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ON THE GLOBAL WELL-POSEDNESS OF ONE-DIMENSIONAL FLUID MODELS WITH NONLOCAL VELOCITY

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In this paper, we consider one-dimensional fluid models with nonlocal velocity, and establish the global well-posedness of classical solutions to such models with large initial data in suitable Sobolev spaces.

1. Introduction and main results

The Euler equations are the classical model for the motion of an ideal incompressible fluid. When it comes to the issue of global existence and regularity of solutions to the three-dimensional (3D) Euler equations, the question of whether solutions of the 3D Euler equations blow up in finite time from smooth data with finite energy is a profound, as of yet unanswered, question, except for the axi-symmetric case (see the recent work by Elgindi and Jeong [2018]). The key reason is due to the stretching term $\omega\mathcal{T}(\omega)$ in the vorticity ω of the 3D Euler equations, namely,

$$\partial_t\omega + (u\cdot\nabla)\omega = \omega\mathcal{T}(\omega),$$

where $\mathcal{T}(\omega)$ is a singular integral operator of ω.

In order to get a better understanding of the nonlinear and nonlocal structure of the 3D Euler equations, considerable one-dimensional (1D) fluid models were proposed and analyzed. These models arise in different contexts to characterize nonlocal and nonlinear behaviors. In order to present these models, we first recall some properties of the 1D Hilbert transform. The 1D Hilbert transform $\mathcal{H}$ is defined by

$$\mathcal{H}f(x) = \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy,$$

where p.v. stands for Cauchy's principal value. For the 1D Hilbert transform, we have the following identities (see, e.g., [Chae et al. 2005; Stein 1970; Pandey

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1996]):

$$\Lambda f = \mathcal{H}\partial_x f = \partial_x(\mathcal{H} f), \tag{1-1}$$

$$\mathcal{H}\big(\partial_x f(\mathcal{H}(\partial_x f))\big) = \tfrac{1}{2}\big((\Lambda f)^2 - (\partial_x f)^2\big), \tag{1-2}$$

where the Zygmund operator $\Lambda := (-\partial_{xx})^{\frac{1}{2}}$ and the more general fractional Laplacian operators $\Lambda^\gamma := (-\partial_{xx})^{\frac{\gamma}{2}}$ with $\gamma \in (0,2)$ are defined by

$$\Lambda^\gamma f(\xi) = C_\gamma \,\text{p.v.}\int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{1+\gamma}}\,dy$$

for a normalized constant $C_\gamma > 0$. For general $\gamma \in \mathbb{R}$, the operator Λ^γ can also be defined through a Fourier transform, namely,

$$\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \hat{f}(\xi).$$

Now we are in the position to review the 1D fluid models. The first model is due to Constantin, Lax and Majda [Constantin et al. 1985]

$$\partial_t w - w\mathcal{H}w = 0. \tag{1-3}$$

In [Constantin et al. 1985], the authors proved that most of the solutions blow-up in finite time (see [Silvestre and Vicol 2016] for four essentially different proofs of this fact, each one based on a different mathematical argument). From the point of view of the existence of a smooth solution, (1-3) can be seen as a simplified model of the 3D Euler equations. Inspired by this work, De Gregorio [1990] proposed and investigated the following model

$$\partial_t w + u\partial_x w - \partial_x u w = 0, \quad \partial_x u = \mathcal{H}w.$$

We remark that at present the question of the smooth solution global existence or finite time blow-up from smooth initial data remains unknown. Later, Córdoba, Córdoba, and Fontelos [Córdoba et al. 2005] investigated the following equation

$$\partial_t w - u\partial_x w = 0, \quad u = \mathcal{H}w, \tag{1-4}$$

where they showed that, for a generic family of initial data, local smooth solutions to Equation (1-4) may blow up in finite time (see [Baker et al. 1996; Morlet 1998] for the local existence of smooth solution). By adding a fractional dissipation term $\Lambda^\gamma w$ to Equation (1-4), namely

$$\partial_t w - u\partial_x w + \Lambda^\gamma w = 0, \quad u = \mathcal{H}w, \tag{1-5}$$

it is shown [Córdoba et al. 2005] that Equation (1-5) is globally well-posed for $\gamma > 1$, as well as in the case $\gamma = 1$ with small initial data. Subsequently, the global well-posedness for the case $\gamma = 1$ with general initial data was obtained by Dong [2008]. Here it is worthwhile to mention that Li and Rodrigo [2008] proved the formation of singularities in finite time of solutions when $0<\gamma<\frac{1}{2}$, by adapting the method developed in [Córdoba et al. 2005]. The inviscid and viscous blowup proofs have been revisited in [Kiselev 2010] by using elementary and elegant methods. However, it is not known if finite time blow up is possible for the case $\frac{1}{2}\le\gamma<1$. Escher and Yin [2008] considered the b-equation in the form

$$\partial_t w+u\partial_x w+b\partial_x u w=0,\quad u=(1-\partial_{xx})^{-1}w.$$

They obtained the global well-posedness for nonnegative initial data for general b or for general initial data with $b\in(0,\ 1]$ or $b=-\frac{1}{2n}$ for each $n\in\mathbb{N}$.

Many other closely related hydrodynamical models with nonlocal velocities have also been proposed and analyzed in the literature. We do not attempt to exhaust the literature. One can refer to [Alibaud et al. 2007; Bae and Granero-Belinchón 2015; Carrillo et al. 2012; Castro and Córdoba 2009; Córdoba et al. 2006; De Gregorio 1996; Do 2014; Do et al. 2016; Dong and Li 2014; Hou and Luo 2013; Granero-Belinchón and Orive-Illera 2014; Kiselev et al. 2007; Lazar 2016; Lazar and Lemarié-Rieusset 2016; Li and Rodrigo 2011; Okamoto et al. 2008] for details.

This paper studies the global in time solvability of the Cauchy problem for the following 1D fluid model with nonlocal velocity

$$\begin{cases}\partial_t w+u\partial_x w+\delta\partial_x u w+\nu\Lambda^\gamma w=0,\\ u=(1-\partial_{xx})^{-\beta}w,\end{cases}\tag{1-6}$$

where $\delta\in\mathbb{R},\ \nu,\gamma,\beta\ge 0$. We make the convention that by $\gamma=0$ we mean that there is no dissipation in the first equation of (1-6). We remark that there is ample physical motivation justifying consideration of the nonlocal Equation (1-6). For example, when $\beta=0$, (1-6) is equivalent to the Burgers' equation with the fractional Laplacian (see, e.g., [Biler et al. 1998; Kiselev et al. 2008]). When $\beta=1,\ \nu=0$ and $\delta=2$, (1-6) reduces to the following inviscid Camassa–Holm equation [Camassa and Holm 1993; Constantin and Escher 1998]

$$\begin{cases}\partial_t w+u\partial_x w+2\partial_x u w=0,\\ w=u-\partial_{xx}u,\end{cases}\tag{1-7}$$

which was originally proposed as a model for the shallow-water waves, and has attracted considerable attention for more than twenty years because of its many remarkable properties, for example conservation laws and complete integrability, existence of peaked solitons and multipeakons, well-posedness and breaking waves [Camassa and Holm 1993; Fuchssteiner and Fokas 1981/82; Alber et al. 1994; Constantin 2000; Constantin and Escher 1998; Li and Olver 2000]. For the system (1-6) with $\delta=2$ and $\beta=1$, it is globally well-posed for $\gamma=2$ [Xin and Zhang 2000], while the wave-breaking phenomena may appear for $\beta=0$ [Constantin and Escher 1998; Constantin 2000]. Moreover, (1-6) is also closely related to the following generalized Proudman–Johnson equation [Okamoto and Zhu 2000; Wunsch 2011] (by denoting $w=\partial_{xx}f$)

$$\partial_{txx}f+f\partial_{xxx}f+\delta\partial_x f\partial_{xx}f=\nu\partial_{xxxx}f.$$

A multidimensional version of (1-6) is the Navier–Stokes-α system given by

$$\begin{cases}\partial_t w+(u\cdot\nabla)w+w\cdot\nabla u^{\mathbb{T}}-\nu\Delta w+\nabla p=0,\\ w=u-\alpha^2\Delta u,\quad \nabla\cdot w=0.\end{cases}\tag{1-8}$$

The above system (1-8) was derived through a variational formulation and a Lagrangian averaging, and also viewed as a filtered Navier–Stokes equations with the parameter α in the filter, which obeys a modified Kelvin circulation theorem along filtered velocities [Holm et al. 1998].

Very recently, the authors of [Bae et al. 2018] showed the global existence of several weak solutions of (1-6) depending on the range of δ, γ and β. Here we remark that Bae, Chae and Okamoto [Bae et al. 2017] established the local well-posedness results, the blow-up criteria and the global well-posedness of (1-6) for smooth initial data. More precisely, the results in [Bae et al. 2017] can be stated as follows.

♦ The model (1-6) with $\delta=0$.

(1) Local well-posedness and blow-up criterion in terms of

$$\int_0^t(\|\partial_x u(s)\|_{\mathrm{BMO}}+\|\partial_x w(s)\|_{\mathrm{BMO}})\,ds.$$

(2) Global well-posedness with $\nu=0$ and $\beta\in\left(\frac{1}{2},1\right]$.

(3) Global well-posedness with $\nu>0$, $\gamma\in(1,2]$ and $\beta\geq\max\left\{\frac{2-\gamma}{4},\frac{4-3\gamma}{4}\right\}$.

♦♦ The model (1-6) with $\delta\neq 0$ and $\beta\geq\frac{1}{2}$.

(4) Local well-posedness and blow-up criterion in terms of

$$\int_0^t(\|w(s)\|_{\mathrm{BMO}}+\|\partial_x w(s)\|_{\mathrm{BMO}})\,ds.$$

(5) Global well-posedness with $\nu>0$, $\gamma\in(1,2]$, $\beta\geq\frac{1}{2}$ and $\delta\in\left(0,\frac{1}{2}\right]$ or $\delta=2$.

In this paper, we also consider the following model

$$\begin{cases}\partial_t w+u\partial_x w+\delta\partial_x uw+\nu\Lambda^\gamma w=0,\\ u=\mathcal{H}w.\end{cases}\tag{1-9}$$

In recent years, the study of the model (1-9) has been attracting many mathematicians, and many results have been achieved, because the model (1-9) can be viewed as a 1D model of the dissipative surface quasigeostrophic equation (see [Constantin et al. 1994]). One particular feature of the model (1-9) with $\delta = 1,\ \nu > 0$ is related with the complex inviscid Burgers equation (see [Castro and Córdoba 2008]). There has been considerable work concerning the mathematical studies on the model (1-9) with $\nu = 0$ or $\nu > 0$, see for example [Baker et al. 1996; Castro and Córdoba 2008; Chae et al. 2005; Córdoba et al. 2005; Li and Rodrigo 2008; Morlet 1998] for the singularity formation results. Recently, Bae, Chae and Okamoto [Bae et al. 2017] established the local well-posedness results, the blow-up criteria and the global well-posedness of (1-9) for smooth initial data.

♣ The model (1-9) with $\delta > 0$ and $\gamma > 1$.

(1) Local well-posedness and blow-up criterion in terms of

$$\int_0^t \big(\|\Lambda^{\frac{1}{2}} w(s)\|_{L^4}^4 + \|w(s)\|_{L^\infty}^{\frac{\gamma}{\gamma-1}} + \|\Lambda w(s)\|_{L^\infty} + \|\partial_x w(s)\|_{L^\infty}\big)\, ds.$$

(2) Global well-posedness with $\nu > 0,\ \gamma \geq \frac{1}{1-\delta}$ with $\delta \in \big(0, \frac{1}{2}\big)$.

♣♣ The model (1-9) with $\delta > 0,\ \gamma = 1$ and nonnegative initial data.

(3) Local well-posedness and blow-up criterion in terms of

$$\int_0^t (\|\Lambda w(s)\|_{\mathrm{BMO}} + \|\partial_x w(s)\|_{\mathrm{BMO}})\, ds.$$

(4) Global well-posedness with $\nu > 0,\ \delta \geq \frac{1}{2}$ and $\gamma = 1$.

The main goal of this paper is to improve the results of [Bae et al. 2017]. More precisely, we consider the viscous equations to see when these equations have global classical solutions depending on the parameters δ, γ and β. Our first result reads as follows.

Theorem 1.1. *If β, γ and δ satisfy one of the following three conditions*:

(1) $\frac{1-\gamma+\delta}{2} < \beta < f(\delta,\gamma)$ *with* $0 < \gamma < 1 + \delta$ *and* $\delta \in \big(0, \frac{1}{2}\big]$;

(2) $\beta \geq 0$ *with* $1 + \delta \leq \gamma \leq 2$ *and* $\delta \in \big(0, \frac{1}{2}\big]$;

(3) $\beta \geq \frac{2-\gamma}{2}$ *with* $0 < \gamma < 1$ *or* $\beta \geq \frac{3-2\gamma}{2}$ *with* $1 \leq \gamma \leq \frac{3}{2}$ *or* $\beta \geq 0$ *with* $\frac{3}{2} \leq \gamma \leq 2$ *for the case* $\delta = 2$,

then for $w_0 \in H^m(\mathbb{R})$ *with* $m \geq 2$, *there exists a unique global solution of* (1-6) *such that for any given* $T > 0$

$$w \in C([0,\ T];\ H^m(\mathbb{R})),$$

where $f(\delta,\ \gamma)$ *is given by*

$$f(\delta,\ \gamma) := \min\left\{\frac{1 + (3\delta - 1)\gamma}{2(1 - 2\delta)},\ \frac{1+\delta}{2},\ \frac{2 + (3\gamma + 2)\delta}{2 + \gamma}\right\}.$$

Remark. In the above theorem, we need β smaller than a complicated explicit function $f(\delta, \gamma)$ given by (2-17). It is a technical assumption. It is commonly believed that the diffusion term is always a good term and the larger the power β is, the better the effects it produces. Therefore, our result strongly suggests that all the cases $\beta > \frac{1-\gamma+\delta}{2}$ should be globally well-posed.

Considering the model (1-9), we have the following two theorems.

Theorem 1.2. *Consider (1-9) with $\gamma = 1$, namely,*

$$\begin{cases} \partial_t w + u\partial_x w + \delta\partial_x u w + \nu\Lambda w = 0, \\ u = \mathcal{H}w. \end{cases} \tag{1-10}$$

Let $\nu = 1$ and $\delta \in \left(0, \frac{1}{2}\right)$. Then for $0 \le w_0 \in H^m(\mathbb{R})$ with $m \ge 2$, there exists a unique global solution of (1-10) satisfying for any given $T > 0$,

$$w \in C([0, T]; H^m(\mathbb{R}))$$

provided that $\left(\frac{1}{2} - \delta\right)\|w_0\|_{L^\infty}$ is sufficiently small.

Theorem 1.3. *Let $\nu = 1$ and $\gamma \ge 1 + \delta$ with $\delta \in \left(0, \frac{1}{2}\right]$. Then for $w_0 \in H^m(\mathbb{R})$ with $m \ge 2$, there exists a unique global solution of (1-9) such that for any given $T > 0$,*

$$w \in C([0, T]; H^m(\mathbb{R})).$$

Our last result concerns the global regularity result of (1-6) with $\delta = 0$.

Theorem 1.4. *Consider (1-6) with $\delta = 0$, namely,*

$$\begin{cases} \partial_t w + u\partial_x w + \nu\Lambda^\gamma w = 0, \\ u = (1 - \partial_{xx})^{-\beta} w. \end{cases} \tag{1-11}$$

Let $\nu = 1$ and $\beta \ge \frac{1-\gamma}{2}$ with $0 \le \gamma \le 1$ or $\beta \ge 0$ with $1 < \gamma \le 2$. Then for $w_0 \in H^m(\mathbb{R})$ with $m \ge 2$, there exists a unique global solution of (1-11) such that for any given $T > 0$,

$$w \in C([0, T]; H^m(\mathbb{R})).$$

Remark. Based on a novel nonlocal weighted inequality, Dong and Li [2014] also obtained a global well-posedness result for $\beta \ge \frac{1-\gamma}{2}$ with $0 < \gamma \le 1$. Quite differently, our argument relies heavily on the Hölder estimate and the differentiability of the drift-diffusion equation. Moreover, in the full supercritical range $0 \le \beta < \frac{1-\gamma}{2}$ with $0 < \gamma \le 1$, the formation of singularities in finite time for a class of smooth initial data was established in [Dong and Li 2014].

