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FREENESS CHARACTERIZATIONS ON FREE CHAOS SPACES

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We deal with characterizing the freeness and asymptotic freeness of free multiple integrals with respect to a free Brownian motion or a free Poisson process. We obtain three characterizations of freeness, in terms of contraction operators, covariance conditions, and free Malliavin gradients. We show how these characterizations can be used in order to obtain limit theorems, transfer principles, and asymptotic properties of converging sequences.

1. Introduction

A classical result in probability theory asserts that one can decompose any functional of a Brownian motion W as an infinite sum of multiple integrals. That is, to any square integrable random variable F measurable with respect to W, one can associate a unique sequence of symmetric and square integrable kernels $\{f_n : n \geq 0\}$ such that

$$F = \sum_{n=0}^{\infty} I_n^W(f_n).$$

The set of all multiple Wiener–Itô integrals of the form $I_n^W(f)$, the so-called *n-th Wiener chaos* of W, thus plays a fundamental role in modern stochastic analysis. Analyzing its many rigid properties (notably those related to independence and normal approximation) has become a subject in its own right, and has grown into a mature and widely applicable mathematical theory.

Among the most striking results about Wiener chaos are the following two theorems, which will play a central role in the present paper. The first one characterizes independence of multiple Wiener–Itô integrals.

Theorem 1.1 [Üstünel and Zakai 1989]. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then $I_n^W(f)$ and $I_m^W(g)$ are independent if and only if, for almost all $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1} \in \mathbb{R}_+$,*

$$\int_0^{\infty} f(x_1, \ldots, x_{n-1}, u) g(y_1, \ldots, y_{m-1}, u)\, du = 0.$$

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The second result is nowadays one of the most central tools of analysis on Wiener chaos, as it represents a drastic simplification with respect to the method of moments for the normal approximation of sequences of multiple Wiener–Itô integrals.

Theorem 1.2 [Nualart and Peccati 2005]. *A unit-variance sequence in a Wiener chaos of fixed order converges in law to the standard Gaussian distribution if and only if the corresponding sequence of fourth moments converges to* 3.

Since its introduction by Voiculescu in the eighties in order to solve some longstanding conjectures about von Neumann algebras of free groups, free probability theory has become a vivid and powerful branch of mathematics, with many applications (including signal processing, channel capacity estimation and nuclear physics) and deep connections with other mathematical fields (like operator algebra, theory of random matrices or combinatorics). Free probability has many parallels with the usual probability theory (hence its name), and the study of these links often brings a new point of view which may then enrich the theory of both worlds (classical and free).

Starting from the free independence property, a genuine stochastic calculus with respect to the free Brownian motion (the free analog of the classical Brownian motion) has emerged within the last twenty years, following the route paved by the seminal paper of Biane and Speicher [1998]. In particular, a common property of the classical and free settings is the possibility of expanding the space as a sum of free chaos, giving rise to the so-called *Wigner chaos*. By their very construction, these free chaos play in the free world a similar role as Wiener chaos in the classical setting. It is thus natural to investigate the similarities and differences between these two mathematical objects. For instance, do we have an analog of Theorem 1.2 in the free world? The answer is "yes", and is given by the following theorem.

Theorem 1.3 [Kemp et al. 2012]. *A unit-variance sequence in a Wigner chaos of fixed order converges in law to the semicircular distribution if and only if the corresponding sequence of fourth moments converges to* 2.

Shortly after the publication of [Kemp et al. 2012], many other results in the spirit of Theorem 1.3 have been added to the literature, including the following ones (the list is not exhaustive).

In [Nourdin et al. 2013], it is shown that component-wise convergence to the semicircular distribution is equivalent to joint convergence, thus extending to the free probability setting a seminal result by Peccati and Tudor [2005].

In [Nourdin and Peccati 2013], a noncentral counterpart of Theorem 1.3 is provided. More precisely, it is shown that any adequately rescaled sequence $\{F_n : n \geq 0\}$ of self-adjoint operators living inside a fixed Wigner chaos of even order converges in distribution to a centered free Poisson random variable with rate $\lambda > 0$ if and only if $\varphi(F_n^4) - 2\varphi(F_n^3) \to 2\lambda^2 - \lambda$ (where φ is the relevant tracial state).

In [Nourdin and Poly 2012], convergence in law of any sequence belonging to the second Wigner chaos is characterized by means of the convergence of only a finite number of cumulants.

In [Deya and Nourdin 2012], making use of heavy combinatorics it is shown that any adequately rescaled sequence $\{F_n : n \geq 0\}$ of self-adjoint operators living inside a fixed Wigner chaos converges in distribution to the tetilla law $\mathcal{T}$ if and only if

$$\varphi(F_n^4) \to \varphi(\mathcal{T}^4) \quad \text{and} \quad \varphi(F_n^6) \to \varphi(\mathcal{T}^6)$$

(where φ is the relevant tracial state). Note that this finding is not an extension of a result known in the classical probability theory, as the existence of such a result in the classical setting is still an open problem.

In [Bourguin and Peccati 2014], a class of sufficient conditions, ensuring that a sequence of multiple integrals with respect to a free Poisson measure converges to a semicircular limit, is established, thus providing an analog of Theorem 1.3 in the context of free Poisson chaos.

In [Bourguin 2015], a fourth moment type condition is given, for an element of a free Poisson chaos of arbitrary order to converge to a free centered Poisson distribution.

In [Arizmendi and Jaramillo 2014], an estimate for the Kolmogorov distance between a freely infinitely divisible distribution and the semicircle distribution is given, in terms of the difference between the fourth moment and 2.

In [Bourguin 2016], a multidimensional counterpart of the aforementioned central limit theorem on the free Poisson chaos is given.

In [Bourguin and Campese 2018], a quantitative version of Theorem 1.3 is derived, using free stochastic analysis as well as a new biproduct formula for bi-integrals.

In the present paper, our main goal is to provide characterizations of free independence on the Wigner and free Poisson chaos, as well as investigate the similarities and dissimilarities between classical and free chaos, as far as (possibly asymptotic) independence properties are concerned.

Our first set of investigations yields a characterization of freeness on the Wigner and free Poisson chaos, in terms of contractions, covariances, or free Malliavin gradient, thus providing a suitable extension of Theorem 1.1 (and related results) to the free setting. Most of our results turn out to be similar to the classical setting, with the notable exception of the characterization of freeness in terms of the free Malliavin gradient, this last fact illustrating a fundamental difference between the classical and the free cases.

Our second set of investigations is concerned again with the independence property, but this time in an asymptotic context. Here, the problem is to find what conditions need to be imposed on *limits* of multiple integrals to be free.

The remainder of this paper is organized as follows: Section 2 contains a short introduction to free probability theory, with a special emphasis to the material needed for the rest of the paper. Section 3 is devoted to the characterization of freeness on the Wigner and free Poisson chaos, in terms of contractions, covariances, or free Malliavin gradient. This section also provides several lemmas which will be used to prove our main results in the following sections. In Section 4, we study different characterizations of asymptotic freeness, in several contexts. We devote Section 5 to the study of transfer principles between classical and free chaos. Finally, Section 6 contains auxiliary results that are used throughout the paper.

2. Preliminaries

Elements of free probability. In the following, a short introduction to free probability theory is provided. For a thorough and complete treatment, see [Nica and Speicher 2006; Voiculescu et al. 1992; Hiai and Petz 2000]. Let $(\mathscr{A}, \varphi)$ be a tracial W^*-probability space, that is $\mathscr{A}$ is a von Neumann algebra with involution $*$ and $\varphi : \mathscr{A} \to \mathbb{C}$ is a unital linear functional assumed to be weakly continuous, positive (meaning that $\varphi(X) \geq 0$ whenever X is a nonnegative element of $\mathscr{A}$), faithful (meaning that $\varphi(XX^*) = 0 \Rightarrow X = 0$ for every $X \in \mathscr{A}$) and tracial (meaning that $\varphi(XY) = \varphi(YX)$ for all $X, Y \in \mathscr{A}$). The self-adjoint elements of $\mathscr{A}$ will be referred to as random variables. The noncommutative space $L^2(\mathscr{A}, \varphi)$ denotes the completion of $\mathscr{A}$ with respect to the norm $\|X\|_2 = \sqrt{\varphi(XX^*)}$.

Recall the definition of freeness (see [Nica and Speicher 2006, Definition 5.3] and [Nica and Speicher 2006, Remarks 5.4] or [Tao 2012, Definition 2.5.18]) for a collection of noncommutative random variables living on an appropriate noncommutative probability space $(\mathscr{A}, \varphi)$.

Definition 2.1. A collection of random variables $X_1, \dots, X_n$ on $(\mathscr{A}, \varphi)$ is said to be free if

$$\varphi\big([P_1(X_{i_1}) - \varphi(P_1(X_{i_1}))] \cdots [P_m(X_{i_m}) - \varphi(P_m(X_{i_m}))]\big) = 0$$

whenever $P_1, \dots, P_m$ are polynomials and $i_1, \dots, i_m \in \{1, \dots, n\}$ are indices with no two adjacent i_j equal.

