ON THE EKELAND–HOFER SYMPLECTIC CAPACITIES
OF THE REAL BIDISC

LUCA BARACCO, MARTINO FASSINA AND STEFANO PINTON

ON THE EKELAND–HOFER SYMPLECTIC CAPACITIES OF THE REAL BIDISC

LUCA BARACCO, MARTINO FASSINA AND STEFANO PINTON

In $\mathbb{C}^2$ with the standard symplectic structure we consider the bidisc $D^2 \times D^2$ constructed as the product of two open real discs of radius 1. We compute explicit values for the first, second and third Ekeland–Hofer symplectic capacity of $D^2 \times D^2$. We discuss some applications to questions of symplectic rigidity.

1. Introduction and main result

The first striking result about a nontrivial obstruction to the existence of a symplectic embedding was obtained by Gromov [1985]. He proved that one can symplectically embed a sphere into a cylinder only if the radius of the sphere is less than or equal to the radius of the cylinder. Since this celebrated nonsqueezing theorem appeared, many similar results of symplectic rigidity have been obtained for a variety of domains. For instance, McDuff [2009; 2011] studied symplectic embeddings of even-dimensional open ellipsoids into one another (see also [Hutchings 2011a; 2011b; McDuff and Schlenk 2012]), and Guth [2008] gave asymptotic results on when a complex polydisc can be symplectically embedded into another one. A useful tool to tackle these questions is given by global symplectic invariants for symplectic manifolds called capacities.

A symplectic capacity is a functor c that assigns to every symplectic manifold (M, ω) of dimension $2n$ a nonnegative (possibly infinite) number $c(M, \omega)$ that satisfies the following conditions:

- (monotonicity) If there exists a symplectic embedding of (M_1, ω_1) into (M_2, ω_2), then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$.

- (conformality) If $\lambda > 0$, then $c(M, \lambda\omega) = \lambda c(M, \omega)$.

- (local nontriviality) For the open unit ball $B \subset \mathbb{R}^{2n}$ we have $c(B, \omega_0) > 0$.

- (nontriviality) For the open cylinder $Z = \{z \in \mathbb{C}^n : |z_1| < 1\}$ we have $c(Z, \omega_0) < \infty$.

Here ω_0 denotes the standard symplectic structure on $\mathbb{R}^{2n}$. The reader can consult any standard textbook in the subject such as [McDuff and Salamon 2017, Chapter 12] for an extensive treatment of this topic.

The first nontrivial capacity arose in Gromov's proof of the nonsqueezing theorem, and over the years many more such symplectic invariants have been constructed [Ekeland and Hofer 1989; 1990; Hofer and Zehnder 1990; Floer et al. 1990; Viterbo 1992; Gutt and Hutchings 2018].

In this paper we consider the construction of Ekeland and Hofer (which is recalled in Section 2C). For any subset of a symplectic vector space, they define an infinite sequence c_n of symplectic capacities. These quantities are notoriously difficult to compute explicitly, and precise values appear in the literature only for very special classes of domains, such as ellipsoids and polydiscs [Ekeland and Hofer 1990]. The main purpose of this work is to compute some of these capacities for the real bidisc $D^2 \times D^2$, which we now introduce.

In the complex space $\mathbb{C}^2$ with coordinates $z_j = x_j + iy_j$, $j = 1, 2$, endowed with the standard symplectic structure $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, consider the real bidisc

$$(1\text{-}1) \qquad D^2 \times D^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid x_1^2 + x_2^2 < 1,\ y_1^2 + y_2^2 < 1\}.$$

Our main result is the following (see Theorems 4.2, 4.4 and 4.6).

Main Theorem. *For the unit real bidisc $D^2 \times D^2$ we have*

$$c_1(D^2 \times D^2) = 4, \quad c_2(D^2 \times D^2) = 3\sqrt{3}, \quad c_3(D^2 \times D^2) = 8.$$

The fact that $c_1(D^2 \times D^2) = 4$ is known [Artstein-Avidan and Ostrover 2014]. The referee pointed out to us that the values of $c_2(D^2 \times D^2)$ and $c_3(D^2 \times D^2)$ can also be obtained from the work of Ramos [2017], which uses very different techniques than the ones developed in this paper (see Remarks 4.5 and 4.9).

The Ekeland–Hofer capacities are closely related to the existence of closed Hamiltonian orbits. The key feature of our computation is the use of Hamiltonians modeled on the gauge function of the domain, rather than Hamiltonians that are quadratic at infinity, as in the original definition of Ekeland and Hofer. This same idea already appears in [Berestycki et al. 1985] in the context of smooth domains, and in [Fan 1992] for the case of Lipschitz domains. The different choice of Hamiltonians does not affect the computation of the capacities, provided that no periodic orbits of period 1 are introduced at infinity. This feature is easily achieved by rescaling, since the closed periodic orbits of the Hamiltonians in question can be computed explicitly. The advantage in modeling the Hamiltonians on the gauge function is that we can then exploit the symmetries of the domain to simplify the estimates involved in the computations of the capacities.

We remark that our strategy can be easily adapted to compute the Ekeland–Hofer capacities of other domains. For instance, for the product of two real spheres in $\mathbb{C}^3$

one just needs to repeat the same computations while taking an additional coordinate into account.

We now point out how the results of this paper are related to recent work of Gutt and Hutchings [2018]. They use positive S^1-equivariant symplectic homology to introduce a sequence c_k of symplectic capacities for star-shaped domains in $\mathbb{R}^{2n}$. Their capacities c_k share many properties with the Ekeland–Hofer capacities, and in particular the remarkable "product property" (see Section 5). Combining the results of [Gutt and Hutchings 2018] with Ramos' insight [2017] that the real bidisc is a toric domain, one can see that the Gutt–Hutchings capacities of $D^2 \times D^2$ are the same as the Ekeland–Hofer capacities that we obtain in our Main Theorem. Our computations therefore support the conjecture made by Gutt and Hutchings [2018] that their capacities c_k are always equal to the Ekeland–Hofer capacities.

The organization of this paper is as follows. In Section 2 we recall some background material and set the notation. In Section 3 we show how one can approximate the bidisc $D^2 \times D^2$ with a sequence of smooth convex domains. The main result is then proved in Section 4, while in Section 5 we present some applications to questions of symplectic embedding. It is the authors' hope that further applications to symplectic rigidity of these explicit values of the capacities will be found in the future.

2. Background

2A. *Basic definitions and notation.* Let $\mathbb{C}^n$ denote the standard complex vector space of dimension n with variables $z_j = x_j + iy_j$. We endow $\mathbb{C}^n$ with the Euclidean scalar product

$$\langle z, w \rangle := \mathrm{Re}\left(\sum_{j=1}^{n} z_j \overline{w}_j \right)$$

and the standard symplectic form

$$\omega := \frac{1}{2i} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

All symplectic embeddings considered in this paper will be with respect to the standard symplectic form.

Let Ω be a bounded subset of $\mathbb{C}^n$ with smooth boundary $\partial\Omega$. We denote by $T\mathbb{C}^n$ the (real) tangent bundle of $\mathbb{C}^n$ and by $T\partial\Omega$ the tangent bundle of $\partial\Omega$. We write $Y \in T\mathbb{C}^n$ to mean that Y is a local section of $T\mathbb{C}^n$. Consider a smooth defining equation ρ of $\partial\Omega$ such that $|d\rho(z)| \neq 0$ for $z \in \partial\Omega$. The *characteristic vector field* of $\partial\Omega$ is the unique vector field X such that

$$\omega(X, Y) = Y(\rho) \quad \text{for all } Y \in T\mathbb{C}^n.$$

The characteristic vector field X is tangent to $\partial\Omega$ and its restriction to $\partial\Omega$ does

not depend on the choice of ρ. Moreover, X generates the kernel of the restricted form $\omega|_{T\partial\Omega}$.

Let J denote the standard complex structure on $\mathbb{C}^n$. For all $z \in \partial\Omega$ we have $X(z) = J\nabla\rho(z)$, where $\nabla\rho$ is the Euclidean gradient of ρ. Hence an integral curve of X on $\partial\Omega$ is the solution of a system

$$(2\text{-}1) \qquad \begin{cases} \dot{z} = J\nabla\rho(z), \\ z(0) = z_0, \end{cases}$$

where $z_0 \in \partial\Omega$. These integral curves are called the *characteristics* of $\partial\Omega$.

Of particular interest in symplectic geometry is the study of closed characteristics, that is, the solutions to (2-1) for which there exists a time $t_0 > 0$ such that $z(t_0) = z_0$. Let T be the smallest such t_0. The image $\{z(t),\, 0 \le t \le T\}$ is called an *orbit* and T the *period* of the orbit.

If $\gamma : [0, T] \to \partial\Omega$ is a closed characteristic, the *action* of γ is defined to be

$$\mathcal{A}(\gamma) := -\frac{1}{2}\int_0^T \langle J\dot{\gamma}, \gamma\rangle\, dt.$$

If Ω is a bounded convex subset of $\mathbb{C}^n$ and $\{\gamma_i\}_{i\in I}$ is the set of closed characteristics of $\partial\Omega$, we can define the *action spectrum* of Ω as the set

$$\Sigma(\Omega) := \{|k\mathcal{A}(\gamma_i)|, k \in \mathbb{N}, i \in I\}.$$

Following Ekeland and Hofer [1990], we will see how it is possible to choose some elements of $\Sigma(\Omega)$ called capacities that are symplectic invariants (see Section 2C). We now describe how to adapt the concepts introduced above to nonsmooth domains.