Remark. For simplicity, we just consider the initial data in H^m for the integer $m \ge 2$. As a matter of fact, one can also prove the same results for the initial data in H^s with $s > \frac{3}{2}$ although we shall not do it here.

The rest of this paper is divided into four sections and an appendix. Section 2 provides the proof of Theorem 1.1 while Section 3 proves the proof of Theorem 1.2. The proof of Theorem 1.3 is presented in Section 4. Next, Section 5 is devoted to the proof of Theorem 1.4. Finally, the definitions of Besov space and the fractional Gagliardo–Nirenberg inequality are collected in the Appendix.

2. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In what follows, C stands for some real positive constant which may be different in each occurrence and may depend on t, initial data and so on. We shall write $C(\lambda_1, \lambda_2, \dots, \lambda_k)$ as the constant C depends on the quantities $\lambda_1, \lambda_2, \dots, \lambda_k$. We begin with the local well-posedness result and blow-up criterion.

Proposition 2.1. *Let $\delta \in \mathbb{R}$, $\nu > 0$, $\beta > \frac{1-\gamma}{2}$ with $0 < \gamma \le 1$ or $\beta \ge 0$ with $1 < \gamma \le 2$. Then for $w_0 \in H^m(\mathbb{R})$ with $m \ge 2$, there exists $T^* = T^*(\|w_0\|_{H^m})$ such that a unique solution of (1-6) belongs to*

$$w \in C([0, T^*); H^m(\mathbb{R})).$$

Moreover, we have the following blow-up criterion

$$\limsup_{t \to \tilde{T}} \|w(t)\|_{H^m} = \infty \quad \textit{if and only if} \tag{2-1}$$
$$\int_0^{\tilde{T}} \left(\|\partial_x w(s)\|_{L^\infty} + \|\partial_x u(s)\|_{L^\infty}^2 + \|w(s)\|_{L^\infty}^2 + \|w(s)\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}} \right) ds = \infty.$$

Proof of Proposition 2.1. Applying Λ^k on (1-6), taking its L^2 inner product with $\Lambda^k w$, and summing over $k = 0, 1, \dots, m$ yields

$$\begin{aligned} \tfrac{1}{2}\frac{d}{dt}\|w(t)\|_{H^m}^2 + \|\Lambda^{\frac{\gamma}{2}} w\|_{H^m}^2 &= -\sum_{k=0}^m \int_{\mathbb{R}} \Lambda^k (u \partial_x w) \Lambda^k w \, dx - \delta \sum_{k=0}^m \int_{\mathbb{R}} \Lambda^k (\partial_x u w) \Lambda^k w \, dx \\ &:= N_1 + N_2. \end{aligned} \tag{2-2}$$

Now we recall the Kato and Ponce [1988] commutator estimate and the bilinear estimate

$$\|[\Lambda^s, f] g\|_{L^p} \le C\big(\|\partial_x f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\big), \tag{2-3}$$
$$\|\Lambda^s (fg)\|_{L^p} \le C\big(\|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\big), \tag{2-4}$$

with $s > 0,\ 1 < p_2,\ p_3 < \infty,\ 1 < p_1,\ p_4 \le \infty$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

We point out that (2-3) and (2-4) are also true if one replaces Λ by $\mathcal{J}$, where

$$\mathcal{J} := (1 - \partial_{xx})^{\frac{1}{2}}.$$

By (2-3), we arrive at

$$\begin{aligned}
N_1 &= -\sum_{k=0}^{m} \int_{\mathbb{R}} \Lambda^k (u \partial_x w) \Lambda^k w\, dx + \sum_{k=0}^{m} \int_{\mathbb{R}} (u \partial_x \Lambda^k w) \Lambda^k w\, dx \\
&\qquad - \sum_{k=0}^{m} \int_{\mathbb{R}} (u \partial_x \Lambda^k w) \Lambda^k w\, dx \\
&= -\sum_{k=0}^{m} \int_{\mathbb{R}} [\Lambda^k, u] \partial_x w\, \Lambda^k w\, dx + \frac{1}{2} \sum_{k=0}^{m} \int_{\mathbb{R}} \partial_x u \Lambda^k w \Lambda^k w\, dx \\
&\le C \sum_{k=0}^{m} \|[\Lambda^k, u] \partial_x w\|_{L^2} \|\Lambda^k w\|_{L^2} + C \sum_{k=0}^{m} \|\partial_x u\|_{L^\infty} \|\Lambda^k w\|_{L^2}^2 \\
&\le C \sum_{k=0}^{m} (\|\partial_x u\|_{L^\infty} \|\Lambda^k w\|_{L^2} + \|\partial_x w\|_{L^\infty} \|\Lambda^k u\|_{L^2}) \|\Lambda^k w\|_{L^2} \\
&\qquad + C \sum_{k=0}^{m} \|\partial_x u\|_{L^\infty} \|\Lambda^k w\|_{L^2}^2 \\
&\le C (\|\partial_x u\|_{L^\infty} + \|\partial_x w\|_{L^\infty}) \|w\|_{H^m}^2.
\end{aligned} \tag{2-5}$$

Thanks to (2-4), it yields

$$\begin{aligned}
N_2 &= -\delta \sum_{k=1}^{m} \int_{\mathbb{R}} \Lambda^{k-\frac{\gamma}{2}} (\partial_x u w) \Lambda^{k+\frac{\gamma}{2}} w\, dx - \int_{\mathbb{R}} \partial_x u w w\, dx \\
&\le C \sum_{k=1}^{m} \|\Lambda^{k-\frac{\gamma}{2}} (\partial_x u w)\|_{L^2} \|\Lambda^{k+\frac{\gamma}{2}} w\|_{L^2} + C \|\partial_x u\|_{L^\infty} \|w\|_{H^m}^2 \\
&\le C \sum_{k=1}^{m} \Big(\|\partial_x u\|_{L^\infty} \|\Lambda^{k-\frac{\gamma}{2}} w\|_{L^2} \\
&\qquad + \|w\|_{L^\infty} \|\Lambda^{k-\frac{\gamma}{2}} \partial_x u\|_{L^2} \Big) \|\Lambda^{k+\frac{\gamma}{2}} w\|_{L^2} + C \|\partial_x u\|_{L^\infty} \|w\|_{H^m}^2 \\
&\le C \big(\|\partial_x u\|_{L^\infty} \|w\|_{H^{m-\frac{\gamma}{2}}} + \|w\|_{L^\infty} \|w\|_{H^{m+1-2\beta-\frac{\gamma}{2}}} \big) \|\Lambda^{\frac{\gamma}{2}} w\|_{H^m} \\
&\qquad + C \|\partial_x u\|_{L^\infty} \|w\|_{H^m}^2
\end{aligned} \tag{2-6}$$

$$\leq C\Big(\|\partial_x u\|_{L^\infty}\|w\|_{H^m}+\|w\|_{L^\infty}\|w\|_{H^m} \\ +\|w\|_{L^\infty}\|w\|_{H^m}^{\frac{4\beta+2\gamma-2}{\gamma}}\|w\|_{H^{m+\frac{\gamma}{2}}}^{\frac{2-4\beta-\gamma}{\gamma}}\Big)\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}+C\|\partial_x u\|_{L^\infty}\|w\|_{H^m}^2$$
$$\leq C\big(\|\partial_x u\|_{L^\infty}\|w\|_{H^m}+\|w\|_{L^\infty}\|w\|_{H^m}\big)\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m} \\ +\|w\|_{L^\infty}\|w\|_{H^m}^{\frac{4\beta+2\gamma-2}{\gamma}}\big(\|w\|_{L^2}+\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}\big)^{\frac{2-4\beta-\gamma}{\gamma}}\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m} \\ +C\|\partial_x u\|_{L^\infty}\|w\|_{H^m}^2$$
$$\leq \tfrac{1}{2}\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}^2+C\big(1+\|\partial_x u\|_{L^\infty}^2+\|w\|_{L^\infty}^2+\|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big)\|w\|_{H^m}^2,$$

where we have used the following facts (see Lemma A.3)

$$\|w\|_{H^{m+1-2\beta-\frac{\gamma}{2}}}\leq C\|w\|_{H^m}^{\frac{4\beta+2\gamma-2}{\gamma}}\|w\|_{H^{m+\frac{\gamma}{2}}}^{\frac{2-4\beta-\gamma}{\gamma}},\quad \text{if } \frac{1-\gamma}{2}<\beta\leq\frac{2-\gamma}{4};$$
$$\|w\|_{H^{m+1-2\beta-\frac{\gamma}{2}}}\leq C\|w\|_{H^m},\qquad \text{if } \beta>\frac{2-\gamma}{4}.$$

We remark that if $1<\gamma\leq 2$, then the above two interpolation inequalities hold for $\beta\geq 0$. Gathering the upper estimates leads to

$$\frac{d}{dt}\|w(t)\|_{H^m}^2 \\ \leq C\big(1+\|\partial_x w\|_{L^\infty}+\|\partial_x u\|_{L^\infty}^2+\|w\|_{L^\infty}^2+\|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big)\|w\|_{H^m}^2. \tag{2-7}$$

By the embedding, it is easy to show

$$\frac{d}{dt}\|w(t)\|_{H^m}^2\leq C\|w\|_{H^m}^{2+\eta}$$

with some constant $\eta>0$. By the simple ODE theorem, we have

$$\|w(t)\|_{H^m}\leq\frac{\|w_0\|_{H^m}^{\eta}}{1-C\eta\|w_0\|_{H^m}^{\eta}},\quad \text{for all } 0\leq t<\frac{1}{C\eta\|w_0\|_{H^m}^{\eta}}.$$

In the functional space $H^m(\mathbb{R})$ with $m\geq 2$, the uniqueness is easy to obtain. Actually, let w_1 and w_2 be two solutions of (1-6), and let $\bar{w}:=w_1-w_2$ and $\bar{u}:=u_1-u_2$. Then, $(\bar{w},\bar{u})$ satisfies

$$\begin{cases}\partial_t\bar{w}+u_1\partial_x\bar{w}+\bar{u}\partial_x w_2+\delta\partial_x u_1\bar{w}+\delta\partial_x\bar{u}w_2+\nu\Lambda^\gamma\bar{w}=0,\\ \bar{u}=(1-\partial_{xx})^{-\beta}\bar{w}.\end{cases}$$

Taking the L^2 product of the equation with $\bar{w}$ and using the above argument yields

$$\tfrac{1}{2}\frac{d}{dt}\|\bar{w}(t)\|_{L^2}^2+\nu\|\Lambda^{\frac{\gamma}{2}}\bar{w}\|_{L^2}^2$$
$$=-\int_{\mathbb{R}}u_1\partial_x\bar{w}\,\bar{w}\,dx-\int_{\mathbb{R}}\bar{u}\partial_x w_2\bar{w}\,dx-\delta\int_{\mathbb{R}}\partial_x u_1\bar{w}\bar{w}\,dx-\delta\int_{\mathbb{R}}\partial_x\bar{u}w_2\bar{w}\,dx$$

$$\le \frac{\nu}{2}\|\Lambda^{\frac{\gamma}{2}}\bar{w}\|_{L^2}^2 + C\big(\|\partial_x u_1\|_{L^\infty} + \|\partial_x w_2\|_{L^\infty} + \|w_2\|_{H^m}^{C(\beta,\gamma)}\big)\|\bar{w}\|_{L^2}^2$$
$$\le \frac{\nu}{2}\|\Lambda^{\frac{\gamma}{2}}\bar{w}\|_{L^2}^2 + C\big(1 + \|w_1\|_{H^m}^{C(\beta,\gamma)} + \|w_2\|_{H^m}^{C(\beta,\gamma)}\big)\|\bar{w}\|_{L^2}^2,$$

where $C(\beta,\gamma) \ge 1$. Let us say some words about the estimate of the term $-\delta\int_{\mathbb{R}}\partial_x\bar{u}w_2\bar{w}\,dx$. Actually, it can be bounded by

$$-\delta\int_{\mathbb{R}}\partial_x\bar{u}w_2\bar{w}\,dx \le C\|\partial_x^{\frac{\gamma}{2}}(w_2\bar{w})\|_{L^2}\|\partial_x^{1-\frac{\gamma}{2}}\bar{u}\|_{L^2}$$
$$\le C\big(\|w_2\|_{L^\infty}\|\partial_x^{\frac{\gamma}{2}}\bar{w}\|_{L^2} + \|\bar{w}\|_{L^p}\|\partial_x^{\frac{\gamma}{2}}w_2\|_{L^{\frac{2p}{p-2}}}\big)\|\mathcal{J}^{1-\frac{\gamma}{2}-2\beta}\bar{w}\|_{L^2}$$
$$\le C\big(\|w_2\|_{L^\infty}\|\Lambda^{\frac{\gamma}{2}}\bar{w}\|_{L^2} + \|\bar{w}\|_{L^2}^{1-\rho_1}\|\Lambda^{\frac{\gamma}{2}}\bar{w}\|_{L^2}^{\rho_1}\|w_2\|_{H^m}\big)$$
$$\times\big(\|\bar{w}\|_{L^2} + \|\bar{w}\|_{L^2}^{1-\rho_2}\|\Lambda^{\frac{\gamma}{2}}\bar{w}\|_{L^2}^{\rho_2}\big)$$
$$\le \frac{\nu}{2}\|\Lambda^{\frac{\gamma}{2}}\bar{w}\|_{H^m}^2 + C(1 + \|w_2\|_{H^m}^{\varphi(\beta,\gamma)})\|\bar{w}\|_{L^2}^2,$$

where we have applied the same argument adopted in proving (2-6). This implies that

$$\frac{d}{dt}\|\bar{w}(t)\|_{L^2}^2 \le C\big(1 + \|w_1\|_{H^m}^{C(\beta,\gamma)} + \|w_2\|_{H^m}^{C(\beta,\gamma)}\big)\|\bar{w}\|_{L^2}^2.$$

Due to $\bar{w}(0) = 0$, the uniqueness follows directly from the Gronwall inequality. To obtain the blow-up criterion, we deduce from (2-7) that

$$\frac{d}{dt}\|w(t)\|_{H^m}^2 \le C\big(1 + \|\partial_x w\|_{L^\infty} + \|\partial_x u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2 + \|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big)\|w\|_{H^m}^2, \tag{2-8}$$

which together with the Gronwall inequality yields

$$\|w(t)\|_{H^m} \le \|w_0\|_{H^m}\exp\bigg[C\int_0^t\big(1 + \|\partial_x w\|_{L^\infty} + \|\partial_x u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2 + \|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big)\,ds\bigg].$$

Consequently, we derive the blow-up criterion expressed in (2-1). The proof of Proposition 2.1 is completed. □

Proof of Theorem 1.1. Now we are ready to prove Theorem 1.1. The proof of Theorem 1.1 is divided into three cases. Let us begin with the first case.

Case 1. $\quad \dfrac{1-\gamma+\delta}{2} < \beta < f(\delta,\gamma), \quad$ with $0 < \gamma < 1+\delta$ and $\delta \in \big(0, \frac{1}{2}\big]$.

The proof of Case 1 of Theorem 1.1 is divided into three steps.