Let $X \in \mathscr{A}$. The p-th moment of X is given by the quantity $\varphi(X^p)$, $p \in \mathbb{N}_0$. Now assume that X is a self-adjoint bounded element of $\mathscr{A}$ (in other words, X is a bounded random variable), and write $\rho(X) = \|X\| \in [0, \infty)$ to indicate the *spectral radius* of X.

Definition 2.2. The *law* (or *spectral measure*) of X is defined as the unique Borel probability measure μ_X on the real line such that $\int_{\mathbb{R}} P(t)\, d\mu_X(t) = \varphi(P(X))$ for every polynomial $P \in \mathbb{R}[X]$. A consequence of this definition is that μ_X has support in $[-\rho(X), \rho(X)]$.

The existence and uniqueness of μ_X in such a general framework are proved, e.g., in [Tao 2012, Theorem 2.5.8] (see also [Nica and Speicher 2006, Proposition 3.13]). Note that, since μ_X has compact support, the measure μ_X is completely determined by the sequence $\{\varphi(X^p) : p \geq 1\}$.

Let $\{X_k : k \geq 1\}$ be a sequence of noncommutative random variables, each possibly belonging to a different noncommutative probability space $(\mathscr{A}_k, \varphi_k)$.

Definition 2.3. The sequence $\{X_k : k \geq 1\}$ is said to converge in distribution to a limiting noncommutative random variable X_∞ (defined on $(\mathscr{A}_\infty, \varphi_\infty)$), if $\varphi_k(P(X_k)) \xrightarrow[k\to+\infty]{} \varphi_\infty(P(X_\infty))$ for every polynomial $P \in \mathbb{R}[X]$.

If X_k, X_∞ are bounded (and therefore the spectral measures $\mu_{X_k}, \mu_{X_\infty}$ are well-defined), this last relation is equivalent to saying that

$$\int_{\mathbb{R}} P(t)\,\mu_{X_k}(dt) \xrightarrow[k\to+\infty]{} \int_{\mathbb{R}} P(t)\,\mu_{X_\infty}(dt).$$

An application of the method of moments yields immediately that, in this case, one has also that μ_{X_k} weakly converges to μ_{X_∞}, that is $\mu_{X_k}(f) \xrightarrow[k\to+\infty]{} \mu_{X_\infty}(f)$, for every $f : \mathbb{R} \to \mathbb{R}$ bounded and continuous (note that no additional uniform boundedness assumption is needed).

In this paper, we will also deal with *joint* convergences in law, for sequences $\{X_k = (X_k^1, \ldots, X_k^d) : k \geq 1\}$ of noncommutative random vectors, each possibly belonging to a different noncommutative probability space $(\mathscr{A}_k, \varphi_k)$.

Definition 2.4. The vector-valued sequence $\{X_k = (X_k^1, \ldots, X_k^d) : k \geq 1\}$ is said to converge jointly in distribution to a limiting noncommutative random vector $X_\infty = (X_\infty^1, \ldots, X_\infty^d)$ (defined on $(\mathscr{A}_\infty, \varphi_\infty)$), if any moment in the variables $X_k^1, \ldots, X_k^d$ converges, as $k \to \infty$, to the corresponding moments in $X_\infty^1, \ldots, X_\infty^d$; otherwise stated, $(X_k^1, \ldots, X_k^d) \xrightarrow{\text{law}} (X_\infty^1, \ldots, X_\infty^d)$ if for any $r \in \mathbb{N}$ and positive integers $i_1, \ldots, i_r$, as $k \to \infty$, one has

$$\varphi_k[X_k^{i_1} \cdots X_k^{i_r}] \to \varphi_\infty[X_\infty^{i_1} \cdots X_\infty^{i_r}].$$

Let us now define the two main processes we will deal with in this paper, namely the free Brownian motion and the free Poisson process.

Definition 2.5. (1) The centered semicircular distribution with variance $t > 0$, denoted by $\mathcal{S}(0, t)$, is the probability distribution given by

$$\mathcal{S}(0, t)(dx) = (2\pi t)^{-1}\sqrt{4t - x^2}\,\mathbb{1}_{[-2\sqrt{t}, 2\sqrt{t}]}(x)\,dx.$$

(2) A free Brownian motion S consists of

(i) a filtration $\{\mathscr{A}_t : t \geq 0\}$ of von Neumann subalgebras of $\mathscr{A}$ (in particular, $\mathscr{A}_s \subset \mathscr{A}_t$ for $0 \leq s < t$),

(ii) a collection $S = \{S_t : t \geq 0\}$ of self-adjoint operators in $\mathscr{A}$ such that

(a) $S_0 = 0$ and $S_t \in \mathscr{A}_t$ for all $t \geq 0$,

(b) for all $t \geq 0$, S_t has a semicircular distribution with mean zero and variance t,

(c) for all $0 \leq u < t$, the increment $S_t - S_u$ is free with respect to $\mathscr{A}_u$, and has a semicircular distribution with mean zero and variance $t - u$.

Definition 2.6. (1) The free Poisson distribution with rate $\lambda > 0$, denoted by $P(\lambda)$, is the probability distribution defined as follows:

(i) if $\lambda \in (0, 1]$, then $P(\lambda) = (1 - \lambda)\delta_0 + \lambda\widetilde{\nu}$, and

(ii) if $\lambda > 1$, then $P(\lambda) = \widetilde{\nu}$, where δ_0 stands for the Dirac mass at 0. Here,

$$\widetilde{\nu}(dx) = (2\pi x)^{-1}\sqrt{4\lambda - (x - 1 - \lambda)^2}\mathbb{1}_{[(1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2]}(x)dx.$$

(2) A free Poisson process N consists of

(i) a filtration $\{\mathscr{A}_t : t \geq 0\}$ of von Neumann subalgebras of $\mathscr{A}$ (in particular, $\mathscr{A}_s \subset \mathscr{A}_t$ for $0 \leq s < t$),

(ii) a collection $N = \{N_t : t \geq 0\}$ of self-adjoint operators in $\mathscr{A}_+$ ($\mathscr{A}_+$ denotes the cone of positive operators in $\mathscr{A}$) such that

(a) $N_0 = 0$ and $N_t \in \mathscr{A}_t$ for all $t \geq 0$,

(b) for all $t \geq 0$, N_t has a free Poisson distribution with rate t, and

(c) for all $0 \leq u < t$, the increment $N_t - N_u$ is free with respect to $\mathscr{A}_u$, and has a free Poisson distribution with rate $t - u$. $\hat{N}$ will denote the collection of random variables $\hat{N} = \{\hat{N}_t = N_t - t\mathbf{1} : t \geq 0\}$, where $\mathbf{1}$ stands for the unit of $\mathscr{A}$. $\hat{N}$ will be referred to as a compensated free Poisson process.

Remark 2.7. In the sequel, $\mathfrak{M}$ will stand for either the free Brownian motion S or the compensated free Poisson process $\hat{N}$.

We continue with some definitions that will play a crucial role in the rest of the paper. For every integer $n \geq 1$, the space $L^2(\mathbb{R}_+^n; \mathbb{C}) = L^2(\mathbb{R}_+^n)$ denotes the collection of all complex-valued functions on $\mathbb{R}_+^n$ that are square-integrable with respect to the Lebesgue measure on $\mathbb{R}_+^n$.

Definition 2.8. Let n be a natural number and let f be a function in $L^2(\mathbb{R}_+^n)$.

(1) The adjoint of f is the function $f^*(t_1, \ldots, t_n) = \overline{f(t_n, \ldots, t_1)}$.

(2) The function f is called mirror-symmetric if $f = f^*$, i.e., if

$$f(t_1, \ldots, t_n) = \overline{f(t_n, \ldots, t_1)}$$

for almost all $(t_1, \ldots, t_n) \in \mathbb{R}_+^n$ with respect to the product Lebesgue measure.

(3) The function f is called (fully) symmetric if it is real-valued and, for any permutation σ in the symmetric group $\mathfrak{S}_n$, it holds that $f(t_1, \dots, t_n) = f(t_{\sigma(1)}, \dots, t_{\sigma(n)})$ for almost all $(t_1, \dots, t_n) \in \mathbb{R}_+^n$ with respect to the product Lebesgue measure.

Definition 2.9. Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$. Let $p \leq n \wedge m$ be a natural number. The p-th nested contraction $f \overset{p}{\frown} g$ of f and g is the $L^2(\mathbb{R}_+^{n+m-2p})$ function defined by nested integration of the middle p variables in $f \otimes g$:

$$(f \overset{p}{\frown} g)(t_1, \dots, t_{n+m-2p}) = \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_1, \dots, s_p) \times g(s_p, \dots, s_1, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p.$$

In the case where $p = 0$, the function $f \overset{0}{\frown} g$ is just given by $f \otimes g$.

Similarly, we define the star contraction $f \star_k^j g$ of f and g.

