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ON THE NONEXISTENCE OF S^6 TYPE COMPLEX THREEFOLDS IN ANY COMPACT HOMOGENEOUS COMPLEX MANIFOLDS WITH THE COMPACT LIE GROUP G_2 AS THE BASE MANIFOLD

DANIEL GUAN

In a recent preprint Professor Etesi asked a question: could one find a complex three dimensional submanifold S in a compact complex seven dimensional homogeneous space with the compact real 14 dimensional Lie group G_2 as the base manifold, such that S is diffeomorphic to the six dimensional sphere S^6? We apply a result of Tits on compact complex homogeneous space, or of H. C. Wang and Hano–Kobayashi on the classification of compact complex homogeneous manifolds with a compact reductive Lie group to give an answer to his question. In particular, we show that one could not obtain a complex structure of S^6 in his way.

1. Introduction

Let M be a complex manifold, h be an Hermitian metric. For a compact complex manifold, there is always some Hermitian metric h by the partition of the unity argument. If h is an Hermitian metric and G is a compact Lie group acting on M biholomorphically, then by taking average on G, we can always assume that h is invariant under G.

A compact complex homogeneous space with an invariant Hermitian structure was classified by H. C. Wang [1954], see also [Hano and Kobayashi 1960]. In fact, they classified the compact complex homogeneous spaces with compact Lie groups. In particular, an Hermitian manifold is a Riemannian manifold. The identity component of the Riemannian isometric group for a compact Riemannian manifold is a compact Lie group. So is the identity component of the Hermitian isometric group for a compact Hermitian manifold.

Therefore, we have:

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Lemma 1. *If $M = G/H$ is a compact homogeneous Riemannian manifold with G connected, then G is a subgroup of a compact Lie group. In particular, both G and H are reductive with compact semisimple parts.*

We then have (see [Hano and Kobayashi 1960, Theorem B]):

Lemma 2. *Any compact Hermitian homogeneous manifold is a complex torus bundle over a rational (therefore simply connected) projective homogeneous space.*

One could also see [Guan 1994, page 66, Remark] for a detailed understanding of this fibration.

There is also a similar fibration [Tits 1971] closely related to Lemma 2 for any general compact complex homogeneous space:

Proposition 3. *Let $M = G/H$ be a compact complex homogeneous space such that G is a complex Lie group, with H a complex Lie subgroup and $M = G/H$ is the complex quotient. Then there is a complex fibration $G/H \to G/N$ such that $N = \mathrm{Norm}_G(H^0)$ and G/N is a rational projective homogeneous space.*

Here, H^0 is the identity component of H and

$$\mathrm{Norm}_G(H^0) = \{g \in G \mid gH^0g^{-1} \subset H^0\}$$

is the normalizer of H^0 in G.

Let $G = G_2$ be the compact real 14 dimensional Lie group of type G_2, $G^{\mathbb{C}}$ be the complex simple Lie group such that the Lie algebra of $G^{\mathbb{C}}$ is the complexification of the Lie algebra of G. Let B be the Borel subgroup such that B is connected and the Lie algebra of B contains all the negative root vector spaces and the given Cartan subalgebra. Then $G^{\mathbb{C}}/B$ has a complex dimension 6. We notice that the Cartan subalgebra has a complex dimension 2. So, let U be the maximal nilpotent subgroup of B, which is generated by the all the negative root spaces; then B/U is a complex two dimensional algebraic torus $(\mathbb{C}^*)^2$. Let $\pi : B \to B/U$ be the quotient map. Given any complex one dimensional line subspace $l = \mathbb{C}$ in B/U, $S_l = \pi^{-1}(l)$ is a complex codimension one normal subgroup of B. Then $G^{\mathbb{C}}/S_l$ is diffeomorphic to G_2. Therefore, it gives a complex structure on G_2.