Step 1. This step is to derive the following bound

$$\|w(t)\|_{L^{\frac{1}{\delta}}}^{\frac{1}{\delta}} + \|w(t)\|_{L^2}^2 + \int_0^t \left(\left\| \Lambda^{\frac{\gamma}{2}} |w(s)|^{\frac{1}{2\delta}} \right\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}} w(s)\|_{L^2}^2 \right) ds \leq C(t, w_0). \tag{2-9}$$

As $\delta \in \left(0, \frac{1}{2}\right]$, we have $\frac{1}{\delta} \in [2, \infty)$. Now multiplying (1-6) by $|w|^{\frac{1}{\delta}-2} w$ and integrating it over the whole space, it yields

$$\delta \frac{d}{dt} \|w(t)\|_{L^{\frac{1}{\delta}}}^{\frac{1}{\delta}} + \int_{\mathbb{R}} \Lambda^{\gamma} w (|w|^{\frac{1}{\delta}-2} w) \, dx = - \int_{\mathbb{R}} (u \partial_x w) |w|^{\frac{1}{\delta}-2} w \, dx - \delta \int_{\mathbb{R}} (\partial_x u w) |w|^{\frac{1}{\delta}-2} w \, dx.$$

Turning now to the lower bound (see for example [Ju 2005]), we compute

$$\int_{\mathbb{R}} \Lambda^{\gamma} w (|w|^{\frac{1}{\delta}-2} w) \, dx \geq \widetilde{C} \int_{\mathbb{R}} (\Lambda^{\frac{\gamma}{2}} |w|^{\frac{1}{2\delta}})^2 \, dx = \widetilde{C} \left\| \Lambda^{\frac{\gamma}{2}} |w|^{\frac{1}{2\delta}} \right\|_{L^2}^2.$$

Direct computation implies

$$\begin{aligned} - \int_{\mathbb{R}} (u \partial_x w) |w|^{\frac{1}{\delta}-2} w \, dx - \delta \int_{\mathbb{R}} (\partial_x u w) |w|^{\frac{1}{\delta}-2} w \, dx &= -\delta \int_{\mathbb{R}} u \partial_x (|w|^{\frac{1}{\delta}}) \, dx - \delta \int_{\mathbb{R}} \partial_x u |w|^{\frac{1}{\delta}} \, dx \\ &= 0. \end{aligned}$$

Consequently, we arrive at

$$\frac{d}{dt} \|w(t)\|_{L^{\frac{1}{\delta}}}^{\frac{1}{\delta}} + \left\| \Lambda^{\frac{\gamma}{2}} |w|^{\frac{1}{2\delta}} \right\|_{L^2}^2 \leq 0.$$

Integrating the above differential equation yields

$$\|w(t)\|_{L^{\frac{1}{\delta}}}^{\frac{1}{\delta}} + \int_0^t \left\| \Lambda^{\frac{\gamma}{2}} |w(s)|^{\frac{1}{2\delta}} \right\|_{L^2}^2 ds \leq \|w_0\|_{L^{\frac{1}{\delta}}}^{\frac{1}{\delta}}. \tag{2-10}$$

Next we multiply (1-6) by w and integrate it over the whole space to get

$$\begin{aligned} \tfrac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}} w\|_{L^2}^2 &= - \int_{\mathbb{R}} (u \partial_x w) w \, dx - \delta \int_{\mathbb{R}} (\partial_x u w) w \, dx \\ &= \left(\tfrac{1}{2} - \delta\right) \int_{\mathbb{R}} (\partial_x u w) w \, dx. \end{aligned} \tag{2-11}$$

We remark that if $\delta = \frac{1}{2}$, then the term at the right hand side of (2-11) is absent. We thus focus on $\delta \in \left(0, \frac{1}{2}\right)$. For $\delta \in \left(0, \frac{1}{2}\right)$ and $\beta > \frac{1-\gamma+\delta}{2}$, it admits the following bound by (2-4):

$$
\begin{aligned}
\left(\tfrac{1}{2}-\delta\right)\int_{\mathbb{R}}(\partial_x u w) w\,dx
&\le C\|\mathcal{J}^{1-\frac{\gamma}{2}}u\|_{L^{\frac{2}{1-2\delta}}}\|\mathcal{J}^{\frac{\gamma}{2}}(ww)\|_{L^{\frac{2}{1+2\delta}}}\\
&\le C\|w\|_{H^{1-2\beta-\frac{\gamma}{2}+\delta}}\|w\|_{L^{\frac{1}{\delta}}}\|\mathcal{J}^{\frac{\gamma}{2}}w\|_{L^2}\\
&\le C\|w\|_{L^2}^{1-\vartheta}\|w\|_{H^{\frac{\gamma}{2}}}^{\vartheta}\|w\|_{L^{\frac{1}{\delta}}}\|\mathcal{J}^{\frac{\gamma}{2}}w\|_{L^2}\\
&\le C\big(\|w\|_{L^2}+\|w\|_{L^2}^{1-\vartheta}\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^{\vartheta}\big)\|w\|_{L^{\frac{1}{\delta}}}\|\mathcal{J}^{\frac{\gamma}{2}}w\|_{L^2}\\
&\le C\big(\|w\|_{L^2}+\|w\|_{L^2}^{1-\vartheta}\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^{\vartheta}\big)\|w\|_{L^{\frac{1}{\delta}}}(\|w\|_{L^2}+\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2})\\
&\le \frac{1}{4}\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^2+C(1+\|w\|_{L^{\frac{1}{\delta}}}^{\frac{2}{1-\vartheta}})\|w\|_{L^2}^2,
\end{aligned}
$$

where we have applied the following fact:

$$
\begin{aligned}
&\|w\|_{H^{1-2\beta-\frac{\gamma}{2}+\delta}}\le C\|w\|_{L^2}^{1-\vartheta}\|w\|_{H^{\frac{\gamma}{2}}}^{\vartheta}, && \tfrac{1-\gamma+\delta}{2}<\beta<\tfrac{2-\gamma+2\delta}{4},\\
&\|w\|_{H^{1-2\beta-\frac{\gamma}{2}+\delta}}\le C\|w\|_{L^2}, && \beta\ge\tfrac{2-\gamma+2\delta}{4}.
\end{aligned}
$$

As a result, we get

$$
\frac{d}{dt}\|w(t)\|_{L^2}^2+\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^2\le C(1+\|w\|_{L^{\frac{1}{\delta}}}^{\frac{2}{1-\vartheta}})\|w\|_{L^2}^2.
$$

We deduce from (2-10) and the Gronwall inequality that

$$
\|w(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{\gamma}{2}}w(s)\|_{L^2}^2\,ds\le C(t,w_0). \tag{2-12}
$$

This concludes the desired bound (2-9).

Step 2. This step is to establish the following bound:

$$
\|\partial_x w(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{\gamma}{2}}\partial_x w(s)\|_{L^2}^2\,ds\le C(t,w_0). \tag{2-13}
$$

Applying ∂_x on (1-6) yields

$$
\partial_t\partial_x w+\nu\Lambda^{\gamma}\partial_x w=-(\delta+1)\partial_x u\partial_x w-u\partial_{xx}w-\delta\partial_{xx}uw. \tag{2-14}
$$

Taking the L^2 inner product of (2-14) with $\partial_x w$ gives

$$
\begin{aligned}
\tfrac{1}{2}\frac{d}{dt}\|\partial_x w(t)\|_{L^2}^2+\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2
&=-(\delta+1)\int_{\mathbb{R}}\partial_x u\partial_x w\partial_x w\,dx-\int_{\mathbb{R}}u\partial_{xx}w\partial_x w\,dx-\delta\int_{\mathbb{R}}\partial_{xx}uw\partial_x w\,dx\\
&=-\delta\int_{\mathbb{R}}\partial_{xx}uw\partial_x w\,dx-(\delta+\tfrac{1}{2})\int_{\mathbb{R}}\partial_x u\partial_x w\partial_x w\,dx:=N_3+N_4.
\end{aligned} \tag{2-15}
$$

For $\delta \in \left(0, \frac{1}{2}\right)$ and $\beta > \frac{1-\gamma+\delta}{2}$, the Gagliardo–Nirenberg inequality gives directly

$$\begin{aligned} N_3 &= -\frac{\delta}{2}\int_{\mathbb{R}} \partial_{xxx}u(ww)\,dx \\ &\le C\|\mathcal{J}^{1-\frac{\gamma}{2}}\partial_x u\|_{L^{\frac{2}{1-2\delta}}}\|\mathcal{J}^{1+\frac{\gamma}{2}}(ww)\|_{L^{\frac{2}{1+2\delta}}} \\ &\le C\|\partial_x w\|_{H^{1-2\beta-\frac{\gamma}{2}+\delta}}\|w\|_{L^{\frac{1}{\delta}}}\|\mathcal{J}^{1+\frac{\gamma}{2}}w\|_{L^2} \\ &\le C\big(\|\partial_x w\|_{L^2}+\|\partial_x w\|_{L^2}^{1-\vartheta}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^{\vartheta}\big)\|w\|_{L^{\frac{1}{\delta}}}\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2} \\ &\le C\big(\|\partial_x w\|_{L^2}+\|\partial_x w\|_{L^2}^{1-\vartheta}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^{\vartheta}\big)\|w\|_{L^{\frac{1}{\delta}}}(\|\partial_x w\|_{L^2}+\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}) \\ &\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2+C(1+\|w\|_{L^{\frac{1}{\delta}}}^{\frac{2}{1-\vartheta}})\|\partial_x w\|_{L^2}^2, \end{aligned}$$

As for $\delta = \frac{1}{2}$, we modify the above estimate as

$$\begin{aligned} N_3 &= -\frac{\delta}{2}\int_{\mathbb{R}} \partial_{xxx}u(ww)\,dx \\ &\le C\|\partial_x u\|_{B^{1-\frac{\gamma}{2}}_{\infty,2}}\|ww\|_{B^{1+\frac{\gamma}{2}}_{1,2}} \\ &\le C\|\partial_x w\|_{H^{1-2\beta-\frac{\gamma}{2}+\delta}}\|w\|_{L^{\frac{1}{\delta}}}\|\mathcal{J}^{1+\frac{\gamma}{2}}w\|_{L^2} \\ &\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2+C(1+\|w\|_{L^{\frac{1}{\delta}}}^{\frac{2}{1-\vartheta}})\|\partial_x w\|_{L^2}^2, \end{aligned}$$

where we have applied the fact (see [Chae 2004, Lemma 2.2]) that

$$\|ww\|_{B^{1+\frac{\gamma}{2}}_{1,2}} \le C\|w\|_{L^2}\|\mathcal{J}^{1+\frac{\gamma}{2}}w\|_{L^2},$$

(see the Appendix for the definition of the Besov space $B^s_{p,r}$). By integrating by parts, the term N_4 can be written as

$$N_4 = \left(\delta+\tfrac{1}{2}\right)\int_{\mathbb{R}} \partial_{xx}uw\partial_x w\,dx + \left(\delta+\tfrac{1}{2}\right)\int_{\mathbb{R}} \partial_x uw\partial_{xx}w\,dx := N_{41}+N_{42}.$$

The term N_{41} admits the same bound as N_3, namely,

$$N_{41} \le \tfrac{1}{8}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2+C(1+\|w\|_{L^{\frac{1}{\delta}}}^{\frac{2}{1-\vartheta}})\|\partial_x w\|_{L^2}^2.$$

By means of (2-4), one thus deduces from the Gagliardo–Nirenberg inequality that

$$\begin{aligned} N_{42} &\le C\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}\|\mathcal{J}^{1-\frac{\gamma}{2}}(\partial_x uw)\|_{L^2} \\ &\le C\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}\big(\|\mathcal{J}^{1-\frac{\gamma}{2}}\partial_x u\|_{L^{\frac{2}{1-2\delta}}}\|w\|_{L^{\frac{1}{\delta}}}+\|\mathcal{J}^{1-\frac{\gamma}{2}}w\|_{L^{p_1}}\|\partial_x u\|_{L^{\frac{2p_1}{p_1-2}}}\big) \end{aligned}$$

$$
\begin{aligned}
&\le C\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}\|\mathcal{J}^{1-\frac{\gamma}{2}}\partial_x u\|_{L^{\frac{2}{1-2\delta}}}\|w\|_{L^{\frac{1}{\delta}}}\\
&\qquad\qquad + C\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}\|\mathcal{J}^{1-\frac{\gamma}{2}}w\|_{L^{p_1}}\|\mathcal{J}^{1-2\beta}w\|_{L^{\frac{2p_1}{p_1-2}}}\\
&\le \tfrac{1}{16}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 + C(1+\|w\|_{L^{\frac{1}{\delta}}}^{\frac{2}{1-\vartheta}})\|\partial_x w\|_{L^2}^2\\
&\qquad + C\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}(\|w\|_{L^{\frac{1}{\delta}}}^{1-\lambda_1}\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^{\lambda_1})(\|w\|_{L^{\frac{1}{\delta}}}^{1-\lambda_2}\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^{\lambda_2}),
\end{aligned}
$$

where $2 < p_1 < \infty$ and

$$
\lambda_1 = \frac{2\delta+2-\gamma-\frac{2}{p_1}}{2\delta+1+\gamma} \in [0,1), \quad \lambda_2 = \frac{2\delta+1-4\beta+\frac{2}{p_1}}{2\delta+1+\gamma} \in [0,1). \tag{2-16}
$$

In order to ensure the Gagliardo–Nirenberg inequality applied above, λ_1 and λ_2 should further satisfy

$$
\lambda_1 \ge \frac{2-\gamma}{2+\gamma}, \quad \lambda_2 \ge \max\left\{0, \frac{2-4\beta}{2+\gamma}\right\}.
$$

Tedious computations yields that λ_1, λ_2 would work as long as

$$
\frac{1-3\gamma}{4} < \beta < \min\left\{\frac{1+(3\delta-1)\gamma}{2(1-2\delta)}, \frac{1+\delta}{2}, \frac{2+(3\gamma+2)\delta}{2+\gamma}\right\} := f(\delta,\gamma). \tag{2-17}
$$

Let us say some words about (2-17). Actually, recalling (2-16) and keeping in mind the restriction

$$
1 > \lambda_1 \ge \frac{2-\gamma}{2+\gamma}, \quad 1 > \lambda_2 \ge \max\left\{0, \frac{2-4\beta}{2+\gamma}\right\},
$$

it is not difficult to check that

$$
\begin{aligned}
&1 > \lambda_1 \ge \frac{2-\gamma}{2+\gamma} \Longrightarrow \frac{1-2\gamma}{2} < \frac{1}{p_1} < \frac{2+(4\delta-1)\gamma}{2(2+\gamma)},\\
&1 > \lambda_2 \ge \max\left\{0, \frac{2-4\beta}{2+\gamma}\right\}\\
&\qquad\Longrightarrow \max\left\{\frac{4\beta-1-2\delta}{2}, \frac{(1-2\delta)(4\beta+\gamma)}{2(2+\gamma)}\right\} < \frac{1}{p_1} < \frac{4\beta+\gamma}{2}.
\end{aligned}
$$

Keeping in mind $2 < p_1 < \infty$, we therefore obtain that p_1 should be satisfied

$$
\underline{D} < \frac{1}{p_1} < \overline{D},
$$

where

$$
\begin{aligned}
\underline{D} &:= \max\left\{\frac{1-2\gamma}{2}, \frac{4\beta-1-2\delta}{2}, \frac{(1-2\delta)(4\beta+\gamma)}{2(2+\gamma)}\right\},\\
\overline{D} &:= \min\left\{\frac{2+(4\delta-1)\gamma}{2(2+\gamma)}, \frac{4\beta+\gamma}{2}, \frac{1}{2}\right\}.
\end{aligned}
$$

By direct computations, we can show that

$$\frac{1-2\gamma}{2} < \frac{4\beta+\gamma}{2} \Rightarrow \beta > \frac{1-3\gamma}{4},$$
$$\frac{4\beta-1-2\delta}{2} < \frac{2+(4\delta-1)\gamma}{2(2+\gamma)} \Rightarrow \beta < \frac{2+(3\gamma+2)\delta}{2+\gamma},$$
$$\frac{4\beta-1-2\delta}{2} < \frac{1}{2} \Rightarrow \beta < \frac{1+\delta}{2},$$
$$\frac{(1-2\delta)(4\beta+\gamma)}{2(2+\gamma)} < \frac{2+(4\delta-1)\gamma}{2(2+\gamma)} \Rightarrow \beta < \frac{1+(3\delta-1)\gamma}{2(1-2\delta)},$$
$$\frac{(1-2\delta)(4\beta+\gamma)}{2(2+\gamma)} < \frac{1}{2} \Rightarrow \beta < \frac{1+\delta\gamma}{2(1-2\delta)}.$$

Consequently, if β satisfies the following restriction, then the above p_1 would work

$$\frac{1-3\gamma}{4} < \beta < \min\left\{\frac{1+(3\delta-1)\gamma}{2(1-2\delta)}, \frac{1+\delta}{2}, \frac{2+(3\gamma+2)\delta}{2+\gamma}\right\}.$$

This is the desired (2-17). Now concerning $\beta > \frac{1-\gamma+\delta}{2}$, we know

$$\lambda_1 + \lambda_2 < 1.$$

It is easy to observe that

$$\frac{1-3\gamma}{4} < \frac{1-\gamma+\delta}{2} < f(\delta, \gamma), \quad \text{for all } \gamma > 0,\ \delta > 0.$$

As a result, we have

$$\frac{1-\gamma+\delta}{2} < \beta < f(\delta,\gamma).$$

As a consequence, for $\frac{1-\gamma+\delta}{2} < \beta < f(\delta, \gamma)$ we obtain

$$N_{42} \le \tfrac{1}{8}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 + C(1+\|w\|_{L^{\frac{1}{\delta}}}^{\frac{2}{1-\vartheta}})\|\partial_x w\|_{L^2}^2 + C\|w\|_{L^{\frac{1}{\delta}}}^{\frac{4\beta+3\gamma-1}{2\beta+\gamma-1+\delta}}.$$

We eventually obtain

$$\frac{d}{dt}\|\partial_x w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 \le C(1+\|w\|_{L^{\frac{1}{\delta}}}^{\frac{2}{1-\vartheta}})\|\partial_x w\|_{L^2}^2 + C\|w\|_{L^{\frac{1}{\delta}}}^{\frac{4\beta+3\gamma-1}{2\beta+\gamma-1+\delta}}.$$

Combining (2-10), (2-12) and the Gronwall inequality, we immediately have the desired bound (2-13).

