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GROUND STATE SOLUTIONS OF POLYHARMONIC EQUATIONS WITH POTENTIALS OF POSITIVE LOW BOUND

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The purpose of this paper is threefold. First, we establish the critical Adams inequality on the whole space with restrictions on the norm

$$(\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} + \tau \|u\|_{\frac{n}{m}}^{\frac{n}{m}})^{\frac{m}{n}}$$

for any $\tau > 0$. Second, we prove a sharp concentration-compactness principle for singular Adams inequalities and a new Sobolev compact embedding in $W^{m,2}(\mathbb{R}^{2m})$. Third, based on the above results, we give sufficient conditions for the existence of ground state solutions to the following polyharmonic equation with singular exponential nonlinearity

$$(-\Delta)^m u + V(x)u = \frac{f(x,u)}{|x|^\beta} \quad \text{in } \mathbb{R}^{2m}, \tag{0-1}$$

where $0 < \beta < 2m$, $V(x)$ has a positive lower bound and $f(x,t)$ behaves like $\exp(\alpha|t|^2)$ as $t \to +\infty$. Furthermore, when $\beta = 0$, in light of the principle of the symmetric criticality and the radial lemma, we also derive the existence of nontrivial weak solutions by assuming $f(x,t)$ and $V(x)$ are radially symmetric with respect to x and $f(x,t) = o(t)$ at origin. Thus our main theorems extend the recent results on bi-Laplacian in $\mathbb{R}^4$ by Chen, Li, Lu and Zhang (2018) to $(-\Delta)^m$ in $\mathbb{R}^m$.

1. Introduction and main results

The standard Sobolev space $W_0^{k,p}(\Omega)$ is defined by the completion of $C_c^\infty(\Omega)$ equipped with the norm

$$\|u\|_{W^{k,p}} = \left(\|u\|_p^p + \sum_{j=1}^{k} \|\nabla^j u\|_p^p\right)^{\frac{1}{p}},$$

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where Ω denotes a smooth bounded domain in $\mathbb{R}^n$. Basically, the Sobolev continuous embeddings state that

$$W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for } 1 \le q \le \frac{np}{n-kp},\ kp < n.$$

However, in the limiting case $kp = n$, many examples show that $W_0^{k,\frac{n}{k}}(\Omega) \not\subseteq L^\infty(\Omega)$. In this case, the Trudinger–Moser inequality and the Adams inequality serve as appropriate replacements. Research concerning the sharp constant for the Trudinger–Moser inequality could be traced back to the 1960s and 1970s. Trudinger [1967] proved there exists a constant $\alpha > 0$ such that the following inequality holds (also see [Pohozaev 1965; Yudovich 1961]):

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \frac{1}{|\Omega|} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}}\, dx \le C_0. \tag{1-1}$$

Nevertheless, the best constant for (1-1) is unknown. Later, Moser [1971] established the sharp version of inequality (1-1) which can be stated as follows:

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \frac{1}{|\Omega|} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}}\, dx \le C_0, \tag{1-2}$$

where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$ is the sharp constant in the sense that if α_n is replaced by any larger number, the supremum would become infinity. ω_{n-1} denotes the area of the surface of the unit ball in $\mathbb{R}^n$. Estimate (1-2) is now referred as the Trudinger–Moser inequality and plays an important role in geometric analysis and partial differential equations (e.g., see [Moser 1973]). For more results of Trudinger–Moser inequalities on compact Riemannian manifolds, one can refer to [Li 2001; 2006; Li and Ndiaye 2007]. If we replace Ω with $\mathbb{R}^n$, the Trudinger–Moser inequality (1-2) makes no sense. Instead, a subcritical Trudinger–Moser type inequality was proved by Adachi and Tanaka [2000]. By replacing the Dirichlet norm with the standard Sobolev norm in $W^{1,n}(\mathbb{R}^n)$, Cao [1992] (for $n = 2$), Panda [1996] and J. M. do Ó [2014] (for general n) constructed the Trudinger–Moser inequality in the whole space which states that for any $\alpha < \alpha_n$,

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\, ||u||_{W^{1,n}(\mathbb{R}^n)} \le 1} \int_{\mathbb{R}^n} \Phi_{n,1}(\alpha |u(x)|^{\frac{n}{n-1}})\, dx \le C_n, \tag{1-3}$$

$$\text{where } \Phi_{n,1}(t) := e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}.$$

However, they did not prove the criticality of this inequality. Later, Ruf [2005] (for the case $n = 2$), Li and Ruf [2008] (for the general case $n \ge 3$) proved that Trudinger–Moser inequality (1-3) still holds in the critical case $\alpha = \alpha_n$ by using the symmetrization argument and the blow-up procedure. Both the critical and subcritical Trudinger–Moser inequalities on $\mathbb{R}^n$ given in the aforementioned works are based on the Pólya–Szegő inequality and symmetrization argument which is not available in other non-Euclidean settings. Lam and Lu [2012c] developed a symmetrization-free argument on the Heisenberg group and established the critical Trudinger–Moser inequality (see also Lam, Lu and Tang [Lam et al. 2014] for the subcritical Trudinger–Moser inequality without using symmetrization argument). In fact, the critical and subcritical Trudinger–Moser inequalities are proved equivalent by Lam, Lu and Zhang [Lam et al. 2017b], where they also establish relationships between supremums of the critical and subcritical Trudinger–Moser inequalities. Such a relationship has been used to establish the existence of extremal functions for subcritical Trudinger–Moser inequalities on the entire space $\mathbb{R}^n$ [Lam et al. 2019].

The above Trudinger–Moser inequalities and its generalizations are often applied to derive the existence of weak solutions for the following n-Laplacian equations:

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u)+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^{\beta}}+\varepsilon h(x), \tag{1-4}$$

where $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is continuous and behaves like $\exp(\alpha|t|^{\frac{n}{n-1}})$ as $t\to+\infty$, $h(x)$ belongs to the dual space of $W^{1,n}(\mathbb{R}^n)$. Adding some appropriate assumptions on $V(x)$, one can see that the compact embedding

$$E=\left\{u:\int_{\mathbb{R}^n}|\nabla u|^n+V(x)|u|^n dx<+\infty\right\}\hookrightarrow L^p(\mathbb{R}^n)\quad\text{for } p\geq n$$

becomes admissible. The authors of [Adimurthi and Yang 2010; Alves and Figueiredo 2009; do Ó et al. 2014; Lam and Lu 2013a; Yang 2012b] carried out the standard mountain-pass procedure to obtain nontrivial weak solutions of (1-4). When $V(x)$ is constant, there is a long way to go yet. In order to overcome the possible failure of the Palais–Smale compactness condition which is caused by the absence of a compact embedding $W^{1,n}(\mathbb{R}^n)\hookrightarrow L^n(\mathbb{R}^n)$, Masmoudi and Sani [2015] applied a method involved with a constrained minimization argument and the sharp Trudinger–Moser inequality with the exact growth condition to investigate the existence of ground state solutions for (1-4) in the case of $V(x)=1$, $f(x,u)=f(u)$, $\beta=\varepsilon=0$. By assuming $f(x,t)=o(t)$, J. M. do Ó et al. [2014] employed a modified form of the Trudinger–Moser inequality and rearrangement inequalities to give sufficient conditions for the existence of ground state solutions. We also note that Lam and Lu [2014; 2013a] investigated the n-Laplacian equation and polyharmonic operators without the Ambrosetti–Rabinowitz condition. For more results about the Trudinger–Moser inequality and its application, we refer the reader to [Adimurthi and Sandeep 2007; Adimurthi and Yang 2010; Atkinson and

Peletier 1986; Carleson and Chang 1986; de Figueiredo et al. 2002; do Ó 1996; Lam and Lu 2012a; Panda 1996; Silva and Soares 1999].

D. Adams [1988] established the sharp Trudinger–Moser inequality with higher order derivatives. More precisely, he proved that

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\mathbb{R}^n),\,\|\nabla^m u\|_{\frac{n}{m}}\le 1}\int_\Omega e^{\beta_{n,m}|u|^{\frac{n}{n-m}}}\,dx<\infty,$$

where

$$\beta_{n,m}=\begin{cases}\dfrac{n}{\omega_{n-1}}\left[\dfrac{\pi^{\frac{n}{2}}2^m\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)}\right]^{\frac{n}{n-m}}, & \text{if } m \text{ is odd,}\\[2ex] \dfrac{n}{\omega_{n-1}}\left[\dfrac{\pi^{\frac{n}{2}}2^m\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}\right]^{\frac{n}{n-m}}, & \text{if } m \text{ is even.}\end{cases}$$

and

$$\nabla^m u=\begin{cases}\Delta^{\frac{m}{2}}, & \text{if } m \text{ is even,}\\ \nabla\Delta^{\frac{m-1}{2}}, & \text{if m is odd.}\end{cases}$$

The above inequality was extended by Tarsi [2012] to a larger space $W_N^{m,n/m}(\Omega)$ containing the Sobolev space $W_0^{n,n/m}(\Omega)$ as a closed subspace, where $W_N^{m,n/m}(\Omega)$ is given by

$$W_N^{m,\frac{n}{m}}(\Omega):=\left\{u\in W^{n,\frac{n}{m}}(\Omega)\mid \Delta^j u=0 \text{ on } \partial\Omega \text{ for } 0\le j\le\left[\tfrac{m-1}{2}\right]\right\}.$$

Sharp singular Adams inequalities on $W_N^{m,n/m}(\Omega)$ were also established by Lam and Lu [2012d]. We also mention that existence results concerning extremals of the Adams inequality in the case $n=2m=4$ were established by Lu and Yang [2009]. Li, Lu and Q. Yang [Li et al. 2018a; Lu and Yang 2017] proved the Hardy–Adams inequalities on hyperbolic spaces as a borderline case of the higher order Hardy–Sobolev–Mazya inequalities established by Lu and Q. Yang [2019] on upper half spaces.

