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LIOUVILLE-TYPE THEOREMS FOR WEIGHTED p-HARMONIC 1-FORMS AND WEIGHTED p-HARMONIC MAPS

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In this paper, we obtain Bochner–Weitzenböck formulas for the weighted Hodge Laplacian operator acting on differential forms and more generally on vector bundle-valued weighted p-harmonic forms. Applying these formulas, we prove Liouville-type theorems for weighted L^q p-harmonic 1-forms and for weighted p-harmonic maps in a weighted complete noncompact manifold with nonnegative Bakry–Émery Ricci curvature, where $q = 2p - 2$ or $q = p$.

1. Introduction

The celebrated Liouville theorem states that every positive harmonic function on $\mathbb{R}^n$ is constant. There have been a lot of effort over the years to generalize the classical Liouville theorem into complete noncompact Riemannian manifolds. Huber [1957] proved that any negative subharmonic function on a complete surface with nonnegative curvature is constant. Yau [1975] proved that any positive harmonic function on a noncompact Riemannian manifold with nonnegative Ricci curvature is constant. See also [Greene and Wu 1979; Hildebrandt 1982; Karp 1982] for further related results. Moreover, Yau [1976] obtained an L^p-Liouville type theorem. More precisely, he proved that, for $1 < p < \infty$, any L^p harmonic function on a complete Riemannian manifold is constant. Given a harmonic function f on a Riemannian manifold M, we note that the differential df is obviously a harmonic 1-form on M. In the case where M is a complete noncompact Riemannian manifold, it is natural to consider L^2 harmonic forms on M because L^2-Hodge theory remains valid in complete noncompact manifolds as classical Hodge theory works well in compact manifolds. It turned out that the theory of L^2 harmonic 1-forms is useful to investigate the geometry and topology at infinity. For example, Li and Tam [1992] proved that if the space of L^2 harmonic 1-forms on a complete Riemannian manifold M is trivial, then M must have at most one nonparabolic end. Cao, Shen,

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and Zhu [Cao et al. 1997] also obtained an interesting topological result which says that if M is a complete Riemannian manifold with all ends of infinite volume supporting a Sobolev inequality and if the space of L^2 harmonic 1-forms is trivial, then M must have only one end. Their argument using the space of L^2 harmonic 1-forms to study the geometry and topology at infinity has been extended in various ways. We refer the readers to [Dung and Seo 2012; 2017; Li and Wang 2002; 2004; Lin 2015; Pigola et al. 2005; Seo 2010; 2014; Vieira 2016; Yun 2002] for recent developments on this topic.

In this paper, we study Liouville-type properties on p-harmonic 1-forms and p-harmonic maps in weighted manifolds. Given a smooth Riemannian manifold (M, g) and a smooth function $f : M \to \mathbb{R}$, a *weighted manifold* (or a smooth metric measure space, also known as a manifold with density) is a triple $M_f := (M, g, e^{-f} dv_g)$, where dv_g is the volume form induced by the metric g. Since the geometry of weighted manifolds were developed by Bakry and Émery [1985], it has been intensively studied by many authors (for instance, see [Lott 2003; Lott and Villani 2009; Sturm 2006a; 2006b; Wei and Wylie 2009]). Moreover, it turned out that the study of weighted manifolds is closely related with that of self-shrinkers and gradient Ricci solitons.

An important geometric quantity on a weighted manifold M_f known as *Bakry–Émery Ricci curvature* is defined by

$$\mathrm{Ric}_f^M = \mathrm{Ric} + \mathrm{Hess}(f),$$

where $\mathrm{Hess}(f)$ denotes the Hessian of f. Obviously, the Bakry–Émery Ricci curvature is a generalization of Ricci curvature. In a weighted manifold, there is a useful elliptic differential operator, the so-called *f-Laplacian*, Δ_f which is defined by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle.$$

The f-Laplacian is a natural generalization of the Laplace–Beltrami operator Δ as it is self-adjoint with respect to the weighted measure $e^{-f} dv_g$, i.e.,

$$\int_M v \Delta_f u e^{-f} \, dv_g = \int_M (u \Delta_f v) e^{-f} \, dv_g$$

and

$$\int_M (v \Delta_f u) \, e^{-f} \, dv_g = -\int_M \langle \nabla u, \nabla v \rangle e^{-f} \, dv_g$$

for $u, v \in C_0^\infty(M)$.

On the other hand, for a smooth map $\varphi : (M^n, g, e^{-f} dv_g) \to (N^m, h)$ from an f-weighted manifold into a Riemannian manifold, and for a bounded domain

$\Omega \subset M$, the *f-weighted p-energy* $\mathcal{E}_{f,p}(\varphi;\Omega)$ with $p > 1$ of φ over Ω is defined by

$$\mathcal{E}_{f,p}(\varphi;\Omega) = \frac{1}{p}\int_\Omega |d\varphi|^p e^{-f}\, dv_g, \tag{1-1}$$

where $|d\varphi|$ denotes the Hilbert–Schmidt norm of $d\varphi$ induced by the metrics g and h. Namely, if $\{e_i\}$ is a local frame on M, $|d\varphi|$ is given by

$$|d\varphi|^2 = \sum_{i=1}^{n} \langle d\varphi(e_i), d\varphi(e_i)\rangle \tag{1-2}$$

so that

$$|d\varphi|^2 = \mathrm{tr}_g\, \varphi^* h = \langle g, \varphi^* h\rangle.$$

A smooth map $\varphi : (M, g, e^{-f} dv_g) \to (N, h)$ is called *f-weighted p-harmonic* if it is a critical point of the f-weighted p-energy functional $\mathcal{E}_{f,p}(\varphi;\Omega)$ for any bounded domain $\Omega \subset M$. It can be easily shown that when φ is C^2-regular, the Euler–Lagrange equation for the f-weighted p-energy $\mathcal{E}_{f,p}$ is the f-weighted p-harmonic map equation

$$\tau_{f,p}(\varphi) = -\delta_f(|d\varphi|^{p-2} d\varphi) = |d\varphi|^{p-2}\tau_f(\varphi) + d\varphi(\nabla|d\varphi|^{p-2}) = 0. \tag{1-3}$$

Here $\delta_f = \delta + i_{\nabla f}$ is the adjoint operator of the exterior derivative d with respect to the measure $e^{-f} dv_g$, $i_{\nabla f}$ denotes the interior product with the vector ∇f, $\tau_f(\varphi) = \tau(\varphi) - i_{\nabla f} d\varphi$ and $\tau(\varphi)$ is the classical tension field of φ. In the case where $p = 2$ and f is a constant function, Schoen and Yau [1976] obtained the following well-known Liouville-type theorem for harmonic maps between complete Riemannian manifolds.

