \Psi^{i}{}_{\underset{j+l}{\downarrow}}{}^{j-1}\Psi^{j+i}{}_{\underset{j+l+3}{\downarrow}}\Psi^{j+l-1}{}_{\underset{j}{\downarrow}}\Psi_{j+l+2}z_\lambda = 0.$$

We now suppose that we are not in the listed cases and that we lie in the *exceptional cases*.

First let $r = 1$, $s = k - j + 3$, and $l = j - 2$. Similarly to above, we obtain

$$\psi_{w(r,s)} \cdot \Psi^{j-1}{}_{\underset{j-l}{\uparrow}}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda = \cdots = \Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-7}{}_{\underset{j-3}{\downarrow}}\Psi_{2j-5}\Psi_{2j-6}\Psi^{2j-4}{}_{\underset{j-2}{\downarrow}}\Psi^{2j-3}{}_{\underset{j-1}{\downarrow}}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda$$

$$= \Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-7}{}_{\underset{j-3}{\downarrow}}\Psi_{2j-5}\Psi_{2j-4}\Psi_{2j-6}\underline{\Psi_{2j-5}\Psi_{2j-6}}\Psi^{2j-4}{}_{\underset{j-2}{\downarrow}}\Psi^{2j-3}{}_{\underset{j-1}{\downarrow}}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda$$

$$= \Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-7}{}_{\underset{j-3}{\downarrow}}\Psi^{2j-5}{}_{\underset{j-2}{\downarrow}}\Psi_{2j-4}\underline{\Psi_{2j-3}\Psi_{2j-4}}\Psi^{2j-5}{}_{\underset{j-1}{\downarrow}}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda$$

$$= \Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-7}{}_{\underset{j-3}{\downarrow}}\Psi^{2j-5}{}_{\underset{j-2}{\downarrow}}\Psi^{2j-4}{}_{\underset{j-1}{\downarrow}}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda.$$

If $i \geqslant j - 3$, then this expression is clearly reduced. If $i < j - 3$, we have

$$\Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-7}{}_{\underset{j-3}{\downarrow}}\Psi^{2j-5}{}_{\underset{j-2}{\downarrow}}\Psi^{2j-4}{}_{\underset{j-1}{\downarrow}}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda$$

$$= \Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-7}{}_{\underset{j-3}{\downarrow}}\Psi^{2j-5}{}_{\underset{j-2}{\downarrow}}\Psi^{2j-4}{}_{\underset{j+i+1}{\downarrow}}\Psi_{j+i}\underline{\Psi_{j+i-1}\Psi_{j+i}}\Psi^{j+i-2}{}_{\underset{j}{\downarrow}}\Psi^{j+i-1}{}_{\underset{j}{\downarrow}}z_\lambda$$

$$= \Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-7}{}_{\underset{j-3}{\downarrow}}\Psi^{2j-5}{}_{\underset{j+i-1}{\downarrow}}\Psi_{j+i-2}\underline{\Psi_{j+i-3}\Psi_{j+i-2}}\Psi^{j+i-4}{}_{\underset{j-2}{\downarrow}}\Psi^{j+i-3}{}_{\underset{j-1}{\downarrow}}\Psi^{2j-4}{}_{\underset{j}{\downarrow}}z_\lambda$$

$$\vdots$$

$$= \Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-2i-5}{}_{\underset{j-i-2}{\downarrow}}\Psi^{2j-2i-3}{}_{\underset{j-i+1}{\downarrow}}\Psi_{j-i}\underline{\Psi_{j-i-1}\Psi_{j-i}}\Psi^{2j-2i-1}{}_{\underset{j-i+1}{\downarrow}}\ldots\Psi^{2j-5}{}_{\underset{j-1}{\downarrow}}\Psi^{2j-4}{}_{\underset{j}{\downarrow}}z_\lambda$$

$$= \Psi_1\Psi^{3}{}_{\underset{2}{\downarrow}}\Psi^{5}{}_{\underset{4}{\downarrow}}\ldots\Psi^{2j-2i-7}{}_{\underset{j-i-3}{\downarrow}}\Psi^{2j-2i-5}{}_{\underset{j-i-1}{\downarrow}}\Psi^{2j-2i-3}{}_{\underset{j-i}{\downarrow}}\Psi^{2j-2i-1}{}_{\underset{j-i+1}{\downarrow}}\ldots\Psi^{2j-5}{}_{\underset{j-1}{\downarrow}}\Psi^{2j-4}{}_{\underset{j}{\downarrow}}\Psi_{j-i-2}z_\lambda$$

$$= 0.$$

If instead $l = j - 1$, we similarly obtain

$$\psi_{w(r,s)} \cdot \Psi^{j-1}{}_{\underset{j-l}{\uparrow}}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda = \eta \cdot \Psi^{2j-3}{}_{\underset{j-1}{\downarrow}}\Psi^{j+i}{}_{\underset{j}{\downarrow}}z_\lambda,$$

which is clearly reduced if $i \geqslant j - 2$. If $i < j - 2$, then the expression becomes zero.

Let $r = 1$, $s = k - j + 2$, and $l = j - 1$. Then the above expression becomes

$$\eta \cdot \Psi^{2j-3}_{\substack{\downarrow \\ j-1}} \Psi^{j+i}_{\substack{\downarrow \\ j}} z_\lambda,$$

which is clearly reduced if $i \geqslant j - 2$. If $i < j - 2$, then the expression becomes zero, as before.

Now let $r = 2$, $s = k - j + 2$, and $l = j - 1$. Then the above expression becomes

$$\eta \cdot \Psi^{2j-2}_{\substack{\downarrow \\ j-1}} \Psi^{j+i}_{\substack{\downarrow \\ j}} z_\lambda,$$

which is clearly reduced if $i \geqslant j - 1$. If $i < j - 2$, then the expression becomes zero, while for $i = j - 2$ we get

$$\eta \cdot \Psi^{2j-4}_{\substack{\downarrow \\ j-1}} \Psi^{2j-2}_{\substack{\downarrow \\ j}} z_\lambda.$$

Finally, let $r > 2$, $s = k - j + 1$, and $l = j - 1$. Then the above expression becomes

$$\eta \cdot \Psi^{k+j-r+2}_{\substack{\downarrow \\ j-1}} \Psi^{j+i}_{\substack{\downarrow \\ j}} z_\lambda,$$

which is clearly reduced if $i \geqslant k - r + 3$. If $j - 2 \leqslant i \leqslant k - r + 2$, then

$$\eta \cdot \Psi^{k+j-r+2}_{\substack{\downarrow \\ j-1}} \Psi^{j+i}_{\substack{\downarrow \\ j}} z_\lambda = \eta \cdot \Psi^{k+j-r+2}_{\substack{\downarrow \\ j+i+1}} \Psi_{j+i} \Psi_{j+i-1} \Psi_{j+i} \Psi^{j+i-2}_{\substack{\downarrow \\ j-1}} \Psi^{j+i-1}_{\substack{\downarrow \\ j}} z_\lambda$$
$$= \eta \cdot \Psi^{j+i-2}_{\substack{\downarrow \\ j-1}} \Psi^{k+j-r+2}_{\substack{\downarrow \\ j}} z_\lambda,$$

which is reduced. If $i < j - 2$, then the expression becomes zero.

We summarize the exceptional cases. First suppose $r = 1$, $s = k - j + 3$, and $l = j - 2$. Then

$$\psi_{w_{(1,s)}} \cdot \Psi^{j-1}_{\substack{\uparrow \\ 2}} \Psi^{j+i}_{\substack{\downarrow \\ j}} z_\lambda = \begin{cases} 0 & \text{if } i \leqslant j - 4, \\ v_{1_s} & \text{if } i = j - 3, \\ v_{2_{s-1}} & \text{if } i = j - 2, \\ v_{(k-i+2)_{s-2}} & \text{if } i \geqslant j - 1. \end{cases}$$

If $s = k - j + 2$ or $s = k - j + 3$ and $l = j - 1$, then

$$\psi_{w_{(1,s)}} \cdot \Psi^{j-1}_{\substack{\uparrow \\ 1}} \Psi^{j+i}_{\substack{\downarrow \\ j}} z_\lambda = \begin{cases} 0 & \text{if } i \leqslant j - 3, \\ v_{1_{k-j+2}} & \text{if } i = j - 2, \\ v_{2_{k-j+1}} & \text{if } i = j - 1, \\ v_{(k-i+2)_{k-j}} & \text{if } i \geqslant j. \end{cases}$$

Now suppose that $r = 2$. If $s = k - j + 2$ and $l = j - 1$, then

$$\psi_{w_{(2,s)}} \cdot \Psi^{j-1}_{\uparrow 1} \Psi^{j+i}_{\downarrow j} z_\lambda = \begin{cases} 0 & \text{if } i \leqslant j - 3, \\ v_{2_s} & \text{if } i = j - 2, \\ v_{1_{s-1}} & \text{if } i = j - 1, \\ v_{2_{s-2}} & \text{if } i = j, \\ v_{(k-i+2)_{s-3}} & \text{if } i \geqslant j + 1. \end{cases}$$

Finally, suppose that $s = k - j + 1$. If $r \in \{3, \ldots, k - j + 3\}$ and $l = j - 1$, then

$$\psi_{w_{(r,s)}} \cdot \Psi^{j-1}_{\uparrow 1} \Psi^{j+i}_{\downarrow j} z_\lambda = \begin{cases} 0 & \text{if } i \leqslant j - 3, \\ v_{r_{k-i-1}} & \text{if } j - 2 \leqslant i \leqslant k - r + 1, \\ v_{2_{r-2}} & \text{if } i = k - r + 2, \\ v_{1_{r-3}} & \text{if } i = k - r + 3, \\ v_{2_{r-4}} & \text{if } i = k - r + 4, \\ v_{(k-i+2)_{r-5}} & \text{if } i \geqslant k - r + 5. \end{cases}$$

In the nonexceptional cases, we determine the coefficient $\alpha \in \mathbb{Z}$ in

$$\psi_{w_{(r,s)}} \cdot \varphi(z_\lambda) = \alpha v_{r_s}.$$

If $s \leqslant k - j$, we obtain

$$\psi_{w_{(r,s)}} \cdot \varphi(z_\lambda) = \psi_{w_{(r,s)}} \cdot \sum_{\substack{0 \leqslant i \leqslant k-1 \\ 0 \leqslant l \leqslant j-1}} (j - l)(k - i) \Psi^{j-i}_{\uparrow j-l} \Psi^{j+i}_{\downarrow j} z_\lambda$$

$$= \sum_{l=0}^{j-1} -2(j-l)(k-l)v_{r_s} + \sum_{l=0}^{j-1}(j-l)(k-l-1)v_{r_s} + \sum_{l=1}^{j-1}(j-l)(k-l+1)v_{r_s}$$

$$= -j(k+1)v_{r_s}.$$

Similarly, we also obtain $\alpha = -j(k+1)$ in the other nonexceptional cases.

By combining this with the exceptional cases, we now determine α_{r_s} where

$$\left(\sum_{s=1}^{k-j+3} \tfrac{1}{2}s(s+1)\psi_{w_{(1,s)}} + \sum_{s=1}^{k-j+2} s(s+1)\psi_{w_{(2,s)}} + \sum_{s=1}^{k-j+1}\sum_{r=3}^{s+2}(r-2)(s+2)\psi_{w_{(r,s)}} \right) \cdot \varphi(z_\lambda)$$

$$= \sum_{s=1}^{k-j+3} \alpha_{1_s} v_{1_s} + \sum_{s=1}^{k-j+2} \alpha_{2_s} v_{2_s} + \sum_{s=1}^{k-j+1}\sum_{r=3}^{s+2}\alpha_{r_s} v_{r_s}.$$

We first let $r = 1$. If $s = k - j + 3$, we know from above that

$$\alpha_{1_s} v_{1_s} = \tfrac{1}{2} s(s+1) \psi_{w_{(1,s)}} \cdot \left(\sum_{\substack{0 \leqslant i \leqslant k-1 \\ 0 \leqslant l \leqslant j-3}} (j-l)(k-i) \Psi'^{\,j-1}_{\uparrow\;\overset{1}{}} \Psi'^{\,j+i}_{\downarrow\;j} z_\lambda + 2s \Psi'^{\,j-1}_{\uparrow\;1} \Psi'^{\,2j-3}_{\downarrow\;j} z_\lambda \right)$$

$$= \tfrac{1}{2} s(s+1)(-j(k+1)v_{1_s} + 2s v_{1_s})$$

$$= -(j-2)(k+3) \cdot \tfrac{1}{2} s(s+1) v_{1_s};$$

hence $\alpha_{1_s} = -(j-2)(k+3) \cdot \tfrac{1}{2} s(s+1)$, as required.

If $s = k - j + 2$, then we have

$$\alpha_{1_s} v_{1_s} = \tfrac{1}{2} s(s+1) \psi_{w_{(1,s)}} \cdot \left(\sum_{\substack{0 \leqslant i \leqslant k-1 \\ 0 \leqslant l \leqslant j-2}} (j-l)(k-i) \Psi'^{\,j-1}_{\uparrow\;\overset{1}{}} \Psi'^{\,j+i}_{\downarrow\;j} z_\lambda + s \Psi'^{\,j-1}_{\uparrow\;1} \Psi'^{\,2j-2}_{\downarrow\;j} z_\lambda \right)$$
$$\qquad\qquad + \tfrac{1}{2}(s+1)(s+2) \psi_{w_{(1,s+1)}} \cdot s \Psi'^{\,j-1}_{\uparrow\;1} \Psi'^{\,2j-2}_{\downarrow\;j} z_\lambda$$

$$= \tfrac{1}{2} s(s+1)(-j(k+1) + s) v_{1_s} + \tfrac{1}{2} s(s+1)(s+2) v_{1_s}$$

$$= -(j-2)(k+3) \cdot \tfrac{1}{2}(s+1)(s+2) v_{1_s}.$$

If $s = k - j + 1$, then we have

$$\alpha_{1_s} v_{1_s} = \tfrac{1}{2} s(s+1) \psi_{w_{(1,s)}} \cdot \varphi(z_\lambda) + (s+1)(s+2) \psi_{w_{(2,s+1)}} \cdot s \Psi'^{\,j-1}_{\uparrow\;1} \Psi'^{\,2j-1}_{\downarrow\;j} z_\lambda$$

$$= -(j-2)(k+3) \cdot \tfrac{1}{2} s(s+1) v_{1_s}.$$

For $1 \leqslant s \leqslant k - j$, we have

$$\alpha_{1_s} v_{1_s} = \tfrac{1}{2} s(s+1) \psi_{w_{(1,s)}} \cdot \varphi(z_\lambda) + (s+1)(k-j+3) \psi_{w_{(s+3,k-j+1)}} \cdot s \Psi'^{\,j-1}_{\uparrow\;1} \Psi'^{\,j+k-s}_{\downarrow\;j} z_\lambda$$

$$= -(j-2)(k+3) \cdot \tfrac{1}{2} s(s+1) v_{1_s}.$$

We now suppose that $r = 2$. If $s = k - j + 2$, then

$$\alpha_{2_s} v_{2_s} = s(s+1) \psi_{w_{(2,s)}} \cdot \sum_{\substack{0 \leqslant i \leqslant k-1 \\ 0 \leqslant l \leqslant j-2}} (j-l)(k-i) \Psi'^{\,j-1}_{\uparrow\;1} \Psi'^{\,j+i}_{\downarrow\;j} z_\lambda$$

$$\quad + \tfrac{1}{2}(s+1)(s+2) \psi_{w_{(1,s+1)}} \cdot 2s \Psi'^{\,j-1}_{\uparrow\;2} \Psi'^{\,2j-2}_{\downarrow\;j} z_\lambda + s(s+1) \psi_{w_{(2,s)}} \cdot s \Psi'^{\,j-1}_{\uparrow\;1} \Psi'^{\,2j-2}_{\downarrow\;j} z_\lambda$$

$$= -s(s+1) j(k+1) v_{2_s} + s(s+1)(s+2) v_{2_s} + s^2(s+1) v_{2_s}$$

$$= -(j-2)(k+3) \cdot s(s+1) v_{2_s}.$$

If $s = k - j + 1$, then

$$
\begin{aligned}
\alpha_{2_s} v_{2_s} &= s(s+1)\psi_{w_{(2,k-j+1)}} \cdot \varphi(z_\lambda) + \tfrac{1}{2}(s+1)(s+2)\psi_{w_{(1,s+1)}} \cdot s\Psi_{1\uparrow}^{j-1} \Psi_{j\downarrow}^{2j-1} z_\lambda \\
&\quad + \tfrac{1}{2}(s+2)(s+3)\psi_{w_{(1,s+2)}} \cdot s\Psi_{1\uparrow}^{j-1} \Psi_{j\downarrow}^{2j-1} z_\lambda + s(s+2)\psi_{w_{(s+2,s)}} \cdot s\Psi_{1\uparrow}^{j-1} \Psi_{j\downarrow}^{2j-1} z_\lambda \\
&= -j(k+1)s(s+1)v_{2_s} + s(s+2)(\tfrac{1}{2}(s+1) + \tfrac{1}{2}(s+3) + s)v_{2_s} \\
&= -(j-2)(k+3) \cdot s(s+1)v_{2_s}.
\end{aligned}
$$

If $s = k - j$, then

$$
\begin{aligned}
\alpha_{2_s} v_{2_s} &= s(s+1)\psi_{w_{(2,s)}} \cdot \varphi_{z_\lambda} + (s+2)(s+3)\psi_{w_{(2,s+2)}} \cdot s\Psi_{1\uparrow}^{j-1} \Psi_{j\downarrow}^{2j} z_\lambda \\
&\qquad\qquad + s(s+3)\psi_{w_{(s+2,s+1)}} \cdot s\Psi_{1\uparrow}^{j-1} \Psi_{j\downarrow}^{2j} z_\lambda \\
&= -j(k+1)s(s+1)v_{2_s} + s(s+3)(2s+2)v_{2_s} \\
&= -(j-2)(k+3) \cdot s(s+1)v_{2_s}.
\end{aligned}
$$

If $s \in \{1, \ldots, k - j - 1\}$, then

$$
\begin{aligned}
\alpha_{2_s} v_{2_s} &= s(s+1)\psi_{w_{(2,s)}} \cdot \varphi(z_\lambda) + s(k-j+3)\varphi_{w_{(s+2,k-j+1)}} \cdot s\Psi_{1\uparrow}^{j-1} \Psi_{j\downarrow}^{j+k-s} z_\lambda \\
&\qquad\qquad + (s+2)(k-j+3)\psi_{w_{(s+4,k-j+1)}} \cdot s\Psi_{1\uparrow}^{j-1} \Psi_{j\downarrow}^{j+k-s} z_\lambda \\
&= -j(k+1)s(s+1)v_{2_s} + s(k-j+3)(2s+2)v_{2_s} \\
&= -(j-2)(k+3) \cdot s(s+1)v_{2_s}.
\end{aligned}
$$