After Adams, establishing Adams type inequalities in higher order Sobolev space $W^{m,n/m}(\mathbb{R}^n)$ has attracted much attention. Ogawa and Ozawa [1991] (for $n=2m$) and Ozawa [1995] (for general n, m) proved that there exist positive constants α and C_α such that

$$\int_{\mathbb{R}^n}\Phi_{n,m}(\alpha|u|^{\frac{n}{n-m}})\,dx\le C_\alpha,\quad \text{for all } u\in W^{m,\frac{n}{m}}(\mathbb{R}^n),\ |u|_{m,n}\le 1,$$

where

$$\Phi_{n,m}(t)=e^t-\sum_{j=0}^{j_{\frac{n}{m}}-2}\frac{t^j}{j!},\quad j_{\frac{n}{m}}=\min\left\{j\in\mathbb{N}: j\ge\tfrac{n}{m}\right\}$$

and $|u|_{m,n}$ is given by $|u|_{m,n}=\|(I-\Delta)^{m/2}u\|_{n/m}$. Kozono et al. [2006] studied the sharp constant problem by applying O'Neil's results on the rearrangement of convolution functions. In fact, they proved that there exists a constant $\beta^*_{n.m} \le \beta(n,m)$, particularly, $\beta^*_{2m.m} = \beta(2m,m)$ such that if $\beta < \beta^*_{n.m}$, then

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\mathbb{R}^n),\, |u|_{m,n}\le 1} \int_{\mathbb{R}^n} \Phi_{n,m}(\beta|u|^{\frac{n}{n-m}})\,dx < \infty.$$

Ruf and Sani [2013] established the sharp Adams type inequality for the critical case $\beta = \beta_{n,m}$ when m is an even integer, where the Talenti's comparison principle plays an important role in their proof. Lam and Lu [2012b; 2012d] proved the above inequality for all integers m (including fractional order γ). More precisely, they showed

$$\sup_{\|(I-\Delta)^{\frac{m-1}{2}}u\|^{\frac{n}{m}}_{\frac{n}{m}} + \|\nabla(I-\Delta)^{\frac{m-1}{2}}u\|^{\frac{n}{m}}_{\frac{n}{m}}\le 1} \int_{\mathbb{R}^n} \Phi_{n,m}(\beta_{n,m}|u|^{\frac{n}{n-m}})\,dx \le C(m,n),$$

for any odd integer m. Lam and Lu [2013b] further developed a rearrangement-free approach. This method can help us to get rid of the symmetrization or the comparison principle argument. Using this method, Lam and Lu established the sharp Adams inequality which can be stated as follows:

$$\sup_{u\in W^{\gamma,p}(\mathbb{R}^n),\, \|(\tau I-\Delta)^{\frac{\gamma}{2}}u\|_p\le 1} \int_{\mathbb{R}^n} \exp(\beta_0(n,\gamma)|u|^{p'})\,dx \le C(n,\gamma), \tag{1-5}$$

$$\text{where}\quad 0<\gamma<n,\quad p=\frac{n}{\gamma}\quad\text{and}\quad \beta_0(n,\gamma) = \frac{n}{\omega_{n-1}}\left(\frac{\pi^{\frac{n}{2}}2^\gamma\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)}\right)^{p'}.$$

Recently, Lam and Lu [2013b] obtained the Adams inequality involved with the norm $(\|\Delta u\|^{n/2}_{n/2} + \|u\|^{n/2}_{n/2})^{2/n}$. Later, Fontana and Morpurgo [2015] extended Lam and Lu's results to higher order derivatives. They proved that there exists some constant $C_{m,n}$ such that

$$\sup_{\|\nabla^m u\|^{\frac{n}{m}}_{\frac{n}{m}} + \|u\|^{\frac{n}{m}}_{\frac{n}{m}}\le 1} \int_{\mathbb{R}^n} \Phi_{n,m}(\beta_{n,m}|u|^{\frac{n}{n-m}})\,dx \le C_{m,n}. \tag{1-6}$$

Note that in (0-1), we assume $V(x)$ has a positive lower bound, thus we need an Adams inequality involved with the norm $(\|\nabla^m u\|^{n/m}_{n/m} + \tau\|u\|^{n/m}_{n/m})^{m/n}$. We utilize the change of variable to obtain the following result.

Theorem 1.1. *For any $\tau > 0$ and $0 < \alpha \le \beta_{n,m}$, there exists some constant $C_{m,n}$ such that for $u \in W^{m,n/m}(\mathbb{R}^n)$ with $\|\nabla^m u\|^{n/m}_{n/m} + \tau\|u\|^{n/m}_{n/m} \le 1$,*

$$\int_{\mathbb{R}^n} \Phi_{n,m}(\alpha|u|^{\frac{n}{n-m}})\,dx \le C_{m,n}. \tag{1-7}$$

Theorem 1.2. *For any $\tau > 0$, $0 \le t < n$ and $0 < \alpha < \beta_{n,m}$, there exists some constant $C_{m,n}$ such that for $u \in W^{m,n/m}(\mathbb{R}^n)$ with $\|\nabla^m u\|_{n/m}^{n/m} + \tau\|u\|_{n/m}^{n/m} \le 1$,*

$$\int_{\mathbb{R}^n} \frac{\Phi_{n,m}\big(\alpha\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx \le C_{m,n}. \tag{1-8}$$

Remark 1.3. In fact, the inequality (1-8) still holds in the case of $\alpha = \beta_{n,m}$. However, in order to prove the concentration-compactness principle for the Adams inequality, we only need inequality (1-5). For convenience, we also give the proof of the critical case of Theorem 1.2.

The purpose of proving such inequalities is to prove the following sharp version of concentration-compactness principle for weighted Adams inequalities in $W^{m,2}(\mathbb{R}^{2m})$. For simplicity, we define a new function space

$$E = \left\{u \in W^{m,2}(\mathbb{R}^{2m}) : \|u\|_E^2 = \int_{\mathbb{R}^{2m}} |\nabla^m u|^2 + V(x)|u|^2\,dx < \infty\right\},$$

where $V(x) \ge c_0$ $(c_0 > 0)$.

Theorem 1.4. *For $0 \le t < 2m$, assume $\{u_k\}_k$ is a sequence in E satisfying $\|u_k\|_E^2 = 1$ and $u_k \rightharpoonup u \not\equiv 0$ in E. If*

$$0 < p < p_{2m,m}(u) := \frac{1}{1-\|u\|_E^2},$$

then

$$\sup_k \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big)pu_k^2\big)}{|x|^t}\,dx < \infty, \tag{1-9}$$

$$\text{where} \quad \Phi_{2m,m}(t) = e^t - 1 \quad \text{and} \quad \beta_{2m,m} = \frac{2m}{\omega_{2m-1}}\pi^{2m}2^{2m}.$$

Furthermore, for any positive constant c, if $V(x) = c$, the constant $p_{2m,m}(u)$ is sharp in the sense that if $p \ge p_{2m,m}(u)$, the supremum will become infinite.

Remark 1.5. Theorem 1.4 is an extension of do Ó's result [2014] which relies heavily on the Pólya–Szegő inequality. Therefore, the methods they used cannot be applied to obtain the concentration-compactness principle of the Adams inequality on $\mathbb{R}^n$ or the Trudinger–Moser inequality in settings where a rearrangement argument fails such as the Heisenberg group $\mathbb{H}^n$. Recently, Li, Lu and Zhu [Li et al. 2018b] developed a symmetrization-free approach and established Lions concentration-compactness of the singular Trudinger–Moser inequality on the Heisenberg group $\mathbb{H}$. The method is rearrangement-free and can be easily applied to other settings. In the present paper, we use a different approach to prove Theorem 1.4. This is due to the fact that the Sobolev space $W^{m,2}(\mathbb{R}^n)$ we are dealing with is a Hilbert space. Analyzing the energy loss when taking the weak limit is an essential part in proving concentration-compactness principle and for the Hilbert space, the weak limit is relatively simple and with the help of the Brezis–Lieb lemma (Lemma 3.1), we are able to develop a different proof.

Remark 1.6. Nguyen [2016] took advantage of the Talenti comparison theorem to obtain inequality (1-9) in the case of $V(x) = 1$ and $t = 0$. However, they did not verify the sharpness of $p_{2m,m}(u)$. By constructing a proper sequence, we also verify that the supremum in (1-9) becomes infinite if $p \geq p_{2m,m}(u)$.

Recently, another improved version of the sharp Adams inequality was investigated by Lam, Lu and Tang [Lam et al. 2017a] in the spirit of Lions' work [1985]. Their result can be stated as follows:

$$\sup_{\substack{u\in W^{2,m}(\mathbb{R}^{2m}) \\ \int_{\mathbb{R}^{2m}} |\Delta u|^m + \tau |u|^m \, dx \leq 1}} \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,2}\Bigl(\frac{1/(2^{m-1}\alpha)}{(1+\|\Delta u\|_m^m)^{\frac{1}{m-1}}} |u|^{\frac{m}{m-1}}\Bigr)}{|x|^\beta} \, dx \leq C(m,\beta,\tau) \tag{1-10}$$

for $0 \leq \beta < 2m,\ \tau > 0$ and $0 \leq \alpha \leq \bigl(1 - \frac{\beta}{2m}\bigr)\beta_{2m,2}$. It is easy to verify that the above inequality is stronger than the general Adams inequalities in $W^{2,m}(\mathbb{R}^{2m})$.

Adams inequalities (1-5) and (1-10) are often used to study nonlinear equations related to the Bessel potential. Bao, Lam and Lu [Bao et al. 2016] considered polyharmonic equations of the form

$$(I - \Delta)^m u = f(x,u) \quad \text{in } \mathbb{R}^{2m}. \tag{1-11}$$

Yang [2012a] exploited the following bi-Laplacian equation with small perturbation

$$\Delta^2 u - \operatorname{div}(a(x)\nabla u) + b(x)u = \frac{f(x,u)}{|x|^\beta} + \varepsilon h(x) \quad \text{in } \mathbb{R}^4, \tag{1-12}$$

where $f(x,u)$ has exponential growth and $V(x)$ satisfies $\lim_{|x|\to+\infty} V(x) = +\infty$.

Recently, Chen, Li, Lu and Zhang [Chen et al. 2018] considered the following equation in $\mathbb{R}^4$:

$$(-\Delta)^2 u + V(x)u = \frac{f(x,u)}{|x|^\beta} \quad \text{in } \mathbb{R}^4, \quad 0 < \beta < 4, \tag{1-13}$$

where $V(x) \geq c_0$ and $f(x,t)$ satisfies some critical exponential growth. They established the existence of the ground state solutions.

