

*Pacific  
Journal of  
Mathematics*

**ON THE FINE EXPANSION OF  
THE UNIPOTENT CONTRIBUTION OF  
THE GUO–JACQUET TRACE FORMULA**

PIERRE-HENRI CHAUDOUARD

# ON THE FINE EXPANSION OF THE UNIPOTENT CONTRIBUTION OF THE GUO–JACQUET TRACE FORMULA

PIERRE-HENRI CHAUDOUARD

**For a useful class of functions (containing functions whose one finite component is essentially a matrix coefficient of a supercuspidal representation), we establish three results about the unipotent contribution of the Guo–Jacquet relative trace formula for the pair  $(\mathrm{GL}_n(D), \mathrm{GL}_n(E))$ . First we get a fine expansion in terms of global nilpotent integrals. Second we express these nilpotent integrals in terms of zeta integrals. Finally we prove that they satisfy certain homogeneity properties. The proof is based on a new kind of truncation introduced in a previous article.**

## 1. Introduction

### 1.1. *The statement.*

**1.1.1.** Let  $E/F$  be a quadratic extension of number fields. Let  $\varepsilon \in E^\times$  such that  $\mathrm{trace}_{E/F}(\varepsilon) = 0$ . Let  $n \geq 1$  be an integer. Let  $G = \mathrm{GL}(2n, F)$  and  $H = \mathrm{GL}(n, E)$  viewed as an algebraic group over  $F$ . The  $F$ -basis  $(1, \varepsilon)$  identifies  $E^n$  with  $F^{2n}$  and gives an embedding  $H \subset G$ . We identify  $\varepsilon$  with a scalar matrix in  $\mathrm{GL}(n, E)$ . Then  $H$  is the centralizer of  $\varepsilon$  in  $G$ . Moreover, the quotient  $G/H$  is a symmetric space which is easily identified with the subvariety  $S \subset G$  of  $g \in G$  such that  $\varepsilon^{-1}g\varepsilon = g^{-1}$ . The group  $H$  acts on  $S$  by conjugation.

**1.1.2.** Let  $\mathbb{A}$  be the ring of adèles of  $F$  and  $|\cdot|$  be the product of local normalized absolute values. We identify  $\varepsilon$  with a central element in  $\mathrm{GL}(n, E)$ . For any cuspidal automorphic form  $\varphi$  on  $G(\mathbb{A})$ , one can define its  $H$ -period as

$$\mathcal{P}_H(\varphi) = \int_{H(F) \backslash H(\mathbb{A})^1} \varphi(h) dh,$$

where  $H(\mathbb{A})^1 \subset H(\mathbb{A})$  is the kernel of the morphism  $h \mapsto |\det(h)|$  (see [Section 2.1.4](#)) and  $dh$  is an invariant measure. A cuspidal automorphic representation is said to

*MSC2010:* 11F70.

*Keywords:* Guo–Jacquet trace formula, relative trace formula, unipotent contribution, orbital integrals, distinction problems, periods, automorphic forms.

be  $H$ -distinguished if  $\mathcal{P}_H$  induces a nonzero linear form on its underlying space. A fundamental question is to understand the interplay between the  $H$ -distinction, the Jacquet–Langlands functoriality and distinction by some other related subgroups: a beautiful answer is given by the so-called Guo–Jacquet conjecture (see the conjecture in [Guo 1996] extrapolating on results of Waldspurger and Jacquet).

**1.1.3.** A promising tool to study this problem is the so-called Guo–Jacquet trace formula (a specific example of a relative trace formula) based on the seminal work of Jacquet [1986]. A simple form of it has already been used successfully (see [Feigon et al. 2018]). It consists in expressing geometrically and spectrally the integral

$$(1-1-3-1) \quad \int_{H(F)\backslash H(\mathbb{A})^1} \sum_{\gamma \in S(F)} \Phi(h^{-1}\gamma h) dh,$$

where  $\Phi \in C_c^\infty(S(\mathbb{A}))$ . In general, (1-1-3-1) is not convergent. To remedy this problem, the simple trace formula introduces severe restrictions at two places on the function  $\Phi$  so that on the spectral side only the cuspidal spectrum appears and on the geometric side only regular semisimple  $\gamma$ 's contribute to the rational sum in (1-1-3-1). However, for the purpose of convergence, it suffices to impose a certain local condition at one place on  $\Phi$ . When it is satisfied, we shall say that  $\Phi$  is very cuspidal. In the paper, this is the local condition (3-2-2-1) below. Here it suffices to say that the condition is satisfied if the local component of  $\Phi$  at a finite place  $v$  is obtained by integration of a matrix coefficient of a supercuspidal representation of  $G(F_v)/Z(F_v)$  (see Remark 3.2.2.1,  $Z$  is the center of  $G$ ).

For a very cuspidal  $\Phi$ , as a part of (1-1-3-1), we have the (convergent) unipotent contribution defined by

$$(1-1-3-2) \quad \int_{H(F)\backslash H(\mathbb{A})^1} \sum_{X \in \mathcal{N}(F)} \Phi(h^{-1} \exp(X)h) dh,$$

where  $\mathcal{N}(F)$  is the cone of nilpotent matrices in the tangent space  $\mathfrak{s}$  of  $S$  at identity (given by matrices  $X$  of size  $2n$  such that  $\varepsilon X + X\varepsilon = 0$ ) and  $\exp$  is the usual exponential. We emphasize that besides regular semisimple terms the unipotent contribution is certainly the most important contribution of the geometric side of the Guo–Jacquet trace formula to understand (for general terms there should be some kind of Jordan decomposition).

**1.1.4.** The goal of the paper is to prove the following properties of the unipotent contribution for a very cuspidal  $\Phi$ .

**Theorem 1.1.4.1** (see Theorem 3.2.4.1 for a more precise and general statement). *Let  $\Phi \in C_c^\infty(G(\mathbb{A}))$  be a very cuspidal function. Let  $\mathcal{O} \in \mathcal{N}(F)/H(F)$  be a nilpotent orbit and let  $v$  be a place of  $F$ .*

(1) For any  $t \in F_v^\times$ , the integral

$$J_{\mathcal{O}}^t(\Phi) = \int_{[H]^1} \sum_{X \in \mathcal{O}} \Phi(h^{-1} \exp(t^{-1}X)h) dh$$

is absolutely convergent. In particular we get  $J_{\mathcal{O}}^1(\Phi)$  for  $t = 1$ .

(2) We have

$$\int_{H(F) \backslash H(\mathbb{A})^1} \sum_{X \in \mathcal{N}(F)} \Phi(h^{-1} \exp(X)h) dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} J_{\mathcal{O}}^1(\Phi).$$

(3) There exists a bound  $\eta > 0$  such that for any  $t$  in  $F_v^\times$  such that  $|t|_v < \eta$ , we have

$$J_{\mathcal{O}}^t(\Phi) = \lim_{s \rightarrow 0^+} s \theta_{\mathcal{O}}(s) Z_{\mathcal{O}}(\Phi, s),$$

where

- $\theta_{\mathcal{O}}(s)$  is some holomorphic function which is defined for  $s \in \mathbb{C}$  such that its real part satisfies  $\Re(s) > 0$  and which does not depend on  $\Phi$ ,
- the function  $Z_{\mathcal{O}}(\Phi, s)$  is a zeta function attached to  $\Phi$  of the variable  $s \in \mathbb{C}$ , holomorphic for  $\Re(s) > 0$ .

(4) Let  $\eta$  be the above bound. For any  $t_0, t$  in  $F_v^\times$  such that  $|t_0|_v < \eta$  and  $|t|_v \leq 1$  we have

$$J_{\mathcal{O}}^{tt_0}(\Phi) = |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}^{t_0}(\Phi).$$

**Remark 1.1.4.2.** Assertions (1) and (2) provide a “fine expansion” of the unipotent contribution that is an expansion according conjugacy classes. Assertion (3) gives a way to compute each nilpotent contribution; moreover the zeta function  $Z_{\mathcal{O}}(\Phi, s)$  admits a Eulerian product so it is possible to give an expression of each nilpotent contribution in terms of local objects.

**Remark 1.1.4.3.** In the paper, the theorem is in fact stated and proved in the broader situation where  $G = \text{GL}_n(D)$ , where  $D$  is a quaternion algebra containing  $E$ . It should also hold in the situation where  $G$  is the multiplicative group of an  $F$ -simple central algebra containing  $E$  and  $H$  is the centralizer of  $E$  in  $G$ . The methods of the article should also apply in this context but we have not written the details.

We have the following corollary.

**Corollary 1.1.4.4** (see [Corollary 3.2.4.2](#)). *Let  $\Phi \in C_c^\infty(G(\mathbb{A}))$  be a very cuspidal function. Let  $v$  be a place of  $F$ . When  $t \in F_v^\times$  goes to 0, the expression*

$$\int_{[H]^1} \sum_{X \in \mathcal{N}(F)} \Phi(h^{-1} \exp(t^{-1}X)h) dh$$

is equivalent to

$$\text{vol}([H]^1)\Phi(1).$$

**Remark 1.1.4.5.** One of the motivations for this result is that it plays a role in Xue’s approach of local distinction problems. For example, [Corollary 1.1.4.4](#) is used to prove the existence of  $H$ -distinguished cuspidal automorphic representations of  $G$  with local supercuspidal components at some places (see [Corollary 6.2](#) of [[Xue 2019](#)]). Another potential application of [Theorem 1.1.4.1](#) and its corollary is the establishment of some kind of (relative) Weyl’s law.

**1.2. The methods.**

**1.2.1.** The Guo–Jacquet trace formula in our context should share many properties with the usual Arthur–Selberg trace formula. Recall that the unipotent contribution of the Arthur–Selberg trace formula has a fine expansion in terms of local unipotent integrals (although some global constants remain unknown; for some progress on these questions see [[Chaudouard 2017; 2018a](#)]). As a consequence it does satisfy general homogeneity properties (see [[Arthur 1985](#)]). We could have tried to prove analogs of several results of Arthur in our context (mainly the results of [[1978; 1988; 1985](#)]). In this paper we choose to offer a rather simple proof of [Theorem 1.1.4.1](#) based on ideas developed in [[Chaudouard 2018b](#)]. The payoff is that we get a (in principle) computable expression (rather than unknown global coefficients).