Finally, suppose that $r \in \{3, \ldots, k - j + 3\}$. If $s = k - j + 1$, then

$$
\begin{aligned}
\alpha_{r_s} v_{r_s} &= (r-2)(s+2)\psi_{w_{(r,s)}} \cdot \sum_{\substack{0 \leqslant i \leqslant k-1 \\ 0 \leqslant l \leqslant j-2}} (j-l)(k-i)\Psi_{j-l\uparrow}^{j-1} \Psi_{j\downarrow}^{j+i} z_\lambda \\
&\qquad + \tfrac{1}{2}(s+2)(s+3)\psi_{w_{(1,s+2)}} \cdot 2(r-2)\Psi_{2\uparrow}^{j-1} \Psi_{j\downarrow}^{j+k-r+2} z_\lambda \\
&\qquad + (r-2)(s+2)\psi_{w_{(r,s)}} \cdot (s+1)\Psi_{1\uparrow}^{j-1} \Psi_{j\downarrow}^{2j-2} z_\lambda \\
&= -(j-2)(k+3) \cdot (r-2)(s+2)v_{r_{k-j+1}}.
\end{aligned}
$$

If $s = k - j$, then

$$\alpha_{r_s} v_{r_s} = (r-2)(s+2)\psi_{w_{(r,s)}} \cdot \varphi(z_\lambda)$$
$$+ \tfrac{1}{2}(s+2)(s+3)\psi_{w_{(1,s+2)}} \cdot (r-2)\Psi^{j-1}_{1\uparrow} \Psi^{j+k-r+2}_{j\downarrow} z_\lambda$$
$$+ \tfrac{1}{2}(s+3)(s+4)\psi_{w_{(1,s+3)}} \cdot (r-2)\Psi^{j-1}_{1\uparrow} \Psi^{j+k-r+2}_{j\downarrow} z_\lambda$$
$$+ (r-2)(s+3)\psi_{w_{(r,s+1)}} \cdot (s+1)\Psi^{j-1}_{1\uparrow} \Psi^{2j-1}_{j\downarrow} z_\lambda$$
$$= (r-2)(s+2)(-j(k+1)+2(s+3))v_{r_s}$$
$$= -(j-2)(k+3) \cdot (r-2)(s+2)v_{r_s}.$$

If $s = k - j - 1$, then

$$\alpha_{r_s} v_{r_s} = (r-2)(s+2)\psi_{w_{(r,s)}} \cdot \varphi(z_\lambda) + (s+3)(s+4)\psi_{w_{(2,s+3)}} \cdot (r-2)\Psi^{j-1}_{1\uparrow} \Psi^{j+k-r+2}_{j\downarrow} z_\lambda$$
$$+ (r-2)(s+4)\psi_{w_{(r,s+2)}} \cdot (k-j)\Psi^{j-1}_{1\uparrow} \Psi^{2j-1}_{j\downarrow} z_\lambda$$
$$= -(j-2)(k+3) \cdot (r-2)(s+2)v_{r_s}.$$

If $s \in \{1, \ldots, k - j - 2\}$, then

$$\alpha_{r_s} v_{r_s} = (r-2)(s+2)\psi_{w_{(r,s)}} \cdot \varphi z_\lambda$$
$$+ (r-2)(k-j+3)\psi_{w_{(r,k-j+1)}} \cdot (s+1)\Psi^{j-1}_{1\uparrow} \Psi^{j+k-s-1}_{j\downarrow} z_\lambda$$
$$+ (s+3)(k-j+3)\psi_{w_{(s+5,k-j+1)}} \cdot (r-2)\Psi^{j-1}_{1\uparrow} \Psi^{j+k-r+2}_{j\downarrow} z_\lambda$$
$$= (r-2)(-j(k+1)(s+2) + (k-j+3)(2s+4))v_{r_s}$$
$$= -(j-2)(k+3) \cdot (r-2)(s+2)v_{r_s}.$$

Hence,

$$\alpha_{r_s} = -(j-2)(k+3) \cdot \begin{cases} \tfrac{1}{2}s(s+1) & \text{if } r = 1, \\ s(s+1) & \text{if } r = 2, \\ (r-2)(s+2) & \text{if } r \geqslant 3, \end{cases}$$

for all s, as required. $\qquad\square$

Corollary 4.18. *Let* $\lambda = ((ke), (je))$ *or* $((1^{ke}), (1^{je}))$ *with* $k, j > 1$.

(i) *If* $j + k$ *is odd, then* S_λ *is decomposable.*

(ii) *If* $j + k$ *is even and* char $\mathbb{F} \neq 2$, *then* S_λ *is decomposable.*

Proof. By Theorems 4.16 and 4.17 parts (i) and (ii), there is an endomorphism φ of S_λ with two eigenvalues differing by $j - k - 2$. If $j = k$, then $j + k$ is even and these eigenvalues differ by 2, and hence are distinct if char $\mathbb{F} \neq 2$.

If $k > j > 2$, then by part (iii), we have a third eigenvalue differing from the second by $j - k - 4$. If char $\mathbb{F} \mid (j - k - 2)$, then the first two eigenvalues are equal,

but the third is distinct if char $\mathbb{F} \neq 2$. If char $\mathbb{F} = 2$, then char $\mathbb{F} \nmid (j - k - 2)$ when $j + k$ is odd.

If $k > j = 2$, then by part (iv), we have 0 as our third eigenvalue. If k is odd, we have at least two distinct eigenvalues in any characteristic. Likewise, if k is even and char $\mathbb{F} \neq 2$, we have at least two distinct eigenvalues.

Thus, in each case, the generalized eigenspace decomposition of S_λ has at least two direct summands, which are easily seen to be $\mathscr{R}_n^\Lambda$-modules. $\qquad\square$

By applying Theorem 2.8 and the method in the proofs of Propositions 4.3 and 4.4, and combining with Theorem 3.8 and Proposition 4.14 and Corollary 4.15, we obtain Theorem 4.1.

5. Quantum characteristic two

We now focus our attention on the case where $e = 2$.

Thankfully most of the difficult work here is already done, and we are able to use results from [Speyer 2014; Murphy 1980; Fayers and Speyer 2016] on Specht modules over the level-one cyclotomic quiver Hecke algebra to obtain a lot of decomposable Specht modules with little effort. We collect the results we will need.

Theorem 5.1 [Murphy 1980, Theorem 4.5]. *If $e = \text{char } \mathbb{F} = 2$, $a \geqslant b$, and either λ or λ' is $(a, 1^b)$, then S_λ is decomposable if and only if $a + b$ is odd and $a - 1 \not\equiv b \pmod{2^L}$, where $2^{L-1} \leqslant b < 2^L$.*

Theorem 5.2 [Speyer 2014, Theorem 6.8]. *If $e = 2$, char $\mathbb{F} \neq 2$, $a > b$, and either λ or λ' is $(a, 1^b)$, then S_λ is decomposable if and only if $a + b$ is odd and either*

 (i) *$a > b \geqslant 4$ or*

 (ii) *$a > b = 2$ or 3, and char $\mathbb{F} \nmid \lceil \frac{a}{2} \rceil$.*

When looking for decomposable Specht modules, the cases $\kappa = (0, 1)$ and $\kappa = (0, 0)$ must be treated separately. However, our first result is independent of this.

Theorem 5.3. *Let κ be arbitrary, with corresponding $\Lambda = \Lambda_\kappa$. Let μ be a hook partition of n such that S_μ is a decomposable $\mathscr{R}_n^{\Lambda_0}$-module (see Theorems 5.1 and 5.2). Then for any partition ν of m, the Specht modules $S_{(\mu,\nu)}$ and $S_{(\nu,\mu)}$ are decomposable $\mathscr{R}_{m+n}^\Lambda$-modules.*

Proof. Since S_μ is decomposable, it has a nontrivial idempotent endomorphism φ determined by

$$\varphi(z_\mu) = \sum_{T \in \text{Std}(\mu)} a_T v_T \quad \text{for some } a_T \in \mathbb{F}.$$

Via the embedding $\mathscr{R}_m \hookrightarrow \mathscr{R}_{m+n}$, it is easy to see that φ also defines an idempotent endomorphism of $S_{(\nu,\mu)}$. For the other Specht module,

$$\hat{\varphi}(z_{(\mu,\nu)}) = \sum_{\mathsf{T} \in \operatorname{Std}(\mu)} a_{\mathsf{T}} \operatorname{shift}_m(\psi_{\mathsf{T}}) z_{(\mu,\nu)}$$

is a nontrivial idempotent endomorphism of $S_{(\mu,\nu)}$, where $\operatorname{shift}_m : \mathscr{R}_m \hookrightarrow \mathscr{R}_{m+n}$ is the *shift map* [Fayers and Speyer 2016, §2.6.1]. $\qquad\square$

Remark. Theorem 5.3 readily extends to higher levels; i.e., we can use Theorems 5.1 and 5.2 to construct many decomposable Specht modules in higher levels, regardless of the chosen κ. Similarly, for any finite e, we may embed the bihooks of Theorem 4.1 into higher levels whenever we have a repeat in the e-multicharge.

The following result is the $e = 2$ extension of Theorem 4.1.

Theorem 5.4. *Let $\kappa = (0, 0)$ and $\lambda = ((2k+a, 1^b), (2j+a, 1^b)), ((b+1, 1^{2j+a-1}), (b+1, 1^{2k+a-1})), ((2k + a, 1^b), (a, 1^{2j+b})),$ or $((a, 1^{2k+b}), (2j + a, 1^b))$ for some $j, k \geqslant 1, 0 < a \leqslant 2,$ and $0 \leqslant b < 2$ with $a + b \neq 2,$ or for $a = b = 0$.*

(i) *For $j, k > 1$, if $j + k$ is even and $\operatorname{char} \mathbb{F} \neq 2$, or if $j + k$ is odd, then S_λ is decomposable.*

(ii) *If $j = 1$ or $k = 1$, then S_λ is decomposable if and only if $\operatorname{char} \mathbb{F} \nmid j + k$.*

Proof. We extend Theorem 4.1 and use the fact that $S_{((2k),(2j))} \cong S_{((2k),(1^{2j}))}$ when $e = 2$ and $\kappa_1 = \kappa_2$. (In general, their Specht presentations only differ in the idempotent relation, and if $e = 2$, then the residues in the second components coincide.) $\qquad\square$

Conjecture 5.5. *Let $e = 2$. If $\kappa_1 \neq \kappa_2$, then Theorem 5.3 provides a complete list of decomposable Specht modules indexed by bihooks. If $\kappa_1 = \kappa_2$, and $\operatorname{char} \mathbb{F} = 0$, then Theorems 3.8, 5.3, and 5.4 provide a complete list of decomposable Specht modules indexed by bihooks.*

Remark. For $\operatorname{char} \mathbb{F} \in \{0, 2, 3, 5\}$, we have checked in GAP that Conjecture 5.5 holds for all (i) $n \leqslant 14$ if $\kappa_1 \neq \kappa_2$ and (ii) $n \leqslant 13$ if $\kappa_1 = \kappa_2$. If $\kappa_1 = \kappa_2$ and $\operatorname{char} \mathbb{F} = 2$, we note that the decomposables appearing in the remark after Conjecture 4.2 are also decomposable here. However, we have also found further examples of decomposable Specht modules not accounted for. Namely the bihooks $((3, 1^2), (3)), ((7, 1^2), (3)), ((5, 1^4), (3)), ((7), (3, 1^2)),$ and $((5, 1^4), (5))$ if $\operatorname{char} \mathbb{F} = 2$, $((5, 1^2), (3)), ((5, 1^2), (5)),$ and $((5, 1^2), (7))$ if $\operatorname{char} \mathbb{F} = 3$, and $((9, 1^2), (3))$ if $\operatorname{char} \mathbb{F} = 5$, along with all bihooks obtained from these by conjugating, transposing, and our induction arguments, index decomposable Specht modules if $\kappa_1 = \kappa_2$. This list is exhaustive for $\operatorname{char} \mathbb{F} \in \{0, 2, 3, 5\}$ and $n \leqslant 13$.

6. Proof of Propositions 3.5 and 4.12

In this section, we complete the long calculations necessary in proving Propositions 3.5 and 4.12. We begin by setting some notation which will hopefully help the reader follow the calculations.

For any reduced expression $w = s_{r_1} \ldots s_{r_m} \in \mathфrak{S}_n$, observe that

$$\begin{aligned}
\psi_r \psi_{r\pm1} \psi_r \psi_w z_\lambda &= \psi_r \psi_{r\pm1} \psi_r \psi_w e(\boldsymbol{i}_\lambda) z_\lambda \\
&= \psi_r \psi_{r\pm1} \psi_r e(s_{r_1} \ldots s_{r_m} \cdot \boldsymbol{i}_\lambda) \psi_w z_\lambda \\
&= \psi_r \psi_{r\pm1} \psi_r e(i_1, \ldots, i_r, i_{r+1}, i_{r+2}, \ldots, i_n) \psi_w z_\lambda.
\end{aligned}$$

Since the KLR "braid relations" only depend on the residues i_r, i_{r+1}, and i_{r+2} of the idempotent $e(s_{r_1} \ldots s_{r_m} \cdot \boldsymbol{i}_\lambda)$, we will instead write the above expression as

$$(\psi_r \psi_{r\pm1} \psi_r (i_r, i_{r+1}, i_{r+2})) \psi_w z_\lambda.$$

Similarly, since $y_{r+1} \psi_r e(\boldsymbol{i}) \psi_w z_\lambda$ only depends on the residues i_r and i_{r+1}, we will write this expression as $(y_{r+1} \psi_r (i_r, i_{r+1})) \psi_w z_\lambda$.

In fact, whenever we apply the KLR relations in our computations, we will analogously abbreviate idempotents to the two or three necessary, consecutive residues, to help the reader identify which relations are being applied, and which case of the relation is applicable.

For both propositions, we will break the calculation apart with several preliminary lemmas. First, we will focus on Proposition 3.5.

Proof of Proposition 3.5.

Lemma 6.1. *Suppose that $e > 3$.*

(i) *Let $1 \leqslant r \leqslant a - 2$ and $1 \leqslant s \leqslant e - r - 1$. Then*

$$\psi_{2e-2r-s} \psi \underset{e-r}{\overset{2e-2r-s-1}{\downarrow}} \underset{e-r+1}{\overset{2e-2r-s}{\downarrow}} \ldots \underset{e}{\overset{2e-r-s-1}{\downarrow}} z_\lambda = 0.$$

(ii) *Let $1 \leqslant r \leqslant a - 1$. Then*

$$\psi \underset{2e-2r+1}{\overset{2e-r-1}{\downarrow}} \psi \underset{e-r+2}{\overset{2e-2r+1}{\downarrow}} \underset{e-r+3}{\overset{2e-2r+2}{\downarrow}} \ldots \underset{e}{\overset{2e-r-1}{\downarrow}} z_\lambda = \psi \underset{e-r+2}{\overset{2e-2r}{\downarrow}} \underset{e-r+3}{\overset{2e-2r+1}{\downarrow}} \ldots \underset{e}{\overset{2e-r-2}{\downarrow}} z_\lambda.$$

(iii) *Let $1 \leqslant r \leqslant a - 1$. Then*

$$\psi \underset{b+2}{\overset{e+b+1}{\downarrow}} \underset{b+3}{\overset{e+b+2}{\downarrow}} \ldots \underset{e-r}{\overset{2e-r-1}{\downarrow}} \psi \underset{e-r+1}{\overset{2e-2r}{\downarrow}} \underset{e-r+2}{\overset{2e-2r+1}{\downarrow}} \ldots \underset{e}{\overset{2e-r-1}{\downarrow}} z_\lambda = \psi \underset{b+2}{\overset{2b+2}{\downarrow}} \underset{b+3}{\overset{2b+3}{\downarrow}} \ldots \underset{e}{\overset{e+b}{\downarrow}} z_\lambda.$$

Proof. (i) We argue by reverse induction on s. For the base case, we let $s = e-r-1$. Then

$$\psi_{2e-2r-s}\,\psi\overset{2e-2r-s-1}{\underset{e-r}{\downarrow}}\overset{2e-2r-s}{\underset{e-r+1}{\downarrow}}\cdots\overset{2e-r-s-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= (\psi_{e-r+1}\psi_{e-r}\psi_{e-r+1}(a-r-1, a-r, 0))\psi_{e-r+2}\psi_{e-r+3}\ldots\psi_e z_\lambda$$

$$= \psi_{e-r}\psi_{e-r+1}\ldots\psi_e\psi_{e-r}z_\lambda$$

$$= 0.$$

Now suppose that $s < e-r-1$. Then

$$\psi_{2e-2r-s}\,\psi\overset{2e-2r-s-1}{\underset{e-r}{\downarrow}}\overset{2e-2r-s}{\underset{e-r+1}{\downarrow}}\cdots\overset{2e-r-s-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= (\psi_{2e-2r-s}\psi_{2e-2r-s-1}\psi_{2e-2r-s}(a-r-1, a-r, a-r-s-1))\psi\overset{2e-2r-s-2}{\underset{e-r}{\downarrow}}\overset{2e-2r-s-1}{\underset{e-r+1}{\downarrow}}$$
$$\cdot\,\psi\overset{2e-2r-s+1}{\underset{e-r+2}{\downarrow}}\overset{2e-2r-s+2}{\underset{e-r+3}{\downarrow}}\cdots\overset{2e-r-s-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= \psi_{2e-2r-s-1}\psi_{2e-2r-s}\cdots\psi_{2e-r-s-1}\psi_{2e-2r-s-1}\psi\overset{2e-2r-s-2}{\underset{e-r}{\downarrow}}\overset{2e-2r-s-1}{\underset{e-r+1}{\downarrow}}\cdots\overset{2e-r-s-2}{\underset{e}{\downarrow}} z_\lambda$$

$$= 0 \quad\text{by the induction hypothesis.}$$

(ii) We argue by induction on r. For the base case, when $r = 1$, both sides of the inequality are equal to z_λ by definition. Thus, we may assume that $r > 1$. Then

$$\psi\overset{2e-r-1}{\underset{2e-2r+1}{\downarrow}}\psi\overset{2e-2r+1}{\underset{e-r+2}{\downarrow}}\overset{2e-2r+2}{\underset{e-r+3}{\downarrow}}\cdots\overset{2e-r-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= \psi\overset{2e-r-1}{\underset{2e-2r+2}{\downarrow}}(\psi^2_{2e-2r+1}(a-r+1, a-r-1))\psi\overset{2e-2r}{\underset{e-r+2}{\downarrow}}\psi\overset{2e-2r+2}{\underset{e-r+3}{\downarrow}}\overset{2e-2r+3}{\underset{e-r+4}{\downarrow}}\cdots\overset{2e-r-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= \psi\overset{2e-2r}{\underset{e-r+2}{\downarrow}}\psi\overset{2e-r-1}{\underset{2e-2r+2}{\downarrow}}\psi\overset{2e-2r+2}{\underset{e-r+3}{\downarrow}}\overset{2e-2r+3}{\underset{e-r+4}{\downarrow}}\cdots\overset{2e-r-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= \psi\overset{2e-2r}{\underset{e-r+2}{\downarrow}}\overset{2e-2r+1}{\underset{e-r+3}{\downarrow}}\cdots\overset{2e-r-2}{\underset{e}{\downarrow}} z_\lambda \quad\text{by the induction hypothesis.}$$