Motivated by the work [Chen et al. 2018], we will study the existence of ground state solutions for the following polyharmonic equations with singular nonlinear term

$$(-\Delta)^m u + V(x)u = \frac{f(x,u)}{|x|^\beta} \quad \text{in } \mathbb{R}^{2m}, \quad 0 < \beta < 2m, \tag{1-14}$$

where $V(x) \geq c_0$ and $f(x,t)$ has critical exponential growth. Furthermore, we assume that $f(x,t)$ satisfies the following conditions.

(H_0) The nonlinearity $f(x,t) : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(x,t) = 0$ at $(x,0)$, and has exponential growth as $t \to +\infty$, which means there exists a constant $\alpha_0 > 0$ such that

$$\lim_{t\to+\infty} f(x,t)e^{-\alpha|t|^2} = \begin{cases} 0 & \text{for all } \alpha > \alpha_0, \\ +\infty & \text{for all } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^{2m}$.

(H_1) There exist constants $\alpha_0, b_1, b_2 > 0$ such that for any $(x,t) \in \mathbb{R}^{2m} \times (0,+\infty)$,

$$0 < f(x,t) \leq b_1 t + b_2 \Phi_{2m,m}(\alpha_0 t^2), \quad \text{where } \Phi_{2m,m}(t) = e^t - 1.$$

(H_2) There exist constants t_0 and $M_0 > 0$ such that

$$0 < F(x,t) := \int_0^t f(x,s)\,ds \leq M_0 f(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^{2m} \times [t_0,+\infty).$$

(H_3) There exists a constant $\theta > 2$ such that for all $x \in \mathbb{R}^{2m}$ and $t > 0$,

$$0 < \theta F(x,t) \leq f(x,t)t.$$

(H_4) $\displaystyle\limsup_{t\to 0+} \frac{2F(x,t)}{|t|^2} < \lambda_\beta$ uniformly in $x \in \mathbb{R}^{2m}$,

$$\text{where} \quad \lambda_\beta = \inf_{u\in E} \frac{\int_{\mathbb{R}^{2m}} |\nabla^m u|^2 + V(x)|u|^2\,dx}{\int_{\mathbb{R}^{2m}} |u|^2/|x|^\beta\,dx}.$$

(H_5) There exist constants $p > 2$ and C_p such that for all $(x,t) \in \mathbb{R}^{2m} \times (0,+\infty)$,

$$f(x,t) \geq C_p t^{p-1},$$

$$\text{where} \quad C_p > \left(\frac{\beta_{2m,m}\left(1-\frac{\beta}{2m}\right)}{\alpha_0}\right)^{\frac{(2-p)}{2}} \left(\frac{p-2}{p}\right)^{\frac{p-2}{2}} S_p^p$$

$$\text{and} \quad S_p^2 := \inf_{u\in E} \frac{\int_{\mathbb{R}^{2m}} |\nabla^m u|^2 + V(x)|u|^2\,dx}{\left(\int_{\mathbb{R}^{2m}} |u|^p/|x|^\beta\,dx\right)^{\frac{2}{p}}}.$$

(H_6) The function $\frac{f(x,t)}{t}$ is increasing for $t > 0$.

By (H_2) and (H_3), we can get that for all $(x,t) \in \mathbb{R}^{2m} \times [0,+\infty)$, there exists $\mu > 0$ such that

$$0 < F(x,t) \leq \mu f(x,t).$$

This result together with (H_1) and the singular Adams inequality in $W^{m,2}(\mathbb{R}^{2m})$ yields the boundedness of $F(x,u)$ and $f(x,u)v$ in $L^1(\mathbb{R}^{2m}, |x|^{-\beta}dx)$ for any

$u, v \in E$. Hence, one can easily find the functional related with polyharmonic equation (1-14), given by

$$I_\beta(u) = \tfrac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^{2m}} \frac{F(x,u)}{|x|^\beta}\, dx,$$

is well defined. With standard calculations, it is easy to obtain that $I_\beta \in C^1(E, \mathbb{R})$ and

$$I'_\beta(u)v = \int_{\mathbb{R}^{2m}} (\nabla^m u \nabla^m v + V(x)uv)\, dx - \int_{\mathbb{R}^{2m}} \frac{f(x,u)v}{|x|^\beta}\, dx, \quad u, v \in E.$$

Since the weak solutions of (1-14) are equivalent to the critical points of functional I_β, we focus our attention on critical points of functional I_β. Equation (1-14) is different from Equations (1-11) and (1-12). Unlike Bao, Lam and Lu's result [Bao et al. 2016], we do not necessarily assume that $f(x,t)$ satisfies some periodicity conditions. Moreover, the presence of potential $V(x)$ of (1-14) makes it difficult to directly apply Yang's argument in [Yang 2012a]. Thus a new compactness embedding in $W^{m,2}(\mathbb{R}^{2m})$ must be established and we observe that the weight term $1/|x|^\beta$ provides a good control to the integral away from zero, which enables us to establish the following compactness result.

Theorem 1.7. *The Sobolev space $W^{m,2}(\mathbb{R}^{2m})$ can be compactly embedded into $L^q(\mathbb{R}^{2m}, |x|^{-s}\, dx)$ when $q \geq 2$ and $0 < s < 2m$.*

Remark 1.8. In view of $E \hookrightarrow W^{m,2}(\mathbb{R}^{2m})$ and Theorem 1.7, we can derive that E can be compactly embedded into $L^q(\mathbb{R}^{2m}, |x|^{-s}\, dx)$ for $q \geq 2$ and $0 < s < 2m$.

With the help of Theorem 1.7, our next result will concern the existence of the ground state solution of polyharmonic equation (1-14).

Theorem 1.9. *Assume $f(x,t)$ satisfies* (H_1)–(H_6), *then* (1-14) *has a ground state solution.*

In the case of $\beta = 0$, (1-14) becomes the following nonsingular polyharmonic equation

$$(-\triangle)^m u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^{2m}. \tag{1-15}$$

The existence of the ground state solution of (1-15) cannot be obtained immediately from Theorem 1.9 due to the absence of compactness embedding. There is a common constrained minimization theory to deal with this problem. Unfortunately, this method crucially depends on the rearrangement inequality which is not obvious available in $W^{m,2}(\mathbb{R}^{2m})$. In order to overcome this difficulty, we use the principle of the symmetric criticality of the Hilbert space. By assuming $f(x,t)$ and $V(x)$ are radially symmetric with respect to x, one can carry out the same process as what we do in Theorem 1.9 to derive a nontrivial weak solution of the polyharmonic equation with nonsingular nonlinearity (1-15). However, whether there exists a ground state solution to (1-15) is still open. In a very recent work of Chen, Lu and Zhu [Chen et al. 2019], they made the first attempt in this direction. They derive the existence of ground state solutions to (1-15) when $m = 2$, V is a trapping potential and

$$f(x,u) = u\exp(2u^2).$$

Theorem 1.10. *Under the assumptions of Theorem 1.9, if we additionally assume that $V(x)$ and $f(x,t)$ are radially symmetric in x, $f(x,t) = o(t)$ at origin, then polyharmonic equation with nonsingular linearity* (1-15) *has a nontrivial weak solution.*

The plan of the paper is as follows. In Section 2, we employ the change of variable to establish some weighted Adams inequalities in $W^{m,2}(\mathbb{R}^{2m})$ involved with the norm $(\|\nabla^m u\|_{n/m}^{n/m} + \tau\|u\|_{n/m}^{n/m})^{m/n}$ for any $\tau > 0$. Sections 3 and 4 are devoted to the concentration-compactness principle for the weighted Adams inequality and a new compactness embedding in $W^{m,2}(\mathbb{R}^{2m})$. As an immediate application of Theorem 1.4, in Section 5, we give sufficient conditions to guarantee the existence of ground state solutions for the polyharmonic equation with singular exponential nonlinearity term. Finally, in Section 6, we also derive the existence of a nontrivial weak solution for the polyharmonic equation (1-15) through the principle of the symmetric criticality.

2. Proof of Theorems 1.1 and 1.2

In this section, we will utilize a change of variable to establish Adams inequality (1-7).

Proof. For any $\tau > 0$, $0 < \alpha \le \beta_{n,m}$ and $u \in W^{m,2}(\mathbb{R}^{2m})$ with

$$\int_{\mathbb{R}^n} |\nabla^m u|^{\frac{n}{m}} + \tau|u|^{\frac{n}{m}}\,dx \le 1,$$

we denote a new function $v(x)$ given by $v(\tau^{1/n}x) = u(x)$. Then direct computations yield that

$$\int_{\mathbb{R}^n} |\nabla^m v|^{\frac{n}{m}} + |v|^{\frac{n}{m}}\,dx = \int_{\mathbb{R}^n} |\nabla^m u|^{\frac{n}{m}} + \tau|u|^{\frac{n}{m}}\,dx \le 1$$

and

$$\int_{\mathbb{R}^n} \Phi_{n,m}(\alpha|u|^{\frac{n}{n-m}})\,dx = \frac{1}{\tau}\int_{\mathbb{R}^n} \Phi_{n,m}(\alpha|v|^{\frac{n}{n-m}})\,dx.$$

Combining this with inequality (1-6), we obtain

$$\sup_{\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}+\tau\|u\|_{\frac{n}{m}}^{\frac{n}{m}}\le 1}\int_{\mathbb{R}^n}\Phi_{n,m}(\alpha|u|^{\frac{n}{n-m}})\,dx \le \frac{1}{\tau}\sup_{\|\nabla^m v\|_{\frac{n}{m}}^{\frac{n}{m}}+\|v\|_{\frac{n}{m}}^{\frac{n}{m}}\le 1}\int_{\mathbb{R}^n}\Phi_{n,m}(\alpha|v|^{\frac{n}{n-m}})\,dx \le C(m,n).$$

Next, it suffices to show that inequality (1-8) still holds for any $\tau > 0$ and $0 \le t < n$. In fact, we have

$$\int_{\mathbb{R}^n}\frac{\Phi_{n,m}\big(\alpha\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx \le \int_{|x|\le 1}\frac{\Phi_{n,m}\big(\alpha\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx+\int_{|x|\ge 1}\Phi_{n,m}\big(\alpha(1-\tfrac{t}{n})|u|^{\frac{n}{n-m}}\big)\,dx. \tag{2-1}$$

This together with (1-7) and the Hölder inequality leads to

$$\sup_{\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}+\tau\|u\|_{\frac{n}{m}}^{\frac{n}{m}}\le 1}\int_{\mathbb{R}^n}\frac{\Phi_{n,m}\big(\alpha\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx \lesssim 1$$

for any $0 < \alpha < \beta_{n,m}$.