**1.2.2.** First one reduces the problem to the case of the infinitesimal situation where  $H$  acts on the tangent space  $\mathfrak{s}$ . This reduction is the content of the final section, [Section 3](#). We introduce in [Section 2.3.3](#) the set of weakly cuspidal functions  $f \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$  that satisfy some vanishing conditions. For such functions, we have

$$(1-2-2-1) \quad \int_{[H]^1} \sum_{X \in \mathcal{N}(F)} f(\text{Ad}(h^{-1})X) dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} \int_{[H]^1} \sum_{X \in \mathcal{O}} f(\text{Ad}(h^{-1})X) dh;$$

on the right-hand side we sum terms that satisfy the nontrivial convergence result (see [Theorem 2.3.4.1](#))

$$\int_{[H]^1} \left| \sum_{X \in \mathcal{O}} f(\text{Ad}(h^{-1})X) \right| dh < \infty.$$

However these terms are difficult to compute since one cannot permute the sum over  $X$  and the integral. This is where enters the truncation of [[Chaudouard 2018b](#)]. It enables to recover each term as a limit

$$\lim_{s \rightarrow 0} s J_{\mathcal{O}}(f, s),$$

where  $J_{\mathcal{O}}(f, s)$  is a holomorphic function for  $s \in \mathbb{C}$  of real part  $\Re(s) > 0$  (see [Theorem 2.5.2.1](#)). The point is that  $J_{\mathcal{O}}(f, s)$  can be expressed in terms of a zeta

integral under a mild assumption on the support of  $f$  (see [Theorem 2.6.4.1](#)). For this zeta integral, the homogeneity property is easy to check (see [Lemma 2.6.3.1](#)).

## 2. Infinitesimal situation

### 2.1. Notation.

**2.1.1.** Let  $F$  be a number field and  $\tau \in F$  such that  $E = F[\sqrt{\tau}]$  is a quadratic extension. Let  $\sigma$  be the generator of the Galois group  $\text{Gal}(E/F)$ . For each place  $v$  of  $F$ , let  $|\cdot|_v$  be the normalized absolute value.

Let  $\mathbb{A}$  be the ring of adèles of  $F$ . Let  $\mathbb{A}^\times$  be the multiplicative group of  $\mathbb{A}$ . For any  $x = (x_v)_v \in \mathbb{A}^\times$ , we shall denote by  $|x|$  the product  $\prod_v |x|_v$  over all places of  $F$ .

For any algebraic group  $G$  over  $F$ , we shall denote the quotient  $G(F)\backslash G(\mathbb{A})$  by  $[G]$ .

**2.1.2.** Let  $V_F$  be a vector space of dimension  $n$  over  $F$ . Let  $V = V_F \otimes_F E$ . By abuse of notation, we denote by  $\sigma$  the  $F$ -automorphism of  $V$  given by  $\text{Id}_{V_F} \otimes_F \sigma$ . Let  $\mathfrak{s} \subset \text{End}_F(V)$  be the  $F$ -subspace of  $\sigma$ -linear endomorphisms of  $V$ , namely the space of maps  $X$  of  $V$  into itself such that  $X \circ \sigma$  is  $E$ -linear.

**2.1.3.** Let  $H$  be the algebraic group of automorphisms of the  $E$ -vector space  $V$ . By restriction of scalars, we will view  $H$  as an  $F$ -group. The group  $H$  is provided with a Galois  $F$ -automorphism still denoted by  $\sigma$ . The group  $H$  acts by conjugation on  $\text{End}_F(V)$ : we denote by  $\text{Ad}$  the restriction of this action on  $\mathfrak{s}$ . This action is defined over  $F$ . By differentiation, we get an action  $\mathfrak{h} \times \mathfrak{s} \rightarrow \mathfrak{s}$  denoted by  $(X, Y) \mapsto [X, Y]$ . For any  $X \in \mathfrak{s}$ , let  $H_X$  be the centralizer of  $X$  in  $H$ .

**2.1.4.** By abuse of notation, we simply denote by  $\det$  the  $F$ -morphism  $H \rightarrow \mathbb{G}_{m,F}$  given by  $\text{Norm}_{E/F} \circ \det$ . Let  $H(\mathbb{A})^1$  be the kernel of  $h \mapsto |\det(h)|$ . We shall denote by  $[H]^1$  the quotient  $H(F)\backslash H(\mathbb{A})^1$ .

**2.1.5. Parabolic decomposition.** We fix an ordered  $F$ -basis  $(e_1, \dots, e_n)$  of  $V_F$ . By a (standard) parabolic subgroup  $P$ , we mean a subgroup which stabilizes an incomplete standard flag of  $E$ -subspaces

$$V_0 = (0) \subsetneq V_1 \subsetneq V_1 \oplus V_2 \subsetneq \dots \subsetneq V_1 \oplus \dots \oplus V_r,$$

where for  $1 \leq i \leq r$  the  $E$ -vector space  $V_i$  is generated by vectors

$$e_{d_i+1}, e_{d_i+2}, \dots, e_{d_i+\dim(V_i)}, \quad \text{where } d_i = \sum_{j=1}^{i-1} \dim(V_j).$$

Then we have a standard Levi decomposition  $P = MN$ , where  $N$  is the unipotent radical of  $P$  and  $M$  is the common stabilizer of the subspaces  $V_i$ . We define  $\mathfrak{s}_M$  and  $\mathfrak{s}_N$  to be the subspaces of  $X \in \mathfrak{s}$  such that  $XV_i \subset V_i$  and  $X(V_1 \oplus \dots \oplus V_i) \subset$

$V_1 \oplus \cdots \oplus V_{i-1}$ , respectively, for all  $1 \leq i \leq r$ . Let  $\mathfrak{s}_P = \mathfrak{s}_M \oplus \mathfrak{s}_N$ . The groups  $P$ ,  $M$  and  $N$  act respectively on  $\mathfrak{s}_P$ ,  $\mathfrak{s}_M$  and  $\mathfrak{s}_N$ .

The stabilizer of the complete standard flag is denoted by  $B$ . We have a standard Levi decomposition  $B = TN_B$ .

**2.1.6. Maximal compact subgroup.** Thanks to the basis of [Section 2.1.5](#), we identify  $H(\mathbb{A})$  with  $\mathrm{GL}(n, \mathbb{A}_E)$ . Let  $K \subset H(\mathbb{A})$  be the maximal compact subgroup corresponding to the standard maximal compact subgroup of  $\mathrm{GL}(n, \mathbb{A}_E)$ . For parabolic subgroups  $P = MN$ , we have the Iwasawa decomposition  $H(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})K$ .

**2.1.7. Haar measures.** We fix a Haar measure on  $\mathbb{A}_E^\times$  the multiplicative group of the adèles of  $E$ . For any  $n \geq 1$ , we take on  $(\mathbb{A}_E^\times)^n$  the product of the Haar measure. On any unipotent group  $N$  over  $F$ , we take the Haar measure on  $N(\mathbb{A})$  such that the quotient measure on  $[N]$  gives the total volume 1. The Haar measure on the standard compact maximal of  $\mathrm{GL}(n, \mathbb{A}_E)$  is such that the total volume is 1. Finally we require that the Haar measure on  $\mathrm{GL}(n, \mathbb{A}_E)$  is compatible with the Iwasawa decomposition, which normalizes the measure if we view  $(\mathbb{A}_E^\times)^n$  as a minimal Levi subgroup. In this way, we get a normalization of the Haar measure on  $H(\mathbb{A})$ . The Haar measure on  $H(\mathbb{A})^1$  is such that the quotient measure on  $H(\mathbb{A})/H(\mathbb{A})^1 \simeq \mathbb{R}_+^\times$  is the usual Haar measure. Then we get quotient measures on  $[H]$  and  $[H]^1$ .

**2.1.8. Categorical quotient.** Let  $C : \mathfrak{s} \rightarrow \mathfrak{c} = \mathfrak{s}/H$  be the categorical quotient. The quotient  $\mathfrak{c}$  can be identified with the standard  $n$ -dimensional affine space over  $F$  in such a way that  $c$  is the map that associates to any  $X \in \mathfrak{s}$  the coefficients of the characteristic polynomial of  $X^2 \in \mathrm{End}_E(V)$  (the coefficients are in fact defined over  $F$ ). For any  $c \in \mathfrak{c}$ , let  $\mathfrak{s}_c$  be the fiber of  $C$  above  $c$ . In particular, if  $c$  corresponds to the polynomial  $t^n$ , we denote by  $\mathcal{N}$  the fiber  $\mathfrak{s}_c$ . Then an element  $X \in \mathfrak{s}$  belongs to  $\mathcal{N}$  if and only if the  $E$ -endomorphism  $X^2$  of  $V$  is nilpotent in the usual sense. By abuse, in the following,  $\mathcal{N}$  will be called the nilpotent cone and the elements of  $\mathcal{N}$  will be called the nilpotent elements of  $\mathfrak{s}$ .

## 2.2. Induction of nilpotent orbits.

**2.2.1. Classification of nilpotent orbits.** It is given by the following lemma.

**Lemma 2.2.1.1.** *For any  $X \in \mathcal{N}$ , there exists an  $E$ -basis of  $V$  such that the matrix of  $X$  is in the Jordan normal form.*

*There are finitely many orbits of  $H$  on  $\mathcal{N}$  and they are classified by their Jordan normal form.*

*Proof.* One can find a proof in [\[Guo 1997, Lemma 2.3\]](#). Alternatively, one can observe that  $V$  has a  $E[t]$ -module structure of  $t^{2n}$ -torsion given by  $(P, v) \mapsto P(X)v$  which boils down to the classification of torsion modules over a principal ideal domain.  $\square$

**2.2.2. Induction à la Lusztig–Spaltenstein.** We state and prove in our context some results about induction of nilpotent orbits analogous to those of [Lusztig and Spaltenstein 1979].

Let  $P = MN$  be a parabolic subgroup of  $H$  as above. Let  $X \in \mathfrak{s}_M$  be a nilpotent element. Let  $\mathcal{O}_X^M$  be the  $M$ -orbit of  $X$ . The variety  $\mathcal{O}_X^M \oplus \mathfrak{s}_N$  is irreducible. Since there are finitely many nilpotent orbits, there is a unique nilpotent  $H$ -orbit  $\mathcal{O}$  such that

$$\mathcal{O} \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)$$

is a Zariski open dense subset of  $\mathcal{O}_X^M \oplus \mathfrak{s}_N$ . We denote  $\mathcal{O}$  by  $I_P(X)$  and we shall call it *the induced orbit*.

