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**THE TWO-DIMENSIONAL ANALOGUE OF THE
LORENTZIAN CATENARY AND THE DIRICHLET PROBLEM**

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THE TWO-DIMENSIONAL ANALOGUE OF THE LORENTZIAN CATENARY AND THE DIRICHLET PROBLEM

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We generalize in Lorentz–Minkowski space \mathbb{L}^3 the two-dimensional analogue of the catenary of Euclidean space. We solve the Dirichlet problem when the bounded domain is mean convex and the boundary data has a spacelike extension to the domain. We also classify all singular maximal surfaces of \mathbb{L}^3 invariant by a uniparametric group of translations and rotations.

1. Introduction and motivation

The purpose of this paper is to investigate the physical problem of characterizing the surfaces in Lorentz–Minkowski space with lowest gravity center and solve the corresponding Dirichlet problem. The existence of a variety of causal vectors in the Lorentzian setting leads to several issues that need to be fixed. Firstly, we recall this problem in the Euclidean space in order to motivate our definitions. Let \mathbb{R}^2 be the Euclidean plane with canonical coordinates (x, y) where the y -axis indicates the gravity direction. Consider the physical problem of finding the curve in the halfplane $y > 0$ with the lowest gravity center. If the curve is $y = u(x)$, then u satisfies the equation

$$(1) \quad \frac{u''}{1+u'^2} = \frac{1}{u}.$$

The solution of this equation is known as the catenary

$$u(x) = \frac{1}{a} \cosh(ax + b), \quad a, b \in \mathbb{R}, a \neq 0.$$

The equation (1) can be expressed in terms of the curvature κ of the curve as

$$(2) \quad \kappa = \frac{\langle \mathbf{n}, \vec{a} \rangle}{y},$$

where \mathbf{n} is the unit normal vector and $\vec{a} = (0, 1)$. In particular, (2) prescribes the angle that the vector \mathbf{n} makes with the vertical direction.

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The generalization in Euclidean 3-space \mathbb{R}^3 of the property of the catenary is to find surfaces in the halfspace $z > 0$ with the lowest gravity center. If (x, y, z) denote the canonical coordinates of \mathbb{R}^3 and z indicates the direction of the gravity, these surfaces characterize by means of the equation

$$H = \frac{\langle N, \vec{a} \rangle}{z},$$

where H is the mean curvature of the surface and $\vec{a} = (0, 0, 1)$. The surface is called the *two-dimensional analogue of the catenary* ([Böhme et al. 1980; Dierkes and Huisken 1990]). Historically, this problem goes back to early works of Lagrange and Poisson on the equation that models a heavy surface in a vertical gravitational field. If we embed \mathbb{R}^2 as the xz -plane by identifying the y -axis of \mathbb{R}^2 with the z -axis of \mathbb{R}^3 , and we rotate the catenary with respect to the x -axis, we obtain the catenoid $a^2(y^2 + z^2) = \cosh^2(x)$, which is the only nonplanar rotational minimal surface of \mathbb{R}^3 .

More generally, given a constant $\alpha \in \mathbb{R}$, a surface in the halfspace $z > 0$ is called a singular minimal surface if it satisfies

$$(3) \quad H = \alpha \frac{\langle N, \vec{a} \rangle}{z}.$$

The theory of singular minimal surfaces has been intensively studied in the works of Bemelmans, Dierkes and Huisken, among others (see, for example, [Bemelmans and Dierkes 1987; Böhme et al. 1980; Dierkes 1988a; 2003; Dierkes and Huisken 1990; López 2018a; 2018b; 2019b; Nitsche 1986]). Now that we have presented the problem in the Euclidean space, we proceed to generalize it in the Lorentz–Minkowski space. As in the Euclidean case, we begin with the one-dimensional case. Let \mathbb{L}^2 be the Lorentz–Minkowski plane defined as the affine (x, y) -plane \mathbb{R}^2 endowed with the metric $dx^2 - dy^2$. Here we use the usual terminology of the Lorentz–Minkowski space: see [O’Neill 1983] as a general reference and [López 2014] for curves and surfaces in Lorentz–Minkowski space. In what follows, we will assume that for a given set, the causal character is the same in all its points, that is, we do not admit the existence of points with different causal character.

A first issue is that the notion of gravity in \mathbb{L}^2 does not make sense because the y -coordinate represents the time in the Lorentzian context. Thus we need to view the initial problem as a problem of finding curves in \mathbb{L}^2 with prescribed angle between the normal vector and a fixed direction, such as it was shown in (2). There appear two new issues. Firstly there are three types of curves in \mathbb{L}^2 owing to its causal character, namely, spacelike, timelike and lightlike and the behavior of each of these curves is completely different. Because we are only interested in the Riemannian sense, we will only consider spacelike curves.