Theorem [Schoen and Yau 1976]. *Let M be a complete Riemannian manifold of nonnegative Ricci curvature and let N be a complete Riemannian manifold of nonpositive sectional curvature. Then, for any constant function f, every harmonic map $u : M \to N$ with finite 2-energy $\mathcal{E}_{f,2}(u)$ must be constant.*

Recently, Rimoldi and Veronelli [2013] generalized Schoen and Yau's Liouville-type theorem for harmonic maps into f-weighted 2-harmonic maps between complete Riemannian manifolds. More precisely, they showed that if

$$u : (M^n, g, e^{-f} dv_g) \to (N^m, h)$$

is an f-weighted 2-harmonic map from a complete Riemannian manifold M with nonnegative Bakry–Émery Ricci curvature into a complete Riemannian manifold with nonpositive sectional curvature and if the f-weighted 2-energy $\mathcal{E}_{f,2}(u)$ is finite, then the harmonic map u must be constant. See also [Hua et al. 2017; Nakauchi 1998; Takeuchi 1991; Zhang and Wang 2016] for related previous results. In this paper, we extend their result into f-weighted p-harmonic maps.

The organization of this paper is the following. In Section 2 we derive a Bochner–Weitzenböck formula for the weighted Hodge Laplacian Δ_f on differential forms. Applying this formula, we are able to show a Liouville-type property of weighted L^q p-harmonic 1-forms on a complete noncompact weighted manifold with nonnegative Bakry–Émery Ricci curvature (see Theorem 2.4 for $q = 2p-2$ and Theorem 2.5 for $q = p$). In Section 3 we obtain a Bochner–Weitzenböck formula for vector bundle-valued weighted p-harmonic forms (Lemma 3.1), which is an extension of our previous results in Section 2. In Section 4 we prove Liouville-type theorems for weighted p-harmonic maps. In fact, we prove that if u is a weighted p-harmonic map from a complete noncompact weighted manifold with nonnegative Bakry–Émery Ricci curvature into a Riemannian manifold with nonpositive sectional curvature and if u has finite weighted q-energy, then u must be constant (see Theorem 4.1 for $q = 2p-2$ and Theorem 4.2 for $q = p$).

2. Weighted p-harmonic forms

Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold and let $f : M \to \mathbb{R}$ be a smooth function on M. We consider differential forms on the f-weighted manifold $(M, g, e^{-f} dv_g)$ and derive a Bochner–Weitzenböck formula for the weighted Hodge Laplacian. Recall that the formal adjoint of the exterior derivative d with respect to the measure $e^{-f} dv_g$ is given by the formula

$$\delta_f = \delta + i_{\nabla f}.$$

Then the f-Hodge Laplacian Δ_f on differential forms is defined by

$$\Delta_f = -(d\delta_f + \delta_f d).$$

Lemma 2.1 (Bochner–Weitzenböck formula). *Let $(M, g, e^{-f} dv_g)$ be an f-weighted manifold. If ω is a differential 1-form on M, then*

$$\tfrac{1}{2}\Delta_f |\omega|^{2p-2} = \langle |\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\nabla(|\omega|^{p-2}\omega)|^2 + |\omega|^{2p-4}\mathrm{Ric}_f^M(\omega^\sharp, \omega^\sharp). \tag{2-1}$$

Here $\omega^\sharp$ is the dual vector field to ω.

Proof. It is well-known (see [Chang and Sung 2011]) that

$$\begin{aligned}\tfrac{1}{2}\Delta|\omega|^{2p-2} &= \tfrac{1}{2}\Delta||\omega|^{p-2}\omega|^2 \\ &= \langle |\omega|^{p-2}\omega, \Delta(|\omega|^{p-2}\omega)\rangle + |\nabla(|\omega|^{p-2}\omega)|^2 + |\omega|^{2p-4}\mathrm{Ric}(\omega^\sharp, \omega^\sharp).\end{aligned}$$

Using the definition of the f-weighted Laplacian $\Delta_f = \Delta - \langle \nabla f, \nabla\cdot\rangle$, we have

$$\tfrac{1}{2}\Delta_f|\omega|^{2p-2} = \tfrac{1}{2}\Delta|\omega|^{2p-2} - \tfrac{1}{2}\langle \nabla f, \nabla|\omega|^{2p-2}\rangle.$$

Since $\mathrm{Ric}_f^M = \mathrm{Ric} + \mathrm{Hess}(f)$ and $\Delta_f = \Delta - di_{\nabla f} - i_{\nabla f}d$, we get

$$\begin{aligned}\tfrac{1}{2}\Delta_f|\omega|^{2p-2} = \langle|\omega|^{p-2}\omega,\, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\nabla(|\omega|^{p-2}\omega)|^2 + |\omega|^{2p-4}\mathrm{Ric}_f^M(\omega^\sharp, \omega^\sharp)\\ + \langle|\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega)\rangle + \langle|\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega)\rangle\\ - |\omega|^{2p-4}\mathrm{Hess}(f)(\omega^\sharp, \omega^\sharp) - \tfrac{1}{2}\langle\nabla f, \nabla(|\omega|^{2p-2})\rangle.\end{aligned}$$

We claim that

$$\begin{aligned}\text{(2-2)}\quad \langle|\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega)\rangle + \langle|\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega)\rangle\\ - |\omega|^{2p-4}\mathrm{Hess}(f)(\omega^\sharp, \omega^\sharp) - \tfrac{1}{2}\langle\nabla f, \nabla(|\omega|^{2p-2})\rangle = 0.\end{aligned}$$

Let $\{e_1, \dots, e_n\}$ be a local geodesic frame at a point p in M and $\{\theta_1, \dots, \theta_n\}$ its dual coframe. Let $\{\theta_{ij}\}$ be the connection 1-form vanishing at the point p. Writing $\omega = \omega^i\theta_i$ with Einstein convention, we have

$$|\omega|^{2p-4}\mathrm{Hess}(f)(\omega^\sharp, \omega^\sharp) = |\omega|^{2p-4}\omega^i\omega^j f_{ij}.$$

Since

$$di_{\nabla f}(|\omega|^{p-2}\omega) = f_i\omega^i d|\omega|^{p-2} + |\omega|^{p-2}\omega^i f_{ij}\theta_j + |\omega|^{p-2} f_i\omega^i_{;j}\theta_j,$$

we have

$$\begin{aligned}&\langle|\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega)\rangle\\ &\qquad = |\omega|^{p-2} f_i\omega^i\langle\omega,\, d|\omega|^{p-2}\rangle + |\omega|^{2p-4}\omega^i\omega^j f_{ij} + |\omega|^{2p-4}\omega^j f_i\omega^i_{;j}.\end{aligned}$$

Here the semicolon means the covariant differentiation. Moreover,

$$\begin{aligned}d(|\omega|^{p-2}\omega) &= d|\omega|^{p-2}\wedge\omega + |\omega|^{p-2}d\omega\\ &= d|\omega|^{p-2}\wedge\omega + |\omega|^{p-2}\,d\omega^i\wedge\theta_i\end{aligned}$$

which gives

$$\begin{aligned}\text{(2-3)}\quad & i_{\nabla f}d(|\omega|^{p-2}\omega)\\ &= d|\omega|^{p-2}(\nabla f)\omega - \omega^i f_i d|\omega|^{p-2} + |\omega|^{p-2}\omega^i_{;j}f_j\theta_i - |\omega|^{p-2}\omega^i_{;j}f_i\theta_j.\end{aligned}$$