**Proposition 2.2.2.1.** *The intersection  $I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)$  is a single  $P$ -orbit.*

*Proof.* Let  $Z \in \mathfrak{s}_N$  such that the element  $Y = X + Z$  belongs to  $I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)$ . Let  $\mathcal{B}$  be the variety of complete flags of  $E$ -subspaces of  $V$ . As usual, we view  $\mathcal{B}$  as an  $F$ -variety. Let  $\mathcal{B}_Y$  be the subvariety of complete flags  $V_\bullet$  such that  $YV \subset V$ .

It is not difficult to compute the dimensions of  $\mathcal{B}_Y$  and  $H_Y$  in terms of the Jordan decomposition of  $Y$  (one can use the method of [Spaltenstein 1982]). Then one can check the equality

$$(2-2-2-1) \quad \dim(H_Y) = 2 \dim(\mathcal{B}_Y) + \dim(T).$$

By using the group  $B$  as a base point, we get an equivariant map  $\pi_1 : H \rightarrow \mathcal{B}$ . Let  $\pi_2$  be the map from  $H$  to the  $H$ -orbit  $\mathcal{O}_Y$  of  $Y$  given by  $h \mapsto h^{-1}Yh$ . We have  $\pi_2^{-1}(\mathcal{O}_Y \cap \mathfrak{s}_B) = \pi_1^{-1}(\mathcal{B}_Y)$ . The dimension of the fibers of  $\pi_1$  and  $\pi_2$  are respectively  $\dim(B)$  and  $\dim(H_Y)$ . We deduce that

$$(2-2-2-2) \quad \dim(\mathcal{O}_Y \cap \mathfrak{s}_B) + \dim(H_Y) = \dim(\mathcal{B}_Y) + \dim(B).$$

Combining (2-2-2-1) and (2-2-2-2), we get

$$(2-2-2-3) \quad \dim(\mathcal{B}_Y) + \dim(\mathcal{O}_Y \cap \mathfrak{s}_B) = \dim(N_B).$$

We can also define a variety  $\mathcal{B}_X^M$  relative to  $M$  and  $X$ . We have an identification

$$\mathcal{B}_X^M \simeq \mathcal{B}_Y^P = \mathcal{B}_Y \cap \mathcal{B}^P,$$

where  $\mathcal{B}^P$  is the variety of the complete flags that refine the flag associated to  $P$ . In particular,  $\dim(\mathcal{B}_X^M) \leq \dim(\mathcal{B}_Y)$ . As before, one proves:

$$\dim(\mathcal{B}_X^M) + \dim(\mathcal{O}_X^M \cap \mathfrak{s}_{M \cap N_B}) = \dim(N_B \cap M).$$

Following the proof of theorem 1.3 in [Lusztig and Spaltenstein 1979], we have

$$\begin{aligned} \dim(\mathcal{B}_Y) + \dim(\mathcal{O}_Y \cap \mathfrak{s}_B) &\geq \dim(\mathcal{B}_X^M) + \dim((\mathcal{O}_X^M \oplus \mathfrak{s}_N) \cap \mathfrak{s}_B) \\ &= \dim(\mathcal{B}_X^M) + \dim(\mathcal{O}_X^M \cap \mathfrak{s}_{M \cap N_B}) + \dim(\mathfrak{s}_N) \\ &= \dim(N_B \cap M) + \dim(\mathfrak{s}_N) \\ &= \dim(N_B). \end{aligned}$$

By (2-2-2-3), all the inequalities are in fact equalities. In particular, we deduce  $\dim(\mathcal{B}_Y) = \dim(\mathcal{B}_X^M)$  and thus  $\dim(H_Y) = \dim(M_X)$  where  $M_X$  is the centralizer of  $X$  in  $M$ .

Let  $\mathcal{O}_Y^P$  be the  $P$ -orbit of  $Y$  and  $P_Y$  be its centralizer in  $P$ . We have

$$\begin{aligned} \dim(I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)) &\geq \dim(\mathcal{O}_Y^P) = \dim(P) - \dim(P_Y) \\ &\geq \dim(P) - \dim(H_Y) \\ &= \dim(P) - \dim(M_X) \\ &= \dim(\mathcal{O}_X^M \oplus \mathfrak{s}_N) \\ &= \dim(I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)). \end{aligned}$$

Thus we have  $\dim(\mathcal{O}_Y^P) = \dim(I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N))$ . But one gets the same result for any  $Y' \in I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)$ . Thus the orbits  $\mathcal{O}_Y^P$  and  $\mathcal{O}_{Y'}^P$  must intersect by irreducibility of  $I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)$ .  $\square$

**2.2.3.** The following lemma is a variant of [Chaudouard 2018a, lemme 2.9.1].

**Lemma 2.2.3.1.** *There exists a finite family of polynomial maps  $(\Phi_i)_{i \in I}$  on  $\mathfrak{s}_M \oplus \mathfrak{s}_N$  such that for any nilpotent  $M$ -orbit  $\mathcal{O}$  in  $\mathfrak{s}_M$ , there exists  $I_{\mathcal{O}} \subset I$  that satisfies the following two properties:*

- (1) *For any  $X \in \mathcal{O}$  and  $Y \in \mathfrak{s}_N$ , one has  $X + Y \in I_P(X)$  if and only if there exists  $i \in I_{\mathcal{O}}$  such that  $P_i(X, Y) \neq 0$ .*
- (2) *For any  $X \in \mathcal{O}$ , there exists  $i \in I_{\mathcal{O}}$  such that the restriction of  $P_i(X, \cdot)$  to  $\mathfrak{s}_N$  is nontrivial.*

*Proof.* Let  $\mathcal{O}$  be an nilpotent  $M$ -orbit in  $\mathfrak{s}_M$ . Let  $X \in \mathcal{O}$  and  $Y \in \mathfrak{s}_N$ . Let  $\mathcal{O}'$  be the  $P$ -orbit of  $X + Y$ . By Proposition 2.2.2.1, we have  $X + Y \in I_P(X)$  if and only if  $\mathcal{O}'$  is a dense open subset of  $\mathcal{O} \oplus \mathfrak{s}_N$ . The latter condition holds if and only if we have the following equality between tangent spaces:

$$(2-2-3-4) \quad [\mathfrak{p}, X + Y] = [\mathfrak{m}_P, X] \oplus \mathfrak{s}_N.$$

One always has the inclusion  $[\mathfrak{p}, X + Y] \subset [\mathfrak{m}_P, X] \oplus \mathfrak{s}_N$ . The dimension  $d_{\mathcal{O}} = \dim([\mathfrak{m}_P, X] \oplus \mathfrak{s}_N)$  does depend only on  $\mathcal{O}$ . Thus the condition (2-2-3-4) holds if and only if the rank of the  $F$ -map  $\mathfrak{p} \rightarrow \mathfrak{s}_P$  given by  $Z \mapsto [Z, X + Y]$  is at least  $d_{\mathcal{O}}$ .

This condition defines a dense Zariski open subset of  $\mathfrak{s}_P$  which does intersect nontrivially  $X + \mathfrak{s}_N$  since, for any nilpotent  $X \in \mathfrak{s}_M$ , there exists  $Y \in \mathfrak{s}_N$  such that  $X + Y \in I_P(X)$ . The statement is then clear.  $\square$

**2.3. Nilpotent expansion and the homogeneity property.**

**2.3.1.** For any  $F$ -vector space  $W$ , we denote by  $\mathcal{S}(W(\mathbb{A}))$  the space of complex Schwartz–Bruhat functions on  $W(\mathbb{A})$  and  $C_c^\infty(W(\mathbb{A})) \subset \mathcal{S}(W(\mathbb{A}))$  the subspace of smooth compactly supported functions.

**2.3.2.** Let  $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ . For any parabolic subgroup  $P = MN$  of  $H$  (see Section 2.1.5) and any  $x \in H(\mathbb{A})$ , we define the constant term of  $f$  by

$$f_{P,x}(X) = \int_{\mathfrak{s}_N(\mathbb{A})} f(\text{Ad}(x)(X + U)) dU,$$

where the Haar measure on  $\mathfrak{s}_N(\mathbb{A})$  is normalized in such a way that the quotient  $\mathfrak{s}_N(\mathbb{A})/\mathfrak{s}_N(F)$  equipped with the quotient measure by the counting measure is of volume 1. This formula defines a function  $f_{P,x}$  in the Schwartz–Bruhat space  $\mathcal{S}(\mathfrak{s}_M(\mathbb{A}))$ .

**2.3.3.** We say that  $f$  is *weakly cuspidal* if  $f_{P,x}$  vanishes on the subset  $\mathcal{N}(\mathbb{A}) \cap \mathfrak{s}_M(\mathbb{A})$  for any proper parabolic subgroup  $P \subsetneq H$  and any  $x \in H(\mathbb{A})$ .

**2.3.4. A convergence result.** Let  $\mathcal{N}(F)/H(F)$  be the finite set of  $H(F)$ -orbits on  $\mathcal{N}(F)$ . Let  $\mathcal{O} \in \mathcal{N}(F)/H(F)$  be a nilpotent orbit. For any  $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$  and  $h \in H(\mathbb{A})$ , we define

$$k_{\mathcal{O}}(f, h) = \sum_{X \in \mathcal{O}} f(\text{Ad}(h^{-1})X).$$

**Theorem 2.3.4.1.** *Let  $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$  be a weakly cuspidal function. The integral*

$$J_{\mathcal{O}}(f) = \int_{[H]^1} k_{\mathcal{O}}(f, h) dh$$

*is absolutely convergent.*

The proof will be given in Section 2.4.4 below. We have the following corollary which is also a simple consequence of a much more general result of Li [2020, Theorem 1.1].

**Corollary 2.3.4.2.** *Let  $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$  be a weakly cuspidal function. The integral*

$$\int_{[H]^1} \left| \sum_{X \in \mathcal{N}(F)} f(\text{Ad}(h^{-1})X) \right| dh$$

*is convergent.*

**2.3.5. Homogeneity of nilpotent integrals.** Let  $v$  be a place of  $F$ . For any  $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$  and any  $t \in F_v^\times$ , let  $f_t$  be the function in  $\mathcal{S}(\mathfrak{s}(\mathbb{A}))$  defined by  $f_t(X) = f(t^{-1}X)$ .

