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Inspired by work of Leung and Wan (*J.* Geom. Anal. 17:2 (2007) 343–364), we study the mean curvature flow in hyper-Kähler manifolds starting from hyper-Lagrangian submanifolds, a class of middle-dimensional submanifolds, which contains the class of complex Lagrangian submanifolds. For each hyper-Lagrangian submanifold, we define a new energy concept called the *twistor energy* by means of the associated twistor family (i.e., 2-sphere of complex structures). We will show that the mean curvature flow starting at any hyper-Lagrangian submanifold with sufficiently small twistor energy will exist for all time and converge to a complex Lagrangian submanifold for one of the hyper-Kähler complex structure. In particular, our result implies some kind of energy gap theorem for hyper-Kähler manifolds which have no complex Lagrangian submanifolds.

1. Introduction

Let $(M, \bar{g})$ be a hyper-Kähler $4n$-manifold, i.e., the holonomy group is contained in $\mathrm{Sp}(n)$. Or equivalently, there exist distinct, $\bar{g}$-compatible complex structures $\{J_d\}_{d=1,2,3}$ which satisfy the *quaternion relations*:

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -\mathrm{Id}.$$

Then each hyper-Kähler manifold M admits a 2-sphere of complex structures called the *twistor family*

$$\sum_d c_d J_d \quad \text{for } (c_1, c_2, c_3) \in \mathbb{S}^2 \subset \mathbb{R}^3.$$

Throughout this paper, we assume that $(M, \bar{g})$ has bounded geometry (i.e., the injectivity radius, curvatures and derivatives of the curvatures are uniformly bounded). Typical examples of hyper-Kähler manifolds are a K3 surface and a compact torus $\mathbb{T}^4$ (in fact, any Calabi–Yau 4-manifold is hyper-Kähler since $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ and these are only compact 4-dimensional examples). Beauville [1983] constructed two distinct deformation classes of hyper-Kähler's in $4n$-dimension for every $n > 1$.

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Moreover, Grady [1999; 2003] constructed two additional deformation classes in dimensions 12 and 20. Each deformation class has representatives which are moduli spaces of semistable sheaves on projective K3 surfaces or abelian surfaces or modifications of such moduli spaces.

In this paper, we show the existence and convergence result for the mean curvature flow (MCF) in hyper-Kähler manifolds when the initial data is very small. There is no doubt that for studying the MCF, Lagrangian is one of the good class of submanifolds in a Kähler–Einstein manifold. Indeed, from Smoczyk's result [1996], the Lagrangian property is preserved under the MCF, and it gives a lot of benefits for computations of evolution equations, by identifying the extrinsic normal bundle with the intrinsic tangent bundle via the complex structure. Nevertheless, we would like to consider another class of submanifolds, called "hyper-Lagrangian submanifolds" as displayed below. This class includes Lagrangian submanifolds in hyper-Kähler 4-manifolds.

1A. ***Main result.*** A natural counterpart of the Lagrangian condition in hyper-Kähler manifolds is the "complex Lagrangian": for $J \in \mathbb{S}^2$, let Ω_J be a holomorphic symplectic form (i.e., nondegenerate J-holomorphic 2-form) with respect to J. For a $2n$-dimensional real submanifold $L \subset M$, we say that L is *complex Lagrangian* if $\Omega_J|_L = 0$ for some $J \in \mathbb{S}^2$. From a basic fact of hyper-Kähler geometry, we find that there exists a J-orthogonal element $K \in \mathbb{S}^2$ such that Ω_J can be expressed as

$$\Omega_J = \bar{\omega}_{JK} - \sqrt{-1}\bar{\omega}_K,$$

where $\bar{\omega}_{JK} = \bar{g}(JK\cdot\,,\cdot)$, $\bar{\omega}_K = \bar{g}(K\cdot\,,\cdot)$ are real symplectic forms for JK and K respectively. So the condition $\Omega_J|_L = 0$ means that two symplectic forms $\bar{\omega}_{JK}$ and $\bar{\omega}_K$ vanish at the same time for any J-orthogonal $K \in \mathbb{S}^2$.

However, this "bi-Lagrangian" condition is so strong that any complex Lagrangian submanifold L in M automatically becomes a (minimal) complex submanifold (see [Hitchin 1999]). So, following the idea of Leung and Wan [2007], we relax the assumption by using rich geometry on M. We say that L is *hyper-Lagrangian* if $\Omega_{\Psi(x)}|_L = 0$ at every point $x \in L$ for some varying complex structure $\Psi : L \to \mathbb{S}^2$. Then this map Ψ is called the *complex phase*. In particular, complex Lagrangian is a special case when we can take Ψ as a constant map. Leung and Wan [2007] showed that if the initial submanifold L_0 is hyper-Lagrangian, then $L_t := F_t(L)$ is still hyper-Lagrangian under the MCF $F_t : L \to M$, and then the complex phase Ψ_t evolves according to the coupled flow

$$\begin{cases} \frac{d}{dt}F_t = H_t, \\ \frac{d}{dt}\Psi_t = \Delta_t\Psi_t, \end{cases} \tag{1-1}$$

where $\Delta_t\Psi_t$ denotes the tension field of Ψ_t with respect to the evolving metric $g_t := F_t^*\bar{g}$. We will call (1-1) the *hyper-Lagrangian mean curvature flow* (HLMCF).

Like other success stories of coupled flows (cf. [Müller 2012; Smoczyk 2000]), the two geometric flows (1-1) can interact with each other to reveal better properties than either had by itself. For any hyper-Lagrangian submanifold $F : L \to M$, we introduce the *twistor energy* of L as the Dirichlet energy of the complex phase Ψ with respect to the induced metric $g := F^*\bar{g}$:

$$\mathcal{T}(L) := \int_L |\nabla\Psi|^2 \, d\mu,$$

where $d\mu$ denotes the Riemannian volume of g. Intuitively, the twister energy measures the deviation from L being complex Lagrangian. We can show that any hyper-Lagrangian submanifold which is "almost" complex Lagrangian can be deformed to a genuine one in the following sense:

Theorem 1.1 (convergence of the HLMCF). *Let $(M, \bar{g})$ be a hyper-Kähler $4n$-manifold with bounded geometry. Suppose L is a hyper-Lagrangian submanifold with the complex phase Ψ_0 which is smoothly immersed into M. Then for any V_0, Λ_0 and $\delta_0 > 0$, there exists $\varepsilon_0 = \varepsilon_0(n, V_0, \Lambda_0, \delta_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M)) > 0$ such that if L satisfies*

$$\mathrm{Vol}(L_0) \leqslant V_0, \quad |A|(0) \leqslant \Lambda_0, \quad \lambda_1(\Delta_L)(0) \geqslant \delta_0, \quad \mathcal{T}(L_0) \leqslant \varepsilon_0,$$

then the hyper-Lagrangian mean curvature flow (1-1) *starting from L converges smoothly, exponentially fast to a complex Lagrangian submanifold in M for one of the hyper-Kähler complex structures on M.*

In the above theorem, we need not assume that M has a complex Lagrangian submanifold, so it also gives an existence result for such a submanifold as well as the stability along the MCF. Although generic K3 surfaces do not have holomorphic curves at all, it is also interesting to understand this situation from a geometric analytic point of view. Applying our theorem, one can immediately see that the twistor energy causes some gap: for any V_0, Λ_0 and $\delta_0 > 0$ we define

$$\mathcal{L}(V_0, \Lambda_0, \delta_0) := \left\{ L \subset M \;\middle|\; \begin{array}{l} L \text{ is a hyper-Lagrangian submanifold,} \\ \mathrm{Vol}(L) \leqslant V_0,\ |A| \leqslant \Lambda_0,\ \lambda_1(\Delta_L) \geqslant \delta_0 \end{array} \right\}.$$

Then we have the following:

Corollary 1.2 (energy gap theorem). *Assume a $4n$-dimensional hyper-Kähler manifold M with bounded geometry has no complex Lagrangian submanifolds. Then for any V_0, Λ_0 and $\delta_0 > 0$, there exists a constant $c = c(n, V_0, \Lambda_0, \delta_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M)) > 0$ such that*

$$\inf_{L \in \mathcal{L}(V_0, \Lambda_0, \delta_0)} \mathcal{T}(L) \geqslant c.$$

The proof of Theorem 1.1 is based on [Li 2012] for the Lagrangian mean curvature flow (LMCF). In [Li 2012], the crucial step is to establish the exponential estimate for the L^2-norm of mean curvature vector H by using the fact that each submanifold L_t is Lagrangian, which is not valid for our case. Instead, we take an alternative approach from the view point of the theory of harmonic map flow. A key observation is that the L^2-norm of H is bounded by the twistor energy (see Proposition 2.4):

$$\int_{L_t} |H_t|^2 \, d\mu_t \leqslant 2\mathcal{T}(L_t).$$

So the problem comes down to establishing the exponential estimate for the twistor energy, which is indeed possible along the same line as the usual harmonic map flow (see Lemma 3.4). Note that for the harmonic map flow into positively curved targets, the flow possibly forms singularities in finite time even if it has small initial Dirichlet energy [Chen and Ding 1990]. We overcome this by showing Proposition 2.5.