A second issue is the choice of the axis with respect to which we measure the angle of the normal vector \mathbf{n} . Notice that in the Euclidean plane both axes are indistinct but in \mathbb{L}^2 the y -axis and the x -axis are not interchangeable by a rigid motion. Thus arises the problem of which axis is to be fixed. Since for a spacelike curve, the vector \mathbf{n} is timelike, we will measure the angle between \mathbf{n} and the y -axis, which is also timelike. This is also justified because it makes sense to define the angle between two timelike vectors ([O'Neill 1983, p. 144]). After all these considerations, let us proceed.

Let $\gamma = \gamma(s)$ be a spacelike curve parametrized by the arc-length $s \in I$ and contained in the halfplane $y > 0$ of \mathbb{L}^2 . The curvature κ of γ is defined by $\gamma''(s) = \kappa(s)\mathbf{n}(s)$ where \mathbf{n} is a unit normal vector of γ . Here we are assuming $\kappa \neq 0$. Motivated by (2), we ask for those spacelike curves of \mathbb{L}^2 that satisfy the same equation (2) where $\mathbf{a} = (0, 1)$. If γ is a graph $y = u(x)$, then $\gamma(x) = (x, u(x))$, which is not parametrized by the arc-length. Then $\mathbf{n} = (u', 1)/\sqrt{1 - u'^2}$, $\langle \mathbf{n}, \vec{\mathbf{a}} \rangle = -1/\sqrt{1 - u'^2}$ and

$$\kappa(x) = -\frac{1}{1 - u'^2} \langle \gamma''(x), \mathbf{n}(x) \rangle = \frac{u''(x)}{(1 - u'(x)^2)^{3/2}}.$$

Let us observe that $u'^2 < 1$ because γ is a spacelike curve. The equation (2) is now

$$(4) \quad \frac{u''}{1 - u'^2} = -\frac{1}{u},$$

which will be the Lorentzian model of (1) that we are looking for. The spacelike condition $u'^2 - 1 < 0$ is an extra hypothesis compared to the Euclidean case. For example, $u(x) = \sinh(x)$, with $u > 0$, solves (4), but $u'^2 > 1$. So, the corresponding curve $y = u(x)$ is a timelike curve. In contrast, because we are assuming that the curve is spacelike, the right solution of (4) is

$$(5) \quad u(x) = \frac{1}{a} \sin(ax + b), \quad x \in \left(-\frac{b}{a}, \pi - \frac{b}{a}\right),$$

where $a \neq 0$, $a, b \in \mathbb{R}$. This curve will be the analogue catenary in \mathbb{L}^2 . As in the Euclidean case, we introduce a constant $\alpha \in \mathbb{R}$ and we consider the analogous equation of (2), namely,

$$(6) \quad \kappa = \alpha \frac{\langle \mathbf{n}, \vec{\mathbf{a}} \rangle}{\langle p, \vec{\mathbf{a}} \rangle} = -\alpha \frac{\langle \mathbf{n}, \vec{\mathbf{a}} \rangle}{y},$$

where $p = (x, y) \in \mathbb{L}^2$. For instance, the curve (5) is the solution for $\alpha = -1$.

Following the same steps as in the Euclidean setting, we embed \mathbb{L}^2 in the Lorentz–Minkowski 3-space \mathbb{L}^3 . Here \mathbb{L}^3 is the affine Euclidean 3-space endowed with the metric $dx^2 + dy^2 - dz^2$. Then \mathbb{L}^2 is identified with the xz -plane, the y -axis of \mathbb{L}^2 with the z -axis of \mathbb{L}^3 and the vector $(0, 1) \in \mathbb{L}^2$ with $\vec{\mathbf{a}} = (0, 0, 1)$. Definitively, the objects of our study in this paper are described in the following definition.

Definition 1.1. Let α be a nonzero real number. A spacelike surface S in the halfspace $z > 0$ of \mathbb{L}^3 is called an α -singular maximal surface if it satisfies

$$(7) \quad H(p) = \alpha \frac{\langle N(p), \vec{a} \rangle}{\langle p, \vec{a} \rangle} = -\alpha \frac{\langle N(p), \vec{a} \rangle}{z}, \quad (p \in S),$$

where N is a unit normal vector field on S and H is the mean curvature.

Here H is the trace of the second fundamental form of S , that is, the sum of the principal curvatures. We will omit the constant α if it is understood from the context. Recently, these surfaces have been studied in [Martínez and Martínez-Triviño 2020] relating the Riemannian and the Lorentzian settings by means of a Calabi type correspondence.

In view of (2), and as a motivation of this paper, the case $\alpha = -1$ in (7) is the corresponding *two-dimensional analogue of the Lorentzian catenary*. Other known examples appear when $\alpha = 2$ because in such a case, the surface is a minimal surface in the steady state space ([López 2017]). Another special example is the hyperbolic plane $\mathbb{H}^2(r) = \{p \in \mathbb{L}^3 : \langle p, p \rangle = -r^2, z > 0\}$, $r > 0$. This surface has mean curvature $H = 2/r$ for $N(p) = p/r$. It is clear that $\mathbb{H}^2(r)$ satisfies (7) for $\alpha = 2$. Even more, $\mathbb{H}^2(r)$ satisfies (7) for any vector \vec{a} .

