Pacific Journal of Mathematics

ON THE COMMUTATIVITY OF COSET PRESSURE

BING LI AND WEN-CHIAO CHENG

Volume 305 No. 1 March 2020

PACIFIC JOURNAL OF MATHEMATICS
Vol. 305, No. 1, 2020
dx.doi.org/10.2140/pjm.2020.305.219

ON THE COMMUTATIVITY OF COSET PRESSURE

BING LI AND WEN-CHIAO CHENG

We establish the conditional entropy with a coset partition and coset pressure for subadditive potentials via separated sets on a compact metric group. Analogues of the variational principle and thermodynamic formalism are shown in this system. The major finding of this study consists of its presentation of the commutativity proposition for those conjugate invariants.

1. Introduction

A basic issue in the theory of dynamical systems is the study of the complexity of orbits. This has led to the development of many different subjects in mathematics. In 1958, Kolmogorov applied the notion of entropy from information theory to ergodic theory. Since then, the notion of entropy has played an important role in understanding the complexity of various dynamical systems. The two main types of entropy are measure-theoretic (or metric) entropy and topological entropy. The former measures the maximal loss of information of the iteration of finite partitions in a measure preserving transformation. The latter measures the maximal exponential growth rate of orbits for an arbitrary topological dynamical system. These two notions are connected by the so-called variational principle. This relation states that the topological entropy is the supremum of the metric entropies for all invariant probability measures of a given topological system, and has received considerable attention.

As a natural generalization of topological entropy, topological pressure is a quantity which belongs to one of the concepts in the thermodynamic formalism. The thermodynamic formalism itself is a generalization of the concepts of statistical physics to the area of mathematical dynamical systems theory. Ruelle [1973] first introduced the concept of topological pressure of additive potentials for expansive dynamical systems. Walters [1982] then extended this concept to the compact space with the continuous transformation. Moreover, in some cases, the values of entropy functions can be expressed as the topological pressure of certain functions related to dynamical systems. Numerous nonlinear physical problems involve a complicated discrete dynamical system. Topological pressure contains information

Cheng is the corresponding author.
MSC2010: 28D20.
Keywords: thermodynamic formalism, coset pressure, conditional entropy, commutativity.

on the dynamics of the system; these dynamics can be extracted by varying the potential energy function. Related studies include [Falconer 1988; Gelfert and Wolf 2008; Huang et al. 2008; Huang and Yi 2007; Molaei 2008; Pollner and Vattay 1996; Spandl 2008]. The framework presented by Bowen [1971] has caused topological pressure to become a fundamental tool for studying the multifractal formalism of dimension theory, especially for examining nonconformal dynamical systems in statistical mechanics; see [Falconer 1988; Pesin 1997].

Furthermore, the notions of topological entropy and topological pressure introduced above are applied to autonomous dynamical systems. Kolyada, Misiurewicz, and Snoha [Kolyada et al. 1999; Kolyada and Snoha 1996] introduced topological entropy for a nonautonomous dynamical system given by a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous self-maps of a compact metric space. More precisely, Kolyada and Snoha [1996] showed that the topological entropy of the composition of two continuous self-maps of a compact metric space does not depend on the order in which functions compose, that is, $h_{\text{top}}(S \circ T) = h_{\text{top}}(T \circ S)$. This fact demonstrates that the dynamics of $S \circ T$ must exhibit some common features with that of $T \circ S$, although $S \circ T$ and $T \circ S$ do not usually coincide. Kong, Cheng, and Li [Kong et al. 2015] generalized topological entropy to the topological pressure of $(X;\, f_{1,\infty})$. They analyzed those basic pressure propositions concerning a nonautonomous dynamical system $(X;\, f_{1,\infty})$ given by a compact metric space X and a sequence $f_{1,\infty} = \{f_n\}_{n=1}^{\infty}$ of continuous self-maps of X. They also showed that, for any continuous maps T and S from a compact metric space into itself, the maps $T \circ S$ and $S \circ T$ have the same topological pressure. Different propositions regarding nonautonomous dynamical systems also can be found in [Cánovas 2011; Kuang et al. 2013].