(iii) We argue by reverse induction on r. For the base case, when $r = a-1$, both sides of the inequality are equal by definition, so there is nothing to prove. So we assume that $r < a-1$. Then

$$\psi\overset{e+b+1}{\underset{b+2}{\downarrow}}\overset{e+b+2}{\underset{b+3}{\downarrow}}\cdots\overset{2e-r-1}{\underset{e-r}{\downarrow}}\psi\overset{2e-2r}{\underset{e-r+1}{\downarrow}}\overset{2e-2r+1}{\underset{e-r+2}{\downarrow}}\cdots\overset{2e-r-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= \psi\overset{e+b+1}{\underset{b+2}{\downarrow}}\overset{e+b+2}{\underset{b+3}{\downarrow}}\cdots\overset{2e-r-2}{\underset{e-r-1}{\downarrow}}\psi\overset{2e-r-1}{\underset{2e-2r+1}{\downarrow}}(\psi_{2e-2r}\psi_{2e-2r-1}\psi_{2e-2r}(a-r-1, a-r, a-r-1))$$
$$\cdot\,\psi\overset{2e-2r-2}{\underset{e-r}{\downarrow}}\overset{2e-2r-1}{\underset{e-r+1}{\downarrow}}\psi\overset{2e-2r+1}{\underset{e-r+2}{\downarrow}}\overset{2e-2r+2}{\underset{e-r+3}{\downarrow}}\cdots\overset{2e-r-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= \psi\overset{e+b+1}{\underset{b+2}{\downarrow}}\overset{e+b+2}{\underset{b+3}{\downarrow}}\cdots\overset{2e-r-2}{\underset{e-r-1}{\downarrow}}\psi\overset{2e-r-1}{\underset{2e-2r+1}{\downarrow}}\psi\overset{2e-r-1}{\underset{2e-2r-1}{\uparrow}}\psi_{2e-2r-1}\psi\overset{2e-2r-2}{\underset{e-r}{\downarrow}}\overset{2e-2r-1}{\underset{e-r+1}{\downarrow}}\cdots\overset{2e-r-2}{\underset{e}{\downarrow}} z_\lambda$$
$$+\, \psi\overset{e+b+1}{\underset{b+2}{\downarrow}}\overset{e+b+2}{\underset{b+3}{\downarrow}}\cdots\overset{2e-r-2}{\underset{e-r-1}{\downarrow}}\psi\overset{2e-r-1}{\underset{e-r}{\downarrow}}(\psi\overset{2e-2r-1}{\underset{2e-2r+1}{\downarrow}}\psi\overset{2e-2r+1}{\underset{e-r+2}{\downarrow}}\overset{2e-2r+2}{\underset{e-r+3}{\downarrow}}\cdots\overset{2e-r-1}{\underset{e}{\downarrow}} z_\lambda)$$

$$= 0 + \psi\overset{e+b+1}{\underset{b+2}{\downarrow}}\overset{e+b+2}{\underset{b+3}{\downarrow}}\cdots\overset{2e-r-2}{\underset{e-r-1}{\downarrow}}\psi\overset{2e-r-1}{\underset{e-r}{\downarrow}}\psi\overset{2e-2r}{\underset{e-r+1}{\downarrow}}\overset{2e-2r+1}{\underset{e-r+2}{\downarrow}}\cdots\overset{2e-r-1}{\underset{e}{\downarrow}} z_\lambda \quad\text{by parts (i) and (ii)}$$

$$= \psi\overset{2b+2}{\underset{b+2}{\downarrow}}\overset{2b+3}{\underset{b+3}{\downarrow}}\cdots\overset{e+b}{\underset{e}{\downarrow}} z_\lambda \quad\text{by the induction hypothesis.} \qquad\square$$

Lemma 6.2. *Suppose that $e > 3$.*

(i) *Let $1 \leqslant r \leqslant b$, $0 \leqslant s \leqslant a - 3$, and $0 \leqslant k \leqslant b$. Then*

$$
\psi_{\substack{\downarrow \\ 2b-r+s+3-k}}^{2b-r+s+2} \psi_{\substack{\downarrow \\ b-r+s+2}}^{e+b-r} \psi_{\substack{\downarrow \\ b-r+s+3}}^{2b-r+s+3-k} \psi_{\substack{\downarrow \\ b-r+s+4}}^{2b-r+s+4} {}_{\substack{\downarrow \\ b-r+s+5}}^{2b-r+s+5} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+2}}^{b-r+s+1} \psi_{\substack{\downarrow \\ b-r+3}}^{e-r} {}_{\substack{\downarrow \\ b-r+4}}^{e-r+1} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda
$$
$$
= \psi_{\substack{\downarrow \\ 2b-r+s+2-k}}^{2b-r+s+2} \psi_{\substack{\downarrow \\ b-r+s+2}}^{e+b-r} \psi_{\substack{\downarrow \\ b-r+s+3}}^{2b-r+s+2-k} \psi_{\substack{\downarrow \\ b-r+s+4}}^{2b-r+s+4} {}_{\substack{\downarrow \\ b-r+s+5}}^{2b-r+s+5} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+2}}^{b-r+s+1} \psi_{\substack{\downarrow \\ b-r+3}}^{e-r} {}_{\substack{\downarrow \\ b-r+4}}^{e-r+1} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda .
$$

(ii) *Let $1 \leqslant r \leqslant b$ and $0 \leqslant s \leqslant a - 3$. Then*

$$
\psi_{\substack{\downarrow \\ b-r+2}}^{2b-r+2} {}_{\substack{\downarrow \\ b-r+3}}^{2b-r+3} \cdots \downarrow {}_{\substack{\downarrow \\ b-r+s+1}}^{2b-r+s+1} \psi_{\substack{\downarrow \\ b-r+s+2}}^{e+b-r} \psi_{\substack{\downarrow \\ b-r+s+3}}^{2b-r+s+3} {}_{\substack{\downarrow \\ b-r+s+4}}^{2b-r+s+4} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+2}}^{b-r+s+1} \psi_{\substack{\downarrow \\ b-r+3}}^{e-r} {}_{\substack{\downarrow \\ b-r+4}}^{e-r+1} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda
$$
$$
= \psi_{\substack{\downarrow \\ b-r+2}}^{2b-r+2} {}_{\substack{\downarrow \\ b-r+3}}^{2b-r+3} \cdots \downarrow {}_{\substack{\downarrow \\ b-r+s+2}}^{2b-r+s+2} \psi_{\substack{\downarrow \\ b-r+s+3}}^{e+b-r} \psi_{\substack{\downarrow \\ b-r+s+4}}^{2b-r+s+4} {}_{\substack{\downarrow \\ b-r+s+5}}^{2b-r+s+5} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+2}}^{b-r+s+2} \psi_{\substack{\downarrow \\ b-r+3}}^{e-r} {}_{\substack{\downarrow \\ b-r+4}}^{e-r+1} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda .
$$

(iii) *Let $1 \leqslant r \leqslant b$. Then*

$$
\psi_{\substack{\downarrow \\ b-r+2}}^{e+b-r} \psi_{\substack{\downarrow \\ b-r+3}}^{2b-r+3} {}_{\substack{\downarrow \\ b-r+4}}^{2b-r+4} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+3}}^{e-r} {}_{\substack{\downarrow \\ b-r+4}}^{e-r+1} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda = \psi_{\substack{\downarrow \\ b-r+2}}^{2b-r+2} {}_{\substack{\downarrow \\ b-r+3}}^{2b-r+3} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+2}}^{e-r-1} {}_{\substack{\downarrow \\ b-r+3}}^{e-r} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda .
$$

(iv) *The equation in part (i) also holds if $r = b + 1$, $s = 0$, and $0 \leqslant k \leqslant b - 1$. That is,*

$$
\psi_{\substack{\downarrow \\ b+2-k}}^{b+1} \psi_{\substack{\downarrow \\ 1}}^{e-1} \psi_{\substack{\downarrow \\ 2}}^{b+2-k} \psi_{\substack{\downarrow \\ 3}}^{b+3} {}_{\substack{\downarrow \\ 4}}^{b+4} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ 2}}^{a-1} {}_{\substack{\downarrow \\ 3}}^{a} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda
$$
$$
= \psi_{\substack{\downarrow \\ b+1-k}}^{b+1} \psi_{\substack{\downarrow \\ 1}}^{e-1} \psi_{\substack{\downarrow \\ 2}}^{b+1-k} \psi_{\substack{\downarrow \\ 3}}^{b+3} {}_{\substack{\downarrow \\ 4}}^{b+4} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ 2}}^{a-1} {}_{\substack{\downarrow \\ 3}}^{a} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda .
$$

(v) *We have*

$$
\psi_{\substack{\downarrow \\ 1}}^{e-1} {}_{\substack{\downarrow \\ 2}}^{e} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e+b-1} \psi_{\substack{\downarrow \\ b+2}}^{2b+2} {}_{\substack{\downarrow \\ b+3}}^{2b+3} \cdots \downarrow {}_{\substack{\downarrow \\ e}}^{e+b} z_\lambda = \psi_{\substack{\downarrow \\ 2}}^{b+1} \psi_{\substack{\downarrow \\ 1}}^{e-1} \psi_2 \psi_{\substack{\downarrow \\ 3}}^{b+3} {}_{\substack{\downarrow \\ 4}}^{b+4} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ 2}}^{a-1} {}_{\substack{\downarrow \\ 3}}^{a} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda .
$$

Proof. (i) We have

$$
\psi_{\substack{\downarrow \\ 2b-r+s+3-k}}^{2b-r+s+2} \psi_{\substack{\downarrow \\ b-r+s+2}}^{e+b-r} \psi_{\substack{\downarrow \\ b-r+s+3}}^{2b-r+s+3-k} \psi_{\substack{\downarrow \\ b-r+s+4}}^{2b-r+s+4} {}_{\substack{\downarrow \\ b-r+s+5}}^{2b-r+s+5} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+2}}^{b-r+s+1} \psi_{\substack{\downarrow \\ b-r+3}}^{e-r} {}_{\substack{\downarrow \\ b-r+4}}^{e-r+1} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda
$$
$$
= \psi_{\substack{\downarrow \\ 2b-r+s+3-k}}^{2b-r+s+2} \psi_{\substack{\downarrow \\ 2b-r+s+4-k}}^{e+b-r} (\psi_{2b-r+s+3-k} \psi_{2b-r+s+2-k} \psi_{2b-r+s+3-k} e(\boldsymbol{j}))
$$
$$
\cdot \psi_{\substack{\downarrow \\ b-r+s+2}}^{2b-r+s-1-k} \psi_{\substack{\downarrow \\ b-r+s+3}}^{2b-r+s+2-k} \psi_{\substack{\downarrow \\ b-r+s+4}}^{2b-r+s+4} {}_{\substack{\downarrow \\ b-r+s+5}}^{2b-r+s+5} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+2}}^{b-r+s+1} \psi_{\substack{\downarrow \\ b-r+3}}^{e-r} {}_{\substack{\downarrow \\ b-r+4}}^{e-r+1} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda
$$

where positions $2b - r + s + 2 - k$, $2b - r + s + 3 - k$, and $2b - r + s + 4 - k$ of $\boldsymbol{j}$ are positions $b - r + 2$, $b + s + 2$, and $e + b + 1 - k$ of $\boldsymbol{i}_\lambda$, respectively; i.e., the corresponding residues are $e - b + r - 1$, $s + 1$, and $e - b + k$, respectively. Since $1 \leqslant s + 1 \leqslant a - 2$ and $a \leqslant e - b + r - 1 \leqslant e - 1$, it is clear that the corresponding braid relation never produces an error term, and the result follows:

$$
= \psi_{\substack{\downarrow \\ 2b-r+s+2-k}}^{2b-r+s+2} \psi_{\substack{\downarrow \\ b-r+s+2}}^{e+b-r} \psi_{\substack{\downarrow \\ b-r+s+3}}^{2b-r+s+2-k} \psi_{\substack{\downarrow \\ b-r+s+4}}^{2b-r+s+4} {}_{\substack{\downarrow \\ b-r+s+5}}^{2b-r+s+5} \cdots \downarrow \psi_{\substack{\downarrow \\ e}}^{e+b} \psi_{\substack{\downarrow \\ b-r+2}}^{b-r+s+1} \psi_{\substack{\downarrow \\ b-r+3}}^{e-r} {}_{\substack{\downarrow \\ b-r+4}}^{e-r+1} \cdots \downarrow {}_{\substack{\downarrow \\ b+1}}^{e-2} z_\lambda .
$$

(ii) Apply part (i) for $k = 0$, then $k = 1$, and so on up to and including the case $k = b$.

(iii) Apply part (ii) for $s = 0$, then $s = 1$, and so on up to and including the case $s = a - 3$.

(iv) The proof is identical to that of part (i), except now we notice that the third residue of the relevant triple is $e - b \leqslant e - b + k \leqslant e - 1$, while the second is 1.

(v) Apply part (iii) for $r = 1$, then $r = 2$, and so on up to and including the case $r = b$, to yield

$$\psi^{e-1}_{1} \psi^{e\ e+1}_{2\ 3} \cdots {}^{e+b-1}_{b+1} \psi^{2b+2\ 2b+3}_{b+2\ b+3} \cdots {}^{e+b}_{e} z_\lambda = \psi^{e-1}_{1} \psi^{b+2\ b+3}_{2\ 3} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda.$$

Then, applying part (iv) for $k = 0$, then $k = 1$, and so on up to and including the case $k = b - 1$ gives

$$\psi^{e-1}_{1} \psi^{b+2\ b+3}_{2\ 3} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda = \psi^{b+1}_{2} \psi^{e-1}_{1} \psi_2 \psi^{b+3\ b+4}_{3\ 4} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda. \qquad \square$$

Lemma 6.3. *Suppose that $e > 3$.*

(i) *For $3 \leqslant x \leqslant a - 1$, we have*

$$\psi^{e-1}_{b+x} \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{b+x-1}_{x} \psi^{b+x\ b+x+1}_{x\ x+1} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda$$

$$= \psi^{e-1}_{b+x+1} \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{b+x}_{x+1} \psi^{b+x+1\ b+x+2}_{x+1\ x+2} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda.$$

(ii) *We have*

$$\psi^{b+1}_{2} \psi^{e-1}_{3} \psi^{b+3\ b+4}_{3\ 4} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda = \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{e-1}_{a} \psi^{e\ e+1}_{a\ a+1} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda.$$

Proof. (i) We have

$$\psi^{e-1}_{b+x} \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{b+x-1}_{x} \psi^{b+x\ b+x+1}_{x\ x+1} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda$$

$$= \psi^{e-1}_{b+x+1} \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{b+x-2}_{x-1} (\psi_{b+x} \psi_{b+x-1} \psi_{b+x}(0, x-1, -b))$$

$$\cdot \psi^{b+x-1}_{x} \psi^{b+x+1\ b+x+2}_{x+1\ x+2} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda$$

$$= \psi^{e-1}_{b+x+1} \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{b+x-2}_{x-1} \psi_{b+x-1} \psi_{b+x} (\psi_{b+x-1} \psi_{b+x-2} \psi_{b+x-1}(0, x-1, -b+1))$$

$$\cdot \psi^{b+x-3}_{x} \psi^{b+x-2}_{x} \psi^{b+x+1\ b+x+2}_{x+1\ x+2} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda$$

$$\vdots$$

$$= \psi^{e-1}_{b+x+1} \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{b+x-2}_{x-1} \psi^{b+x-1}_{x+1} \psi^{b+x}_{x+2} (\psi_{x+1} \psi_x \psi_{x+1}(0, x-1, -1))$$

$$\cdot \psi_x \psi^{b+x+1\ b+x+2}_{x+1\ x+2} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda$$

$$= \psi^{e-1}_{b+x+2} \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{b+x}_{x+1} (\psi_x^2(x-1, 0)) \psi^{b+x+1\ b+x+2}_{x+1\ x+2} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda$$

$$= \psi^{e-1}_{b+x+1} \psi^{b+1\ b+2}_{2\ 3} \cdots {}^{b+x}_{x+1} \psi^{b+x+1\ b+x+2}_{x+1\ x+2} \cdots {}^{e+b}_{e} \psi^{a-1\ a}_{2\ 3} \cdots {}^{e-2}_{b+1} z_\lambda.$$

(ii) By applying part (i) repeatedly, we have

$$\psi_{\substack{b+1\\2}}\psi_{\substack{e-1\\3}}\psi_{\substack{b+3\,b+4\\3\ \ 4}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda = \psi_{\substack{e-1\\b+3}}\psi_{\substack{b+1\\2}}\psi_{\substack{b+2\\3}}\psi_{\substack{b+3\,b+4\\3\ \ 4}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda$$

$$= \psi_{\substack{e-1\\b+4}}\psi_{\substack{b+1\,b+2\,b+3\\2\ \ 3\ \ 4}}\psi_{\substack{b+4\\4}}\psi_{\substack{b+5\\5}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda$$

$$\vdots$$

$$= \psi_{\substack{b+1\,b+2\\2\ \ 3}}\dots\psi_{\substack{e-1\\a}}\psi_{\substack{e\,e+1\\a\,a+1}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda. \qquad \square$$

Lemma 6.4. *Suppose that $e > 3$. For $0 \leqslant x \leqslant b-1$ and $1 \leqslant y \leqslant b-x+1$, we have*

$$\psi_{a+2x+y}\psi_{\substack{e+x+y\ e+x+y+1\\a+x+y\ \ a+x+y+1}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda = 0.$$

Proof. We fix x, and prove the statement by reverse induction on y. If $y = b-x+1$, the expression is

$$\psi_{e+x+1}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda = 0.$$

If $y < b-x+1$, then

$$\psi_{a+2x+y}\psi_{\substack{e+x+y\ e+x+y+1\\a+x+y\ \ a+x+y+1}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda$$

$$= \psi_{\substack{e+x+y\\a+2x+y+2}}(\psi_{a+2x+y}\psi_{a+2x+y+1}\psi_{a+2x+y}(\alpha, -x, -x-1))$$

$$\cdot\,\psi_{\substack{a+2x+y-1\ e+x+y+1\\a+x+y\ \ \ \ a+x+y+1}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda$$

$$= \psi_{\substack{e+x+y\\a+x+y}}\psi_{a+2x+y+1}\psi_{\substack{e+x+y+1\ e+x+y+2\\a+x+y+1\ \ a+x+y+2}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda$$

$$= 0 \quad \text{by the induction hypothesis,}$$

where $\alpha = -y-x-1$ if $y \leqslant b-x-1$ and $\alpha = a-1$ if $y = b-x$. $\qquad \square$

Lemma 6.5. *Suppose that $e > 3$. For $0 \leqslant x \leqslant b-1 < e-2$ or $0 \leqslant x \leqslant b-2 = e-3$, we have*

(i) $$(\psi_{\substack{a+2x-1\\a+x}}\psi_{\substack{a+2x-1\\a+x}})\psi_{\substack{e+x+1\ e+x+2\\a+x+1\ a+x+2}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda$$

$$= \psi_{\substack{e+x+1\ e+x+2\\a+x+1\ a+x+2}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda;$$

(ii) $$\psi_{\substack{b+1\,b+2\\2\ \ 3}}\dots\psi_{\substack{e-2\\a-1}}\psi_{\substack{e-1\\a+x\ a+x+1}}\dots\psi_{\substack{e\\a+2x}}\psi_{\substack{e+x-1\\a+x\ a+x+1}}\psi_{\substack{e+x\ e+x+1\\a+x\ a+x+1}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda$$

$$= -\psi_{\substack{b+1\,b+2\\2\ \ 3}}\dots\psi_{\substack{e-2\\a-1}}\psi_{\substack{e-1\\a+(x+1)\ a+(x+1)+1}}\dots\psi_{\substack{e\\a+2(x+1)}}\psi_{\substack{e+(x+1)-1\\a+(x+1)\ a+(x+1)+1}}\psi_{\substack{e+(x+1)\ e+(x+1)+1\\a+(x+1)\ \ a+(x+1)+1}}\dots\psi_{\substack{e+b\\e}}\psi_{\substack{a-1\,a\\2\ \ 3}}\dots\psi_{\substack{e-2\\b+1}}z_\lambda.$$