Let $\Omega \subset \mathbb{C}^n$ be a convex bounded domain and let $p \in \partial\Omega$. We say that a unit vector $n(p)$ is *normal* at p for $\partial\Omega$ if

$$(2\text{-}2) \qquad \langle n(p), x - p\rangle \le 0 \quad \text{for all } x \in \Omega.$$

If p is a smooth boundary point, then $n(p)$ is the usual exterior normal vector. If $\partial\Omega$ is not smooth at p, then there could be more than one choice for a normal vector. In this case, we let $n(p)$ denote the set of all vectors satisfying (2-2).

Definition 2.1. Let Ω be a convex domain in $\mathbb{C}^n$. We say that $z : \mathbb{R} \to \Omega$ is a *characteristic* if $z(t)$ has right and left derivative $\dot{z}^{\pm}(t)$ for all t, and

$$\dot{z}^{\pm}(t) \in Jn(z(t)).$$

Note that, for Ω smooth, this definition coincides with the definition of a characteristic given before.

2B. *The action spectrum of the real bidisc.* We now turn our attention to the domain that is the main object of interest of this paper: the real bidisc $D^2 \times D^2$. Recall that $D^2 \times D^2$ was defined in (1-1) as the product of two open real discs of radius 1. The next proposition, which follows from the work of Artstein-Avidan and Ostrover [2014], describes the closed characteristics of $D^2 \times D^2$ and its action spectrum $\Sigma(D^2 \times D^2)$.

Proposition 2.2. *The unitary real bidisc $D^2 \times D^2$ has infinitely many closed characteristics. The action spectrum is given by the set*

$$(2\text{-}3) \qquad \Sigma(D^2 \times D^2) = \{2n \cos(\theta_{k,n}) \mid k, n \in \mathbb{N}, \theta_{k,n} \in J_n\} \cup \{2n\pi \mid n \in \mathbb{N}\},$$

where $J_n = \{(2k-1)\pi/2n, 1 \le k \le (n-1)/2\}$ if n is odd, and, if n is even, $J_n = \{k\pi/n, 0 \le k \le n/2 - 1\}$.

Remark 2.3. The elements of $\Sigma(D^2 \times D^2)$ are precisely the lengths of all closed billiard orbits in a circle of radius 1.

Remark 2.4. Note that $\min \Sigma(D^2 \times D^2) = 4$. Moreover, the second smallest element in $\Sigma(D^2 \times D^2)$ is $3\sqrt{3}$.

2C. *Ekeland–Hofer symplectic capacities.* Following [Ekeland and Hofer 1990], we recall the definition of these symplectic capacities. We first set up the functional analytical framework. For more details, see also [Ekeland and Hofer 1989, Section II].

Let E be the Hilbert space of all functions $f \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{C}^n)$ such that the Fourier series

$$f(t) = \sum_{k \in \mathbb{Z}} f_k e^{2k\pi it}, \quad f_k \in \mathbb{C}^n,$$

satisfies

$$\sum_{k \in \mathbb{Z}} |k| |f_k|^2 < \infty.$$

The inner product in E is defined by

$$(f, g) := \langle f_0, g_0 \rangle + 2\pi \sum_{k \in \mathbb{Z}} |k| \langle f_k, g_k \rangle.$$

E is the most natural space on which the action functional $\mathcal{A}$ can be defined. It is easy to see that

$$\mathcal{A}(f) = \pi \sum_{k=0}^{+\infty} k(|f_k|^2 - |f_{-k}|^2).$$

Note that there is a natural action $T : \mathbb{S}^1 \to \mathrm{Aut}(E)$ of $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ on E given by the phase shift

$$T_{e^{2\pi i\theta}} f(t) := f(t + \theta).$$

The space E has a natural orthogonal splitting, compatible with the phase shift action, given by

$$E = E^- \oplus E^0 \oplus E^+.$$

Here we have defined

$$E^- = \{f \in E \mid f_k = 0 \text{ for } k \geq 0\},$$
$$E^0 = \{f \in E \mid f_k = 0 \text{ for } k \neq 0\} = \mathbb{C}^n,$$
$$E^+ = \{f \in E \mid f_k = 0 \text{ for } k \leq 0\}.$$

We denote by P^+, P^0 and P^- the corresponding orthogonal projections.

We now need to introduce the notion of the index of a subspace. Let X be a Hilbert space over $\mathbb{C}$, and let $T : S^1 \to \mathrm{Aut}(X)$ be a representation of S^1 on the vector space

$$\mathrm{Aut}(X) := \{f : X \to X \mid f \text{ is a linear isometry}\}.$$

A subset $A \subseteq X$ is called *invariant* if $T(\theta)(A) = A$ for all $\theta \in S^1$. Let Y be another Hilbert space, and $R : S^1 \to \mathrm{Aut}(Y)$ a representation of S^1 on $\mathrm{Aut}(Y)$. A linear map $f : X \to Y$ is called *equivariant* if $f \circ T(\theta) = R(\theta) \circ f$ for all $\theta \in S^1$. Let $A \subseteq X$ be an invariant subset. For every $k \in \mathbb{N}$, we let $\mathcal{F}(A, k)$ denote the collection of functions $f : A \to \mathbb{C}^k \setminus \{0\}$ such that

- f is continuous,
- there exists a positive integer n such that $f(T(\theta)(x)) = e^{2\pi n i \theta} f(x)$ for all $\theta \in S^1$ and for all $x \in A$.

We define the *index* of A as the quantity

(2-4) $\alpha(A) := \min\{k \in \mathbb{N} \mid \mathcal{F}(A, k) \neq \varnothing\}.$

If $\mathcal{F}(A, k) = \varnothing$ for every $k \in \mathbb{N}$, we set $\alpha(A) = +\infty$. Moreover, we set $\alpha(\varnothing) = 0$. Observe that if F is the set of fixed points of X for T, that is,

$$F := \{x \in X \mid T(\theta)(x) = x \text{ for all } \theta \in S^1\},$$

then $A \cap F \neq \varnothing$ implies $\alpha(A) = \infty$.

In the following paragraph we describe a pseudoindex theory in the sense of Benci relative to E^+. In our exposition we mainly follow [Benci 1982] (see also [Ekeland and Hofer 1990]).

Consider the group of homeomorphisms

(2-5) $\Gamma := \{h : E \to E \mid h = e^{\gamma^+} P^+ + e^{\gamma^-} P^- + P^0 + K\}$

such that the following conditions are satisfied:

- K is a compact equivariant operator.

- $\gamma^+, \gamma^- : E \to \mathbb{R}^+$ map bounded sets in precompact sets and are invariant.
- There exists a constant $c > 0$ such that $\mathcal{A}(x) \le 0$ or $\|x\| > c$ implies that $\gamma^+(x) = \gamma^-(x) = 0$ and $K(x) = 0$.

Let $S := \{x \in E \mid \|x\|_E = 1\}$ and let $\xi \subseteq E$ be an invariant subset of E. We define the *pseudoindex* of ξ as

$$\mathrm{ind}(\xi) := \inf\{\alpha(h(\xi) \cap S \cap E^+) \mid h \in \Gamma\}.$$

For the basic properties of the pseudoindex we refer to [Ekeland and Hofer 1990]. In particular, the following result is often useful.

Proposition 2.5 [Ekeland and Hofer 1990, Proposition 1]. *If $V_k \subseteq E^+$ is a finite-dimensional invariant subspace of E^+ of complex dimension k, $\mathrm{ind}(V_k \oplus E^0 \oplus E^-) = k$.*

We need to introduce one last concept before defining the symplectic capacities. We call a smooth function $H : \mathbb{C}^n \to (0, +\infty)$ an *admissible Hamiltonian* for a bounded domain $\Omega \subset \mathbb{C}^n$ if

- H is 0 on some open neighborhood of $\overline{\Omega}$,
- $H(z) = c|z|^2$ for $|z|$ large enough, where $c > \pi$ and $c \notin \mathbb{Z}\pi$.

We denote by $\mathcal{H}(\Omega)$ the set of the admissible Hamiltonians for Ω. For j a positive integer and $H \in \mathcal{H}(\Omega)$, we define a number $c_{H,j} \in (0, +\infty) \cup \{\infty\}$ by

$$c_{H,j} := \inf\{\sup \mathcal{A}_H(\xi) \mid \xi \subset E \text{ is } S^1\text{-invariant and } \mathrm{ind}(\xi) \ge j\}.$$

Here $\mathcal{A}_H : E \to \mathbb{R}$ is the action functional associated to a Hamiltonian H defined by

$$\mathcal{A}_H(f) := \mathcal{A}(f) - \int_0^1 H(f(t))\,dt.$$

Every number $c_{H,j}$ is nonnegative and, if finite, is a critical value of $\mathcal{A}_H$ [Ekeland and Hofer 1990, page 559].