**Theorem 2.3.5.1.** *Let  $f \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$  be a weakly cuspidal compactly supported function. Let  $v$  be a place of  $F$ . There exists a bound  $\eta > 0$  such for any  $t_0, t$  in  $F_v^\times$  such that  $|t_0|_v < \eta$  and  $|t|_v \leq 1$  we have*

$$J_{\mathcal{O}}(f_{t_0}) = |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}(f_{t_0}).$$

**Remark 2.3.5.2.** The proof is given in Section 2.7 but it is built upon several preliminaries results. Indeed to get Theorem 2.3.5.1, the convergence of Theorem 2.3.4.1 is not sufficient. The main difficulty is that we cannot a priori invert the integral and the sum over  $\mathcal{O}$ : this would in general make appear an infinite volume related to a centralizer. We shall rather use a roundabout method inspired by [Chaudouard 2018b]. As a byproduct, we shall obtain an expression of  $J_{\mathcal{O}}(f_{t_0})$  in terms of an explicit zeta integral depending on  $f$  (see Theorem 2.6.4.1). Here we will need a mild assumption on the support of  $f_{t_0}$  hence the bound on  $t_0$ .

For future reference, let's state a simple corollary.

**Corollary 2.3.5.3.** *Under the hypothesis of Theorem 2.3.5.1, for any  $t \in F_v^\times$  such that  $|t|_v \leq 1$  we have*

$$\int_{[H]^1} \sum_{X \in \mathcal{N}(F)} f_{t_0}(\text{Ad}(h^{-1})X) dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}(f_{t_0}).$$

In particular, when  $t \rightarrow 0$  the expression above is equivalent to

$$J_{(0)}(f_{t_0}) = \text{vol}([H]^1) f(0),$$

where  $(0)$  is the orbit of 0.

**2.4. Refined convergence results.**

**2.4.1.** Let  $\mathcal{O} \in \mathcal{N}(F)/H(F)$  be a nilpotent orbit. Let  $P = MN$  be standard parabolic subgroup of  $H$  (see Section 2.1.5). We borrow notations from [Chaudouard 2018b]; to do this, we identify  $H$  with  $\text{GL}(n, E)$  and  $P$  with a standard parabolic subgroup of  $\text{GL}(n, E)$  by the choice of the basis of Section 2.1.5. We follow [Chaudouard 2018b, §§1.6 and 2.7] (relative to the base field  $E$ ). We have the Harish-Chandra map from  $H_P$  from  $H(\mathbb{A})$  to some real vector space  $a_P$ . Restricted to  $P(\mathbb{A})$  this is a morphism and we denote by  $M(\mathbb{A})^1$  the intersection of  $M(\mathbb{A})$  with the kernel of  $H_P$ . Let  $\tau_P$  be the characteristic function of the acute Weyl chamber in  $a_P$  associated to  $P$ . Let  $Z_P$  be the maximal central  $F$ -split torus in  $M$  and let  $A_P$  be the neutral component of the group of  $\mathbb{R}$ -points of the split component of  $\text{Res}_{F/\mathbb{Q}}(Z_P)$ . We have  $M(\mathbb{A}) = M(\mathbb{A})^1 A_P$ . The function  $F^P$  is the characteristic function of some

compact subset of  $A_P M(F)N(\mathbb{A}) \backslash H(\mathbb{A})$ . Note that both  $F^P$  and  $H_P$  depend on the choice of the compact subgroup  $K$  of [Section 2.1.6](#).

**Theorem 2.4.1.1.** *Let  $f \in S(\mathfrak{s}(\mathbb{A}))$  be a weakly cuspidal function. The integral*

$$\int_{P(F) \backslash H(\mathbb{A})^1} F^P(h) \tau_P(H_P(h)) k_{\mathcal{O}}(f, h) dh$$

*is absolutely convergent.*

The proof will be given in [Section 2.4.2](#) below.

**Remark 2.4.1.2.** [Theorem 2.4.1.1](#) is a simple analog of [[Chaudouard 2018a](#), proposition 3.5.1, corollaire 3.2.2].

**2.4.2. Proof of [Theorem 2.4.1.1](#).** It follows the lines of the proof of [[Chaudouard 2018a](#), théorème 3.2.1]. For the reader’s convenience, we will sketch the main steps and the simplifications in our case. The case where  $P = H$  is obvious since then  $h \mapsto F^P(h) \tau_P(H_P(h)) = F^H(h)$  is compactly supported on  $[H]^1$ . So we assume  $P \subsetneq G$ . We denote by  $\Delta_P$  the set of simple roots of  $Z_P$  in  $N$  and  $\rho_P$  the half-sum of roots of  $Z_P$  on  $N$ . It is easy to see that [Theorem 2.4.1.1](#) is a direct consequence of the following majorization.

**Lemma 2.4.2.1.** *Let  $f \in S(\mathfrak{s}(\mathbb{A}))$  be a weakly cuspidal function. Let  $\Omega$  be a compact subset of  $N(\mathbb{A})M(\mathbb{A})^1 K$ . There exist  $\varepsilon > 0$  and  $c_0 > 0$  such that*

$$\exp(-\langle 2\rho_P, H_P(a) \rangle) |k_{\mathcal{O}}(f, ah)| \leq c_0 \cdot \prod_{\alpha \in \Delta_P} \alpha(a)^{-\varepsilon}$$

for all  $h \in \Omega$  and all  $a \in A_P$  such that  $\tau_P(H_P(a)) = 1$ .

[Lemma 2.4.2.1](#) itself is a straightforward consequence of

**Lemma 2.4.2.2.** *Let  $f$  and  $\Omega$  be as in [Lemma 2.4.2.1](#). Let  $\alpha \in \Delta_P$ . There exists  $c_0 > 0$  such that*

$$\exp(-\langle 2\rho_P, H_P(a) \rangle) |k_{\mathcal{O}}(f, ah)| \leq c_0 \cdot \alpha(a)^{-1}$$

for all  $h \in \Omega$  and all  $a \in A_P$  such that  $\tau_P(H(a)) = 1$ .

**2.4.3. Proof of [Lemma 2.4.2.2](#).** The simple root  $\alpha \in \Delta_P$  defines a maximal parabolic subgroup  $Q$  that contains  $P$ . We denote  $Q = LR$  be the standard Levi decomposition of  $Q$  where  $R$  is the unipotent radical of  $Q$ . Let  $\bar{R}$  be the unipotent radical of the opposite parabolic subgroup.

Let’s denote by  $Y \mapsto Y_{\bar{R}}$  the projection of  $\mathfrak{s}$  on  $\mathfrak{s}_{\bar{R}}$  according to the decomposition  $\mathfrak{s} = \mathfrak{s}_R \oplus \mathfrak{s}_L \oplus \mathfrak{s}_{\bar{R}}$ . To prove [Lemma 2.4.2.2](#), we split the sum  $\sum_{X \in \mathcal{O}} f(\text{Ad}(ah)^{-1} X)$  into the following three contributions:

(2-4-3-1) 
$$\sum_{X \in \mathcal{O}, X_{\bar{R}} \neq 0} f(\text{Ad}(ah)^{-1} X);$$

$$(2-4-3-2) \quad \sum_{X \in \mathcal{O} \cap \mathfrak{s}_Q(F)} f(\text{Ad}(ah)^{-1}X) - \sum_{\substack{X \in \mathfrak{s}_L(F) \\ I_Q(X) = \mathcal{O}}} \sum_{Y \in \mathfrak{s}_R(F)} f(\text{Ad}(ah)^{-1}(X + Y));$$

$$(2-4-3-3) \quad \sum_{\substack{X \in \mathfrak{s}_L(F) \\ I_Q(X) = \mathcal{O}}} \sum_{Y \in \mathfrak{s}_R(F)} f(\text{Ad}(ah)^{-1}(X + Y)).$$

For the contribution (2-4-3-1) we have a better majorization (see proof of [Chaudouard 2018a, lemme 3.8.2]): for any integer  $k \geq 1$ , there exists  $c_0 > 0$  such that

$$\exp(-\langle 2\rho_P, H_P(a) \rangle) \left| \sum_{X \in \mathcal{O}, X_{\bar{R}} \neq 0} f(\text{Ad}(ah)^{-1}X) \right| \leq c_0 \cdot \alpha(a)^{-k}$$

for all  $h \in \Omega$  and all  $a \in A_P$  such that  $\tau_P(H(a)) = 1$ . For the contribution (2-4-3-3), we have the same kind of majorization. But to see it, we need to introduce a nontrivial continuous additive character  $F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$  and a nondegenerate bilinear form on  $\mathfrak{s}$  given by  $\langle X, Y \rangle = \text{trace}(XY)$  (this is the trace of an  $F$ -endomorphism of  $V_E$ ). Then, using the Poisson summation formula for the sum over  $\mathfrak{s}_R(F)$  and the fact that  $f$  is weakly cuspidal, one gets

$$\begin{aligned} & \sum_{\substack{X \in \mathfrak{s}_L(F) \\ I_Q(X) = \mathcal{O}}} \sum_{Y \in \mathfrak{s}_R(F)} f(\text{Ad}(ah)^{-1}(X + Y)) \\ &= \sum_{\substack{X \in \mathfrak{s}_L(F) \\ I_Q(X) = \mathcal{O}}} \sum_{\substack{Y \in \mathfrak{s}_{\bar{R}}(F) \\ Y \neq 0}} \int_{\mathfrak{s}_R(\mathbb{A})} f(\text{Ad}(ah)^{-1}(X + U)) \psi(\langle Y, U \rangle) dU. \end{aligned}$$

At this point we can conclude as in the proof of [Chaudouard 2018a, lemme 3.8.3]. The most difficult contribution is thus (2-4-3-2) which can be written as

$$\sum_{\substack{X \in \mathfrak{s}_L(F) \\ I_Q(X) \neq \mathcal{O}}} \sum_{\substack{Y \in \mathfrak{s}_R(F) \\ X+Y \in \mathcal{O}}} f(\text{Ad}(ah)^{-1}X) - \sum_{\substack{X \in \mathfrak{s}_L(F) \\ I_Q(X) = \mathcal{O}}} \sum_{\substack{Y \in \mathfrak{s}_R(F) \\ X+Y \neq \mathcal{O}}} f(\text{Ad}(ah)^{-1}(X + Y)).$$

But then the argument is that used in [Chaudouard 2018a, §3.12] with Lemma 2.2.3.1 above playing the role of [Chaudouard 2018a, lemme 2.9.1].

**2.4.4. Proof of Theorem 2.3.4.1.** Theorem 2.3.4.1 is a straightforward consequence of Theorem 2.4.1.1 and the equality for any  $h \in H(\mathbb{A})$

$$(2-4-4-4) \quad \sum_P \sum_{\delta \in P(F) \backslash H(F)} F^P(\delta h) \tau_P(H_P(\delta h)) = 1,$$

where the sum is over all standard parabolic subgroups  $P$  of  $H$  (see [Chaudouard 2018b, proposition 2.5.1]).