Essentially, the thermodynamic formalism can be described as a rigorous study of certain mathematical structures inspired in thermodynamics. Balibrea, López and Peña [Balibrea et al. 1999b] presented the commutativity proposition for topological pressure by using thermodynamic formalism. The commutativity proposition for the sequence entropy on the interval also can be obtained in [Balibrea et al. 1999a]. Here we pursue the commutativity for coset pressure and conditional entropy function. The outline of the paper is as follows. Section 2 establishes basic entropy with a coset partition and shows the variational principle. Then we discuss the commutativity of conditional entropy. Section 3 studies coset pressure with a sequence of subadditive potential functions and demonstrates the thermodynamic formalism. Then, along with [Balibrea et al. 1999b] and based on thermodynamic formalism, the main result of the commutativity proposition also holds for the coset pressure.

2. Commutativity of entropy

In dynamical systems and ergodic theory it is understood that a reasonable measure-theoretic or topological entropy should be a measure of the uncertainty of the system and that they should be invariant under measurable or topological change of coordinates, respectively. This section reviews the concept of conditional entropy in the context of a probability space and topological entropy in the context of the compact metric space. We then show the commutativity property of conditional entropy as follows.

Assume that (X, d) is a compact metric space with metric d and $T : X \to X$ is a continuous selfmap. For any $n \in \mathbb{N}$, first we define the Bowen metric on X as

$$d_n(x, y) = \max_{0 \le i \le n-1} d(T^i x, T^i y) \quad \text{for all } x, y \in X.$$

Let K be a compact subset of X. For any $n \in \mathbb{N}$ and $\epsilon > 0$, a subset F of X is said to be an (n, ϵ)-spanning set of K with respect to T if for all $x \in K$, there exists $y \in F$ with $d_n(x, y) \le \epsilon$, i.e.,

$$K \subset \bigcup_{y \in F} \bigcap_{i=0}^{n-1} T^{-i} \bar{B}(T^i y; \epsilon),$$

where $B(T^i y; \epsilon)$ represents the open ball with center $T^i y$ and radius ϵ in the metric d, and $\bar{B}(T^i y; \epsilon)$ is the corresponding closed ball. Let $r(n, \epsilon, K)$ denote the smallest cardinality of (n, ϵ)-spanning set for K with respect to T. A subset E of K is said to be (n, ϵ)-separated with respect to T if $x, y \in E, x \ne y$, implies $d_n(x, y) > \epsilon$. In other words, for $x \in E$, the set $\bigcap_{i=0}^{n-1} T^{-i} \bar{B}(T^i x; \epsilon)$ contains no other points of E except x itself. Let $s(n, \epsilon, K)$ be the largest cardinality of a (n, ϵ)-separated subset of K with respect to T.

In the compact metric space, the topological entropy of a set K introduced by Bowen and defined by the separated or spanning sets can also be given using open covers. Let α be an open cover of X and denote by $\aleph(\alpha|_K)$ the number of sets in a finite subcover of α with the smallest cardinality for K. The entropy of α on K is defined by $H(\alpha|_K) = \log \aleph(\alpha|_K)$, and the topological entropy of T for K is as follows:

$$\begin{aligned} h_{\text{top}}(T|K) &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, K) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, K) \\ &= \sup_{\text{open cover } \beta} \limsup_{n \to \infty} \frac{1}{n} \log \aleph\big(\textstyle\bigvee_{i=0}^{n-1} T^{-i} \beta|_K\big), \end{aligned}$$

where

$$\textstyle\bigvee_{i=0}^{n-1} T^{-i} \beta = \{A_{i_0} \cap T^{-1} A_{i_1} \cap \cdots \cap T^{-(n-1)} A_{i_{n-1}} : A_{i_j} \in \beta\}.$$

We assume throughout that $(X, d, \cdot)$ is a compact group with metric d and group product $\cdot$. $T : X \to X$ is a continuous endomorphism, and B is a closed T-invariant subgroup of X, where T-invariant means $T^{-1}(B) = B$. For convenience, some more notations are as follows:

- $[B]$ is the coset partition, i.e.,

$$[B] = \{B \cdot x : x \in X\},$$

  where $B \cdot x = \{y \cdot x : y \in B\}$. It is easy to check that the collection of sets $B \cdot x$ forms a partition of X.

- $\mathcal{M}(X)$ is the set of all Borel probability measures on X. $\mathcal{M}(X, T)$ is the subset of $\mathcal{M}(X)$ with all T-invariant measures. $\mathcal{E}(X, T)$ is the subset of $\mathcal{M}(X, T)$ with all T-invariant ergodic measures.