Thus

$$\begin{aligned}\langle|\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega)\rangle = |\omega|^p d|\omega|^{p-2}(\nabla f) - |\omega|^{p-2}\omega^i f_i\langle\omega, d|\omega|^{p-2}\rangle\\ + |\omega|^{2p-4}\omega^i_{;j}\omega^i f_j - |\omega|^{2p-4}\omega^i_{;j}f_i\omega^j.\end{aligned}$$

Next we have

$$\begin{aligned}\tfrac{1}{2}\langle\nabla f, \nabla|\omega|^{2p-2}\rangle &= \tfrac{1}{2}\langle\nabla f, \nabla(|\omega|^p\cdot|\omega|^{p-2})\rangle\\ &= \tfrac{1}{2}|\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle + \tfrac{1}{2}|\omega|^{p-2}\langle\nabla f, \nabla|\omega|^p\rangle.\end{aligned}$$

Since

$$\nabla|\omega|^2 = 2\omega^i\omega^i_{;j}e_j \quad\text{and}\quad \nabla|\omega|^p = \nabla(|\omega|^2)^{p/2} = \frac{p}{2}|\omega|^{p-2}\nabla|\omega|^2,$$

we get

$$\tfrac{1}{2}|\omega|^{p-2}\langle\nabla f, \nabla|\omega|^p\rangle = \frac{p}{4}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle = \frac{p}{2}|\omega|^{2p-4} f_j\omega^i\omega^i_{;j}$$

and

$$|\omega|^{2p-4} f_j\omega^i_{;j}\omega^i = \tfrac{1}{2}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle.$$

Thus the left-hand side of (2-2) becomes

$$\begin{aligned}|\omega|^p d|\omega|^{p-2}(\nabla f)+|\omega|^{2p-4} f_j\omega^i_{;j}\omega^i-\tfrac{1}{2}|\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle-\tfrac{1}{2}|\omega|^{p-2}\langle\nabla f, \nabla|\omega|^p\rangle \\ = \tfrac{1}{2}|\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle + \frac{2-p}{4}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle.\end{aligned}$$

Since

$$\nabla|\omega|^{p-2} = \nabla(|\omega|^2)^{(p-2)/2} = \frac{p-2}{2}(|\omega|^2)^{(p-2)/2-1}\nabla|\omega|^2 = \frac{p-2}{2}|\omega|^{p-4}\nabla|\omega|^2,$$

the left-hand side of (2-2) vanishes, which completes the proof of Lemma 2.1. □

As a consequence of Lemma 2.1, we have the following.

Corollary 2.2. *Let ω be a differential 1-form on a weighted manifold $(M, g, e^{-f}dv_g)$. Then*

$$|\omega|^{p-1}\Delta_f|\omega|^{p-1} \geq \langle|\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\omega|^{2p-4}\operatorname{Ric}^M_f(\omega^\sharp, \omega^\sharp).$$

Proof. Since

$$\tfrac{1}{2}\Delta_f|\omega|^{2p-2} = |\omega|^{p-1}\Delta_f|\omega|^{p-1} + |\nabla|\omega|^{p-1}|^2, \tag{2-4}$$

it follows from Lemma 2.1 that

$$\begin{aligned}|\omega|^{p-1}\Delta_f|\omega|^{p-1} + |\nabla|\omega|^{p-1}|^2 \\ = \langle|\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\nabla(|\omega|^{p-2}\omega)|^2 + |\omega|^{2p-4}\operatorname{Ric}^M_f(\omega^\sharp, \omega^\sharp).\end{aligned}$$

From the generalized Kato type inequality, we have

$$|\nabla|\omega|^{p-1}|^2 = \big|\nabla|\,|\omega|^{p-2}\omega|\big|^2 \leq |\nabla(|\omega|^{p-2}\omega)|^2.$$

Thus we get

$$|\omega|^{p-1}\Delta_f|\omega|^{p-1} \geq \langle|\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\omega|^{2p-4}\operatorname{Ric}^M_f(\omega^\sharp, \omega^\sharp). \qquad \square$$

Let $\phi : M \to \mathbb{R}$ be a harmonic function. Since

$$d(d\phi) = 0 \quad \text{and} \quad \Delta\phi = \delta(d\phi) = 0,$$

the differential $d\phi$ is a harmonic 1-form. Similarly, if $\phi : M \to \mathbb{R}$ is a p-harmonic function, then

$$\Delta_p\phi = \operatorname{div}(|\nabla\phi|^{p-2}\nabla\phi) = 0,$$

which is equivalent to the equation

$$\delta(|d\phi|^{p-2}d\phi) = 0.$$

In fact, this is the Euler–Lagrange equation of the p-energy functional $\mathcal{E}_p(\phi) = \frac{1}{p}\int_M |d\phi|^p \, dv_g$. Using this observation, one can define a p-harmonic form ω on M as follows [Chang and Sung 2011]:

$$d\omega = 0 \quad \text{and} \quad \delta(|\omega|^{p-2}\omega) = 0,$$

which shows that, for any p-harmonic function ϕ on M, its differential $d\phi$ is a p-harmonic 1-form. Motivated by this notion of p-harmonic differential forms in [Chang and Sung 2011] and weighted harmonic forms in [Vieira 2013], we give the definition of weighted p-harmonic forms on a weighted manifold.

Definition 2.3. A differential form ω on M is *f-weighted p-harmonic* if ω satisfies

$$d\omega = 0 \quad \text{and} \quad \delta_f(|\omega|^{p-2}\omega) = 0.$$

When f is constant, we note that the above definition of f-weighted p-harmonic forms is equivalent to the definition of p-harmonic forms in the sense of [Chang and Sung 2011]. Consider an f-weighted L_f^{2p-2} p-harmonic 1-form ω on a weighted manifold M_f with nonnegative Bakry–Émery Ricci curvature, where the L_f^{2p-2} norm of ω is given by

$$\int_M |\omega|^{2p-2} e^{-f} \, dv_g < \infty.$$

Then we have the following Liouville-type theorem for weighted p-harmonic 1-forms.