1B. ***Examples and relation to other results.*** This paper is entirely written for hyper-Lagrangian submanifolds of arbitrary dimension. But after we posted the preprint, we noticed Qiu and Sun's result [2019] which states that every hyper-Lagrangian except surface must be a complex Lagrangian, so the concept of the hyper-Lagrangian is meaningful only when $n = 1$. However, we emphasize that our results are new even when $n = 1$. Contrary to the higher-dimensional case, the concept of hyper-Lagrangian surface is universal and enables us to make a systematic study of several conditions for submanifolds preserved under the MCF. We can see that every surface L in a hyper-Kähler 4-manifold M admits a canonical complex phase map $\Psi : L \to \mathbb{S}^2$ defined by

$$J_\Psi e_1 = e_2, \quad J_\Psi e_3 = -e_4,$$

where $\{e_1, e_2, e_3, e_4\}$ is any oriented orthonormal frame on TM such that $\{e_1, e_2\}$ is an oriented frame on TL and $\{e_3, e_4\}$ is an orthonormal frame for the normal bundle. Indeed, the map Ψ is independent of the choice for such a frame. In the following, we will explain each class of submanifolds separately while considering what shape each complex phase is (see also [Leung and Wan 2007]).

1B.1. *Symplectic mean curvature flow.* First, we consider symplectic surfaces. Yau asked (for instance, see [Wang 2001]) "*how can a symplectic submanifold be deformed to a holomorphic one?*" Since a symplectic surface remains symplectic along the MCF in a Kähler–Einstein surface (see [Chen and Li 2001; Wang 2001]), one expects that the symplectic mean curvature flow (SMCF) is applicable to Yau's question. It seems that the convergence of the SMCF with small initial data has not been accomplished yet in the general case, whereas we know several partial results. For instance, our theorem generalizes Han and Sun's result [2012, Corollary 4.6]: we express Ψ as a map $\boldsymbol{a} : L \to \mathbb{R}^3$, i.e., $\boldsymbol{a}$ is a coefficient of Ψ with respect to $\{J_d\}$,

$$J_\Psi = \sum_d a_d J_d, \quad \boldsymbol{a} := (a_1, a_2, a_3).$$

By using the quaternion relations, we see that

$$\cos\alpha := \bar{\omega}_{J_3}(e_1, e_2) = \bar{g}(J_3 e_1, e_2) = a_3. \tag{1-2}$$

Hence the condition that L is symplectic with respect to $\bar{\omega}_{J_3}$ is equivalent to saying that the image $\Psi(L)$ is contained in the hemisphere

$$\mathbb{S}^2_+ := \{(c_1, c_2, c_3) \in \mathbb{S}^2 \subset \mathbb{R}^3 | c_3 > 0\}.$$

Then the (local) angle α defined by (1-2) is called the *Kähler angle*. Applying the maximum principle to the evolution equation of $\boldsymbol{a}$, we find that the hemisphere condition is preserved under the HLMCF (see Corollary 3.2), which is essentially a restatement of the fact as explained above that *if the initial surface is symplectic, then the surface is still symplectic along the mean curvature flow*. In [Han and Sun 2012], they showed the convergence of the SMCF under the stronger assumption that the ambient Kähler surface M has zero sectional curvature and the initial L^2-norm of A is very small. Also there is a convergence result for the SMCF in Kähler–Einstein surfaces with positive Ricci curvature by Han and Li [2005], where the positivity of the extrinsic curvature was essentially used. Theorem 1.1 indicates that the MCF method is still valid for Yau's question, and makes the first step in this direction.

1B.2. *Lagrangian mean curvature flow.* Next, we explain the Lagrangian case. If L is Lagrangian with respect to $\bar{\omega}_{J_o}$ for a fixed $J_o \in \mathbb{S}^2$, then without loss of generality, we may assume $J_3 = J_o$. By the Lagrangian condition, we find that L has the J_3-orthogonal complex phase J_Ψ which can be expressed as

$$J_\Psi(x) = \cos\theta(x) J_1 + \sin\theta(x) J_2 \tag{1-3}$$

for some multivalued function $\theta : L \to \mathbb{R}$. Moreover, the functions θ and $\bar{\omega}_{J_o}$ are related by the formula

$$i_H \bar{\omega}_{J_o} = d\theta. \tag{1-4}$$

So θ is nothing but the *Lagrangian angle*. In particular, we often consider the following special cases:

(1) The form $i_H \bar{\omega}_{J_o}$ is exact, or equivalently, θ is a single-valued function.

(2) The submanifold L is *almost calibrated*, i.e., L satisfies (1) and $\cos\theta > 0$.

As it is Lagrangian, these two conditions are preserved under the MCF [Smoczyk 1999; Chen and Li 2001; Wang 2001]. The convergence result for the LMCF with small initial data was obtained by Li [2012, Theorem 1.2]. He showed this similar convergence result to Theorem 1.1 under the assumption (1) (but, we need not assume (2)) and that the initial L^2-norm of H is very small. So Theorem 1.1 is still meaningful even if L_0 is Lagrangian since we need not assume (1) in our theorem.

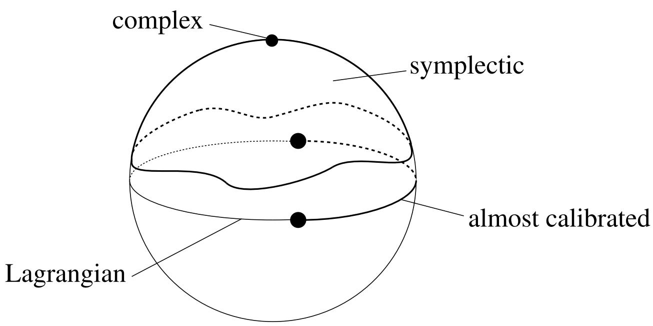

Figure 1. Image of the complex phase Ψ in $\mathbb{S}^2$.

Finally, we again emphasize the benefit of the hyper-Lagrangian submanifolds. In fact, the hyper-Lagrangian structure gives one a comprehensive view point to understand the concepts of symplectic surfaces or (almost calibrated) Lagrangian submanifolds in hyper-Kähler 4-manifolds. Figure 1 shows the correspondence between each of these concepts and the image of the complex phase map $\Psi : L \to \mathbb{S}^2$.

1B.3. *Holomorphic curves in K3.* On any polarized K3 surface (M, H) (with $H \not\simeq \mathcal{O}_M$), it is known that there exists at least one holomorphic curve which belongs to the linear system $|mH|$ for all $m \geqslant 1$ (Bogomolov, Mumford, Mori and Mukai [Mori and Mukai 1983]). Due to the Lefschetz theorem, the existence of such an H is equivalent to saying that the *Néron–Severi lattice*

$$\mathrm{NS}(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z})$$

is nonempty. Moreover, Chen [1999] proved the existence of infinitely many holomorphic curves on general K3 surfaces. Then we can take any small perturbation of the holomorphic curves as an initial data in Theorem 1.1.

1C. ***Organization of the paper.*** Our article will be organized as follows. We will first recall some results discovered by Leung and Wan [2007] and prove formulas relating the mean curvature vector (or second fundamental form) with the complex phase which are needed in the rest of the article. In Section 3, we study the behavior of the twistor energy and first eigenvalue along the HLMCF, and then establish some parabolic estimates. Finally, we give the proof of Theorem 1.1 in the last part of Section 3.

2. Hyper-Lagrangian submanifolds

In this section, we recall some results about hyper-Lagrangian submanifolds studied in [Leung and Wan 2007]. Let M be a hyper-Kähler $4n$-manifold and $L \subset M$ a real submanifold of dimension $2n$. In this section, we show the indices $(i, j, \alpha, \beta, \text{etc.})$

run in the following manner:

$$i, j = 1, \dots, 2n, \quad \alpha, \beta = 2n+1, \dots, 4n, \quad A, B = 1, \dots, 4n,$$
$$\nu, \lambda = 1, \dots n, \quad \mu, \rho = n+1, \dots, 2n.$$

Definition 2.1. A submanifold L is called hyper-Lagrangian if $\Omega_{\Psi(x)}|_L = 0$ at every point $x \in L$ for some $\Psi : L \to \mathbb{S}^2$. Then Ψ is called the complex phase. In particular, a hyper-Lagrangian submanifold is called complex Lagrangian if we can take Ψ as a constant map.