- For any partition ξ of X,

$$\xi^n = \bigvee_{i=0}^{n-1} T^{-i}\xi = \{A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-(n-1)}A_{i_{n-1}} : A_{i_j} \in \xi, 0 \le j \le n-1\}.$$

Under the same assumption as above, the topological entropy of T for $B \cdot x$ is defined to be

$$\begin{aligned} h_{\text{top}}(T \mid B \cdot x) &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, B \cdot x) \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon, B \cdot x) \\ &= \sup_{\text{open cover } \beta} \limsup_{n \to \infty} \frac{1}{n} \log \aleph\Big(\bigvee_{i=0}^{n-1} T^{-i}\beta|_{B \cdot x}\Big). \end{aligned}$$

The conditional entropy is usually defined as follows. Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $\mathcal{A}$ and $\mathcal{C}$ be two partitions of $(X, \mathcal{B}, \mu)$ with

$$\mathcal{A} = \{A_1, \ldots, A_k\} \quad \text{and} \quad \mathcal{C} = \{C_1, \ldots, C_p\}.$$

The entropy of $\mathcal{A}$ given $\mathcal{C}$ is the number

$$H_\mu(\mathcal{A} \mid \mathcal{C}) = -\sum_{j=1}^{p} \mu(C_j) \sum_{i=1}^{k} \frac{\mu(A_i \cap C_j)}{\mu(C_j)} \log \frac{\mu(A_i \cap C_j)}{\mu(C_j)}$$

omitting the j-terms when $\mu(C_j) = 0$. Here, the summation of each quantity is 1, i.e., $\sum_{j=1}^{p} \mu(C_j) = 1$.

Next, let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure preserving transformation of probability space $(X, \mathcal{B}, \mu)$ (i.e., if $A \in \mathcal{B}$, then $T^{-1}A \in \mathcal{B}$ and $\mu(T^{-1}A) = \mu(A)$). For any finite partition α of X, we consider the refinement α^n of α. Again, B is a closed T-invariant subgroup. Then $B \cdot x$ is closed and $[B] = T^{-1}[B]$. Therefore each element in the partition $[B]$ belongs to $\mathcal{B}$. Then we consider the conditional entropy given by the coset partition $[B]$, defined by $H_\mu(\alpha^n \mid [B]) = H_\mu\big(\bigvee_{i=0}^{n-1} T^{-i}\alpha \mid [B]\big)$.

Good references for these entropy invariants are [Brucks and Bruin 2004; Katok and Hasselblatt 1995], which contain many of the relevant earlier references.

Lemma 2.1 [Cheng 2006]. *The sequence* $a_n = H_\mu(\alpha^n \mid [B])$ *is subadditive*, *that is*,

$$a_{n+m} = H_\mu(\alpha^{n+m} \mid [B]) \le H_\mu(\alpha^n \mid [B]) + H_\mu(\alpha^m \mid [B]) = a_n + a_m.$$

The conditional entropy of α given $[B]$ with respect to T is the value

$$\begin{aligned} h_\mu(T \mid [B], \alpha) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha^n \mid [B]) \\ &= \inf_{n\ge 1} \frac{1}{n} H_\mu(\alpha^n \mid [B]) \end{aligned}$$

and the conditional entropy of T with respect to μ and $[B]$ is defined to be

$$h_\mu(T \mid [B]) = \sup_\alpha h_\mu(T \mid [B], \alpha),$$

where α ranges over all finite partitions of X.

The well-known variational principle for entropy shows the relationship between topological entropy and metric entropy. We only can obtain the variational inequality for the invariant partition. However, under the condition of coset partition and using Misiurewicz's technique and more advanced ergodic analysis, a similar type of local variational principle can be obtained as follows.