Proof. (i) Firstly, it is obvious that the statement holds when $x = 0$, since the term $\psi^{a-1}_{\uparrow a} \psi^{a-1}_{\downarrow a}$ is trivial. Now assuming that $x > 0$, we have

$$
\psi^{a+2x-1}_{\uparrow a+x} \psi^{a+2x-1}_{\downarrow a+x} \psi^{e+x+1\ e+x+2}_{\downarrow a+x+1\ a+x+2} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
= \psi^{a+2x-2}_{\uparrow a+x} (\psi^2_{a+2x-1}(-x-1,-x+1)) \psi^{a+2x-2}_{\downarrow a+x} \psi^{e+x+1\ e+x+2}_{\downarrow a+x+1\ a+x+2} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
= \psi^{a+2x-3}_{\uparrow a+x} (\psi^2_{a+2x-2}(-x-1,-x+2)) \psi^{a+2x-3}_{\downarrow a+x} \psi^{e+x+1\ e+x+2}_{\downarrow a+x+1\ a+x+2} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
\vdots
$$

$$
= (\psi^2_{a+x}(-x-1,0)) \psi^{e+x+1\ e+x+2}_{\downarrow a+x+1\ a+x+2} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
= \psi^{e+x+1\ e+x+2}_{\downarrow a+x+1\ a+x+2} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda.
$$

(ii) Applying Lemma 6.4, we have

$$
\psi^{b+1\ b+2}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow a-1} \psi^{e-1}_{\downarrow a+x}{}^{e}_{a+x+1} \cdots^{e+x-1}_{\downarrow a+2x} \psi^{e+x\ e+x+1}_{\downarrow a+x\ a+x+1} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
= \psi^{b+1\ b+2}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow a-1} \psi^{e-1}_{\downarrow a+x}{}^{e}_{a+x+1} \cdots^{e+x-2}_{\downarrow a+2x-1} \psi^{e+x-1}_{\downarrow a+2x+1} \psi_{a+2x} \psi^{e+x\ e+x+1}_{\downarrow a+x\ a+x+1} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
= \psi^{b+1\ b+2}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow a-1} \psi^{e-1}_{\downarrow a+x}{}^{e}_{a+x+1} \cdots^{e+x-2}_{\downarrow a+2x-1} \psi^{e+x-1}_{\downarrow a+2x+1}{}^{e+x}_{a+2x+2}
$$

$$
\cdot (\psi_{a+2x}\psi_{a+2x+1}\psi_{a+2x}(-x-1,-x,-x-1))
$$

$$
\cdot \psi^{a+2x-1}_{\downarrow a+x} \psi^{e+x+1\ e+x+2}_{\downarrow a+x+1\ a+x+2} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
= 0 - \psi^{b+1\ b+2}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow a-1} \psi^{e-1}_{\downarrow a+x}{}^{e}_{a+x+1} \cdots^{e+x-2}_{\downarrow a+2x-1}
$$

$$
\cdot \psi^{e+x-1}_{\downarrow a+2x+1}{}^{e+x}_{a+2x+2} \psi^{a+2x-1}_{\downarrow a+x} \psi^{e+x+1\ e+x+2}_{\downarrow a+x+1\ a+x+2} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
= -\psi^{b+1\ b+2}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow a-1} \psi^{e-1}_{\downarrow a+x+1}{}^{e}_{a+x+2} \cdots^{e+x}_{\downarrow a+2x+2} (\psi^{a+2x-1}_{\uparrow a+x} \psi^{a+2x-1}_{\downarrow a+x})
$$

$$
\cdot \psi^{e+x+1\ e+x+2}_{\downarrow a+x+1\ a+x+2} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda
$$

$$
= -\psi^{b+1\ b+2}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow a-1} \psi^{e-1}_{\downarrow a+(x+1)}{}^{e}_{a+(x+1)+1} \cdots^{e+(x+1)-1}_{\downarrow a+2(x+1)} \psi^{e+(x+1)\ e+(x+1)+1}_{\downarrow a+(x+1)\ a+(x+1)+1} \cdots^{e+b}_{\downarrow e} \psi^{a-1\ a}_{\downarrow 2\ 3} \cdots^{e-2}_{\downarrow b+1} z_\lambda,
$$

by part (i). $\qquad\square$

Lemma 6.6. *Suppose that $e > 3$. For $2 \leqslant x \leqslant b+1$, we have*

$$
\psi^{a+x-3}_{\uparrow x} \psi^{a+x-3\ a+x-2}_{\downarrow x\ x+1} \cdots^{e-2}_{\downarrow b+1} z_\lambda = \psi^{a+x-2\ a+x-1}_{\downarrow x+1\ x+2} \cdots^{e-2}_{\downarrow b+1} z_\lambda.
$$

Proof. Firstly, if $a < 3$, then the left-hand side and the right-hand side are both equal to the generator z_λ. If $a \geqslant 3$, we have

$$\psi \overset{a+x-3}{\underset{x}{\uparrow}} \psi \overset{a+x-3}{\underset{x}{\downarrow}} \overset{a+x-2}{\underset{x+1}{\downarrow}} \cdots \overset{e-2}{\underset{b+1}{\downarrow}} z_\lambda$$

$$= \psi \overset{a+x-4}{\underset{x}{\uparrow}} (\psi^2_{a+x-3}(-x+1, a-2)) \psi \overset{a+x-4}{\underset{x}{\downarrow}} \psi \overset{a+x-2}{\underset{x+1}{\downarrow}} \overset{a+x-1}{\underset{x+2}{\downarrow}} \cdots \overset{e-2}{\underset{b+1}{\downarrow}} z_\lambda$$

$$= \psi \overset{a+x-5}{\underset{x}{\uparrow}} (\psi^2_{a+x-4}(-x+1, a-3)) \psi \overset{a+x-5}{\underset{x}{\downarrow}} \psi \overset{a+x-2}{\underset{x+1}{\downarrow}} \overset{a+x-1}{\underset{x+2}{\downarrow}} \cdots \overset{e-2}{\underset{b+1}{\downarrow}} z_\lambda$$

$$\vdots$$

$$= (\psi^2_x(-x+1, 1)) \psi \overset{a+x-2}{\underset{x+1}{\downarrow}} \overset{a+x-1}{\underset{x+2}{\downarrow}} \cdots \overset{e-2}{\underset{b+1}{\downarrow}} z_\lambda$$

$$= \psi \overset{a+x-2}{\underset{x+1}{\downarrow}} \overset{a+x-1}{\underset{x+2}{\downarrow}} \cdots \overset{e-2}{\underset{b+1}{\downarrow}} z_\lambda. \qquad \square$$

Repeated application of Lemma 6.6 yields the following corollary.

Corollary 6.7. *Suppose that $e > 3$. We have*

$$\psi \overset{e-2}{\underset{b+1}{\uparrow}} \overset{e-3}{\underset{b}{\uparrow}} \cdots \overset{a-1}{\underset{2}{\uparrow}} \psi \overset{a-1}{\underset{2}{\downarrow}} \overset{a}{\underset{3}{\downarrow}} \cdots \overset{e-2}{\underset{b+1}{\downarrow}} z_\lambda = z_\lambda.$$

We are now ready to prove Proposition 3.5, using the above lemmas.

Proof of Proposition 3.5. In order to prove that $\varphi(v_T) = (-1)^{b+1} 2 v_T$, it suffices to prove that

$$\psi \overset{2e-1}{\underset{e}{\downarrow}} \cdot \psi \overset{e}{\underset{1}{\downarrow}} \overset{e+1}{\underset{2}{\downarrow}} \cdots \overset{2e-1}{\underset{e}{\downarrow}} z_\lambda = (-1)^{b+1} 2 \psi \overset{2e-1}{\underset{e}{\downarrow}} z_\lambda \quad \text{or} \quad (-1)^{e-1} 2 \psi \overset{2e-1}{\underset{e}{\downarrow}} z_\lambda.$$

We first suppose that $e = 3$ and $\lambda = ((2, 1), (2, 1))$ or $((1^3), (1^3))$. Then we have

$$\psi \overset{5}{\underset{3}{\downarrow}} \cdot \psi \overset{3}{\underset{1}{\downarrow}} \psi \overset{4}{\underset{2}{\downarrow}} \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda = \psi \overset{5}{\underset{4}{\downarrow}} (\psi^2_3(0, 1)) \psi \overset{2}{\underset{1}{\downarrow}} \psi \overset{4}{\underset{2}{\downarrow}} \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda = \psi \overset{5}{\underset{4}{\downarrow}} (y_3 - y_4) \psi \overset{2}{\underset{1}{\downarrow}} \psi \overset{4}{\underset{2}{\downarrow}} \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda.$$

The first term becomes

$$\psi \overset{5}{\underset{4}{\downarrow}} (y_3 \psi_2(0, 2)) \psi_1 \psi \overset{4}{\underset{2}{\downarrow}} \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda = \psi \overset{5}{\underset{4}{\downarrow}} \psi_2(y_2 \psi_1(0, 0)) \psi \overset{4}{\underset{2}{\downarrow}} \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda$$

$$= \psi \overset{5}{\underset{4}{\downarrow}} \psi_2(\psi_1 y_1 + 1) \psi \overset{4}{\underset{2}{\downarrow}} \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda$$

$$= \psi \overset{5}{\underset{4}{\downarrow}} \psi_4(\psi_2 \psi_3 \psi_2(2, 0, 2)) \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda$$

$$= \psi \overset{5}{\underset{4}{\downarrow}} \psi_4(\psi_3 \psi_2 \psi_3 - 1) \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda$$

$$= \psi \overset{5}{\underset{4}{\downarrow}} \psi \overset{4}{\underset{2}{\downarrow}} \psi_5(\psi_3 \psi_4 \psi_3(1, 0, 2)) z_\lambda - \psi_5(\psi^2_4(2, 1)) \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda$$

$$= \psi \overset{5}{\underset{4}{\downarrow}} \psi \overset{4}{\underset{2}{\downarrow}} \psi_5(\psi_4 \psi_3 \psi_4) z_\lambda - \psi_5(y_5 - y_4) \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda$$

$$= 0 - \psi_5(y_5 \psi_5(1, 1)) \psi \overset{4}{\underset{3}{\downarrow}} z_\lambda + \psi^2_5(y_4 \psi_4(1, 2)) \psi_3 z_\lambda$$

$$= -\psi_5(\psi_5 y_6 - 1) \psi \overset{4}{\underset{3}{\downarrow}} z_\lambda + \psi^2_5 \psi \overset{4}{\underset{3}{\downarrow}} y_5 z_\lambda$$

$$= 0 + \psi \overset{5}{\underset{3}{\downarrow}} z_\lambda + 0.$$

The second term becomes

$$
-\psi^{5}_{4}\psi^{2}_{1}(y_4\psi_4(2,1))\psi^{3}_{2}\psi^{5}_{3}z_\lambda = -\psi^{5}_{4}\psi^{2}_{1}\psi^{4}_{2}(y_5\psi_5(1,1))\psi^{4}_{3}z_\lambda
$$

$$
= -\psi^{5}_{4}\psi^{2}_{1}\psi^{4}_{2}(\psi_5 y_6 - 1)\psi^{4}_{3}z_\lambda
$$

$$
= -0 + \psi^{5}_{4}\psi^{2}_{1}(\psi_4\psi_3\psi_4(2,1,2))\psi_2\psi_3 z_\lambda
$$

$$
= \psi^{5}_{4}\psi^{2}_{1}(\psi_3\psi_4\psi_3 - 1)\psi_2\psi_3 z_\lambda
$$

$$
= \psi^{5}_{4}\psi^{2}_{1}\psi_3\psi_4(\psi_3\psi_2\psi_3(2,1,0))z_\lambda - \psi^{5}_{4}(\psi_2\psi_1\psi_2(0,2,0))\psi_3 z_\lambda
$$

$$
= \psi^{5}_{4}\psi^{2}_{1}\psi_3\psi_4(\psi_2\psi_3\psi_2)z_\lambda - \psi^{5}_{4}(\psi_1\psi_2\psi_1 - 1)\psi_3 z_\lambda
$$

$$
= 0 - 0 + \psi^{5}_{3}z_\lambda.
$$

We now suppose that either $e = 3$ and $\lambda = ((3),(3))$, or $e > 3$ and $a > 2$. Then we have

$$
\psi^{2e-1}_{e}\cdot\psi^{e}_{1}\psi^{e+1}_{2}\ldots^{2e-1}_{e}z_\lambda = \psi^{2e-1}_{e+1}(\psi^2_e(0,a-1))\psi^{e-1}_{1}\psi^{e+1}_{2}\psi^{e+2}_{3}\ldots^{2e-1}_{e}z_\lambda
$$

$$
= \psi^{2e-1}_{e+2}\psi^{e-1}_{1}(\psi^2_{e+1}(-1,a-1))\psi^{e}_{2}\psi^{e+2}_{3}\psi^{e+3}_{4}\ldots^{2e-1}_{e}z_\lambda
$$

$$
= \psi^{2e-1}_{e+3}\psi^{e-1}_{1}\psi^{e}_{2}(\psi^2_{e+2}(-2,a-1))\psi^{e+1}_{3}\psi^{e+3}_{4}\ldots^{2e-1}_{e}z_\lambda
$$

$$
\vdots
$$

$$
= \psi^{2e-1}_{e+b+1}\psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-2}_{b}(\psi^2_{e+b}(a,a-1))\psi^{e+b-1}_{b+1}\psi^{e+b+1}_{b+2}\psi^{e+b+2}_{b+3}\ldots^{2e-1}_{e}z_\lambda
$$

$$
= \psi^{2e-1}_{e+b+1}\psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-2}_{b}(y_{e+b+1}-y_{e+b})\psi^{e+b-1}_{b+1}\psi^{e+b+1}_{b+2}\psi^{e+b+2}_{b+3}\ldots^{2e-1}_{e}z_\lambda.
$$

It thus suffices to show that

$$
y_{e+b+1}\psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-1}_{b+1}\psi^{e+b+1}_{b+2}\psi^{e+b+2}_{b+3}\ldots^{2e-1}_{e}z_\lambda = (-1)^{b+1}\psi^{e+b}_{e}z_\lambda
$$

$$
= -y_{e+b}\psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-1}_{b+1}\psi^{e+b+1}_{b+2}\psi^{e+b+2}_{b+3}\ldots^{2e-1}_{e}z_\lambda.
$$

We will prove the first equality here, for which we have set up the relevant computational lemmas — the second equality may be proved in a similar manner.

$$
y_{e+b+1}\psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-1}_{b+1}\psi^{e+b+1}_{b+2}\psi^{e+b+2}_{b+3}\ldots^{2e-1}_{e}z_\lambda
$$

$$
= \psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-1}_{b+1}(y_{e+b+1}\psi_{e+b+1}(1,a-1))\psi^{e+b}_{b+2}\psi^{e+b+2}_{b+3}\psi^{e+b+3}_{b+4}\ldots^{2e-1}_{e}z_\lambda
$$

$$
= \psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-1}_{b+1}\psi^{e+b+1}_{b+2}(y_{e+b+2}\psi_{e+b+2}(2,a-1))\psi^{e+b+1}_{b+3}\psi^{e+b+3}_{b+4}\psi^{e+b+4}_{b+5}\ldots^{2e-1}_{e}z_\lambda
$$

$$
\vdots
$$

$$
= \psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-1}_{b+1}\psi^{e+b+1}_{b+2}\psi^{e+b+2}_{b+3}\ldots^{2e-2}_{e-1}(y_{2e-1}\psi_{2e-1}(a-1,a-1))\psi^{2e-2}_{e}z_\lambda
$$

$$
= -\psi^{e-1}_{1}\psi^{e}_{2}\ldots^{e+b-1}_{b+1}\psi^{e+b+1}_{b+2}\psi^{e+b+2}_{b+3}\ldots^{2e-2}_{e-1}\psi^{2e-2}_{e}z_\lambda.
$$

If $e = 3$, then this expression becomes, as required,

$$-\psi^2_{\downarrow 1}\psi^4_{\downarrow 2}\psi^4_{\downarrow 3}z_{((3),(3))} = -\psi^2_{\downarrow 1}(\psi_4\psi_3\psi_4(1,2,1))\psi_2\psi_3 z_{((3),(3))}$$

$$= -\psi^2_{\downarrow 1}(\psi_3\psi_4\psi_3 + 1)\psi_2\psi_3 z_{((3),(3))}$$

$$= -\psi^2_{\downarrow 1}\psi_3\psi_4(\psi_3\psi_2\psi_3(1,2,0))z_{((3),(3))} - (\psi_2\psi_1\psi_2(0,1,0))\psi_3 z_{((3),(3))}$$

$$= -\psi^2_{\downarrow 1}\psi_3\psi_4\psi_2\psi_3\psi_2 z_{((3),(3))} - (\psi_1\psi_2\psi_1 + 1)\psi_3 z_{((3),(3))}$$

$$= 0 - 0 - \psi_3 z_{((3),(3))}.$$

If $e > 3$, then the above expression becomes

$$-\psi^{e-1\;\;e}_{\downarrow\;\downarrow}{}_{1\;\;2}\cdots{}^{e+b-1}_{\downarrow}{}_{b+1}\,\psi^{2b+2\;2b+3}_{\downarrow\;\;\;\downarrow}{}_{b+2\;\;b+3}\cdots{}^{e+b}_{\downarrow}{}_{e}\,z_\lambda \quad \text{by Lemma 6.1(iii)}$$

$$= -\psi^{b+1}_{\downarrow}{}_{2}\,\psi^{e-1}_{\downarrow}{}_{3}\,\psi_2\psi^{b+3\;b+4}_{\downarrow\;\;\;\downarrow}{}_{3\;\;\;4}\cdots{}^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda \quad \text{by Lemma 6.2(v)}$$

$$= -\psi^{b+1}_{\downarrow}{}_{2}\,\psi^{e-1}_{\downarrow}{}_{3}(\psi_2\psi_1\psi_2(0,1,0))\psi^{b+3\;b+4}_{\downarrow\;\;\;\downarrow}{}_{3\;\;\;4}\cdots{}^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$= -\psi^{b+1}_{\downarrow}{}_{2}\,\psi^{e-1}_{\downarrow}{}_{3}(\psi_1\psi_2\psi_1 + 1)\psi^{b+3\;b+4}_{\downarrow\;\;\;\downarrow}{}_{3\;\;\;4}\cdots{}^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$= 0 - \psi^{b+1}_{\downarrow}{}_{2}\,\psi^{e-1}_{\downarrow}{}_{3}\,\psi^{b+3\;b+4}_{\downarrow\;\;\;\downarrow}{}_{3\;\;\;4}\cdots{}^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda.$$

Now, we handle the case $b = 0$ separately. In this case, the above expression is

$$-\psi^{b+1}_{\downarrow}{}_{2}\,\psi^{e-1}_{\downarrow}{}_{3}\,\psi^{b+3\;b+4}_{\downarrow\;\;\;\downarrow}{}_{3\;\;\;4}\cdots{}^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda = -\psi^{e-1}_{\downarrow}{}_{3}\,\psi^{e}_{\uparrow}{}_{3}z_\lambda$$

$$= -\psi^{e-1}_{\downarrow}{}_{4}(\psi^2_3(2,0))\psi^{e}_{\uparrow}{}_{4}z_\lambda$$

$$= -\psi^{e-1}_{\downarrow}{}_{5}(\psi^2_4(3,0))\psi^{e}_{\uparrow}{}_{5}z_\lambda$$

$$\vdots$$

$$= -(\psi^2_{e-1}(-2,0))\psi_e z_\lambda$$

$$= -\psi_e z_\lambda,$$

in which case our proof is complete here.