**2.5. A limit formula for nilpotent contributions.**

**2.5.1.** In this section, we will get an expression for  $J_{\mathcal{O}}(f)$  as the residue at  $s = 0$  of a function  $J_{\mathcal{O}}(f, s)$  when  $f$  is a weakly cuspidal function. In the next section, under a mild condition on  $f$ , the function  $J_{\mathcal{O}}(f, s)$  is expressed in terms of a zeta integral. In our context, this is a simple analog of results in [Chaudouard 2018b]. The homogeneity of  $J_{\mathcal{O}}(f)$  is then an easy consequence of this result. Once again, we borrow notations from [Chaudouard 2018b]: we will denote by  $E^H$  the function  $E^{\text{GL}_E(n)}$  defined in [Chaudouard 2018b, §3.2 équation (3.2.1)]: it is a characteristic function on  $[H]$ .

**2.5.2.** Let  $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$  and  $\mathcal{O} \in \mathcal{N}(F)/H(F)$  be a nilpotent orbit.

**Theorem 2.5.2.1.** *Assume that  $f$  is weakly cuspidal. Then the integral*

$$J_{\mathcal{O}}(f, s) = \int_{[H]} E^H(h)k_{\mathcal{O}}(f, h)|\det(h)|^s dh$$

is absolutely convergent for  $s \in \mathbb{C}$  with  $\Re(s) > 0$ . Moreover,

$$\lim_{s \rightarrow 0^+} sJ_{\mathcal{O}}(f, s) = J_{\mathcal{O}}(f),$$

where  $\lim_{s \rightarrow 0^+}$  means that the limit is taken over complex numbers  $s$  such that  $\Re(s) > 0$ .

*Proof.* It is a variation on the proof of théorème 4.6.1 of [Chaudouard 2018b]. In fact, the hypothesis “weakly cuspidal” makes the situation even simpler.

One shows that for a standard parabolic subgroup  $P \subset H$  the integral

$$J_{\mathcal{O}}^P(f, s) = \int_{P(F) \backslash H(\mathbb{A})} E^H(h)F^P(h)\tau_P(H_P(h))k_{\mathcal{O}}(f, h)|\det(h)|^s dh$$

is absolutely convergent for  $s \in \mathbb{C}$  with  $\Re(s) > 0$ . The convergence is a straightforward consequence of Theorem 2.4.1.1 and the fact that the function  $E^H(h)$  has a simple expression when

$$F^P(h)\tau_P(H_P(h)) = 1,$$

see [ibid., (3.2.1)]. Moreover (as in the proof of [ibid., proposition 4.5.1]), one has

$$\lim_{s \rightarrow 0^+} sJ_{\mathcal{O}}^P(f, s) = \int_{P(F) \backslash H(\mathbb{A})^1} F^P(h)\tau_P(H_P(h))k_{\mathcal{O}}(f, h) dh.$$

One gets the theorem by adding the contributions of the various parabolic subgroups  $P$ , see (2-4-4-4). □

## 2.6. Computation of a nilpotent integral.

**2.6.1.** Let  $X \in \mathcal{N}(F)$ . Using the proof of [Lemma 2.2.1.1](#), one can show that there exist

- an integer  $r \geq 1$ ;
- a decomposition

$$V_E = \bigoplus_{1 \leq i \leq j \leq r} V_j^i,$$

$d_j = \dim(V_j^i)$  does not depend on  $i$ ;

- a basis  $(e_{k,j}^i)_{1 \leq k \leq d_j}$  of  $V_j^i$ ;
- in the basis  $(e_{k,j}^i)_{1 \leq i \leq j \leq r, 1 \leq k \leq d_j}$  we have

$$Xe_{k,j}^i = \begin{cases} e_{k,j}^{i-1} & \text{if } i > 1, \\ 0 & \text{if } i = 1. \end{cases}$$

We may order the basis by  $e_{k,j}^i < e_{k',j'}^{i'}$  if and only if one of the following conditions are satisfied:

- $i < i'$ ;
- $i = i'$  and  $j > j'$ ;
- $i = i'$ ,  $j = j'$  and  $k < k'$ .

In this basis, the matrix of  $X$  is given by

$$\begin{pmatrix} 0_{d_1+\dots+d_r} & I_{d_2+\dots+d_r} & 0 & \cdots & 0 & 0 \\ & 0\{\vdots & \vdots & & \vdots & \vdots \\ & 0_{d_2+\dots+d_r} & & & 0 & 0 \\ & & \ddots & & \vdots & \vdots \\ & & & \ddots & I_{d_{r-1}+d_r} & 0 \\ & & & & 0\{\vdots & \vdots \\ & & & & 0_{d_{r-1}+d_r} & I_{d_r} \\ & & & & & 0\{\vdots \\ & & & & & 0_{d_r} \end{pmatrix}.$$

In the following we may replace  $X$  by a conjugate. So we can and we shall assume that the ordered basis  $(e_{k,j}^i)$  is the basis chosen in [Section 2.1.5](#). Thanks to this basis, we shall also identify the group  $H(F)$  with  $\mathrm{GL}(n, E)$  and the  $F$ -space  $\mathfrak{s}$  with the space  $\mathfrak{gl}(n, E)$  of  $n \times n$  matrices with coefficients in  $E$ . Note that through these identifications, the action  $\mathrm{Ad}$  of the group  $H(F)$  on  $\mathfrak{s}$  is the  $\sigma$ -conjugation of  $\mathrm{GL}(n, E)$  on  $\mathfrak{gl}(n, E)$  (given by  $(y, Y) \mapsto yY\sigma(y)^{-1}$ ).

2.6.2. Let

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n},$$

where

- $\mathfrak{m} = \bigoplus_{1 \leq i \leq j \leq r} \text{Hom}_E(V_j^i, V_j^i)$ ;
- $\mathfrak{n} = \bigoplus \text{Hom}_E(V_j^i, V_{j'}^{i'})$ , where the sum is over  $1 \leq i \leq j \leq r$  and  $1 \leq i' \leq j' \leq r$  such that  $i > i'$  or  $i = i'$  and  $j < j'$ .

Let  $M$  and  $N$  be the  $F$ -subgroups of  $H$  of Lie algebras  $\mathfrak{m}$  and  $\mathfrak{n}$ . We get a parabolic subgroup  $P = MN$  with a Levi decomposition. Recall that  $H_X$  denotes the centralizer of  $X$  in  $H$ . Let  $M_X = M \cap H_X$  and  $N_X = N \cap H_X$ . One has  $H_X \subset P$  and  $H_X = M_X N_X$  is a Levi decomposition. The groups  $M_X$  and  $M$  can be respectively identified to  $\prod_{1 \leq j \leq r} \text{GL}_E(V_j^j)$  and  $\prod_{1 \leq i \leq j \leq r} \text{GL}_E(V_j^i)$ . The inclusion  $M_X \subset M$  is given by the diagonal embedding of  $\text{GL}_E(V_j^j)$  in  $\prod_{1 \leq i \leq j} \text{GL}_E(V_j^i)$ : by “diagonal”, we mean that we identify  $\text{GL}_E(V_j^j)$  to  $\text{GL}_E(V_j^i)$  via the  $F$ -isomorphism  $V_j^j \simeq V_j^i$  given by  $X^{j-i}$ . Let

$$\mathfrak{u}_X = \mathfrak{s} \cap \left( \bigoplus \text{Hom}_F(V_j^i, V_{j'}^{i'}) \right),$$

where the sum is over  $1 < i \leq j \leq r$  and  $1 \leq i' \leq j' \leq r$  such that  $i - 1 > i'$  or  $i = i' + 1$  and  $j < j'$ .

**Lemma 2.6.2.1.** *The map  $n \mapsto nXn^{-1} - X$  induces an isomorphism from  $N_X \backslash N$  onto  $\mathfrak{u}_X$ .*

*Proof.* It is similar to the proof of [Chaudouard 2017, proposition 4.5.1].  $\square$

**2.6.3. Zeta integral.** Let  $L = \prod_{1 \leq i < j \leq r} \text{GL}_E(V_j^i)$ . An element  $A$  of  $L$  is written  $(A_{i,j})_{1 \leq i < j \leq r}$  with  $A_{i,j} \in \text{GL}_E(V_j^i)$ . For any  $A \in L(\mathbb{A})$ , let

$$\Delta_A = \begin{pmatrix} 0_{d_1+\dots+d_r} & \Delta_1(A) & 0 & \cdots & 0 & 0 \\ & \text{o}\{\vdots & \vdots & & \vdots & \vdots \\ & & 0_{d_2+\dots+d_r} & \ddots & 0 & 0 \\ & & & \ddots & \Delta_{r-2}(A) & 0 \\ & & & & \text{o}\{\vdots & \vdots \\ & & & & 0_{d_{r-1}+d_r} & \Delta_{r-1}(A) \\ & & & & & \text{o}\{\vdots \\ & & & & & 0_{d_r} \end{pmatrix},$$

where

$$\Delta_i(A) = \begin{pmatrix} A_{i,r} \\ A_{i,r-1} \\ \cdots \\ A_{i,i+1} \end{pmatrix}$$

for any  $1 \leq i \leq r - 1$ .

For any  $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$  let's define

$$(2-6-3-1) \quad f_X^K(A) = \int_{\mathfrak{u}_X(\mathbb{A})} \int_K f(k^{-1}(U + \Delta_A)\sigma(k)) dU dk$$

and for  $s \in \mathbb{C}$  such that  $\Re(s) > 0$  the zeta function

$$(2-6-3-2) \quad Z_X(f, s) = \int_{L(\mathbb{A})} f_X^K(A) \delta(A, s) dA$$

with

$$(2-6-3-3) \quad \delta(A, s) = \prod_{1 \leq i < j \leq r} |\det(A_{i,j})|^{d_i + \dots + d_j + (j-i)s}$$

and  $dA$  is a Haar measure on  $L(\mathbb{A})$  (normalized as in [Section 2.1.7](#)). Recall that  $\det$  is a shortcut for  $N_{E/F} \circ \det$  (see [Section 2.1.4](#)).

As in [[Chaudouard 2018b](#), §8.3], the integral is convergent and defines a holomorphic function on the domain  $\Re(s) > 0$ .