We can now define the j-th *Ekeland–Hofer symplectic capacity* of Ω as

$$c_j(\Omega) := \inf_{H \in \mathcal{H}(\Omega)} c_{H,j}.$$

Remark 2.6. In our computation of the Ekeland–Hofer symplectic capacities, we will not use Hamiltonians with quadratic behavior at infinity, as in the definition. We will use instead Hamiltonians of the form $H(z) = f(r(z))$, where r is the gauge function of the domain, and f is linear at infinity (see [Berestycki et al. 1985; Fan 1992]). This choice does not affect the values obtained in the computation of the capacities, as one can easily prove by combining the following two facts:

- For Hamiltonians H_1 and H_2 we have

$$H_1 \le H_2 \Longrightarrow \mathcal{A}_{H_1} \ge \mathcal{A}_{H_2} \Longrightarrow c_{H_1,k} \le c_{H_2,k}.$$

- For every f linear at infinity there exists $H \in \mathcal{H}(\Omega)$ such that $f(r) \le H$.
  Similarly, for any $H \in \mathcal{H}(\Omega)$ there exists f such that $H \le f(r)$.

Observe also that if we choose f so that the Hamiltonian $f(r)$ has no periodic solutions of period 1 at infinity, then $\mathcal{A}_{f(r)}$ satisfies the Palais–Smale condition. This implies that $c_{f(r),k}$, if it is finite, is a critical value of $\mathcal{A}_{f(r)}$ (see [Aebischer et al. 1994, page 71]).

The next theorem, from [Hofer and Zehnder 1990], characterizes the first symplectic capacity as the infimum of the action spectrum.

Theorem 2.7. *Let Ω be a smoothly bounded convex domain in $\mathbb{C}^n$ and let $\alpha = \min \Sigma(\Omega)$. Then $c_1(\Omega) = \alpha$.*

Note that we cannot apply Theorem 2.7 directly to the real bidisc, since its boundary $\partial(D^2 \times D^2)$ is not smooth. In the next section we show how to overcome this difficulty by appropriately approximating $D^2 \times D^2$ with smooth domains.

3. Approximation with smooth domains

We start by constructing a decreasing sequence of smooth convex domains $\mathcal{D}_n$ converging to $D^2 \times D^2$. Let $g : \mathbb{R} \to [0, +\infty)$ be a convex, increasing function such that $g(1) = 1$ and $g(s) = 0$ for $s < 0$. Consider the following subsets of $\mathbb{C}^2$:

$$(3\text{-}1) \qquad \mathcal{D}_n := \{z \in \mathbb{C}^2 \mid g(n(x_1^2 + x_2^2 - 1)) + g(n(y_1^2 + y_2^2 - 1)) \le 1\}.$$

The domains defined in (3-1) are smooth and convex. Moreover, they satisfy the following properties:

- $\mathcal{D}_n \supset \mathcal{D}_{n+1}$ for all $n \in \mathbb{N}$.
- $\bigcap_{n=1}^{+\infty} \mathcal{D}_n = D^2 \times D^2$.
- For all $n \in \mathbb{N}$, we have

$$\frac{1}{\sqrt{1 + \frac{1}{n}}} \mathcal{D}_n \subset D^2 \times D^2.$$

In particular, by the properties of the capacities, for any choice of positive integers k and n, the following double inequality holds:

$$\frac{1}{1 + \frac{1}{n}} c_k(\mathcal{D}_n) \le c_k(D^2 \times D^2) \le c_k(\mathcal{D}_n).$$

Proposition 3.2 shows how the closed characteristics of the real bidisc $D^2 \times D^2$ are approximated by the closed characteristics of the approximating domains $\mathcal{D}_n$. Before stating the result, it is convenient to give the following definition.

Definition 3.1. For all $M > 0$ and $\varepsilon > 0$ we define

$$\Sigma_{M,\varepsilon}(\mathcal{D}_n) := (\Sigma(\mathcal{D}_n) \cap [0, M]) \setminus \bigcup_{k=1}^{\infty} [2k\pi - \varepsilon, 2k\pi + \varepsilon].$$

Here $\Sigma(\mathcal{D}_n)$ denotes the action spectrum of $\mathcal{D}_n$.

Proposition 3.2. *Let $\alpha \in \Sigma(D^2 \times D^2)$ and suppose that $\alpha = \mathcal{A}(\gamma)$, where γ is a nongliding closed characteristic of $D^2 \times D^2$. Then there exists a sequence γ_n, where each γ_n is a closed characteristic of $\mathcal{D}_n$, such that γ_n converges to γ and $\mathcal{A}(\gamma_n)$ converges to α. In particular, for all $M > 0$ and $\varepsilon > 0$, we have*

$$d(\Sigma_{M,\varepsilon}(\mathcal{D}_n), \Sigma_{M,\varepsilon}(D^2 \times D^2)) \to 0 \quad \text{for } n \to +\infty,$$

where $d(\cdot, \cdot)$ is the Hausdorff distance between sets.

Proof. Note that, for every n, we can decompose the boundary $\partial \mathcal{D}_n$ of $\mathcal{D}_n$ into three components

(3-2)
$$X_n^3 := \left\{ (z_1, z_2) \in \mathbb{C}^2, \ x_1^2 + x_2^2 - 1 < 0, \ y_1^2 + y_2^2 = 1 + \frac{1}{n} \right\}$$
$$Y_n^3 := \left\{ (z_1, z_2) \in \mathbb{C}^2, \ y_1^2 + y_2^2 - 1 < 0, \ x_1^2 + x_2^2 = 1 + \frac{1}{n} \right\}$$

and

(3-3)
$$T_n^3 := \partial \mathcal{D}_n \setminus (X_n^3 \cup Y_n^3).$$

To find the closed characteristics of $\mathcal{D}_n$, we consider the system of differential equations

(3-4)
$$\begin{cases} \dot{x} = -g'(n(|y|^2 - 1))2ny, \\ \dot{y} = g'(n(|x|^2 - 1))2nx, \end{cases}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$. We first note that $\det(x, y) = x_1 y_2 - x_2 y_1$ is constant along the solutions of (3-4). It is convenient to use polar coordinates in the planes defined by the variables x and y, respectively. We recall the notation already introduced in the proof of Proposition 2.2:

$$r_1 e^{i\varphi_1} := r_1(\cos \varphi_1, \sin \varphi_1) = (x_1, x_2),$$
$$r_2 e^{i\varphi_2} := r_2(\cos \varphi_2, \sin \varphi_2) = (y_1, y_2).$$

We can now rewrite the system (3-4) as

(3-5)
$$\begin{cases} (\dot{r}_1 + i r_1 \dot{\varphi}_1) e^{i\varphi_1} = -g'(n(r_2^2 - 1))2n r_2 e^{i\varphi_2}, \\ (\dot{r}_2 + i r_2 \dot{\varphi}_2) e^{i\varphi_2} = g'(n(r_1^2 - 1))2n r_1 e^{i\varphi_1}, \end{cases}$$

from which we obtain the two systems

(3-6)
$$
\begin{cases}
r_1 \dot{\varphi}_1 = g'(n(r_2^2 - 1))2nr_2 \sin(\varphi_1 - \varphi_2), \\
r_2 \dot{\varphi}_2 = g'(n(r_1^2 - 1))2nr_2 \sin(\varphi_1 - \varphi_2), \\
r_1 r_2 \sin(\varphi_1 - \varphi_2) = \det(x, y) = \text{const}
\end{cases}
$$

and

(3-7)
$$
\begin{cases}
r_1 \dot{r}_1 = -g'(n(r_2^2 - 1))2nr_3, \\
r_2 \dot{r}_2 = g'(n(r_1^2 - 1)2nr_3, \\
\dot{r}_3 = 2n(g'(n(r_1^2 - 1))r_1^2 - g'(n(r_2^2 - 1))r_2^2).
\end{cases}
$$

Here $r_3 := x \cdot y = x_1 y_1 + x_2 y_2 = r_1 r_2 \cos(\varphi_1 - \varphi_2)$.

Clearly the characteristics in $\mathcal{D}_n$ are straight segments when they lie on X_n^3 or Y_n^3. We now want to understand the behavior of the characteristics at their intersections with T_n^3. Let us consider the Cauchy problem (3-4) with initial data $x(0) = (1, 0)$ and $y(0) = \sqrt{1 + 1/n}(\cos\theta_0, \sin\theta_0)$ for $\theta_0 \in (0, \pi/2)$. The corresponding solution of (3-4) enters X_n^3 (this can be seen by inspection of (3-7), since $r_3(0) > 0$ implies that r_1 is decreasing and r_2 is constant). Reasoning as in Proposition 2.2, we see that the solution reaches the point

$$
(x_1, x_2) = (\cos(\pi + 2\theta_0), \sin(\pi + 2\theta_0)), \quad (y_1, y_2) = \sqrt{1 + \tfrac{1}{n}}(\cos\theta_0, \sin\theta_0).
$$

The solution then enters T_n^3 at a time t_0. From (3-7) we see that $r_3(t_0) < 0$, r_1 increases and r_2 decreases. Let $T > 0$ be the smallest positive real number such that $\dot{r}_3(t_0 + T) = 0$. By symmetry, we see that $r_1(t) = r_2(2T + 2t_0 - t)$, $r_2(t) = r_1(2T + 2t_0 - t)$ and $r_3(t) = r_3(2T + 2t_0 - t)$. In particular, this tells us that the solution eventually leaves T_n^3. By the third equation in (3-6), the angle between $x(t_0 + 2T)$ and $y(t_0 + 2T)$ is the same as the angle between $x(t_0)$ and $y(t_0)$ but $\varphi_1(t_0) \neq \varphi_1(t_0 + 2T)$. We want to compute $\Delta\varphi := \varphi_1(t_0 + 2T) - \varphi_1(t_0) = \varphi_2(t_0 + 2T) - \varphi_2(t_0)$. By (3-6),