Now suppose that $b > 0$. Then by Lemma 6.3 followed by repeated application of Lemma 6.5(ii),

$$-\psi^{b+1}_{\downarrow}{}_{2}\,\psi^{e-1}_{\downarrow}{}_{3}\,\psi^{b+3\;b+4}_{\downarrow\;\;\;\downarrow}{}_{3\;\;\;4}\cdots{}^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$= -\psi^{b+1\;b+2}_{\downarrow\;\;\;\downarrow}{}_{2\;\;\;3}\cdots{}^{e-1}_{\downarrow}{}_{a}\,\psi^{e\;\;e+1}_{\downarrow\;\downarrow}{}_{a\;a+1}\cdots{}^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$= (-1)^2\psi^{b+1\;b+2}_{\downarrow\;\;\;\downarrow}{}_{2\;\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{a-1}\,\psi^{e-1\;\;e}_{\downarrow\;\;\downarrow}{}_{a+1\;a+2}\,\psi^{e+1\;e+2}_{\downarrow\;\;\;\downarrow}{}_{a+1\;a+2}\cdots{}^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$\vdots$$

$$= (-1)^b\psi^{b+1\;b+2}_{\downarrow\;\;\;\downarrow}{}_{2\;\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{a-1}\,\psi_{e-1}\psi_e\ldots\psi_{e+b-2}\psi^{e+b-1\;e+b}_{\downarrow\;\;\;\;\downarrow}{}_{e-1\;\;\;e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$= (-1)^{b+1}\psi^{b+1\;b+2}_{\downarrow\;\;\;\downarrow}{}_{2\;\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{a-1}\,\psi^{e+b}_{\downarrow}{}_{e}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$= (-1)^{b+1}\psi^{e+b}_{\downarrow}{}_{e}\,\psi^{b+1\;b+2}_{\downarrow\;\;\;\downarrow}{}_{2\;\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{a-1}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$= (-1)^{b+1}\psi^{e+b}_{\downarrow}{}_{e}\,\psi^{e-2\;e-3}_{\uparrow\;\;\;\uparrow}{}_{b+1\;\;b}\cdots{}^{a-1}_{\uparrow}{}_{2}\,\psi^{a-1\;a}_{\downarrow\;\downarrow}{}_{2\;\;3}\cdots{}^{e-2}_{\downarrow}{}_{b+1}z_\lambda$$

$$= (-1)^{b+1}\psi^{e+b}_{\downarrow}{}_{e}\,z_\lambda \quad \text{by Corollary 6.7, completing the proof.}$$

If $e > 3$ and $\lambda = ((2, 1^{e-2}), (2, 1^{e-2}))$ or $((1^e), (1^e))$, then we have

$$\psi \overset{2e-1}{\underset{e}{\downarrow}} \cdot \psi \overset{e\ e+1}{\underset{1\ 2}{\downarrow\ \downarrow}} \dots \overset{2e-1}{\underset{e}{\downarrow}} z_\lambda = \psi \overset{2e-1}{\underset{e+1}{\downarrow}} (\psi_e^2(0, 1)) \psi \overset{e-1}{\underset{1}{\downarrow}} \psi \overset{e+1\ e+2}{\underset{2\ 3}{\downarrow\ \downarrow}} \dots \overset{2e-1}{\underset{e}{\downarrow}} z_\lambda$$

$$= \psi \overset{2e-1}{\underset{e+1}{\downarrow}} (y_e - y_{e+1}) \psi \overset{e-1}{\underset{1}{\downarrow}} \psi \overset{e+1\ e+2}{\underset{2\ 3}{\downarrow\ \downarrow}} \dots \overset{2e-1}{\underset{e}{\downarrow}} z_\lambda,$$

and the proof may be finished in a similar manner to the other cases. $\qquad\square$

Proof of Proposition 4.12. Let $\lambda = ((ke), (je))$ for some $j, k \geqslant 1$. In order to prove Proposition 4.12, we now look at the action of the KLR generators $\psi_1, \dots, \psi_{n-1}$ on an arbitrary basis element $v_T \in S_\lambda$, where T does not necessarily lie in $\mathcal{T}_e$.

Lemma 6.8. *Let* $T \in \mathrm{Std}(\lambda)$, $v_T = v(a_1, \dots, a_{je})$, $1 \leqslant r < n$, *and* $1 \leqslant s < je$ *such that* $r \not\equiv 2s \pmod{e}$.

(i) *If* $a_s = r$ *and* $a_{s+1} = r + 1$, *then* $\psi_r v(a_1, \dots, a_{je}) = 0$.

(ii) *If* s *is maximal such that* $a_s \leqslant r - 1$, *and* $r, r + 1 \notin \{a_1, \dots, a_{je}\}$, *then* $\psi_r v(a_1, \dots, a_{je}) = 0$.

Proof. We proceed by induction on $r - s$ on both of the statements.

(i) For $r = s$, we observe that

$$\psi_r v(a_1, \dots, a_{je}) = \psi_r \psi \overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \overset{a_{ke}-1}{\underset{ke}{\downarrow}} z_\lambda = \psi \overset{a_{r+2}-1}{\underset{r+2}{\downarrow}} \dots \overset{a_{ke}-1}{\underset{ke}{\downarrow}} \psi_r z_\lambda = 0.$$

Now assuming that $r > s$, we have

$\psi_r v(a_1, \dots, a_{je})$
$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} (\psi_r \psi_{r-1} \psi_r (s - 1, s, r - s - 1)) \psi \overset{r-2\ r-1}{\underset{s\ s+1}{\downarrow\ \downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} \psi_{r-1} \psi_r \psi_{r-1} \psi \overset{r-2\ r-1}{\underset{s\ s+1}{\downarrow\ \downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} \psi_{r-1} \psi_r \psi_{r-1} v(1, \dots, s - 1, r - 1, r, a_{s+2}, \dots, a_{je})$$

which is 0 by induction if $r \not\equiv 2s + 1 \pmod{e}$ or if $r = s + 1$ as ψ_{r-1} commutes through to the right. If $r \equiv 2s + 1 \pmod{e}$ and $r > s + 1$, then we continue:

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} \psi_{r-1} \psi_r (\psi_{r-1} \psi_{r-2} \psi_{r-1}(s - 1, s, s - 1)) \psi \overset{r-3\ r-2}{\underset{s\ s+1}{\downarrow\ \downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} \psi_{r-1} \psi_r (\psi_{r-2} \psi_{r-1} \psi_{r-2} + 1) \psi \overset{r-3\ r-2}{\underset{s\ s+1}{\downarrow\ \downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda.$$

The first term becomes

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} \psi_{r-1} \psi_r \psi_{r-2} \psi_{r-1} \psi_{r-2} \psi \overset{r-3\ r-2}{\underset{s\ s+1}{\downarrow\ \downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} \psi_{r-1} \psi_r \psi_{r-2} \psi_{r-1} \psi_{r-2} v(1, \dots, s - 1, r - 2, r - 1, a_{s+2}, \dots, a_{je})$$
$$= 0 \quad \text{by induction.}$$

If $s = je - 1$, then the second term becomes

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_{je-2}-1}{\underset{je-2}{\downarrow}} \psi_{r-1} \psi_r \psi \overset{r-3}{\underset{je-1}{\downarrow}} \overset{r-2}{\underset{je}{\downarrow}} z_\lambda = \psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_{je-2}-1}{\underset{je-2}{\downarrow}} \psi_{r-1} \psi \overset{r-3}{\underset{je-1}{\downarrow}} \overset{r-2}{\underset{je}{\downarrow}} \psi_r z_\lambda = 0.$$

If $s < je - 1$, then the second term becomes

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} \psi_{r-1} \psi \overset{r-3}{\underset{s}{\downarrow}} \psi_r \psi \overset{r-2}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \ldots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$

$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_{s-1}-1}{\underset{s-1}{\downarrow}} \psi_{r-1} \psi \overset{r-3}{\underset{s}{\downarrow}} \psi_r v(1, \ldots, s, r-1, a_{s+2}, \ldots, a_{je})$$

$$= 0$$

by the inductive hypothesis of (ii) as $a_{s+1} \leqslant r - 1$, $a_{s+2} \geqslant r + 2$, and $r \not\equiv 2(s+1)$ (mod e).

(ii) For $r = s + 1$, we have

$$\psi_{s+1} v(a_1, \ldots, a_{je}) = \psi_{s+1} \psi \overset{a_{s+1}-1}{\underset{s+1}{\downarrow}} \ldots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda.$$

We observe that the first $s+2$ residues in the residue sequence of $s_{s+1} s \overset{a_{s+1}-1}{\underset{s+1}{\downarrow}} \ldots s \overset{a_{je}-1}{\underset{je}{\downarrow}} \mathrm{T}_\lambda$ are $0, 1, \ldots, s-1, 1, 0$. There exists no $\mathrm{S} \in \mathrm{Std}(\lambda)$ with such a residue sequence, and hence $\psi_{s+1} v(a_1, \ldots, a_{je}) = 0$.

Now assuming that $r > s+1$, we argue by induction on $\ell(w_\mathrm{T})$. For the base case, the minimal length is obtained when $s = je - 1$ and $v_\mathrm{T} = v(1, 2, \ldots, je - 1, r - 2)$. Then

$$\psi_r v(1, 2, \ldots, je - 1, r + 2) = (\psi_r \psi_{r+1} \psi_r (-1, r, r+1)) \psi \overset{r-1}{\underset{je}{\downarrow}} z_\lambda$$

$$= \psi_{r+1} \psi_r \psi_{r+1} \psi \overset{r-1}{\underset{je}{\downarrow}} z_\lambda$$

$$= \psi_{r+1} \psi_r \psi \overset{r-1}{\underset{je}{\downarrow}} \psi_{r+1} z_\lambda$$

$$= 0 \quad \text{since } r + 1 \not\equiv -1 \text{ (mod } e) \text{ by our residue hypothesis.}$$

Now for $\ell(w_\mathrm{T})$ arbitrary, we have

$$\psi_r v(a_1, \ldots, a_{je})$$
$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_s-1}{\underset{s}{\downarrow}} \psi \overset{a_{s+1}-1}{\underset{r+2}{\downarrow}} (\psi_r \psi_{r+1} \psi_r (s, r-s-1, r-s)) \psi \overset{r-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \ldots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda,$$

where $r - s \not\equiv s$ (mod e).

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_s-1}{\underset{s}{\downarrow}} \psi \overset{a_{s+1}-1}{\underset{r+2}{\downarrow}} \psi_{r+1} \psi_r \psi_{r+1} \psi \overset{r-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \ldots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$

$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_s-1}{\underset{s}{\downarrow}} \psi \overset{a_{s+1}-1}{\underset{r+2}{\downarrow}} \psi_{r+1} \psi_r \psi_{r+1} v(1, \ldots, s, r, a_{s+2}, \ldots, a_{je}) = 0$$

by the inductive hypothesis if $r + 1 \not\equiv 2(s+1) \pmod{e}$. If $r + 1 \equiv 2s + 2 \pmod{e}$, then

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s+1}-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{r+3}{\downarrow}} (\psi_{r+1}\psi_{r+2}\psi_{r+1}(s+1, s, s+1))\psi \overset{r}{\underset{s+2}{\downarrow}} \psi \overset{a_{s+3}-1}{\underset{s+3}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$

$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s+1}-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{r+3}{\downarrow}} (\psi_{r+2}\psi_{r+1}\psi_{r+2} + 1)\psi \overset{r}{\underset{s+2}{\downarrow}} \psi \overset{a_{s+3}-1}{\underset{s+3}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda.$$

If $s = je - 2$, then this becomes

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{je-1}-1}{\underset{je-1}{\downarrow}} \psi \overset{a_{je}-1}{\underset{r+3}{\downarrow}} (\psi_{r+2}\psi_{r+1}\psi_{r+2} + 1)\psi \overset{r}{\underset{je}{\downarrow}} z_\lambda = 0.$$

If $s < je - 2$, then the first term becomes

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s+1}-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{r+3}{\downarrow}} \psi_{r+2}\psi_{r+1}\psi_{r+2}\psi \overset{r}{\underset{s+2}{\downarrow}} \psi \overset{a_{s+3}-1}{\underset{s+3}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$

$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_{s+1}-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{r+3}{\downarrow}} \psi_{r+2}\psi_{r+1}\psi_{r+2}v(1, \dots, s+1, r+1, a_{s+3}, \dots, a_{je})$$

$$= 0 \quad \text{by induction as } r + 2 \not\equiv 2s + 4 \pmod{e}.$$

Now, the second term becomes

$$\psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_s-1}{\underset{s}{\downarrow}} \psi \overset{a_{s+1}-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{r+3}{\downarrow}} \psi \overset{r}{\underset{s+2}{\downarrow}} \psi \overset{a_{s+3}-1}{\underset{s+3}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$

$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_s-1}{\underset{s}{\downarrow}} \psi \overset{a_{s+1}-1}{\underset{r+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{r+3}{\downarrow}} \psi_r \psi \overset{r-1}{\underset{s+1}{\downarrow}} \overset{r}{\underset{s+2}{\downarrow}} \psi \overset{a_{s+3}-1}{\underset{s+3}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$

$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \dots \overset{a_s-1}{\underset{s}{\downarrow}} \psi \overset{a_{s+1}-1}{\underset{r+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{r+3}{\downarrow}} \psi_r v(1, \dots, s, r, r+1, a_{s+3}, \dots, a_{je})$$

$$= 0 \quad \text{by the inductive hypothesis on (i), as } r \not\equiv 2s + 2 \pmod{e}. \quad \square$$

Corollary 6.9. *Let $1 \leqslant r < n$ and $1 \leqslant s < je$ with $r \geqslant s + 1$ and $r \equiv 2s \pmod{e}$. Then*

(i) $\psi_r v(1, \dots, s, r+2, a_{s+2}, \dots, a_{je}) = v(1, \dots, s, r, a_{s+2}, \dots, a_{je})$,

(ii) $\psi_r v(1, \dots, s-1, r, r+1, a_{s+2}, \dots, a_{je}) = v(1, \dots, s-1, r-1, r, a_{s+2}, \dots, a_{je})$.

Proof. (i) We have

$$\psi_r v(1, \dots, s, r+2, a_{s+2}, \dots, a_{je})$$
$$= \psi_r \psi \overset{r+1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= (\psi_r \psi_{r+1}\psi_r(s, s-1, s))\psi \overset{r-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= (\psi_{r+1}\psi_r \psi_{r+1} + 1)\psi \overset{r-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \dots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi_{r+1}\psi_r \psi_{r+1} v(1, \dots, s, r, a_{s+2}, \dots, a_{je}) + v(1, \dots, s, r, a_{s+2}, \dots, a_{je}),$$

and $\psi_{r+1} v(1, \dots, s, r, a_{s+2}, \dots, a_{je}) = 0$ by Lemma 6.8(ii).

(ii) We have

$$\psi_r v(1, \ldots, s-1, r, r+1, a_{s+2}, \ldots, a_{je})$$
$$= \psi_r \psi \overset{r-1}{\underset{s}{\downarrow}} \overset{r}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \ldots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= (\psi_r \psi_{r-1} \psi_r (s-1, s, s-1)) \psi \overset{r-2}{\underset{s}{\downarrow}} \overset{r-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \ldots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= (\psi_{r-1} \psi_r \psi_{r-1} + 1) \psi \overset{r-2}{\underset{s}{\downarrow}} \overset{r-1}{\underset{s+1}{\downarrow}} \psi \overset{a_{s+2}-1}{\underset{s+2}{\downarrow}} \ldots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi_{r-1} \psi_r \psi_{r-1} v(1, \ldots, s-1, r-1, r, a_{s+2}, \ldots, a_{je})$$
$$+ v(1, \ldots, s-1, r-1, r, a_{s+2}, \ldots, a_{je}),$$

and the first term is 0 by Lemma 6.8(i) since $r-1 \not\equiv 2s \pmod e$. □

Lemma 6.10. *Let* $1 \leqslant s \leqslant i \leqslant r < n$, $s+r-i \leqslant je$, $a_{s-1} < i$, *and* $i \not\equiv x \pmod e$ *for any* $x \in \{2s-2, 2s-1, \ldots, 2s+r-i\}$. *Then*

$$\psi \overset{r}{\underset{i}{\downarrow}} v(a_1, \ldots, a_{s-1}, i+1, i+2, \ldots, r+1, a_{s+r-i+1}, \ldots, a_{je})$$
$$= v(a_1, \ldots, a_{s-1}, i, i+1, \ldots, r, a_{s+r-i+1}, \ldots, a_{je}).$$

Proof. Suppose that $a_{s+l-i-1} < l$ for all $l \in \{i, i+1, \ldots, r\}$. Then the result follows directly from the KLR relations since, for all l,

$$\psi_l v(a_1, \ldots, a_{s+l-i-1}, l+1, a_{s+l-i+1}, a_{s+l-i+2}, \ldots, a_{je})$$
$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_{s+l-i-1}-1}{\underset{s+l-i-1}{\downarrow}} \psi_l^2 (s+l-i-1, i-s) \psi \overset{l-1}{\underset{s+l-i}{\downarrow}} \psi \overset{a_{s+l-i+1}-1}{\underset{s+l-i+1}{\downarrow}} \overset{a_{s+l-i+1}-1}{\underset{s+l-i+1}{\downarrow}} \ldots \overset{a_{je}-1}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi \overset{a_1-1}{\underset{1}{\downarrow}} \ldots \overset{a_{s+l-i-1}-1}{\underset{s+l-i-1}{\downarrow}} v(1, \ldots, s+l-i-1, l, a_{s+l-i+1}, a_{s+l-i+2}, \ldots, a_{je})$$

if $i \not\equiv 2s+l-i-2, 2s+l-i-1, 2s+l-i \pmod e$. □

Corollary 6.11. *Suppose that* $1 \leqslant s \leqslant i \leqslant r < n$, $s+r-i < je$, $a_{s-1} \leqslant i-2$, $i \equiv 2s \pmod e$, *and* $r-i+2 < e$. *Then*

$$\psi \overset{r}{\underset{i}{\downarrow}} v(a_1, \ldots, a_{s-1}, i, i+1, i+2, \ldots, r+1, a_{s+r-i+2}, \ldots, a_{je})$$
$$= v(a_1, \ldots, a_{s-1}, i-1, i, i+1, \ldots, r, a_{s+r-i+2}, \ldots, a_{je}).$$

Proof. Since $i \equiv 2s \pmod e$, we apply Corollary 6.9(ii) to give us

$$\psi \overset{r}{\underset{i+1}{\downarrow}} \psi_i v(a_1, \ldots, a_{s-1}, i, i+1, i+2, \ldots, r+1, a_{s+r-i+2}, \ldots, a_{je})$$
$$= \psi \overset{r}{\underset{i+1}{\downarrow}} v(a_1, \ldots, a_{s-1}, i-1, i, i+2, \ldots, r+1, a_{s+r-i+2}, \ldots, a_{je}).$$

We now obtain our desired result by applying Lemma 6.10 since $i+1 \not\equiv x \pmod e$ for all $x \in \{2s, 2s+1, \ldots, 2s+r-i+1\}$ (note that x runs over $r-i+2 < e$ terms). □

Lemma 6.12. *Let* $1 \leqslant r \leqslant je$. *If* $r \not\equiv 1 \pmod e$, *then* $y_r \psi_r \psi_{r+1} \ldots \psi_{je} z_\lambda = 0$.