Let $\Phi : L \to \mathbb{S}^2$ be a smooth map such that $\Phi(x)$ is orthogonal to $\Psi(x)$ for each $x \in L$. We can take a special orthonormal frame $\{e_i\}$ for TL satisfying

$$J_\Psi e_{2\nu-1} = e_{2\nu}.$$

Then $\{e_{i+2n} := J_\Phi e_i\}$ is an orthonormal frame for the normal bundle satisfying

$$J_\Psi e_{2\mu-1} = -e_{2\mu}.$$

Then $\{e_A\}$ defines a frame of TM. For a hyper-Lagrangian submanifold L with the complex phase Ψ, we denote the associated almost-complex structure by J_Ψ. Then the complex phase J_Ψ acts on TL, and determines an almost-complex structure on L. However, hyper-Lagrangian is a strong condition which imposes a lot of restrictions on the structural equations. For instance, let $\{\varphi_{AB}\}$ be the connection forms with respect to $\{e_A\}$, i.e., $\overline{\nabla} e_A = \varphi_{AB} e_B$. Then the structure theorem of hyper-Lagrangian submanifolds (see [Leung and Wan 2007, Theorem 4.1]) implies

$$\begin{aligned} &\varphi_{2\nu-1,2\lambda-1} = \varphi_{2\nu,2\lambda}, \quad &\varphi_{2\nu,2\lambda-1} = -\varphi_{2\nu-1,2\lambda}, \\ &\varphi_{2\mu-1,2\rho-1} = -\varphi_{2\mu,2\rho}, \quad &\varphi_{2\mu,2\rho-1} = \varphi_{2\mu-1,2\rho}. \end{aligned} \tag{2-1}$$

As a consequence, we obtain the following:

Theorem 2.2 [Leung and Wan 2007, Corollary 4.2]. *The complex phase Ψ induces an integrable Kähler structure $(J_\Psi, \bar{g}|_L)$ on L with holomorphic normal bundle.*

We set

$$e'_\nu = \tfrac{1}{2}(e_{2\nu-1} - \sqrt{-1}e_{2\nu}), \quad e''_\nu = \tfrac{1}{2}(e_{2\nu-1} + \sqrt{-1}e_{2\nu}),$$

$$e'_\mu = \tfrac{1}{2}(e_{2\mu-1} + \sqrt{-1}e_{2\mu}), \quad e''_\mu = \tfrac{1}{2}(e_{2\mu-1} - \sqrt{-1}e_{2\mu}).$$

Then $\{e'_\nu, e'_\mu\}$ defines a complex basis referred to as the *canonical frame adapted to* (Ψ, Φ). Correspondingly, we take the basis $\{\zeta_A\}$ dual to $\{e_A\}$ and set

$$\zeta'_\nu = \zeta_{2\nu-1} + \sqrt{-1}\zeta_{2\nu}, \quad \zeta''_\nu = \zeta_{2\nu-1} - \sqrt{-1}\zeta_{2\nu},$$

$$\zeta'_\mu = \zeta_{2\mu-1} - \sqrt{-1}\zeta_{2\mu}, \quad \zeta''_\mu = \zeta_{2\mu-1} + \sqrt{-1}\zeta_{2\mu}.$$

With this basis, Ω_Ψ can be written as

$$\Omega_\Psi = -\sqrt{-1}\sum_{\nu,\mu} \zeta'_\nu \wedge \zeta'_\mu.$$

Leung and Wan [2007, Theorem 4.5] found the formula relating the mean curvature vector H and the complex phase Ψ as follows:

Proposition 2.3. *We have*

$$i_H \Omega_\Psi + 2\sqrt{-1}\partial\Psi = 0. \tag{2-2}$$

In particular, the above proposition shows that a hyper-Lagrangian submanifold L is minimal if and only if the complex phase Ψ is antiholomorphic. Meanwhile, by using the formula (2-2), one can obtain a bound for $|H|$ by means of the energy density of the complex phase Ψ:

Proposition 2.4. *We have*

$$|H|^2 \leqslant 2|\nabla\Psi|^2.$$

Proof. For a fixed $x \in L$, we set

$$J'_1 = J_\Psi(x), \quad J'_2 = J_\Phi(x), \quad J'_3 = J'_1 J'_2.$$

We would like to call it the *canonical basis adapted to* (Ψ, Φ) *at* x. Then we set the coefficient $\boldsymbol{a}' = (a'_1, a'_2, a'_3)$ as $J_\Psi = \sum_d a'_d J'_d$. We take a local representation of Ψ:

$$\Theta(p) = \frac{a'_1(p) + \sqrt{-1}a'_2(p)}{1 - a'_3(p)}$$

via stereographic projection. Then the formula (2-2) yields that

$$i_H \Omega_\Psi + 2\sqrt{-1}\partial\Theta = 0 \quad \text{at } x.$$

From the construction, we know that

$$a'_1(x) = 1, \quad a'_2(x) = a'_3(x) = 0, \quad \Theta(x) = 1.$$

Also since L is hyper-Lagrangian with the complex phase Ψ, the derivative $\overline{\nabla} J_\Psi$ is spanned by J'_2 and J'_3 at x, so

$$da'_1|_x = 0.$$

Thus we have

$$\begin{aligned}
\partial\Theta|_x &= \sqrt{-1}\partial a'_2|_x + \partial a'_3|_x, \\
|\partial\Theta|^2 &\leqslant 2(|\partial a'_2|^2 + |\partial a'_3|^2) = |da'_2|^2 + |da'_3|^2 = |\nabla \boldsymbol{a}'|^2
\end{aligned}$$

at x. On the other hand, if we set $H = -\sum_\alpha H^\alpha e_\alpha$, one can easily observe that

$$i_H\Omega_\Psi = -\sqrt{-1}\sum_\mu (H^{2\mu-1} - \sqrt{-1}H^{2\mu})\zeta'_\mu,$$

$$|i_H\Omega_\Psi|^2 = 2|H|^2.$$

So we have

$$|H|^2 = 2|\partial\Theta|^2 \leqslant 2|\nabla \boldsymbol{a}'|^2.$$

We note that $\boldsymbol{a}'$ and Θ heavily depend on the choice of the basis (J'_1, J'_2, J'_3) whereas $\boldsymbol{a}$ only depends on the background basis (J_1, J_2, J_3). However, the point is that the norm $|\nabla\boldsymbol{a}'|^2$ is independent of the choice of an orthogonal basis (J'_1, J'_2, J'_3) since the Euclidean metric on $\mathbb{R}^3$ is invariant under the standard $O(3)$-action. So we have $|\nabla\boldsymbol{a}'| = |\nabla\boldsymbol{a}| = |\nabla\Psi|$ and $|H|^2 \leqslant 2|\nabla\Psi|^2$. □

We also note that the quantity $|\nabla\Psi|$ has the following three equivalent definitions:

- We regard the complex phase Ψ as a map $\boldsymbol{a} : L \to \mathbb{S}^2 \subset \mathbb{R}^3$, and define $|\nabla\Psi|$ as the energy density of $\boldsymbol{a}$:

$$|\nabla\boldsymbol{a}|^2 = \sum_d |\nabla a_d|_g^2.$$

- We define $|\nabla\Psi|$ as the energy density of $\Psi : L \to \mathbb{S}^2$, i.e., a map into $\mathbb{S}^2$ (also see (3-2)).
- We define $|\nabla\Psi|$ as the norm of the covariant derivative of J_Ψ along L:

$$|\overline{\nabla} J_\Psi|^2 = \sum_{i,A,B} \bar{g}((\overline{\nabla}_i J)(e_A), e_B)^2,$$

  where $\overline{\nabla}$ denotes the Levi-Civita connection on the ambient space $(M, \bar{g})$. Then, taking into account the fact that $\{J_d\}$ is parallel and $\langle J_d, J_e\rangle_{\bar{g}} = 4n\delta_{de}$, we have $\overline{\nabla} J_\Psi = \sum_d da_d \otimes J_d$ and $|\overline{\nabla} J_\Psi| = 2\sqrt{n}|\nabla\boldsymbol{a}|$.