(3-8)
$$
\Delta\varphi = \int_0^{2T} \frac{g'(n(r_2^2 - 1))2nr_2}{r_1} 2\sin(\varphi_1 - \varphi_2)\, dt
$$
$$
= \int_0^{2T} \frac{g'(n(r_2^2 - 1))2n}{r_1^2} \sqrt{1 + \tfrac{1}{n}} \sin\theta_0\, dt.
$$

Since $r_3 = x \cdot y = r_1 r_2 \cos(\varphi_1 - \varphi_2)$, (3-7) implies

(3-9)
$$
(\dot{r}_1^2) = -g'(n(r_2^2 - 1))2nr_1 r_2 \cos(\varphi_1 - \varphi_2)
$$
$$
= -g'(n(r_2^2 - 1))2n\sqrt{r_1^2 r_2^2 - \left(1 + \tfrac{1}{n}\right)\sin^2\theta_0}.
$$

Using (3-9) inside (3-8) we obtain

$$(3\text{-}10) \qquad \Delta\varphi = -\int_1^{1+\frac{1}{n}} \frac{\sqrt{1+\frac{1}{n}}\sin(\theta_0)}{r_1^2\sqrt{r_1^2 r_2^2 - \left(1+\frac{1}{n}\right)\sin^2(\theta_0)}} \, d(r_1^2).$$

It follows from

$$g\left(n(r_1^2 - 1)\right) + g\left(n(r_2^2 - 1)\right) = 1$$

that

$$(3\text{-}11) \qquad r_2^2 = 1 + \frac{g^{-1}(1 - g(n(r_1^2 - 1)))}{n}.$$

Plugging (3-11) into (3-10) we obtain

$$(3\text{-}12) \quad \Delta\varphi = -\int_1^{1+\frac{1}{n}} \frac{\sqrt{1+\frac{1}{n}}\sin\theta_0}{u\sqrt{u(1 + g^{-1}(1 - g(n(u-1)))/n) - \left(1+\frac{1}{n}\right)\sin^2\theta_0}} \, du.$$

From (3-12) we see that $\Delta\varphi$ is small for $\theta_0 < \pi/2$ and n large. Following the same reasoning as in Proposition 2.2 we can see that after $2m$ straight sides the characteristic hits the point

$$P_{2m} = \left((-1)^m e^{i(2m(\theta + \Delta\varphi))}, (-1)^m\sqrt{1+\frac{1}{n}}\, e^{i((2m+1)\theta + 2m\Delta\varphi)}\right).$$

The characteristic is closed if and only if

$$(3\text{-}13) \qquad \begin{cases} \theta + \Delta\varphi = \frac{k\pi}{m} \text{ for } k = 0, \ldots, \frac{m}{2} - 1 & \text{if } m \text{ is even,} \\ \theta + \Delta\varphi = \frac{2k-1}{2m}\pi \text{ for } k = 1, \ldots, \frac{m-1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

Since $\Delta\varphi$ depends continuously on θ_0 and is small if n is large, then the equations in (3-13) are solvable. In particular, if $\theta_{k,m} \in J_m$ (see Proposition 2.2), for n big enough there exists θ close to $\theta_{k,m}$ such that (3-13) is satisfied. The corresponding characteristics of $\mathcal{D}_n$ and $D^2 \times D^2$ are close to each other and their actions are also close. Note that there could also be some characteristics that are entirely contained in T_n^3. These characteristics are left out by the description above. If n is large, they are close to the gliding trajectories of the bidisc $D^2 \times D^2$ and their actions are close to a multiple of 2π. $\qquad\qquad\square$

4. The Ekeland–Hofer symplectic capacities of the real bidisc

In this section we compute the symplectic capacities of the real bidisc $D^2 \times D^2$. The following result will be the main tool for our computations.

Proposition 4.1. *Let Ω be a smooth convex domain in $\mathbb{C}^n$ containing 0, and let r be its gauge function. For all $c > 0$, let $\Psi_c : E \to \mathbb{R}$ be the functional*

$$(4\text{-}1) \qquad\qquad \Psi_c(\zeta) := \mathcal{A}(\zeta) - c \int_0^1 r(\zeta(t))\, dt.$$

If $W \subset E$ is an invariant subset of pseudoindex at least k such that $\Psi_c|_W \leq 0$, then $c_k(\Omega) \leq c$.

Proof. Let $\varepsilon > 0$ and choose a smooth function $f_\varepsilon : [0, \infty) \to [0, \infty)$ such that

$$(4\text{-}2) \qquad\qquad f_\varepsilon(s) = \begin{cases} 0 & \text{if } s \leq 1, \\ (c + \varepsilon)s & \text{if } s \text{ is large.} \end{cases}$$

Moreover, we require that $0 \leq f_\varepsilon'(s) \leq c + 2\varepsilon$ for all s, and that $f_\varepsilon'(s) \in \Sigma(\Omega)$ only for finitely many numbers $s_1, \ldots, s_m$, which we can assume to be arbitrarily close to 1. Let $f_\varepsilon'(s_j) = \alpha_j \in \Sigma(\Omega)$ and define $H_{f_\varepsilon}(z) := f_\varepsilon(r(z))$. The periodic orbits of $\dot{z} = J \nabla H_{f_\varepsilon}(z)$ of period 1 are obtained by scaling. Namely, they are the curves $\sqrt{s_j}\gamma_j(\alpha_j t)$, where γ_j is the closed characteristic in $\partial\Omega$ such that $\mathcal{A}(\gamma_j) = \alpha_j$. The corresponding critical values of $\mathcal{A}_{H_{f_\varepsilon}}$ are

$$(4\text{-}3) \qquad\qquad \mathcal{A}_{H_{f_\varepsilon}}(\sqrt{s_j}\gamma_j(\alpha_j t)) = s_j\alpha_j - f_\varepsilon(s_j).$$

As $\varepsilon \to 0$, these critical values tend to α_j, and each of them is less than c. Note that

$$\mathcal{A}_{H_{f_\varepsilon}}(\zeta) = \Psi_c(\zeta) + \int_0^1 [cr(\zeta(t)) - f_\varepsilon(r(\zeta(t)))]\, dt$$

$$\leq \Psi_c(\zeta) + \int_0^1 [C - \varepsilon r(\zeta(t))]\, dt \leq \Psi_c(\zeta) + D$$

for some constants C and D. Recalling that $\Psi|_W \leq 0$, then

$$(4\text{-}4) \qquad\qquad\qquad \mathcal{A}_{H_{f_\varepsilon}}|_W \leq C$$

for some new constant C. Equation (4-4) implies $c_{H_{f_\varepsilon},k} < \infty$. Since there are no periodic orbits at infinity of period 1, then $\mathcal{A}_{H_{f_\varepsilon}}$ satisfies the Palais–Smale condition and therefore $c_{H_{f_\varepsilon},k}$ is a critical value of $\mathcal{A}_{H_{f_\varepsilon}}$. Hence $c_{H_{f_\varepsilon},k} \leq c$ by (4-3). Since $c_k(\Omega) \leq c_{H_{f_\varepsilon},k}$, the conclusion follows. $\qquad\square$

In the next theorem we compute the first Ekeland–Hofer capacity of the real bidisc. A different proof of the same result appears in [Artstein-Avidan and Ostrover 2014].

Theorem 4.2. *For the unit real bidisc $D^2 \times D^2$ we have $c_1(D^2 \times D^2) = 4$.*

Proof. Recall that $4 = \min \Sigma(D^2 \times D^2)$ (Remark 2.4). Proposition 3.2 then implies the existence of a sequence of closed characteristics $\gamma_n \subset \partial\mathcal{D}_n$ with $\alpha_n := \mathcal{A}(\gamma_n) \to 4$.

By Theorem 2.7 we have

(4-5)
$$\lim_{n \to +\infty} c_1(\mathcal{D}_n) = 4.$$

Now observe, from the construction of the $\mathcal{D}_n$, that

$$\frac{1}{\sqrt{1 + \frac{1}{n}}} \mathcal{D}_n \subset D^2 \times D^2 \subset \mathcal{D}_n.$$

The basic properties of the capacities then yield the double inequality

$$\frac{1}{1 + \frac{1}{n}} c_1(\mathcal{D}_n) \leq c_1(D^2 \times D^2) \leq c_1(\mathcal{D}_n),$$

from which we can conclude that $c_1(D^2 \times D^2) = 4$ by applying (4-5). $\quad\square$

For the next theorem we need the following simple lemma.

Lemma 4.3. *Let* $f(t) = \sum_{k \in \mathbb{Z}} f_k e^{2k\pi it}$ *be an* L^1*-convergent series. Then, for every integer* n,

$$\int_0^1 |f(t)| \, dt \geq |f_n|.$$

Proof. For every n we have

$$\int_0^1 \left| \sum_{k \in \mathbb{Z}} f_k e^{2k\pi it} \right| dt = \int_0^1 \left| e^{2n\pi it} \sum_{k \in \mathbb{Z}} f_k e^{2(k-n)\pi it} \right| dt$$

$$\geq \left| \int_0^1 \sum_{k \in \mathbb{Z}} f_{k+n} e^{2k\pi it} \, dt \right| = |f_n|. \qquad \square$$

Theorem 4.4. *For the real bidisc* $D^2 \times D^2$ *we have* $c_2(D^2 \times D^2) = 3\sqrt{3}$.