Finally, we prove that inequality (1-8) still holds in the case of $\alpha = \beta_{n,m}$. Following the same line of the proof of Theorem 1 in [Fontana and Morpurgo 2015], we can obtain that for any $u \in W^{m,n/m}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n}|\nabla^m u|^{\frac{n}{m}}+|u|^{\frac{n}{m}}\,dx \le 1$,

$$\int_{\Omega}\frac{\Phi_{n,m}\big(\beta_{n,m}\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx \lesssim (1+|\Omega|^{1-\frac{t}{n}}), \tag{2-2}$$

where Ω is any bounded domain of $\mathbb{R}^n$. Let

$$A := \{x \in \mathbb{R}^n : |u(x)| \ge 1\}.$$

Since $u \in W^{m,n/m}(\mathbb{R}^n)$, it is obvious that A is a bounded domain. Now, we split the integral over $\mathbb{R}^n$ into two parts:

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{\Phi_{n,m}\big(\beta_{n,m}\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx &= \int_{A}\frac{\Phi_{n,m}\big(\beta_{n,m}\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx+\int_{\mathbb{R}^n\setminus A}\frac{\Phi_{n,m}\big(\beta_{n,m}\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx\\ &= I_1+I_2.\end{aligned}$$

For I_1, using the estimate (2-2), we obtain that

$$I_1 = \int_A \frac{\Phi_{n,m}\big(\beta_{n,m}\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx \lesssim 1.$$

For I_2, since

$$\int_{\mathbb{R}^n\setminus A} \frac{\Phi_{n,m}\big(\beta_{n,m}\big(1-\frac{t}{n}\big)|u|^{\frac{n}{n-m}}\big)}{|x|^t}\,dx \lesssim \int_{\{|x|\le 1,\,|u|\le 1\}} \frac{|u|^{\frac{n}{m}}}{|x|^t}\,dx + \int_{\{|x|\ge 1,\,|u|\le 1\}} \frac{|u|^{\frac{n}{m}}}{|x|^t}\,dx \lesssim 1,$$

we derive that $I_2 \lesssim 1$. Combining the above estimates, we derive inequality (1-8) in the case of $\alpha = \beta_{n,m},\ \tau = 1$. Carrying out the same procedure as the proof of Theorem 1.1, one can conclude that inequality (1-8) still holds for any $\tau > 0$. $\square$

3. The proof of Theorem 1.4

Our purpose in this section is to prove Theorem 1.4. Namely, we will give the proof of the concentration-compactness principle for weighted Adams inequalities. Our proof relies on the following lemmas.

Lemma 3.1 [Brézis and Lieb 1983]. *Let Ω be an open subset of $\mathbb{R}^n$ and $\{u_k\}_k \subseteq L^p(\Omega)$ $(1 \le p < \infty)$. If $\{u_k\}_k$ satisfies the following conditions*:

(i) *$\{u_k\}_k$ is bounded in $L^p(\Omega)$,*

(ii) *$u_k \to u$ almost everywhere in Ω,*

then

$$\lim_{k\to\infty} (\|u_k\|_p^p - \|u_k - u\|_p^p) = \|u\|_p^p.$$

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open domain and $\{f_k\}_k \subseteq W^{m,n/m}(\Omega)$ that strongly converges to f in $W^{m,n/m}(\Omega)$. Then there exists a subsequence $\{f_{k_j}\}_j$ and a positive function $g \in W^{m,n/m}(\Omega)$ such that*

$$f_{k_j}(x) \to f(x) \quad \text{a.e. in } \Omega \text{ as } j \to +\infty,$$

and

$$|f_{k_j}(x)| \le g(x) \quad \text{a.e. in } \Omega \text{ for all } j.$$

Remark 3.3. Since the proof of Lemma 3.2 is similar to that of Proposition 1 in [do Ó et al. 2009], we omit the details.

Proof of Theorem 1.4. At first, we show the proof of inequality (1-9). It follows from the semicontinuity of the norm in E that

$$\|u\|_E^2 \le \liminf_k \|u_k\|_E^2 = 1.$$

We carry out the process by considering the following two cases.

Case 1. $\|u\|_E^2 = 1$. Applying the Brezis–Lieb lemma (Lemma 3.1) on the Hilbert space E, one can show that $u_k \to u$ strongly in E. In light of Lemma 3.2, we can find a subsequence $\{u_{k_j}\}_j$ and a positive function $v \in E$ such that $|u_{k_j}(x)| \le v(x)$. Then it follows that

$$\text{(3-1)}\quad \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big)pu_{k_j}^2\big)}{|x|^t}\,dx \le \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big)pv^2\big)}{|x|^t}\,dx < \infty.$$

Case 2. $0 < \|u\|_E^2 < 1$. Defining $\Psi(X) = \Phi_{2m,m}\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big)pX\big)$ for notational convenience, one can write that

$$\begin{aligned}\text{(3-2)}\quad \sup_k \int_{\mathbb{R}^{2m}} \frac{\Psi(u_k^2)}{|x|^t}\,dx &\le \sup_k \int_{\mathbb{R}^{2m}} \frac{\Psi\big((1+\varepsilon)(u_k-u)^2 + C_\varepsilon u^2\big)}{|x|^t}\,dx \\ &= \sup_k \int_{\mathbb{R}^{2m}} \frac{\Psi((1+\varepsilon)(u_k-u)^2)\Psi(C_\varepsilon u^2)}{|x|^t}\,dx \\ &\quad + \sup_k \int_{\mathbb{R}^{2m}} \frac{\Psi((1+\varepsilon)(u_k-u)^2)}{|x|^t}\,dx + \sup_k \int_{\mathbb{R}^{2m}} \frac{\Psi(C_\varepsilon u^2)}{|x|^t}\,dx \\ &=: I_1 + I_2 + I_3,\end{aligned}$$

where we use the elementary inequality which states

$$(a+b)^2 \le (1+\varepsilon)a^2 + C_\varepsilon b^2 \quad \text{for } a, b \ge 0 \text{ and } \varepsilon > 0.$$

For I_1, as an immediately consequence of the Hölder inequality and the singular Adams inequality, we can derive that

$$\text{(3-3)}\qquad I_1 \lesssim \left(\sup_k \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big)pr(1+\varepsilon)(u_k-u)^2\big)}{|x|^t}\,dx\right)^{\frac{1}{r}},$$

where r is sufficiently close to 1. Noting that $u_k \rightharpoonup u$ weakly in E and E is a Hilbert space, one can apply the Brezis–Lieb lemma to derive that

$$\|u_k-u\|_E^2 = \|u_k\|_E^2 - \|u\|_E^2 = 1 - \|u\|_E^2,$$

which yields that

$$\beta_{2m,m}\big(1-\tfrac{t}{2m}\big)pr(1+\varepsilon)(\|\nabla^m(u_k-u)\|_2^2 + c_0\|u_k-u\|_2^2) < \beta_{2m,m}\big(1-\tfrac{t}{2m}\big).$$

Combining this with Theorem 1.2 with $\tau = c_0$, we conclude that $I_1 < +\infty$. Similarly, we can obtain that $I_2 < +\infty$. Thus, we accomplish the proof of inequality (1-9).

Next, we are ready to show that $p_{2m,m}(u)$ is sharp when $V(x)$ is constant. Without loss of generality, we assume $V(x) = 1$. The idea of proving this sharpness follows from the result of do Ó et al. [2014]. Similarly, we construct a sequence $\{u_k\}_k \subseteq W^{m,2}(\mathbb{R}^{2m})$ and a function $u \in W^{m,2}(\mathbb{R}^{2m})$ such that

$$\|u_k\| = 1, \quad u_k \rightharpoonup u \not\equiv 0 \text{ in } W^{m,2}(\mathbb{R}^{2m}), \quad \|u\| = \delta < 1,$$

but

$$\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big)p_{2m,m}(u)u_k^2\big)}{|x|^t}\,dx \to \infty.$$

We denote a sequence $\{w_k\}_k \subseteq W^{m,2}(\mathbb{R}^{2m})$ by

$$w_k(x) = \begin{cases} \frac{1}{(2m-2)!!(2m)^{\frac{1}{2}}}\omega_{2m-1}^{-\frac{1}{2}}k^{\frac{1}{2}} & \text{if } |x| \in [0, re^{\frac{-k}{2m}}],\\ \omega_{2m-1}^{-\frac{1}{2}}\frac{(2m)^{\frac{1}{2}}}{(2m-2)!!}\ln(\frac{r}{|x|})k^{-\frac{1}{2}} & \text{if } |x| \in [re^{\frac{-k}{2m}}, r],\\ 0 & \text{if } |x| \in [r, +\infty), \end{cases}$$

where $r > 0$ to be chosen later. Simple calculations show that

$$w_k \rightharpoonup 0 \text{ in } W^{m,2}(\mathbb{R}^{2m}), \quad \|\nabla^m w_k\|_2^2 = 1, \quad \|w_k\|_2^2 = O(k^{-1}).$$

Next, we define a new function $u : \mathbb{R}^{2m} \to \mathbb{R}$ given by

$$u(x) = \begin{cases} A & \text{if } |x| \in [0, \frac{2R}{3}],\\ \big(1-\big(\frac{2}{3}\big)^m\big)^{-1}\big(A - \frac{A}{R^m}|x|^m\big) & \text{if } |x| \in [\frac{2R}{3}, R],\\ 0 & \text{if } |x| \in [R, +\infty), \end{cases}$$

where $R = 3r$ and A is a positive constant which needs to be chosen later. Then

$$\begin{aligned} \|u\|^2 &= \|u\|_2^2 + \|\nabla^m u\|_2^2 \\ &= \frac{\omega_{2m-1}}{2m}\big(\tfrac{2}{3}R\big)^{2m}A^2 \\ &\quad + \omega_{2m-1}\int_{\frac{2R}{3}}^{R}\Big(\big(1-\big(\tfrac{2}{3}\big)^m\big)^{-1}\Big(A - \frac{A}{R^m}r^m\Big)\Big)^2 r^{2m-1}\,dr \\ &\qquad + \big(1-\big(\tfrac{2}{3}\big)^m\big)^{-2}\Big(\frac{A}{R^m}\Big)^2\omega_{2m-1}\int_{\frac{2R}{3}}^{R}\frac{m!!(3m-2)!!}{(2m-2)!!}r^{2m-1}\,dr \\ &= CA^2. \end{aligned} \tag{3-4}$$