On the other hand, we extend a similar property that has the catenary in Euclidean space. Indeed, we take the catenary (5) and we rotate with respect to the x -axis. The rotations that leave the x -axis pointwise fixed are described by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

For a curve $z = u(x)$, namely, $\gamma(x) = (x, 0, u(x))$, $x \in I \subset \mathbb{R}$, contained in the xz -plane, the corresponding rotational surface S is parametrized by

$$(8) \quad X(x, \theta) = (x, u(x) \sinh \theta, u(x) \cosh \theta), \quad \theta \in \mathbb{R}.$$

If $u(x) = \sin(ax + b)/a$, it is not difficult to see that the corresponding rotational surface (8) has zero mean curvature, that is, S is a maximal surface of \mathbb{L}^3 . This surface is called the catenoid of second kind or the hyperbolic catenoid in the literature.

Remark 1.2. If we rotate the curve $u(x) = \sinh(ax + b)/a$, the timelike solution of (4), with respect to the x -axis, the rotational surface is a timelike surface with zero mean curvature ([López 2000]). Similarly, any vertical straight line is a timelike curve that satisfies (2) and if we rotate with respect to the x -axis, we obtain a (timelike) plane parallel to the yz -plane, which has zero mean curvature everywhere.

As a conclusion, the generalization in \mathbb{L}^3 of the two-dimensional analogue of the catenary, or more generally, singular maximal surfaces in Lorentz–Minkowski

space \mathbb{L}^3 , is carried out for spacelike surfaces and the angle between N and \vec{a} is measured with respect to the (timelike) z -axis. We have also discussed that there are other possibilities to generalize the initial problem in \mathbb{L}^3 , although all of them less justified, such as for example, changing the axis $\vec{a} = (0, 0, 1)$ to $(1, 0, 0)$ (spacelike) or $(1, 0, 1)$ (lightlike). Also, we may consider timelike surfaces and measuring the angle between N with respect to an axis of \mathbb{L}^3 .

In this paper we will also be interested in solving the Dirichlet problem of the singular maximal surface equation. Since a spacelike surface is locally the graph of a function $z = u(x, y)$, the nonparametric form of (7) is

$$(9) \quad \operatorname{div} \frac{Du}{\sqrt{1 - |Du|^2}} = \alpha \frac{1}{u\sqrt{1 - |Du|^2}},$$

with the spacelike condition $|Du| < 1$. The left-hand side of this equation is the mean curvature of the graph $z = u(x, y)$ computed with respect to the upwards orientation

$$N = \frac{1}{\sqrt{1 - |Du|^2}}(Du, 1).$$

Comparing (9) with the Riemannian case [Dierkes 1988a; 1988b; 2003; López 2019a]), this equation is not uniformly elliptic and, as a consequence, we must ensure that $|Du|$ is bounded away from 1.

This paper is organized as follows. In Section 2 we classify all singular maximal surfaces that are invariant by a uniparametric group of translations and of rotations. In Section 3 we describe the solutions of (7) that are invariant by rotations about the z -axis and finally, in Section 4 we solve the Dirichlet problem associated to (9) for mean convex domains and arbitrary boundary data.

2. Invariant singular maximal surfaces

In this section we classify and describe all singular maximal surfaces that are invariant by a uniparametric group of translations or of rotations of \mathbb{L}^3 . Firstly, we notice that some transformations of the affine Euclidean space \mathbb{R}^3 preserve the singular maximal surface equation. To fix the terminology, a vector $\vec{v} \in \mathbb{R}^3$ is called horizontal direction if it is parallel to the xy -plane and it is called vertical if it is parallel to the z -axis.

It is clear that a solution of (7) is invariant by a translation along a horizontal direction, that is, if S is an α -singular maximal surface, then $S + \vec{v}$ is also an α -singular maximal surface, where \vec{v} is a horizontal vector of \mathbb{R}^3 . Similarly, the same property holds if we rotate S with respect to a vertical direction because the term $\langle N, \vec{a} \rangle$ and the denominator z in (7) are invariant by this type of rotation.

Finally, if $\lambda > 0$ is a positive real number, and $T_\lambda(p) = p_0 + \lambda(p - p_0)$ is the dilation with center $p_0 \in \mathbb{R}^2 \times \{0\}$, then $T_\lambda(S)$ is an α -singular maximal surface.

Remark 2.1. We point out that a rigid motion of \mathbb{L}^3 does not preserve (7) in general because the denominator z may change in general by the motion.

As we have announced, a natural source of examples of singular maximal surfaces of \mathbb{L}^3 is found in the class of invariant surfaces by a uniparametric group of rigid motions. The key point is that (7), which locally is the partial differential (9), changes into an ordinary differential equation. In particular, by standard theory, there always is a solution for any initial conditions.