Proof. Let W be the following invariant subset of E:

$$W := \left\{ (\alpha, \beta) e^{2\pi it} + \gamma([\alpha : \beta]) \left(\bar{\alpha} \frac{\alpha^3}{|\alpha|^3}, \bar{\beta} \frac{\alpha^3}{|\alpha|^3} \right) e^{4\pi it} \,\middle|\, \alpha, \beta \in \mathbb{C} \right\} \oplus E^0 \oplus E^-,$$

where $\gamma : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{R}$ is a nonnegative continuous function which is nonzero only in a neighborhood of the two points $[1 : i]$ and $[1 : -i]$. We will specify later how γ is chosen. We now prove that W has pseudoindex 2. First note that

$$W \cap S \cap E^+ =$$
$$\left\{ (\alpha, \beta) e^{2\pi it} + \gamma([\alpha:\beta]) \left(\bar{\alpha} \frac{\alpha^3}{|\alpha|^3}, \bar{\beta} \frac{\alpha^3}{|\alpha|^3} \right) e^{4\pi it} \,\middle|\, \alpha, \beta \in \mathbb{C}, \, (|\alpha|^2 + |\beta|^2)(1 + \gamma^2) = 1 \right\},$$

and therefore $W \cap S \cap E^+$ has index 2. Assume now by contradiction that there exists $h \in \Gamma$ such that $F := h(W) \cap E^+ \cap S$ has index strictly smaller than 2. Then,

by the properties of the index, there exists an open neighborhood of U of F in E such that $\alpha(U) = \alpha(F)$. Let $E_k = \{f \in E \mid f_j = 0 \text{ for } |j| > k\}$ and denote by $Q_k : E \to E_k$ the corresponding orthogonal projection. We claim, for k large, that

$$(4\text{-}6) \qquad\qquad Q_k(h(W \cap E_k)) \cap S \cap E^+ \subset U.$$

Assume by contradiction that (4-6) is false. Then there exists a sequence of functions $f_k \in E_k \cap W$ such that $Q_k(h(f_k)) \in E^+ \cap S$ and $Q_k(h(f_k)) \notin U$. Recalling the structure of the homeomorphism h (see (2-5)), we have

$$(4\text{-}7) \qquad \begin{cases} f_k^0 + e^{\gamma^-(f_k)} f_k^- + (P^- + P^0) Q_k K(f_k) = 0, \\ \|e^{\gamma^+(f_k)} f_k^+ + P^+ Q_k K(f_k)\| = 1. \end{cases}$$

Note that f_k must be a bounded sequence, otherwise we have $K(f_k) = 0$, which together with (4-7) implies $\|f_k^+\| = 1$ and $f_k^0 = f_k^- = 0$, thus giving a contradiction. We can therefore assume that $K(f_k)$ and $\gamma^\pm(f_k)$ converge. Hence, by (4-7), the sequences f_k^- and f_k^0 also converge. Furthermore, the sequence f_k^+ converges as well, since it lies in a finite-dimensional space. We therefore have that f_k converges to some element $f_\infty \in W$ with $h(f_\infty) \in E^+ \cap S$ and $h(f_\infty) \notin U$, which is a contradiction.

In order to apply [Fadell et al. 1982, Proposition 3.3] as done in [Ekeland and Hofer 1990, page 558], we consider the following "truncated" set: for $M > 0$ let

$$W_M := \left\{ (\alpha, \beta) e^{2\pi i t} + \chi \left(\frac{|\alpha|^2 + |\beta|^2 + \|f^0 + f^-\|^2}{M} \right) \gamma([\alpha : \beta]) \right.$$
$$\left. \times \left(\bar{\alpha} \frac{\alpha^3}{|\alpha|^3}, \bar{\beta} \frac{\alpha^3}{|\alpha|^3} \right) e^{4\pi i t} + f^0 + f^- \,\middle|\, \alpha, \beta \in \mathbb{C}, f^0 \in E^0, f^- \in E^- \right\},$$

where χ is a smooth function such that $\chi(t) = 1$ for $t < 1$ and $\chi(t) = 0$ for $t > \frac{3}{2}$. Note that W_M coincides with W inside the ball of radius M in E and that $h(W_M) \cap E^+ \cap S = h(W) \cap E^+ \cap S$. Consider now the equivariant map

$$(4\text{-}8) \quad \phi : \mathbb{C}^2 \oplus E^0 \oplus (E^- \cap E_k) \to W_M,$$
$$\phi(\alpha, \beta, f^0, f_k^-) = \chi \left(\frac{|\alpha|^2 + |\beta|^2 + \|f^0 + f^-\|^2}{M} \right) \gamma([\alpha : \beta]) \left(\bar{\alpha} \frac{\alpha^3}{|\alpha|^3}, \bar{\beta} \frac{\alpha^3}{|\alpha|^3} \right) e^{4\pi i t}$$
$$+ (\alpha, \beta) e^{2\pi i t} + f^0 + f_k^-.$$

By [Fadell et al. 1982, Proposition 3.3] applied to the map $Q_k h(\phi)$, we obtain

$$\alpha(Q_k(W_M) \cap E^+ \cap S) \geq 2.$$

The conclusion that $\operatorname{ind}(W) = 2$ is achieved by taking M large enough.

Now, let r be the gauge function of $D^2 \times D^2$ and $\Psi_c : E \to \mathbb{R}$ the functional defined in (4-1):

$$\Psi_c(\zeta) := \mathcal{A}(\zeta) - c \int_0^1 r(\zeta(t)) \, dt.$$

We recall that

$$r(z_1, z_2) = \frac{|z_1|^2 + |z_2|^2}{2} + \frac{|\mathrm{Re}(z_1^2 + z_2^2)|}{2}.$$

We will prove that $\Psi_c|_W < 0$ for $c = 4\sqrt{2}$.

Let $v = (\zeta_1, \zeta_2) \in W$. Then

$$v = v_1 e^{2\pi it} + v_2 e^{4\pi it} + \sum_{k=0}^{+\infty} w_{-k} e^{-2k\pi it},$$

where $v_1 = (\alpha, \beta)$ and $v_2 = \gamma \alpha^3 / |\alpha|^3 (\bar{\alpha}, \bar{\beta})$ for some $\alpha, \beta \in \mathbb{C}$. We have

$$(4\text{-}9) \quad \Psi_c(v) = \left(\pi - \frac{c}{2}\right)|v_1|^2 + \left(2\pi - \frac{c}{2}\right)|v_2|^2 - \sum_{k=1}^{+\infty} k\pi |w_{-k}|^2$$

$$- \frac{c}{2} \sum_{k=0}^{+\infty} |w_{-k}|^2 - \frac{c}{2} \int_0^1 \left| \mathrm{Re}(\zeta_1^2(t) + \zeta_2^2(t)) \right| dt.$$

To estimate the integral on the right side of (4-9) we compute the Fourier coefficients of order 4 and 6 of $\mathrm{Re}\big(\zeta_1^2(t) + \zeta_2^2(t)\big)$. We denote them respectively by I_4 and I_6.

$$(4\text{-}10) \quad 2I_4 = v_1^2 + 2(v_2 \cdot w_0) + \overline{(w_{-1})^2 + 2(w_{-2} \cdot w_0) + 2(w_{-3} \cdot v_1) + 2(w_{-4} \cdot v_2)},$$

$$(4\text{-}11) \quad 2I_6 = 2(v_2 \cdot v_1) + \overline{2(w_{-1} \cdot w_{-2}) + 2(w_0 \cdot w_{-3}) + 2(v_1 \cdot w_{-4}) + 2(v_2 \cdot w_{-5})}.$$

By Lemma 4.3 applied to I_6 and (4-11) we get

$$(4\text{-}12) \quad \int_0^1 \left| \mathrm{Re}\big(\zeta_1^2(t) + \zeta_2^2(t)\big) \right| dt$$

$$\geq |I_6| \geq |v_2 \cdot v_1| - |w_{-1} \cdot w_{-2}| - |w_0 \cdot w_{-3}| - |v_1 \cdot w_{-4}| - |v_2 \cdot w_{-5}|.$$

Since all the functionals on the right side of (4-12), which we are going to estimate, are homogeneous, it is not restrictive to assume that $|\alpha|^2 + |\beta|^2 = 1$. In particular, $|v_2 \cdot v_1| = |\gamma|$. Applying the Cauchy–Schwarz inequality we obtain the estimates

$$(4\text{-}13) \quad |w_{-1} \cdot w_{-2}| \leq \tfrac{1}{2}(|w_{-1}|^2 + |w_{-2}|^2) \quad \text{and} \quad |w_0 \cdot w_{-3}| \leq \tfrac{1}{2}(|w_0|^2 + |w_{-3}|^2).$$

For the terms $|v_1 \cdot w_{-4}|$ and $|v_2 \cdot w_{-5}|$ we give small and large constants to obtain

$$2|v_1 \cdot w_{-4}| \leq \frac{\frac{c}{2} + 4\pi}{\frac{c}{4}} |w_{-4}|^2 + \frac{\frac{c}{4}}{\frac{c}{2} + 4\pi},$$

$$(4\text{-}14)$$

$$2|v_2 \cdot w_{-5}| \leq \frac{\frac{c}{2} + 5\pi}{\frac{c}{4}} |w_{-5}|^2 + \frac{\frac{c}{4}}{\frac{c}{2} + 5\pi} \gamma^2,$$

where we have used that $|v_1| = 1$ and $|v_2|^2 = \gamma^2$. Combining the estimate for the integral in (4-12) with (4-13) and (4-14) and plugging into (4-9), we see that

$$(4\text{-}15) \quad \Psi_c(v) < \left(\pi - \frac{c}{2} - \frac{c}{2}\gamma + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 4\pi} \right) + \left(2\pi - \frac{c}{2} + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 5\pi} \right)\gamma^2 + \cdots,$$

where the dots stand for negative terms that arise after absorbing the terms that involve w_{-4} and w_{-5} on the right side of (4-14) with the terms $\left(-4\pi - \frac{c}{2}\right)|w_{-4}|^2$ and $\left(-5\pi - \frac{c}{2}\right)|w_{-5}|^2$ on the right side of (4-9). The right side of (4-15) is negative if

$$(4\text{-}16) \quad \left(2\pi - \frac{c}{2} + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 5\pi} \right)\gamma^2 - \frac{c}{2}\gamma + \left(\pi - \frac{c}{2} + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 4\pi} \right) < 0.$$