Theorem 2.4. *Let $(M, g, e^{-f} dv_g)$ be a complete noncompact f-weighted manifold with nonnegative Bakry–Émery Ricci tensor, $\mathrm{Ric}_f^M \geq 0$. Suppose that f is a bounded function. If ω is an f-weighted L_f^{2p-2} p-harmonic 1-form on M for $p > 1$, then ω vanishes.*

Proof. Since ω is an f-weighted p-harmonic 1-form, we have

$$\delta_f(|\omega|^{p-2}\omega) = 0.$$

Thus Corollary 2.2 together with curvature condition implies

$$|\omega|^{p-1}\Delta_f|\omega|^{p-1} \geq \langle |\omega|^{p-2}\omega, \delta_f d(|\omega|^{p-2}\omega)\rangle. \tag{2-5}$$

Fix a point $p \in M$ and choose a cut-off function η satisfying

$$0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } B_p(r), \quad \mathrm{supp}(\eta) \subset B_p(2r), \quad \text{and} \quad |\nabla\eta| \leq \frac{1}{r}. \tag{2-6}$$

Here $B_p(r)$ denotes the geodesic ball of radius r centered at p. Multiplying (2-5) by η^2 and integrating it over M with respect to the measure $e^{-f}dv_g$, we obtain

$$\begin{aligned}
&\int_M \eta^2|\omega|^{p-1}\Delta_f|\omega|^{p-1}e^{-f}\,dv_g \\
&= -\int_M \eta^2|\nabla|\omega|^{p-1}|^2e^{-f}\,dv_g - 2\int_M \eta|\omega|^{p-1}\langle\nabla\eta, \nabla|\omega|^{p-1}\rangle e^{-f}\,dv_g \\
&\le -\int_M \eta^2|\nabla|\omega|^{p-1}|^2e^{-f}\,dv_g + \tfrac{1}{2}\int_M \eta^2|\nabla|\omega|^{p-1}|^2e^{-f}\,dv_g \\
&\qquad\qquad + 2\int_M |\omega|^{2p-2}|\nabla\eta|^2e^{-f}\,dv_g \\
&= -\tfrac{1}{2}\int_M \eta^2|\nabla|\omega|^{p-1}|^2e^{-f}\,dv_g + 2\int_M |\omega|^{2p-2}|\nabla\eta|^2e^{-f}\,dv_g.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\int_M \eta^2\langle|\omega|^{p-2}\omega, \delta_f d(|\omega|^{p-2}\omega)\rangle e^{-f}\,dv_g \\
&= \int_M \langle d(\eta^2|\omega|^{p-2}\omega), d(|\omega|^{p-2}\omega)\rangle e^{-f}\,dv_g \\
&= 2\int_M \eta|\omega|^{p-2}\langle d\eta\wedge\omega, d(|\omega|^{p-2}\omega)\rangle e^{-f}\,dv_g + \int_M \eta^2|d(|\omega|^{p-2}\omega)|^2e^{-f}\,dv_g \\
&\ge -\int_M \eta^2|d(|\omega|^{p-2}\omega)|^2e^{-f}\,dv_g - \int_M |\nabla\eta|^2|\omega|^{2p-2}e^{-f}\,dv_g \\
&\qquad\qquad + \int_M \eta^2|d(|\omega|^{p-2}\omega)|^2e^{-f}\,dv_g \\
&= -\int_M |\nabla\eta|^2|\omega|^{2p-2}e^{-f}\,dv_g.
\end{aligned}$$

Therefore

$$\tfrac{1}{2}\int_M \eta^2|\nabla|\omega|^{p-1}|^2e^{-f}\,dv_g \le 3\int_M |\nabla\eta|^2|\omega|^{2p-2}e^{-f}\,dv_g \le \frac{3}{r^2}\int_M |\omega|^{2p-2}e^{-f}\,dv_g.$$

Since ω is an f-weighted L^{2p-2} harmonic 1-form, we obtain

$$\nabla|\omega|^{p-1} = 0$$

by letting $r\to\infty$. Hence $|\omega|^{p-1}$ is constant. Since $\mathrm{Ric}_f^M \ge 0$ and f is bounded, the f-volume of (M, g) is infinite (see [Wei and Wylie 2009], for example). Therefore we see that $\omega = 0$. □

Using the Bochner–Weitzenböck formula, we can also prove the following.

Theorem 2.5. *Let $(M, g, e^{-f}dv_g)$ be a complete noncompact f-weighted manifold with nonnegative Bakry–Émery Ricci tensor. Suppose that f is a bounded function. For $p \ge 2$, if ω is an f-weighted L_f^p p-harmonic 1-form on M, then $\omega = 0$.*

Proof. Since $\delta_f(|\omega|^{p-2}\omega) = 0$, Corollary 2.2 and the curvature condition implies

$$\begin{aligned} |\omega|\Delta_f|\omega|^{p-1} &\geq \langle \omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\omega|^{2p-4}\operatorname{Ric}_f^M(\omega^\sharp, \omega^\sharp) \\ &\geq \langle \omega, \delta_f d(|\omega|^{p-2}\omega)\rangle. \end{aligned} \tag{2-7}$$

Fix a point $p \in M$ and choose a cut-off function η satisfying (2-6). Multiplying (2-7) by η^2 and integrating it over M with respect to the measure $e^{-f}\,dv_g$, we obtain

$$\int_M \eta^2|\omega|\Delta_f|\omega|^{p-1}e^{-f}\,dv_g \geq \int_M \eta^2\langle \omega, \delta_f d(|\omega|^{p-2}\omega)\rangle e^{-f}\,dv_g. \tag{2-8}$$

Then the left-hand side of (2-8) is given by

$$\begin{aligned} &\int_M \eta^2|\omega|\Delta_f|\omega|^{p-1}e^{-f}\,dv_g \\ &= -\int_M \eta^2\langle \nabla|\omega|, \nabla|\omega|^{p-1}\rangle e^{-f}\,dv_g - 2\int_M \eta|\omega|\langle \nabla\eta, \nabla|\omega|^{p-1}\rangle e^{-f}\,dv_g \\ &= -(p-1)\int_M \eta^2|\omega|^{p-2}\big|\nabla|\omega|\big|^2 e^{-f}\,dv_g \\ &\qquad\qquad - 2(p-1)\int_M \eta|\omega|^{p-1}\langle \nabla\eta, \nabla|\omega|\rangle e^{-f}\,dv_g. \end{aligned} \tag{2-9}$$

Note that

$$|\omega|^{p-2}\big|\nabla|\omega|\big|^2 = \frac{4}{p^2}\big|\nabla|\omega|^{p/2}\big|^2$$

and

$$|\omega|^{p-1}\nabla|\omega| = |\omega|^{p/2}\cdot|\omega|^{p/2-1}\nabla|\omega| = \frac{2}{p}|\omega|^{p/2}\nabla|\omega|^{p/2}. \tag{2-10}$$

Substituting these two identities into (2-9), we obtain

$$\begin{aligned} &\int_M \eta^2|\omega|\Delta_f|\omega|^{p-1}e^{-f}\,dv_g \\ &= -\frac{4(p-1)}{p^2}\int_M \eta^2\big|\nabla|\omega|^{p/2}\big|^2 e^{-f}\,dv_g \\ &\qquad\qquad - \frac{4(p-1)}{p}\int_M \eta|\omega|^{p/2}\langle \nabla\eta, \nabla|\omega|^{p/2}\rangle e^{-f}\,dv_g \\ &\leq -\frac{4(p-1)}{p^2}\int_M \eta^2\big|\nabla|\omega|^{p/2}\big|^2 e^{-f}\,dv_g \\ &\quad + \frac{2(p-1)}{p}\bigg\{\varepsilon\int_M \eta^2\big|\nabla|\omega|^{p/2}\big|^2 e^{-f}\,dv_g + \frac{1}{\varepsilon}\int_M |\omega|^p|\nabla\eta|^2 e^{-f}\,dv_g\bigg\}, \end{aligned} \tag{2-11}$$

where we used Young's inequality in the last inequality for arbitrary $\varepsilon > 0$.