Surfaces invariant by translations. We begin the study of the surfaces invariant by a uniparametric group of translations. Since the rulings generated by this group are straight lines contained in the surface, and the surface is spacelike, then any ruling is a spacelike line. Thus the vector generating the group of translation must be spacelike. Let \vec{v} be a unit spacelike vector and consider a surface S invariant by the group of translations generated by \vec{v} . Then S parametrizes as $X(s, t) = \gamma(s) + t\vec{v}$, where γ is a planar spacelike curve of \mathbb{L}^3 contained in a (timelike) orthogonal plane to \vec{v} . The equation (7) gives

$$\kappa \det(\gamma', \vec{v}, \mathbf{n}) = \alpha \frac{\det(\gamma', \vec{v}, \vec{a})}{\gamma_3 + tv_3},$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and $\vec{v} = (v_1, v_2, v_3)$. We consider the orientation in γ so $\gamma' \times \vec{v} = \mathbf{n}$. Since \mathbf{n} is a unit timelike vector, the above equation is now

$$(10) \quad \kappa(\gamma_3 + tv_3) + \alpha \langle \mathbf{n}, \vec{a} \rangle = 0.$$

This is a polynomial equation on t , hence

$$\kappa v_3 = 0, \quad \kappa \gamma_3 + \alpha \langle \mathbf{n}, \vec{a} \rangle = 0.$$

Since $\kappa \neq 0$, we deduce that $v_3 = 0$ and $\kappa \gamma_3 + \alpha \langle \mathbf{n}, \vec{a} \rangle = 0$. Then \vec{v} is a horizontal vector and γ is a planar curve contained in a vertical plane. After a horizontal translation and a rotation about the z -axis, we assume that this plane is the xz -plane which can be identified with \mathbb{L}^2 . Furthermore, the equation $\gamma_3 + \alpha \langle \mathbf{n}, \vec{a} \rangle = 0$ means that γ satisfies, as a planar curve of \mathbb{L}^2 , the one-dimensional singular maximal surface equation (6). The converse of this result is immediate.

Proposition 2.2. *Let S be an α -singular maximal surface of \mathbb{L}^3 invariant by a uniparametric group of translations generated by \vec{v} and denote by γ its generatrix. Then \vec{v} is a horizontal vector, γ is contained in a plane orthogonal to \vec{v} and γ , as a planar curve, satisfies (6). Conversely, if γ is a curve in \mathbb{L}^2 that satisfies (6) and, if we embed this curve in the xz -plane as usual, then the surface $X(s, t) = \gamma(s) + t(0, 1, 0)$ is an α -singular maximal surface.*

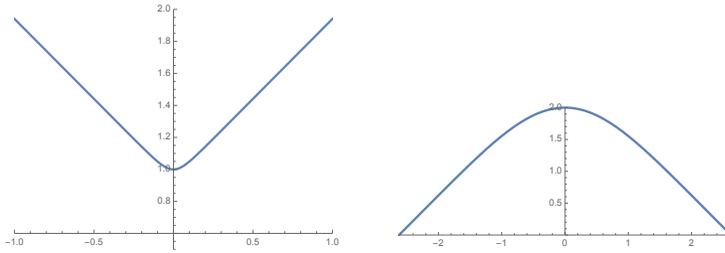


Figure 1. Solutions of (11). Left: $\alpha = 1$. Right: $\alpha = -2$.

In view of this proposition, consider the one-dimensional case of (7). Let $\gamma(s) = (x(s), y(s))$ be a spacelike curve in \mathbb{L}^2 that satisfies (6). Since γ is spacelike, then $x'^2 - y'^2 > 0$, in particular, $x'(s) \neq 0$ for every s and thus γ is globally the graph of a function $u = u(x)$, $x \in I \subset \mathbb{R}$. The equation (6) now becomes

$$(11) \quad \frac{u''}{1 - u'^2} = \alpha \frac{1}{u}, \quad u > 0, \quad u'^2 < 1.$$

It is possible to find some explicit solutions of (11) by simple quadratures. In the introduction we have seen that if $\alpha = -1$, the solution is $u(x) = \sin(ax + b)/a$, where $a \neq 0$, $a, b \in \mathbb{R}$ and where x is defined in some interval to ensure that $u > 0$. If $\alpha = 1$, it is easy to find that the solution of (11) is

$$u(x) = \frac{1}{a} \sqrt{1 + a^2 x^2 + 2abx + b^2}, \quad a, b \in \mathbb{R}, \quad a > 0.$$

After a change of variable, this function u is written as $u(x) = \sqrt{1 + a^2 x^2}/a$, $a > 0$. It is immediate that u is the upper branch of the hyperbola $a^2(x^2 - y^2) = -1$. This curve, viewed as a planar curve in \mathbb{L}^2 , has nonzero constant curvature $\kappa = a$. The generated surface by Proposition 2.2 is the right-cylinder of $a^2(x^2 - z^2) = -1$.

Remark 2.3. As for the catenary $u(x) = \sin(ax + b)/a$, if we rotate the curve $u(x) = \sqrt{1 + a^2 x^2}/a$ with respect to the x -axis, we obtain the hyperbolic plane $\mathbb{H}^2(1/a)$.