**Lemma 2.6.3.1.** *Let  $\mathcal{O}$  be the  $H$ -orbit of  $X$ . Let  $v$  be a place of  $F$ . For any  $t \in F_v^\times$ , we have*

$$Z_X(f_t, s) = |t|_v^{\dim(\mathcal{O})/2 + cs} Z_X(f, s),$$

where  $c = \sum_{1 \leq j \leq r} j(j-1)d_j$ .

**Remark 2.6.3.2.** As usual in this paper,  $\dim(\mathcal{O})$  is the dimension of  $\mathcal{O}$  over  $F$ .

*Proof.* To begin with, we have the homogeneity property with the exponent:

$$2 \sum_{1 \leq i < j \leq r} d_j(d_i + \dots + d_j + (j-i)s) + \dim(\mathfrak{u}_X).$$

Indeed, by the change of variables  $U \mapsto t^{-1}U$  in the integral (2-6-3-1), we have

$$(f_t)_X^K(A) = |t|_v^{\dim(\mathfrak{u}_X)} f_X^K(t^{-1}A).$$

Then we have by the change of variables  $A \mapsto t^{-1}A$  in the integral (2-6-3-2),

$$Z_X(f_t, s) = |t|_v^{\dim(\mathfrak{u}_X) + e} Z_X(f, s),$$

where  $e$  satisfies  $\delta(tA, s) = |t|^e \delta(A, s)$ . By the definition (2-6-3-3) of  $\delta(A, s)$  we can compute

$$e = 2 \sum_{1 \leq i < j \leq r} d_j(d_i + \dots + d_j + (j-i)s).$$

The factor 2 is due to our definition of  $\det$  (see [Section 2.1.4](#)).

Since we have

$$2 \sum_{1 \leq i < j \leq r} d_j(j-i)s = \sum_{1 < j \leq r} j(j-1)d_j s,$$

it suffices to show that

$$2 \sum_{1 \leq i < j \leq r} d_j(d_i + \cdots + d_j) + \dim(\mathfrak{u}_X) = \dim(\mathcal{O})/2.$$

One can compute  $\dim(H_X)$  as follows

$$\begin{aligned} \dim(H_X)/2 &= \sum_{1 \leq j, j' \leq r} d_j d_{j'} \min(j, j') \\ &= 2 \sum_{1 \leq i < j \leq r} i d_i d_j + \sum_{1 \leq j \leq r} j d_j^2 \\ &= 2 \sum_{1 \leq i < j \leq r} d_j(d_i + \cdots + d_j) - 2 \sum_{1 \leq j \leq r} (j-1)d_j^2 + \sum_{1 \leq j \leq r} j d_j^2 \\ &= 2 \sum_{1 \leq i < j \leq r} d_j(d_i + \cdots + d_j) - \sum_{1 \leq j \leq r} j d_j^2 + 2 \sum_{1 \leq j \leq r} d_j^2 \\ &= 2 \sum_{1 \leq i < j \leq r} d_j(d_i + \cdots + d_j) - \dim(M)/2 + \dim(M_X). \end{aligned}$$

Using the equality  $\dim(\mathfrak{u}_X) = \dim(N/N_X)$  (see [Lemma 2.6.2.1](#)), we get

$$\begin{aligned} 2 \sum_{1 \leq i < j \leq r} d_j(d_i + \cdots + d_j) + \dim(\mathfrak{u}_X) &= \dim(H_X)/2 + \dim(M)/2 - \dim(M_X) + \dim(N) - \dim(N_X) \\ &= (\dim(M) + 2 \dim(N))/2 - \dim(H_X)/2 \\ &= (\dim(H) - \dim(H_X))/2 = \dim(\mathcal{O})/2. \quad \square \end{aligned}$$

**2.6.4. Computation of the nilpotent integral.** We will denote by  $\theta_X(s)$  the function defined in [[Chaudouard 2018b](#), §7.2] relative to the field  $E$  and the datum  $(d_1, \dots, d_r)$  (see [Section 2.6.1](#)). The main property of  $\theta_X(s)$  we retain is that it is holomorphic for  $s \in \mathbb{C}$  such that  $\Re(s) > 0$ .

**Theorem 2.6.4.1.** *Let  $f \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$  and  $v$  a place of  $F$ . There exists a bound  $\eta > 0$  such that for any  $t$  in  $F_v^\times$  such that  $|t|_v < \eta$  we have*

$$J_{\mathcal{O}}(f_t, s) = \theta_X(s) \cdot Z_X(f_t, s).$$

*Proof.* It is analogous to the proof of theorem 9.1.1 of [[Chaudouard 2018b](#)]. By the Iwasawa decomposition  $H(\mathbb{A}) = P(\mathbb{A})K$ , we can write

$$g = mnk, \quad \text{with } m = (m_i^j) \in M \simeq \prod_{1 \leq i \leq j \leq r} \mathrm{GL}_E(V_j^i), \quad n \in N(\mathbb{A}), \quad k \in K.$$

Let  $R = \text{GL}_E(d_1 + \dots + d_r)$  and  $r(g) \in R(\mathbb{A})$  be the matrix extracted from the first  $d_1 + \dots + d_r$  rows and columns of  $mn$ . As in [Chaudouard 2018b, p. 115], one shows that  $E^G(g) = 1$  if and only if  $E^R(r(g)) = 1$ . The main new ingredient in our context is the observation that  $E^G(g) = E^G(\sigma(g))$ . Then the same kind of computations as those of [Chaudouard 2018b, p. 116] leads to the statement (see also [ibid., remarque 9.1.2]).  $\square$

**2.7. Proof of Theorem 2.3.5.1.** Let  $f \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$  be a weakly cuspidal function. Recall that we have defined integrals  $J_{\mathcal{O}}(f, s)$  (see Theorem 2.5.2.1). Let  $\eta$  be the bound given by Theorem 2.6.4.1. Let  $t_0, t$  in  $F_v^\times$  such that  $|t_0|_v < \eta$  and  $|t|_v \leq 1$ . By Theorem 2.6.4.1,  $J_{\mathcal{O}}(f_{t_0}, s)$  and  $J_{\mathcal{O}}(f_t, s)$  can be expressed in terms of a zeta function for which we have a homogeneity property (see Lemma 2.6.3.1). We deduce that we have for the constant  $c$  of Lemma 2.6.3.1

$$J_{\mathcal{O}}(f_{t_0}, s) = |t|_v^{\dim(\mathcal{O})/2+cs} J_{\mathcal{O}}(f_t, s).$$

Taking the product with  $s$  and then the limit on  $s \rightarrow 0^+$  given by Theorem 2.5.2.1, we get the result.

### 3. The unipotent contribution

#### 3.1. Algebraic situation.

**3.1.1.** We follow the notation of Section 2.1. Let  $D$  be a quaternion algebra over  $F$  equipped with a fixed embedding  $E \hookrightarrow D$ . Note that  $D$  may be split. Then  $V_D = V \otimes_E D$  is a right  $D$ -module. Let  $G = \text{Aut}_D(V_D)$  viewed as an  $F$ -group. Let  $\varepsilon \in G$  given by the left multiplication by  $\sqrt{\tau}$ . Let  $\theta$  the involution of  $\text{End}_F(V_D)$  given by  $\theta(X) = \varepsilon X \varepsilon^{-1}$ . Let  $H' \subset G$  be the subgroup fixed by  $\theta$ . Let  $S' \subset G$  be the  $F$ -variety of automorphism  $g \in \text{Aut}_D(V_D)$  such that  $g\varepsilon = \varepsilon g^{-1}$ . The map

$$(3-1-1-1) \quad \rho : g \mapsto g\theta(g)^{-1}$$

induces an isomorphism from  $G/H'$  onto  $S'$  (see [Guo 1997]). The action of  $G$  by left translations on  $G/H'$  gives an action of  $G$  by  $\theta$ -conjugation for which  $\rho$  is equivariant: we have  $\rho(g_1 g_2) = g_1 \rho(g_2) \theta(g_1)^{-1}$ . This action induces an action by conjugation of  $H'$  on  $S'$ .

**3.1.2.** Let  $\mathcal{U} \subset G$  be the unipotent variety. We define  $\mathcal{U}_{S'} = \mathcal{U} \cap S'$  and  $\mathcal{U}_G = \rho^{-1}(\mathcal{U}_{S'})$ . One knows [Guo 1997, Lemma 3.2] that  $\mathcal{U}_G(F) = H'(F)\mathcal{U}_{S'}(F)H'(F)$ .

**3.1.3.** The tangent space of  $S'$  at  $\text{Id}_{V_D}$  is denoted by  $\mathfrak{s}'$ : it is the space of  $X \in \text{End}_D(V_D)$  such that  $X\varepsilon + \varepsilon X = 0$ . Let  $\mathcal{N}_V$  be the cone of nilpotent elements in  $\text{End}_D(V_D)$ . Let  $\mathcal{N}' = \mathcal{N}_V \cap \mathfrak{s}'$ . The usual exponential map denoted by  $\exp$  induces an isomorphism from  $\mathcal{N}_V$  to  $\mathcal{U}$  and also from  $\mathcal{N}'$  to  $\mathcal{U}_{S'}$ .

**3.1.4.** The map  $\text{End}_E(V) \rightarrow \text{End}_D(V_D)$  given by  $\varphi \mapsto \varphi \otimes \text{Id}_D$  gives an identification of  $H$  with  $H'$  and  $\mathfrak{s}$  with  $\mathfrak{s}'$  and  $\mathcal{N}$  with  $\mathcal{N}'$ . Hence we can freely use all the results of Section 2 for the action of  $H'$  on  $\mathfrak{s}'$ .

To simplify the notation, we will suppress the superscript  $'$  and we will not distinguish between  $H$  and  $H'$ ,  $S$  and  $S'$ , etc. For  $v$  a place of  $F$ , the measures used on the groups of  $F_v$ -points are Haar measures (we do not need any normalization).

**3.2. Main results.**

**3.2.1.** Let  $v$  be a place of  $F$ . For any  $\Phi \in C_c^\infty(G(F_v))$ , there is a unique function, denoted by  $\Phi_S \in C_c^\infty(S(F_v))$  such that

$$\Phi_S(\rho(x)) = \int_{H(F_v)} \Phi(xh) dh.$$

The map  $\Phi \mapsto \Phi_S$  is a surjection from  $C_c^\infty(G(F_v))$  onto  $C_c^\infty(S(F_v))$ .