For $c = 4\sqrt{2}$ the inequality in (4-16) is satisfied by the solutions to

$$0.44 - 2.82\gamma + 3.56\gamma^2 < 0.$$

In particular, $\Psi_c(v) < 0$ for

$$(4\text{-}17) \qquad\qquad\qquad 0.22 \leq \gamma \leq 0.58.$$

Using I_4 in place of I_6, we get that

$$(4\text{-}18) \quad \int_0^1 \left| \mathrm{Re}\big(\zeta_1^2(t) + \zeta_2^2(t)\big) \right| dt$$

$$\geq |I_4| \geq |\alpha^2 + \beta^2| - |v_2 \cdot w_0| - |w_0 \cdot w_{-2}| - |v_1 \cdot w_{-3}| - |v_2 \cdot w_{-4}| - \frac{|w_{-1}|^2}{2}.$$

We first estimate

$$2|v_2 \cdot w_0| \leq \frac{\frac{3}{4}c + 4\pi}{\frac{c}{2} + 2\pi}|w_0|^2 + \frac{\frac{c}{2} + 2\pi}{\frac{3}{4}c + 4\pi}\gamma^2,$$

$$2|w_0 \cdot w_{-2}| \leq \frac{\frac{c}{2} + 2\pi}{\frac{c}{4}}|w_{-2}|^2 + \frac{\frac{c}{4}}{\frac{c}{2} + 2\pi}|w_0|^2,$$

(4-19)

$$2|v_1 \cdot w_{-3}| \leq \frac{\frac{c}{2} + 3\pi}{\frac{c}{4}}|w_{-3}|^2 + \frac{\frac{c}{4}}{\frac{c}{2} + 3\pi},$$

$$2|v_2 \cdot w_{-4}| \leq \frac{\frac{c}{2} + 4\pi}{\frac{c}{4}}|w_{-4}|^2 + \frac{\frac{c}{4}}{\frac{c}{2} + 4\pi}\gamma^2.$$

We have used again that $|v_1|^2 = 1$ and $|v_2|^2 = \gamma^2$. Combining (4-19) with (4-18) and replacing inside (4-9) we obtain

$$\Psi_c(v) \leq \left(\pi - \frac{c}{2} - \frac{c}{4}|\alpha^2 + \beta^2| + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 3\pi} \right) + \left(2\pi - \frac{c}{2} + \frac{c}{4}\left(\frac{\frac{c}{2} + 2\pi}{\frac{3}{4}c + 4\pi} + \frac{\frac{c}{4}}{\frac{c}{2} + 4\pi} \right) \right)\gamma^2 + \cdots$$

Once again, the dots indicate negative terms which come after absorption with the negative terms involving $|w_k|^2$ for $k = 0, -2, -3, -4$. Hence $\Psi_c(v) < 0$ if

$$(4\text{-}20) \quad \left(\pi - \frac{c}{2} + \frac{(\frac{c}{4})^2}{\frac{c}{2}+3\pi}\right) - \frac{c}{4}|\alpha^2+\beta^2| + \left(2\pi - \frac{c}{2} + \frac{c}{4}\left(\frac{\frac{c}{2}+2\pi}{\frac{3}{4}c+4\pi}\right) + \frac{\frac{c}{4}}{\frac{c}{2}+4\pi}\right)\gamma^2 < 0.$$

For $c = 4\sqrt{2}$, the inequality (4-20) is satisfied by the solutions to

$$(4\text{-}21) \qquad\qquad 0.477 - 1.41|\alpha^2 + \beta^2| + 4.352\gamma^2 < 0.$$

Letting $\gamma_0 = 0.23$ by (4-17) and solving the corresponding equation to (4-21), we can define $\delta_0 := (0.47 + 4.22 \cdot 0.23^2)/1.41 = 0.496$. We then choose a continuous function γ such that $0 \le \gamma \le \gamma_0$ and

$$(4\text{-}22) \qquad\qquad \gamma([\alpha : \beta]) = \begin{cases} \gamma_0 & \text{if } |\alpha^2 + \beta^2| < \delta_0 = 0.502, \\ 0 & \text{if } |\alpha^2 + \beta^2| > 0.6. \end{cases}$$

With this choice, we conclude that $\Psi_c|_W < 0$. The conclusion remain valid even if we choose a constant c slightly smaller than $4\sqrt{2}$.

Now let $\mathcal{D}_n$ be the approximating sets for $D^2 \times D^2$ constructed in Proposition 3.2. Note that

$$(4\text{-}23) \qquad\qquad \frac{\mathcal{D}_n}{\sqrt{1+\frac{1}{n}}} \subseteq D^2 \times D^2 \subseteq \mathcal{D}_n.$$

The first inclusion in (4-23) implies

$$(4\text{-}24) \qquad\qquad \frac{r_n}{1+\frac{1}{n}} \ge r,$$

where r_n is the gauge function of the set $\mathcal{D}_n$. For each n, let Ψ_c^n be defined as

$$\Psi_c^n(\zeta) := \mathcal{A}(\zeta) - c \int_0^1 \frac{r_n(\zeta(t))}{1+\frac{1}{n}}\, dt, \quad \zeta \in E.$$

By (4-24), we have $\Psi_c^n \le \Psi_c \le 0$. Propositions 4.1 and 3.2 then imply

$$\lim_{n \to +\infty} \frac{c_2(\mathcal{D}_n)}{1+\frac{1}{n}} = 3\sqrt{3}.$$

This is because $\Sigma(\mathcal{D}_n \cap [0, 4\sqrt{2} - \varepsilon]) = \{\alpha_1^n, \alpha_2^n\}$ for $\varepsilon > 0$ and n large, where α_1^n and α_2^n are two sequences converging to 4 and $3\sqrt{3}$, respectively (Proposition 3.2). Together with the fact that $c_2(\mathcal{D}_n)/(1 + \frac{1}{n}) < 4\sqrt{2}$ we have that either $c_2(\mathcal{D}_n) = \alpha_1^n$ or $c_2(\mathcal{D}_n) = \alpha_2^n$. It cannot be $c_2(\mathcal{D}_n) = \alpha_1^n = c_1(\mathcal{D}_n)$, otherwise the set of characteristics of $\mathcal{D}_n$ of action α_1^n would have index 2, and this is not the case. Hence $c_2(\mathcal{D}_n) = \alpha_2^n \to 3\sqrt{3}$, and this gives the conclusion. $\qquad\square$

Remark 4.5. One can deduce that $c_2(D^2 \times D^2) = 3\sqrt{3}$ from the work of Ramos [2017]. Let $E(a, b)$ denote the ellipsoid

$$(4\text{-}25) \qquad E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \ \Big| \ \pi\left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\right) \le 1 \right\}.$$

It follows from [Ramos 2017, Corollary 9] that $E(4-\varepsilon, 4+\delta)$ embeds symplectically into the bidisc $D^2 \times D^2$ for some $\varepsilon, \delta > 0$. Hence

$$4 + \delta \le c_2(D^2 \times D^2) \le 3\sqrt{3}.$$

Since $3\sqrt{3}$ is the only number in the spectrum $\Sigma(D^2 \times D^2)$ with this property, we conclude that $c_2(D^2 \times D^2) = 3\sqrt{3}$.