Proof. We proceed by induction on $\ell(w_T)$, where the minimal length is obtained when $r = je$.

For $r = je$, we have

$$(y_{je}\psi_{je}(-1,0))z_\lambda = \psi_{je}y_{je+1}z_\lambda = 0.$$

Now assuming that $r < je$,

$$(y_r\psi_r(r-1,0))\psi_{r+1}\psi_{r+2}\ldots\psi_{je}z_\lambda = \psi_r y_{r+1}\psi_{r+1}\psi_{r+2}\psi_{r+3}\ldots\psi_{je}z_\lambda = 0$$

by induction if $r \not\equiv 0 \pmod{e}$. If $r \equiv 0 \pmod{e}$, then this term becomes

$$\psi_r(y_{r+1}\psi_{r+1}(0,0))\psi_{r+2}\psi_{r+3}\ldots\psi_{je}z_\lambda = \psi_r(\psi_{r+1}y_{r+2}+1)\psi_{r+2}\psi_{r+3}\ldots\psi_{je}z_\lambda.$$

The second term becomes $\psi_{r+2}\psi_{r+3}\ldots\psi_{je}(\psi_r z_\lambda) = 0$, whilst the first term is 0 by induction. $\square$

Lemma 6.13. *Let $1 \leqslant s < i \leqslant r < n$ and $s < je$, and suppose that $r \equiv 2s \pmod{e}$ and $r - i + 2 < e$. Then*

$$\psi_{\substack{r\\\uparrow\\i}}v(1,\ldots,s,r+2,a_{s+2},\ldots,a_{je}) = v(1,\ldots,s,i,a_{s+2},\ldots,a_{je}).$$

Proof. The proof is similar to the proof of Corollary 6.11. $\square$

Lemma 6.14. *Let $0 \leqslant s \leqslant je - e$ and $v_T = v(a_1,\ldots,a_{je})$.*

(i) *If $a_{s+e} = r$ for some $1 \leqslant r \leqslant n$ such that $r \not\equiv 2s, 2s+1 \pmod{e}$ and $r-1, r+1, r+2, r+3, \ldots, r+e-2 \notin \{a_1,\ldots,a_{je}\}$, then $y_{r-1}v_T = 0$.*

(ii) *If $a_{s+e} = r$ for some $1 \leqslant r \leqslant n$ such that $r \not\equiv 2s, 2s+1 \pmod{e}$ and $r+1, r+2, r+3, \ldots, r+e-2 \notin \{a_1,\ldots,a_{je}\}$, then $y_r v_T = 0$.*

(iii) *If for some $1 \leqslant r < n$, we have $a_{s+i} = r - e + i$ for all $i \in \{1,\ldots,e-1\}$, $a_{s+e} = r+1, r \equiv 2s \pmod{e}$, and $r+2, r+3, \ldots, r+e \notin \{a_{s+e+1},\ldots,a_{je}\}$, then $\psi_r v_T = 0$.*

Proof. We proceed by simultaneous induction on $r-s$ on each of the three statements. Note that we apply Corollary 6.9 without further reference.

(i) Our base case is when $r = s+e+1$, so that $s \not\equiv 0, 1 \pmod{e}$ and $a_{s+e-1} = s+e-1$. We prove this by induction on $\ell(w_T)$. For the base case, the minimal length is obtained when $s + e + 1 = je$. We thus have

$$y_{r-1}v_T = (y_{je-1}\psi_{je-1}(-2,0))\psi_{\substack{a_{je}-1\\\downarrow\\je}}z_\lambda$$

$$= \psi_{je-1}\psi_{\substack{a_{je}-1\\\downarrow\\je+1}}(y_{je}\psi_{je}(-1,0))z_\lambda = \psi_{je-1}\psi_{\substack{a_{je}-1\\\downarrow\\je+1}}\psi_{je}y_{je+1}z_\lambda = 0.$$

Now suppose that $s + e + 1 < je$, and assume without loss of generality that $v_T = v(1, \ldots, s+e-1, s+e+1, s+2e, \ldots, je+e-1)$. Then we have

$$
\begin{aligned}
y_{r-1} v_T &= (y_{s+e} \psi_{s+e}(s-1, 0)) \psi \overset{s+2e-1}{\underset{s+e+1}{\downarrow}} \overset{s+2e}{\underset{s+e+2}{\downarrow}} \cdots \overset{je+e-2}{\underset{je}{\downarrow}} z_\lambda \\
&= \psi_{s+e} y_{s+e+1} \psi \overset{s+2e-1}{\underset{s+e+1}{\downarrow}} \overset{s+2e}{\underset{s+e+2}{\downarrow}} \cdots \overset{je+e-2}{\underset{je}{\downarrow}} z_\lambda \\
&= \psi_{s+e} \psi \overset{s+2e-1}{\underset{s+e+2}{\downarrow}} y_{s+e+1} \psi_{s+e+1} \psi \overset{s+2e}{\underset{s+e+2}{\downarrow}} \overset{s+2e+1}{\underset{s+e+3}{\downarrow}} \cdots \overset{je+e-2}{\underset{je}{\downarrow}} z_\lambda \\
&= \psi_{s+e} \psi \overset{s+2e-1}{\underset{s+e+2}{\downarrow}} y_{s+e+1} v(1, \ldots, s+e, s+e+2, s+2e+1, \ldots, je+e-3) \\
&= 0 \quad \text{by induction if } s \not\equiv -1 \pmod{e}.
\end{aligned}
$$

If $s \equiv -1 \pmod{e}$, then

$$
\begin{aligned}
\psi_{s+e} \psi \overset{s+2e-1}{\underset{s+e+2}{\downarrow}} (y_{s+e+1} \psi_{s+e+1}(-1, 0)) \psi \overset{s+2e}{\underset{s+e+2}{\downarrow}} \overset{s+2e+1}{\underset{s+e+3}{\downarrow}} \cdots \overset{je+e-2}{\underset{je}{\downarrow}} z_\lambda \\
= \psi_{s+e} \psi \overset{s+2e-1}{\underset{s+e+1}{\downarrow}} y_{s+e+2} \psi \overset{s+2e}{\underset{s+e+2}{\downarrow}} \overset{s+2e+1}{\underset{s+e+3}{\downarrow}} \cdots \overset{je+e-2}{\underset{je}{\downarrow}} z_\lambda \\
= \psi_{s+e} \psi \overset{s+2e-1}{\underset{s+e+1}{\downarrow}} \psi \overset{s+2e}{\underset{s+e+3}{\downarrow}} (y_{s+e+2} \psi_{s+e+2}(0,0)) \psi \overset{s+2e+1}{\underset{s+e+3}{\downarrow}} \overset{s+2e+2}{\underset{s+e+4}{\downarrow}} \cdots \overset{je+e-2}{\underset{je}{\downarrow}} z_\lambda \\
= \psi_{s+e} \psi \overset{s+2e-1}{\underset{s+e+1}{\downarrow}} \psi \overset{s+2e}{\underset{s+e+3}{\downarrow}} (\psi_{s+e+2} y_{s+e+3} - 1) \psi \overset{s+2e+1}{\underset{s+e+3}{\downarrow}} \overset{s+2e+2}{\underset{s+e+4}{\downarrow}} \cdots \overset{je+e-2}{\underset{je}{\downarrow}} z_\lambda \\
= \psi_{s+e} \psi \overset{s+2e-1}{\underset{s+e+1}{\downarrow}} \psi \overset{s+2e}{\underset{s+e+2}{\downarrow}} \psi \overset{s+2e+1}{\underset{s+e+4}{\downarrow}} \overset{s+2e+2}{\underset{s+e+5}{\downarrow}} \cdots \overset{je+e-2}{\underset{je+1}{\downarrow}} y_{s+e+3} \psi_{s+e+3} \psi_{s+e+4} \cdots \psi_{je} z_\lambda \\
\qquad - \psi_{s+e} \psi \overset{s+2e-1}{\underset{s+e+2}{\downarrow}} \psi \overset{s+2e}{\underset{s+e+3}{\downarrow}} \psi \overset{s+2e+1}{\underset{s+e+3}{\downarrow}} \overset{s+2e+2}{\underset{s+e+4}{\downarrow}} \cdots \overset{je+e-2}{\underset{je}{\downarrow}} \psi_{s+e+1} z_\lambda
\end{aligned}
$$

$= 0$ by [Lemma 6.12](#).

Next, we assume that $r > s + e + 1$, and again argue by induction on $\ell(w_T)$. For the base case, the minimal length is obtained when $s + e = je$ and $v_T = v(1, \ldots, je-1, r)$. Then

$$
y_{r-1} v_T = (y_{r-1} \psi_{r-1}(-1, r-1)) \psi \overset{r-2}{\underset{je}{\downarrow}} z_\lambda = 0.
$$

Now for $s + e < je$, we may assume by induction (on $\ell(w_T)$) that $v_T = v(1, \ldots, s+e-1, r, r+e-1, r+e, \ldots, r+je-s-2)$ and we have

$$
\begin{aligned}
y_{r-1} v_T &= (y_{r-1} \psi_{r-1}(s-1, r-s-1)) \psi \overset{r-2}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{s+e+1}{\downarrow}} \overset{r+e-1}{\underset{s+e+2}{\downarrow}} \cdots \overset{r+je-s-3}{\underset{je}{\downarrow}} z_\lambda \\
&= \psi \overset{r-1}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{r+1}{\downarrow}} y_r \psi \overset{r}{\underset{s+e+1}{\downarrow}} \psi \overset{r+e-1}{\underset{s+e+2}{\downarrow}} \overset{r+e}{\underset{s+e+3}{\downarrow}} \cdots \overset{r+je-s-3}{\underset{je}{\downarrow}} z_\lambda \\
&= \psi \overset{r-1}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{r+1}{\downarrow}} y_r v(1, \ldots, s+e, r+1, r+e, \ldots, r+je-s-2), \\
&= 0 \quad \text{by induction if } r \not\equiv 2s+2 \pmod{e}.
\end{aligned}
$$

If $r \equiv 2s + 2 \pmod{e}$, then we have

$$\psi^{r-1}_{\ s+e} \psi^{r+e-2}_{\ \ r+1}(y_r\psi_r(s, s+1))\psi^{r-1}_{\ s+e+1}\psi^{r+e-1}_{\ s+e+2}{}^{r+e}_{s+e+3}\cdots^{r+je-s-3}_{\quad je} z_\lambda$$

$$= \psi^{r-1}_{\ s+e}\psi^{r+e-2}_{\ s+e+1}y_{r+1}\psi^{r+e-1}_{\ s+e+2}{}^{r+e}_{s+e+3}\cdots^{r+je-s-3}_{\quad je} z_\lambda$$

$$= 0 \quad \text{if } s+e = je - 1.$$

Now suppose that $s + e < je - 1$. Then the above term becomes

$$\psi^{r-1}_{\ s+e}\psi^{r+e-2}_{\ s+e+1}\psi^{r+e-1}_{\ r+2}(y_{r+1}\psi_{r+1}(s+1, s+1))\psi^{r}_{\ s+e+2}\psi^{r+e}_{\ s+e+3}{}^{r+e+1}_{s+e+4}\cdots^{r+je-s-3}_{\quad je} z_\lambda$$

$$= \psi^{r-1}_{\ s+e}\psi^{r+e-2}_{\ s+e+1}\psi^{r+e-1}_{\ r+2}(\psi_{r+1}y_{r+2} - 1)\psi^{r}_{\ s+e+2}\psi^{r+e}_{\ s+e+3}{}^{r+e+1}_{s+e+4}\cdots^{r+je-s-3}_{\quad je} z_\lambda$$

$$= 0 \quad \text{if } s+e = je - 2.$$

If $s + e < je - 2$, the first term becomes

$$\psi^{r-1}_{\ s+e}\psi^{r+e-2}_{\ s+e+1}{}^{r+e-1}_{s+e+2}\psi^{r+e}_{\ r+3}y_{r+2}\psi^{r+2}_{\ s+e+3}\psi^{r+e+1}_{\ s+e+4}{}^{r+e+2}_{s+e+5}\cdots^{r+je-s-3}_{\quad je} z_\lambda$$

$$= \psi^{r-1}_{\ s+e}\psi^{r+e-2}_{\ s+e+1}{}^{r+e-1}_{s+e+2}\psi^{r+e}_{\ r+3}y_{r+2}$$

$$\cdot v(1, \ldots, s+e+2, r+3, r+e+2, \ldots, r+je-s-2)$$

$$= 0 \quad \text{by induction since } r \not\equiv 2s+3, 2s+4 \pmod{e}.$$

Then the second term above becomes

$$-\psi^{r-1}_{\ s+e}\psi^{r+e-2}_{\ s+e+1}\psi^{r}_{\ s+e+2}\psi^{r+e-1}_{\ r+3}\psi^{r+e}_{\ r+4}\psi_{r+2}\overbrace{\psi_{r+3}\psi_{r+2}}\psi^{r+1}_{\ s+e+3}\psi^{r+e+1}_{\ s+e+4}{}^{r+e+2}_{s+e+5}\cdots^{r+je-s-3}_{\quad je} z_\lambda,$$

which is zero if $s + e = je - 3$. If $s + e < je - 3$, then we continue

$$-\psi^{r-1}_{\ s+e}\psi^{r+e-2}_{\ s+e+1}\psi^{r}_{\ s+e+2}\psi^{r+1}_{\ s+e+3}\psi^{r+e-1}_{\ r+3}{}^{r+e}_{r+4}\psi^{r+e+1}_{\ s+e+4}{}^{r+e+2}_{s+e+5}\cdots^{r+je-s-3}_{\quad je} z_\lambda$$

$$= -\psi^{r-1}_{\ s+e}(\psi^{r+e-2}_{\ r+2}{}^{r+e-1}_{r+3}{}^{r+e}_{r+4})\psi_{r+1}(\overline{\psi_r\psi_{r-1}\psi_r})$$

$$\cdot \psi^{r-2}_{\ s+e+1}\psi^{r-1}_{\ s+e+2}\psi^{r+1}_{\ s+e+3}\psi^{r+e+1}_{\ s+e+4}{}^{r+e+2}_{s+e+5}\cdots^{r+je-s-3}_{\quad je} z_\lambda$$

$$= -\psi^{r-1}_{\ s+e}(\psi^{r+e-2}_{\ r+2}{}^{r+e-1}_{r+3}{}^{r+e}_{r+4})\psi_{r+1}$$

$$\cdot v(1, \ldots, s+e, r-1, r, r+2, r+e+2, \ldots, r+je-s-2)$$

$$= 0 \quad \text{by the inductive hypothesis of part (iii) if } e = 3.$$

We apply Lemma 6.13 without further reference. If $e > 3$, we have

$$-\psi^{r-1}_{\downarrow s+e}(\psi^{r+e-2\,r+e-1\,r+e}_{\;\downarrow r+2\;\downarrow r+3\;\downarrow r+4})\psi^{r-2\;r-1}_{\downarrow s+e+1\;\downarrow s+e+2}(\psi^2_{r+1}(s+2,s))\psi^{r}_{\downarrow s+e+3}\psi^{r+e+1\,r+e+2}_{\;\downarrow s+e+4\;\downarrow s+e+5}\cdots {}^{r+je-s-3}_{\downarrow je}z_\lambda$$

$$= -\psi^{r-1}_{\downarrow s+e}(\psi^{r+e-2\,r+e-1\,r+e\,r+e+1}_{\;\downarrow r+3\;\downarrow r+4\;\downarrow r+5\;\downarrow r+6})$$

$$\cdot\,\psi^{r+4}_{\uparrow r+2}v(1,\ldots,s+e,r-1,r,r+1,r+6,r+e+3,\ldots,r+je-s-2)$$

$$= -\psi^{r-1}_{\downarrow s+e}(\psi^{r+e-2\,r+e-1\,r+e\,r+e+1}_{\;\downarrow r+3\;\downarrow r+4\;\downarrow r+5\;\downarrow r+6})$$

$$\cdot\,\psi_{r+2}v(1,\ldots,s+e,r-1,r,r+1,r+3,r+e+3,\ldots,r+je-s-2)$$

$$= 0 \quad \text{by the inductive hypothesis of part (iii) if } e = 4.$$

If $e > 4$, then we have

$$-\psi^{r-1}_{\downarrow s+e}(\psi^{r+e-2\,r+e-1\,r+e\,r+e+1}_{\;\downarrow r+3\;\downarrow r+4\;\downarrow r+5\;\downarrow r+6})\psi^{r-2\;r-1\;r}_{\downarrow s+e+1\;\downarrow s+e+2\;\downarrow s+e+3}$$

$$\cdot\,(\psi^2_{r+2}(s+3,s))\psi^{r+1}_{\downarrow s+e+4}\psi^{r+e+2\,r+e+3}_{\;\downarrow s+e+5\;\downarrow s+e+6}\cdots {}^{r+je-s-3}_{\downarrow je}z_\lambda$$

$$= -\psi^{r-1}_{\downarrow s+e}(\psi^{r+e-2\,r+e-1}_{\;\downarrow r+4\;\downarrow r+5}\cdots {}^{r+e+2}_{\downarrow r+8})$$

$$\cdot\,\psi^{r+6}_{\uparrow r+3}v(1,\ldots,s+e,r-1,r,r+1,r+2,r+8,r+e+4,\ldots,r+je-s-2)$$

$$= -\psi^{r-1}_{\downarrow s+e}(\psi^{r+e-2\,r+e-1}_{\;\downarrow r+4\;\downarrow r+5}\cdots {}^{r+e+2}_{\downarrow r+8})$$

$$\cdot\,\psi_{r+3}v(1,\ldots,s+e,r-1,r,r+1,r+2,r+4,r+e+4,\ldots,r+je-s-2)$$

$$= 0 \quad \text{by the inductive hypothesis of part (iii) if } e = 5.$$

Continuing in this fashion, we eventually obtain

$$-\psi^{r-1}_{\downarrow s+e}\psi^{r+2e-4}_{\uparrow r+e-2}\cdot v(1,\ldots,s+e,r-1,r,\ldots,r+e-3,r+2e-2,r+2e-1,\ldots,je+r-s-2)$$

$$= -\psi^{r-1}_{\downarrow s+e}\psi_{r+e-2}$$

$$\cdot\,v(1,\ldots,s+e,r-1,r,\ldots,r+e-3,r+e-1,r+2e-1,\ldots,je+r-s-2)$$

$$= 0 \quad \text{by the inductive hypothesis of part (iii).}$$

(ii) If $r = s + e$, then the term $\psi^{r-1}_{\downarrow s+e}$ is trivial so that

$$y_r v_{\mathrm{T}} = y_{s+e}\psi^{a_{s+e+1}-1}_{\;\downarrow s+e+1}\cdots {}^{a_{je}-1}_{\downarrow je}z_\lambda = \psi^{a_{s+e+1}-1}_{\;\downarrow s+e+1}\cdots {}^{a_{je}-1}_{\downarrow je}y_{s+e}z_\lambda = 0.$$