**3.2.2. Very cuspidal test functions.** We shall say that  $\Phi \in C_c^\infty(G(F_v))$  is *very cuspidal* if one has

$$\int_{N(F_v)} \Phi(xny) dn = 0$$

for any parabolic subgroup  $P \subsetneq G$  and any  $x, y \in G(F_v)$ . Here  $N$  is the unipotent radical of  $P$ .

**Remark 3.2.2.1.** Assume  $v$  is finite. Let  $\tilde{\Phi}$  be a matrix coefficient of a supercuspidal representation of  $G(F_v)/Z(F_v)$  where  $Z$  is the center of  $G$ . Let  $\Phi \in C_c^\infty(G(F_v))$  such that

$$\int_{Z(F_v)} \Phi(gz) dz = \tilde{\Phi}(g)$$

for any  $g \in G(F_v)/Z(F_v)$ . Then  $\Phi$  is very cuspidal.

Note that any very cuspidal function  $\Phi \in C_c^\infty(G(F_v))$  is such that

$$(3-2-2-1) \quad \int_{N(F_v)/N_H(F_v)} \Phi_S(\rho(xn)) dn = 0,$$

where  $N_H = N \cap H$  and  $dn$  is the quotient of the Haar measures on  $N(F_v)$  and  $N_H(F_v)$ .

**3.2.3. Global setting.** We also have a surjective map  $\Phi \mapsto \Phi_S$  from  $C_c^\infty(G(\mathbb{A}))$  onto  $C_c^\infty(S(\mathbb{A}))$  given by

$$\Phi_S(\rho(x)) = \int_{H(\mathbb{A})} \Phi(xh) dh$$

for any  $x \in G(\mathbb{A})$ .

We shall say that  $\Phi \in C_c^\infty(G(\mathbb{A}))$  is *very cuspidal* if there is a place  $v$  such that if one writes  $\mathbb{A} = F_v \times \mathbb{A}^v$ , one has  $\Phi = \Phi_v \otimes \Phi^v$  with  $\Phi^v \in C_c^\infty(G(\mathbb{A}^v))$  and  $\Phi_v \in C_c^\infty(G(F_v))$  is very cuspidal.

**3.2.4.** Let's define for  $\Phi \in C_c^\infty(G(\mathbb{A}))$  and  $x, y \in G(\mathbb{A})$

$$K_{\mathcal{U}_G}(\Phi, x, y) = \sum_{\gamma \in \mathcal{U}_G(F)} \Phi(x^{-1}\gamma y)$$

and for  $h \in H(\mathbb{A})$

$$K_{\mathcal{U}_S}(\Phi, h) = \sum_{\gamma \in \mathcal{U}_S(F)} \Phi_S(h^{-1}\gamma h).$$

We have the simple relation for  $h \in H(\mathbb{A})$

$$(3-2-4-2) \quad \int_{[H]} K_{\mathcal{U}_G}(\Phi, x, h) dx = K_{\mathcal{U}_S}(\Phi, h).$$

**Theorem 3.2.4.1.** *Let  $\Phi \in C_c^\infty(G(\mathbb{A}))$  be a very cuspidal function. Let  $\mathcal{O} \in \mathcal{N}(F)/H(F)$  and let  $v$  be a place of  $F$ .*

(1) *For any  $t \in F_v^\times$ , the integral*

$$J_{\mathcal{O}}^t(\Phi) = \int_{[H]^1} \sum_{X \in \mathcal{O}} \Phi_S(h^{-1} \exp(t^{-1}X)h) dh$$

*is absolutely convergent. For  $t = 1$ ,  $J_{\mathcal{O}}^1(\Phi)$  is denoted by  $J_{\mathcal{O}}(\Phi)$ .*

(2) *(Fine expansion) We have*

$$\int_{[H]^1} K_{\mathcal{U}_S}(\Phi, h) dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} J_{\mathcal{O}}(\Phi),$$

*where the left-hand side is absolutely convergent.*

(3) *There exists a bound  $\eta > 0$  such that for any  $t$  in  $F_v^\times$  such that  $|t|_v < \eta$ , we have*

$$J_{\mathcal{O}}^t(\Phi) = \lim_{s \rightarrow 0^+} s \theta_X(s) Z_X(f_t, s),$$

*where  $X \in \mathcal{O}$  is the element considered in Section 2.6.1,  $\theta_X(s)$  (defined in Section 2.6.4) is holomorphic for  $\Re(s) > 0$  and does not depend on  $\Phi$ , the zeta function  $Z_X(f_t, s)$  is defined in Section 2.6.3 relative to any function  $f_t \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$  such that  $f_t(Y) = \Phi_S(\exp(t^{-1}Y))$ .*

(4) *Let  $\eta$  be the bound of (3). For any  $t_0, t$  in  $F_v^\times$  such that  $|t_0|_v < \eta$  and  $|t|_v \leq 1$  we have*

$$J_{\mathcal{O}}^{tt_0}(\Phi) = |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}^{t_0}(\Phi).$$

*Proof.* [Theorem 3.2.4.1](#) is deduced from similar results on  $\mathfrak{s}$  and the (standard) descent procedure to  $\mathfrak{s}$  that is explained in [Section 3.2.5](#). More precisely, assertion (1) results from [Theorem 2.3.4.1](#). Assertion (2) is a consequence of assertion (1) and the finiteness of nilpotent orbits. Assertion (3) is a combination of [Theorem 2.5.2.1](#) and [2.6.4.1](#). Finally, assertion (4) is a consequence of assertion (3) (see [Section 2.7](#)).  $\square$

Let's state a corollary which is a straightforward consequence of [Theorem 3.2.4.1](#).

**Corollary 3.2.4.2.** (We use notation of [Theorem 3.2.4.1](#).) *For any  $t_0, t$  in  $F_v^\times$  such that  $|t_0|_v < \eta$  and  $|t|_v \leq 1$  we have*

$$\int_{[H]^1} \left| \sum_{X \in \mathcal{N}(F)} \Phi_S(h^{-1} \exp((tt_0)^{-1} X)h) \right| dh < \infty$$

and

$$\int_{[H]^1} \sum_{X \in \mathcal{N}(F)} \Phi_S(h^{-1} \exp((tt_0)^{-1} X)h) dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}^{t_0}(\Phi).$$

In particular, when  $t \in F_v^\times$  goes to 0, the expression

$$\int_{[H]^1} \sum_{X \in \mathcal{N}(F)} \Phi_S(h^{-1} \exp(t^{-1} X)h) dh$$

is equivalent to

$$\text{vol}([H]^1) \int_{H(\mathbb{A})} \Phi(h) dh.$$

**Remark 3.2.4.3.** For  $\Phi$  very cuspidal, we have the equality

$$\int_{[H]^1} \int_{[H]} K_{\mathcal{U}_G}(\Phi, x, y) dx dy = \int_{[H]^1} K_{\mathcal{U}_S}(\Phi, h) dh,$$

where the left-hand side is at least conditionally convergent. One can prove in fact that it is absolutely convergent using mixed truncation operators (in the sense of the seminal paper [[Jacquet et al. 1999](#)]) built upon the combinatorics of [[Li 2020](#)].

**3.2.5. Descent to the tangent space.** Let  $\mathcal{V}_0$  be a finite set of places of  $F$  containing the archimedean places and a fixed place denoted by  $v_0$ . Let  $A \subset F$  the ring of integers outside  $\mathcal{V}_0$ . We assume that  $\mathcal{V}_0$  is large enough such that all objects  $G, H, S, \mathcal{U}_S, \mathcal{N}$  come naturally from  $A$ -schemes by base change. We assume also that the exponential (denoted by  $\exp$ ) induces an isomorphism of  $A$ -scheme from  $\mathcal{N}$  to  $\mathcal{U}_S$ . For  $v \notin \mathcal{V}_0$ , let  $\mathcal{O}_v \subset F_v$  be the ring of integers.

Let

$$\Phi = \Phi_0 \otimes \Phi_1 \otimes \Phi_2,$$

where  $\Phi_0 \in C_c^\infty(G(F_0))$  (with  $F_0 = F_{v_0}$ ) is a very cuspidal function,  $\Phi_1$  is a test function on

$$F_1 = \prod_{v \in \mathcal{V}_0 \setminus \{v_0\}} F_v$$

and  $\Phi_2$  is the characteristic function of  $\prod_{v \notin \mathcal{V}_0} G(\mathcal{O}_v)$ .

Let  $c : \mathfrak{s} \rightarrow \mathfrak{c} = \mathfrak{s}/H$  be the categorical quotient. For all  $v \in \mathcal{V}_0$ , we fix an open subset  $\omega_v^b \subset \mathfrak{c}(F_v)$  containing  $c(0)$  such that the exponential map  $\exp$  is well-defined and induces an analytic diffeomorphism from  $\omega_v = c^{-1}(\omega_v^b)$  onto an open subset  $\Omega_v \subset S(F_v)$ . Let  $\zeta_v$  be a smooth function on  $\mathfrak{c}(F_v)$  with compact support included in  $\omega_v^b$  and with value 1 in a neighborhood of  $c(0)$ .

Let's define  $\zeta_0 = \zeta_{v_0}$  and  $\omega_0 = \omega_{v_0}$  and also  $\zeta_1 = \prod_{v \in \mathcal{V}_0 \setminus \{v_0\}} \zeta_v$  and  $\omega_1 = \prod_{v \in \mathcal{V}_0 \setminus \{v_0\}} \omega_v$ . We define functions  $f_0 \in C_c^\infty(\mathfrak{s}(F_0))$  and  $f_1 \in C_c^\infty(\mathfrak{s}(F_1))$  by

$$f_i(X) = \begin{cases} \zeta_i(X)\Phi_{i,S}(\exp(X)) & \text{if } X \in \omega_i; \\ 0 & \text{otherwise;} \end{cases}$$

for any  $i \in \{0, 1\}$  and  $X \in \mathfrak{s}(F_i)$ . Let  $f = f_0 \otimes f_1 \otimes f_2 \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$ , where  $f_2$  is the characteristic function of  $\prod_{v \notin \mathcal{V}_0} \mathfrak{s}(\mathcal{O}_v)$ .

We have for all  $X \in \mathcal{N}(\mathbb{A})$

$$\Phi_S(h^{-1} \exp(X)h) = f(\text{Ad}(h^{-1})(X)).$$

To complete the proof of [Theorem 3.2.4.1](#) it suffices to check the next lemma.