On the other hand, applying the divergence theorem with respect to the measure $e^{-f}\,dv_g$, the right-hand side of (2-8) becomes

$$\int_M \eta^2 \langle \omega, \delta_f d(|\omega|^{p-2}\omega)\rangle e^{-f}\,dv_g = \int_M \langle d(\eta^2\omega), d(|\omega|^{p-2}\omega)\rangle e^{-f}\,dv_g.$$

Since

$$|d(\varphi\omega)| = |d\varphi \wedge \omega| \le |d\varphi|\,|\omega|$$

for any smooth function $\varphi : M \to \mathbb{R}$ and any closed 1-form ω (see Lemma 13 in [Pigola et al. 2008]), using (2-10) and Young's inequality again gives

$$\begin{aligned}\text{(2-12)}\quad \big|\langle d(\eta^2\omega), d(|\omega|^{p-2}\omega)\rangle\big| &\le |d(\eta^2\omega)|\big|d(|\omega|^{p-2}\omega)\big| \\ &\le |d\eta^2|\,|\omega|^2\big|d|\omega|^{p-2}\big| \\ &= 2\eta|\omega|^2|\nabla\eta|\big|\nabla|\omega|^{p-2}\big| \\ &= 2(p-2)\eta|\nabla\eta|\,|\omega|^{p-1}\big|\nabla|\omega|\big| \\ &= \frac{4(p-2)}{p}\eta|\nabla\eta|\,|\omega|^{p/2}\big|\nabla|\omega|^{p/2}\big| \\ &\le \frac{2(p-2)}{p}\Big(\delta\eta^2\big|\nabla|\omega|^{p/2}\big|^2 + \frac{1}{\delta}|\nabla\eta|^2|\omega|^p\Big)\end{aligned}$$

for any $\delta > 0$. Therefore

$$\begin{aligned}\text{(2-13)}\quad \int_M \eta^2\langle \omega, \delta_f d(|\omega|^{p-2}\omega)\rangle e^{-f}\,dv_g \ge &-\frac{2(p-2)}{p}\delta\int_M \eta^2\big|\nabla|\omega|^{p/2}\big|^2 e^{-f}\,dv_g \\ &- \frac{2(p-2)}{p}\frac{1}{\delta}\int_M |\nabla\eta|^2|\omega|^p e^{-f}\,dv_g.\end{aligned}$$

Combining (2-8), (2-11) and (2-13), we obtain

$$\begin{aligned}\Big(\frac{4(p-1)}{p^2} - \frac{2(p-1)}{p}\varepsilon - \frac{2(p-2)}{p}\delta\Big)&\int_M \eta^2\big|\nabla|\omega|^{p/2}\big|^2 e^{-f}\,dv_g \\ &\le \Big(\frac{2(p-1)}{p}\frac{1}{\varepsilon} + \frac{2(p-2)}{p}\frac{1}{\delta}\Big)\int_M |\nabla\eta|^2|\omega|^p e^{-f}\,dv_g.\end{aligned}$$

Choose ε and δ sufficiently small so that

$$\frac{4(p-1)}{p^2} - \frac{2(p-1)}{p}\varepsilon - \frac{2(p-2)}{p}\delta > 0.$$

Since ω is an L^p_f p-harmonic 1-form, as r tends to infinity, we see

$$\nabla|\omega|^{p/2} = 0,$$

which implies that $\omega \equiv 0$ as in the proof of Theorem 2.4. □

Remark 2.6. In Theorems 2.4 and 2.5, the boundedness on the weighted function f is only needed to guarantee that the weighted volume of $(M, g, e^{-f}dv_g)$ is infinite. In fact, we prove that any f-weighted L_f^{2p-2} p-harmonic 1-form with $p > 1$ or L_f^p p-harmonic 1-form with $p \geq 2$ on a complete noncompact f-weighted manifold with nonnegative Bakry–Émery Ricci tensor has constant length, which implies that ω is f-harmonic. Thus applying the standard Bochner formula for f-harmonic 1-forms (see Lemma 2.1 with $p = 2$, [Lott 2003] or [Vieira 2013]), one can see that ω is parallel without the assumption that f is bounded. This result leads to applications in gradient steady Ricci solitons or, more generally, to applications in weighted manifolds with infinite weighted volumes (see [Vieira 2013]). Recall that a gradient steady Ricci soliton is a manifold (M, g) together with a smooth function f satisfying $\mathrm{Ric}_f^M = 0$.

Furthermore, if we assume that Ric_f^M is nonnegative and positive at a point, it is easy to see, from Corollary 2.2, that ω vanishes without assuming the boundedness of f. This property leads to applications in gradient shrinking Ricci solitons satisfying $\mathrm{Ric}_f^M = \lambda g$ for some positive constant λ as follows.

Corollary 2.7. *Let $(M, g, e^{-f}dv_g)$ be a complete gradient shrinking Ricci soliton satisfying* $\mathrm{Ric} + \mathrm{Hess}(f) = \lambda g$ *with* $\lambda > 0$, *constant. Then if ω is an L_f^{2p-2} $(p > 1)$ or L_f^p $(p \geq 2)$ p-harmonic 1-form on M, then $\omega = 0$.*

Proof. The proof follows from the argument in Remark 2.6. □

In case of gradient steady Ricci solitons, we also have the following same vanishing property.

Corollary 2.8. *Let $(M, g, e^{-f}dv_g)$ be a complete gradient steady Ricci soliton satisfying* $\mathrm{Ric} + \mathrm{Hess}(f) = 0$. *Then if ω is an L_f^{2p-2} $(p > 1)$ or L_f^p $(p \geq 2)$ p-harmonic 1-form on M, then $\omega = 0$.*

Proof. For $q = 2p - p$ or $q = p$, applying the same argument as in the proofs of Theorems 2.4 and 2.5, we see that $|\omega| \equiv C$ for some constant C. Thus

$$\int_M |\omega|^q e^{-f} dv_g = C^q \,\mathrm{Vol}_f(M),$$

where $\mathrm{Vol}_f(M)$ denotes the f-weighted volume of M.

On the other hand, it is well-known that the scalar curvature of a gradient steady Ricci soliton is nonnegative and $|\nabla f|$ is bounded by a positive constant (see [Cao 2010] for example). Moreover, Munteanu and Wang [2011] proved that the first eigenvalue of f-Laplacian Δ_f on the nontrivial gradient steady Ricci solitons is positive. Therefore, applying the result by Vieira [2013], we get $\mathrm{Vol}_f(M) = \infty$. This shows that $\omega = 0$. □

3. Vector bundle-valued weighted p-harmonic forms

In this section, we extend the notions discussed in Section 2 including the Bochner–Weitzenböck formula to vector bundles over a weighted manifold.