Picking A satisfying $\|u\| = \delta < 1$, a direct application of the Hölder inequality yields that

$$\begin{aligned}\|v_k\|_2^2 &:= \|u + (1-\delta^2)^{\frac{1}{2}} w_k\|_2^2 \\ &= \int_{\mathbb{R}^{2m}} |u + (1-\delta^2)^{\frac{1}{2}} w_k|^2 dx \\ &= \int_{\mathbb{R}^{2m}} u^2 + 2(1-\delta^2)^{\frac{1}{2}} u w_k + (1-\delta^2) w_k^2 dx \\ &= \|u\|_2^2 + \eta_k,\end{aligned} \tag{3-5}$$

where

$$\eta_k = \left(\frac{1}{m} A (1-\delta^2)^{\frac{1}{2}} \frac{(2m)^{\frac{1}{2}}}{(2m-2)!!} \omega_{2m-1}^{\frac{1}{2}} \left(r^{2m} e^{-k} \frac{k}{2m} + \frac{r^{2m}}{2m} - \frac{r^{2m}}{2m} e^{-k}\right)\right) k^{-\frac{1}{2}} + O(k^{-1}).$$

It is clear that $\nabla^m u$ and $\nabla^m w_k$ have disjoint supports, so

$$\|\nabla^m v_k\|_2^2 = \|\nabla^m u\|_2^2 + (1-\delta^2) \quad \text{and} \quad \|v_k\|^2 = 1 + \eta_k. \tag{3-6}$$

Let $u_k = v_k/(1+\eta_k)^{1/2}$; one can easily see that

$$\|u_k\| = 1 \quad \text{and} \quad u_k \rightharpoonup u \text{ in } W^{m,2}(\mathbb{R}^{2m}).$$

Consequently,

$$\begin{aligned}&\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big) p_{2m,m}(u) u_k^2\big)}{|x|^t} dx \\ &\geq \int_{B_{re^{\frac{-k}{2m}}}} \frac{\exp\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big)(1-\delta^2)^{-1} u_k^2\big)}{|x|^t} dx - \int_{B_{re^{\frac{-k}{2m}}}} \frac{1}{|x|^t} dx \\ &= \int_{B_{re^{\frac{-k}{2m}}}} \frac{\exp\big(\beta_{2m,m}\big(1-\frac{t}{2m}\big)((1+\eta_k)^{-\frac{1}{2}} (A + (1-\delta^2)^{\frac{1}{2}} w_k))^2 (1-\delta^2)^{-1}\big)}{|x|^t} dx + C \\ &= \int_{B_{re^{\frac{-k}{2m}}}} \frac{\exp\Big(\big(1-\frac{t}{2m}\big)\Big((1+\eta_k)^{-\frac{1}{2}} \Big(\frac{\beta_{2m,m}^{1/2} A}{(1-\delta^2)^{1/2}} + k^{\frac{1}{2}}\Big)\Big)^2\Big)}{|x|^t} dx + C \\ &\gtrsim \exp\left(\left(1-\frac{t}{2m}\right)\left(\left((1+\eta_k)^{-\frac{1}{2}} \left(\frac{\beta_{2m,m}^{\frac{1}{2}} A}{(1-\delta^2)^{\frac{1}{2}}} + k^{\frac{1}{2}}\right)\right)^2 - k\right)\right) r^{2m-t} + C \to +\infty,\end{aligned} \tag{3-7}$$

where $r < 1$ is selected in such a way that $\eta_k < \frac{\beta_{2m,m}^{1/2} A}{(1-\delta^2)^{1/2}} k^{-\frac{1}{2}}$. Then Theorem 1.4 is completed. □

4. the proof of Theorem 1.7

In this section, we begin with a simple fact that the norm $(\|\nabla^m u\|_2^2+\|u\|_2^2)^{1/2}$ and the standard Sobolev norm given by

$$\|u\|_{W^{m,2}}=\left(\sum_{j=0}^{j=m}\|\nabla^j u\|_2^2\right)^{\frac{1}{2}}$$

is equivalent. In fact, for any $u\in\mathcal{C}_c^\infty(\mathbb{R}^{2m})$, through Caffarelli–Kohn–Nirenberg inequalities [Lin 1986], one can derive that

$$(*)\qquad \int_{\mathbb{R}^{2m}}|\nabla^j u|^2\,dx\le\left(\int_{\mathbb{R}^{2m}}|u|^2\,dx\right)^{\frac{j}{m}}\left(\int_{\mathbb{R}^{2m}}|\Delta u|^2\,dx\right)^{1-\frac{j}{m}}.$$

Then a simple density argument implies that $(*)$ also holds for $u\in W^{m,2}(\mathbb{R}^{2m})$. Now, we are in a position to show that a Sobolev space equipped with the norm $(\|\nabla^m u\|_2^2+\|u\|_2^2)^{1/2}$ can be compactly embedded into $L^p(\mathbb{R}^{2m},|x|^{-\beta}dx)$ for any $p\ge 2$ and $0<\beta<2m$.

Proof. Continuous embedding is an easy consequence of the Adams inequality (1-5). Our aim is to show that the above continuous embedding is compact. In light of $W^{m,2}(\mathbb{R}^{2m})\hookrightarrow\hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^{2m})$ for $q\ge 1$, one can find a subsequence $\{u_{k_j}\}_j$ such that

$$u_{k_j}(x)\to u(x),\quad \text{strongly in } L^q(B_R(0)) \text{ for any } R>0,$$
$$u_{k_j}(x)\to u(x),\quad \text{for almost every } x\in\mathbb{R}^{2m}.$$

Therefore, our purpose is to show that

$$u_{k_j}\to u\quad\text{in } L^q(\mathbb{R}^{2m},|x|^{-s}dx). \tag{4-1}$$

Applying the Egorov theorem, we obtain that for any $B_R(0)$ and $\delta>0$,

$$\text{there exists } E_\delta\subset B_R(0) \text{ satisfying } m(E_\delta)<\delta,$$

such that

$$u_{k_j} \text{ uniformly converges to } u \text{ in } B_R(0)\setminus E_\delta.$$

Hence, it follows that

$$\begin{aligned}(4\text{-}2)\quad &\lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{j\to+\infty}\int_{\mathbb{R}^{2m}}\frac{|u_{k_j}-u|^q}{|x|^s}\,dx\\ &=\lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{j\to+\infty}\int_{E_\delta}\frac{|u_{k_j}-u|^q}{|x|^s}\,dx\\ &\qquad+\lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{j\to+\infty}\int_{B_R(0)\setminus E_\delta}\frac{|u_{k_j}-u|^q}{|x|^s}\,dx\\ &\qquad\qquad+\lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{j\to+\infty}\int_{\mathbb{R}^{2m}\setminus B_R(0)}\frac{|u_{k_j}-u|^q}{|x|^s}\,dx\\ &=: I_1+I_2+I_3.\end{aligned}$$

By the Hölder inequality and the Sobolev continuous embedding, one can derive that

$$
\begin{aligned}
\text{(4-3)} \qquad I_1 &\le \lim_{\delta\to 0}\lim_{j\to+\infty}\left(\int_{E_\delta} 1\,dx\right)^{\frac{1}{t}}\left(\int_{E_\delta}\frac{|u_{k_j}-u|^{qt'}}{|x|^{st'}}\,dx\right)^{\frac{1}{t'}} \\
&\lesssim \lim_{\delta\to 0}\sup_j \|u_{k_j}\|^q (m(E_\delta))^{\frac{1}{t}} \\
&= 0,
\end{aligned}
$$

where $t > 1$ and $st' < 2m$. For I_2, the uniform convergence of u_{k_j} in $B_R(0)\backslash E_\delta$ yields that $I_2 = 0$. For I_3, the Sobolev continuous embedding $W^{m,2}(\mathbb{R}^{2m}) \hookrightarrow L^q(\mathbb{R}^{2m})$ for $q \ge 2$ yields that

$$
\begin{aligned}
\text{(4-4)} \qquad I_3 &\le \lim_{R\to+\infty}\lim_{\delta\to 0}\lim_{j\to+\infty}\frac{1}{R^s}\int_{\mathbb{R}^{2m}\backslash B_R(0)} |u_{k_j}-u|^q\,dx \\
&\lesssim \lim_{R\to+\infty}\sup_j \|u_{k_j}\|^q \frac{1}{R^s} \\
&= 0.
\end{aligned}
$$

Thus, we have accomplished the proof of Theorem 1.7. □

As a direct result of Theorem 1.7 and Remark 1.8, we can easily see that the best constant S_p $(p \ge 2)$ in (H_3) could be achieved (one can refer to [Zhang and Chen 2018] for details).

5. The proof of Theorem 1.9

This section is devoted to the proof of Theorem 1.9. We carry out the proof in three parts. In Part 1, we use the mountain-pass theorem without the Palais–Smale compactness condition to derive the existence of weak solutions of (1-14) satisfying hypotheses (H_1)–(H_4). Therefore, in Part 2, we utilize the method combining the concentration-compactness principle and the new compactness theorem in $W^{m,2}(\mathbb{R}^{2m})$ to verify that the functional I_β satisfies the Palais–Smale compactness condition. Part 3 is devoted to showing that the critical point of the functional I_β is actually a ground state solution of polyharmonic equation (1-14). Before starting the proof, we need a couple of important lemmas for which we omit the proofs.