On the other hand, if $G = G_2$ is a complex homogeneous space itself, then its complexification $G^{\mathbb{C}}$ acts on G also. And $G = G^{\mathbb{C}}/H$. By Lemma 2, the Hano–Kobayashi fiber bundle is $G^{\mathbb{C}}/H \to G^{\mathbb{C}}/B$. The torus is just the Cartan subgroup of G. By earlier works, e.g., [Guan 2002], we know that the Tits fibration and the HK fibration are the same and therefore, $B = \mathrm{Norm}_{G^{\mathbb{C}}}(H^0)$. But $\dim_{\mathbb{C}} B/H = 1$ and H^0 is normal in B. We see that H contains all the subgroups generated by the negative root spaces. That is, $U \subset H$.

Proposition 4. *Let G be a complex manifold with the base manifold being G_2 such that $G = G_2$ acts on itself biholomorphically. Then there is a holomorphic* *fibration from G into a projective manifold* $F : G \to G^{\mathbb{C}}/B$ *such that each fiber is a complex one dimensional complex torus. If* S *is a complex three-dimensional compact submanifold of* G, *then either* $F(S)$ *is complex three dimensional or* $F(S)$ *is complex two dimensional and* $S = F^{-1}(F(S))$.

Therefore, if S is diffeomorphic to S^6, by the result in [Campana et al. 1998; Campana et al. 2020], the algebraic dimension of a complex structure on S is zero. This implies that S can not be a complex submanifold of G.

However, in this paper, we would like to give an argument which is mildly independent of the result from Campana et al.

We shall prove:

Main Theorem. *The conjugate orbit described in both [Etesi 2015] and [Chaves and Rigas 1991] can not be a complex submanifold of* G_2.

Recall that $S^6 = G_2/\operatorname{SU}(3)$. The Cartan subgroup of SU(3) is also a Cartan subgroup of G_2. The Cartan subgroup of SU(3) has elements $\operatorname{diag}(a, b, c)$ with $abc = 1$. The center of SU(3) has elements $aI = \operatorname{diag}(a, a, a)$ with $a^3 = 1$. This implies the center of SU(3) has the structure $C = \mathbb{Z}/(3\mathbb{Z}) = \mathbb{Z}_3$. Let A be one of the generator of $C = \mathbb{Z}_3$. Then $S = \{gAg^{-1} \mid g \in G_2\}$. Since A is in the center of SU(3), it is not difficult to see that SU(3) fixes A.

A description of the Lie algebra of G_2 can be also found in the Section 4 of [Guan 2006], in which the Lie algebra of SU(3) is simply generated by the root spaces with the short roots $e_i - e_j$ with $\{i, j\} \in \{1, 2, 3\}$.

2. Proofs

Proof of Proposition 4. If $F(S)$ in Proposition 4 has two complex dimensions, let $p \in F(S)$ be a regular point for F; then $F^{-1}(p) \cap S$ is a complex one-dimensional manifold. As a closed complex one-dimensional submanifold of $F^{-1}(p)$, it can only be the whole complex torus. Since the fiber bundle is locally free and S a complex submanifold, the regular points for F are a dense set, by continuity we have the last part of Proposition 4 ☐

Proof of Main Theorem. Assume that S is a complex submanifold, we shall get a contradiction.

If $F(S)$ is complex three dimensional, by $F(G)$ being projective, the pullback of the Kähler form ω is a nonzero H^2 class since it introduces a nonzero measure ω^3 on S. A contradiction.

If $F(S)$ is of complex dimension two, the conjugate orbit of Λ in [Etesi 2015] or [Chaves and Rigas 1991] is S. Here, Λ is one of the two nonidentity elements in the center of SU(3) and therefore is in the given Cartan subgroup of SU(3), which is exactly the Cartan subgroup T of G_2. But $S = F^{-1}(F(S))$, we have that the whole Cartan subgroup of G_2 is $F^{-1}(F(\Lambda))$. Therefore, S could be described as $S_t = \{gtg^{-1} \mid g \in G_2\}$ for any $t \in T$. But $S_t = G/G_t$ with G_t the centralizer $\{g \in G \mid gt = tg\}$. However, in general, $G_t = T$. That is S_t is real 12 dimensional. A contradiction. ☐

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