Theorem 2.2 [Cheng 2006] (variational principle). *Let* $T : X \to X$ *be a continuous endomorphism of the compact group* $(X, d, \cdot\,)$ *with closed* T*-invariant subgroup* B*. Then we have*

$$\sup_{x\in X} h_{\text{top}}(T \mid B\cdot x) = \sup_{\mu\in\mathcal{M}(X,T)} h_\mu(T \mid [B]).$$

Next, we investigate the invariant measure for the measurable selfmap. Assume that $S : X \to X$ and $T : X \to X$ are functions. The composition of S and T is given by $S \circ T(x) = S(T(x))$. It is obvious that $\mathcal{M}(X, S) \cap \mathcal{M}(X, T) \subseteq \mathcal{M}(X, S \circ T)$. Let $\widetilde{S} : \mathcal{M}(X) \to \mathcal{M}(X)$ be defined as follows. For any $\mu \in \mathcal{M}(X)$, the measure $\widetilde{S}\mu$ is defined by $\widetilde{S}\mu(E) = \mu(S^{-1}(E))$ for all measurable set $E \subset X$. If μ is an invariant measure of S, then $\widetilde{S}\mu = \mu$. If $\mu \in \mathcal{M}(X, S) \cap \mathcal{M}(X, S \circ T)$, then $\widetilde{S}\mu = \mu$ and hence, $\widetilde{T}\mu = \mu$. Thus μ is an invariant measure for S and T.

Lemma 2.3 [Balibrea et al. 1999b]. *Given the same conditions as indicated above*, *we can see that* $\widetilde{S \circ T} = \widetilde{S} \circ \widetilde{T}$.

Lemma 2.4 [Walters 1982]. *Given the same conditions as indicated above and that* $\phi : X \to R$ *is a continuous real-value function*, *we can see that*

$$\int \phi \, d(\widetilde{S}\mu) = \int \phi \circ S \, d\mu.$$

Lemma 2.5 [Balibrea et al. 1999b]. *Given the same conditions as indicated above, the maps*

$$\widetilde{T}\,|_{\mathcal{M}(X,S\circ T)} : \mathcal{M}(X, S\circ T) \to \mathcal{M}(X, T\circ S),$$
$$\widetilde{S}\,|_{\mathcal{M}(X,T\circ S)} : \mathcal{M}(X, T\circ S) \to \mathcal{M}(X, S\circ T)$$

are well defined bijections.

Theorem 2.6. *Assume that $S, T : X \to X$ are continuous endomorphisms and let $\mu \in \mathcal{M}(X, S\circ T)$. B is a closed S, T-invariant subgroup of X, with coset partition $[B]$. Then, we have*

$$h_\mu(S\circ T \mid [B]) = h_{\widetilde{T}\mu}(T\circ S \mid [B]).$$

Proof. Assume $\mathcal{A}$ is a finite partition of X, then we have that

$$H_\mu(S^{-1}\mathcal{A} \mid [B]) = H_{\widetilde{S}\mu}(\mathcal{A} \mid [B])$$

and clearly that

$$h_\mu(S\circ T \mid [B], \mathcal{A}) = h_\mu(S\circ T \mid [B], (S\circ T)^{-1}\mathcal{A}).$$

Moreover,

$$\begin{aligned} H_\mu\big(\textstyle\bigvee_{i=0}^{n-1}(S\circ T)^{-i}\mathcal{A} \mid [B]\big) &= H_\mu\big(T^{-1}\textstyle\bigvee_{i=0}^{n-2}(S\circ T)^{-i}(S^{-1}\mathcal{A}) \mid [B]\big) \\ &= H_\mu\big(T^{-1}\textstyle\bigvee_{i=0}^{n-2}(S\circ T)^{-i}(S^{-1}\mathcal{A}) \mid T^{-1}[B]\big) \\ &= H_{\widetilde{T}\mu}\big(\textstyle\bigvee_{i=0}^{n-2}(S\circ T)^{-i}(S^{-1}\mathcal{A}) \mid [B]\big). \end{aligned}$$

We can divide by n and let n go to ∞ to obtain

$$h_\mu(S\circ T \mid [B], \mathcal{A}) = h_{\widetilde{T}\mu}(T\circ S \mid [B], S^{-1}\mathcal{A}).$$

This implies $h_\mu(S\circ T \mid [B], \mathcal{A}) = h_{\widetilde{T}\mu}(T\circ S \mid [B], S^{-1}\mathcal{A}) \le h_{\widetilde{T}\mu}(T\circ S \mid [B])$. Furthermore, since $\mu \in \mathcal{M}(X, S\circ T)$, we have $\widetilde{S}\widetilde{T}\mu = \mu$, and

$$h_{\widetilde{T}\mu}(T\circ S \mid [B], \mathcal{A}) = h_\mu(S\circ T) \mid [B], T^{-1}\mathcal{A}) \le h_\mu(S\circ T \mid [B]).$$

Thus $h_\mu(S\circ T \mid [B]) = h_{\widetilde{T}\mu}(T\circ S \mid [B])$. □

We have $\widetilde{T}\mu = \mu$ if $\mu \in \mathcal{M}(X, T)$. Therefore the commutativity for conditional entropy holds under some situations.