Lemma 5.1 [Badiale and Serra 2011]. *Let X be a Hilbert space, $\varphi \in C^2(X,\mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\| > r$ and $b := \inf_{\|u\|=r}\varphi(u) > \varphi(0) \ge \varphi(e)$. Define*

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]}\varphi(g(s)),$$

where

$$\Gamma := \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}.$$

Then there exists a sequence $\{u_k\}_k \in X$ *such that* $\varphi(u_k) \to c,\ \varphi'(u_k) \to 0$ *as* $k \to +\infty$.

Remark 5.2. In the case of $p = 2$, one can use the property of the Hilbert space to replace $u_k \to u$ almost everywhere in Ω with $u_k \rightharpoonup u$.

Now, we are ready to start the proof of Theorem 1.9.

Part 1. In this part, we first check that $I_\beta(u)$ satisfies geometric conditions without the Palais–Smale compactness condition.

Lemma 5.3. *Assume* (H$_1$)–(H$_4$) *hold. Then*

(i) *there exist constants* $\delta, \rho > 0$ *such that* $I_\beta(u) \geq \delta$ *for any* $\|u\|_E = \rho$,

(ii) *there exists* $e \in E$ *such that* $\|e\|_E > \rho$, *but* $I_\beta(e) < 0$.

Proof. According to (H$_4$), there exist positive constants ε, δ such that for any $|t| \leq \delta$,

$$F(x,t) \leq \tfrac{1}{2}(\lambda_\beta - \varepsilon)|t|^2 \quad \text{for } x \in \mathbb{R}^{2m}. \tag{5-1}$$

Moreover, by (H$_1$), we derive that for any $|t| \geq \delta$ and $x \in \mathbb{R}^{2m}$, there exists constants c_1, c_2 such that

$$F(x,t) \leq c_1|t|^2 + c_2|t|\Phi_{2m,m}(\alpha_0|t|^2) \leq C_\delta|t|^3\Phi_{2m,m}(\alpha_0|t|^2), \quad \text{where} \quad C_\delta = \frac{c_1}{\delta\Phi_{2m,m}(\alpha_0|\delta|^2)} + \frac{c_2}{\delta^2}. \tag{5-2}$$

Then it follows from (5-1) and (5-2) that

$$F(x,t) \leq \tfrac{1}{2}(\lambda_\beta - \varepsilon)|t|^2 + C|t|^3\Phi_{2m,m}(\alpha_0|t|^2) \quad \text{for all } (x,t) \in \mathbb{R}^{2m} \times \mathbb{R}. \tag{5-3}$$

For sufficiently small $\|u\|_E$, we claim that the following inequality holds:

$$\int_{\mathbb{R}^{2m}} |u|^3 \frac{\Phi_{2m,m}(\alpha_0|u|^2)}{|x|^\beta}\, dx \leq C\|u\|_E^3. \tag{5-4}$$

For the continuity of our work, let us postpone the proof of (5-4).

Suppose (5-4) holds, we can combine (5-3) and (5-4) to arrive at

$$\begin{aligned} I_\beta(u) &= \tfrac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^{2m}} \frac{F(x,u)}{|x|^\beta}\, dx \\ &\geq \tfrac{1}{2}\|u\|_E^2 - \tfrac{1}{2}(\lambda_\beta - \varepsilon)\int_{\mathbb{R}^{2m}} \frac{|u|^2}{|x|^\beta}\, dx - C\int_{\mathbb{R}^4} |u|^3 \frac{\Phi_{2m,m}(\alpha_0|u|^2)}{|x|^\beta}\, dx \\ &\geq \tfrac{1}{2}\|u\|_E^2 - \tfrac{1}{2}\frac{\lambda_\beta - \varepsilon}{\lambda_\beta}\|u\|_E^2 - C\|u\|_E^3 \\ &= \|u\|_E^2\Big(\frac{\varepsilon}{2\lambda_\beta} - C\|u\|_E\Big). \end{aligned} \tag{5-5}$$

When $\|u\|_E \leq \varepsilon/(2C\lambda_\beta)$, inequality (i) holds.

Now, we give the proof of inequality (5-4). By applying the Hölder inequality and considering the level sets of the function, one can obtain that for $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned}(5\text{-}6)\quad & \int_{\mathbb{R}^{2m}} |u|^3 \frac{\Phi_{2m,m}(\alpha_0|u|^2)}{|x|^\beta}\,dx \\ &\lesssim \left(\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(p\alpha_0|u|^2)}{|x|^\beta}\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^{2m}} \frac{|u|^{3p'}}{|x|^\beta}\,dx\right)^{\frac{1}{p'}} \\ &\lesssim \left(\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(p\alpha_0|u|^2)}{|x|^\beta}\,dx\right)^{\frac{1}{p}} \|u\|_E^3,\end{aligned}$$

where the last inequality comes from the Sobolev continuous embedding $E \hookrightarrow L^q(\mathbb{R}^{2m}, |x|^{-\beta}dx)$. Pick $p > 1$ sufficiently close to 1 such that

$$p\alpha_0\|u\|^2 \le \beta_{2m,m}\big(1 - \tfrac{\beta}{2m}\big)$$

due to the fact that $\|u\| \le \|u\|_E$ is sufficiently small. The singular Adams inequalities in $\mathbb{R}^{2m}$ yield that

$$(5\text{-}7)\qquad \int_{\mathbb{R}^{2m}} |u|^3 \frac{\Phi_{2m,m}(\alpha_0|u|^2)}{|x|^\beta}\,dx \le C\|u\|_E^3.$$

For (ii), it suffices to show that for a fixed $u \in E$,

$$I_\beta(su) \to -\infty \quad \text{as } s \to +\infty.$$

Without loss of generality, we may assume u has bounded support Ω. Through (H_3), one finds that for any $t > 0$,

$$\frac{\partial}{\partial t}(\ln F(x,t)) \ge \frac{\theta}{t},$$

which leads to the result $F(x,t) \ge F(x,t_0)t_0^{-\theta}t^\theta$ for some $t_0 > 0$. Therefore, there exist positive constants c_1, c_2 such that

$$F(x,t) \ge c_1 t^\theta - c_2 \quad \text{for } (x,t) \in \Omega \times [0,\infty).$$

Then,

$$\begin{aligned}(5\text{-}8)\qquad I_\beta(su) &= \frac{s^2}{2}\|u\|_E^2 - \int_\Omega \frac{F(x,su)}{|x|^\beta}\,dx \\ &\le \frac{s^2}{2}\|u\|_E^2 - c_1 s^\theta \int_\Omega \frac{|u|^\theta}{|x|^\beta}\,dx + c_3|\Omega|^{1-\frac{\beta}{2m}}.\end{aligned}$$

This inequality together with $\theta > 2$ implies that

$$I_\beta(su) \to -\infty \quad \text{as } s \to +\infty.$$

The proof of Lemma 5.3 is finished. □

Lemma 5.3 shows that the functional I_β satisfies geometric conditions of the mountain-pass theorem which yields that there exists a Palais–Smale sequence $\{u_k\}_k$ which satisfies $I_\beta(u_k) \to c_\beta$ and $I'_\beta(u_k) \to 0$ as $k \to +\infty$, where

$$c_\beta = \inf_{g\in\Gamma} \max_{s\in[0,1]} I_\beta(g(s)), \quad \Gamma := \{g \in C([0,1], E) : g(0) = 0,\ I(g(1)) < 0\}.$$

Lemma 5.4. *Assume* (H$_1$), (H$_2$) *and* (H$_3$) *hold. Let* $\{u_k\}_k \subset E$ *be an arbitrary Palais–Smale sequence, i.e.*,

$$I_\beta(u_k) \to c_\beta, \quad I'_\beta(u_k) \to 0, \quad \text{as } k \to +\infty.$$

Then there exists a subsequence of $\{u_k\}_k$ (still denoted by $\{u_k\}_k$) *and* $u \in E$ *such that*

$$\begin{cases} \dfrac{f(x,u_k)}{|x|^\beta} \to \dfrac{f(x,u)}{|x|^\beta} & \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^{2m}), \\ \dfrac{F(x,u_k)}{|x|^\beta} \to \dfrac{F(x,u)}{|x|^\beta} & \text{strongly in } L^1(\mathbb{R}^{2m}). \end{cases}$$

Furthermore, *u is a weak solution of* (1-14).

Proof. At first, we prove that

$$\int_{\mathbb{R}^{2m}} \frac{F(x,u_k)}{|x|^\beta}\,dx \le C \quad \text{and} \quad \int_{\mathbb{R}^{2m}} \frac{f(x,u_k)u_k}{|x|^\beta}\,dx \le C. \tag{5-9}$$

Let $\{u_k\}_k$ denote a Palais–Smale sequence of the function I_β, i.e.,

$$\tfrac{1}{2}\|u_k\|_E^2 - \int_{\mathbb{R}^{2m}} \frac{F(x,u_k)}{|x|^\beta}\,dx \to c_\beta \quad \text{as } k \to \infty \tag{5-10}$$

and

$$|I'(u_k)v| \le \tau_k \|v\|_E \quad \text{for all } v \in E, \tag{5-11}$$

where $\tau_k \to 0$ as $k \to \infty$. Moreover, taking $v = u_k$ in (5-11), we get

$$\int_{\mathbb{R}^{2m}} \frac{f(x,u_k)u_k}{|x|^\beta}\,dx - \|u_k\|_E^2 \le \tau_k \|u_k\|_E. \tag{5-12}$$

This together with (5-10) and (H$_4$) leads to

$$\begin{aligned} \theta c_\beta + \tau_k\|u_k\|_E &\ge \Big(\frac{\theta}{2} - 1\Big)\|u_k\|_E^2 - \int_{\mathbb{R}^{2m}} \frac{[\theta F(x,u_k) - f(x,u_k)u_k]}{|x|^\beta}\,dx \\ &\ge \Big(\frac{\theta-2}{2}\Big)\|u_k\|_E^2. \end{aligned}$$

Thus, $\|u_k\|_E$ is bounded. Combine this with (5-10) and (5-12), we can get (5-9). Since $\|u_k\|_E$ is bounded. Thanks to Theorem 1.7, we can assume that up to a sequence,

$$\begin{aligned} u_k &\rightharpoonup u, && \text{weakly in } E, \\ u_k &\to u, && \text{strongly in } L^q(\mathbb{R}^{2m}, |x|^{-\beta}\,dx) \text{ for all } q \geq 2, \\ u_k(x) &\to u(x), && \text{for almost every } x \in \mathbb{R}^{2m}. \end{aligned}$$