Definition 2.10. Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$. Let $k \in \{1, \dots, n \wedge m\}$ and $j \in \{0, \dots, k\}$ be two natural numbers. We set

$$(f \star_k^j g)(t_1, \dots, t_{n+m-2k+j}) = \int_{\mathbb{R}_+^{k-j}} f(t_1, \dots, t_{n-k+j}, s_{k-j}, \dots, s_1) \times g(s_1, \dots, s_{k-j}, t_{n-k+1}, \dots, t_{n+m-2k+j}) ds_1 \cdots ds_{k-j}.$$

For $f \in L^2(\mathbb{R}_+^n)$, we denote by $I_n^S(f)$ the multiple Wigner integral of f with respect to the free Brownian motion as introduced in [Biane and Speicher 1998]. The space $L^2(\mathcal{S}, \varphi) = \{I_n^S(f) : f \in L^2(\mathbb{R}_+^n), n \geq 0\}$ is a unital $*$-algebra, with product rule given, for any $n, m \geq 1$, $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$, by

$$I_n^S(f) I_m^S(g) = \sum_{p=0}^{n \wedge m} I_{n+m-2p}^S(f \overset{p}{\frown} g) \tag{1}$$

and involution $I_n^S(f)^* = I_n^S(f^*)$. For a proof of (1), see [Biane and Speicher 1998].

Similarly, we can define free Poisson multiple integrals with respect to $\hat{N}$ (these integrals were studied in [Bourguin and Peccati 2014], and we refer to this reference for details). The space $L^2(\mathcal{N}, \varphi) = \{I_n^{\hat{N}}(f) : f \in L^2(\mathbb{R}_+^n), n \geq 0\}$ is a unital $*$-algebra, with product rule given, for any $n, m \geq 1$, $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$, by

$$I_n^{\hat{N}}(f) I_m^{\hat{N}}(g) = \sum_{p=0}^{n \wedge m} I_{n+m-2p}^{\hat{N}}(f \overset{p}{\frown} g) + \sum_{p=1}^{n \wedge m} I_{m+n-2p+1}^{\hat{N}}(f \star_p^{p-1} g) \tag{2}$$

and involution $I_n^{\hat{N}}(f)^* = I_n^{\hat{N}}(f^*)$. For a proof of this formula, see [Bourguin and Peccati 2014].

Furthermore, as is well known, both Wigner and free Poisson multiple integrals of different orders are orthogonal in $L^2(\mathscr{A},\varphi)$, whereas for two integrals of the same order, the Wigner isometry holds:

$$\varphi(I_n^{\mathfrak{M}}(f)I_n^{\mathfrak{M}}(g)^*) = \langle f, g\rangle_{L^2(\mathbb{R}_+^n)}. \tag{3}$$

Remark 2.11. (1) Observe that it follows from the definition of the involution on the algebras $L^2(\mathcal{S},\varphi)$ and $L^2(\mathcal{N},\varphi)$ that operators of the type $I_n^{\mathfrak{M}}(f)$ are self-adjoint if and only if f is mirror-symmetric.

(2) In what follows, we will use the notation I_n^S, $I_n^{\hat{N}}$, I_n^W and $I_n^{\hat{\eta}}$ to denote multiple Wigner integrals, multiple free Poisson integrals, multiple Wiener integrals, and multiple classical Poisson integrals, respectively.

Bi-integrals and free gradient operator. In this particular subsection, we only focus on the Wigner case, as the tools we are about to introduce do not exist in the context of free Poisson processes.

Let $(\mathscr{A},\varphi)$ be a W^*-probability space. An $\mathscr{A}\otimes\mathscr{A}$-valued stochastic process $t\mapsto U_t$ is called a biprocess. For $p\geq 1$, U is an element of $\mathscr{B}_p$, the space of L^p-biprocesses, if its norm

$$\|U\|_{\mathscr{B}_p}^2 = \int_0^\infty \|U_t\|_{L^p(\mathscr{A}\otimes\mathscr{A},\varphi\otimes\varphi)}^2\,dt$$

is finite.

Let n, m be two positive integers and $f = g\otimes h\in L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m)$. Then, the Wigner bi-integral $[I_n^S\otimes I_m^S](f)$ is defined as

$$[I_n^S\otimes I_m^S](f) = I_n^S(g)\otimes I_m^S(h).$$

From the Wigner isometry for multiple integrals, we obtain the so-called Wigner bisometry: for $f\in L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m)$ and $g\in L^2(\mathbb{R}_+^{n'})\otimes L^2(\mathbb{R}_+^{m'})$ having the form of a tensor product,

$$\varphi\otimes\varphi([I_n^S\otimes I_m^S](f)[I_{n'}^S\otimes I_{m'}^S](g)^*) = \begin{cases}\langle f,g\rangle_{L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m)} & \text{if } n=n' \text{ and } m=m',\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

Formula (4) is then extended linearly to generic elements $f\in L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m)\cong L^2(\mathbb{R}_+^{n+m})$, where the symbol $\cong$ denotes an isomorphic identification.

A crucial tool in the analysis of Wigner integrals is the product formula (1), and a biproduct formula for bi-integrals was recently obtained in [Bourguin and Campese 2018], which will be a crucial tool in the sequel. It makes use of a new type of contraction, referred to in [Bourguin and Campese 2018] as bi-contractions, defined as follows. Let n_1, m_1, nf_2, m_2 be positive integers. Let

$f \in L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1}) \cong L^2(\mathbb{R}_+^{n_1+m_1})$ and $g \in L^2(\mathbb{R}_+^{n_2}) \otimes L^2(\mathbb{R}_+^{m_2}) \cong L^2(\mathbb{R}_+^{n_2+m_2})$ and let $p \le n_1 \wedge n_2$, $r \le m_1 \wedge m_2$ be natural numbers. The (p,r)-bicontraction $f \overset{p,r}{\frown} g$ is the $L^2(\mathbb{R}_+^{n_1+n_2-2p}) \otimes L^2(\mathbb{R}_+^{m_1+m_2-2r}) \cong L^2(\mathbb{R}_+^{n_1+n_2+m_1+m_2-2p-2r})$ function defined by

$$
\begin{aligned}
&f \overset{p,r}{\frown} g(t_1, \dots, t_{n_1+n_2+m_1+m_2-2p-2r}) \\
&\quad = \int_{\mathbb{R}_+^{p+r}} f\big(t_1, \dots, t_{n_1-p}, s_p, \dots, s_1, y_1, \dots, y_r, \\
&\qquad\qquad\qquad\qquad t_{n_1+n_2+m_2-2p-r+1}, \dots, t_{n_1+n_2+m_1+m_2-2p-2r}\big) \\
&\qquad \times g(s_1, \dots, s_p, t_{n_1-p+1}, \dots, t_{n_1+n_2+m_2-2p-r}, y_r, \dots, y_1)\, ds_1 \cdots ds_p\, dy_1 \cdots dy_r.
\end{aligned}
$$

Remark 2.12. Observe that these bicontractions have the following properties (for a proof, see [Bourguin and Campese 2018]). For $n_1, m_1, n_2, m_2 \in \mathbb{N}$, let $f \in L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1}) \cong L^2(\mathbb{R}_+^{n_1+m_1})$ and $g \in L^2(\mathbb{R}_+^{n_2}) \otimes L^2(\mathbb{R}_+^{m_2}) \cong L^2(\mathbb{R}_+^{n_2+m_2})$ be fully symmetric functions. Furthermore, let $p \le n_1 \wedge n_2$ and $r \le m_1 \wedge m_2$ be natural numbers such that $p + r = p' + r'$. Then, the following holds.

(1) $f \overset{p,r}{\frown} g \cong f \overset{p+r}{\frown} g$.

(2) $f \overset{p,r}{\frown} g = f \overset{p',r'}{\frown} g$.

(3) $\|f \overset{p,r}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n_1+n_2-2p}) \otimes L^2(\mathbb{R}_+^{m_1+m_2-2r})} = \|f \overset{p+r}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n_1+n_2+m_1+m_2-2p-2r})}$.

(4) $f \overset{n_1,m_1}{\frown} f = \|f\|^2_{L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1})} 1 \otimes 1$, which is a constant in $L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1})$.

We introduce $\sharp$ to be the associative action of $\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$ (where $\mathscr{A}^{\mathrm{op}}$ denotes the opposite algebra) on $\mathscr{A} \otimes \mathscr{A}$, as

$$
(A \otimes B) \sharp (C \otimes D) = (AC) \otimes (DB). \tag{5}
$$

We also write $\sharp$ to denote the action of $\mathscr{A} \otimes L^2(\mathbb{R}_+) \otimes \mathscr{A}^{\mathrm{op}}$ on $\mathscr{A} \otimes L^2(\mathbb{R}_+) \otimes \mathscr{A}$, as

$$
(A \otimes f \otimes B) \sharp (C \otimes g \otimes D) = (AC) \otimes fg \otimes (DB).
$$

Using the bicontractions definition, the biproduct formula for Wigner bi-integrals proved in [Bourguin and Campese 2018] can be stated as follows.