Step 3. This step is to establish the following global H^m-bound

$$\|w(t)\|_{H^m} \le C(t, w_0).$$

It follows from (2-2) that

$$\tfrac{1}{2}\frac{d}{dt}\|w(t)\|_{H^m}^2+\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}^2=N_1+N_2.$$

Thanks to (2-6), we obtain for $\beta>\frac{1-\gamma}{2}$

$$N_2\le\tfrac{1}{4}\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}^2+C\big(1+\|\partial_x u\|_{L^\infty}^2+\|w\|_{L^\infty}^2+\|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big)\|w\|_{H^m}^2.$$

According to (2-5) and the Gagliardo–Nirenberg inequalities, one thus deduces

$$\begin{aligned}
N_1&\le C\sum_{k=0}^{m}\big(\|\partial_x u\|_{L^\infty}\|\Lambda^k w\|_{L^2}+\|\partial_x w\|_{L^\infty}\|\Lambda^k u\|_{L^2}\big)\|\Lambda^k w\|_{L^2}\\
&\qquad\qquad\qquad\qquad+C\sum_{k=0}^{m}\|\partial_x u\|_{L^\infty}\|\Lambda^k w\|_{L^2}^2\\
&\le C\|\partial_x w\|_{L^\infty}\|\mathcal{J}^m u\|_{L^2}\|\mathcal{J}^m w\|_{L^2}+C\|\partial_x u\|_{L^\infty}\|\mathcal{J}^m w\|_{L^2}^2\\
&\le C\big(\|\partial_x w\|_{L^2}^{\frac{2m+\gamma-3}{2m+\gamma-2}}\|\mathcal{J}^{m+\frac{\gamma}{2}}w\|_{L^2}^{\frac{1}{2m+\gamma-2}}\big)\\
&\quad\times\big(\|\partial_x w\|_{L^2}^{\frac{\gamma+4\beta}{2m+\gamma-2}}\|\mathcal{J}^{m+\frac{\gamma}{2}}w\|_{L^2}^{\frac{2m-4\beta-2}{2m+\gamma-2}}\big)\|\mathcal{J}^m w\|_{L^2}+C\|\partial_x u\|_{L^\infty}\|\mathcal{J}^m w\|_{L^2}^2\\
&\le\tfrac{1}{4}\|\mathcal{J}^{m+\frac{\gamma}{2}}w\|_{L^2}^2+C\|\partial_x w\|_{L^2}^2\|\mathcal{J}^m w\|_{L^2}^{\frac{2(2m+\gamma-2)}{2m+4\beta+2\gamma-3}}+C\|\partial_x u\|_{L^\infty}\|\mathcal{J}^m w\|_{L^2}^2\\
&\le\tfrac{1}{4}\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}^2+C\|w\|_{L^2}+C\|\partial_x w\|_{L^2}^2(1+\|w\|_{H^m}^2)+C\|\partial_x u\|_{L^\infty}\|w\|_{H^m}^2\\
&\le\tfrac{1}{4}\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}^2+C\big(\|w\|_{L^2}+\|\partial_x w\|_{L^2}^2+\|\partial_x u\|_{L^\infty}\big)(1+\|w\|_{H^m}^2),
\end{aligned}$$

where we have used the following Gagliardo–Nirenberg inequalities:

$$\begin{aligned}
&\|\partial_x w\|_{L^\infty}\le C\|\partial_x w\|_{L^2}^{\frac{2m+\gamma-3}{2m+\gamma-2}}\|\mathcal{J}^{m+\frac{\gamma}{2}}w\|_{L^2}^{\frac{1}{2m+\gamma-2}},\\
&\|\mathcal{J}^m u\|_{L^2}\le C\|\mathcal{J}^{m-2\beta}w\|_{L^2}\le C\|\partial_x w\|_{L^2}^{\frac{\gamma+4\beta}{2m+\gamma-2}}\|\mathcal{J}^{m+\frac{\gamma}{2}}w\|_{L^2}^{\frac{2m-4\beta-2}{2m+\gamma-2}}
\end{aligned}$$

and the fact that

$$\frac{2(2m+\gamma-2)}{2m+4\beta+2\gamma-3}\le 2$$

due to $\beta\ge\frac{1-\gamma}{4}$. Combining the above estimates implies that

$$\begin{aligned}
\text{(2-18)}\quad&\frac{d}{dt}\|w(t)\|_{H^m}^2+\|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}^2\\
&\le C\big(1+\|w\|_{L^2}+\|\partial_x w\|_{L^2}^2+\|\partial_x u\|_{L^\infty}^2+\|w\|_{L^\infty}^2+\|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big)\|w\|_{H^m}^2.
\end{aligned}$$

It is easy to show that if $\beta>\frac{1-\gamma}{4}$, then it holds

$$\|\partial_x u\|_{L^\infty}\le C\|\partial_x w\|_{H^{\frac{\gamma}{2}}}.$$

This along with (2-13) gives

$$\int_0^t \|\partial_x u(s)\|_{L^\infty}^2\,ds \le C(t, w_0).$$

According to (2-12) and (2-13), we obtain

$$\|w\|_{L^\infty} \le C\|w\|_{L^2}^{\frac{1}{2}}\|\partial_x w\|_{L^2}^{\frac{1}{2}} \le C(t, w_0). \tag{2-19}$$

Thanks to (2-19), one has

$$\int_0^t \left(\|w(s)\|_{L^\infty}^2 + \|w(s)\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\right) ds \le C(t, w_0).$$

Consequently, we conclude

$$\int_0^t \left(1 + \|w\|_{L^2} + \|\partial_x w\|_{L^2}^2 + \|\partial_x u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2 + \|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\right) ds \le C(t, w_0).$$

Taking advantage of the above bound, one gets by applying the Gronwall inequality to (2-18) that

$$\|w(t)\|_{H^m} \le C(t, w_0).$$

We thus conclude the desired result of Theorem 1.1 under the Case 1.

Case 2. $\beta \ge 0$ with $1+\delta \le \gamma \le 2$ and $\delta \in \left(0, \tfrac{1}{2}\right]$.

According to Step 1 of the Case 1, it is not difficult to conclude that for $\delta \in \left(0, \tfrac{1}{2}\right]$

$$\|w(t)\|_{L^{\frac{1}{\delta}}}^{\frac{1}{\delta}} + \int_0^t \left\|\Lambda^{\frac{\gamma}{2}}|w(s)|^{\frac{1}{2\delta}}\right\|_{L^2}^2 ds \le C(t, w_0). \tag{2-20}$$

Moreover, if it further satisfies $\beta > 0$ with $1+\delta \le \gamma \le 2$, then one deduces from Step 1 of the Case 1,

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\gamma}{2}} w(s)\|_{L^2}^2\,ds \le C(t, w_0). \tag{2-21}$$

Now let us verify that (2-21) holds true for $\beta = 0$ with $1+\delta \le \gamma \le 2$. It should be mentioned that in this case $\beta = 0$, we always have the relation $u = w$. Multiplying (1-6) by Λw, using the Gagliardo–Nirenberg inequality and the fact $\gamma \ge 1+\delta$, it leads to

$$\tag{2-22}\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}}w(t)\|_{L^2}^2+\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2 &= -\int_{\mathbb{R}}(u\partial_x w)\Lambda w\,dx-\delta\int_{\mathbb{R}}(\partial_x u w)\Lambda w\,dx\\ &= -\int_{\mathbb{R}}(w\partial_x w)\Lambda w\,dx-\delta\int_{\mathbb{R}}(\partial_x w w)\Lambda w\,dx\\ &\le C\|w\|_{L^\infty}\|\Lambda w\|_{L^2}\|\partial_x w\|_{L^2}\\ &\le C\|w\|_{L^\infty}\|\Lambda w\|_{L^2}^2\\ &\le C\big\||w|^{\frac{1}{2\delta}}\big\|_{L^\infty}^{2\delta}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2(\gamma-1)}{\gamma}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{2}{\gamma}}\\ &\le C\big\||w|^{\frac{1}{2\delta}}\big\|_{L^2}^{\frac{2\delta(\gamma-1)}{\gamma}}\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^{\frac{2\delta}{\gamma}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2(\gamma-1)}{\gamma}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{2}{\gamma}}\\ &\le C\|w\|_{L^{\frac{1}{\delta}}}^{\frac{\gamma-1}{\gamma}}\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^{\frac{2\delta}{\gamma}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2(\gamma-1)}{\gamma}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{2}{\gamma}}\\ &\le \frac{1}{2}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2+C\|w\|_{L^{\frac{1}{\delta}}}\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^{\frac{2\delta}{\gamma-1}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2\\ &\le \frac{1}{2}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2+C\|w\|_{L^{\frac{1}{\delta}}}\big(1+\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^2\big)\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2.\end{aligned}$$

This implies

$$\frac{d}{dt}\|\Lambda^{\frac{1}{2}}w(t)\|_{L^2}^2+\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2\le C\|w\|_{L^{\frac{1}{\delta}}}\big(1+\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^2\big)\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2.$$

Recalling (2-20) and the Gronwall inequality, we may deduce

$$\tag{2-23}\|\Lambda^{\frac{1}{2}}w(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{\gamma+1}{2}}w(s)\|_{L^2}^2\,ds\le C(t,w_0).$$

We get from (1-6) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2+\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^2 &= -\int_{\mathbb{R}}(w\partial_x w)w\,dx-\delta\int_{\mathbb{R}}(\partial_x w w)w\,dx\\ &\le C\|\Lambda^{\frac{1}{2}}w\|_{L^2}\|\Lambda^{\frac{1}{2}}(ww)\|_{L^2}\\ &\le C\|w\|_{L^\infty}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2\\ &\le C\|w\|_{L^2}^{\frac{\gamma}{\gamma+1}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{1}{\gamma+1}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2\\ &\le C\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2(1+\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2)(1+\|w\|_{L^2}^2).\end{aligned}$$

In view of the Gronwall inequality and the estimate (2-23), this easily yields

$$\tag{2-24}\|w(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{\gamma}{2}}w(s)\|_{L^2}^2\,ds\le C(t,w_0).$$

It is not difficult to show that noting the proof of (2-22) and using (2-21), the desired estimate (2-23) also holds true for the case $\beta > 0$ with $1+\delta \le \gamma \le 2$. Therefore, for $\beta \ge 0$ with $1+\delta \le \gamma \le 2$, we have

$$\|w(t)\|_{L^p} \le \|w(t)\|_{L^2}^{\frac{2}{p}} \|\Lambda^{\frac{1}{2}} w(t)\|_{L^2}^{\frac{p-2}{p}} \le C(t, w_0), \quad \text{for all } 2 \le p < \infty. \tag{2-25}$$

Multiplying Equation (1-6) by $\Lambda^2 w$ and integrating the resultant over the whole space, it implies

$$\begin{aligned} \tfrac{1}{2}\frac{d}{dt}\|\Lambda w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 &= -\int_{\mathbb{R}} (u\partial_x w)\Lambda^2 w\,dx - \delta\int_{\mathbb{R}} (\partial_x u w)\Lambda^2 w\,dx \\ &= -\int_{\mathbb{R}} \Lambda(u\partial_x w)\Lambda w\,dx - \delta\int_{\mathbb{R}} (\partial_x u w)\Lambda^2 w\,dx \\ &= -\int_{\mathbb{R}} [\Lambda, u]\partial_x w\Lambda w\,dx - \int_{\mathbb{R}} u\partial_x \Lambda w\Lambda w\,dx - \delta\int_{\mathbb{R}} (\partial_x u w)\Lambda^2 w\,dx \\ &= -\int_{\mathbb{R}} [\Lambda, u]\partial_x w\Lambda w\,dx + \tfrac{1}{2}\int_{\mathbb{R}} \partial_x u\Lambda w\Lambda w\,dx - \delta\int_{\mathbb{R}} (\partial_x u w)\Lambda^2 w\,dx. \end{aligned} \tag{2-26}$$

By (2-3) and the Gagliardo–Nirenberg inequality, we infer

$$\begin{aligned} -\int_{\mathbb{R}} [\Lambda, u]\partial_x w\Lambda w\,dx &\le \|[\Lambda, u]\partial_x w\|_{L^2}\|\Lambda w\|_{L^2} \\ &\le C\big(\|\partial_x u\|_{L^4}\|\Lambda w\|_{L^4} + \|\Lambda u\|_{L^4}\|\partial_x w\|_{L^4}\big)\|\Lambda w\|_{L^2} \\ &\le C\|\Lambda u\|_{L^4}\|\Lambda w\|_{L^4}\|\Lambda w\|_{L^2} \\ &\le C\|\Lambda u\|_{L^2}^{\frac{2\gamma-1}{2\gamma}} \|\Lambda^{\frac{\gamma+2}{2}} u\|_{L^2}^{\frac{1}{2\gamma}} \|\Lambda w\|_{L^2}^{\frac{2\gamma-1}{2\gamma}} \|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^{\frac{1}{2\gamma}} \|\Lambda w\|_{L^2} \\ &\le C\|\Lambda w\|_{L^2}^{\frac{2\gamma-1}{\gamma}} \|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^{\frac{1}{\gamma}} \|\Lambda w\|_{L^2} \\ &\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|\Lambda w\|_{L^2}^{\frac{2\gamma}{2\gamma-1}} \|\Lambda w\|_{L^2}^2 \\ &\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}} w\|_{L^2}^{\frac{2(\gamma-1)}{2\gamma-1}} \|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^{\frac{2}{2\gamma-1}} \|\Lambda w\|_{L^2}^2 \\ &\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}} w\|_{L^2}^{\frac{2(\gamma-1)}{2\gamma-1}} (1 + \|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^2)\|\Lambda w\|_{L^2}^2. \end{aligned}$$