By hypothesis (H_1), through similar arguments to Lemma 2.1 in [de Figueiredo et al. 1995], we derive that

$$\frac{f(x,u_k)}{|x|^\beta} \to \frac{f(x,u)}{|x|^\beta} \quad \text{strongly in} \quad L^1_{\text{loc}}(\mathbb{R}^{2m}). \tag{5-13}$$

To show the convergence of $\int_{\mathbb{R}^{2m}} \frac{F(x,u_k)}{|x|^\beta}\,dx$, one can write

$$\begin{aligned} \int_{\mathbb{R}^{2m}} & \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx \\ &= \int_{B_R} \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx + \int_{\mathbb{R}^{2m}\setminus B_R} \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx. \end{aligned}$$

According to (H_2) and (H_3), there exists a positive constant R_0 such that

$$\frac{F(x,u_k)}{|x|^\beta} \leq \frac{R_0 f(x,u_k)}{|x|^\beta} \quad \text{for all } x \in \mathbb{R}^{2m}. \tag{5-14}$$

Together with the generalized Lebesgue dominated convergence theorem, we can get that

$$\lim_{R\to+\infty} \lim_{k\to+\infty} \int_{B_R} \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx = 0. \tag{5-15}$$

Thus, it suffices to check that

$$\lim_{R\to+\infty} \lim_{k\to+\infty} \int_{\mathbb{R}^{2m}\setminus B_R} \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx = 0. \tag{5-16}$$

By dividing the integral into two parts, we arrive at

$$\begin{aligned} \int_{\mathbb{R}^{2m}\setminus B_R} \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx &= \int_{\{|x|\geq R\}\bigcap\{|u_k|>A\}} \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx \\ &\quad + \int_{\{|x|\geq R\}\bigcap\{|u_k|\leq A\}} \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx \\ &=: I_A + II_A. \end{aligned}$$

For I_A, it follows from (5-9) that

$$\int_{\{|x|\geq R\}\bigcap\{|u_k|>A\}} \frac{|F(x,u_k)|}{|x|^\beta}\,dx \leq \frac{R_0}{A}\int_{\{|x|\geq R\}\bigcap\{|u_k|>A\}} \frac{|f(x,u_k)u_k|}{|x|^\beta}\,dx \lesssim \frac{R_0}{A}.$$

Thus, $\lim_{A\to+\infty}\lim_{R\to+\infty}\lim_{k\to+\infty} I_A = 0$.

For II_A, applying hypothesis (H_1) and Theorem 1.7, one can derive that

$$\begin{aligned}\lim_{A\to+\infty}\lim_{R\to+\infty}\lim_{k\to+\infty} II_A &\leq \lim_{A\to+\infty}\lim_{R\to+\infty}\lim_{k\to+\infty} C(\alpha_0,A)\int_{\{|x|\geq R\}\bigcap\{|u_k|\leq A\}} \frac{|u_k|^2}{|x|^\beta}\,dx \\ &\leq \lim_{A\to+\infty}\lim_{R\to+\infty}\lim_{k\to+\infty} \frac{C(\alpha_0,A)}{R^{\beta/2}}\sup_k \|u_k\|_E^2 \\ &= 0.\end{aligned} \tag{5-17}$$

Hence,

$$\lim_{k\to+\infty}\int_{\mathbb{R}^{2m}} \frac{|F(x,u_k)-F(x,u)|}{|x|^\beta}\,dx = 0. \tag{5-18}$$

A simple application of (5-13) shows that

$$\int_{\mathbb{R}^{2m}} (\nabla^m u \nabla^m \varphi + u\varphi)\,dx - \int_{\mathbb{R}^{2m}} \frac{f(x,u)}{|x|^\beta}\varphi\,dx = 0, \quad \text{for all } \varphi\in C_0^\infty(\mathbb{R}^{2m}).$$

Thus, u is a weak solution of polyharmonic equation (1-14). □

Part 2. This part is devoted to showing that the Palais–Smale sequence $\{u_k\}_k$ satisfies the Palais–Smale condition in light of the concentration-compactness principle. We begin with a crucial fact:

$$0 < c_\beta < \Big(1-\frac{\beta}{2m}\Big)\frac{\beta_{2m,m}}{2\alpha_0}.$$

Recall that we have shown the attainability of S_p in Section 4, so there exists a function u such that

$$\int_{\mathbb{R}^{2m}} \frac{|u|^p}{|x|^\beta}\,dx = 1 \quad \text{and} \quad \|u\|_E = S_p.$$

Through the definition of c_β, we get

$$0 < c_\beta \leq \max_{t\geq 0} I_\beta(tu) = \max_{t\geq 0}\bigg(\frac{t^2}{2}S_p^2 - \int_{\mathbb{R}^{2m}} \frac{F(x,tu)}{|x|^\beta}\,dx\bigg).$$

According to the definition of C_p, we can obtain that

$$c_\beta \le \max_{t\ge 0}\left(\frac{t^2}{2}S_p^2 - t^p\frac{C_p}{p}\right) = \frac{(p-2)}{2p}\frac{S_p^{2p/(p-2)}}{C_p^{2/(p-2)}} < \frac{\beta_{2m,m}\left(1-\frac{\beta}{2m}\right)}{2\alpha_0}. \tag{5-19}$$

Now, we are in a position to verify that $\{u_k\}_k$ satisfies the Palais–Smale compactness condition. We discuss this by the following two cases.

Case 1. ($c_\beta \ne 0, u = 0$). We first claim that there exists some $q > 1$ such that

$$\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\alpha_0|u_k|^2)^q}{|x|^\beta}\,dx \le C \quad \text{for all } k \in \mathbb{N}.$$

Since $u = 0$, one can employ Lemma 5.4 to drive that

$$\int_{\mathbb{R}^{2m}} \frac{F(x,u_k)}{|x|^\beta}\,dx \to \int_{\mathbb{R}^{2m}} \frac{F(x,u)}{|x|^\beta}\,dx = 0. \tag{5-20}$$

Together with (5-10), we obtain that

$$\|u_k\|_E^2 \to 2c_\beta \quad \text{as } k \to \infty. \tag{5-21}$$

Take $q > 1$ sufficiently close to 1 such that

$$\alpha_0 q\|u_k\|^2 \le \alpha_0 q\|u_k\|_E^2 \le \beta_0 < \left(1-\frac{\beta}{2m}\right)\beta_{2m,m}. \tag{5-22}$$

Then, it follows that

$$\begin{aligned}\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\alpha_0|u_k|^2)^q}{|x|^\beta}\,dx &\le \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\alpha_0|u_k|^2)^q}{|x|^\beta}\,dx \\ &\lesssim \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\left(\beta_0\left(\frac{u_k}{\|u_k\|}\right)^2\right)}{|x|^\beta}\,dx \\ &\lesssim 1.\end{aligned} \tag{5-23}$$

Combining hypothesis (H_1), the Hölder inequality and (5-23), one can derive that

$$\begin{aligned}&\left|\int_{\mathbb{R}^{2m}} \frac{f(x,u_k)u_k}{|x|^\beta}\,dx\right| \\ &\lesssim \int_{\mathbb{R}^{2m}} \frac{|u_k|^2}{|x|^\beta}\,dx + \left(\int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}(\alpha_0|u_k|^2)^q}{|x|^\beta}\,dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^{2m}} \frac{|u_k|^{q'}}{|x|^\beta}\,dx\right)^{\frac{1}{q'}} \\ &\lesssim \left(\int_{\mathbb{R}^{2m}} \frac{|u_k-u|^2}{|x|^\beta}\,dx\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^{2m}} \frac{|u_k-u|^{q'}}{|x|^\beta}\,dx\right)^{\frac{1}{q'}},\end{aligned} \tag{5-24}$$

where $q > 1$ close enough to 1 and $\frac{1}{p} + \frac{1}{p'} = 1$.

Thanks to Theorem 1.7 again, we arrive at

$$\int_{\mathbb{R}^{2m}} \frac{f(x,u_k)u_k}{|x|^\beta}\,dx \to 0 \quad \text{as } k \to \infty.$$

Taking $I'_\beta(u_k)u_k \to 0$ into consideration, we get $\lim_{k\to\infty} \|u_k\|_E \to 0$, which is a contradiction with $c_\beta > 0$.

Case 2. ($c_\beta \neq 0, u \neq 0$). We claim that $\lim_{k\to\infty} \|u_k\|_E = \|u\|_E$. We argue this by contradiction. Suppose $\lim_{k\to\infty} \|u_k\|_E > \|u\|_E$, and define

$$v_k := \frac{u_k}{\|u_k\|_E} \quad \text{and} \quad v_0 := \frac{u}{\lim_{k\to\infty} \|u_k\|_E}.$$

We claim that for $q > 1$ sufficiently close to 1, there exists a constant $\beta_0 > 0$ such that the following inequality holds.

$$q\alpha_0 \|u_k\|_E^2 \le \beta_0 < \frac{\beta_{2m,m}\left(1-\frac{\beta}{2m}\right)}{1-\|v_0\|_E^2}. \tag{5-25}$$

Indeed,

$$\begin{aligned}\lim_{k\to\infty} \|u_k\|_E^2 (1-\|v_0\|_E^2) &= \lim_{k\to\infty} \|u_k\|_E^2 \left(1 - \frac{\|u\|_E^2}{\lim_{k\to\infty} \|u_k\|_E^2}\right) \\ &= 2c_\beta + 2\int_{\mathbb{R}^{2m}} \frac{F(x,u)}{|x|^\beta}\,dx - 2I_\beta(u) - 2\int_{\mathbb{R}^{2m}} \frac{F(x,u)}{|x|^\beta}\,dx \\ &< \frac{\beta_{2m,m}\left(1-\frac{\beta}{2m}\right)}{\alpha_0},\end{aligned} \tag{5-26}$$

where we apply $I_\beta(u) \ge 0$. Then it follows from the above estimate and Theorem 1.4 that

$$\int_{\mathbb{R}^{2m}} \frac{(\Phi_{2m,m}(\alpha_0|u_k|^2))^q}{|x|^\beta}\,dx \le C \int_{\mathbb{R}^{2m}} \frac{\Phi_{2m,m}\left(\beta_0\left|\frac{u_k}{\|u_k\|_E}\right|^2\right)}{|x|^\beta}\,dx \lesssim 1. \tag{5-27}$$