As for the relation to the second fundamental form A, we have the following:

Proposition 2.5. *In the canonical frame adapted to* (Ψ, Φ), *the quantity* $|\overline{\nabla} J_\Psi|^2$ *is expressed as*

$$|\overline{\nabla} J_\Psi|^2 = 4\sum_{i,\nu,\mu}[(h^{2\mu-1}_{2\nu,i} - h^{2\mu}_{2\nu-1,i})^2 + (h^{2\mu-1}_{2\nu-1,i} + h^{2\mu}_{2\nu,i})^2],$$

where $h^\alpha_{ij} := \bar{g}(e_i, \overline{\nabla}_j e_\alpha)$. *In particular, we have*

$$|\nabla\Psi| \leqslant c(n)|A|.$$

Proof. Set $J_{i,A,B} := \bar{g}(\overline{\nabla}_i J_\Psi(e_A), e_B)$ for simplicity. We compute

$$
\begin{aligned}
(\overline{\nabla} J_\Psi)(e_{2\nu-1}) &= \overline{\nabla}(e_{2\nu}) - J_\Psi(\overline{\nabla} e_{2\nu-1}) \\
&= \sum_j \varphi_{2\nu,j} e_j + \sum_\alpha \varphi_{2\nu,\alpha} e_\alpha - J_\Psi\biggl(\sum_j \varphi_{2\nu-1,j} e_j + \sum_\alpha \varphi_{2\nu-1,\alpha} e_\alpha\biggr).
\end{aligned}
$$

By using (2-1), we know that the first and third terms cancel each other out. So

$$
(\overline{\nabla} J_\Psi)(e_{2\nu-1}) = \sum_\mu [(\varphi_{2\nu,2\mu-1} - \varphi_{2\nu-1,2\mu}) e_{2\mu-1} + (\varphi_{2\nu,2\mu} + \varphi_{2\nu-1,2\mu-1}) e_{2\mu}],
$$

and hence

$$
J_{i,2\nu-1,j} = 0, \quad J_{i,2\nu-1,2\mu-1} = -h^{2\mu-1}_{2\nu,i} + h^{2\mu}_{2\nu-1,i}, \quad J_{i,2\nu-1,2\mu} = -h^{2\mu}_{2\nu,i} - h^{2\mu-1}_{2\nu-1,i}.
$$

In the same way, we can compute other terms by using (2-1) as follows:

$$
\begin{aligned}
J_{i,2\nu,j} &= 0, & J_{i,2\nu,2\mu-1} &= h^{2\mu}_{2\nu,i} + h^{2\mu-1}_{2\nu-1,i}, & J_{i,2\nu,2\mu} &= -h^{2\mu-1}_{2\nu,i} + h^{2\mu}_{2\nu-1,i}, \\
J_{i,2\mu-1,\alpha} &= 0, & J_{i,2\mu-1,2\nu-1} &= h^{2\mu-1}_{2\nu,i} - h^{2\mu}_{2\nu-1,i}, & J_{i,2\mu-1,2\nu} &= -h^{2\mu}_{2\nu,i} - h^{2\mu-1}_{2\nu-1,i}, \\
J_{i,2\mu,\alpha} &= 0, & J_{i,2\mu,2\nu-1} &= h^{2\mu-1}_{2\nu-1,i} + h^{2\mu}_{2\nu,i}, & J_{i,2\mu,2\nu} &= h^{2\mu-1}_{2\nu,i} - h^{2\mu}_{2\nu-1,i}.
\end{aligned}
$$

So we obtain the desired formula. □

3. Hyper-Lagrangian mean curvature flow

3A. ***Evolution of the coefficient vector.*** We regard the complex phase Ψ as a map into $\mathbb{S}^2 \subset \mathbb{R}^3$ and write $\boldsymbol{a} = (a_1, a_2, a_3)$. We compute the evolution equation of $\boldsymbol{a}$ when Ψ evolves along the generalized harmonic map flow $\frac{d}{dt}\Psi = \Delta_t \Psi$.

Lemma 3.1. *Along the HLMCF, $\boldsymbol{a}$ satisfies*

$$
\left(\frac{d}{dt} - \Delta_t\right)\boldsymbol{a} = |\nabla \boldsymbol{a}|^2 \boldsymbol{a}. \tag{3-1}
$$

Proof. We take a polar coordinate (θ, φ) of $\mathbb{S}^2$ and express $\boldsymbol{a}$ as

$$
\boldsymbol{a} = \begin{pmatrix} \cos\Psi^\theta \sin\Psi^\varphi \\ \sin\Psi^\theta \sin\Psi^\varphi \\ \cos\Psi^\varphi \end{pmatrix},
$$

where we write $\Psi^\theta = \theta \circ \Psi$, $\Psi^\varphi = \varphi \circ \Psi$ for simplicity. Then

$$
\frac{d}{dt}\Psi = \frac{d}{dt}\Psi^\theta \cdot \frac{\partial}{\partial\theta} \circ \Psi + \frac{d}{dt}\Psi^\varphi \cdot \frac{\partial}{\partial\varphi} \circ \Psi.
$$

Let $(x^1, \dots, x^{2n})$ be a local coordinate in L. Recall the definition of the tension field of Ψ:

$$\Delta\Psi = \sum_{i,j=1}^{n} g^{ij}\hat{\nabla}_i\hat{\nabla}_j\Psi^\theta \cdot \frac{\partial}{\partial\theta}\circ\Psi + \sum_{i,j=1}^{n} g^{ij}\hat{\nabla}_i\hat{\nabla}_j\Psi^\varphi \cdot \frac{\partial}{\partial\varphi}\circ\Psi \in C^\infty(\Psi^{-1}T\mathbb{S}^2),$$

where $\hat{\nabla}$ denotes the canonical connection on $\Psi^{-1}T\mathbb{S}^2$ associated to g and the standard metric $\tilde{g}$ on $\mathbb{S}^2$. Then

$$\hat{\nabla}_i\hat{\nabla}_j\Psi^\alpha = \nabla_i\nabla_j\Psi^\alpha + \sum_{\beta,\gamma=\theta,\varphi} \tilde{\Gamma}^\alpha_{\beta\gamma}(\Psi)\frac{\partial\Psi^\beta}{\partial x^i}\cdot\frac{\partial\Psi^\gamma}{\partial x^j}, \qquad \alpha = \theta, \varphi,$$

where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ denotes the Christoffel symbol with respect to $\tilde{g}$. We can easily compute

$$\begin{aligned} &\tilde{g}_{\theta\theta} = \sin^2\varphi, && \tilde{g}_{\theta\varphi} = 0, && \tilde{g}_{\varphi\varphi} = 1, \\ &\tilde{\Gamma}^\theta_{\theta\theta} = \tilde{\Gamma}^\varphi_{\varphi\varphi} = 0, && \tilde{\Gamma}^\theta_{\theta\varphi} = \frac{\cos\varphi}{\sin\varphi}, && \tilde{\Gamma}^\varphi_{\theta\theta} = -\sin\varphi\cos\varphi. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{d}{dt}\Psi^\theta &= \sum_{i,j=1}^{n} g^{ij}\hat{\nabla}_i\hat{\nabla}_j\Psi^\theta = \Delta\Psi^\theta + \frac{\cos\Psi^\varphi}{\sin\Psi^\varphi}\cdot\langle\nabla\Psi^\theta, \nabla\Psi^\varphi\rangle_g, \\ \frac{d}{dt}\Psi^\varphi &= \sum_{i,j=1}^{n} g^{ij}\hat{\nabla}_i\hat{\nabla}_j\Psi^\varphi = \Delta\Psi^\varphi - \sin\Psi^\varphi\cos\Psi^\varphi\cdot|\nabla\Psi^\theta|_g^2. \end{aligned}$$

Since

$$\begin{aligned} \Delta a_3 &= -\sin\Psi^\varphi\cdot\Delta\Psi^\varphi - \cos\Psi^\varphi\cdot|\nabla\Psi^\varphi|_g^2, \\ |\nabla \boldsymbol{a}|^2 &= \sin^2\Psi^\varphi\cdot|\nabla\Psi^\theta|_g^2 + |\nabla\Psi^\varphi|_g^2, \end{aligned} \tag{3-2}$$

we have

$$\frac{d}{dt}a_3 = -\sin\Psi^\varphi\cdot\frac{d}{dt}\Psi^\varphi = \Delta a_3 + |\nabla\boldsymbol{a}|^2 a_3.$$

We can compute the evolution equation of a_1 and a_2 in the similar way. □

Applying the maximum principle to (3-1), we obtain

Corollary 3.2 (also see [Leung and Wan 2007, Theorem 5.1]). *If L_0 satisfies $a_3 > c$ for some constant $c \in (0, 1)$ then $a_3 > c$ holds along the HLMCF L_t for all $t \in [0, T]$. In particular, the hemisphere condition $\Psi(L) \subset \mathbb{S}^2_+$ is preserved under the HLMCF.*

3B. *L^2-estimates.* Let $L \subset M$ be a hyper-Lagrangian submanifold with the complex phase Ψ.

Definition 3.3. We define the twistor energy of L as the Dirichlet energy of the complex phase:

$$\mathcal{T}(L) := \int_L |\nabla\Psi|^2 \, d\mu.$$

By using (3-1), we can obtain the exponential estimate for the twistor energy:

Lemma 3.4 (exponential estimate for the twistor energy). *For the HLMCF L_t,*

$$\frac{d}{dt}\mathcal{T}(L_t) \leqslant \big(-2\lambda_1(t) + C(n)\max_{L_t}|H||A| + 2\max_{L_t}|\nabla\Psi|^2\big) \cdot \mathcal{T}(L_t),$$

where $\lambda_1(t) > 0$ denotes the first eigenvalue of the Laplacian Δ_t.