Similarly, the second term admits the same bound

$$\begin{aligned} \tfrac{1}{2}\int_{\mathbb{R}} \partial_x u\Lambda w\Lambda w\,dx &\le C\|\Lambda u\|_{L^4}\|\Lambda w\|_{L^4}\|\Lambda w\|_{L^2} \\ &\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}} w\|_{L^2}^{\frac{2(\gamma-1)}{2\gamma-1}} (1 + \|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^2)\|\Lambda w\|_{L^2}^2. \end{aligned}$$

According to (2-4) and the Gagliardo–Nirenberg inequality, one concludes

$$
\begin{aligned}
&-\delta\int_{\mathbb{R}}(\partial_x u w)\Lambda^2 w\,dx\\
&\le C\|\Lambda^{\frac{2-\gamma}{2}}(\partial_x u w)\|_{L^2}\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}\\
&\le C\big(\|\partial_x u\|_{L^\infty}\|\Lambda^{\frac{2-\gamma}{2}}w\|_{L^2}+\|w\|_{L^{\frac{4}{\gamma-1}}}\|\Lambda^{\frac{4-\gamma}{2}}u\|_{L^{\frac{4}{3-\gamma}}}\big)\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}\\
&\le C\Big(\|\Lambda u\|_{L^2}^{\frac{\gamma-1}{\gamma}}\|\Lambda^{\frac{\gamma+2}{2}}u\|_{L^2}^{\frac{1}{\gamma}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{1}{\gamma}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{\gamma-1}{\gamma}}\\
&\qquad\qquad+\|w\|_{L^{\frac{4}{\gamma-1}}}\|\Lambda u\|_{L^2}^{\frac{3(\gamma-1)}{2\gamma}}\|\Lambda^{\frac{\gamma+2}{2}}u\|_{L^2}^{\frac{3-\gamma}{2\gamma}}\Big)\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}\\
&\le C\Big(\|\Lambda w\|_{L^2}^{\frac{\gamma-1}{\gamma}}\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}^{\frac{1}{\gamma}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{1}{\gamma}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{\gamma-1}{\gamma}}\\
&\qquad\qquad+\|w\|_{L^{\frac{4}{\gamma-1}}}\|\Lambda w\|_{L^2}^{\frac{3(\gamma-1)}{2\gamma}}\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}^{\frac{3-\gamma}{2\gamma}}\Big)\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}\\
&\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}^2+C\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2}{\gamma-1}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2\|\Lambda w\|_{L^2}^2+C\|w\|_{L^{\frac{4}{\gamma-1}}}^{\frac{4\gamma}{3(\gamma-1)}}\|\Lambda w\|_{L^2}^2\\
&\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}^2+C\big(\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2}{\gamma-1}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2+\|w\|_{L^{\frac{4}{\gamma-1}}}^{\frac{4\gamma}{3(\gamma-1)}}\big)\|\Lambda w\|_{L^2}^2.
\end{aligned}
$$

Plugging the above estimates into (2-26) yields that

$$
\begin{aligned}
&\frac{d}{dt}\|\Lambda w(t)\|_{L^2}^2+\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}^2\\
&\qquad\le C\big(\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2(\gamma-1)}{2\gamma-1}}+\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2}{\gamma-1}}+\|w\|_{L^{\frac{4}{\gamma-1}}}^{\frac{4\gamma}{3(\gamma-1)}}\big)(1+\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2)\|\Lambda w\|_{L^2}^2.
\end{aligned}
$$

Owing to (2-23)–(2-25) and the Gronwall inequality, one thus obtains

$$
\|\Lambda w(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{\gamma+2}{2}}w(s)\|_{L^2}^2\,ds\le C(t,w_0). \tag{2-27}
$$

By using the Gagliardo–Nirenberg inequality

$$
\|\partial_x w\|_{L^\infty}\le C\|\Lambda w\|_{L^2}^{\frac{\gamma-1}{\gamma}}\|\Lambda^{\frac{\gamma+2}{2}}w\|_{L^2}^{\frac{1}{\gamma}},\quad \|w\|_{L^\infty}\le C\|w\|_{L^2}^{\frac{1}{2}}\|\Lambda w\|_{L^2}^{\frac{1}{2}},
$$

it follows from (2-23), (2-24), (2-25) and (2-27) that

$$
\int_0^t\|\partial_x u(s)\|_{L^\infty}\,ds\le C\int_0^t\|\partial_x w(s)\|_{L^\infty}\,ds\le C(t,w_0),\quad \|w(t)\|_{L^\infty}\le C(t,w_0).
$$

Obviously, it implies

$$
\int_0^t\big(\|\partial_x w(s)\|_{L^\infty}+\|\partial_x u(s)\|_{L^\infty}^2+\|w(s)\|_{L^\infty}^2+\|w(s)\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big)\,ds\le C(t,w_0).
$$

Hence, recalling the blow-up criterion (2-1), we conclude the desired result of Theorem 1.1 under the Case 2.

Case 3. The case $\delta = 2$.

In this case, taking L^2 inner product of (1-6) with u gives (see [Bae et al. 2017, (4.13)])

$$\tfrac{1}{2}\frac{d}{dt}\|(1-\partial_{xx})^{-\frac{\beta}{2}}w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}}(1-\partial_{xx})^{-\frac{\beta}{2}}w\|_{L^2}^2 = 0,$$

which leads to

$$\|(1-\partial_{xx})^{-\frac{\beta}{2}}w(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^{\frac{\gamma}{2}}(1-\partial_{xx})^{-\frac{\beta}{2}}w(s)\|_{L^2}^2\,ds = \|(1-\partial_{xx})^{-\frac{\beta}{2}}w_0\|_{L^2}^2. \tag{2-28}$$

Thanks to $\beta \geq \frac{2-\gamma}{2}$, we deduce

$$\int_0^t \|\partial_x u(s)\|_{L^2}^2\,ds \leq C\int_0^t \|(1-\partial_{xx})^{-\frac{\beta}{2}}w(s)\|_{H^{\frac{\gamma}{2}}}^2\,ds \leq C(t,w_0). \tag{2-29}$$

Recalling (2-11) and the Gagliardo–Nirenberg inequality, it can be obtained that

$$\begin{aligned}
\tfrac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^2 &= -\int_{\mathbb{R}}(u\partial_x w)w\,dx - 2\int_{\mathbb{R}}(\partial_x uw)w\,dx \\
&= -\tfrac{3}{2}\int_{\mathbb{R}}\partial_x uww\,dx \\
&\leq C\|\partial_x u\|_{L^2}\|w\|_{L^4}^2 \\
&\leq C\|\partial_x u\|_{L^2}\|w\|_{L^2}^{2-\frac{1}{\gamma}}\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^{\frac{1}{\gamma}} \\
&\leq \tfrac{1}{2}\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^2 + C\|\partial_x u\|_{L^2}^{\frac{2\gamma}{2\gamma-1}}\|w\|_{L^2}^2 \\
&\leq \tfrac{1}{2}\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^2 + C(1+\|\partial_x u\|_{L^2}^2)\|w\|_{L^2}^2.
\end{aligned}$$

On the other hand, one has an alternative estimate for $\beta \geq \frac{3-2\gamma}{2}$

$$\begin{aligned}
\tfrac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 &+ \|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^2 \\
&= -\frac{3}{2}\int_{\mathbb{R}}\partial_x uww\,dx \\
&\leq C\|\partial_x u\|_{B^{-\frac{\gamma}{2}}_{\infty,2}}\|ww\|_{B^{\frac{\gamma}{2}}_{1,2}} \\
&\leq C\|(1-\partial_{xx})^{-\frac{\beta}{2}}w\|_{H^{\frac{\gamma}{2}}}\|w\|_{L^2}\|\mathcal{J}^{\frac{\gamma}{2}}w\|_{L^2} \\
&\leq \tfrac{1}{2}\|\Lambda^{\frac{\gamma}{2}}w\|_{L^2}^2 + C\big(1+\|(1-\partial_{xx})^{-\frac{\beta}{2}}w\|_{H^{\frac{\gamma}{2}}}^2\big)\|w\|_{L^2}^2.
\end{aligned} \tag{2-30}$$

This implies that for $\beta \geq \min\{\frac{2-\gamma}{2}, \frac{3-2\gamma}{2}\}$

$$\frac{d}{dt}\|w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}} w\|_{L^2}^2 \leq C(1+\|\partial_x u\|_{L^2}^2)\|w\|_{L^2}^2$$

or

$$\frac{d}{dt}\|w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}} w\|_{L^2}^2 \leq C\big(1+\|(1-\partial_{xx})^{-\frac{\beta}{2}} w\|_{H^{\frac{\gamma}{2}}}^2\big)\|w\|_{L^2}^2.$$

By (2-29) or (2-28) and the Gronwall inequality, it follows that

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\gamma}{2}} w(s)\|_{L^2}^2\, ds \leq C(t, w_0). \tag{2-31}$$

The remainder proof is divided into two cases: $0<\gamma<1$ and $1 \leq \gamma \leq 2$.

For the case $1 \leq \gamma \leq 2$, we keep in mind (2-15) that

$$\tfrac{1}{2}\frac{d}{dt}\|\partial_x w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 = N_3 + N_4.$$

For $\beta > \frac{3-2\gamma}{4}$, we obtain

$$\begin{aligned}
N_3 &= -\frac{\delta}{2}\int_{\mathbb{R}} \partial_{xxx} u(ww)\, dx \\
&\leq C\|\partial_x u\|_{B_{\infty,2}^{1-\frac{\gamma}{2}}}\|ww\|_{B_{1,2}^{1+\frac{\gamma}{2}}} \\
&\leq C\|\partial_x w\|_{H^{\frac{3}{2}-2\beta-\frac{\gamma}{2}}}\|w\|_{L^2}\|\mathcal{J}^{1+\frac{\gamma}{2}} w\|_{L^2} \\
&\leq C\big(\|\partial_x w\|_{L^2}+\|\partial_x w\|_{L^2}^{1-\widetilde{\vartheta}}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^{\widetilde{\vartheta}}\big)\|w\|_{L^2}\|\mathcal{J}^{\frac{\gamma}{2}}\partial_x w\|_{L^2} \\
&\leq C\big(\|\partial_x w\|_{L^2}+\|\partial_x w\|_{L^2}^{1-\widetilde{\vartheta}}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^{\widetilde{\vartheta}}\big)\|w\|_{L^{\frac{1}{\delta}}}(\|\partial_x w\|_{L^2}+\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}) \\
&\leq \tfrac{1}{4}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 + C(1+\|w\|_{L^2}^{2/(1-\widetilde{\vartheta})})\|\partial_x w\|_{L^2}^2,
\end{aligned}$$

where we have applied the facts

$$\begin{aligned}
\|\partial_x w\|_{H^{\frac{3}{2}-2\beta-\frac{\gamma}{2}}} &\leq C\|\partial_x w\|_{L^2}^{1-\widetilde{\vartheta}}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^{\widetilde{\vartheta}}, \quad && \tfrac{3-2\gamma}{4} < \beta < \tfrac{3-\gamma}{4}, \\
\|\partial_x w\|_{H^{\frac{3}{2}-2\beta-\frac{\gamma}{2}}} &\leq C\|\partial_x w\|_{L^2}, && \beta \geq \tfrac{3-\gamma}{4}.
\end{aligned}$$

It should be remarked that if $\frac{3}{2} \leq \gamma \leq 2$, the above facts are true for $\beta \geq 0$. On the other hand, according to (2-30), we infer that for $\beta \geq \frac{3-2\gamma}{2}$

$$N_4 \leq \tfrac{1}{2}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 + C\|(1-\partial_{xx})^{-\frac{\beta}{2}} w\|_{H^{\frac{\gamma}{2}}}^2\|\partial_x w\|_{L^2}^2.$$

Combining the above estimates and making use of the Gronwall inequality yields

$$\|\partial_x w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\gamma}{2}}\partial_x w(s)\|_{L^2}^2\, ds \leq C(t, w_0). \tag{2-32}$$

The above estimates allows us to deduce

$$\begin{aligned} \|w\|_{L^\infty} &\le C\|w\|_{L^2}^{\frac{1}{2}}\|\partial_x w\|_{L^2}^{\frac{1}{2}} \le C, \\ \int_0^t \|\partial_x u(s)\|_{L^\infty}^2\,ds &\le C\int_0^t \|\partial_x w(s)\|_{H^{\frac{\gamma}{2}}}^2\,ds \le C. \end{aligned} \tag{2-33}$$

Due to $1 \le \gamma \le 2$, it is easy to observe

$$\int_0^t \|\partial_x w(s)\|_{\mathrm{BMO}}^2\,ds \le \int_0^t \|\Lambda^{\frac{\gamma}{2}}\partial_x w(s)\|_{L^2}^2\,ds \le C(t, w_0), \quad \text{if } \gamma = 1;$$
$$\int_0^t \|\partial_x w(s)\|_{L^\infty}^2\,ds \le \int_0^t \|\partial_x w(s)\|_{H^{\frac{\gamma}{2}}}^2\,ds \le C(t, w_0), \qquad \text{if } 1 < \gamma \le 2.$$

Recalling (2-8), one has

$$\begin{aligned} \frac{d}{dt}\|w(t)\|_{H^m}^2 &\\ \le C\big(1 + \|\partial_x w\|_{L^\infty} &+ \|\partial_x u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2 + \|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big)\|w\|_{H^m}^2. \end{aligned} \tag{2-34}$$

Here we would like to state that it suffices to consider the case $\gamma = 1$. Otherwise, we directly have

$$\int_0^t \|\partial_x w(s)\|_{L^\infty}\,ds \le C\int_0^t (\|\partial_x w(s)\|_{L^2} + \|\Lambda^{\frac{\gamma}{2}}\partial_x w(s)\|_{L^2})\,ds \le C(t, w_0).$$

To bound the term $\|\partial_x w\|_{L^\infty}$ for the case $\gamma = 1$, we need the following version of the logarithmic Sobolev inequality (see for example [Brézis and Gallouet 1980])

$$\|f\|_{L^\infty} \le C\big(1 + \|\Lambda^{\frac{\gamma}{2}} f\|_{L^2}\sqrt{\ln(e + \|f\|_{H^s})}\big), \quad s > \tfrac{1}{2}. \tag{2-35}$$

Applying (2-35) to (2-34), we find that the following is immediate

$$\begin{aligned} \frac{d}{dt}\|w(t)\|_{H^m} \le C\big(1 + \|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 + \|\partial_x u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2 + \|w\|_{L^\infty}^{\frac{\gamma}{2\beta+\gamma-1}}\big) \\ \times \ln(e + \|w\|_{H^m})\|w\|_{H^m}. \end{aligned}$$

This together with (2-32), (2-33) and the Gronwall inequality leads to the bound

$$\|w(t)\|_{H^m} \le C(t, w_0).$$

For the case $0 < \gamma < 1$, we first recall (2-15) that

$$\tfrac{1}{2}\frac{d}{dt}\|\partial_x w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 = N_3 + N_4.$$

For $\beta \geq \frac{2-\gamma}{2}$, one thus deduces from the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
N_3 &\leq C\|\partial_{xx}u\|_{L^{\frac{2}{\gamma}}}\|\partial_x w\|_{L^{\frac{2}{1-\gamma}}}\|w\|_{L^2} \\
&\leq C\|\mathcal{J}^{\frac{1-\gamma}{2}}\partial_{xx}u\|_{L^2}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}\|w\|_{L^2} \\
&\leq C\|\mathcal{J}^{\frac{1-\gamma}{2}+1-2\beta}\partial_x w\|_{L^2}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}\|w\|_{L^2} \\
&\leq C\big(\|\partial_x w\|_{L^2} + \|\partial_x w\|_{L^2}^{1-\theta_3}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^{\theta_3}\big)\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}\|w\|_{L^2} \\
&\leq \tfrac{1}{4}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 + C(1+\|w\|_{L^2}^{\frac{2}{1-\theta_3}})\|\partial_x w\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
N_4 &\leq C\|\partial_x u\|_{L^{\frac{2}{\gamma}}}\|\partial_x w\|_{L^{\frac{4}{2-\gamma}}}^2 \\
&\leq C\|w\|_{H^{\frac{3-4\beta-\gamma}{2}}}\|\partial_x w\|_{L^2}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2} \\
&\leq C\|w\|_{H^{\frac{\gamma}{2}}}\|\partial_x w\|_{L^2}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2} \\
&\leq \tfrac{1}{4}\|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 + C\|w\|_{H^{\frac{\gamma}{2}}}^2\|\partial_x w\|_{L^2}^2.
\end{aligned}$$

This further leads to

$$\frac{d}{dt}\|\partial_x w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2 \leq C(1+\|w\|_{L^2}^{\frac{2}{1-\theta_3}} + \|w\|_{H^{\frac{\gamma}{2}}}^2)\|\partial_x w\|_{L^2}^2.$$

It follows from (2-31) and the Gronwall inequality

$$\|\partial_x w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\gamma}{2}}\partial_x w(s)\|_{L^2}^2\,ds \leq C(t, w_0).$$

Finally, the left part of the proof proceeds by the same manner as that of Case 1. Thus, to avoid redundancy, the details are omitted here. We thus conclude the desired result of Theorem 1.1 under the Case 3. □

3. The proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. It should be noted that in Theorem 1.2, we have restriction on the sign of initial data w_0. The following is the local well-posedness result and blow-up criterion (see [Bae et al. 2017, Theorem 5.3]).