Under hypothesis (H_1), the Hölder inequality gives that

$$\begin{aligned}&\left|\int_{\mathbb{R}^{2m}} \frac{f(x,u_k)(u_k-u)}{|x|^\beta}\,dx\right| \\ &\le b_1 \left(\int_{\mathbb{R}^{2m}} \frac{|u_k|^2}{|x|^\beta}\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2m}} \frac{|u_k-u|^2}{|x|^\beta}\,dx\right)^{\frac{1}{2}} \\ &\qquad + b_2 \left(\int_{\mathbb{R}^{2m}} \frac{|u_k-u|^{q'}}{|x|^\beta}\,dx\right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^{2m}} \frac{(\Phi_{2m,m}(\alpha_0|u_k|^2))^q}{|x|^\beta}\,dx\right)^{\frac{1}{q}}.\end{aligned} \tag{5-28}$$

Thanks to Theorem 1.7, we derive the following conclusion with inequalities (5-27) and (5-28):

$$\int_{\mathbb{R}^{2m}} \frac{f(x,u_k)(u_k-u)}{|x|^\beta}\,dx \to 0.$$

Together with $I'_\beta(u_k)(u_k-u)\to 0$, we get

$$\int_{\mathbb{R}^{2m}} \nabla^m u_k(\nabla^m u_k - \nabla^m u)\,dx + \int_{\mathbb{R}^{2m}} V(x)u_k(u_k-u)\,dx \to 0.$$

Since $u_k \rightharpoonup u$ in E, we have

$$\int_{\mathbb{R}^{2m}} \nabla^m u(\nabla^m u_k - \nabla^m u)\,dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^{2m}} V(x)u(u_k-u)\,dx \to 0.$$

Therefore

$$\begin{aligned}\lim_{k\to+\infty}\|u_k-u\|_E^2 &= \lim_{k\to+\infty}\int_{\mathbb{R}^{2m}}(\nabla^m u_k-\nabla^m u)(\nabla^m u_k-\nabla^m u)\,dx \\ &\quad + \lim_{k\to+\infty}\int_{\mathbb{R}^{2m}} V(x)(u_k-u)(u_k-u)\,dx \\ &= 0,\end{aligned} \tag{5-29}$$

which arrives at a contradiction with $\lim_{k\to\infty}\|u_k\|_E > \|u\|_E$.

Part 3. In this part, we show that the critical point of functional I_β is actually a ground state solution for the singular polyharmonic equation (1-14). Define

$$m = \inf_{u\in s} I_\beta(u) \quad \text{and} \quad S := \{u\in E : u\not\equiv 0 \text{ and } I'_\beta(u)=0\}.$$

For all $w\in S$, pick t_0 sufficiently large such that $I_\beta(t_0w)<0$. Denote $h:(0,+\infty)\to\mathbb{R}$ by $h(t)=I_\beta(tw)$ and $g:[0,1]\to E$ by $g(t)=tt_0w$. It is easy to check that

$$h'(t) = I'_\beta(tw)w = t\|w\|_E^2 - \int_{\mathbb{R}^{2m}} \frac{f(x,tw)w}{|x|^\beta}\,dx, \quad \text{for all } t>0.$$

Combine this with $I'_\beta(w)w=0$, we easily see that

$$h'(t) = t\int_{\mathbb{R}^{2m}}\left(\frac{f(x,w)}{w}-\frac{f(x,tw)}{tw}\right)\frac{w^2}{|x|^\beta}\,dx,$$

which implies that $h'(t)>0$ for $t\in(0,1)$ and $h'(t)<0$ for $t>1$ under hypothesis (H_6). Thus,

$$c_\beta \le \max_{t\in[0,1]} I_\beta(g(t)) \le \max_{t\ge 0} I_\beta(tw) = I_\beta(w),$$

which concludes the proof of Theorem 1.9.

6. The proof of Theorem 1.10

In this section, we will investigate the existence of the nontrivial weak solutions for nonsingular polyharmonic equation (1-15). The presence of the constant potential $V(x)$ makes it hard to follow the same line of reasoning as for Theorem 1.9. In order to overcome this difficulty, we need to use the principle of symmetric criticality. We first introduce some background knowledge about the principle of symmetric criticality.

Definition 6.1. The action of a topological group G on a normed space X is a continuous map

$$G \times X \to X : \quad [g, u] \mapsto gu$$

such that

$$1 \cdot u = u, \qquad (gh)u = g(hu), \qquad u \longmapsto gu \quad \text{is linear.}$$

The action is isometric if

$$\|gu\| = \|u\|$$

The space of invariant points is defined by

$$Fix(G) := \{u \in X : gu = u, \forall g \in G\}.$$

A function $\varphi : X \to \mathbb{R}$ is invariant if $\varphi \circ g = \varphi$ for every $g \in G$.

Lemma 6.2 (principle of symmetric criticality [Badiale and Serra 2011]). *Assume that the action of the topological group G on the Hilbert space X is isometric. If $\varphi \in C^1(X, \mathbb{R})$ is invariant and if u is a critical point of φ restricted to* $\mathrm{Fix}(G)$, *then u is also a critical point of φ.*

Lemma 6.3. *For $q \geq 2$, $W_r^{m,2}(\mathbb{R}^{2m})$ can be compactly embedded into $L^q(\mathbb{R}^{2m})$ for any $q > 2$.*

Remark 6.4. Through applying the radial lemma, one can easily get Lemma 6.3 with a slight modification of the proof of Theorem 1.7.

Now, we are in a position to prove Theorem 1.10. The functional related with (1-15) is given by $I(u) = \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^{2m}} F(x, u)\, dx$. Based on Lemma 6.2, we can restrict the functional I to the subspace E_r of E, where E_r is the set of all radial functions in E. It follows from same reasoning as for Lemma 5.3 that functional I satisfies the geometric conditions which imply that there exists a sequence $\{u_k\}_k \in E_r$ such that $I(u_k) \to c_0$, $I'(u_k) \to 0$ as $k \to +\infty$. Furthermore, we also can obtain

$$\begin{aligned} u_k &\rightharpoonup u_0, && \text{in } E_r, \\ u_k &\to u_0, && \text{in } L^q(\mathbb{R}^{2m}) \text{ for all } q > 2, \\ u_k(x) &\to u_0(x), && \text{almost everywhere in } \mathbb{R}^{2m}. \end{aligned}$$

We will use a new method based on Lemma 6.3 to prove that

$$\int_{\mathbb{R}^{2m}} F(x, u_k)\, dx \to \int_{\mathbb{R}^{2m}} F(x, u)\, dx.$$

By splitting the integral into three parts, we have

$$\begin{aligned} \text{(6-1)} \quad & \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \int_{\mathbb{R}^{2m}} |F(x, u_k) - F(x, u)|\, dx \\ &= \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \int_{B_R} |F(x, u_k) - F(x, u)|\, dx \\ &\qquad + \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \int_{|x|>R,\, |u_k|>A} |F(x, u_k) - F(x, u)|\, dx \\ &\qquad\qquad + \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \int_{|x|>R,\, |u_k|\le A} |F(x, u_k) - F(x, u)|\, dx \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For I_1, it directly follows from (5-13), (5-14) for the case $\beta = 0$. For I_2, in view of hypotheses (H_2) and (H_3), we have

$$\begin{aligned} \text{(6-2)} \qquad I_2 &= \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \int_{|x|>R,\, |u_k|>A} |F(x, u_k) - F(x, u)|\, dx \\ &\lesssim \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \int_{|x|>R,\, |u_k|>A} |F(x, u_k)|\, dx \\ &\lesssim \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \frac{1}{A} \int_{|x|>R,\, |u_k|>A} |f(x, u_k) u_k|\, dx \\ &= 0. \end{aligned}$$

For I_3, combining the hypothesis $f(x,t) = o(t)$ and Lemma 6.3, one can obtain that for any $\varepsilon > 0$,

$$\begin{aligned} \text{(6-3)} \qquad I_3 &= \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \int_{|x|>R,\, |u_k|\le A} |F(x, u_k) - F(x, u)|\, dx \\ &\lesssim \lim_{R\to\infty} \lim_{k\to\infty} \lim_{A\to\infty} \int_{|x|>R,\, |u_k|\le A} |F(x, u_k)|\, dx \\ &\lesssim \varepsilon \|u_k\|_E^2 + \lim_{R\to\infty} \lim_{k\to\infty} \int_{|x|>R} |u_k|^3\, dx \\ &\lesssim \varepsilon \|u_k\|_E^2, \end{aligned}$$

which leads to $I_3 = 0$. Carrying out similar steps as we did in Section 4 (Part 1), one can easily see that u is a weak solution of (1-15).

Next, we show u_k satisfies the Palais–Smale compactness condition and u is a critical point of functional I restricted in E_r. The process of proof follows from the similar argument of Section 4 (Part 2) as long as we can verify that

$$\left|\int_{\mathbb{R}^{2m}} f(x,u_k)(u_k-u)\,dx\right| \to 0.$$

Since $f(x,t)=o(t)$ at the origin, through hypothesis (H_1) and the Hölder inequality, we derive that for any $\varepsilon > 0$, it holds that

$$\begin{aligned}\left|\int_{\mathbb{R}^{2m}} f(x,u_k)(u_k-u)\,dx\right| &\le \varepsilon\left(\int_{\mathbb{R}^{2m}} |u_k|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^{2m}} |u_k-u|^2\,dx\right)^{\frac{1}{2}} \\ &\quad + C_\varepsilon\left(\int_{\mathbb{R}^{2m}} |u_k-u|^{q'}\,dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^{2m}} (\Phi_{2m,m}(\alpha_0|u_k|^2))^q\,dx\right)^{\frac{1}{q}}.\end{aligned} \tag{6-4}$$

Letting $k \to \infty$ and $\varepsilon \to 0$, we arrives at the desired conclusion. Finally applying the principle of symmetric criticality again, we see that u is also a critical point of I in E.

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