Proposition. *For $n_1, m_1, n_2, m_2 \in \mathbb{N}$, let $f \in L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1}) \cong L^2(\mathbb{R}_+^{n_1+m_1})$ and $g \in L^2(\mathbb{R}_+^{n_2}) \otimes L^2(\mathbb{R}_+^{m_2}) \cong L^2(\mathbb{R}_+^{n_2+m_2})$. Then*

$$
[I^S_{n_1} \otimes I^S_{m_1}](f) \sharp [I^S_{n_2} \otimes I^S_{m_2}](g) = \sum_{p=0}^{n_1 \wedge n_2} \sum_{r=0}^{m_1 \wedge m_2} [I^S_{n_1+n_2-2p} \otimes I^S_{m_1+m_2-2r}](f \overset{p,r}{\frown} g). \tag{6}
$$

Finally, the free gradient operator $\nabla : L^2(\mathcal{S}, \varphi) \to \mathscr{B}_2$ is a densely defined and closable operator whose action on Wigner integrals is given by

$$\nabla_t I_n^S(f) = \sum_{k=1}^{n} [I_{k-1}^S \otimes I_{n-k}^S](f_t^{(k)}),$$

where $f_t^{(k)}(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{k-1}, t, x_k, \dots, x_{n-1})$ is viewed as an element of $L^2(\mathbb{R}_+^{k-1}) \otimes L^2(\mathbb{R}_+^{n-k})$. We also define the pairing $\langle \cdot, \cdot \rangle$ between $\mathscr{B}_2 \times \mathscr{B}_2$ and $L^2(\mathscr{A} \otimes \mathscr{A}, \varphi \otimes \varphi)$ to be

$$\langle \cdot, \cdot \rangle : \mathscr{B}_2 \times \mathscr{B}_2 \mapsto L^2(\mathscr{A} \otimes \mathscr{A}, \varphi \otimes \varphi), \quad \langle U, V \rangle = \int_{\mathbb{R}_+} U_s \sharp V_s^* ds. \tag{7}$$

3. Characterizations of freeness

In this section, we are interested in providing several characterizations of freeness between two multiple integrals. We will derive those characterizations in terms of contractions, covariances and free Malliavin gradients respectively.

Characterization in terms of contractions. Recall the well-known characterization of independence of multiple Wiener–Itô integrals by Üstünel and Zakai [1989] in terms of the first contraction of the associated kernels.

Theorem 3.1 [Üstünel and Zakai 1989]. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then, $I_n^W(f)$ and $I_m^W(g)$ are independent if and only if $f \otimes_1 g = 0$ almost everywhere (for the definition of $\otimes_1$, see the first point of Remark 3.2 below).*

Remark 3.2. • In Theorem 3.1 and throughout the text, the notation $\otimes_r$ stands for the usual r-th contraction operator, defined as follows: if $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ are symmetric and if $r \in \{1, \dots, n \wedge m\}$, we set

$$\begin{aligned}(f \otimes_r g)(t_1, \dots, t_{n+m-2r}) = \int_{\mathbb{R}_+^r} & f(t_1, \dots, t_{n-r}, x_1, \dots, x_r) \\ & \times g(t_{n-r+1}, \dots, t_{n+m-2r}, x_1, \dots, x_r) dx_1 \dots dx_r.\end{aligned}$$

• In the context of a multiple Wiener–Itô integral $I_n^W(f)$, note that one can always assume without loss of generality that the kernel f is symmetric, as $I_n^W(f) = I_n^W(\tilde{f})$, where $\tilde{f}$ denotes the symmetrization of the function f given by

$$\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

with $\mathfrak{S}_n$ the symmetric group of $\{1, \dots, n\}$.

A natural question is to ask whether or not the characterization of independence of Üstünel and Zakai has a counterpart in the free setting. It turns out that a similar characterization of freeness holds on both the Wigner and the free Poisson space, which is the first result of this paper.

Theorem 3.3. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then*:

(i) *$I_n^S(f)$ and $I_m^S(g)$ are free if and only if $f \overset{1}{\frown} g = 0$ almost everywhere.*

(ii) *$I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free if and only if $f \star_1^0 g = 0$ almost everywhere.*

Proof. First, assume that $I_n^{\mathfrak{M}}(f)$ and $I_m^{\mathfrak{M}}(g)$ are free. Then, by Definition 2.1, it holds that, in particular

$$\begin{aligned}\varphi\big([I_n^{\mathfrak{M}}(f)^2 - \varphi(I_n^{\mathfrak{M}}(f)^2)][I_m^{\mathfrak{M}}(g)^2 - \varphi(I_m^{\mathfrak{M}}(g)^2)]\big)\\ = \varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2) - \varphi(I_n^{\mathfrak{M}}(f)^2)\varphi(I_m^{\mathfrak{M}}(g)^2) = 0.\end{aligned}$$

Observe that

$$\begin{aligned}\varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2) &= \sum_{p=0}^{n}\sum_{r=0}^{m} \varphi(I_{2n-2p}^{\mathfrak{M}}(f \overset{p}{\frown} f) I_{2m-2r}^{\mathfrak{M}}(g \overset{r}{\frown} g))\\ &\quad + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n}\sum_{r=1}^{m} \varphi(I_{2n-2p+1}^{\mathfrak{M}}(f \star_p^{p-1} f) I_{2m-2r+1}^{\mathfrak{M}}(g \star_r^{r-1} g))\\ &= \sum_{p=0}^{n}\sum_{r=0}^{m} \varphi(I_{2p}^{\mathfrak{M}}(f \overset{n-p}{\frown} f) I_{2r}^{\mathfrak{M}}(g \overset{m-r}{\frown} g))\\ &\quad + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=0}^{n-1}\sum_{r=0}^{m-1} \varphi(I_{2p+1}^{\mathfrak{M}}(f \star_{n-p}^{n-p-1} f) I_{2r+1}^{\mathfrak{M}}(g \star_{m-r}^{m-r-1} g)).\end{aligned}$$

Using the isometry property (3), we get

$$\begin{aligned}\varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2) &= \sum_{p=0}^{n\wedge m} \langle f \overset{n-p}{\frown} f, g \overset{m-p}{\frown} g\rangle_{L^2(\mathbb{R}_+^{2p})}\\ &\quad + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=0}^{(n\wedge m)-1} \langle f \star_{n-p}^{n-p-1} f, g \star_{m-p}^{m-p-1} g\rangle_{L^2(\mathbb{R}_+^{2p+1})}\\ &= \sum_{p=0}^{n\wedge m} \| f \overset{p}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p})} + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n\wedge m} \| f \star_p^{p-1} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})}\\ &= \|f\|^2_{L^2(\mathbb{R}_+^n)} \|g\|^2_{L^2(\mathbb{R}_+^m)} + \sum_{p=1}^{n\wedge m} \| f \overset{p}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p})}\\ &\quad + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n\wedge m} \| f \star_p^{p-1} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})}.\end{aligned}$$

Recalling that $\varphi(I_n^{\mathfrak{M}}(f)^2) = \|f\|^2_{L^2(\mathbb{R}_+^n)}$ and $\varphi(I_m^{\mathfrak{M}}(g)^2) = \|g\|^2_{L^2(\mathbb{R}_+^m)}$ yields

$$\text{(8)}\quad \varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2) - \varphi(I_n^{\mathfrak{M}}(f)^2)\varphi(I_m^{\mathfrak{M}}(g)^2) \\ = \sum_{p=1}^{n\wedge m} \|f \overset{p}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p})} + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n\wedge m} \|f \star_p^{p-1} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})}.$$

As the left-hand side of the above equality is zero, the fact that $f \overset{1}{\frown} g = 0$ a.e. in the Wigner case and $f \star_1^0 g = 0$ a.e. in the free Poisson case follows.

Conversely, assume that $f \overset{1}{\frown} g = 0$ a.e. in the Wigner case and that $f \star_1^0 g = 0$ a.e. in the free Poisson case. According to Definition 2.1 together with the linearity of the functional φ, we must prove that, for any natural number ℓ and for any natural numbers $k_1, \dots, k_{2\ell}$,

$$\varphi([I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \\ \cdots [I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})]) = 0.$$

Remark 3.4. Observe that we only consider an even number of powers k. This comes from the tracial property of the functional φ together with the condition that no two adjacent indices i_j can be equal in Definition 2.1. Indeed, if we consider an odd number of powers k, we would have

$$\varphi([I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \\ \cdots [I_n^{\mathfrak{M}}(f)^{k_{2\ell+1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell+1}})]) \\ = \varphi([I_n^{\mathfrak{M}}(f)^{k_{2\ell+1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell+1}})][I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})] \\ [I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \cdots [I_m^{\mathfrak{M}}(g)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})]),$$

where the first two indices would be the same in the framework of Definition 2.1.