Remark 2.4. Similarly as in the case $\alpha = -1$, there is a timelike solution of (11) by replacing the spacelike condition $u'^2 < 1$ by $u'^2 > 1$. The solution is now $u(x) = \sqrt{a^2 x^2 - 1}/a$, where $a > 0$ and $x > 1/a$. The function u is the positive part of the hyperbola $x^2 - y^2 = 1/a^2$, which is a timelike curve. If we rotate about the x -axis, the generated surface is $x^2 + y^2 - z^2 = 1/a^2$. This surface is the (upper part of) the de Sitter space $\mathbb{S}_1^2(1/a) = \{p \in \mathbb{L}^3 : \langle p, p \rangle = 1/a^2\}$. This surface satisfies (7) when $\alpha = 2$ and plays the same role as the hyperbolic plane in the family of timelike surfaces of \mathbb{L}^3 .

We now describe the geometric properties of the solutions of (11). See Figure 1.

Theorem 2.5. *Let $u = u(x)$ be a solution of (11), $x \in I$, where $I \subset \mathbb{R}$ is the maximal domain of u . Then u is symmetric about a vertical line and $I = \mathbb{R}$ if $\alpha > 0$ or I is a bounded interval if $\alpha < 0$. Furthermore:*

(1) *Case $\alpha > 0$. The function u is convex with a unique global minimum,*

$$\lim_{r \rightarrow \infty} u(r) = \infty \quad \text{and} \quad \lim_{r \rightarrow \infty} u'(r) = 1.$$

(2) *Case $\alpha < 0$. The function u is concave with a unique global maximum. If $I = (-b, b)$, then*

$$\lim_{r \rightarrow b} u(r) = 0 \quad \text{and} \quad \lim_{r \rightarrow b} u'(r) = -1.$$

Proof. If u has a critical point at $r = r_0$, then $u''(r_0) = \alpha/u(r_0)$ has the same sign as α . Hence, there is one critical point at most that will be a global minimum (resp. maximum) if $\alpha > 0$ (resp. $\alpha < 0$).

Claim: There exists a critical point of u .

Suppose now that the claim is proved and we finish the proof of the theorem. After a change in the variable x , we suppose that $x = 0$ is the critical point, $u'(0) = 0$. Then u is the solution of (11) with initial conditions $u(0) = u_0 > 0$ and $u'(0) = 0$. It is clear that $u(-s)$ is also a solution of the same initial value problem, so $u(s) = u(-s)$ by uniqueness. This proves that u is symmetric about the y -axis.

Multiplying (11) by u' , we obtain a first integral

$$(12) \quad \frac{1}{1 - u'^2} = \mu u^{2\alpha},$$

for some positive constant $\mu > 0$.

(1) Case $\alpha > 0$. Since $u(x) \geq u_0$, we deduce from (11) that u' and u'' are bounded functions and this implies the maximal domain is \mathbb{R} . Since u is a convex function, then $u(r) \rightarrow \infty$ as $r \rightarrow \infty$ and from (12), we conclude that $u'(r) \rightarrow 1$ as $r \rightarrow \infty$.

(2) Case $\alpha < 0$. By symmetry, $I = (-b, b)$ for some $b \leq \infty$. Since u is a positive concave function, then $b < \infty$. Using the concavity of u again, and because $u'^2 < 1$, then the graph of u must meet the x -axis, that is, $\lim_{r \rightarrow b} u(r) = 0$. From (12), we deduce $\lim_{r \rightarrow b} u'(r)^2 = 1$, and by concavity, $\lim_{r \rightarrow b} u'(r) = -1$.

We now prove the claim. The proof is by contradiction. Assume that the sign of u' is constant and denote $I = (a, b)$ with $-\infty \leq a < b \leq \infty$.

(1) Case $\alpha > 0$. We suppose that $u' > 0$ in I (similar argument if u' is negative). Since u is increasing and u' and u'' are bounded near $r = b$, we deduce that $b = \infty$ by standard theory. If $-\infty < a$, then $\lim_{r \rightarrow a} u(r) = 0$ because otherwise we could extend u beyond $r = a$ because u' and u'' would be bounded functions. Therefore $\lim_{r \rightarrow a} u'(r)^2 = 1$ by (12). Since $u' > 0$, this limit is just 1. This is a contradiction

because u' is an increasing function and we would have $u' > 1$ in I , which is not possible by the spacelike condition.

Thus $a = -\infty$. Since u is increasing and $u > 0$ in \mathbb{R} , we find $\lim_{r \rightarrow -\infty} u(r) = c \geq 0$. Because $u' > 0$ and $u'' > 0$, then $\lim_{r \rightarrow -\infty} u'(r) = \lim_{r \rightarrow -\infty} u''(r) = 0$. However, by (11), and letting $r \rightarrow -\infty$, we have that $u''(r)$ goes to $\alpha/c \neq 0$ if $c > 0$ or to ∞ if $c = 0$, obtaining a contradiction.