Lemma 2.7. *Assume that $S, T : X \to X$ are continuous endomorphisms and let $\mu \in \mathcal{M}(X, S) \cap \mathcal{M}(X, T)$. If B is a closed S, T-invariant subgroup of X, with coset partition $[B]$, then we have*

$$h_\mu(S\circ T \mid [B]) = h_\mu(T\circ S \mid [B]).$$

3. Commutativity of pressure

Topological pressure is an important generalization of topological entropy. It is well known that the topological pressure with a potential function plays a fundamental role in the study of the Hausdorff dimension of repellers and the hyperbolic set. It roughly measures the orbit structure complexity of the iterated map on the potential function. During this section, along with the discussion of the approximation provided in the study by Cao, Feng and Huang [Cao et al. 2008], we define the subadditivity of a sequence of functions and coset pressure in the following. Then we state the thermodynamic formalism and use this formalism to demonstrate the commutativity of coset pressure.

A sequence $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ of functions on X is called subadditive if each f_n is a positive real-valued continuous function on X with

$$f_{n+m}(x) \le f_n(x) f_m(T^n x) \quad \text{for all } x \in X, m, n \in \mathbb{N}.$$

Assume $[B]$ is a coset partition of this compact group X and $T : X \to X$ is a continuous endomorphism. Therefore the coset pressure is defined as follows:

$$P_n(T, \mathcal{F}, \epsilon, B \cdot x) = \sup_E \left\{ \sum_{y \in E} f_n(y) : E \text{ is an } (n, \epsilon)\text{-separated subset of } B \cdot x \right\}.$$

and

$$P(T, \mathcal{F}, \epsilon, B \cdot x) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(T, \mathcal{F}, \epsilon, B \cdot x).$$

Then

$$P(T, \mathcal{F}, [B], \epsilon) = \sup_{x \in X} P(T, \mathcal{F}, \epsilon, B \cdot x).$$

It is easy to show that $P(T, \mathcal{F}, [B], \epsilon)$ is a decreasing function of ϵ. Therefore the coset pressure of T with respect to $\mathcal{F}$ and $[B]$ is defined as follows:

$$P(T, \mathcal{F}, [B]) = \lim_{\epsilon \to 0} P(T, \mathcal{F}, [B], \epsilon).$$

Furthermore, we consider a sequence of positive real-valued functions $\{f_n\}$ on X. For a T-invariant Borel probability measure μ, denote

$$\mathcal{F}_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int \log f_n \, d\mu.$$

A standard subadditive argument assures the existence of the above limit. When $\mu \in \mathcal{E}(X, T)$, by the subadditive ergodic theorem, the above limit exists μ-almost everywhere without integrating against μ.

The basic propositions of coset pressure are provided in [Zhao and Cheng 2014]. The following thermodynamic formalism is the main finding of coset pressure, and gives the relation among $P(T, \mathcal{F}, [B])$, $h_\mu(T \mid [B])$ and $\mathcal{F}_*(\mu)$. The process of the proof is influenced by the methods in [Cao et al. 2008].

Theorem 3.1. *Let $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ be a sequence of subadditive potential functions on the compact group X and let $T : X \to X$ be a continuous endomorphism with a closed T-invariant subgroup B. Then*

$$P(T,\mathcal{F},[B])=\begin{cases}-\infty \text{ if } \mathcal{F}_*(\mu)=-\infty \text{ for all } \mu\in M(X,T),\\ \sup\{h_\mu(T\mid[B])+\mathcal{F}_*(\mu):\mu\in M(X,T),\mathcal{F}_*(\mu)\neq-\infty\} \text{ otherwise.}\end{cases}$$