Let $q < k$ be two nonnegative integers. For $0 \le q \le k-1$, define the multisets $S_q^k = \{1, \dots, 1, 0, \dots, 0\}$ where the element 1 has multiplicity q and the element 0 has multiplicity $k-q-1$. Such a set is sometimes denoted $\{(1, q), (0, k-q-1)\}$. We denote the group of permutations of the multiset S_q^k by $\mathfrak{S}_q^k$ and its cardinality is given by the multinomial coefficient

$$\binom{k-1}{q, m-q-1} = \frac{(k-1)!}{q!(k-q-1)!} = \binom{k-1}{q}.$$

Observe that in the definition of the group of permutations of a multiset, each permutation yields a different ordering of the elements of the multiset, which is why the cardinality of $\mathfrak{S}_q^k$ is $\binom{k-1}{q}$ and not $(k-1)!$. Using the Wigner and free Poisson product formulas along with Equation (4.1) in [Nourdin and Peccati 2013] and Lemma 4.1 in [Bourguin 2015], we can write

$$I_n^{\mathfrak{M}}(f)^k = \varphi(I_n^{\mathfrak{M}}(f)^k) + \sum_{r=1}^{kn} I_r^{\mathfrak{M}}(a_r(f)) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{r=1}^{kn} I_r^{\mathfrak{M}}(b_r(f)),$$

where

$$a_r(f) = \sum_{(p_1,\dots,p_{k-1})\in A_r} (\cdots((f \overset{p_1}{\frown} f) \overset{p_2}{\frown} f)\cdots f) \overset{p_{k-1}}{\frown} f$$

with

$$A_r = \left\{(p_1, \dots, p_{k-1}) \in \{0, 1, \dots, n\}^{k-1} : kn - 2\sum_{i}^{k-1} p_i = r\right\}$$

and where (recall Definition 2.10 for the contractions appearing below)

$$b_r(f) = \sum_{q=1}^{k-1} \sum_{\pi\in\mathfrak{S}_q^k} \sum_{(p_1,\dots,p_{k-1})\in B_{r,q}^{\pi}} (\cdots((f \star_{p_1}^{p_1-\pi(1)} f) \star_{p_2}^{p_2-\pi(2)} f)\cdots f) \star_{p_{k-1}}^{p_{k-1}-\pi(k-1)} f$$

with, for each $q = 1, \dots, k-1$ and each $\pi \in \mathfrak{S}_q^k$,

$$B_{r,q}^{\pi} = \left\{(p_1, \dots, p_{k-1}) \in \bigotimes_{s=1}^{k-1} \{\pi(s), \dots, n\} : kn + q - 2\sum_{i}^{k-1} p_i = r\right\}.$$

We get that

$$\begin{aligned}
&[I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \\
&\qquad \cdots [I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})] \\
&= \sum_{r_1=1}^{k_1 n} \sum_{r_2=1}^{k_2 m} \cdots \sum_{r_{2\ell-1}=1}^{k_{2\ell-1} n} \sum_{r_{2\ell}=1}^{k_{2\ell} m} I_{r_1}^{\mathfrak{M}}(a_{r_1}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_1}(f)) \\
&\quad \times I_{r_2}^{\mathfrak{M}}(a_{r_2}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_2}(g)) \cdots I_{r_{2\ell-1}}^{\mathfrak{M}}(a_{r_{2\ell-1}}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{2\ell-1}}(f)) \\
&\qquad \times I_{r_{2\ell}}^{\mathfrak{M}}(a_{r_{2\ell}}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{2\ell}}(g)).
\end{aligned}$$

At this point, observe that the assumptions that $f \overset{1}{\frown} g = 0$ a.e in the Wigner case and $f \star_1^0 g = 0$ a.e in the free Poisson case imply, by Lemmas 6.1 and 6.2 respectively, that for any given $i = 1, \dots, 2\ell-1$, the contractions between $(a_{r_i}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_i}(f))$ and $(a_{r_{i+1}}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{i+1}}(g))$ resulting from using the appropriate product formula iteratively will all be zero a.e. except for the ones of order zero corresponding to the tensor product operation (it is the only contraction that can be nonzero under both the Wigner and free Poisson case assumptions).

Remark 3.5. Note that for the above argument to hold, we need to assume that the functions f and g are symmetric in order to be able to freely reorder variables appearing in the contractions of $a_{r_i}(f)$ and $a_{r_j}(g)$ (as well as in the contractions of $b_{r_{i+1}}(f)$ and $b_{r_{j+1}}(g)$) so that the assumptions $f \overset{1}{\frown} g = 0$ a.e. in the Wigner case and $f \star_1^0 g = 0$ a.e. in the free Poisson case can be used to deduce that the resulting contractions will all be zero.

Hence, keeping only the nonzero terms in the above expression yields

$$\begin{aligned}
&[I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \\
&\qquad \cdots [I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})] \\
&= \sum_{r_1=1}^{k_1 n} \sum_{r_2=1}^{k_2 m} \cdots \sum_{r_{2\ell-1}=1}^{k_{2\ell-1} n} \sum_{r_{2\ell}=1}^{k_{2\ell} m} I^{\mathfrak{M}}_{r_1+\cdots+r_{2\ell}}((a_{r_1}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_1}(f)) \\
&\qquad \otimes (a_{r_2}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_2}(g)) \otimes \cdots \otimes (a_{r_{2\ell-1}}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{2\ell-1}}(f)) \\
&\qquad\qquad \otimes (a_{r_{2\ell}}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{2\ell}}(g))).
\end{aligned}$$

As the quantity $r_1 + \cdots + r_{2\ell}$ is strictly positive, applying φ to the above expression yields

$$\begin{aligned}
&\varphi\big([I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \\
&\qquad \cdots [I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})]\big) = 0,
\end{aligned}$$

which is the desired result. □

Observe that the above characterization of freeness is stated and proven for symmetric kernels only. A natural question is whether or not this characterization continues to hold in the more general case of a mirror-symmetric kernel. We provide a negative answer to this question, proving that our characterization is exhaustive. Concretely, we will exhibit two mirror-symmetric kernels $f, g \in L^2([0,2]^3)$ such that $\|f \overset{1}{\frown} g\|_{L^2([0,2]^3)} = 0$ but $I_3^S(f)$ and $I_3^S(g)$ are not free.

Indeed, consider $f = \mathbb{1}_{[0,1]\times[0,2]\times[0,1]}$ and $g = \mathbb{1}_{[1,2]\times[0,2]\times[1,2]}$. It is readily checked that $f \overset{1}{\frown} g = 0$. On the other hand, using the product formula (1) iteratively, we can write

$$\begin{aligned}
I_3^S(f)^7 &= \sum_{(r_1,\ldots,r_6)\in C} I^S_{21-2r_1-\cdots-2r_6}((((((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f) \overset{r_3}{\frown} f) \overset{r_4}{\frown} f) \overset{r_5}{\frown} f) \overset{r_6}{\frown} f) \\
I_3^S(g)^7 &= \sum_{(r_1,\ldots,r_6)\in C} I^S_{21-2r_1-\cdots-2r_6}((((((g \overset{r_1}{\frown} g) \overset{r_2}{\frown} g) \overset{r_3}{\frown} g) \overset{r_4}{\frown} g) \overset{r_5}{\frown} g) \overset{r_6}{\frown} g),
\end{aligned}$$

where

$$\begin{aligned}
C = \{(r_1, \ldots, r_6) \in \{0,1,2,3\}^6 : r_2 \le 6 - 2r_1, \\
r_3 \le 9 - 2r_1 - 2r_2, \ldots, r_6 \le 18 - 2r_1 - \cdots - 2r_5\}.
\end{aligned}$$

Using the Wigner isometry (3), we deduce that $\varphi(I_3^S(f)^7) = 0$ and $\varphi(I_3^S(g)^7) = 0$, as well as (the functions f and g being positive)

$$\varphi(I_3^S(f)^7 I_3^S(g)^7) \geq \Big\langle \big(((((f \overset{2}{\frown} f) \overset{2}{\frown} f) \overset{1}{\frown} f) \overset{1}{\frown} f) \overset{1}{\frown} f\big) \overset{3}{\frown} f, \\ \big(((((g \overset{2}{\frown} g) \overset{2}{\frown} g) \overset{1}{\frown} g) \overset{1}{\frown} g) \overset{1}{\frown} g\big) \overset{3}{\frown} g \Big\rangle_{L^2([0,2])} = 32 \neq 0.$$

Consequently, according to the definition of freeness given in Definition 2.1, $I_3^S(f)$ and $I_3^S(g)$ are not free.

Remark 3.6. The same counterexample would also yield the same conclusion in the free Poisson case (replacing the Wigner integrals by free Poisson ones) as it is also the case that $f \star_1^0 g = 0$ and as the first part of the free Poisson product formula (2) is the same as the Wigner product formula used above.

However, even if establishing a characterization of freeness in terms of contractions in the mirror-symmetric case is not possible, we can still give a sufficient condition for freeness, which is the object of the following result.

Theorem 3.7. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be mirror-symmetric functions.*

(i) *If dealing with Wigner integrals, assume that $f^{(\sigma)} \overset{1}{\frown} g^{(\pi)} = 0$ almost everywhere for all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$, where*

$$f^{(\sigma)}(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad x_1, \dots, x_n \in \mathbb{R}_+,$$

and a similar definition for $g^{(\pi)}$. Then, $I_n^S(f)$ and $I_m^S(g)$ are free.

(ii) *If dealing with free Poisson integrals, assume that $f^{(\sigma)} \star_1^0 g^{(\pi)} = 0$ almost everywhere for all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$. Then, one has that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free.*

Proof. Apply the same strategy as in the proof of Theorem 3.3 with the stronger assumptions. □

Characterization in terms of covariances. The next result is a free analog of [Rosiński and Samorodnitsky 1999, Corollary 5.2], which is itself a consequence of Theorem 3.1 by Üstünel and Zakai.