**Lemma 3.2.5.1.** *The function  $f$  is weakly cuspidal (in the sense of [Section 2.3.3](#)).*

*Proof.* Let  $X \in \mathfrak{s}_M(F_0) \cap \mathcal{N}(F_0)$ . Clearly, it suffices to prove that

$$\int_{\mathfrak{s}_N(F_0)} f_0(\text{Ad}(x)(X + U)) dU = 0$$

for any  $x \in H(F_v)$ . But for any  $U \in \mathfrak{s}_N(F_0)$  we have

$$f_0(\text{Ad}(x)(X + U)) = \Phi_{0,S}(x \exp(X + U)x^{-1}).$$

Let's write

$$\exp(X + U) = \exp(X/2)(\exp(-X/2) \exp(X + U) \exp(-X/2))\theta(\exp(X/2))^{-1}.$$

We observe that  $U \mapsto \exp(-X/2) \exp(X + U) \exp(-X/2)$  induces an isomorphism from  $\mathfrak{s}_N$  onto  $S \cap N$ . But the map  $\rho$  of [\(3-1-1-1\)](#) induces an isomorphism from  $N/N_H$  onto  $S \cap N$ . We get a bijection from  $N(F_0)/N_H(F_0)$  onto  $\mathfrak{s}_N(F_0)$ . By a change of variables, we get (up to a constant  $c \neq 0$ )

$$\int_{\mathfrak{s}_N(F_0)} f_0(\text{Ad}(x)(X + U)) dU = c \int_{N(F_0)/N_H(F_0)} \Phi_{0,S}(\rho(x \exp(X/2)n)) dn = 0$$

by the vanishing condition [\(3-2-2-1\)](#). □

## Acknowledgements

We thank Hang Xue for raising the problem and Huajie Li for many useful discussions. We also thank the referee for his/her careful reading.

## References

- [Arthur 1978] J. G. Arthur, “A trace formula for reductive groups, I: Terms associated to classes in  $G(\mathbf{Q})$ ”, *Duke Math. J.* **45**:4 (1978), 911–952. [MR](#) [Zbl](#)
- [Arthur 1985] J. Arthur, “A measure on the unipotent variety”, *Canad. J. Math.* **37**:6 (1985), 1237–1274. [MR](#) [Zbl](#)
- [Arthur 1988] J. Arthur, “The local behaviour of weighted orbital integrals”, *Duke Math. J.* **56**:2 (1988), 223–293. [MR](#) [Zbl](#)
- [Chaudouard 2017] P.-H. Chaudouard, “Sur la contribution unipotente dans la formule des traces d’Arthur pour les groupes généraux linéaires”, *Israel J. Math.* **218**:1 (2017), 175–271. [MR](#) [Zbl](#)
- [Chaudouard 2018a] P.-H. Chaudouard, “Sur certaines contributions unipotentes dans la formule des traces d’Arthur”, *Amer. J. Math.* **140**:3 (2018), 699–752. [MR](#) [Zbl](#)
- [Chaudouard 2018b] P.-H. Chaudouard, “Sur une variante de troncatures d’Arthur”, pp. 85–120 in *Geometric aspects of the trace formula*, edited by W. Müller et al., Springer, 2018. [MR](#) [Zbl](#)
- [Feigon et al. 2018] B. Feigon, K. Martin, and D. Whitehouse, “Periods and nonvanishing of central  $L$ -values for  $GL(2n)$ ”, *Israel J. Math.* **225**:1 (2018), 223–266. [MR](#) [Zbl](#)
- [Guo 1996] J. Guo, “On a generalization of a result of Waldspurger”, *Canad. J. Math.* **48**:1 (1996), 105–142. [MR](#) [Zbl](#)
- [Guo 1997] J. Guo, “Uniqueness of generalized Waldspurger model for  $GL(2n)$ ”, *Pacific J. Math.* **180**:2 (1997), 273–289. [MR](#) [Zbl](#)
- [Jacquet 1986] H. Jacquet, “Sur un résultat de Waldspurger”, *Ann. Sci. École Norm. Sup. (4)* **19**:2 (1986), 185–229. [MR](#) [Zbl](#)
- [Jacquet et al. 1999] H. Jacquet, E. Lapid, and J. Rogawski, “Periods of automorphic forms”, *J. Amer. Math. Soc.* **12**:1 (1999), 173–240. [MR](#) [Zbl](#)
- [Li 2020] H. Li, “An infinitesimal variant of Guo–Jacquet trace formula, II”, preprint, 2020. [arXiv](#)
- [Lusztig and Spaltenstein 1979] G. Lusztig and N. Spaltenstein, “Induced unipotent classes”, *J. London Math. Soc. (2)* **19**:1 (1979), 41–52. [MR](#) [Zbl](#)
- [Spaltenstein 1982] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Mathematics **946**, Springer, 1982. [MR](#) [Zbl](#)
- [Xue 2019] H. Xue, “Epsilon dichotomy for linear models”, preprint, 2019, Available at [https://www.math.arizona.edu/~xuehang/epsilon\\_v2.pdf](https://www.math.arizona.edu/~xuehang/epsilon_v2.pdf).

Received August 23, 2019. Revised October 25, 2019.

PIERRE-HENRI CHAUDOUARD  
 UNIVERSITÉ DE PARIS  
 INSTITUT DE MATHÉMATIQUES DE JUSSIEU - PARIS RIVE GAUCHE  
 F-75013 PARIS  
 FRANCE  
[pierre-henri.chaudouard@imj-prg.fr](mailto:pierre-henri.chaudouard@imj-prg.fr)

# PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

[msp.org/pjm](http://msp.org/pjm)

## EDITORS

Don Blasius (Managing Editor)  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[blasius@math.ucla.edu](mailto:blasius@math.ucla.edu)

Paul Balmer  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[balmer@math.ucla.edu](mailto:balmer@math.ucla.edu)

Wee Teck Gan  
Mathematics Department  
National University of Singapore  
Singapore 119076  
[matgwt@nus.edu.sg](mailto:matgwt@nus.edu.sg)

Sorin Popa  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[popa@math.ucla.edu](mailto:popa@math.ucla.edu)

Paul Yang  
Department of Mathematics  
Princeton University  
Princeton NJ 08544-1000  
[yang@math.princeton.edu](mailto:yang@math.princeton.edu)

Matthias Aschenbrenner  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[matthias@math.ucla.edu](mailto:matthias@math.ucla.edu)

Daryl Cooper  
Department of Mathematics  
University of California  
Santa Barbara, CA 93106-3080  
[cooper@math.ucsb.edu](mailto:cooper@math.ucsb.edu)

Jiang-Hua Lu  
Department of Mathematics  
The University of Hong Kong  
Pokfulam Rd., Hong Kong  
[jhlu@maths.hku.hk](mailto:jhlu@maths.hku.hk)

Vyjayanthi Chari  
Department of Mathematics  
University of California  
Riverside, CA 92521-0135  
[chari@math.ucr.edu](mailto:chari@math.ucr.edu)

Kefeng Liu  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

Jie Qing  
Department of Mathematics  
University of California  
Santa Cruz, CA 95064  
[qing@cats.ucsc.edu](mailto:qing@cats.ucsc.edu)

## PRODUCTION

Silvio Levy, Scientific Editor, [production@msp.org](mailto:production@msp.org)

## SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI  
CALIFORNIA INST. OF TECHNOLOGY  
INST. DE MATEMÁTICA PURA E APLICADA  
KEIO UNIVERSITY  
MATH. SCIENCES RESEARCH INSTITUTE  
NEW MEXICO STATE UNIV.  
OREGON STATE UNIV.

STANFORD UNIVERSITY  
UNIV. OF BRITISH COLUMBIA  
UNIV. OF CALIFORNIA, BERKELEY  
UNIV. OF CALIFORNIA, DAVIS  
UNIV. OF CALIFORNIA, LOS ANGELES  
UNIV. OF CALIFORNIA, RIVERSIDE  
UNIV. OF CALIFORNIA, SAN DIEGO  
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ  
UNIV. OF MONTANA  
UNIV. OF OREGON  
UNIV. OF SOUTHERN CALIFORNIA  
UNIV. OF UTAH  
UNIV. OF WASHINGTON  
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

See inside back cover or [msp.org/pjm](http://msp.org/pjm) for submission instructions.

---

The subscription price for 2020 is US \$520/year for the electronic version, and \$705/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).


---

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

---

PJM peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2020 Mathematical Sciences Publishers

# PACIFIC JOURNAL OF MATHEMATICS

Volume 305    No. 2    April 2020

---

Singular periodic solutions to a critical equation in the Heisenberg group CLAUDIO AFELTRA	385
On a theorem of Hegyvári and Hennecart DAO NGUYEN VAN ANH, LE QUANG HAM, DOOWON KOH, THANG PHAM and LE ANH VINH	407
On the Ekeland–Hofer symplectic capacities of the real bidisc LUCA BARACCO, MARTINO FASSINA and STEFANO PINTON	423
Freeness characterizations on free chaos spaces SOLESNE BOURGUIN and IVAN NOURDIN	447
Dominance order and monoidal categorification of cluster algebras ELIE CASBI	473
On the fine expansion of the unipotent contribution of the Guo–Jacquet trace formula PIERRE-HENRI CHAUDOUARD	539
Strongly algebraic realization of dihedral group actions KARL HEINZ DOVERMANN	563
On commuting billiards in higher-dimensional spaces of constant curvature ALEXEY GLUTSYUK	577
On the arithmetic of a family of twisted constant elliptic curves RICHARD GRIFFON and DOUGLAS ULMER	597
On the nonexistence of $S^6$ type complex threefolds in any compact homogeneous complex manifolds with the compact lie group $G_2$ as the base manifold DANIEL GUAN	641
On $SU(3)$ Toda system with multiple singular sources ALI HYDER, CHANGSHOU LIN and JUNCHENG WEI	645
Convergence of mean curvature flow in hyper-Kähler manifolds KEITA KUNIKAWA and RYOSUKE TAKAHASHI	667
The two-dimensional analogue of the Lorentzian catenary and the Dirichlet problem RAFAEL LÓPEZ	693
Schwarz D-surfaces in $\mathbb{N}\mathbb{I}_3$ HEYONG SHIN, YOUNG WOOK KIM, SUNG-EUN KOH, HYUNG YONG LEE and SEONG-DEOG YANG	721
Compactness of constant mean curvature surfaces in a three-manifold with positive Ricci curvature AO SUN	735
The rational cohomology Hopf algebra of a generic Kac–Moody group ZHAO XU-AN and GAO HONGZHU	757