(2) Case $\alpha < 0$. We suppose that $u' > 0$ in I (similar argument if u' is negative). Since u' and u'' are bounded for r close to b , then $b = \infty$ and by concavity, we deduce that $-\infty < a$. If u is bounded from above with $\lim_{r \rightarrow \infty} u(r) = c > 0$, then $\lim_{r \rightarrow \infty} u'(r) = 0$ and since $u'' < 0$, then $\lim_{r \rightarrow \infty} u''(r) = 0$. By (11), we find $\lim_{r \rightarrow \infty} u''(r) = \alpha/c < 0$, a contradiction. Thus $\lim_{r \rightarrow \infty} u(r) = \infty$. By using (12), we conclude $\lim_{r \rightarrow \infty} u'(r)^2 = 1$, so this limit is 1; this is a contradiction because u' is a decreasing function and we would have $u' > 1$ in the interval I , which is not possible. □

Surfaces of revolution with respect to a spacelike axis and a lightlike axis. The second source of examples of singular maximal surfaces are the surfaces invariant by a uniparametric group of rotations. A difference between the Euclidean and the Lorentzian settings is that in \mathbb{L}^3 there are three types of surfaces of revolution depending on whether the rotational axis is spacelike, timelike or lightlike. Section 3 is devoted to the surfaces of revolution whose rotation axis is timelike because this type of surface will play a special role in the solvability of the Dirichlet problem in Section 4. In this section we investigate the cases where the rotation axis is spacelike and lightlike.

We point out that there is not an a priori relation between the rotation axis L and the vector $\vec{a} = (0, 0, 1)$ of (7). This implies that if we apply a rigid motion to prescribe the rotation axis, then the vector \vec{a} does change; see also Remark 2.1.

First we consider the case that the axis is spacelike.

Proposition 2.6. *Let S be a spacelike surface of \mathbb{L}^3 invariant by the uniparametric group of rotations about a spacelike axis L . Suppose that S satisfies $lrefeqL$ where \vec{a} is a timelike vector. Then either \vec{a} is orthogonal to L , or S is the hyperbolic plane $\mathbb{H}^2(r)$ with \vec{a} an arbitrary timelike vector.*

Proof. After a rigid motion of \mathbb{L}^3 we assume that L is the x -axis. This rigid motion changes the vector \vec{a} in (7) and \vec{a} must be considered an arbitrary (timelike) vector. Let $\vec{a} = (a, b, c)$ denote the new vector \vec{a} in (7) after the rigid motion. Since \vec{a} is timelike, then $c \neq 0$.

Using the expression of a parametrization (8) of S and after some computations, (7) is a polynomial equation on $\{1, \sinh \theta, \cos \theta\}$. Since these functions are linearly independent, all three coefficients (which are functions on the variable s) must

vanish, obtaining

$$\begin{aligned} -c(1 - u'^2 + uu'') + \alpha c(1 - u'^2) &= 0 \\ -b(1 - u'^2 + uu'') + \alpha b(1 - u'^2) &= 0 \\ as(1 - u'^2 + uu'') - \alpha auu'(1 - u'^2) &= 0. \end{aligned}$$

Since $c \neq 0$, we find $a(uu' - s) = b(uu' - s) = 0$. If $uu' - s \neq 0$, then $a = b = 0$, proving that $\vec{a} = (0, 0, c)$, hence L is orthogonal to the x -axis and the result is proved. The other possibility is $uu' - s = 0$. Solving this equation, we find $u(s) = \sqrt{s^2 + r^2}$, $r > 0$. Then $X(s, \theta) = (s, \sqrt{s^2 + r^2} \sinh \theta, \sqrt{s^2 + r^2} \cos \theta)$ and it is immediate that this surface is the hyperbolic plane $\mathbb{H}^2(r)$. \square

As a consequence of Proposition 2.6, and besides the hyperbolic plane as a special case, we can assume that $\vec{a} = (0, 0, 1)$ in the singular maximal surface (7), and that the rotation axis is the x -axis. In such a case, the proof of Proposition 2.6 gives immediately that (7) is

$$\frac{u''}{1 - u'^2} = (\alpha - 1) \frac{1}{u}.$$

This equation is just (11). Identifying the Lorentzian plane \mathbb{L}^2 with the plane of equation $y = 0$, we have obtained the following result.

Proposition 2.7. *Any rotational α -singular maximal surface in \mathbb{L}^3 about the x -axis is generated by a planar curve in \mathbb{L}^2 that satisfies the one-dimensional $(\alpha - 1)$ -singular maximal surface equation. Conversely, any planar curve in \mathbb{L}^2 that satisfies (11) is the generating curve of an $(\alpha + 1)$ -singular maximal surface invariant by all rotations about the x -axis.*

Example 2.8. We know that the solution of (11) for $\alpha = 1$ is the hyperbola $u(x) = \sqrt{1 + a^2 x^2}/a$, $a > 0$. As a consequence of Proposition 2.7, the only 2-singular maximal surface that is invariant by the rotations about the x -axis is the surface $x^2 + y^2 - z^2 = -1/a^2$, $z > 0$. This surface is the hyperbolic plane $\mathbb{H}^2(1/a)$. Another solution of (11) appeared in the introduction for $\alpha = -1$. Then the surface generated is the hyperbolic catenoid of \mathbb{L}^3 .