Corollary 3.8. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then, $I_n^{\mathfrak{M}}(f)$ and $I_m^{\mathfrak{M}}(g)$ are free if and only if their squares are uncorrelated, i.e., if and only if*

$$\mathrm{Cov}(I_n^{\mathfrak{M}}(f)^2, I_m^{\mathfrak{M}}(g)^2) = 0.$$

Proof. First, assume that $I_n^{\mathfrak{M}}(f)$ and $I_m^{\mathfrak{M}}(g)$ are free. Then, by Definition 2.1,

$$\varphi\big([I_n^{\mathfrak{M}}(f)^2-\varphi(I_n^{\mathfrak{M}}(f)^2)][I_m^{\mathfrak{M}}(g)^2-\varphi(I_m^{\mathfrak{M}}(g)^2)]\big) = \varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2)-\varphi(I_n^{\mathfrak{M}}(f)^2)\varphi(I_m^{\mathfrak{M}}(g)^2)=0.$$

As $\mathrm{Cov}(I_n^{\mathfrak{M}}(f)^2, I_m^{\mathfrak{M}}(g)^2)=\varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2)-\varphi(I_n^{\mathfrak{M}}(f)^2)\varphi(I_m^{\mathfrak{M}}(g)^2)$, the desired conclusion follows.

Conversely, assume that $\mathrm{Cov}(I_n^{\mathfrak{M}}(f)^2, I_m^{\mathfrak{M}}(g)^2)=0$. Using (8), it holds that

$$\mathrm{Cov}(I_n^{\mathfrak{M}}(f)^2, I_m^{\mathfrak{M}}(g)^2) = \sum_{p=1}^{n\wedge m}\|f\overset{p}{\frown}g\|^2_{L^2(\mathbb{R}_+^{n+m-2p})}+\mathbb{1}_{\{\mathfrak{M}=\hat{N}\}}\sum_{p=1}^{n\wedge m}\|f\star_p^{p-1}g\|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})},$$

which implies that all the contraction norms appearing on the right-hand side of the above equality are zero. In particular, in the Wigner case, $\|f\overset{1}{\frown}g\|^2_{L^2(\mathbb{R}_+^{n+m-2})}=0$, and in the free Poisson case, $\|f\star_1^0 g\|^2_{L^2(\mathbb{R}_+^{n+m-1})}=0$, which, by Theorem 3.3 implies that $I_n^{\mathfrak{M}}(f)$ and $I_m^{\mathfrak{M}}(g)$ are free. □

Characterization in terms of free Malliavin gradients. In the context of Wiener integrals, Üstünel and Zakai [1989, Proposition 2] proved that a necessary condition for two Wiener integrals $I_n^W(f)$ and $I_m^W(g)$ to be independent was that the inner product of their Malliavin derivatives was zero almost surely. More precisely, their statement reads as follows.

Theorem 3.9 [Üstünel and Zakai 1989]. *A necessary condition for the independence of $I_n^W(f)$ and $I_m^W(g)$ is*

$$\langle DI_n^W(f), DI_m^W(g)\rangle_{L^2(\mathbb{R}_+)}=0 \quad a.s. \tag{9}$$

However, they were also able to show that this condition is not sufficient and hence cannot provide a proper characterization of independence of Wiener integrals. The technical reason for this is that this condition implies that only the symmetrization of the first contraction of f and g be zero almost everywhere, which in turn does not necessarily imply that the first contraction itself be zero almost everywhere. As the latter is an equivalent statement to independence, the sufficiency of (9) fails.

In the free case, a free version of the Malliavin calculus (with respect to the free Brownian motion) has been developed by Biane and Speicher [1998], and it is a natural question to ask whether it can be used to provide a characterization of freeness for Wigner integrals.

Remark 3.10. In this subsection, we only focus on Wigner integrals and not on the free Poisson case. The reason for this is that there is no free Malliavin calculus available for free Poisson random measures, which is what would be needed to explore similar statements in the free Poisson case.

The following result is the main result of this subsection, which is a characterization of freeness in terms of the free gradient operator for Wigner integrals with symmetric kernels. It is worth noting that, as opposed to the case of Wiener integrals studied by Üstünel and Zakai, we are able to provide a positive answer to the question of characterizing freeness in terms of free gradients, which illustrates a fundamental difference between the classical case and the free case.

Theorem 3.11. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then, $I_n^S(f)$ and $I_m^S(g)$ are free if and only if*

$$\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle = 0 \text{ in } L^2(\mathcal{A}\otimes\mathcal{A}, \varphi\otimes\varphi), \tag{10}$$

where the notation $\langle\,\cdot\,,\cdot\,\rangle$ is defined in (7).

Proof. In the following we will use the shorthand $f_s^{(k)}$ to denote the function given by

$$f_s^{(k)}(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{k-1}, s, x_{k+1}, \dots, x_n).$$

Applying the definition of the action of ∇ on Wigner integrals, we get that

$$\begin{aligned}\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle &= \int_{\mathbb{R}_+} (\nabla_s I_n^S(f))\sharp(\nabla_s I_m^S(g))^*\,ds\\ &= \sum_{k=1}^{n}\sum_{q=1}^{m}\int_{\mathbb{R}_+}[I_{k-1}^S\otimes I_{n-k}^S](f_s^{(k)})\sharp([I_{q-1}^S\otimes I_{m-q}^S](g_s^{(q)}))^*\,ds\\ &= \sum_{k=1}^{n}\sum_{q=1}^{m}\int_{\mathbb{R}_+}[I_{k-1}^S\otimes I_{n-k}^S](f_s^{(k)})\sharp[I_{q-1}^S\otimes I_{m-q}^S](g_s^{(q)})\,ds,\end{aligned}$$

where the last equality follows from the full symmetry of the function g. The biproduct formula (6) yields

$$\begin{aligned}&\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle\\ &\quad= \sum_{k=1}^{n}\sum_{q=1}^{m}\int_{\mathbb{R}_+}\sum_{p=0}^{(k\wedge q)-1}\sum_{r=0}^{(n-k)\wedge(m-q)}[I_{k+q-2-2p}^S\otimes I_{n+m-k-q-2r}^S](f_s^{(k)} \overset{p,r}{\frown} g_s^{(q)})\,ds,\end{aligned}$$

and by using a Fubini argument, it follows that

$$\begin{aligned}&\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle\\ &\quad= \sum_{k=1}^{n}\sum_{q=1}^{m}\sum_{p=0}^{(k\wedge q)-1}\sum_{r=0}^{(n-k)\wedge(m-q)}[I_{k+q-2-2p}^S\otimes I_{n+m-k-q-2r}^S]\left(\int_{\mathbb{R}_+} f_s^{(k)} \overset{p,r}{\frown} g_s^{(q)}\,ds\right).\end{aligned}$$

The full symmetry of f and g implies that $f_s^{(k)} = f_s^{(n)}$ for every $1 \le k \le n$ and $g_s^{(q)} = g_s^{(1)}$ for every $1 \le q \le m$. Hence, using Remark 2.12, we get

$$\int_{\mathbb{R}_+} f_s^{(k)} \overset{p,r}{\frown} g_s^{(q)} ds = f \overset{p+r+1}{\frown} g,$$

so that we finally get

$$\text{(11)} \quad \langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle = \sum_{k=1}^{n} \sum_{q=1}^{m} \sum_{p=0}^{(k\wedge q)-1} \sum_{r=0}^{(n-k)\wedge(m-q)} [I_{k+q-2-2p}^S \otimes I_{n+m-k-q-2r}^S](f \overset{p+r+1}{\frown} g).$$

Using the Wigner bisometry (4), we see that the quantity

$$\varphi \otimes \varphi(|\langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle|^2)$$

is just a sum with strictly positive coefficients only involving the contractions norms

$$\|f \overset{1}{\frown} g\|^2_{L^2(R_+^{n+m-2})}, \|f \overset{2}{\frown} g\|^2_{L^2(R_+^{n+m-4})}, \dots, \|f \overset{n\wedge m}{\frown} g\|^2_{L^2(R_+^{n+m-2(n\wedge m)})}.$$

Formally, we have an equality of the type

$$\text{(12)} \qquad \varphi \otimes \varphi(|\langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle|^2) = \sum_{u=1}^{n\wedge m} c_u \|f \overset{u}{\frown} g\|^2_{L^2(R_+^{n+m-2u})},$$

with $c_u > 0$.

Now assume that $I_n^S(f)$ and $I_m^S(g)$ are free. By Theorem 3.3, this is equivalent to $f \overset{1}{\frown} g = 0$ almost everywhere, which by Lemma 6.1 implies that $f \overset{p}{\frown} g = 0$ almost everywhere for all $1 \le p \le n \wedge m$. Using (12), we get (10).

Conversely, assume that

$$\langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle = 0.$$

Then,

$$\varphi \otimes \varphi(|\langle \nabla I_n^S(f), \nabla I_m^S(g) \rangle|^2) = 0.$$

This implies that all the norms appearing in the representation (12) are zero, and in particular that $f \overset{1}{\frown} g = 0$ almost everywhere. Using Theorem 3.3 concludes the proof. □

4. Characterizations of asymptotic freeness

In the asymptotic context, the problem of interest is to find necessary and sufficient conditions for the limits in law of multiple integrals to be free. It is a much more general problem than before, as limits in law of multiple integrals need not be multiple integrals themselves.