Proposition 3.1. *Let $\nu = 1$ and $\delta > 0$. Then for $0 \leq w_0 \in H^m(\mathbb{R})$ with $m \geq 2$, there exists $T^* = T^*(\|w_0\|_{H^m})$ such that a unique solution of (1-10) belongs to*

$$w \in C([0, T^*); H^m(\mathbb{R})).$$

Moreover, we have the following blow-up criterion

$$\limsup_{t\to\tilde{T}} \|w(t)\|_{H^m} = \infty \Leftrightarrow \int_0^{\tilde{T}} (\|\partial_x w(s)\|_{\mathrm{BMO}} + \|\Lambda w(s)\|_{\mathrm{BMO}})\, ds = \infty. \tag{3-1}$$

Next we state the minimum and maximum principles to the system (1-10).

Lemma 3.2. *Let $\nu = 1$ and $\delta > 0$. Then for $0 \le w_0 \in L^\infty(\mathbb{R})$, the solution w of the system (1-10) admits the bound*

$$0 \le w(t,x) \le \|w_0\|_{L^\infty}. \tag{3-2}$$

Proof of Lemma 3.2. The proof is a consequence of many previous works (see, e.g., [Córdoba and Córdoba 2004; Bae and Granero-Belinchón 2015]). For the sake of simplicity, we provide the details. By (1-1), we rewrite (1-10) as

$$\partial_t w + u\partial_x w + \delta\Lambda w w + \Lambda w = 0. \tag{3-3}$$

Let $\overline{x_t}$ and $\underline{x_t}$ be points such that

$$M(t) = \max_{x\in\mathbb{R}} w(t,x) = w(t,\overline{x_t}), \quad m(t) = \min_{x\in\mathbb{R}} w(t,x) = w(t,\underline{x_t}).$$

Since $M(t)$ and $m(t)$ are continuous Lipschitz functions, they are differentiable at almost every t by Rademacher's theorem. Thanks to the definition of Λ, we have

$$\begin{aligned} \frac{d}{dt} m(t) &= -C\delta w(t,\underline{x_t})\, \mathrm{p.v.}\!\int_{\mathbb{R}} \frac{w(t,\underline{x_t}) - w(t,y)}{|\underline{x_t} - y|^2}\, dy \\ &\qquad - C\, \mathrm{p.v.}\!\int_{\mathbb{R}} \frac{w(t,\underline{x_t}) - w(t,y)}{|\underline{x_t} - y|^2}\, dy \\ &\ge \left[-C\delta\, \mathrm{p.v.}\!\int_{\mathbb{R}} \frac{w(t,\underline{x_t}) - w(t,y)}{|\underline{x_t} - y|^2}\, dy\right] m(t). \end{aligned} \tag{3-4}$$

Due to the fact that the quantity in the bracket of (3-4) is nonnegative, we conclude that $m(t)$ is nondecreasing in time if $w_0 \ge 0$ and thus $w(t,x) \ge 0$ for all time. Similarly, one gets

$$\begin{aligned} \frac{d}{dt} M(t) &= -C\delta w(t,\overline{x_t})\, \mathrm{p.v.}\!\int_{\mathbb{R}} \frac{w(t,\overline{x_t}) - w(t,y)}{|\overline{x_t} - y|^2}\, dy \\ &\qquad - C\, \mathrm{p.v.}\!\int_{\mathbb{R}} \frac{w(t,\overline{x_t}) - w(t,y)}{|\overline{x_t} - y|^2}\, dy \\ &\le \left[-C\delta\, \mathrm{p.v.}\!\int_{\mathbb{R}} \frac{w(t,\overline{x_t}) - w(t,y)}{|\overline{x_t} - y|^2}\, dy\right] M(t). \end{aligned} \tag{3-5}$$

Since the quantity in the bracket of (3-5) is nonpositive, we have that $M(t)$ is nonincreasing in time and thus $w(t,x) \le \|w_0\|_{L^\infty}$ for all time. Therefore, this ends the proof of Lemma 3.2. □

Proof of Theorem 1.2. Multiplying Equation (3-3) by w and integrating the resultant over $\mathbb{R}$, it follows from integration by parts and (1-1) that

$$\begin{aligned}\text{(3-6)}\quad \tfrac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2 &= -\tfrac{1}{2}\int_{\mathbb{R}}\mathcal{H}w\partial_x(ww)\,dx-\delta\int_{\mathbb{R}}\Lambda w(ww)\,dx\\ &= \left(\tfrac{1}{2}-\delta\right)\int_{\mathbb{R}}\Lambda w(ww)\,dx.\end{aligned}$$

According to (2-4), it leads to

$$\begin{aligned}\left(\tfrac{1}{2}-\delta\right)\int_{\mathbb{R}}\Lambda w(ww)\,dx &\le C\left(\tfrac{1}{2}-\delta\right)\|\Lambda^{\frac{1}{2}}w\|_{L^2}\|\Lambda^{\frac{1}{2}}(ww)\|_{L^2}\\ &\le C\left(\tfrac{1}{2}-\delta\right)\|\Lambda^{\frac{1}{2}}w\|_{L^2}\|w\|_{L^\infty}\|\Lambda^{\frac{1}{2}}w\|_{L^2}\\ &\le C_1\left(\tfrac{1}{2}-\delta\right)\|w_0\|_{L^\infty}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2,\end{aligned}$$

where $C_1>0$ is an absolute constant. Consequently, we obtain

$$\tfrac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2\le C_1\left(\tfrac{1}{2}-\delta\right)\|w_0\|_{L^\infty}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2.$$

Under the assumption that $\left(\tfrac{1}{2}-\delta\right)\|w_0\|_{L^\infty}$ is sufficiently small, one deduces

$$\frac{d}{dt}\|w(t)\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2\le 0,$$

which yields

$$\|w(t)\|_{L^2}^2+\int_0^t\|\Lambda^{\frac{1}{2}}w(s)\|_{L^2}^2\,ds\le\|w_0\|_{L^2}^2.$$

Now multiplying Equation (3-3) by Λw, integrating the resultant over $\mathbb{R}$ and taking advantage of (1-1)–(1-2), we infer

$$\begin{aligned}\text{(3-7)}\quad \tfrac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}}w(t)\|_{L^2}^2+\|\Lambda w\|_{L^2}^2 &= -\int_{\mathbb{R}}\mathcal{H}w\partial_x w\Lambda w\,dx-\delta\int_{\mathbb{R}}w(\Lambda w)^2\,dx\\ &= -\int_{\mathbb{R}}\mathcal{H}w\partial_x w\mathcal{H}\partial_x w\,dx-\delta\int_{\mathbb{R}}w(\Lambda w)^2\,dx\\ &= \int_{\mathbb{R}}w\mathcal{H}(\partial_x w\mathcal{H}\partial_x w)\,dx-\delta\int_{\mathbb{R}}w(\Lambda w)^2\,dx\\ &= \left(\tfrac{1}{2}-\delta\right)\int_{\mathbb{R}}w(\Lambda w\Lambda w)\,dx-\tfrac{1}{2}\int_{\mathbb{R}}w(\partial_x w\partial_x w)\,dx\\ &\le \left(\tfrac{1}{2}-\delta\right)\int_{\mathbb{R}}w(\Lambda w\Lambda w)\,dx\\ &\le C\left(\tfrac{1}{2}-\delta\right)\|w\|_{L^\infty}\|\Lambda w\|_{L^2}^2\\ &\le C_2\left(\tfrac{1}{2}-\delta\right)\|w_0\|_{L^\infty}\|\Lambda w\|_{L^2}^2,\end{aligned}$$

where $C_2 > 0$ is an absolute constant and we have used $w \geq 0$ due to (3-2). Thanks to the assumption that $\left(\frac{1}{2}-\delta\right)\|w_0\|_{L^\infty}$ is sufficiently small, we have

$$\frac{d}{dt}\|\Lambda^{\frac{1}{2}}w(t)\|_{L^2}^2 + \|\Lambda w\|_{L^2}^2 \leq 0,$$

which yields

$$\|\Lambda^{\frac{1}{2}}w(t)\|_{L^2}^2 + \int_0^t \|\Lambda w(s)\|_{L^2}^2\,ds \leq \|\Lambda^{\frac{1}{2}}w_0\|_{L^2}^2. \tag{3-8}$$

Multiplying Equation (3-3) by $\Lambda^2 w$ and integrating the resultant over the whole space, it implies

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|\Lambda w(t)\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}w\|_{L^2}^2 &= -\int_{\mathbb{R}} \mathcal{H}w\,\partial_x w\,\Lambda^2 w\,dx - \delta\int_{\mathbb{R}} w\Lambda w\Lambda^2 w\,dx\\ &= -\int_{\mathbb{R}} \Lambda(\mathcal{H}w\,\partial_x w)\Lambda w\,dx - \delta\int_{\mathbb{R}} w\Lambda w\Lambda^2 w\,dx\\ &= -\int_{\mathbb{R}}[\Lambda,\mathcal{H}w]\partial_x w\Lambda w\,dx - \int_{\mathbb{R}}\mathcal{H}w\,\partial_x\Lambda w\Lambda w\,dx - \delta\int_{\mathbb{R}} w\Lambda w\Lambda^2 w\,dx\\ &= -\int_{\mathbb{R}}[\Lambda,\mathcal{H}w]\partial_x w\Lambda w\,dx + \tfrac{1}{2}\int_{\mathbb{R}}\mathcal{H}\partial_x w\Lambda w\Lambda w\,dx - \delta\int_{\mathbb{R}} w\Lambda w\Lambda^2 w\,dx.\end{aligned} \tag{3-9}$$

On account of (2-3) and the Gagliardo–Nirenberg inequality, it is not difficult to check

$$\begin{aligned}-\int_{\mathbb{R}}[\Lambda,\mathcal{H}w]\partial_x w\Lambda w\,dx &\leq \|[\Lambda,\mathcal{H}w]\partial_x w\|_{L^2}\|\Lambda w\|_{L^2}\\ &\leq C\big(\|\mathcal{H}\partial_x w\|_{L^4}\|\Lambda w\|_{L^4} + \|\mathcal{H}\Lambda w\|_{L^4}\|\partial_x w\|_{L^4}\big)\|\Lambda w\|_{L^2}\\ &\leq C\|\Lambda w\|_{L^4}^2\|\Lambda w\|_{L^2}\\ &\leq C\|\Lambda w\|_{L^2}\|\Lambda^{\frac{3}{2}}w\|_{L^2}\|\Lambda w\|_{L^2}\\ &\leq \tfrac{1}{8}\|\Lambda^{\frac{3}{2}}w\|_{L^2}^2 + C\|\Lambda w\|_{L^2}^2\|\Lambda w\|_{L^2}^2.\end{aligned}$$

Direct computation yields

$$\begin{aligned}\tfrac{1}{2}\int_{\mathbb{R}}\mathcal{H}\partial_x w\Lambda w\Lambda w\,dx &\leq C\|\mathcal{H}\partial_x w\|_{L^4}\|\Lambda w\|_{L^4}\|\Lambda w\|_{L^2}\\ &\leq C\|\Lambda w\|_{L^4}^2\|\Lambda w\|_{L^2}\\ &\leq C\|\Lambda w\|_{L^2}\|\Lambda^{\frac{3}{2}}w\|_{L^2}\|\Lambda w\|_{L^2}\\ &\leq \tfrac{1}{8}\|\Lambda^{\frac{3}{2}}w\|_{L^2}^2 + C\|\Lambda w\|_{L^2}^2\|\Lambda w\|_{L^2}^2.\end{aligned}$$

Recalling $w \geq 0$ and using the following pointwise inequality (see [Córdoba and Córdoba 2004; Ju 2005] for example)

$$f\Lambda f \geq \tfrac{1}{2}\Lambda(f^2),$$

we thus have

$$-\delta\int_{\mathbb{R}} w\Lambda w\Lambda^2 w\,dx \leq -\frac{\delta}{2}\int_{\mathbb{R}} \Lambda w\Lambda w\Lambda w\,dx \leq \tfrac{1}{8}\|\Lambda^{\frac{3}{2}}w\|_{L^2}^2 + C\|\Lambda w\|_{L^2}^2\|\Lambda w\|_{L^2}^2.$$

Combining all the above estimates, it follows that

$$\frac{d}{dt}\|\Lambda w(t)\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}w\|_{L^2}^2 \leq C\|\Lambda w\|_{L^2}^2\|\Lambda w\|_{L^2}^2.$$

Thanks to (3-8) and the Gronwall inequality, we deduce

$$\|\Lambda w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}w(s)\|_{L^2}^2\,ds \leq C(t, w_0). \tag{3-10}$$

By the classical embedding $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow \mathrm{BMO}(\mathbb{R})$, it follows from (3-10) that

$$\int_0^t (\|\partial_x w(s)\|_{\mathrm{BMO}} + \|\Lambda w(s)\|_{\mathrm{BMO}})\,ds \leq C(t, w_0).$$

Applying the blow-up criterion (3-1), we immediately complete the proof of Theorem 1.2. □

4. The proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We first prove the following local well-posedness result and blow-up criterion.