Proof. First, we recall the evolution of the Riemannian metric on L (for instance, see [Chen and Li 2001]):

$$\frac{d}{dt} g_{ij} = -2H^\alpha h^\alpha_{ij}.$$

By using this and the expression of the energy density as the norm of the coefficient vector $|\nabla\Psi|^2 = |\nabla \boldsymbol{a}|^2$, we compute

$$\begin{aligned}\frac{d}{dt}&\int_L |\nabla \boldsymbol{a}|^2 \, d\mu_t \\ &= 2\int_L \langle \nabla \frac{d}{dt}\boldsymbol{a}, \nabla \boldsymbol{a}\rangle \, d\mu_t + \int_L \sum_d \frac{d}{dt} g^{ij} \nabla_i a_d \nabla_j a_d \, d\mu_t - \int_L |\nabla \boldsymbol{a}|^2 |H|^2 \, d\mu_t.\end{aligned}$$

We estimate each term separately. The first term is

$$\begin{aligned} 2\int_L \langle \nabla \frac{d}{dt}\boldsymbol{a}, \nabla \boldsymbol{a}\rangle \, d\mu_t &= 2\int_L \langle \nabla((\Delta + |\nabla \boldsymbol{a}|^2)\boldsymbol{a}), \nabla \boldsymbol{a}\rangle \, d\mu_t \\ &= -2\int_L |\Delta \boldsymbol{a}|^2 \, d\mu_t - 2\int_L |\nabla \boldsymbol{a}|^2 \langle \boldsymbol{a}, \Delta \boldsymbol{a}\rangle \, d\mu_t \\ &\leqslant -2\lambda_1 \int_L |\nabla \boldsymbol{a}|^2 \, d\mu_t + 2\int_L |\nabla \boldsymbol{a}|^4 \, d\mu_t \\ &\leqslant -2\lambda_1 \int_L |\nabla \boldsymbol{a}|^2 \, d\mu_t + 2\max_{L_t}|\nabla \boldsymbol{a}|^2 \int_L |\nabla \boldsymbol{a}|^2 \, d\mu_t, \end{aligned}$$

where we used the formula

$$0 = \left\langle \frac{d}{dt}\boldsymbol{a}, \boldsymbol{a} \right\rangle = \langle (\Delta + |\nabla \boldsymbol{a}|^2)\boldsymbol{a}, \boldsymbol{a}\rangle = \langle \Delta \boldsymbol{a}, \boldsymbol{a}\rangle + |\nabla \boldsymbol{a}|^2,$$

which can be proved easily by differentiating $|\boldsymbol{a}|^2 = 1$ in t. For the second term, we have

$$\left|\int_L \sum_d \frac{d}{dt} g^{ij} \nabla_i a_d \nabla_j a_d \, d\mu_t\right| = \left|2 \int_L \sum_d H^\alpha h^\alpha_{ij} \nabla_i a_d \nabla_j a_d \, d\mu_t\right|$$
$$\leqslant C(n) \max_{L_t} |H||A| \cdot \int_L |\nabla \boldsymbol{a}|^2 \, d\mu_t. \qquad \square$$

The above lemma says that we need to control λ_1 in order to obtain a bound for the twistor energy. So we establish the exponential estimate for λ_1 as follows:

Lemma 3.5 (exponential estimate for the first eigenvalue). *Along the HLMCF, the first eigenvalue $\lambda_1(t)$ satisfies*

$$\frac{d}{dt}\lambda_1 \geqslant -\big(\max_{L_t} |H|^2 + C(n) \max_{L_t} |H||A|\big) \cdot \lambda_1.$$

Proof. Let f be an eigenfunction with respect to λ_1, i.e., f satisfies

$$-\Delta_t f = \lambda_1 f, \qquad \int_L f^2 \, d\mu_t = 1.$$

Then the first eigenvalue λ_1 is

$$\lambda_1 = \int_L |\nabla f|^2 \, d\mu_t.$$

Differentiating $\int_L f^2 \, d\mu_t = 1$ in t, we have

$$\int_L \left(2 \frac{d}{dt} f \cdot f - f^2 |H|^2\right) d\mu_t = 0.$$

Thus we can compute

$$\frac{d}{dt}\lambda_1 = 2 \int_L \left\langle \nabla \frac{d}{dt} f, \nabla f \right\rangle d\mu_t + \int_L \frac{d}{dt} g^{ij} \nabla_i f \nabla_j f \, d\mu_t - \int_L |\nabla f|^2 |H|^2 \, d\mu_t$$
$$= -2 \int_L \frac{d}{dt} f \cdot \Delta f \, d\mu_t + 2 \int_L H^\alpha h^\alpha_{ij} \nabla_i f \nabla_j f \, d\mu_t$$
$$+ \int_L f \Delta f \cdot |H|^2 \, d\mu_t + \int_L f \langle \nabla f, \nabla |H|^2 \rangle \, d\mu_t.$$

Using the relation $-\Delta f = \lambda_1 f$, we find that the first term and the third term cancel each other out. The second term can be estimated as

$$\left|2 \int_L H^\alpha h^\alpha_{ij} \nabla_i f \nabla_j f \, d\mu_t\right| \leqslant C(n) \max_{L_t} |H||A| \cdot \lambda_1.$$

The fourth term is

$$\begin{aligned}\int_L f\langle\nabla f, \nabla|H|^2\rangle\, d\mu_t &= -\int_L (f\Delta f + |\nabla f|^2)|H|^2\, d\mu_t \\ &= \lambda_1 \int_L f^2|H|^2\, d\mu_t - \int_L |\nabla f|^2|H|^2\, d\mu_t \\ &\geqslant -\max_{L_t}|H|^2 \cdot \lambda_1.\end{aligned}$$

Thus we obtain the desired result. □

3C. *C^0-estimates.* In order to get the C^0-estimates from L^2, the notion of a noncollapsing geodesic ball is convenient. Roughly speaking, the volume of each geodesic ball in L is bounded from below by that of the Euclidean geodesic ball of the same radius. Let N be a compact Riemannian m-manifold.

Definition 3.6. We say that

(1) A geodesic ball $B(x, \rho)$ in N is called κ-noncollapsed if

$$\frac{\mathrm{Vol}(B(y, s))}{s^m} \geqslant \kappa$$

whenever $B(y, s) \subset B(x, \rho)$.

(2) A compact Riemannian manifold N is called κ-noncollapsed on the scale r if every geodesic ball $B(x, s)$ is κ-noncollapsed for $s \leqslant r$.

Lemma 3.7. *Let (E, h, D) be a vector bundle with a fiber metric h and a compatible connection D over a compact Riemannian manifold N. Assume that N is κ-noncollapsed on the scale r. For any smooth section $\sigma \in C^\infty(E)$, if*

$$|D\sigma| \leqslant \Lambda, \qquad \int_N |\sigma|^2\, d\mu \leqslant \varepsilon \leqslant r^{m+2},$$

then

$$\max_N|\sigma| \leqslant (\Lambda + \kappa^{-1/2})\varepsilon^{1/(m+2)}.$$

Proof. Assume that $|\sigma|$ attains its maximum at a point $x_0 \in N$ and the statement does not hold, i.e.,

$$|\sigma(x_0)| > (\Lambda + \kappa^{-1/2})\varepsilon^{1/(m+2)}.$$

Then by setting $\delta := \varepsilon^{1/(m+2)}$, we get

$$\Lambda\delta = \Lambda\varepsilon^{1/(m+2)} < |\sigma(x_0)|.$$

Thus for any $x \in B(x_0, \delta)$, we have

$$|\sigma(x)| \geqslant |\sigma(x_0)| - \Lambda\delta > 0.$$

Integrating on $B(x_0, \delta)$ yields that

$$\varepsilon \geqslant \int_{B(x_0,\delta)} |\sigma|^2 \, d\mu \geqslant (|\sigma(x_0)| - \Lambda\delta)^2 \operatorname{Vol}(B(x_0, \delta)) \geqslant (|\sigma(x_0)| - \Lambda\delta)^2 \kappa \delta^m,$$

where we used $\delta = \varepsilon^{1/(m+2)} \leqslant r$ and the assumption that N is κ-noncollapsed on the scale r in the last inequality. So putting $\delta = \varepsilon^{1/(m+2)}$ into the above yields that $|\sigma(x_0)| \leqslant (\Lambda + \kappa^{-1/2})\varepsilon^{1/(m+2)}$, contradicting the assumption. □

Now we go back to our situation, so let L_t be the HLMCF in a hyper-Kähler $4n$-manifold M. Lemma 3.7 indicates that it is important to study the evolution of the volume ratio along the flow.