In physics, a continuous function is regarded as a complicated potential function. Thus, setting $\mathcal{F}=\phi$, the usual variational principle is as follows. Furthermore, we can present the commutativity proposition for coset pressure by using thermodynamic formalism.

$$P(T,\phi,[B])=\sup_{\mu\in\mathcal{M}(X,T)}\{h_\mu(T\mid[B])+\int\phi\,d\mu\}.$$

Theorem 3.2. *Assume that both S, $T : X \to X$ are continuous endomorphisms and $\phi : X \to R$ is a continuous real-value function. If B is a closed S, and T-invariant subgroup of X, with coset partition $[B]$. Then, we have*

$$P(S\circ T,\phi,[B])=P(T\circ S,\phi\circ S,[B]).$$

Proof. With the variational principle and Lemmas 2.4, 2.5, we have

$$\begin{aligned}P(T\circ S,\phi\circ S,[B])&=\sup_{\mu\in\mathcal{M}(X,T\circ S)}\{h_\mu(T\circ S)+\int\phi\circ S\,d\mu\}\\ &=\sup_{\mu\in\mathcal{M}(X,T\circ S)}\{h_{\widetilde{S}\mu}(S\circ T)+\int\phi\,d(\widetilde{S}\mu)\}\\ &=\sup_{\widetilde{S}\mu\in\mathcal{M}(X,S\circ T)}\{h_{\widetilde{S}\mu}(S\circ T)+\int\phi\,d(\widetilde{S}\mu)\}\\ &=P(S\circ T,\phi,[B]).\end{aligned}$$

□

Using Theorem 3.2, we have the following lemma:

Lemma 3.3. *If $\mathcal{M}(X, T\circ S)=\mathcal{M}(X, S)$, then*

$$P(T\circ S,\phi,[B])=P(S\circ T,\phi,[B]).$$

Next, assume $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ is a sequence of subadditive potential functions. Let $\mathcal{F}\circ S=\{\log f_n\circ S\}_{n=1}^{\infty}$ and $\mathcal{F}_*\circ S(\mu)=\lim_{n\to\infty}\frac{1}{n}\int\log f_n\circ S\,d\mu$. With the same process as was used for the proof, the next proposition is trivial.

Theorem 3.4. *Assume that $S, T : X \to X$ are continuous endomorphisms and that $\mathcal{F}=\{\log f_n\}_{n=1}^{\infty}$ is a sequence of subadditive potential functions. If B is a closed S, T-invariant subgroup of X, with coset partition $[B]$, then we have*

$$P(S\circ T,\mathcal{F},[B])=P(T\circ S,\mathcal{F}\circ S,[B]).$$

Similarly, using Theorem 3.4, we have the following lemma:

Lemma 3.5. *If* $\mathcal{M}(X, T \circ S) = \mathcal{M}(X, S)$, *then*

$$P(T \circ S, \mathcal{F}, [B]) = P(S \circ T, \mathcal{F}, [B]).$$

Acknowledgements

This work was carried out when Bing Li visited NCTS at the National Taiwan University and Wen-Chiao Cheng visited the University of Paris-Est Cretéil during the summer of 2016. The authors sincerely appreciate the warm hospitality of the hosts. Wen-Chiao Cheng is partially supported by NSC Grant 105-2115-M-034-001. Bing Li is partially supported by NSFC 11671151 and Guangdong Natural Science Foundation 2018B0303110005.

References

[Balibrea et al. 1999a] F. Balibrea, J. S. Cánovas Peña, and V. Jiménez López, "Commutativity and non-commutativity of topological sequence entropy", *Ann. Inst. Fourier* (*Grenoble*) **49**:5 (1999), 1693–1709. MR Zbl

[Balibrea et al. 1999b] F. Balibrea, V. Jiménez López, and J. S. Cánovas Peña, "Some results on entropy and sequence entropy", *Int. J. Bifur. Chaos Appl. Sci. Eng.* **9**:9 (1999), 1731–1742. MR Zbl

[Bowen 1971] R. Bowen, "Entropy for group endomorphisms and homogeneous spaces", *Trans. Amer. Math. Soc.* **153** (1971), 401–414. MR Zbl

[Brucks and Bruin 2004] K. M. Brucks and H. Bruin, *Topics from one-dimensional dynamics*, Lond. Math. Soc. Student Texts **62**, Cambridge Univ. Press, 2004. MR Zbl