We now suppose that $r > s + e$, and assume without loss of generality that $v_T = v(1, \ldots, s+e-1, r, r+e-1, r+e, \ldots, je-s+r-2)$. Then

$$y_r v_T$$
$$= (y_r \psi_{r-1}(s-1, r-s-1)) \psi \overset{r-2}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{s+e+1}{\downarrow}} \overset{r+e-1}{\underset{s+e+2}{\downarrow}} \cdots \overset{je-s+r-3}{\underset{je}{\downarrow}} z_\lambda,$$
$$= \psi_{r-1} y_{r-1} \psi \overset{r-2}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{s+e+1}{\downarrow}} \overset{r+e-1}{\underset{s+e+2}{\downarrow}} \cdots \overset{je-s+r-3}{\underset{je}{\downarrow}} z_\lambda$$
$$= \begin{cases} 0 & \text{if } r = s+e+1, \\ \psi_{r-1} y_{r-1} v(1, \ldots, s+e-1, r-1, r+e-1, \ldots, je-s+r-2) & \text{if } r \geqslant s+e+2 \end{cases}$$
$$= 0 \text{ by induction if } r \not\equiv 2s + 2 \pmod e.$$

If $r \geqslant s + e + 2$ and $r \equiv 2s + 2 \pmod e$, then

$$\psi_{r-1}(y_{r-1}\psi_{r-2}(s-1, s)) \psi \overset{r-3}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{s+e+1}{\downarrow}} \cdots \overset{je-s+r-3}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi_{r-1}\psi_{r-2}(y_{r-2}\psi_{r-3}(s-1, s-1)) \psi \overset{r-4}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{s+e+1}{\downarrow}} \cdots \overset{je-s+r-3}{\underset{je}{\downarrow}} z_\lambda$$
$$= \begin{cases} 0 & \text{if } r = s+e+2, \\ \psi_{r-1}\psi_{r-2}(\psi_{r-3} y_{r-3} + 1) \psi \overset{r-4}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{s+e+1}{\downarrow}} \cdots \overset{je-s+r-3}{\underset{je}{\downarrow}} z_\lambda & \text{if } r \geqslant s+e+3. \end{cases}$$

Assuming $r \geqslant s + e + 3$, the first term of this is

$$\psi_{r-1}\psi_{r-2}\psi_{r-3} y_{r-3} v(1, \ldots, s+e-1, r-3, r+e-1, \ldots, je-s+r-2) = 0$$
$$\text{by induction.}$$

If $je = s + e$, then the second term becomes

$$\psi_{r-1}\psi_{r-2}\psi \overset{r-4}{\underset{je}{\downarrow}} z_\lambda = \psi_{r-1}\psi \overset{r-4}{\underset{je}{\downarrow}} \psi_{r-2} z_\lambda = 0.$$

Now suppose that $je > s + e$. Then the second term becomes

$$\psi_{r-1}\psi \overset{r-4}{\underset{s+e}{\downarrow}} \psi \overset{r+e-2}{\underset{r}{\downarrow}} \psi_{r-2}\psi_{r-1}\psi_{r-2}\psi \overset{r-3}{\underset{s+e+1}{\downarrow}} \psi \overset{r+e-1}{\underset{s+e+2}{\downarrow}} \overset{r+e}{\underset{s+e+3}{\downarrow}} \cdots \overset{je-s+r-3}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi_{r-1}\psi \overset{r+e-2}{\underset{r+1}{\downarrow}} \psi \overset{r-4}{\underset{s+e}{\downarrow}} \overset{r-3}{\underset{s+e+1}{\downarrow}} \psi \overset{r+e-1}{\underset{r+2}{\downarrow}} \psi_r \psi_{r+1}\psi_r \psi \overset{r+e}{\underset{s+e+2}{\downarrow}} \overset{r+e+1}{\underset{s+e+3}{\downarrow}} \overset{r+e+1}{\underset{s+e+4}{\downarrow}} \cdots \overset{je-s+r-3}{\underset{je}{\downarrow}} z_\lambda$$
$$= \psi \overset{r+e-2}{\underset{r+1}{\downarrow}} \psi \overset{r+e-1}{\underset{r+2}{\downarrow}} \psi_{r-1} v(1, \ldots, s+e-1, r-3, r-2, r, r+e+1, \ldots, je-s+r-2)$$
$$= 0 \quad \text{by the inductive hypothesis of part (iii) if } e = 3.$$

From here, the proof concludes in a similar manner to the proof of part (ii).

(iii) Our base case is when $r = s + e$, so that $s \equiv 0 \pmod e$. We prove this by induction on $\ell(w_T)$, and assume without loss of generality that $v_T = v(1, \ldots, s+e-1, s+e+1, s+2e+1, s+2e+2, \ldots, je+e)$. For the base case, the minimal length is obtained when $s + e = je$:

$$\psi_r v_T = (\psi_{s+e}^2(-1, 0)) z_\lambda = (y_{s+e} - y_{s+e+1}) z_\lambda = 0.$$

Now suppose that $s + e < je$. Then we have

$$\psi_r v_{\mathrm{T}} = (\psi_{s+e}^2(-1,0))\psi \begin{smallmatrix} s+2e & s+2e+1 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+1 & s+e+2 & & je \end{smallmatrix} z_\lambda$$

$$= (y_{s+e} - y_{s+e+1})\psi \begin{smallmatrix} s+2e & s+2e+1 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+1 & s+e+2 & & je \end{smallmatrix} z_\lambda$$

$$= 0 - \psi \begin{smallmatrix} s+2e \\ \downarrow \\ s+e+2 \end{smallmatrix} (y_{s+e+1}\psi_{s+e+1}(0,0))\psi \begin{smallmatrix} s+2e+1 & s+2e+2 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+2 & s+e+3 & & je \end{smallmatrix} z_\lambda$$

$$= -\psi \begin{smallmatrix} s+2e \\ \downarrow \\ s+e+2 \end{smallmatrix} (\psi_{s+e+1}y_{s+e+2} - 1)\psi \begin{smallmatrix} s+2e+1 & s+2e+2 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+2 & s+e+3 & & je \end{smallmatrix} z_\lambda.$$

The first term becomes

$$-\psi \begin{smallmatrix} s+2e \\ \downarrow \\ s+e+1 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+1 & s+2e+2 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+3 & s+e+4 & & je+1 \end{smallmatrix} y_{s+e+2}\psi_{s+e+2}\psi_{s+e+3}\cdots\psi_{je}z_\lambda = 0 \quad \text{by Lemma 6.12.}$$

The second term becomes

$$\psi \begin{smallmatrix} s+2e & s+2e+1 \\ \downarrow & \downarrow \\ s+e+3 & s+e+4 \end{smallmatrix} \psi_{s+e+2}\psi_{s+e+3}\psi_{s+e+2}\psi \begin{smallmatrix} s+2e+2 & s+2e+3 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+3 & s+e+4 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e & s+2e+1 & s+2e+2 \\ \downarrow & \downarrow & \downarrow \\ s+e+3 & s+e+5 & s+e+6 \end{smallmatrix} \psi_{s+e+4}\psi_{s+e+5}\psi_{s+e+4}\psi_{s+e+3}\psi \begin{smallmatrix} s+2e+3 & s+2e+4 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+4 & s+e+5 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e & s+2e+1 & s+2e+2 \\ \downarrow & \downarrow & \downarrow \\ s+e+4 & s+e+5 & s+e+6 \end{smallmatrix} \psi_{s+e+3}v(1,\ldots,s+e+2,s+e+4,s+2e+4,\ldots,je+e)$$

$$= 0 \quad \text{by induction if } e = 3.$$

If $e > 3$, then we have

$$\psi \begin{smallmatrix} s+2e & s+2e+1 & s+2e+2 \\ \downarrow & \downarrow & \downarrow \\ s+e+4 & s+e+5 & s+e+6 \end{smallmatrix} (\psi_{s+e+3}^2(2,0))\psi \begin{smallmatrix} s+2e+3 & s+2e+4 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+4 & s+e+5 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e & s+2e+1 \\ \downarrow & \downarrow \\ s+e+4 & s+e+5 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+2 & s+2e+3 \\ \downarrow & \downarrow \\ s+e+7 & s+e+8 \end{smallmatrix} \psi_{s+e+6}\psi_{s+e+7}\psi_{s+e+6}\psi \begin{smallmatrix} s+e+5 \\ \downarrow \\ s+e+4 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+4 & s+2e+5 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+5 & s+e+6 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e \\ \downarrow \\ s+e+4 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+1 & s+2e+2 & s+2e+3 \\ \downarrow & \downarrow & \downarrow \\ s+e+6 & s+e+7 & s+e+8 \end{smallmatrix} (\psi_{s+e+5}^2(3,1))\psi_{s+e+4}\psi \begin{smallmatrix} s+2e+4 & s+2e+5 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+5 & s+e+6 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e & s+2e+1 & s+2e+2 & s+2e+3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ s+e+5 & s+e+6 & s+e+7 & s+e+8 \end{smallmatrix} \psi_{s+e+4}v(1,\ldots,s+e+3,s+e+5,s+2e+5,\ldots,je+e)$$

$$= 0 \quad \text{by induction if } e = 4.$$

If $e > 4$, then we have

$$\psi \begin{smallmatrix} s+2e & s+2e+1 & s+2e+2 & s+2e+3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ s+e+5 & s+e+6 & s+e+7 & s+e+8 \end{smallmatrix} \psi_{s+e+4}^2(3,0)\psi \begin{smallmatrix} s+2e+4 & s+2e+5 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+5 & s+e+6 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e & s+2e+1 & s+2e+2 \\ \downarrow & \downarrow & \downarrow \\ s+e+5 & s+e+6 & s+e+7 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+3 & s+2e+4 \\ \downarrow & \downarrow \\ s+e+9 & s+e+10 \end{smallmatrix} \psi_{s+e+8}\psi_{s+e+9}\psi_{s+e+8}\psi \begin{smallmatrix} s+e+7 \\ \downarrow \\ s+e+5 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+5 & s+2e+6 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+6 & s+e+7 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e & s+2e+1 \\ \downarrow & \downarrow \\ s+e+5 & s+e+6 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+2 & s+2e+3 & s+2e+4 \\ \downarrow & \downarrow & \downarrow \\ s+e+8 & s+e+9 & s+e+10 \end{smallmatrix} (\psi_{s+e+7}^2(4,2))\psi \begin{smallmatrix} s+e+6 \\ \downarrow \\ s+e+5 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+5 & s+2e+6 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+6 & s+e+7 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e \\ \downarrow \\ s+e+7 \end{smallmatrix} \psi \begin{smallmatrix} s+2e+1 & s+2e+2 & s+2e+3 & s+2e+4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ s+e+7 & s+e+8 & s+e+9 & s+e+10 \end{smallmatrix} (\psi_{s+e+6}^2(4,1))\psi_{s+e+5}\psi \begin{smallmatrix} s+2e+5 & s+2e+6 & & je+e-1 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+6 & s+e+7 & & je \end{smallmatrix} z_\lambda$$

$$= \psi \begin{smallmatrix} s+2e & s+2e+1 & & s+2e+4 \\ \downarrow & \downarrow & \cdots & \downarrow \\ s+e+6 & s+e+7 & & s+e+10 \end{smallmatrix} \psi_{s+e+5}v(1,\ldots,s+e+4,s+e+6,s+2e+6,\ldots,je+e)$$

$$= 0 \quad \text{by induction if } e = 5.$$

If $e > 5$, we continue in this way until we obtain

$$\psi_{\underset{s+2e+1}{\overset{s+2e}{\downarrow}}\underset{s+2e+2}{\overset{s+2e+1}{\downarrow}}\cdots\underset{s+3e}{\overset{s+3e-1}{\downarrow}}\psi_{s+2e}v(1,\ldots,s+2e-1,s+2e+1,s+3e+1,\ldots,je+e)=0$$

by induction.

We now suppose that $r > s+e$, and again we will use induction on $\ell(w_T)$, so we may assume without loss of generality that $v_T = v(1,\ldots,s,r-e+1,r-e+2,\ldots,r-1,r+1,r+e+1,r+e+2,\ldots,je+r-s)$.

For the base case, we assume that $s+e = je$ and hence $r \equiv s \equiv 0 \pmod{e}$. Applying Corollary 6.11, we have

$$\psi_r v_T = \psi_r v(1,\ldots,je-e,r-e+1,r-e+2,\ldots,r-1,r+1)$$

$$= \psi_r \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-1}{\overset{r-2}{\downarrow}}\psi_{\overset{r}{\underset{je}{\downarrow}}}z_\lambda$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-1}{\overset{r-2}{\downarrow}}(\psi_r^2(-1,0))\psi_{\overset{r-1}{\underset{je}{\downarrow}}}z_\lambda$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-1}{\overset{r-2}{\downarrow}}(y_r-y_{r+1})\psi_{\overset{r-1}{\underset{je}{\downarrow}}}z_\lambda$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-1}{\overset{r-2}{\downarrow}}y_r\psi_{\overset{r-1}{\underset{je}{\downarrow}}}z_\lambda-0$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-1}{\overset{r-2}{\downarrow}}(y_r\psi_{r-1}(-1,-1))\psi_{\overset{r-2}{\underset{je}{\downarrow}}}z_\lambda$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-1}{\overset{r-2}{\downarrow}}(\psi_{r-1}y_{r-1}+1)\psi_{\overset{r-2}{\underset{je}{\downarrow}}}z_\lambda$$

$$= 0+\psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-1}{\overset{r-2}{\downarrow}}\psi_{\overset{r-2}{\underset{je}{\downarrow}}}z_\lambda \quad \text{by the inductive hypothesis of part (i)}$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-2}{\overset{r-3}{\downarrow}}\psi_{r-2}\psi_{\overset{r-2}{\underset{r-3}{\uparrow}}}\psi_{\underset{je-1}{\overset{r-4}{\downarrow}}\underset{je}{\overset{r-3}{\downarrow}}}z_\lambda$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-3}{\overset{r-4}{\downarrow}}\psi_{\overset{r-3}{\underset{r-4}{\downarrow}}}\psi_{\overset{r-3}{\underset{r-5}{\uparrow}}}\psi_{\underset{je-2}{\overset{r-6}{\downarrow}}\underset{je-1}{\overset{r-5}{\downarrow}}}\psi_{\overset{r-4}{\underset{je}{\downarrow}}}z_\lambda$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}\underset{je-e+2}{\overset{r-e+1}{\downarrow}}\cdots\underset{je-4}{\overset{r-5}{\downarrow}}\psi_{\overset{r-4}{\underset{r-6}{\downarrow}}}\psi_{\overset{r-4}{\underset{r-7}{\uparrow}}}\psi_{\underset{je-3}{\overset{r-8}{\downarrow}}\underset{je-2}{\overset{r-7}{\downarrow}}\underset{je-1}{\overset{r-6}{\downarrow}}}\psi_{\overset{r-5}{\underset{je}{\downarrow}}}z_\lambda$$

$$\vdots$$

$$= \psi_{\underset{je-e+1}{\overset{r-e}{\downarrow}}}\psi_{\overset{r-e+1}{\underset{r-2e+4}{\downarrow}}}\psi_{\overset{r-e+1}{\underset{r-2e+3}{\uparrow}}}\psi_{\underset{je-e+2}{\overset{r-2e+2}{\downarrow}}\underset{je-e+3}{\overset{r-2e+3}{\downarrow}}}\cdots\underset{je-1}{\overset{r-e-1}{\downarrow}}\psi_{\overset{r-e}{\underset{je}{\downarrow}}}z_\lambda$$

$$= \psi_{r-e}\psi_{\overset{r-e-1}{\underset{r-2e+2}{\downarrow}}}\psi_{\overset{r-e}{\underset{r-2e+1}{\uparrow}}}\psi_{\underset{je-e+1}{\overset{r-2e}{\downarrow}}\underset{je-e+2}{\overset{r-2e+1}{\downarrow}}}\cdots\underset{je-1}{\overset{r-e-2}{\downarrow}}\psi_{\overset{r-e}{\underset{je}{\downarrow}}}z_\lambda$$

$$= \psi_{\underset{je-e+1}{\overset{r-2e}{\downarrow}}\underset{je-e+2}{\overset{r-2e+1}{\downarrow}}\cdots\underset{je-1}{\overset{r-e-2}{\downarrow}}(\psi_{r-e}^2(-1,0))\psi_{\overset{r-e-1}{\underset{je}{\downarrow}}}z_\lambda.$$

We repeat the above process $s-j-1$ more times, until we reach

$$(\psi_{je}^2(-1,0))z_\lambda = (y_{je}-y_{je+1})z_\lambda = 0.$$

We now suppose that $s + e < je$. We thus have

$$
(\dagger) \qquad \psi_r v_{\mathrm{T}} = \psi_r \psi^{r-e\ r-e+1}_{\ s+1\ s+2} \cdots^{r-2}_{s+e-1} \psi^{r}_{s+e} \psi \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda
$$

$$
= \psi^{r-e\ r-e+1}_{\ s+1\ s+2} \cdots^{r-2}_{s+e-1} (\psi^2_r(s-1,s)) \psi^{r-1}_{s+e} \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda
$$

$$
= \psi^{r-e\ r-e+1}_{\ s+1\ s+2} \cdots^{r-2}_{s+e-1} (y_r - y_{r+1}) \psi^{r-1}_{s+e} \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda.
$$