Proposition 4.1. *Let $\delta \in \mathbb{R}$ and $\gamma > 1$. Then for $w_0 \in H^m(\mathbb{R})$ with $m \geq 2$, there exists $T^* = T^*(\|w_0\|_{H^m})$ such that a unique solution of (1-11) belongs to*

$$w \in C([0,\ T^*);\ H^m(\mathbb{R})).$$

Moreover, we have the following blow-up criterion

$$\limsup_{t\to\widetilde{T}} \|w(t)\|_{H^m} = \infty \Leftrightarrow \int_0^{\widetilde{T}} (\|w(s)\|_{L^\infty}^{\frac{\gamma}{\gamma-1}} + \|\Lambda w(s)\|_{\mathrm{BMO}})\,ds = \infty. \tag{4-1}$$

Proof of Proposition 4.1. Recalling (2-2), one has

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|w(t)\|_{H^m}^2 + \|\Lambda^{\frac{\gamma}{2}}w\|_{H^m}^2 &= -\sum_{k=0}^{m}\int_{\mathbb{R}} \Lambda^k(u\partial_x w)\Lambda^k w\,dx - \delta\sum_{k=0}^{m}\int_{\mathbb{R}} \Lambda^k(\partial_x u w)\Lambda^k w\,dx\\ &:= N_1 + N_2.\end{aligned}$$

According to (2-5), we have

$$N_1 \le C(\|\partial_x u\|_{L^\infty} + \|\partial_x w\|_{L^\infty})\|w\|_{H^m}^2 \le C(\|\Lambda w\|_{L^\infty} + \|\partial_x w\|_{L^\infty})\|w\|_{H^m}^2.$$

It follows from (2-4) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
N_2 &= -\delta \sum_{k=1}^{m} \int_{\mathbb{R}} \Lambda^{k-\frac12}(\Lambda w w)\Lambda^{k+\frac12} w\,dx - \int_{\mathbb{R}} \Lambda w w w\,dx \\
&\le C \sum_{k=1}^{m} \|\Lambda^{k-\frac12}(\Lambda w w)\|_{L^2}\|\Lambda^{k+\frac12} w\|_{L^2} + C\|\Lambda w\|_{L^\infty}\|w\|_{L^2}^2 \\
&\le C \sum_{k=1}^{m} \Big(\|\Lambda w\|_{L^{2k+1}}\|\Lambda^{k-\frac12} w\|_{L^{\frac{2(2k+1)}{2k-1}}} \\
&\qquad\qquad + \|w\|_{L^\infty}\|\Lambda^{k-\frac12}\Lambda w\|_{L^2}\Big)\|\Lambda^{k+\frac12} w\|_{L^2} + C\|\Lambda w\|_{L^\infty}\|w\|_{H^m}^2 \\
&\le C \sum_{k=1}^{m} \Big(\|w\|_{L^\infty}^{1-\frac{2}{2k+1}}\|\Lambda^{k+\frac12} w\|_{L^2}^{\frac{2}{2k+1}}\|w\|_{L^\infty}^{1-\frac{2k-1}{2k+1}}\|\Lambda^{k+\frac12} w\|_{L^2}^{\frac{2k-1}{2k+1}} \\
&\qquad\qquad + \|w\|_{L^\infty}\|\Lambda^{k-\frac12}\partial_x u\|_{L^2}\Big)\|\Lambda^{k+\frac12} w\|_{L^2} + C\|\Lambda w\|_{L^\infty}\|w\|_{H^m}^2 \\
&\le C\|w\|_{L^\infty}\|w\|_{H^{m+\frac12}}\|w\|_{H^{m+\frac12}} + C\|\Lambda w\|_{L^\infty}\|w\|_{H^m}^2 \\
&\le C\|w\|_{L^\infty}\|w\|_{H^m}^{\frac{2(\gamma-1)}{\gamma}}\|\Lambda^{\frac{\gamma}{2}} w\|_{H^m}^{\frac{2}{\gamma}} + C\|\Lambda w\|_{L^\infty}\|w\|_{H^m}^2 \\
&\le \tfrac12\|\Lambda^{\frac{\gamma}{2}} w\|_{H^m}^2 + C(\|\Lambda w\|_{L^\infty} + \|w\|_{L^\infty}^{\frac{\gamma}{\gamma-1}})\|w\|_{H^m}^2,
\end{aligned}$$

where we have applied the fractional type Gagliardo–Nirenberg inequalities (notice that they can be deduced from [Bahouri et al. 2011, Theorem 2.42])

$$\begin{aligned}
\|\Lambda w\|_{L^{2k+1}} &\le C\|w\|_{L^\infty}^{1-\frac{2}{2k+1}}\|\Lambda^{k+\frac12} w\|_{L^2}^{\frac{2}{2k+1}}, \\
\|\Lambda^{k-\frac12} w\|_{L^{\frac{2(2k+1)}{2k-1}}} &\le C\|w\|_{L^\infty}^{1-\frac{2k-1}{2k+1}}\|\Lambda^{k+\frac12} w\|_{L^2}^{\frac{2k-1}{2k+1}}.
\end{aligned}$$

Summing up the above estimates yields

$$\begin{aligned}
\text{(4-2)}\qquad \frac{d}{dt}\|w(t)\|_{H^m}^2 &\le C\big(\|\partial_x w\|_{L^\infty} + \|\Lambda w\|_{L^\infty} + \|w\|_{L^\infty}^{\frac{\gamma}{\gamma-1}}\big)\|w\|_{H^m}^2 \\
&\le C\|w\|_{H^m}^{\frac{3\gamma-2}{\gamma-1}}.
\end{aligned}$$

Notice that

$$\|\partial_x w\|_{\mathrm{BMO}} = \Big\|\frac{\partial_x}{\Lambda}\Lambda w\Big\|_{\mathrm{BMO}} \le C\|\Lambda w\|_{\mathrm{BMO}}$$

and the following logarithmic Sobolev inequality [Kozono and Taniuchi 2000, Theorem 1]

$$\|f\|_{L^\infty} \le C\big(1+\|f\|_{\mathrm{BMO}}\ln(e+\|f\|_{H^s})\big), \quad s>\tfrac{1}{2}, \tag{4-3}$$

one thus deduces from (4-2) that

$$\frac{d}{dt}\|w(t)\|_{H^m}^2 \le C(1+\|w\|_{L^\infty}^{\frac{\gamma}{\gamma-1}}+\|\Lambda w\|_{\mathrm{BMO}})\ln(e+\|w\|_{H^m}^2)\|w\|_{H^m}^2.$$

Consequently, the local well-posedness and the blow-up criterion hold true. This ends the proof of Proposition 4.1. □

Proof of Theorem 1.3. The proof of Theorem 1.3 can be deduced from that of Case 2 of Theorem 1.1. For the sake of completeness, we provide the details. According to Step 1 of the Case 1, we have that

$$\|w(t)\|_{L^{\frac{1}{\delta}}}^{\frac{1}{\delta}} + \int_0^t \|\Lambda^{\frac{\gamma}{2}}|w(s)|^{\frac{1}{2\delta}}\|_{L^2}^2\,ds \le C(t,w_0). \tag{4-4}$$

In view of (3-7), we conclude

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}}w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2 \\ = \big(\tfrac{1}{2}-\delta\big)\int_{\mathbb{R}} w(\Lambda w\Lambda w)\,dx - \tfrac{1}{2}\int_{\mathbb{R}} w(\partial_x w\partial_x w)\,dx := N.\end{aligned}$$

It follows from the Gagliardo–Nirenberg inequality, we get by direct computation

$$\begin{aligned} N &\le C\|w\|_{L^\infty}\|\Lambda w\|_{L^2}^2 \\ &\le C\big\||w|^{\frac{1}{2\delta}}\big\|_{L^\infty}^{2\delta}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2(\gamma-1)}{\gamma}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{2}{\gamma}} \\ &\le C\big\||w|^{\frac{1}{2\delta}}\big\|_{L^2}^{\frac{2\delta(\gamma-1)}{\gamma}}\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^{\frac{2\delta}{\gamma}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2(\gamma-1)}{\gamma}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{2}{\gamma}} \\ &\le C\|w\|_{L^{\frac{1}{\delta}}}^{\frac{\gamma-1}{\gamma}}\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^{\frac{2\delta}{\gamma}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^{\frac{2(\gamma-1)}{\gamma}}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^{\frac{2}{\gamma}} \\ &\le \tfrac{1}{2}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2 + C\|w\|_{L^{\frac{1}{\delta}}}\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^{\frac{2\delta}{\gamma-1}}\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2 \\ &\le \tfrac{1}{2}\|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2 + C\|w\|_{L^{\frac{1}{\delta}}}\big(1+\big\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\big\|_{L^2}^2\big)\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2, \end{aligned}$$

where in the last line we have used the fact $\gamma \ge 1+\delta$. It should be noted that this is the only place in the proof where we use the main assumption of the theorem, namely $\gamma \ge 1+\delta$. We therefore arrive at

$$\frac{d}{dt}\|\Lambda^{\frac{1}{2}}w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma+1}{2}}w\|_{L^2}^2 \le C\|w\|_{L^{\frac{1}{\delta}}}(1+\|\Lambda^{\frac{\gamma}{2}}|w|^{\frac{1}{2\delta}}\|_{L^2}^2)\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2.$$

Making use of (4-4) and the Gronwall inequality yields

$$\|\Lambda^{\frac{1}{2}} w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\gamma+1}{2}} w(s)\|_{L^2}^2 \, ds \le C(t, w_0). \tag{4-5}$$

Coming back to (3-6) and using (2-4), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma}{2}} w\|_{L^2}^2 &= \left(\tfrac{1}{2} - \delta\right)\int_{\mathbb{R}} \Lambda w w w \, dx \\
&\le C\|\Lambda^{\frac{1}{2}} w\|_{L^2}\|\Lambda^{\frac{1}{2}}(ww)\|_{L^2} \\
&\le C\|w\|_{L^\infty}\|\Lambda^{\frac{1}{2}} w\|_{L^2}^2 \\
&\le C\|w\|_{L^2}^{\frac{\gamma}{\gamma+1}}\|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^{\frac{1}{\gamma+1}}\|\Lambda^{\frac{1}{2}} w\|_{L^2}^2 \\
&\le C\|\Lambda^{\frac{1}{2}} w\|_{L^2}^2(1 + \|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^2)(1 + \|w\|_{L^2}^2),
\end{aligned}$$

which together with the Gronwall inequality and the estimate (4-5) leads to

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\gamma}{2}} w(s)\|_{L^2}^2 \, ds \le C(t, w_0). \tag{4-6}$$

Combining (4-5) and (4-6) gives

$$\|w(t)\|_{L^p} \le \|w(t)\|_{L^2}^{\frac{2}{p}}\|\Lambda^{\frac{1}{2}} w(t)\|_{L^2}^{\frac{p-2}{p}} \le C(t, w_0), \quad \text{for all } 2 \le p < \infty. \tag{4-7}$$

Recalling (3-9), it gives

$$\frac{1}{2}\frac{d}{dt}\|\Lambda w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 \le C\|\Lambda w\|_{L^4}^2\|\Lambda w\|_{L^2} - \delta\int_{\mathbb{R}} w\Lambda w\Lambda^2 w \, dx.$$

Employing the Gagliardo–Nirenberg inequality, it follows that

$$\begin{aligned}
C\|\Lambda w\|_{L^4}^2\|\Lambda w\|_{L^2} &\le C\|\Lambda w\|_{L^2}^{\frac{2\gamma-1}{\gamma}}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^{\frac{1}{\gamma}}\|\Lambda w\|_{L^2} \\
&\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|\Lambda w\|_{L^2}^{\frac{2\gamma}{2\gamma-1}}\|\Lambda w\|_{L^2}^2 \\
&\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}} w\|_{L^2}^{\frac{2(\gamma-1)}{2\gamma-1}}\|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^{\frac{2}{2\gamma-1}}\|\Lambda w\|_{L^2}^2 \\
&\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}} w\|_{L^2}^{\frac{2(\gamma-1)}{2\gamma-1}}(1 + \|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^2)\|\Lambda w\|_{L^2}^2.
\end{aligned}$$

Taking account of (2-4), we conclude that

$$\begin{aligned}
&-\delta\int_{\mathbb{R}} w\Lambda w\Lambda^2 w \, dx \\
&\quad \le C\|\Lambda^{\frac{1}{2}}(w\Lambda w)\|_{L^2}\|\Lambda^{\frac{3}{2}} w\|_{L^2} \\
&\quad \le C\big(\|w\|_{L^\infty}\|\Lambda^{\frac{3}{2}} w\|_{L^2} + \|\Lambda^{\frac{1}{2}} w\|_{L^6}\|\Lambda w\|_{L^3}\big)\|\Lambda^{\frac{3}{2}} w\|_{L^2} \\
&\quad \le C\big(\|w\|_{L^\infty}\|\Lambda^{\frac{3}{2}} w\|_{L^2} + \|w\|_{L^\infty}^{\frac{2}{3}}\|\Lambda^{\frac{3}{2}} w\|_{L^2}^{\frac{1}{3}}\|w\|_{L^\infty}^{\frac{1}{3}}\|\Lambda^{\frac{3}{2}} w\|_{L^2}^{\frac{2}{3}}\big)\|\Lambda^{\frac{3}{2}} w\|_{L^2} \\
&\quad \le C\|w\|_{L^\infty}\|\Lambda^{\frac{3}{2}} w\|_{L^2}^2,
\end{aligned}$$

where we have applied the fractional type Gagliardo–Nirenberg inequalities (see also [Bahouri et al. 2011, Theorem 2.42])

$$\|\Lambda^{\frac{1}{2}} w\|_{L^6} \le C\|w\|_{L^\infty}^{\frac{2}{3}} \|\Lambda^{\frac{3}{2}} w\|_{L^2}^{\frac{1}{3}}, \quad \|\Lambda w\|_{L^3} \le C\|w\|_{L^\infty}^{\frac{1}{3}} \|\Lambda^{\frac{3}{2}} w\|_{L^2}^{\frac{2}{3}}.$$

It follows from the Gagliardo–Nirenberg inequality that

$$\|\Lambda^{\frac{3}{2}} w\|_{L^2} \le C\|\Lambda w\|_{L^2}^{1-\frac{1}{\gamma}} \|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^{\frac{1}{\gamma}},$$
$$\|w\|_{L^\infty} \le C\|w\|_{L^{p_0}}^{\frac{\gamma p_0}{\gamma p_0+2}} \|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^{\frac{2}{\gamma p_0+2}}.$$

Consequently, we deduce

$$\begin{aligned}
-\delta \int_{\mathbb{R}} w\Lambda w \Lambda^2 w\, dx
&\le C\|w\|_{L^{p_0}}^{\frac{\gamma p_0}{\gamma p_0+2}} \|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^{\frac{2}{\gamma p_0+2}} \|\Lambda w\|_{L^2}^{2-\frac{2}{\gamma}} \|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^{\frac{2}{\gamma}} \\
&\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|w\|_{L^{p_0}}^{\frac{\gamma^2 p_0}{(\gamma p_0+2)(\gamma-1)}} \|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^{\frac{2\gamma}{(\gamma p_0+2)(\gamma-1)}} \|\Lambda w\|_{L^2}^2 \\
&\le \tfrac{1}{8}\|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2 + C\|w\|_{L^{p_0}}^{\frac{\gamma^2 p_0}{(\gamma p_0+2)(\gamma-1)}} (1+\|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^2) \|\Lambda w\|_{L^2}^2,
\end{aligned}$$

where in the last line p_0 satisfies $\max\{2, \frac{2-\gamma}{\gamma(\gamma-1)}\} \le p_0 < \infty$. Finally, we get by putting the above estimates together

$$\begin{aligned}
\frac{d}{dt}\|\Lambda w(t)\|_{L^2}^2 + \|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^2
&\le C\big(\|\Lambda^{\frac{1}{2}} w\|_{L^2}^{\frac{2(\gamma-1)}{2\gamma-1}} + \|w\|_{L^{p_0}}^{\frac{\gamma^2 p_0}{(\gamma p_0+2)(\gamma-1)}}\big)(1+\|\Lambda^{\frac{\gamma+1}{2}} w\|_{L^2}^2)\|\Lambda w\|_{L^2}^2.
\end{aligned}$$

With help of (4-5)–(4-7) and the Gronwall inequality, it yields

$$\|\Lambda w(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\gamma+2}{2}} w(s)\|_{L^2}^2\, ds \le C(t, w_0). \tag{4-8}$$

The simple Gagliardo–Nirenberg inequalities allow us to conclude

$$\|\Lambda w\|_{\mathrm{BMO}} \le \|\Lambda w\|_{L^\infty} \le C\|\Lambda w\|_{L^2}^{\frac{\gamma-1}{\gamma}} \|\Lambda^{\frac{\gamma+2}{2}} w\|_{L^2}^{\frac{1}{\gamma}},$$
$$\|w\|_{L^\infty} \le C\|w\|_{L^2}^{\frac{1}{2}} \|\Lambda w\|_{L^2}^{\frac{1}{2}}.$$

Thanks to the above estimates (4-5), (4-7) and (4-8), one infers that

$$\int_0^t \big(\|w(s)\|_{L^\infty}^{\frac{\gamma}{\gamma-1}} + \|\Lambda w(s)\|_{\mathrm{BMO}}\big)\, ds \le C(t, w_0).$$

Therefore, taking advantage of the blow-up criterion (4-1), we immediately complete the proof of Theorem 1.3. □

5. The proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. We begin with the following local well-posedness result and blow-up criterion (see [Bae et al. 2017, Theorem 3.1]).