Lemma 3.8 (volume ratio estimate). *If L_0 is κ_0-noncollapsed on the scale r_0, then for any small geodesic ball $B_t(x, \rho)$ in L_t with radius $\rho \in (0, r_0)$, we have*

$$\operatorname{Vol}(B_t(x, \rho)) \geqslant \kappa_0 e^{-(2n+1)E(t)} \rho^{2n},$$

where $E(t)$ is given by

$$E(t) := \int_0^t (\max_{L_s} |H|^2 + \max_{L_s} |A||H|) \, ds.$$

Proof. Let γ_t be a length-minimizing unit-speed geodesic with respect to $g(t)$ joining p to $q \in B_t(p, \rho)$. Then for every t_0 we have

$$d_t(p, q) = \operatorname{Length}_{g(t)}(\gamma_t) \leqslant \operatorname{Length}_{g(t)}(\gamma_{t_0}),$$

and equality holds when $t = t_0$, which implies that

$$\frac{d}{dt} d_t(p, q)|_{t=t_0} = \frac{d}{dt} \operatorname{Length}_{g(t)}(\gamma_t)|_{t=t_0} = \frac{d}{dt} \operatorname{Length}_{g(t)}(\gamma_{t_0})|_{t=t_0}.$$

Thus we can compute

$$\left| \frac{d}{dt} d_t(p, q) \right| = \left| \frac{1}{2} \int_0^{d_t(p,q)} \frac{dg_t}{dt} \left(\frac{d}{ds}\gamma_t, \frac{d}{ds}\gamma_t \right) ds \right| \leqslant \max_{L_t} |A||H| \cdot d_t(p, q).$$

This implies that

$$e^{-E(t)} d_0(p, q) \leqslant d_t(p, q) \leqslant d_0(p, q) e^{E(t)}, \qquad d\mu_t \geqslant e^{-E(t)} d\mu_0.$$

Since L_0 is κ_0-noncollapsed on the scale r_0, for $\rho \leqslant r_0$, we have

$$\operatorname{Vol}(B_t(p, \rho)) = \int_{B_t(p,\rho)} d\mu_t \geqslant \int_{B_0(p, e^{-E(t)}\rho)} e^{-E(t)} \, d\mu_0 \geqslant \kappa_0 e^{-(2n+1)E(t)} \rho^{2n}. \quad □$$

3D. ***Some parabolic estimates for the HLMCF.*** In this subsection, we prove some parabolic estimates for the HLMCF. The first lemma says that the HLMCF does not change a lot in short time intervals.

Lemma 3.9. *If L_0 satisfies*

$$|A|(0) \leqslant \Lambda, \quad |\nabla\Psi|(0) \leqslant P, \quad \lambda_1(0) \geqslant \delta,$$

then there exists $T = T(n, \Lambda, \overline{\mathrm{Rm}})$ such that the HLMCF L_t satisfies

$$|A|(0) \leqslant 2\Lambda, \quad |\nabla\Psi|(t) \leqslant 2P, \quad \lambda_1(t) \geqslant \tfrac{2}{3}\delta, \quad t \in [0, T].$$

Proof. The estimate of $|A|$ follows from [Han and Sun 2012, Lemma 2.2]. Then the estimate of λ_1 follows from the exponential estimate for λ_1. Finally, we establish the estimate for $|\nabla\Psi|$. By the Bochner identity, the Gauss equation and Proposition 2.5, we can compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta_t\right)|\nabla\Psi|^2 &= -2|\nabla^2\Psi|^2 + \mathrm{Rm}^{\mathbb{S}^2} *(\nabla\Psi)^4 + \overline{\mathrm{Rm}} *(\nabla\Psi)^2 + A^2 * (\nabla\Psi)^2 \\ &\leqslant C(n, \Lambda, \overline{\mathrm{Rm}})|\nabla\Psi|^2.\end{aligned}$$

Applying the maximum principle, we obtain

$$|\nabla\Psi|(t) \leqslant e^{\frac{1}{2}C(n,\Lambda,\overline{\mathrm{Rm}})t}|\nabla\Psi|(0) \leqslant e^{\frac{1}{2}C(n,\Lambda,\overline{\mathrm{Rm}})t}P,$$

so we may take $T \leqslant 2\log 2/(C(n, \Lambda, \overline{\mathrm{Rm}}))$. □

We can obtain not only the usual smoothing estimates for A, but also for Ψ with the help of Proposition 2.5.

Lemma 3.10 (smoothing estimates). *Suppose along the HLMCF, we have*

$$\sup_{L_t}|A| \leqslant \Lambda, \quad t \in [0, T],$$

for some $T > 0$. Then for each $l \geqslant 1$, there exist constants $\Lambda_l = \Lambda_l(n, \Lambda, \overline{\mathrm{Rm}}, T)$ such that

$$\sup_{L_t}|\nabla^l A| \leqslant \frac{\Lambda_l}{t^{l/2}}, \quad t \in (0, T].$$

Further, for any $t_0 \in (0, T]$, there exist constants $P_l = P_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)$ such that

$$\sup_{L_t}|\nabla^l \Psi_*| \leqslant P_l, \quad t \in [t_0, T],$$

where $\Psi_* = \nabla\Psi$ is the differential map of the complex phase $\Psi : L \to \mathbb{S}^2$.

Proof. The estimate of A follows from [Han and Sun 2012, Theorem 3.1]. Then for any $t_0 \in (0, T]$ we have

$$\sup_{L_t} |\nabla^l A| \leqslant \frac{\Lambda_l}{(t_0/2)^{l/2}}, \quad t \in [t_0/2, T].$$

We use this estimate to show the estimate of Ψ_*. Note also that $|\Psi_*|$ has a uniform bound $|\Psi_*| \leqslant c(n)|A| \leqslant c(n)\Lambda$ by Proposition 2.5.

In order to derive the estimate of Ψ_*, we first compute the time derivative of $|\nabla^l \Psi_*|^2$ along the generalized harmonic map flow. A straight calculation shows that for each $l \geqslant 0$ we get the formula

$$\begin{aligned}\frac{d}{dt}\nabla^l \Psi_* = \Delta(\nabla^l \Psi_*) &+ \sum_{r+i+j+k=l} \tilde{\nabla}^r \operatorname{Rm}^{\mathbb{S}^2} *(\Psi_*)^r * \nabla^i \Psi_* * \nabla^j \Psi_* * \nabla^k \Psi_* \\ &+ \sum_{r+i_i+\cdots+i_l+j=l} \overline{\nabla}^r\, \overline{\operatorname{Rm}} * \nabla^{i_1 -1} A * \cdots * \nabla^{i_l-1} A * \nabla^j \Psi_* \\ &+ \sum_{i+j+k=l} \nabla^i A * \nabla^j A * \nabla^k \Psi_*,\end{aligned}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection on $T\mathbb{S}^2$. It follows that for $t \in [t_0/2, T]$ we have

$$\begin{aligned}\text{(3-3)}\quad \frac{d}{dt}|\nabla^l \Psi_*|^2 &= A^2 * (\nabla^l \Psi_*)^2 + 2\left\langle \frac{d}{dt}\nabla^l \Psi_*, \nabla^l \Psi_*\right\rangle \\ &\leqslant \Delta|\nabla^l \Psi_*|^2 - 2|\nabla^{l+1}\Psi_*|^2 + C \sum_{0\leqslant i+j+k\leqslant l} |\nabla^i \Psi_*||\nabla^j \Psi_*||\nabla^k \Psi_*||\nabla^l \Psi_*|,\end{aligned}$$

where $C = C(n, \Lambda, \overline{\operatorname{Rm}}, t_0, T)$ is a constant. From (3-3) we have

$$\frac{d}{dt}|\Psi_*|^2 \leqslant \Delta|\Psi_*|^2 - 2|\nabla\Psi_*|^2 + c_1$$

and

$$\frac{d}{dt}|\nabla\Psi_*|^2 \leqslant \Delta|\nabla\Psi_*|^2 - 2|\nabla^2\Psi_*|^2 + c_2|\nabla\Psi_*|^2 + c_3,$$

where $c_k = c_k(n, \Lambda, \overline{\operatorname{Rm}}, t_0, T)$, $k = 1, 2, 3$, are constants. Set

$$F := (t - t_0/2)|\nabla\Psi_*|^2 + \alpha|\Psi_*|^2,$$

where α is a constant which will be determined later. It is not difficult to see

$$\left(\frac{d}{dt} - \Delta\right)F \leqslant (-2\alpha + 1 + Tc_2)|\nabla\Psi_*|^2 + \alpha c_1 + Tc_3.$$

Then we choose $\alpha = (1 + TC_2)/2$ to get

$$\left(\frac{d}{dt} - \Delta\right) F \leqslant \left(\frac{1 + Tc_2}{2}\right) c_1 + Tc_3.$$

Applying the maximum principle, we have

$$F(t) \leqslant F(0) \leqslant \left(\frac{1 + Tc_2}{2}\right)\Lambda^2 = C_1(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T), \quad t \in [t_0/2, T].$$

Hence we get

$$|\nabla \Psi_*|^2 \leqslant \frac{C_1}{t - t_0/2}, \quad t \in (t_0/2, T].$$

It follows that

$$\sup_{L_t} |\nabla \Psi_*| \leqslant \frac{6C_1}{t_0} = P_1(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T), \quad t \in [2t_0/3, T].$$

This proves the case $l = 1$.