Theorem 4.6. *For the unit real bidisc $D^2 \times D^2$ we have $c_3(D^2 \times D^2) = 8$.*

Proof. Let W be the subspace of E defined by

$$W := E^- \oplus E^0 \oplus \langle (e^{2\pi it}, 0), (0, e^{2\pi it}), (e^{4\pi it}, 0) \rangle.$$

By [Ekeland and Hofer 1990, Proposition 1] the pseudoindex of W is equal to 3. We now prove that for some constant c we have $\Psi_c|_W \le 0$, where Ψ_c is the functional defined in (4-1). For an element $(\zeta_1, \zeta_2) = (\alpha e^{2\pi it} + \gamma e^{4\pi it}, \beta e^{2\pi it}) + w^- + w^0 \in W$, with $\alpha, \beta, \gamma \in \mathbb{C}$, $w^- \in E^-$, $w^0 \in E^0$, we have

$$(4\text{-}26) \quad \Psi_c((\alpha e^{2\pi it} + \gamma e^{4\pi it}, \beta e^{2\pi it}) + w^- + w^0)$$
$$= \mathcal{A}(\zeta) - c \int_0^1 r(\zeta(t)) \, dt$$
$$= -\frac{c}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2) - \|w^-\|_E - \frac{c}{2}\|w^0 + w^-\|_{L^2}^2$$
$$- \frac{c}{2} \int_0^1 |\text{Re}(\zeta_1^2(t) + \zeta_2^2(t))| \, dt + \pi(|\alpha|^2 + |\beta|^2 + 2|\gamma|^2).$$

To give an estimate of the last integral in (4-26), we compute the coefficient I_8 of $e^{8\pi it}$ in the Fourier expansion of $\text{Re}(\zeta_1^2(t) + \zeta_2^2(t))$:

$$I_8 = \gamma^2 + \overline{2(\alpha, \beta) \cdot w_5^-} + 2(\gamma, 0) \cdot w_6^- + 2w^0 \cdot w_4^- + 2w_1^- \cdot w_3^- + w_2^- \cdot w_2^-.$$

We want to find a constant c such that

$$(4\text{-}27) \quad \pi(|\alpha|^2 + |\beta|^2 + 2|\gamma|^2)$$
$$- \frac{c}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2) - \|w^-\|_E - \frac{c}{2}\|w^0 + w^-\|_{L^2}^2 - \frac{c}{4}|I_8| < 0.$$

Applying the Cauchy–Schwarz inequality, we get

$$|I_8| \ge |\gamma|^2 - a_1|w_5^-|^2 - \frac{1}{a_1}|(\alpha, \beta)|^2 - a_2|w_6^-|^2 - \frac{1}{a_2}|\gamma|^2$$
$$- |w^0|^2 - |w_4^-|^2 - |w_1^-|^2 - |w_3^-|^2 - |w_2^-|^2.$$

Choosing a_1 such that $\frac{c}{4}a_1 = \frac{c}{2} + 5\pi$ and a_2 such that $\frac{c}{4}a_2 = \frac{c}{2} + 6\pi$, (4-27) becomes

$$(4\text{-}28) \quad (|\alpha|^2 + |\beta|^2)\left(\pi - \frac{c}{2} - \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 5\pi}\right) + |\gamma|^2\left(2\pi - \frac{c}{2} - \frac{c}{4} + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 6\pi}\right) + \cdots,$$

where the dots indicate other negative terms not involving α, β or γ. The expression in (4-28) is negative if

$$(4\text{-}29) \quad \begin{cases} \pi - \dfrac{c}{2} - \dfrac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 5\pi} < 0, \\[4mm] 2\pi - \dfrac{c}{2} - \dfrac{c}{4} + \dfrac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 6\pi} < 0. \end{cases}$$

Solving the system (4-29), we obtain

$$c > 4\pi \frac{\sqrt{109} - 7}{5}.$$

Considering the approximating domains $\mathcal{D}_n$ and reasoning as in the last part of the proof of Theorem 4.4, we conclude that

$$(4\text{-}30) \qquad c_3(D^2 \times D^2) < 4\pi \frac{\sqrt{109} - 7}{5}.$$

Moreover, Proposition 4.8 implies

$$(4\text{-}31) \qquad c_3(D^2 \times D^2) > c_3(\mathbb{B}^2) = 2\pi.$$

Looking at the spectrum $\Sigma(D^2 \times D^2)$ computed in Proposition 2.2, we see that (4-30) and (4-31) imply $c_3(D^2 \times D^2) = 8$. $\qquad\square$

Note that in the proof of Theorem 4.6 we have used the fact that the capacity c_3 of the real bidisc is strictly greater than the corresponding capacity for the unit ball $\mathbb{B}^2$. This inequality is proved below in Proposition 4.8 for every capacity c_k. The proof relies on the following lemma.

Lemma 4.7. *Let $D_1 \subset D_2$ be two convex smooth subdomains of $\mathbb{C}^n$ such that for some k we have $c_k(D_1) = c_k(D_2) =: c$. Assume also that c is isolated in $\Sigma(D_1)$ and $\Sigma(D_2)$. Then there exists a closed characteristic $\gamma \subset \partial D_1 \cap \partial D_2$ such that $\mathcal{A}(\gamma) = c$.*

Proof. Let r_1 and r_2 be the gauge functions of D_1 and D_2, respectively. Let f be an increasing positive function such that

$$(4\text{-}32) \qquad f(s) = \begin{cases} 0 & \text{if } s \le 1, \\ Cs & \text{if } s \text{ is large,} \end{cases}$$

and $f'(s_0) = c$ only for one $s_0 > 1$. We will specify later how s_0 and C are chosen.

Let $H_i^f := f \circ r_i$ for $i = 1, 2$. Note that

$$(4\text{-}33) \qquad c_k(D_1) = \inf_{H_1}\{c_{H_1}^k\} = \inf_{H_2}\{c_{H_2}^k\} = c_k(D_2),$$

where each infimum is taken over all possible admissible Hamiltonians H_i for D_i. Equation (4-33) together with the fact that c is isolated in the spectra $\Sigma(D_1)$ and $\Sigma(D_2)$ implies that, choosing a function f with C sufficiently large, we obtain

$$c_{H_1^f}^k = c_{H_2^f}^k.$$

Let $\mathcal{A}_{H_i^f}$ be the corresponding Hamiltonian actions. Since $D_1 \subset D_2$, then $r_1 \geq r_2$ and $\mathcal{A}_{H_1^f} \leq \mathcal{A}_{H_2^f}$. Let

$$W_\varepsilon = \{\zeta \in E \mid \mathcal{A}_{H_2^f}(\zeta) \leq c_{H_2^f}^k + \varepsilon\}.$$

W_ε is a closed equivariant set of pseudoindex at least k. We have

$$c_{H_1^f}^k \leq \sup_{\zeta \in W_\varepsilon}\{\mathcal{A}_{H_1^f}(\zeta)\} \leq \sup_{\zeta \in W_\varepsilon}\{\mathcal{A}_{H_2^f}(\zeta)\} = c_{H_2^f}^k + \varepsilon.$$

We now claim that there exist two sequences $\varepsilon_n \to 0$ and $\zeta_n \in W_{\varepsilon_n}$ such that

$$\nabla_E \mathcal{A}_{H_1^f}(\zeta_n) \to 0 \quad \text{and} \quad \mathcal{A}_{H_1^f}(\zeta_n) \to c_{H_1^f}^k.$$

Indeed, assume that this is not true. Then there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$ and for all $\zeta \in W_\varepsilon$ such that $|\mathcal{A}_{H_1^f}(\zeta) - c_{H_1^f}^k| \leq \varepsilon_0$ we have $\|\nabla \mathcal{A}_{H_1^f}(\zeta)\|_E > \delta_0$. Following [Ekeland and Hofer 1990, Lemma 1] and [Ekeland and Hofer 1989, Proposition 2], we denote by Φ_t the flow at the time t in E of the vector field $-\nabla \mathcal{A}_{H_1^f}$ with a suitable cut-off. Choosing ε small enough we have, for $t > \varepsilon/\delta_0^2$, that

$$\sup_{\zeta \in \Phi_t(W_\varepsilon)} \mathcal{A}_{H_1^f}(\zeta) < c_{H_1^f}^k,$$

which gives a contradiction.

We have thus proved that there exists a subsequence of ζ_n converging in E to ζ_0, which is a critical point for $\mathcal{A}_{H_1^f}$ and such that $\mathcal{A}_{H_1^f}(\zeta_0) = c_{H_1^f}^k$. If the image of ζ_0 is not in $\partial D_1 \cap \partial D_2$, then $r_1(\zeta_0(t)) \geq r_2(\zeta_0(t))$ and the inequality is strict in some open interval. This implies that $\mathcal{A}_{H_2^f}(\zeta_0) > c_{H_2^f}^k$ and in turn $\mathcal{A}_{H_2^f}(\zeta_n) > c_{H_2^f}^k + \varepsilon$ for some $\varepsilon > 0$ not depending on n, but this is in contradiction with the definition of W_{ε_n}. $\square$

Proposition 4.8. *For the real bidisc $D^2 \times D^2$ and the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$, we have $c_k(D^2 \times D^2) > c_k(\mathbb{B}^2)$ for every positive integer k.*

Proof. Since the unit ball $\mathbb{B}^2$ centered at 0 is contained in $D^2 \times D^2$ we have, for any k, that $c_k(D^2 \times D^2) \geq c_k(\mathbb{B}^2)$. Choose a smooth domain W containing $\mathbb{B}^2$ and contained in $D^2 \times D^2$ with a discrete action spectrum and having the same

intersection at the boundary with $D^2 \times D^2$ and $\mathbb{B}^2$. If $c_k(D^2 \times D^2) = c_k(\mathbb{B}^2)$ for some k, then Lemma 4.7 implies that there exists a characteristic contained in

$$\partial W \cap \mathbb{B}^2 = \partial(D^2 \times D^2) \cap \mathbb{B}^2$$
$$= \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1^2 + x_2^2 = 1\} \cup \{(iy_1, iy_2) \in \mathbb{C}^2 \mid y_1^2 + y_2^2 = 1\}.$$

Since there are no characteristics in the intersection of the boundaries, then we must have $c_k(D^2 \times D^2) > c_k(\mathbb{B}^2)$. □

Remark 4.9. Let $B(a) = E(a, a)$, that is, $B(a)$ is the Euclidean ball of radius $\sqrt{a/\pi}$. It follows from [Ramos 2017] that the real bidisc $D^2 \times D^2$ can be symplectically embedded into the ellipsoid $E(4, 3\sqrt{3})$ and that the ball $B(4)$ can be symplectically embedded into $D^2 \times D^2$. This implies that $c_3(D^2 \times D^2) = 8$. Proposition 4.8 can also be obtained from [Ramos 2017]. Indeed, $\mathbb{B}^2 = B(\pi)$ has strictly smaller Ekeland–Hofer capacities than $B(4)$, which embeds into $D^2 \times D^2$.