We finish this section considering singular maximal surfaces of revolution about a lightlike axis. Again, we have in mind that if we fix the rotation axis, then the vector \vec{a} in (7) is arbitrary. If the rotation axis is determined by the vector $(1, 0, 1)$, the parametrization of the surface is

$$(13) \quad X(s, t) = \begin{pmatrix} 1 - \frac{t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & 1 + \frac{t^2}{2} \end{pmatrix} \begin{pmatrix} u(s) + s \\ 0 \\ u(s) - s \end{pmatrix}, \quad t \in \mathbb{R},$$

for some function $u = u(s)$, $s \in I \subset \mathbb{R}$. The spacelike condition on the surface is equivalent to $u' > 0$.

Proposition 2.9. *Let S be a spacelike surface of \mathbb{L}^3 invariant by the uniparametric group of rotations about a lightlike axis L . Suppose S satisfies (7) where \vec{a} is a timelike vector. Then either \vec{a} is orthogonal to L , or S is the hyperbolic plane $\mathbb{H}^2(r)$ with \vec{a} an arbitrary vector. More precisely, if L is generated by the vector $(1, 0, 1)$, S is parametrized by (13) and if $\alpha \neq 2$, then $\vec{a} = (1, b, 1)$, $b \neq 0$, and we have the following possibilities:*

- (1) If $\alpha = \frac{3}{2}$, then $u(s) = m \log(s)$, $m > 0$.
- (2) If $\alpha \neq \frac{3}{2}$, then $u(s) = ms^{3-2\alpha}/(3 - 2\alpha)$, $m > 0$.

In particular, hyperbolic planes $\mathbb{H}^2(r)$ are the only α -singular maximal surfaces in \mathbb{L}^3 satisfying (7) with $\vec{a} = (0, 0, 1)$ and invariant by the group of rotations about the lightlike axis generated by the vector $(1, 0, 1)$.

Proof. A straightforward computation of (7) for the surface (13) concludes that this equation is a polynomial equation on t of degree 2. Thus the coefficients corresponding to the variable t must vanish, obtaining

$$\begin{aligned} 2u'((\alpha + 1)s(a + c) + (a - c)(u + \alpha su')) - su''((a - c)u + s(a + c)) &= 0 \\ b(su'' - 2(1 - \alpha)u') &= 0 \\ (a - c)(su'' - 2(1 - \alpha)u') &= 0. \end{aligned}$$

From the second and third equation, if $su'' - 2(1 - \alpha)u' \neq 0$, we have $b = 0$ and $a = c$, obtaining that \vec{a} is a lightlike vector, which is not possible. Thus $su'' - 2(1 - \alpha)u' = 0$. The solution of this equation depends on the value of α .

- (1) Case $\alpha = \frac{3}{2}$. Here, $u(s) = m \log(s)$ with $m > 0$. The first equation yields $(a - c)m^2(1 + \log(s)) = 0$, that is, $a = c$ and $\vec{a} = (a, b, a)$, $b \neq 0$.
- (2) Case $\alpha \neq \frac{3}{2}$. Here, $u(s) = ms^{3-2\alpha}/(3 - 2\alpha)$ with $m > 0$. Now the first equation simplifies into $(a - c)(2 - \alpha)s^{5-4\alpha} = 0$. If $\alpha = 2$, then $u(s) = -m/s$ and it is not difficult to see that this surface is the hyperbolic plane $\mathbb{H}^2(2\sqrt{m})$. If $\alpha \neq 2$, then $a = c$, so $\vec{a} = (a, b, a)$, $b \neq 0$. □

3. Surfaces of revolution about the z -axis

In this section we study the surfaces of revolution with timelike axis L . Again, the same observations as in the previous section hold in the sense that there is not an a priori relation between the vector \vec{a} and the axis L . The first result that we will prove is that, indeed, L must be parallel to the vector \vec{a} .

Proposition 3.1. *Let S be an α -singular maximal surface in \mathbb{L}^3 that is invariant by the uniparametric group of rotations about a timelike axis L . Suppose that S*

satisfies (7) where \vec{a} is now an arbitrary timelike vector. Then either L and \vec{a} are parallel, or S is the hyperbolic plane $\mathbb{H}^2(m)$ and \vec{a} is an arbitrary timelike vector.

Proof. After a rigid motion, we suppose that the rotation axis is the z -axis. Let $\vec{a} = (a, b, c)$ after this motion. The surface S parametrizes as $X(r, \theta) = (r \cos \theta, r \sin \theta, u(r))$, $r \in I \subset \mathbb{R}^+$, $\theta \in \mathbb{R}$, $u > 0$ and $u'^2 < 1$. The computation of (7) gives a polynomial equation on the trigonometric functions $\{1, \sin \theta, \cos \theta\}$. Thus all three coefficients must vanish, obtaining

$$\begin{aligned} a(ru'' + (\alpha + 1)u'(1 - u'^2)) &= 0 \\ b(ru'' + (\alpha + 1)u'(1 - u'^2)) &= 0 \\ c(\alpha r(1 - u'^2) + u(ru'' + u'(1 - u'^2))) &= 0. \end{aligned}$$

If $ru'' - (\alpha + 1)u'(1 - u'^2) \neq 0$, then $a = b = 0$, proving that $\vec{a} = (0, 0, c)$, hence L and \vec{a} are parallel.