For $l \geqslant 2$, our proof is by induction. Assume that the following estimate holds for each $0 \leqslant m \leqslant l - 1$:

$$\sup_{L_t} |\nabla^m \Psi_*| \leqslant \frac{(m+1)(m+2)C_m(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)}{t_0}, \quad t \in [((m+1)/(m+2))t_0, T].$$

Then by (3-3) we have

$$\frac{d}{dt}|\nabla^{l-1}\Psi_*|^2 \leqslant \Delta|\nabla^{l-1}\Psi_*|^2 - 2|\nabla^l \Psi_*|^2 + c_4$$

and

$$\frac{d}{dt}|\nabla^{l}\Psi_*|^2 \leqslant \Delta|\nabla^{l}\Psi_*|^2 - 2|\nabla^{l+1} \Psi_*|^2 + c_5|\nabla^l \Psi_*|^2 + c_6,$$

for $t \in [(l/(l+1))t_0, T]$, where $c_k = c_k(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)$, $k = 4, 5, 6$, are constants which are controlled by the lower order estimates. As for $l = 1$, using the maximum principle we see

$$|\nabla^l \Psi_*|^2 \leqslant \frac{C_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)}{t - (l/(l+1))t_0}, \qquad t \in ((l/(l+1))t_0, T].$$

Therefore we obtain the desired bound

$$|\nabla^l \Psi_*|^2 \leqslant \frac{(l+1)(l+2)C_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)}{t_0} =: P_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)$$

for $t \in [((l+1)/(l+2))t_0, T]$. □

Remark 3.11. From the smoothing estimates, for any $t_0 \in (0, T)$ we have

$$\sup_{L_t}|\nabla^l A| \leqslant \Lambda_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0), \quad \sup_{L_t}|\nabla^l \Psi| \leqslant P_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0), \quad t \in [t_0/2, t_0].$$

In particular, we have bounds for the derivatives $|\nabla^l A|$ and $|\nabla^l \Psi|$ for $l \geqslant 1$ at $t = t_0$. On the other hand, as in the proof of the above lemma, it is not difficult to see that we have bounds which only depend on n, $A(t_0)$ and $\Psi_*(t_0)$ (including their higher order derivatives):

$$\sup_{L_t}|\nabla^l A| \leqslant \Lambda_l(n, A(t_0), \overline{\mathrm{Rm}}), \quad \sup_{L_t}|\nabla^l \Psi| \leqslant P_l(n, A(t_0), \Psi_*(t_0), \overline{\mathrm{Rm}}), \quad t \in [t_0, T].$$

Combining both estimates on $[t_0, T]$, we obtain T-independent estimates

$$\sup_{L_t}|\nabla^l A| \leqslant \Lambda_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0), \qquad \sup_{L_t}|\nabla^l \Psi| \leqslant P_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0), \qquad t \in [t_0, T].$$

We often use this property without mentioning it in later arguments.

3E. *Convergence of the flow.* Now we are ready to prove the main theorem.

Theorem 3.12 (Theorem 1.1). *Let $(M, \bar{g})$ be a hyper-Kähler $4n$-manifold with bounded geometry. Suppose L is a hyper-Lagrangian submanifold with the complex phase Ψ_0 which is smoothly immersed into M. Then for any V_0, Λ_0 and $\delta_0 > 0$, there exists $\varepsilon_0 = \varepsilon_0(n, V_0, \Lambda_0, \delta_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M)) > 0$ such that if L satisfies*

$$\mathrm{Vol}(L_0) \leqslant V_0, \quad |A|(0) \leqslant \Lambda_0, \quad \lambda_1(\Delta_L)(0) \geqslant \delta_0, \quad \mathcal{T}(L_0) \leqslant \varepsilon_0,$$

then the hyper-Lagrangian mean curvature flow starting from L converges smoothly, exponentially fast to a complex Lagrangian submanifold in M for one of the hyper-Kähler complex structure on M.

Proof. Step 1 (reduction from L^2 to C^0): In the first step, we see that after a short period of time, the parabolicity of the flow improves the initial L^2-condition for $\nabla\Psi$ to the C^0-condition. From Proposition 2.5 and Lemma 3.9, we know that L_t satisfies

$$|A|(t) \leqslant 2\Lambda_0, \quad |\nabla\Psi|(t) \leqslant c(n)\Lambda_0, \quad \lambda_1(t) \geqslant \tfrac{2}{3}\delta_0, \quad t \in [0, T_0]$$

for $T_0 = T_0(n, \Lambda_0, \overline{\mathrm{Rm}})$. So Lemma 3.4 implies the following exponential estimate for the twistor energy:

$$\mathcal{T}(L_t) \leqslant e^{ct}\mathcal{T}(L_0) \leqslant \varepsilon_0 e^{ct}, \quad t \in [0, T_0]$$

for some $c = c(n, \Lambda_0) > 0$. Therefore we can choose $t_0 = t_0(n, \Lambda_0) \in (0, T_0]$ so that

$$\mathcal{T}(L_t) \leqslant 2\varepsilon_0, \quad t \in [0, t_0].$$

On the other hand, by the smoothing estimates, we know that for any $l \geqslant 1$,

$$|\nabla^l A|(t) \leqslant C_l(n, \Lambda_0, \overline{\mathrm{Rm}}), \qquad t \in [t_0/2, t_0], \tag{3-4}$$

and also

$$|\nabla^2 \Psi|(t) \leqslant c(n, \Lambda_0, \overline{\mathrm{Rm}}), \qquad t \in [t_0/2, t_0].$$

In order to get the estimate for the energy density $|\nabla \Psi|$, we need to establish the noncollapsing estimate for L_t first. By [Chen and He 2010, Proposition 2.2] and (3-4), we know that the injectivity radius of L is bounded from below along the HLMCF

$$\mathrm{inj}(L_t) \geqslant \iota(n, \Lambda_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M)) > 0, \quad t \in [t_0/2, t_0].$$

Meanwhile, the Gauss equation implies that

$$|\mathrm{Rm}| \leqslant C(\Lambda_0, \overline{\mathrm{Rm}}), \qquad t \in [t_0/2, t_0].$$

So in the same way as the proof of [Li 2012, Theorem 1.1], the volume comparison theorem shows there exist $\kappa = \kappa(n, \Lambda_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M))$ and $r = r(n, \Lambda_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M))$ such that L_t is κ-noncollapsed on the scale r for all $t \in [t_0/2, t_0]$. So Lemma 3.7 implies that

$$|\nabla \Psi|(t) \leqslant (c + \kappa^{-1/2})(2\varepsilon_0)^{\frac{1}{2n+2}} =: \eta, \quad t \in [t_0/2, t_0],$$

where we take ε_0 sufficiently small so that $2\varepsilon_0 \leqslant r^{2n+2}$.

Step 2 (ε_0-regularity): We set

$$\mathcal{A}(\kappa, r, \Lambda, P, \delta) := \left\{ L \subset M \;\middle|\; \begin{array}{l} L \text{ is a hyper-Lagrangian submanifold,} \\ L \text{ is } \kappa\text{-noncollapsed on the scale } r, \\ |A| \leqslant \Lambda, \quad |\nabla \Psi| \leqslant P, \quad \lambda_1(\Delta_L) \geqslant \delta \end{array} \right\}.$$

Without loss of generality, we regard $L_{t_0/2}$ as the initial data of the HLMCF, so

$$L_t \in \mathcal{A}(\kappa, r, \Lambda, \eta, \delta), \quad t \in [0, t_0/2],$$

where $\Lambda := 2\Lambda_0$, $\eta := (c+\kappa^{-1/2})(2\varepsilon_0)^{1/(2n+2)}$, $\delta := \frac{2}{3}\delta_0$. So Lemma 3.9 combining with the volume ratio estimate (see Lemma 3.8) implies that we can choose a small $T^* > 0$ such that

$$L_t \in \mathcal{A}\big(\tfrac{1}{3}\kappa, r, 6\Lambda, 2\eta^{\frac{1}{2n+2}}, \tfrac{1}{3}\delta\big), \quad t \in [0, T^*].$$

Let T^* be the maximal time such that the above estimate holds. Then in order to prove the long-time existence of the flow, it suffices to prove the following ε_0-regularity:

Claim 3.13. There exists a small $\eta > 0$ (and hence small $\varepsilon_0 > 0$) such that

$$L_t \in \mathcal{A}\big(\tfrac{2}{3}\kappa, r, 3\Lambda, \eta^{\frac{1}{2n+2}}, \tfrac{1}{2}\delta\big), \quad t \in [0, T^*].$$

Indeed, if $T^* < \infty$ then from the claim we have $L_t \in \mathcal{A}\big(\frac{2}{3}\kappa, r, 3\Lambda, \eta^{1/(2n+2)}, \frac{1}{2}\delta\big)$ for $t \in [0, T^*]$. By using Lemma 3.9 and the volume ratio estimate again, we find that there exists $\tilde{T} > T^*$ such that $L_t \in \mathcal{A}\big(\frac{1}{3}\kappa, r, 6\Lambda, 2\eta^{1/(2n+2)}, \frac{1}{3}\delta\big)$ for $t \in [0, \tilde{T}]$, contradicting the maximality of T^*.