5. Applications

We now exploit our computations of the capacities of $D^2 \times D^2$ to prove some results of symplectic rigidity. First recall that the complex bidisc $\Delta^2 \subset \mathbb{C}^2$ is defined by

$$\Delta^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid x_1^2 + y_1^2 < 1, x_2^2 + y_2^2 < 1\}.$$

Sukhov and Tumanov [2012] applied techniques from classical complex analysis to prove that there exists no symplectic embedding of $D^2 \times D^2$ into Δ^2. Using symplectic capacities we can easily show that no symplectic embedding is possible in the other direction.

Corollary 5.1. *There is no symplectic embedding of Δ^2 into $D^2 \times D^2$.*

Proof. Assume by contradiction that there is such an embedding

$$\psi : \Delta^2 \to D^2 \times D^2.$$

By the extension after restriction principle, for any $\varepsilon > 0$ there exists a symplectic map with compact support $\psi_\varepsilon : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\psi_\varepsilon|_{\Delta^2_{1-\varepsilon}} = \psi|_{\Delta^2_{1-\varepsilon}}$. Therefore,

$$3\pi(1 - \varepsilon)^2 = c_3(\Delta^2_{1-\varepsilon}) = c_3(\psi_\varepsilon(\Delta^2_{1-\varepsilon})) \leq c_3(D^2 \times D^2) = 8,$$

which gives a contradiction. □

Note that the proof of Corollary 5.1 implies that the complex bidisc cannot be embedded even in a slightly larger real bidisc.

Remark 5.2. Corollary 5.1 can also be obtained without using the Ekeland–Hofer capacities. By [Ramos 2017], the bidisc $D^2 \times D^2$ is a concave toric domain. One can then apply [Gutt and Hutchings 2018, Theorem 1.18] to conclude that the cube

capacity $c_\square$ of $D^2 \times D^2$ is equal to 2. By the very definition of the cube capacity [Gutt and Hutchings 2018, Definition 1.17], Corollary 5.1 follows.

To prove the next rigidity result we need to recall a product property of the Ekeland–Hofer capacities: if $A \subset \mathbb{C}^n$ and $B \subset \mathbb{C}^m$, then

$$c_k(A \times B) = \min_{i+j=k}\{c_i(A) + c_j(B)\}.$$

Here we use the convention that the zero-th capacity is equal to 0, that is, $c_0(A) = c_0(B) = 0$. We denote by Δ_R the standard complex disc of radius R.

Corollary 5.3. *The product $D^2 \times D^2 \times \Delta_R$ is not symplectomorphic to $\Delta^2 \times \Delta_R$ for $R > \sqrt{3\sqrt{3}/2\pi}$.*

Proof. The case $R \geq 1$ is known [Wong 2018, Theorem 4.1], hence let $R < 1$. We have

$$c_2(\Delta^2 \times \Delta_R) = 2\pi R^2.$$

On the other hand,

$$c_2(D^2 \times D^2 \times \Delta_R) = \min\{3\sqrt{3}, 4 + \pi R^2, 2\pi R^2\}.$$

The two capacities are different if $R > \sqrt{3\sqrt{3}/2\pi}$. $\square$

Remark 5.4. The bound in Corollary 5.3 can be improved to $\sqrt{2/\pi}$ using Ekeland–Hofer symplectic capacities of higher order. More precisely, one has to show, for each positive integer n, that $c_{2n-1}(D^2 \times D^2) = 4n$. This can be achieved by arguing in a similar way as in Theorem 4.6, where we computed the value of $c_3(D^2 \times D^2)$. Vinicius Gripp Barros Ramos has informed the authors that the bound $\sqrt{2/\pi}$ can also be obtained using the results in [Gutt and Hutchings 2018].

6. Acknowledgements

References

[Aebischer et al. 1994] B. Aebischer, M. Borer, M. Kälin, C. Leuenberger, and H. M. Reimann, *Symplectic geometry: an introduction based on the seminar in Bern, 1992*, Progress in Mathematics **124**, Birkhäuser, Basel, 1994. MR Zbl

[Artstein-Avidan and Ostrover 2014] S. Artstein-Avidan and Y. Ostrover, "Bounds for Minkowski billiard trajectories in convex bodies", *Int. Math. Res. Not.* **2014**:1 (2014), 165–193. MR Zbl

[Benci 1982] V. Benci, "On critical point theory for indefinite functionals in the presence of symmetries", *Trans. Amer. Math. Soc.* **274**:2 (1982), 533–572. MR Zbl

[Berestycki et al. 1985] H. Berestycki, J.-M. Lasry, G. Mancini, and B. Ruf, "Existence of multiple periodic orbits on star-shaped Hamiltonian surfaces", *Comm. Pure Appl. Math.* **38**:3 (1985), 253–289. MR Zbl

[Ekeland and Hofer 1989] I. Ekeland and H. Hofer, "Symplectic topology and Hamiltonian dynamics", *Math. Z.* **200**:3 (1989), 355–378. MR Zbl

[Ekeland and Hofer 1990] I. Ekeland and H. Hofer, "Symplectic topology and Hamiltonian dynamics, II", *Math. Z.* **203**:4 (1990), 553–567. MR Zbl

[Fadell et al. 1982] E. R. Fadell, S. Y. Husseini, and P. H. Rabinowitz, "Borsuk–Ulam theorems for arbitrary S^1 actions and applications", *Trans. Amer. Math. Soc.* **274**:1 (1982), 345–360. MR Zbl

[Fan 1992] X. L. Fan, "Existence of multiple periodic orbits on star-shaped Lipschitz–Hamiltonian surfaces", *J. Differential Equations* **98**:1 (1992), 91–110. MR Zbl

[Floer et al. 1990] A. Floer, H. Hofer, and C. Viterbo, "The Weinstein conjecture in $P \times \mathbf{C}^l$", *Math. Z.* **203**:3 (1990), 469–482. MR Zbl

[Gromov 1985] M. Gromov, "Pseudo holomorphic curves in symplectic manifolds", *Invent. Math.* **82**:2 (1985), 307–347. MR Zbl

[Guth 2008] L. Guth, "Symplectic embeddings of polydisks", *Invent. Math.* **172**:3 (2008), 477–489. MR Zbl

[Gutt and Hutchings 2018] J. Gutt and M. Hutchings, "Symplectic capacities from positive S^1-equivariant symplectic homology", *Algebr. Geom. Topol.* **18**:6 (2018), 3537–3600. MR Zbl

[Hofer and Zehnder 1990] H. Hofer and E. Zehnder, "A new capacity for symplectic manifolds", pp. 405–427 in *Analysis, et cetera*, edited by P. H. Rabinowitz and E. Zehnder, Academic Press, Boston, 1990. MR Zbl

[Hutchings 2011a] M. Hutchings, "Quantitative embedded contact homology", *J. Differential Geom.* **88**:2 (2011), 231–266. MR Zbl

[Hutchings 2011b] M. Hutchings, "Recent progress on symplectic embedding problems in four dimensions", *Proc. Natl. Acad. Sci. USA* **108**:20 (2011), 8093–8099. MR Zbl

[McDuff 2009] D. McDuff, "Symplectic embeddings of 4-dimensional ellipsoids", *J. Topol.* **2**:1 (2009), 1–22. MR Zbl

[McDuff 2011] D. McDuff, "The Hofer conjecture on embedding symplectic ellipsoids", *J. Differential Geom.* **88**:3 (2011), 519–532. MR Zbl

[McDuff and Salamon 2017] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 3rd ed., Oxford Graduate Texts in Mathematics **27**, Oxford University Press, 2017. MR Zbl

[McDuff and Schlenk 2012] D. McDuff and F. Schlenk, "The embedding capacity of 4-dimensional symplectic ellipsoids", *Ann. of Math.* (2) **175**:3 (2012), 1191–1282. MR Zbl

[Ramos 2017] V. G. B. Ramos, "Symplectic embeddings and the Lagrangian bidisk", *Duke Math. J.* **166**:9 (2017), 1703–1738. MR Zbl

[Sukhov and Tumanov 2012] A. Sukhov and A. Tumanov, "Hartogs figure and symplectic non-squeezing", *Illinois J. Math.* **56**:1 (2012), 221–233. MR Zbl

[Viterbo 1992] C. Viterbo, "Symplectic topology as the geometry of generating functions", *Math. Ann.* **292**:4 (1992), 685–710. MR Zbl

[Wong 2018] Y.-S. Wong, "Symplectic rigidity of real and complex polydiscs", *J. Math. Anal. Appl.* **459**:2 (2018), 852–860. MR Zbl

LUCA BARACCO
DIPARTIMENTO DI MATEMATICA TULLIO LEVI-CIVITA
UNIVERSITÀ DEGLI STUDI DI PADOVA
PADOVA
ITALY

baracco@math.unipd.it

MARTINO FASSINA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, IL
UNITED STATES

fassina2@illinois.edu

STEFANO PINTON
DIPARTIMENTO DI MATEMATICA
POLITECNICO DI MILANO
MILANO
ITALY

stefano.pinton@polimi.it

PACIFIC JOURNAL OF MATHEMATICS

See inside back cover or msp.org/pjm for submission instructions.

PACIFIC JOURNAL OF MATHEMATICS