[Cánovas 2011] J. S. Cánovas, "Li–Yorke chaos in a class of nonautonomous discrete systems", *J. Difference Equ. Appl.* **17**:4 (2011), 479–486. MR Zbl

[Cao et al. 2008] Y.-L. Cao, D.-J. Feng, and W. Huang, "The thermodynamic formalism for sub-additive potentials", *Discrete Contin. Dyn. Syst.* **20**:3 (2008), 639–657. MR Zbl

[Cheng 2006] W.-C. Cheng, "Conditional entropy with a coset partition", *Far East J. Dyn. Syst.* **8**:2 (2006), 193–208. MR Zbl

[Falconer 1988] K. J. Falconer, "A subadditive thermodynamic formalism for mixing repellers", *J. Phys. A* **21**:14 (1988), L737–L742. MR Zbl

[Gelfert and Wolf 2008] K. Gelfert and C. Wolf, "Topological pressure via saddle points", *Trans. Amer. Math. Soc.* **360**:1 (2008), 545–561. MR Zbl

[Huang and Yi 2007] W. Huang and Y. Yi, "A local variational principle of pressure and its applications to equilibrium states", *Israel J. Math.* **161** (2007), 29–74. MR Zbl

[Huang et al. 2008] X. Huang, X. Wen, and F. Zeng, "Topological pressure of nonautonomous dynamical systems", *Nonlinear Dyn. Syst. Theory* **8**:1 (2008), 43–48. MR Zbl

[Katok and Hasselblatt 1995] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyc. Math. Appl. **54**, Cambridge Univ. Press, 1995. MR Zbl

[Kolyada and Snoha 1996] S. Kolyada and L. Snoha, "Topological entropy of nonautonomous dynamical systems", *Random Comput. Dynam.* **4**:2-3 (1996), 205–233. MR Zbl

[Kolyada et al. 1999] S. Kolyada, M. Misiurewicz, and L. Snoha, "Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval", *Fund. Math.* **160**:2 (1999), 161–181. MR Zbl

[Kong et al. 2015] M. Kong, W.-C. Cheng, and B. Li, "Topological pressure for nonautonomous systems", *Chaos Solitons Fractals* **76** (2015), 82–92. MR Zbl

[Kuang et al. 2013] R. Kuang, W.-C. Cheng, and B. Li, "Fractal entropy of nonautonomous systems", *Pacific J. Math.* **262**:2 (2013), 421–436. MR Zbl

[Molaei 2008] M. R. Molaei, "Dynamically defined topological pressure", *J. Dyn. Syst. Geom. Theor.* **6**:1 (2008), 75–81. MR Zbl

[Pesin 1997] Y. B. Pesin, *Dimension theory in dynamical systems: contemporary views and applications*, Univ. Chicago Press, 1997. MR Zbl

[Pollner and Vattay 1996] P. Pollner and G. Vattay, "New method for computing topological pressure", *Phys. Rev. Lett.* **76**:22 (1996), 4155–4158.

[Ruelle 1973] D. Ruelle, "Statistical mechanics on a compact set with Z^v action satisfying expansiveness and specification", *Trans. Amer. Math. Soc.* **185** (1973), 237–251. MR Zbl

[Spandl 2008] C. Spandl, "Computability of topological pressure for sofic shifts with applications in statistical physics", *J. Univ. Comput. Sci.* **14**:6 (2008), 876–895. MR Zbl

[Walters 1982] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Math. **79**, Springer, 1982. MR Zbl

[Zhao and Cheng 2014] Y. Zhao and W.-C. Cheng, "Coset pressure with sub-additive potentials", *Stoch. Dyn.* **14**:1 (2014), art. id. 1350012. MR Zbl

Received December 11, 2016. Revised September 3, 2019.

BING LI
SCHOOL OF MATHEMATICS
SOUTH CHINA UNIVERSITY OF TECHNOLOGY
GUANGZHOU
CHINA
scbingli@scut.edu.cn

WEN-CHIAO CHENG
DEPARTMENT OF APPLIED MATHEMATICS
CHINESE CULTURE UNIVERSITY
YANGMINGSHAN
TAIPEI
TAIWAN
zwq2@faculty.pccu.edu.tw

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2020 is US $520/year for the electronic version, and $705/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 305 No. 1 March 2020