First, we establish an estimate for $|\nabla\Psi|$. We know that

$$\lambda_1(t) \geqslant \tfrac{1}{3}\delta, \quad t \in [0, T^*].$$

So if we choose $\eta > 0$ small so that

$$\lambda_1(t) \geqslant \tfrac{1}{4}\delta + C(n) \cdot 3\Lambda \cdot 2\eta^{\frac{1}{2n+2}} + (2\eta^{\frac{1}{2n+2}})^2, \quad t \in [0, T^*],$$

then the exponential estimate for the twistor energy (see Lemma 3.4) implies

$$\mathcal{T}(L_t) \leqslant e^{-\frac{\delta}{2}t}\mathcal{T}(L_0) \leqslant \eta^2 V_0 e^{-\frac{\delta}{2}t}, \quad t \in [0, T^*].$$

By Lemma 3.9, there exists some $t^* = t^*(n, \Lambda, \overline{\mathrm{Rm}}) \in (0, T^*)$ such that

$$|\nabla\Psi| \leqslant 2\eta \leqslant \eta^{\frac{1}{2n+2}}, \quad t \in [0, t^*],$$

for $\eta \leqslant \frac{1}{2}$. On the other hand, since $|A|(t) \leqslant 6\Lambda$ for $t \in [0, T^*]$, the smoothing estimates imply that

$$|\nabla^2\Psi| \leqslant C(n, \Lambda, \overline{\mathrm{Rm}}), \quad t \in [t^*, T^*].$$

Thus we obtain

$$|\nabla\Psi|(t) \leqslant C(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}}) \cdot \eta^{\frac{1}{n+1}} e^{-\frac{\delta t}{4n+4}}, \qquad t \in [t^*, T^*]. \tag{3-5}$$

So we can choose $\eta > 0$ small so that

$$C(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}}) \cdot \eta^{\frac{1}{2n+2}} \leqslant 1$$

and obtain

$$|\nabla\Psi|(t) \leqslant \eta^{\frac{1}{2n+2}}, \quad t \in [0, T^*].$$

Next, we compute $|A|$. By the smoothing estimates, for any $l \geqslant 1$, we have

$$|\nabla^l A| \leqslant C_l(n, \Lambda, \overline{\mathrm{Rm}}), \quad t \in [t^*, T^*].$$

Thus we also have

$$|\nabla^l H| \leqslant C_l(n, \Lambda, \overline{\mathrm{Rm}}), \quad t \in [t^*, T^*].$$

From Proposition 2.4 and (3-5), we know that $|H|$ also decreases exponentially fast. So integrating by parts, we have

$$\int_{L_t} |\nabla^2 H|^2 \, d\mu_t \leqslant \int_{L_t} |H||\nabla^4 H| \, d\mu_t \leqslant C(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}})\eta^{\frac{1}{n+1}} e^{-\frac{\delta t}{4n+4}}$$

for $t \in [t^*, T^*]$. So we have

$$|\nabla^2 H| \leqslant c(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}})\eta^{\frac{1}{2(n+1)^2}} e^{-\frac{\delta t}{8(n+1)^2}}, \qquad t \in [t^*, T^*].$$

We recall the evolution equation of A along the MCF (see [Chen and Li 2001])

$$\frac{d}{dt} h_{ij}^{\alpha} = \nabla_i \nabla_j H^{\alpha} - H^{\beta} h_{jk}^{\beta} h_{ik}^{\alpha} + H^{\beta} \bar{R}_{\alpha j \beta i} + h_{ij}^{\beta} b_{\alpha}^{\beta},$$

where $b_{\alpha}^{\beta} = \bar{g}\big(\frac{d}{dt} e_{\alpha}, e_{\beta}\big) = \bar{g}(\overline{\nabla}_H e_{\alpha}, e_{\beta})$. Note that b_{α}^{β} is antisymmetric since

$$0 = \frac{d}{dt}(\bar{g}(e_{\alpha}, e_{\beta})) = b_{\alpha}^{\beta} + b_{\beta}^{\alpha}.$$

Then it follows that

$$h_{ij}^{\alpha} h_{ij}^{\beta} b_{\beta}^{\alpha} = 0.$$

So we compute

$$2|A| \frac{d}{dt}|A| = \frac{d}{dt}|A|^2 \leqslant c(n)(|\nabla^2 H||A| + |H||A||\overline{\mathrm{Rm}}| + |H||A|^3).$$

Dividing both sides by $|A|$, we have

$$\frac{d}{dt}|A| \leqslant c(n)(|\nabla^2 H| + |H||\overline{\mathrm{Rm}}| + |H||A|^2). \tag{3-6}$$

Meanwhile, Lemma 3.9 shows that

$$|A|(t) \leqslant 2\Lambda, \quad t \in [0, t^*].$$

So integrating (3-6) in t and using the exponential decay of $|H|$, we have

$$\begin{aligned} |A|(t) &\leqslant |A|(t^*) + c(n) \int_{t^*}^{t} (|\nabla^2 H| + |H||\overline{\mathrm{Rm}}| + |H||A|^2)\, ds \\ &\leqslant 2\Lambda + c(n)\bigg[c\eta^{\frac{1}{2(n+1)^2}} \frac{16(n+1)^2}{\delta} + (C(\overline{\mathrm{Rm}}) + 64\Lambda^2) \cdot c\eta^{\frac{1}{n+1}} \frac{8(n+1)}{\delta}\bigg]. \end{aligned}$$

Thus we can take $\eta > 0$ sufficiently small so that

$$|A|(t) \leqslant 3\Lambda, \quad t \in [0, T^*].$$

Then we establish the estimate for $\lambda_1(t)$. Since $\lambda_1(0) \geqslant \delta$, Lemma 3.9 shows that

$$\lambda_1(t) \geqslant \tfrac{2}{3}\delta, \quad t \in [0, t^*].$$

Thus the exponential estimate for λ_1 combined with the exponential decay of $|H|$ implies that

$$\begin{aligned}\lambda_1(t) &\geqslant \exp\biggl[-\int_{t^*}^{t} (\max_{L_s}|H|^2 + C(n)\max_{L_s}|H|\,|A|)\,ds\biggr]\lambda_1(t^*)\\ &\geqslant \exp\biggl[-c^2\eta^{\frac{2}{n+1}}\frac{4(n+1)}{\delta} - C(n)\cdot 3\Lambda\cdot c\eta^{\frac{1}{n+1}}\frac{8(n+1)}{\delta}\biggr]\lambda_1(t^*).\end{aligned}$$

If we take $\eta > 0$ sufficiently small, then

$$\lambda_1(t) \geqslant \tfrac{1}{2}\delta, \quad t \in [0, T^*].$$

We can prove a noncollapsing estimate of L_t in the same way as λ_1, by using the volume ratio estimate.

Step 3 (exponential convergence of the flow): From Step 2, we have a uniform bound for A. So the standard bootstrapping arguments combined with Simon's theorem [1983] imply the smooth convergence of the MCF $L_t \to L_\infty$. Moreover, we have already seen that for a fixed sufficiently small $\eta > 0$, we have

$$|\nabla\Psi(t)| \leqslant C(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}})\cdot\eta^{\frac{1}{n+1}}e^{-\frac{\delta t}{4n+4}} \searrow 0.$$

In particular, Proposition 2.4 implies that H_t converges exponentially fast to $H_\infty = 0$, and hence L_∞ is minimal.

As for the generalized harmonic map flow, we have also the uniform bounds $|\nabla^l\Psi| \leqslant C_l$ for all $l \geqslant 1$. Thus there exists a subsequence $\{\Psi_{t_i}\}$ which converges to a smooth map $\Psi_\infty : L \to \mathbb{S}^2$ and L_∞ inherits a hyper-Lagrangian structure with the complex phase Ψ_∞. Since $|\nabla\Psi_\infty| = 0$, the map Ψ_∞ should be a constant. Finally, we show that the complex phase Ψ_∞ which arises from the generalized harmonic map flow does not depend on the choice of the subsequence $\{\Psi_{t_i}\}$ by contradiction. So we assume that there exist two distinct constant phase maps Ψ_∞ and Ψ'_∞ which arise in this way. We take a small geodesic ball in $B \subset \mathbb{S}^2$ centered at Ψ_∞ so that $\Psi'_\infty \notin B$. Since $\{\Psi_{t_i}\}$ converges to Ψ_∞ we know that $\Psi_{t_i}(L) \subset B$ for i large enough. We fix such an i and consider the generalized harmonic map flow Ψ'_t starting from the data (L_{t_i}, Ψ_{t_i}). Then a simple maximum principle argument (see Corollary 3.2) shows that $\Psi'_t(L) \subset B$ for all $t \in [0, \infty)$ whereas $\{\Psi'_t\}$ should have a convergent subsequence to $\Psi'_\infty \notin B$, which is a contradiction. This completes the proof. □

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