Let $\pi : E \to M$ be a vector bundle of rank m over a smooth oriented Riemannian manifold (M^n, g). We denote by $\Gamma(E)$ the vector space of smooth sections of E over M. A Riemannian structure on the bundle E is a pair (∇^E, ρ), where ρ is a Riemannian metric on E, ∇^E a connection and $\nabla^E \rho = 0$. Denoting $\rho = \langle \cdot, \cdot \rangle$, the condition $\nabla^E \rho = 0$ means that, for each $X \in \Gamma(TM)$ and $s_1, s_2 \in \Gamma(E)$, we have

$$X \cdot \langle s_1, s_2 \rangle = \langle \nabla^E s_1, s_2 \rangle + \langle s_1, \nabla^E s_2 \rangle.$$

The curvature of the connection ∇^E is the map $R^E : \Lambda^2 TM \otimes \Gamma(E) \to \Gamma(E)$ defined by

$$R^E(X, Y)s = -\nabla^E_X \nabla^E_Y s + \nabla^E_Y \nabla^E_X s + \nabla^E_{[X,Y]} s.$$

Let ω be an l-form on M with values in the vector bundle $\pi : E \to M$. Then, choosing a (local) frame $s_1, \ldots, s_m$ on E, for each $X_1, \ldots, X_l \in \Gamma(TM)$, we can write

$$\omega(X_1, \ldots, X_l) = \sum_{\alpha=1}^{m} a_\alpha s_\alpha$$

for some local smooth functions a_α on M. For the Levi–Civita connection $D^M = D$ on (M, g), the induced connection ∇ on $\Gamma(\Lambda^l T^* M \otimes E)$, the space of smooth l-forms on M with values in the vector bundle $\pi : E \to M$, is given by

$$(\nabla_X \omega)(X_1, \ldots, X_l) = \nabla^E_X(\omega(X_1, \ldots, X_l)) - \sum_{i=1}^{l} \omega(X_1, \ldots, D_X X_i, \ldots, X_l)$$

and its associated curvature is given by

$$\begin{aligned}(R(X, Y)\omega)&(X_1, \ldots X_l) \\ &= R^E(X, Y)(\omega(X_1, \ldots, X_l)) - \sum_{i=1}^{l} \omega(X_1, \ldots, R^M(X, Y)X_i, \ldots, X_l).\end{aligned}$$

For the induced connection ∇, the exterior differential operator

$$d : \Gamma(\Lambda^l T^* M \otimes E) \to \Gamma(\Lambda^{l+1} T^* M \otimes E)$$

is given by

$$(d\omega)(X_1, \ldots, X_{l+1}) = \sum_{i=1}^{l+1} (-1)^{i+1} (\nabla_{X_i} \omega)(X_1, \ldots, \widehat{X_i}, \ldots, X_{l+1}),$$

where the symbol covered by $\widehat{X_i}$ is omitted. The codifferential operator δ is given by

$$(\delta\omega)(X_1,\dots,X_{l-1}) = -\sum_{i=1}^{n}(\nabla_{e_i}\omega)(e_i,X_1,\dots,X_{l-1}),$$

where $\{e_i\}$ is a local frame on M. Finally the Laplacian Δ and the f-weighted Laplacian Δ_f are defined on E-valued differential forms by

$$\Delta = -(d\delta+\delta d) \quad\text{and}\quad \Delta_f = -(d\delta_f+\delta_f d),$$

respectively.

For a vector bundle $\pi : E \to M$ over a weighted manifold $(M, g, e^{-f}dv_g)$, we have the following Bochner–Weitzenböck formula for differential 1-forms on M with values in E.

Lemma 3.1 (Bochner–Weitzenböck formula). *Let $\pi : E \to M$ be a vector bundle of rank m over a smooth oriented Riemannian manifold (M, g), and let $f : M \to \mathbb{R}$ be a smooth function. If ω is an E-valued 1-form on M, then*

$$\begin{aligned}\text{(3-1)}\quad \tfrac{1}{2}\Delta_f|\omega|^{2p-2} &= \langle|\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\nabla(|\omega|^{p-2}\omega)|^2\\ &\quad + |\omega|^{2p-4}\sum_{i=1}^{n}\langle\omega(\mathrm{Ric}_f^M(e_i)),\omega(e_i)\rangle\\ &\quad - |\omega|^{2p-4}\sum_{i,j}\langle R^E(e_i,e_j)\omega(e_i),\omega(e_j)\rangle,\end{aligned}$$

where $\{e_i\}$ is a local frame on M and $\mathrm{Ric}_f^M(e_i)$ is a vector given by

$$\mathrm{Ric}_f^M(e_i) = \sum_{j=1}^{n}\mathrm{Ric}_f^M(e_i,e_j)e_j = \sum_{j=1}^{n}\big[\mathrm{Ric}^M(e_i,e_j)+\mathrm{Hess}(f)(e_i,e_j)\big]e_j.$$

Proof. It is well-known (see [Eells and Lemaire 1983]) that

$$\begin{aligned}\text{(3-2)}\quad \tfrac{1}{2}\Delta|\omega|^{2p-2} &= \langle|\omega|^{p-2}\omega, \Delta(|\omega|^{p-2}\omega)\rangle + |\nabla(|\omega|^{p-2}\omega)|^2\\ &\quad + |\omega|^{2p-4}\sum_{i=1}^{n}\langle\omega(\mathrm{Ric}^M(e_i)),\omega(e_i)\rangle\\ &\quad - |\omega|^{2p-4}\sum_{i,j}\langle R^E(e_i,e_j)\omega(e_i),\omega(e_j)\rangle.\end{aligned}$$

By definition of weighted Laplacian $\Delta_f = \Delta - \langle\nabla f, \nabla\cdot\rangle$, we have

$$\tfrac{1}{2}\Delta_f|\omega|^{2p-2} = \tfrac{1}{2}\Delta|\omega|^{2p-2} - \tfrac{1}{2}\langle\nabla f, \nabla|\omega|^{2p-2}\rangle.$$

Since

$$\mathrm{Ric}_f^M = \mathrm{Ric}^M + \mathrm{Hess}(f) \quad\text{and}\quad \Delta_f = \Delta - di_{\nabla f} - i_{\nabla f}d,$$

we get

$$\begin{aligned}
&\tfrac{1}{2}\Delta_f|\omega|^{2p-2}\\
&= \langle |\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + \langle |\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega)\rangle\\
&\qquad + \langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega)\rangle + |\nabla(|\omega|^{p-2}\omega)|^2\\
&\qquad + |\omega|^{2p-4}\sum_{i=1}^{n}\langle \omega(\mathrm{Ric}_f^M(e_i)), \omega(e_i)\rangle - |\omega|^{2p-4}\langle \omega(\mathrm{Hess}(f)(e_i)), \omega(e_i)\rangle\\
&\qquad - |\omega|^{2p-4}\sum_{i,j}\langle R^E(e_i,e_j)\omega(e_i), \omega(e_j)\rangle - \tfrac{1}{2}\langle \nabla f, \nabla(|\omega|^{2p-2})\rangle.
\end{aligned}$$