Characterization in terms of contractions. In the classical case, the following result holds.

Theorem 4.1 [Nourdin and Rosiński 2014, Theorem 3.1]. *Let n, m be natural numbers and let $\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$ be sequences of symmetric functions. Assume that $(I_n^W(f_k), I_m^W(g_k)) \xrightarrow{\text{law}} (F, G)$ as $k \to \infty$, where F, G are square integrable random variables with laws determined by their moments. Then, F and G are independent if and only if $f_k \otimes_p g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2p})$ for all $p = 1, \ldots, n \wedge m$.*

Remark 4.2. The fact that the limiting random variables in the above theorem need to have laws determined by their moments (a condition that we get automatically in the free setting) has been shown in [Nourdin et al. 2016] to be not necessary. On the other hand, observe that the necessary and sufficient condition for asymptotic independence is not

$$f_k \otimes_1 g_k \xrightarrow[k\to+\infty]{} 0 \ \text{ in } \ L^2(\mathbb{R}_+^{n+m-2}),$$

as one could have expected in view of Theorem 3.1. This weaker condition is necessary but not sufficient in the asymptotic case, as pointed out in [Nourdin and Rosiński 2014, Remark 3.2]. In the free case, the same phenomenon happens in the sense that the condition $f_k \overset{1}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2})$ (in the Wigner case) and $f_k \star_1^0 g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2})$ (in the free Poisson case) will prove to be necessary but not sufficient either, for the same reason.

The following result in the free case is hence rather an analog of the stronger results of [Nourdin et al. 2016] instead of those found in [Nourdin and Rosiński 2014]. In Theorem 4.1 or in the forthcoming Theorem 4.3, note that F and G do not need to have the form of a multiple integral. This implies that sequences of multiple integrals can be used in order to prove the freeness of general random variables in $L^2(\varphi)$ (provided these random variables admit approximating sequences of multiple integrals with symmetric kernels).

Theorem 4.3. *Let n, m be natural numbers and let $\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$ be sequences of symmetric functions such that*

$$(I_n^{\mathfrak{M}}(f_k), I_m^{\mathfrak{M}}(g_k)) \xrightarrow{\text{law}} (F, G) \tag{13}$$

as $k \to \infty$, where F, G are random variables in $L^2(\mathscr{A}, \varphi)$. Then,

(i) *If $\mathfrak{M} = S$, then F and G are free if and only if $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2p})$ for all $p = 1, \ldots, n \wedge m$.*

(ii) *If $\mathfrak{M} = \hat{N}$, then F and G are free if and only if $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2p})$ and $f_k \star_p^{p-1} g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2p+1})$ for all $p = 1, \ldots, n \wedge m$.*

Proof. First, assume that F and G are free. Then, $\mathrm{Cov}(F^2, G^2) = 0$. Using (8) along with assumption (13) yields

$$\mathrm{Cov}(I_n^{\mathfrak{M}}(f_k)^2, I_m^{\mathfrak{M}}(g_k)^2) = \sum_{p=1}^{n\wedge m} \| f_k \overset{p}{\frown} g_k \|^2_{L^2(\mathbb{R}_+^{n+m-2p})} + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n\wedge m} \| f_k \star_p^{p-1} g_k \|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})} \xrightarrow[k\to+\infty]{} \mathrm{Cov}(F^2, G^2) = 0,$$

so that for all $p = 1, \dots, n\wedge m$, $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ (in the Wigner case) and for all $p = 1, \dots, n\wedge m$, $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ and $f_k \star_p^{p-1} g_k \xrightarrow[k\to+\infty]{} 0$ (in the free Poisson case).

Conversely, assume that, for all $p = 1, \dots, n \wedge m$, $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ (in the Wigner case) or that, for all $p = 1, \dots, n \wedge m$, $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ and $f_k \star_p^{p-1} g_k \xrightarrow[k\to+\infty]{} 0$ (in the free Poisson case). As in the proof of Theorem 3.3 (together with assumption (13)), these conditions imply that, for any natural number ℓ and for any natural numbers $k_1, \dots, k_{2\ell}$,

$$\varphi\big([I_n^{\mathfrak{M}}(f_k)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f_k)^{k_1})][I_m^{\mathfrak{M}}(g_k)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g_k)^{k_2})] \cdots [I_n^{\mathfrak{M}}(f_k)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f_k)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g_k)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g_k)^{k_{2\ell}})]\big) \xrightarrow[k\to+\infty]{} 0,$$

which implies that F and G are free as they are determined by their moments. □

Remark 4.4. Observe that the only difference between the proofs of Theorem 3.3 and Theorem 4.3 is the fact that in the nonasymptotic case, we have one additional step which states that the seemingly weaker condition $f \overset{1}{\frown} g = 0$ a.e. implies that, for all $p = 1, \dots, n\wedge m$, $f \overset{p}{\frown} g = 0$ a.e. (in the Wigner case) and that the condition $f \star_1^0 g = 0$ a.e. implies that, for all $p = 1, \dots, n\wedge m$, $f \overset{p}{\frown} g = 0$ and $f \star_p^{p-1} g = 0$ a.e. (in the free Poisson case). Recall that these implications do not necessarily hold true asymptotically, as pointed out in [Nourdin and Rosiński 2014, Remark 3.2]. For instance, the sequence $\{f_k : n \geq 1\} \subset L^2([0, 1]^2)$ given by

$$f_k = \sqrt{k} \sum_{i=0}^{k-1} \mathbb{1}_{[i/k,(i+1)/k]^2}$$

satisfies $f_k \overset{1}{\frown} f_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^2)$, although $f_k \overset{2}{\frown} f_k = 1$ for all k. As we directly assume the asymptotic equivalent of the conclusions of these implications, the same arguments as in the proof of Theorem 3.3 yield the desired conclusion in the proof of Theorem 4.3.

As before with Theorem 3.7, we can give sufficient conditions for the asymptotic freeness of F and G whenever the sequences of multiple integrals have mirror-symmetric kernels instead of symmetric ones.

Theorem 4.5. *Let n, m be natural numbers and let $\{f_k : k \geq 0\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 0\} \subset L^2(\mathbb{R}_+^m)$ be sequences of mirror-symmetric functions. Assume that $(I_n^{\mathfrak{M}}(f_k), I_m^{\mathfrak{M}}(g_k)) \xrightarrow{\text{law}} (U, V)$ and that $f_k^{(\sigma)} \overset{p}{\frown} g_k^{(\pi)} \to 0$ as $k \to \infty$, for all $p = 1, \dots, n \wedge m$ and all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$, where $f_k^{(\sigma)}$ and $g_k^{(\pi)}$ are defined as in Theorem 3.7. Finally, if dealing with free Poisson integrals, assume moreover that $f_k^{(\sigma)} \star_p^{p-1} g_k^{(\pi)} \to 0$ as $k \to \infty$, for all $p = 1, \dots, n \wedge m$ and all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$. Then U and V are free.*

Proof. Using the exact same argument as in the proof of Theorem 3.3, we can obtain that, for any natural number ℓ and for any natural numbers $p_1, \dots, p_{2\ell}$,

$$\varphi\big([I_n^{\mathfrak{M}}(f_k)^{p_1} - \varphi(I_n^{\mathfrak{M}}(f_k)^{p_1})][I_m^{\mathfrak{M}}(g_k)^{p_2} - \varphi(I_m^{\mathfrak{M}}(g_k)^{p_2})] \\ \cdots [I_n^{\mathfrak{M}}(f_k)^{p_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f_k)^{p_{2\ell-1}})][I_m^{\mathfrak{M}}(g_k)^{p_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g_k)^{p_{2\ell}})]\big) \xrightarrow[k \to +\infty]{} 0.$$

Taking the limit as $k \to \infty$,

$$\varphi\big([U^{p_1} - \varphi(U^{p_1})][V^{p_2} - \varphi(V^{p_2})] \cdots [U^{p_{2\ell-1}} - \varphi(U^{p_{2\ell-1}})][V^{p_{2\ell}} - \varphi(V^{p_{2\ell}})]\big) = 0,$$

which concludes the proof. □

Characterization in terms of covariances. Based on Theorem 4.1, Nourdin and Rosiński [2014, Corollary 3.6] obtained the following result that links component-wise convergence and joint convergence of multiple integrals. As before, note that in the following results, the random variables F and G need not have the form of multiple integrals. This implies that sequences of multiple integrals can be used in order to prove the freeness of general random variables in $L^2(\varphi)$ (provided these random variables admit approximating sequences of multiple integrals with symmetric kernels).

Theorem 4.6. *Let n, m be natural numbers and let $\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$ be sequences of symmetric functions such that $I_n^W(f_k) \xrightarrow{\text{law}} F$ and $I_m^W(g_k) \xrightarrow{\text{law}} G$ as $k \to \infty$, where F, G are square integrable independent random variables with laws determined by their moments. If*

$$\mathrm{Cov}(I_n^W(f_k)^2, I_m^W(g_k)^2) \xrightarrow[k \to +\infty]{} 0,$$

then $(I_n^W(f_k), I_m^W(g_k)) \xrightarrow{\text{law}} (F, G)$, as $k \to \infty$.