Proposition 5.1. *Let $\nu \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$. Then for $w_0 \in H^m(\mathbb{R})$ with $m \geq 2$, there exists $T^* = T^*(\|w_0\|_{H^m})$ such that a unique solution of (1-11) belongs to*

$$w \in C([0,\ T^*);\ H^m(\mathbb{R})).$$

Moreover, we have the following blow-up criterion

$$\limsup_{t \to \widetilde{T}} \|w(t)\|_{H^m} = \infty \Leftrightarrow \int_0^{\widetilde{T}} (\|\partial_x u(s)\|_{\mathrm{BMO}} + \|\partial_x w(s)\|_{\mathrm{BMO}})\, ds = \infty.$$

The following lemma plays a crucial role in proving Theorem 1.4 (see [Silvestre 2012a]).

Lemma 5.2. *Consider the following advection fractional-diffusion equation with $0 < \beta \leq 1$ in $\mathbb{R}$*

$$\begin{cases} \partial_t \theta + u \partial_x \theta + \Lambda^\beta \theta = f, \\ \theta(x, 0) = \theta_0(x). \end{cases}$$

Let $T > 0$ be given. Suppose that the θ is bounded and the drift u satisfies $u \in L^\infty((0, T], C^{1-\beta}(\mathbb{R}))$ as well as $f \in L^\infty((0, T] \times \mathbb{R})$. Then the solution θ is Hölder continuous for any positive time $0 < t \leq T$. Moreover, it holds

$$\|\theta\|_{L^\infty((0,T];C^\ell(\mathbb{R}))} \leq C\big(\|\theta\|_{L^\infty([0,T]\times\mathbb{R})} + \|f\|_{L^\infty([0,T]\times\mathbb{R})}\big),$$

where the constant C and $\ell > 0$ depend on β and $\|u\|_{C^{1-\beta}}$ only.

We remark that for the case $\beta \geq 1$, Lemma 5.3 holds true if $u \in L^\infty([0, T] \times \mathbb{R})$.

To prove Theorem 1.4, we shall use the following lemma (see [Silvestre 2012b; Xue and Ye 2018]).

Lemma 5.3. *Consider the following advection fractional-diffusion equation with $0 < \beta \leq 1$ in $\mathbb{R}$*

$$\begin{cases} \partial_t \theta + u \partial_x \theta + \Lambda^\beta \theta = f, \\ \theta(x, 0) = \theta_0(x). \end{cases}$$

Let $T > 0$ be given and

$$u \in L^\infty((0, T], C^{1-\beta+\zeta}(\mathbb{R})), \quad f \in L^\infty((0, T], C^{1-\beta+\zeta}(\mathbb{R})), \quad \text{for any } \zeta \in (0, \beta),$$

then any bounded solution θ in $(0, T] \times \mathbb{R}$ actually belongs to space $C^{1,\zeta}$. Moreover, it holds for any $0 < \nu \leq \zeta$,

$$\|\theta\|_{L^\infty((0,T],C^{1,\nu}(\mathbb{R}))} \leq C\big(\|\theta\|_{L^\infty([0,T]\times\mathbb{R})} + \|f\|_{L^\infty([0,T],C^{1,\zeta}(\mathbb{R}))}\big),$$

where the constant C depends on β and $\|u\|_{C^{1-\beta+\zeta}}$ only.

We remark that for the case $\beta \geq 1$ Lemma 5.3 holds true if $u \in L^\infty([0,T]\times\mathbb{R})$ and $f \in L^\infty([0,T]\times\mathbb{R})$.

Proof of Theorem 1.4. We begin with the case $\beta \geq \frac{1-\gamma}{2}$ with $0 < \gamma \leq 1$. By the maximum principle applied to Equation (1-11) (see Section 2.2 of [Bae et al. 2017]), we have

$$\|w(t)\|_{L^\infty} \leq \|w_0\|_{L^\infty}. \tag{5-1}$$

Recalling (1-11), one has

$$\partial_t w + u\partial_x w + \Lambda^\gamma w = 0. \tag{5-2}$$

Thanks to $\beta \geq \frac{1-\gamma}{2}$, we obtain by direct computation

$$\begin{aligned}\|u\|_{C^{1-\gamma}} &\leq \|(1-\partial_{xx})^{-\beta} w\|_{C^{1-\gamma}}\\ &\leq C\|w\|_{C^{1-\gamma-2\beta}}\\ &\leq C\|w\|_{L^\infty} \leq C(t, w_0).\end{aligned}$$

Applying Lemma 5.2 to (5-2) implies that

$$\|w(t)\|_{C^l} \leq C(t, w_0), \quad \text{for some } l > 0.$$

This allows us to conclude

$$\begin{aligned}\|u\|_{C^\eta} &\leq \|(1-\partial_{xx})^{-\beta} w\|_{C^\eta}\\ &\leq C\|w\|_{C^{\eta-2\beta}}\\ &\leq C\|w(t)\|_{C^l} \leq C(t, w_0),\end{aligned}$$

where $1-\gamma < \eta = \frac{1-\gamma+2\beta+l}{2} \leq l + 2\beta$ due to $\beta \geq \frac{1-\gamma}{2}$ and $l > 0$. Applying Lemma 5.3 to (5-2) implies that

$$\|w(t)\|_{C^{1,\vartheta}} \leq C(t, w_0), \quad \text{for some } \vartheta > 0.$$

Therefore, we have

$$\begin{aligned}\|\partial_x u(s)\|_{\mathrm{BMO}} + \|\partial_x w(s)\|_{\mathrm{BMO}} &\leq C(\|\partial_x u(s)\|_{L^\infty} + \|\partial_x w(s)\|_{L^\infty})\\ &\leq C\|w(t)\|_{C^{1,\vartheta}} \leq C(t, w_0).\end{aligned}$$

By Proposition 5.1, we conclude the proof of the case $\beta \geq \frac{1-\gamma}{2}$ with $0 < \gamma \leq 1$.

Next, we will consider the case $\beta \geq \frac{1}{2}$ with $\gamma = 0$. Moreover, it suffices to deal with $\beta = \frac{1}{2}$ as the remainder case $\beta > \frac{1}{2}$ is relatively easy (see [Bae et al. 2017, Theorem 3.3]). Thus, the corresponding equation is

$$\partial_t w + u\partial_x w = 0, \quad u = (1-\partial_{xx})^{-\frac{1}{2}} w. \tag{5-3}$$

It still holds true that

$$\|w(t)\|_{L^\infty} \le \|w_0\|_{L^\infty}.$$

Applying $\mathcal{J}^m$ on (5-3), taking its L^2 inner product with $\mathcal{J}^m w$ and using (2-3) yields

$$\begin{aligned}
\tfrac{1}{2}\frac{d}{dt}\|w(t)\|_{H^m}^2 & \\
&= -\int_{\mathbb{R}} \mathcal{J}^m(u\partial_x w)\mathcal{J}^m w\,dx \\
&= -\int_{\mathbb{R}} [\mathcal{J}^m, u]\partial_x w\mathcal{J}^m w\,dx - \int_{\mathbb{R}} u\partial_x\mathcal{J}^m w\mathcal{J}^m w\,dx \\
&\le \|\mathcal{J}^m, u]\partial_x w\|_{L^2}\|\mathcal{J}^m w\|_{L^2} + \tfrac{1}{2}\int_{\mathbb{R}} \partial_x u\mathcal{J}^m w\mathcal{J}^m w\,dx \\
&\le C\big(\|\partial_x u\|_{L^\infty}\|w\|_{H^m} + \|\partial_x w\|_{L^{2m}}\|\mathcal{J}^{m-1}w\|_{L^{\frac{2m}{m-1}}}\big)\|w\|_{H^m} \\
&\le C\big(\|\partial_x u\|_{L^\infty}\|w\|_{H^m} + \|w\|_{L^\infty}^{\frac{m-1}{m}}\|\mathcal{J}^m w\|_{L^2}^{\frac{1}{m}}\|w\|_{L^\infty}^{\frac{1}{m}}\|\mathcal{J}^m w\|_{L^2}^{\frac{m-1}{m}}\big)\|w\|_{H^m} \\
&\le C(\|\partial_x u\|_{L^\infty} + \|w\|_{L^\infty})\|w\|_{H^m}^2.
\end{aligned}$$

By means of (4-3), we obtain

$$\begin{aligned}
\frac{d}{dt}\|w(t)\|_{H^m} &\le C(\|\partial_x u\|_{\mathrm{BMO}} + \|w\|_{L^\infty})\ln(e + \|w\|_{H^m})\|w\|_{H^m} \\
&\le C(\|w\|_{\mathrm{BMO}} + \|w\|_{L^\infty})\ln(e + \|w\|_{H^m})\|w\|_{H^m} \\
&\le C\|w\|_{L^\infty}\ln(e + \|w\|_{H^m})\|w\|_{H^m},
\end{aligned}$$

where we use the fact that the pseudodifferential operator $(1-\partial_{xx})^{-\frac{1}{2}}\partial_x$ maps space BMO into itself boundedly and $L^\infty \hookrightarrow \mathrm{BMO}$. By the Gronwall inequality, it implies

$$\|w(t)\|_{H^m} \le C(t, w_0).$$

Finally, let us say some words about the case $\beta \ge 0$ with $1 < \gamma \le 2$. As a matter of fact, the remark of Lemma 5.3 and (5-1) allow us to show that

$$\|w(t)\|_{C^{1,\vartheta}} \le C(t, w_0), \quad \text{for some } \vartheta > 0.$$

Therefore, we conclude the desired result for the case $\beta \ge 0$ with $1 < \gamma \le 2$. This eventually ends the proof of Theorem 1.4. □

Appendix: Besov spaces and fractional Gagliardo–Nirenberg inequality

This appendix provides the definition of Besov spaces and the fractional Gagliardo–Nirenberg inequality. We denote the function spaces of rapidly decreasing functions by $S(\mathbb{R}^n)$, tempered distributions by $S'(\mathbb{R}^n)$, and polynomials by $\mathcal{P}(\mathbb{R}^n)$. Now let us begin with the Littlewood–Paley theory (see for instance [Bahouri et al. 2011]). We choose some smooth radial nonincreasing function χ with values in $[0,1]$ such that $\chi \in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B} := \{\xi \in \mathbb{R}^n : |\xi| \le \frac{4}{3}\}$ and with value 1 on $\{\xi \in \mathbb{R}^n : |\xi| \le \frac{3}{4}\}$, then we set $\varphi(\xi) = \chi\big(\frac{\xi}{2}\big) - \chi(\xi)$. One easily verifies that $\varphi \in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C} := \{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ and satisfies

$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n.$$

Let $h = \mathcal{F}^{-1}(\varphi)$ and $\tilde{h} = \mathcal{F}^{-1}(\chi)$, then we introduce the dyadic blocks Δ_j of our decomposition by setting

$$\Delta_j u = 0, \quad j \le -2; \qquad \Delta_{-1} u = \chi(D) u = \int_{\mathbb{R}^n} \tilde{h}(y) u(x-y)\, dy;$$

$$\Delta_j u = \varphi(2^{-j} D) u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) u(x-y)\, dy, \quad \text{for all } j \in \mathbb{N}.$$

We shall also use the following low-frequency cut-off:

$$S_j u = \chi(2^{-j} D) u = \sum_{-1 \le k \le j-1} \Delta_k u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}(2^j y) u(x-y)\, dy, \quad \text{for all } j \in \mathbb{N}.$$

Meanwhile, we define the homogeneous dyadic blocks as

$$\dot{\Delta}_j u = \varphi(2^{-j} D) u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) u(x-y)\, dy, \quad \text{for all } j \in \mathbb{Z}.$$

Let us recall the definition of homogeneous and inhomogeneous Besov spaces through the dyadic decomposition.

Definition A.1. Let $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\dot{B}^s_{p,r} = \{ f \in S'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) : \|f\|_{\dot{B}^s_{p,r}} < \infty \},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} = \begin{cases} \big(\sum_{j\in\mathbb{Z}} 2^{jrs} \|\dot{\Delta}_j f\|^r_{L^p}\big)^{1/r} & \text{for all } r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p} & \text{for all } r = \infty. \end{cases}$$

Definition A.2. Let $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^n)$ such that

$$B^s_{p,r} = \{ f \in S'(\mathbb{R}^n); \|f\|_{B^s_{p,r}} < \infty \},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \big(\sum_{j\ge -1} 2^{jrs} \|\Delta_j f\|^r_{L^p}\big)^{1/r} & \text{for all } r < \infty, \\ \sup_{j\ge -1} 2^{js} \|\Delta_j f\|_{L^p} & \text{for all } r = \infty. \end{cases}$$

For $s > 0, (p, r) \in [1, +\infty]^2$, the inhomogeneous Besov space norm $B^s_{p,r}$ is equal to

$$\|f\|_{B^s_{p,r}} \approx \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,r}}.$$

For any $s \in \mathbb{R}$ and $1 < p < \infty$, we have

$$B^s_{p,\min\{p,2\}} \hookrightarrow W^{s,p} \hookrightarrow B^s_{p,\max\{p,2\}}.$$

For any noninteger $\sigma > 0$, the Hölder space C^σ is equivalent to $B^\sigma_{\infty,\infty}$.

Let us recall the following fractional type Gagliardo–Nirenberg inequality due to Hajaiej, Molinet, Ozawa and Wang [Hajaiej et al. 2011].

Lemma A.3. *Let* $0 < p, p_0, p_1, q, q_0, q_1 \leq \infty,\ s, s_0, s_1 \in \mathbb{R}$ *and* $0 \leq \vartheta \leq 1$. *Then the following fractional type Gagliardo–Nirenberg inequality*

$$\|v\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \leq C\|v\|^{1-\vartheta}_{\dot{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n)} \|v\|^{\vartheta}_{\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n)}$$

holds for all $v \in \dot{B}^{s_0}_{p_0,q_0} \cap \dot{B}^{s_1}_{p_1,q_1}$ *if and only if*

$$\frac{n}{p} - s = (1-\vartheta)\Big(\frac{n}{p_0} - s_0\Big) + \vartheta\Big(\frac{n}{p_1} - s_1\Big) \quad \text{if } s \leq (1-\vartheta)s_0 + \vartheta s_1,$$
$$\frac{1}{q} \leq \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1} \quad \text{if } p_0 \neq p_1 \text{ and } s = (1-\vartheta)s_0 + \vartheta s_1,$$
$$s_0 \neq s_1 \quad \text{or} \quad \frac{1}{q} \leq \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1} \quad \text{if } p_0 = p_1 \text{ and } s = (1-\vartheta)s_0 + \vartheta s_1$$
$$s_0 - \frac{n}{p_0} \neq s - \frac{n}{p} \quad \text{or} \quad \frac{1}{q} \leq \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1} \quad \text{if } s < (1-\vartheta)s_0 + \vartheta s_1.$$

Remark. Lemma A.3 is also true in the nonhomogeneous framework.

Lemma A.4. *Let* $p, q, r \in (1, \infty),\ \theta \in [0, 1]$ *and* $s, s_1, s_2 \in \mathbb{R}$, *then*

$$\|\Lambda^s u\|_{L^p} \leq C\|u\|^{1-\theta}_{L^q} \|\Lambda^{s_1} u\|^{\theta}_{L^r},$$

where

$$\frac{1}{p} - \frac{s}{n} = \frac{1-\theta}{q} + \theta\Big(\frac{1}{r} - \frac{s_1}{n}\Big), \quad s \leq \theta s_1.$$

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ZHUAN YE
DEPARTMENT OF MATHEMATICS AND STATISTICS
JIANGSU NORMAL UNIVERSITY
XUZHOU, JIANGSU
CHINA
yezhuan815@126.com

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