Applying the inductive hypothesis of part (i), the first term of the last line of ($\dagger$) becomes

$$
\psi^{r-e\ r-e+1}_{\ s+1\ s+2} \cdots^{r-2}_{s+e-1} (y_r \psi_{r-1}(s-1, s-1)) \psi^{r-2}_{s+e} \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda
$$

$$
= \psi^{r-e\ r-e+1}_{\ s+1\ s+2} \cdots^{r-2}_{s+e-1} (\psi_{r-1} y_{r-1} + 1) \psi^{r-2}_{s+e} \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda
$$

$$
= 0 + \psi^{r-e\ r-e+1}_{\ s+1\ s+2} \cdots^{r-3}_{s+e-2} \psi_{r-2}\cancel{\psi_{r-3}\psi_{r-2}} \psi^{r-4}_{s+e-1} \psi^{r-3}_{s+e} \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda
$$

$$
= \psi^{r-e\ r-e+1}_{\ s+1\ s+2} \cdots^{r-4}_{s+e-3} \psi_{r-3}\cancel{\psi_{r-4}\psi_{r-5}\psi_{r-4}} \psi^{r-6}_{s+e-2}{}^{r-5}_{s+e-1} \psi^{r-3}_{s+e} \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda
$$

$$
= \psi^{r-e\ r-e+1}_{r-e-2\ r-e-1} \cdots^{r-4}_{r-6} \psi_{r-3} \psi^{r-e-3\ r-e-2}_{\ s+1\ s+2} \cdots^{r-5}_{s+e-1} \psi^{r-3}_{s+e} \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda
$$

$$
= \psi^{r-e\ r-e+1}_{r-e-2\ r-e-1} \cdots^{r-4}_{r-6} \psi_{r-3}
$$

$$
\cdot v(1, \ldots, s, r-e-2, r-e-1, \ldots, r-4, r-2, r+e+1, \ldots, je+r-s)
$$

$$
= 0 \quad \text{by induction on } r - s \text{ if } e = 3.
$$

If $e > 3$, then by applying [Corollary 6.11](), we have

$$
(\psi^{r-e\ r-e+1}_{r-e-2\ r-e-1} \cdots^{r-4}_{r-6}) \psi^{r-e-3\ r-e-2}_{\ s+1\ s+2} \cdots^{r-5}_{s+e-1} (\cancel{\psi^2_{r-3}(s-1, s-3)}) \psi^{r-4}_{s+e} \psi^{r+e\ r+e+1}_{s+e+1\ s+e+2} \cdots^{je+r-s-1}_{je} z_\lambda
$$

$$
= (\psi^{r-e\ r-e+1}_{r-e-2\ r-e-1} \cdots^{r-5}_{r-7}) \psi^{r-e-3\ r-e-2}_{\ s+1\ s+2} \cdots^{r-8}_{s+e-4} \psi_{r-4}
$$

$$
\cdot \psi^{r-5}_{r-6} v(1, \ldots, s+e-4, r-6, r-5, r-4, r-3, r+e+1, \ldots, je+r-s)
$$

$$
= (\psi^{r-e\ r-e+1}_{r-e-2\ r-e-1} \cdots^{r-5}_{r-7}) \psi^{r-e-3\ r-e-2}_{\ s+1\ s+2} \cdots^{r-8}_{s+e-4}
$$

$$
\cdot \psi_{r-4} v(1, \ldots, s+e-4, r-7, r-6, r-5, r-3, r+e+1, \ldots, je+r-s)
$$

$$
= 0 \quad \text{by induction if } e = 4.
$$

As in parts (ii) and (i), we continue in this fashion for $e > 4$, until we eventually obtain

$$
\psi_{r-e} v(1, \ldots, s, r-2e+1, r-2e+3, \ldots, r-e-1, r-e+1, r+e+1, \ldots, je+r-s) = 0
$$

by induction. Applying the inductive hypothesis of part (ii), the second term of the last line of ($\dagger$) becomes

$$-\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}_{}\ldots\ \psi^{\substack{r-1\\ \downarrow\\ s+e}}\psi^{\substack{r+e\\ \downarrow\\ r+2}}(y_{r+1}\psi_{r+1}(s,s))\psi^{\substack{r\\ \downarrow\\ s+e+1}}\psi^{\substack{r+e+1\ r+e+2\\ \downarrow\ \downarrow\\ s+e+2\ s+e+3}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=-\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}_{}\ldots\ \psi^{\substack{r-1\\ \downarrow\\ s+e}}\psi^{\substack{r+e\\ \downarrow\\ r+2}}(\psi_{r+1}y_{r+2}-1)\psi^{\substack{r\\ \downarrow\\ s+e+1}}\psi^{\substack{r+e+1\ r+e+2\\ \downarrow\ \downarrow\\ s+e+2\ s+e+3}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=-\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}_{}\ldots\ \psi^{\substack{r-1\\ \downarrow\\ s+e}}\psi^{\substack{r+e\\ \downarrow\\ s+e+1}}\psi^{\substack{r+e+1\\ \downarrow\\ r+3}}\,y_{r+2}v(1,\ldots,s+e+1,r+3,r+e+3,\ldots,je+r-s)$$

$$+\psi^{\substack{r+e\\ \downarrow\\ r+3}}\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}\ldots\ \psi^{\substack{r\\ \downarrow\\ s+e+1}}\psi^{\substack{r+e+1\\ \downarrow\\ r+4}}\,\cancel{\psi_{r+2}\psi_{r+3}}\psi_{r+2}\psi^{\substack{r+1\\ \downarrow\\ s+e+2}}\psi^{\substack{r+e+2\ r+e+3\\ \downarrow\ \downarrow\\ s+e+3\ s+e+4}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=0+\psi^{\substack{r+e\ r+e+1\\ \downarrow\ \downarrow\\ r+3\ r+4}}\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}\ldots\ \psi^{\substack{r+1\\ \downarrow\\ s+e+2}}\psi^{\substack{r+e+2\ r+e+3\\ \downarrow\ \downarrow\\ s+e+3\ s+e+4}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=\psi^{\substack{r+e\\ \downarrow\\ r+3}}\psi^{\substack{r+e+1\\ \downarrow\\ r+5}}\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}\ldots\ \psi^{\substack{r+1\\ \downarrow\\ s+e+2}}\psi^{\substack{r+e+2\\ \downarrow\\ r+6}}\,\cancel{\psi_{r+4}\psi_{r+5}}\psi_{r+4}\psi^{\substack{r+3\\ \downarrow\\ s+e+3}}\psi^{\substack{r+e+3\ r+e+4\\ \downarrow\ \downarrow\\ s+e+4\ s+e+5}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=\psi^{\substack{r+e\ r+e+1\ r+e+2\\ \downarrow\ \downarrow\ \downarrow\\ r+4\ r+5\ r+6}}\psi^{\substack{r-e\ r-e+1\ r-e+2\\ \downarrow\ \downarrow\ \downarrow\\ s+1\ s+2\ s+3}}$$

$$\cdot\psi_{r+3}v(1,\ldots,s+3,r-e+4,r-e+5,\ldots,r+2,r+4,r+e+4,\ldots,je+r-s)$$

$$=0\quad\text{by induction if }e=3.$$

If $e>3$, then we have

$$\psi^{\substack{r+e\ r+e+1\ r+e+2\\ \downarrow\ \downarrow\ \downarrow\\ r+4\ r+5\ r+6}}\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}\ldots\ \psi^{\substack{r+1\\ \downarrow\\ s+e+2}}(\psi^2_{r+3}(s+2,s))\psi^{\substack{r+2\\ \downarrow\\ s+e+3}}\psi^{\substack{r+e+3\ r+e+4\\ \downarrow\ \downarrow\\ s+e+4\ s+e+5}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=\psi^{\substack{r+e\ r+e+1\\ \downarrow\ \downarrow\\ r+4\ r+5}}\psi^{\substack{r+e+2\ r+e+3\\ \downarrow\ \downarrow\\ r+7\ r+8}}\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}\ldots\ \psi^{\substack{r+2\\ \downarrow\\ s+e+3}}\cancel{\psi_{r+6}\psi_{r+7}}\psi_{r+6}\psi^{\substack{r+5\\ \downarrow\\ s+e+4}}\psi^{\substack{r+e+4\ r+e+5\\ \downarrow\ \downarrow\\ s+e+5\ s+e+6}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=\psi^{\substack{r+e\\ \downarrow\\ r+4}}\psi^{\substack{r+e+1\ r+e+2\ r+e+3\\ \downarrow\ \downarrow\ \downarrow\\ r+6\ r+7\ r+8}}\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}\ldots\ \psi^{\substack{r+2\\ \downarrow\\ s+e+3}}(\psi^2_{r+5}(s+3,s+1))$$

$$\cdot\psi^{\substack{r+4\\ \downarrow\\ s+e+4}}\psi^{\substack{r+e+4\ r+e+5\\ \downarrow\ \downarrow\\ s+e+5\ s+e+6}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=\psi^{\substack{r+e\ r+e+1\\ \downarrow\ \downarrow\\ r+5\ r+6}}\psi^{\substack{r+e+2\ r+e+3\\ \downarrow\ \downarrow\\ r+7\ r+8}}\psi^{\substack{r-e\ r-e+1\ r-e+2\ r-e+3\\ \downarrow\ \downarrow\ \downarrow\ \downarrow\\ s+1\ s+2\ s+3\ s+4}}\psi_{r+4}$$

$$\cdot v(1,\ldots,s+4,r-e+5,r-e+6,\ldots,r+3,r+5,r+e+5,r+e+6,\ldots,je+r-s)$$

$$=0\quad\text{by induction if }e=4.$$

We continue for $e>4$ in a similar manner until we reach

$$\psi_{r+e}\psi^{\substack{r-e\ r-e+1\\ \downarrow\ \downarrow\\ s+1\ s+2}}\ldots\ ^{\substack{r+e-2\\ \downarrow\\ s+2e-1}}\psi^{\substack{r+e\\ \downarrow\\ s+2e}}\psi^{\substack{r+2e\ r+2e+1\\ \downarrow\ \downarrow\\ s+2e+1\ s+2e+2}}\ldots\ ^{\substack{je+r-s-1\\ \downarrow\\ je}}z_\lambda$$

$$=\psi_{r+e}v(1,\ldots,s,r-e+1,r-e+2,\ldots,r+e-1,$$

$$r+e+1,r+2e+1,r+2e+2,\ldots,je+r-s)$$

$$=0\quad\text{by induction.}\qquad\square$$

Proof of Proposition 4.12. (i) This in fact follows just like the proof of Proposition 3.5, with indices shifted by the corresponding multiples of e. In fact, that proof gives that

$$\psi_{(r+1)e-1}\psi_{(r+1)e-2}\ldots\psi_{re}\Psi_r v=-2\psi_{(r+1)e-1}\psi_{(r+1)e-2}\ldots\psi_{re}v,$$

since there we allow each component to be an arbitrary (small) hook, not just the trivial partition (e). If we follow the proof, setting $b=0$, it may be considerably

shortened and in fact the prefix of generators $\psi_{(r+1)e-1}\psi_{(r+1)e-2}\dots\psi_{re+1}$ is not needed at all—the special case $b = 0$ of that proof ends on page 694.

(ii) Without loss of generality, we will assume that $\Psi_{r+1}\Psi_r v$ is reduced, i.e., v is a linear combination of basis vectors indexed by standard tableaux that have brick r in the second component and bricks $r + 1$ and $r + 2$ in the first. One can show that if brick r is in the first component, the calculation of $\Psi_r v_T$ reduces to applying part (i) of the proposition to basis vectors of the assumed form. If brick r is in the second component but bricks $r + 1$ and $r + 2$ are not both in the first component, then the calculation of $\Psi_{r+1}\Psi_r v_T$ reduces to applying part (iii) of the proposition to basis vectors of the assumed form.

By repeatedly applying Corollary 6.9(i), we have

$$\psi_{re}\Psi_{r+1}\Psi_r v$$

$$= \psi_{re}\left(\psi_{\substack{re+e \, re+e+1 \quad re+2e-1 \\ \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\ re+1 \, re+2 \qquad re+e}}\right)\left(\psi_{\substack{re \quad re+1 \qquad re+e-1 \\ \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\ re-e+1 \, re-e+2 \quad re}}\right)v$$

$$= \psi_{re}\psi_{\substack{re+e \, re+e+1 \quad re+2e-1 \\ \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\ re-e+1 \, re-e+2 \qquad re}}v$$

$$= \psi_{\substack{re+e \\ \downarrow \\ re+2}}\psi_{re}\psi_{re+1}\psi_{re}\psi_{\substack{re-1 \\ \downarrow \\ re-e+1}}\psi_{\substack{re+e+1 \, re+e+2 \quad re+2e-1 \\ \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\ re-e+2 \, re-e+3 \qquad re}}v$$

$$= \psi_{\substack{re+e \\ \downarrow \\ re+3}}\psi_{\substack{re-1 \\ \downarrow \\ re-e+1}}\psi_{\substack{re+e+1 \\ \downarrow \\ re+4}}\psi_{re+2}\psi_{re+3}\psi_{re+2}\psi_{\substack{re+1 \\ \downarrow \\ re-e+2}}\psi_{\substack{re+e+2 \, re+e+3 \quad re+2e-1 \\ \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\ re-e+3 \, re-e+4 \qquad re}}v$$

$$\vdots$$

$$= \left(\psi_{\substack{re+e \\ \downarrow \\ re+3}}\psi_{\substack{re-1 \\ \downarrow \\ re-e+1}}\right)\left(\psi_{\substack{re+e+1 \\ \downarrow \\ re+5}}\psi_{\substack{re+1 \\ \downarrow \\ re-e+2}}\right)\left(\psi_{\substack{re+e+2 \\ \downarrow \\ re+7}}\psi_{\substack{re+3 \\ \downarrow \\ re-e+3}}\right)\dots\left(\psi_{\substack{re+2e-4 \\ \downarrow \\ re+2e-5}}\psi_{\substack{re+2e-9 \\ \downarrow \\ re-3}}\right)$$

$$\cdot\left(\psi_{re+2e-3}\psi_{\substack{re+2e-7 \\ \downarrow \\ re-2}}\right)\left(\psi_{\substack{re+2e-5 \\ \downarrow \\ re-1}}\right)\psi_{re+2e-2}\psi_{re+2e-1}\psi_{re+2e-2}\psi_{\substack{re+2e-3 \\ \downarrow \\ re}}v$$

$$= \left(\psi_{\substack{re+e \\ \downarrow \\ re+3}}\psi_{\substack{re+e+1 \\ \downarrow \\ re+5}}\psi_{\substack{re+e+2 \\ \downarrow \\ re+7}}\dots\psi_{\substack{re+2e-4 \\ \downarrow \\ re+2e-5}}\psi_{re+2e-3}\right)\psi_{\substack{re-1 \\ \downarrow \\ re-e+1}}\psi_{\substack{re+1 \\ \downarrow \\ re-e+2}}\dots\psi_{\substack{re+2e-5 \\ \downarrow \\ re-1}}\psi_{\substack{re+2e-3 \\ \downarrow \\ re}}v.$$

If $e = 3$, this becomes $\psi_{3r+3}\psi_{\substack{3r-1 \\ \downarrow \\ 3r-2}}\psi_{\substack{3r+1 \\ \downarrow \\ 3r-1}}\psi_{\substack{3r+3 \\ \downarrow \\ 3r}}v$. However, if $e > 3$, then by applying Lemma 6.10, we have

$$\left(\psi_{\substack{re+e \\ \downarrow \\ re+3}}\psi_{\substack{re+e+1 \\ \downarrow \\ re+5}}\psi_{\substack{re+e+2 \\ \downarrow \\ re+7}}\dots\psi_{\substack{re+2e-4 \\ \downarrow \\ re+2e-5}}\right)\psi_{\substack{re-1 \\ \downarrow \\ re-e+1}}\psi_{\substack{re+1 \\ \downarrow \\ re-e+2}}\dots\psi_{\substack{re+2e-5 \\ \downarrow \\ re-1}}\psi_{re+2e-3}^2\left(\psi_{\substack{re+2e-4 \\ \downarrow \\ re}}v\right)$$

$$= \left(\psi_{\substack{re+e \\ \downarrow \\ re+3}}\psi_{\substack{re+e+1 \\ \downarrow \\ re+5}}\dots\psi_{\substack{re+2e-5 \\ \downarrow \\ re+2e-7}}\right)\psi_{\substack{re-1 \\ \downarrow \\ re-e+1}}\psi_{\substack{re+1 \\ \downarrow \\ re-e+2}}$$

$$\dots\psi_{\substack{re+2e-7 \\ \downarrow \\ re-2}}\psi_{\substack{re+2e-4 \\ \downarrow \\ re+2e-5}}\psi_{\substack{re+2e-4 \\ \uparrow \\ re+2e-5}}\left(\psi_{\substack{re+2e-6 \, re+2e-5 \\ \downarrow \quad \downarrow \\ re-1 \qquad re}}\right)v$$

$$= \left(\psi_{\substack{re+e \\ \downarrow \\ re+3}}\psi_{\substack{re+e+1 \\ \downarrow \\ re+5}}\dots\psi_{\substack{re+2e-6 \\ \downarrow \\ re+2e-9}}\right)\psi_{\substack{re-1 \\ \downarrow \\ re-e+1}}\psi_{\substack{re+1 \\ \downarrow \\ re-e+2}}$$

$$\dots\psi_{\substack{re+2e-9 \\ \downarrow \\ re-3}}\psi_{\substack{re+2e-5 \\ \downarrow \\ re+2e-7}}\psi_{\substack{re+2e-5 \\ \uparrow \\ re+2e-7}}\left(\psi_{\substack{re+2e-8 \, re+2e-7 \, re+2e-6 \\ \downarrow \quad \downarrow \quad \downarrow \\ re-2 \qquad re-1 \qquad re}}\right)v$$

$$\vdots$$

$$= \psi_{re+e}\psi_{\substack{re-1 \\ \downarrow \\ re-e+1}}\psi_{\substack{re+1 \\ \downarrow \\ re-e+2}}\psi_{\substack{re+e-1 \\ \downarrow \\ re+3}}\psi_{\substack{re+e \\ \uparrow \\ re+3}}\left(\psi_{\substack{re+2 \, re+3 \qquad re+e-2 \\ \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\ re-e+3 \, re-e+4 \qquad re-1}}\right)\psi_{\substack{re+e \\ \downarrow \\ re}}v.$$

For $e \geqslant 3$, the last terms become

$$\psi^{re-1}_{re-e+1} \psi^{re+1}_{re-e+2} {}^{re+2}_{re-e+3} \cdots {}^{re+e-2}_{re-1}(\psi^2_{re+e}(-1,0))\psi^{re+e-1}_{re} v$$
$$= \psi^{re-1}_{re-e+1} \psi^{re+1}_{re-e+2} {}^{re+2}_{re-e+3} \cdots {}^{re+e-2}_{re-1}(y_{re+e} - y_{re+e+1})\psi^{re+e-1}_{re} v.$$

We know from Lemma 4.9 that the second term becomes zero, whilst the first term is

$$\psi^{re-1}_{re-e+1} \psi^{re+1}_{re-e+2} {}^{re+2}_{re-e+3} \cdots {}^{re+e-2}_{re-1}(y_{re+e}\psi_{re+e-1}(-1,-1))\psi^{re+e-2}_{re} v$$
$$= \psi^{re-1}_{re-e+1} \psi^{re+1}_{re-e+2} {}^{re+2}_{re-e+3} \cdots {}^{re+e-2}_{re-1}(\psi_{re+e-1}y_{re+e-1} + 1)\psi^{re+e-2}_{re} v.$$

Now the first term is

$$\psi^{re-1}_{re-e+1} \psi^{re+1}_{re-e+2} {}^{re+2}_{re-e+3} \cdots {}^{re+e-2}_{re-1}\psi_{re+e-1}y_{re+e-1}v(1,\ldots,re-1,re+e-1,a_{re+1},\ldots,a_{re})$$
$$= 0 \quad \text{by Lemma 6.14(i) since } re+e-1 \not\equiv 0,1 \pmod{e}.$$

If $e = 3$, then applying Corollary 6.9(ii) to the second term yields

$$\psi^{3r-1}_{3r-2}\psi^{3r+1}_{3r-1}\psi^{3r+1}_{3r}v = \psi^{3r-1}_{3r-2}\psi_{3r+1}\overline{\psi_{3r}\psi_{3r+1}}\psi_{3r-1}\psi_{3r}v$$
$$= \underline{\psi_{3r-1}\psi_{3r-2}}\psi_{3r-1}\psi_{3r}v = \psi_{3r}v.$$

If $e > 3$, then repeatedly applying Corollary 6.11 to the second term yields

$$\psi^{re-1}_{re-e+1} \psi^{re+1}_{re-e+2} {}^{re+2}_{re-e+3} \cdots {}^{re+e-3}_{re-2}\psi_{re+e-2}\psi^{re+e-2}_{re-3}\psi^{re+e-4\,re+e-3}_{re-1\;\;re}v$$
$$= \psi^{re-1}_{re-e+1} \psi^{re+1}_{re-e+2} {}^{re+2}_{re-e+3} \cdots {}^{re+e-4}_{re-3}\psi^{re+e-3}_{re+e-4}\psi^{re+e-3}_{re+e-5}\psi^{re+e-6\,re+e-5\,re+e-4}_{re-2\;\;re-1\;\;re}v$$

$$\vdots$$

$$= \psi^{re-1}_{re-e+1} \psi^{re+1}_{re-e+4}\psi^{re+1}_{re-e+3}\psi^{re}_{re-e+2}v$$
$$= \psi^{re-1}_{re-e+2}\psi^{re-1}_{re-e+1}\psi_{re}v$$
$$= \psi_{re}v.$$

(iii) The proof proceeds analogously to part (ii), and is omitted for the sake of brevity. $\qquad\square$

Acknowledgements

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