We claim that

$$\begin{aligned}
\text{(3-3)}\quad &\langle |\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega)\rangle + \langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega)\rangle\\
&\qquad - |\omega|^{2p-4}\langle \omega(\mathrm{Hess}(f)(e_i)), \omega(e_i)\rangle - \tfrac{1}{2}\langle \nabla f, \nabla(|\omega|^{2p-2})\rangle = 0.
\end{aligned}$$

Let $\{e_1, \dots, e_n\}$ be a local geodesic frame at a point p in M, and $\{\theta_1, \dots, \theta_n\}$ be its dual coframe. Let $\{\theta_{ij}\}$ be the connection 1-form vanishing at the point p. Let $\{s_1, \dots, s_m\}$ be a local frame on E such that

$$\nabla^E s_\alpha|_p = 0.$$

Then ω can be expressed as

$$\omega = \sum_{\alpha=1}^{m}\sum_{i=1}^{n} a_{i\alpha}\theta_i \otimes s_\alpha$$

so that

$$\omega(e_j) = \sum_\alpha a_{j\alpha}s_\alpha \quad \text{and} \quad |\omega|^2 = \sum_{i,\alpha} a_{i\alpha}^2.$$

Since

$$\begin{aligned}
di_{\nabla f}(|\omega|^{p-2}\omega) &= di_{\nabla f}(|\omega|^{p-2}a_{i\alpha}\theta_i \otimes s_\alpha)\\
&= d(|\omega|^{p-2}f_i a_{i\alpha}s_\alpha)\\
&= f_i a_{i\alpha}d|\omega|^{p-2}\otimes s_\alpha + |\omega|^{p-2}a_{i\alpha}f_{ij}\theta_j \otimes s_\alpha\\
&\qquad + |\omega|^{p-2}f_i a_{i\alpha;j}\theta_j \otimes s_\alpha + |\omega|^{p-2}f_i a_{i\alpha}\nabla^E s_\alpha,
\end{aligned}$$

we have

$$\begin{aligned}
\text{(3-4)}\quad \langle |\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega)\rangle &= |\omega|^{p-2}\langle a_{j\alpha}\theta_j \otimes s_\alpha, di_{\nabla f}(|\omega|^{p-2}\omega)\rangle\\
&= |\omega|^{p-2}a_{j\alpha}f_i a_{i\alpha}d|\omega|^{p-2}(e_j)\\
&\qquad + |\omega|^{2p-4}a_{j\alpha}a_{i\alpha}f_{ij} + |\omega|^{2p-4}a_{j\alpha}f_i a_{i\alpha;j}.
\end{aligned}$$

Moreover

$$\begin{aligned} d(|\omega|^{p-2}\omega) &= a_{i\alpha}(d|\omega|^{p-2}\wedge\theta_i)\otimes s_\alpha + |\omega|^{p-2}(da_{i\alpha}\wedge\theta_i)\otimes s_\alpha \\ &\qquad + |\omega|^{p-2}a_{i\alpha}\theta_{ij}\wedge\theta_j\otimes s_\alpha - |\omega|^{p-2}a_{i\alpha}\theta_i\wedge\nabla^E s_\alpha \\ &= a_{i\alpha}(d|\omega|^{p-2}\wedge\theta_i)\otimes s_\alpha + |\omega|^{p-2}a_{i\alpha;j}(\theta_j\wedge\theta_i)\otimes s_\alpha \end{aligned}$$

which gives

$$\begin{aligned} i_{\nabla f}d(|\omega|^{p-2}\omega) &= d|\omega|^{p-2}(\nabla f)\omega - a_{i\alpha}f_i d|\omega|^{p-2}\otimes s_\alpha \\ &\qquad + |\omega|^{p-2}a_{i\alpha;j}f_j\theta_i\otimes s_\alpha - |\omega|^{p-2}a_{i\alpha;j}f_i\theta_j\otimes s_\alpha. \end{aligned}$$

Thus

$$\begin{aligned} &\langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega)\rangle \\ &= |\omega|^{p-2}\langle a_{j\beta}\theta_j\otimes s_\beta, i_{\nabla f}d(|\omega|^{p-2}\omega\rangle \\ &= |\omega|^p d|\omega|^{p-2}(\nabla f) - |\omega|^{p-2}a_{i\alpha}a_{j\alpha}f_i d|\omega|^{p-2}(e_j) \\ &\qquad + |\omega|^{2p-4}a_{i\alpha;j}a_{i\alpha}f_j - |\omega|^{2p-4}a_{i\alpha;j}f_i a_{j\alpha} \\ &= |\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle - |\omega|^{p-2}a_{i\alpha}a_{j\alpha}f_i d|\omega|^{p-2}(e_j) \\ &\qquad + \tfrac{1}{2}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle - |\omega|^{2p-4}a_{i\alpha;j}f_i a_{j\alpha}. \end{aligned} \tag{3-5}$$

Note that

$$\langle\omega(\mathrm{Hess}(f)(e_i)), \omega(e_i)\rangle = f_{ij}\langle\omega(e_j), \omega(e_i)\rangle = f_{ij}a_{j\alpha}a_{i\alpha}. \tag{3-6}$$

From (3-4), (3-5), and (3-6), it follows that

$$\begin{aligned} &\langle |\omega|^{p-2}\omega, di_{\nabla f}(|\omega|^{p-2}\omega)\rangle + \langle |\omega|^{p-2}\omega, i_{\nabla f}d(|\omega|^{p-2}\omega)\rangle \\ &\qquad - |\omega|^{2p-4}\langle\omega(\mathrm{Hess}(f)(e_i)), \omega(e_i)\rangle \\ &\qquad = |\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle + \tfrac{1}{2}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle. \end{aligned}$$

We observe that

$$\begin{aligned} \tfrac{1}{2}\langle\nabla f, \nabla|\omega|^{2p-2}\rangle &= \tfrac{1}{2}\langle\nabla f, \nabla(|\omega|^p\cdot|\omega|^{p-2})\rangle \\ &= \tfrac{1}{2}|\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle + \tfrac{1}{2}|\omega|^{p-2}\langle\nabla f, \nabla|\omega|^p\rangle. \end{aligned}$$

Since

$$\nabla|\omega|^p = \nabla(|\omega|^2)^{p/2} = \frac{p}{2}|\omega|^{p-2}\nabla|\omega|^2,$$

we have

$$\tfrac{1}{2}|\omega|^{p-2}\langle\nabla f, \nabla|\omega|^p\rangle = \frac{p}{4}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle.$$

Thus the left-hand side of (3-3) becomes

$$\tfrac{1}{2}|\omega|^p\langle\nabla f, \nabla|\omega|^{p-2}\rangle + \frac{2-p}{4}|\omega|^{2p-4}\langle\nabla f, \nabla|\omega|^2\rangle.$$

Using

$$\begin{aligned}\nabla|\omega|^{p-2} &= \nabla(|\omega|^2)^{(p-2)/2}\\ &= \frac{p-2}{2}(|\omega|^2)^{(p-2)/2-1}\nabla|\omega|^2\\ &= \frac{p-2}{2}|\omega|^{p-4}\nabla|\omega|^2,\end{aligned}$$

we see that the left-hand side of (3-3) vanishes, which completes the proof of Lemma 3.1. □

As in the proof of Corollary 2.2, we can easily show the following by using Lemma 3.1.