In the free case, we obtain the following similar result.

Theorem 4.7. *Let n, m be natural numbers and let*

$$\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n) \quad \text{and} \quad \{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$$

be sequences of symmetric functions such that $(I_n^{\mathfrak{M}}(f_k), I_m^{\mathfrak{M}}(g_k)) \xrightarrow{\text{law}} (F, G)$ as $k \to \infty$. Then, F and G are free if and only if

$$\mathrm{Cov}(I_n^{\mathfrak{M}}(f_k)^2, I_m^{\mathfrak{M}}(g_k)^2) \xrightarrow[k \to +\infty]{} 0.$$

Proof. Combine (8) with Theorem 4.3. □

Characterization in terms of free Malliavin gradients. It is also possible to characterize asymptotic freeness in terms of the free gradient quantity appearing in Theorem 3.11. We offer the following statement.

Theorem 4.8. *Let n, m be natural numbers and let $\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$ be sequences of symmetric functions such that*

$$(I_n^S(f_k), I_m^S(g_k)) \xrightarrow{\text{law}} (F, G)$$

as $k \to \infty$, where F, G are random variables in $L^2(\mathscr{A}, \varphi)$. Then, F and G are free if and only if

$$\langle \nabla I_n^S(f_k), \nabla I_m^S(g_k) \rangle \xrightarrow[k\to+\infty]{} 0 \text{ in } L^2(\mathscr{A} \otimes \mathscr{A}, \varphi \otimes \varphi),$$

where the notation $\langle \cdot, \cdot \rangle$ is defined in (7).

Proof. Combine the representation (12) with Theorem 4.3. □

5. Transfer principles

Since the characterizations of freeness we have obtained in Section 3 involve quantities which are similar whatever the context (classical or free, Brownian or Poisson), it is natural to study possible transfer principles from one setting to another one. It is the goal of this section to study these aspects.

Theorem 5.1. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Assume that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free. Then, $I_n^S(f)$ and $I_m^S(g)$ are free. However, the fact that $I_n^S(f)$ and $I_m^S(g)$ are free does not necessarily imply that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free, as illustrated by Example 5.2.*

Proof. By Theorem 3.3, if $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free, then it holds that $f \star_1^0 g = 0$ a.e. Lemma 6.2 guarantees that $f \star_1^0 g = 0$ a.e. implies $f \overset{1}{\frown} g = 0$ a.e. Using Theorem 3.3 again concludes the proof. □

Example 5.2. Let T be a positive real number and let $f, g \in L^2(\mathbb{R}_+)$ be functions defined by

$$f(x) = x \mathbb{1}_{[0,T]}(x) \quad \text{and} \quad g(x) = \Big(x^2 - \frac{3T}{4} x\Big) \mathbb{1}_{[0,T]}(x).$$

Note that

$$f \overset{1}{\frown} g = \langle f, g \rangle_{L^2(\mathbb{R}_+)} = \int_0^T x\Big(x^2 - \frac{3T}{4}x\Big)\,dx = \int_0^T \Big(x^3 - \frac{3T}{4}x^2\Big)\,dx = 0$$

whereas

$$f \star_1^0 g(x) = f(x) \cdot g(x) = \Big(x^3 - \frac{3T}{4}x^2\Big) \mathbb{1}_{[0,T]}(x) \neq 0.$$

Hence, by Theorem 3.3, $I_1^S(f)$ and $I_1^S(g)$ are free but $I_1^{\hat{N}}(f)$ and $I_1^{\hat{N}}(g)$ are not.

Based on Theorems 3.1 and 3.3, we can obtain the following transfer principles between the Wiener and Wigner chaos.

Proposition. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. It holds that $I_n^S(f)$ and $I_m^S(g)$ are free if and only if $I_n^W(f)$ and $I_m^W(g)$ are independent.*

Proof. Observe that as f and g are symmetric functions; it holds that $f \otimes_1 g = f \overset{1}{\frown} g$. Using Theorems 3.1 and 3.3 concludes the proof. □

Remark 5.3. In the classical Poisson case, there is no known characterization of independence in terms of the almost sure nullity of a contraction. By using similar techniques to the ones used in the proof of Theorem 3.3 (using the definition of moment independence in place of the definition of freeness), one can prove that the condition $f \star_1^0 g = 0$ a.e. implies moment independence. However, moment independence only implies $\widetilde{f \star_1^0 g} = 0$ a.e., which is weaker than $f \star_1^0 g = 0$ a.e. Summing up, one can prove that the condition $f \star_1^0 g = 0$ a.e. is sufficient but not necessary and that the condition $\widetilde{f \star_1^0 g} = 0$ a.e. is necessary but not sufficient (the fact that it is not sufficient is illustrated by the counterexample provided in [Rosiński and Samorodnitsky 1999, Example 5.3]). Also pointed out therein is the fact that the squares of multiple Poisson integrals being uncorrelated does not imply that these multiple integrals are independent. This makes it difficult to establish any independence correspondence or transfer principles between the classical and free Poisson chaos. However, it can be pointed out that the freeness of free Poisson multiple integrals implies the freeness of the corresponding Wigner integrals and the independence of the corresponding Wiener integrals.

Despite the above remark, we can still provide the following partial transfer result.

Corollary 5.4. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Assume that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free. Then, $I_n^{\hat{\eta}}(f)$ and $I_m^{\hat{\eta}}(g)$ are moment independent.*

Proof. Assuming $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free, Theorem 3.3 states that $f \star_1^0 g = 0$ a.e., which, as pointed out in Remark 5.3, is a sufficient condition for $I_n^{\hat{\eta}}(f)$ and $I_m^{\hat{\eta}}(g)$ to be moment independent. Conversely, if $I_n^{\hat{\eta}}(f)$ and $I_m^{\hat{\eta}}(g)$ are moment independent and $f \star_1^0 g = 0$ a.e., Theorem 3.3 ensures that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free. □

6. Auxiliary results

This last section contains two auxiliary results that have been used along the proof of Theorem 3.3.

Lemma 6.1. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be mirror-symmetric functions. Assume furthermore that $f \overset{1}{\frown} g = 0$ almost everywhere. Then, for all $p = 1, \dots, n \wedge m$, it holds that $f \overset{p}{\frown} g = 0$ almost everywhere.*

Proof. Observe that, for any $p = 1, \dots, n \wedge m$,

$$\begin{aligned}
&f \overset{p}{\frown} g(t_1, \dots, t_{n+m-2p}) \\
&= \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_p, \dots, s_1) g(s_1, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p \\
&= \int_{\mathbb{R}_+^{p-1}} \left(\int_{\mathbb{R}_+} f(t_1, \dots, t_{n-p}, s_p, \dots, s_1) \right. \\
&\qquad\qquad \left. g(s_1, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \right) ds_2 \cdots ds_p \\
&= \int_{\mathbb{R}_+^{p-1}} f \overset{1}{\frown} g(t_1, \dots, t_{n-p}, s_p, \dots, s_2, s_2, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_2 \cdots ds_p.
\end{aligned}$$

Using the assumption that $f \overset{1}{\frown} g = 0$ a.e., we get $f \overset{p}{\frown} g = 0$ a.e., which concludes the proof. □

Lemma 6.2. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be mirror-symmetric functions. Assume furthermore that $f \star_1^0 g = 0$ almost everywhere. Then, for all $p = 1, \dots, n \wedge m$ and all $r = 2, \dots, n \wedge m$, it holds that $f \overset{p}{\frown} g = 0$ and $f \star_r^{r-1} g = 0$ almost everywhere.*

Proof. Observe that, for any $p = 1, \dots, n \wedge m$,

$$\begin{aligned}
&f \overset{p}{\frown} g(t_1, \dots, t_{n+m-2p}) \\
&\quad = \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_p, \dots, s_1) g(s_1, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p \\
&\quad = \int_{\mathbb{R}_+^p} f \star_1^0 g(t_1, \dots, t_{n-p}, s_p, \dots, s_1, s_2, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p.
\end{aligned}$$

Similarly, it holds that, for any $r = 2, \dots, n \wedge m$,

$$\begin{aligned}
&f \star_r^{r-1} g(t_1, \dots, t_{n+m-2r+1}) \\
&\quad = \int_{\mathbb{R}_+^{r-1}} f(t_1, \dots, t_{n-r+1}, s_{r-1}, \dots, s_1) \\
&\qquad\qquad g(s_1, \dots, s_{r-1}, t_{n-r+1}, \dots, t_{n+m-2r+1})\, ds_1 \cdots ds_{r-1} \\
&\quad = \int_{\mathbb{R}_+^{r-1}} f \star_1^0 g(t_1, \dots, t_{n-r+1}, s_{r-1}, \dots, s_1, s_2, \dots, s_{r-1}, \\
&\qquad\qquad t_{n-r+1}, \dots, t_{n+m-2r+1})\, ds_1 \cdots ds_{r-1}.
\end{aligned}$$

Using the assumption that $f \star_1^0 g = 0$ a.e. concludes the proof. □

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