Corollary 3.2. *Let $\pi : E \to M$ be a vector bundle of rank m over a smooth oriented Riemannian manifold (M, g), and let $f : M \to \mathbb{R}$ be a smooth function. If ω is an E-valued 1-form on M, then*

$$\begin{aligned}|\omega|^{p-1}\Delta_f|\omega|^{p-1} \geq \langle |\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\omega|^{2p-4}\sum_{i=1}^{n}\langle\omega(\mathrm{Ric}_f^M(e_i)), \omega(e_i)\rangle\\ - |\omega|^{2p-4}\sum_{i,j}\langle R^E(e_i, e_j)\omega(e_i), \omega(e_j)\rangle.\end{aligned}$$

4. Weighted p-harmonic maps

In this section, we obtain some Liouville-type theorems for weighted p-harmonic maps as an application of the Bochner–Weitzenböck formula stated in Section 3. The following theorem shows that the same result holds for f-weighted p-harmonic maps with L_f^{2p-2}-finite energy for $p > 1$ as in the case of f-weighted L_f^{2p-2} p-harmonic 1-forms.

Theorem 4.1. *Let $u : (M, g, e^{-f}dv_g) \to (N, h)$ be an f-weighted p-harmonic map from an oriented complete noncompact f-weighted manifold into a Riemannian manifold for $p > 1$. Suppose that f is bounded. Assume that the Bakry–Émery Ricci curvature of M is nonnegative, $\mathrm{Ric}_f^M \geq 0$, and the sectional curvature of N is nonpositive, $K^N \leq 0$. If u has finite f-weighted $(2p-2)$-energy, i.e.,*

$$\int_M |du|^{2p-2}e^{-f}\,dv_g < \infty,$$

then u must be a constant map.

Proof. Let $du = \omega$. Then ω is an f-weighted p-harmonic 1-form with values in the pull-back bundle $u^{-1}TN$. In particular,

$$\delta_f(|\omega|^{p-2}\omega) = 0.$$

From Corollary 3.2 together with curvature conditions, it follows that

$$|\omega|^{p-1}\Delta_f|\omega|^{p-1} \geq \langle |\omega|^{p-2}\omega, \delta_f d(|\omega|^{p-2}\omega)\rangle. \tag{4-1}$$

From this, we can see that the same argument as in the proof of Theorem 2.4 shows $\omega = 0$. □

From Corollary 3.2, it follows that

$$\begin{aligned}|\omega|\Delta_f|\omega|^{p-1} \geq \langle \omega, \Delta_f(|\omega|^{p-2}\omega)\rangle + |\omega|^{p-2}\sum_{i=1}^{n}\langle \omega(\mathrm{Ric}_f^M(e_i)), \omega(e_i)\rangle \\ - |\omega|^{p-2}\sum_{i,j}\langle R^E(e_i, e_j)\omega(e_i), \omega(e_j)\rangle.\end{aligned} \tag{4-2}$$

Applying the same argument as in Theorem 2.5 to (4-2), we are able to prove the following theorem.

Theorem 4.2. *Let $u : (M, g, e^{-f}dv_g) \to (N, h)$ be an f-weighted p-harmonic map from an oriented complete noncompact f-weighted manifold into a Riemannian manifold. Suppose that f is bounded, and $\mathrm{Ric}_f^M \geq 0$ and $K^N \leq 0$. For $p \geq 2$, if u has finite f-weighted p-energy, then u must be a constant map.*

Remark 4.3. In Theorems 4.1 and 4.2, without the boundedness of f, if we assume that Ric_f is nonnegative and positive at a point, we can conclude that any f-weighted p-harmonic map $u : (M, g, e^{-f}dv_g) \to (N, h)$ with finite f-weighted $(2p-2)$-energy or p-energy for $p > 1$ from an oriented complete noncompact f-weighted manifold into a Riemannian manifold of nonpositive sectional curvature, $K^N \leq 0$, must be constant.

Applying the argument in Remark 4.3 to gradient shrinking Ricci solitons, we have the following as in the case of L_f^p p-harmonic 1-forms.

Corollary 4.4. *Let $(M, g, e^{-f}dv_g)$ be a complete noncompact gradient shrinking Ricci soliton satisfying $\mathrm{Ric} + \mathrm{Hess}(f) = \lambda g$ with $\lambda > 0$, constant. If $u : (M, g, e^{-f}dv_g) \to (N, h)$ is an f-weighted p-harmonic map into a Riemannian manifold of nonpositive sectional curvature $K^N \leq 0$ with finite f-weighted $(2p-2)$-energy or p-energy for $p > 1$, then u must be a constant map.*

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Volume 305 No. 1 March 2020

The Poincaré homology sphere, lens space surgeries, and some knots with tunnel number two 1
KENNETH L. BAKER

Fusion systems of blocks of finite groups over arbitrary fields 29
ROBERT BOLTJE, ÇISIL KARAGÜZEL and DENIZ YILMAZ

Torsion points and Galois representations on CM elliptic curves 43
ABBEY BOURDON and PETE L. CLARK

Stability of the positive mass theorem for axisymmetric manifolds 89
EDWARD T. BRYDEN

Index estimates for free boundary constant mean curvature surfaces 153
MARCOS P. CAVALCANTE and DARLAN F. DE OLIVEIRA

A criterion for modules over Gorenstein local rings to have rational Poincaré series 165
ANJAN GUPTA

Generalized Cartan matrices arising from new derivation Lie algebras of isolated hypersurface singularities 189
NAVEED HUSSAIN, STEPHEN S.-T. YAU and HUAIQING ZUO

On the commutativity of coset pressure 219
BING LI and WEN-CHIAO CHENG

Signature invariants related to the unknotting number 229
CHARLES LIVINGSTON

The global well-posedness and scattering for the 5-dimensional defocusing conformal invariant NLW with radial initial data in a critical Besov space 251
CHANGXING MIAO, JIANWEI YANG and TENGFEI ZHAO

Liouville-type theorems for weighted p-harmonic 1-forms and weighted p-harmonic maps 291
KEOMKYO SEO and GABJIN YUN

Remarks on the Hölder-continuity of solutions to parabolic equations with conic singularities 311
YUANQI WANG

Deformation of Milnor algebras 329
ZHENJIAN WANG

Preservation of log-Sobolev inequalities under some Hamiltonian flows 339
BO XIA

Ground state solutions of polyharmonic equations with potentials of positive low bound 353
CAIFENG ZHANG, JUNGANG LI and LU CHEN

0030-8730(2020)305:1;1-R