Suppose now that $ru'' + (\alpha + 1)u'(1 - u'^2) = 0$. Recall that $c \neq 0$ because \vec{a} is a timelike vector. Combining this with the third equation, we find $uu' - r = 0$. Solving this equation we obtain $u(r) = \sqrt{r^2 + m^2}$, $m > 0$, and the corresponding surface is the hyperbolic plane $\mathbb{H}^2(m)$. \square

By Proposition 3.1, and after a horizontal translation, we will assume that the rotation axis is the z -axis and $\vec{a} = (0, 0, 1)$ in (7). We know that $X(r, \theta) = (r \cos \theta, r \sin \theta, u(r))$, where $r \in I \subset \mathbb{R}^+$, $\theta \in \mathbb{R}$ and $u > 0$. By the proof of Proposition 3.1, (7) is written as

$$(14) \quad \frac{u''}{(1 - u'^2)^{3/2}} + \frac{u'}{r\sqrt{1 - u'^2}} = \frac{\alpha}{u\sqrt{1 - u'^2}},$$

or equivalently,

$$(15) \quad \frac{u''}{1 - u'^2} + \frac{u'}{r} = \frac{\alpha}{u}.$$

We are interested in those solutions that meet the z -axis, that is, when $r = 0$ is contained in the domain of the solution. Let us observe that (14) is singular at $r = 0$ and thus the existence of solutions is not a direct consequence of standard ODE theory.

Multiplying (14) by r , and integrating by parts, we wish to establish the existence of a classical solution of

$$(16) \quad \begin{cases} \left(r \frac{u'}{\sqrt{1 - u'^2}} \right)' = r \frac{\alpha}{u\sqrt{1 - u'^2}}, & r \in (0, \delta), \\ u(0) = u_0 > 0, & u'(0) = 0. \end{cases}$$

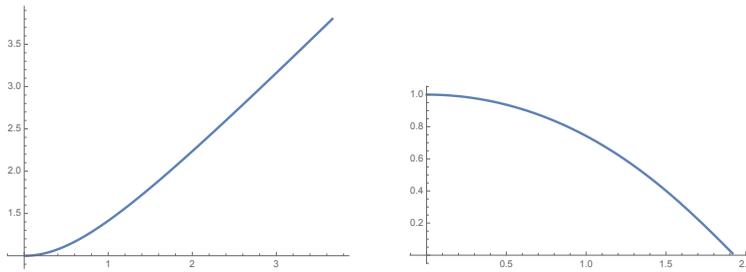


Figure 2. Solutions of (16). Left: case $\alpha > 0$, shown with $\alpha = 2$. Right: case $\alpha < 0$, shown with $\alpha = -1$.

Define the functions $\phi : (-1, 1) \rightarrow \mathbb{R}$ and $f : \mathbb{R}^+ \times (-1, 1) \rightarrow \mathbb{R}$ by

$$\phi(y) = \frac{y}{\sqrt{1-y^2}}, \quad f(x, y) = \frac{\alpha}{x\sqrt{1-y^2}}.$$

Let $\delta > 0$. It is clear that a function $u \in C^2([0, \delta])$ is a solution of (16) if and only if $(r\phi(u'))' = rf(u, u')$ and $u(0) = u_0, u'(0) = 0$. Let $\mathcal{B} = (C^1([0, \delta]), \|\cdot\|)$ be the Banach space of the continuously differentiable functions on $[0, \delta]$ endowed with the usual norm

$$\|u\| = \|u\|_\infty + \|u'\|_\infty.$$

Define the operator $T : \mathcal{B} \rightarrow \mathcal{B}$ by

$$(Tu)(r) = u_0 + \int_0^r \phi^{-1} \left(\int_0^s \frac{t}{s} f(u, u') dt \right) ds.$$

Notice that a fixed point of the operator T is a solution of the initial value problem (16). Indeed, $(Tu)' = \phi^{-1}(\frac{1}{r} \int_0^r tf(u, u') dt)$ and

$$r\phi(Tu') \int_0^r tf(u, u') dt,$$

obtaining the result. Moreover, $Tu(0) = u_0$ and

$$\phi(Tu)'(0) = \lim_{r \rightarrow 0} \frac{1}{r} \int_0^r tf(u, u') dt = \lim_{r \rightarrow 0} rf(u, u') = 0,$$

where in the second identity we have used the L'Hôpital rule. Thus, $(Tu)'(0) = 0$.

The existence of solutions of (16) follows now standard techniques of radial solutions for some equations of mean curvature type ([Bereanu et al. 2009; Corsato et al. 2015]). In Figure 2 we show the solutions of (16) when α is positive and negative.

Theorem 3.2. *The initial value problem (16) has a solution $u \in C^2([0, \delta])$ for some $\delta > 0$ that depends continuously on the initial data.*

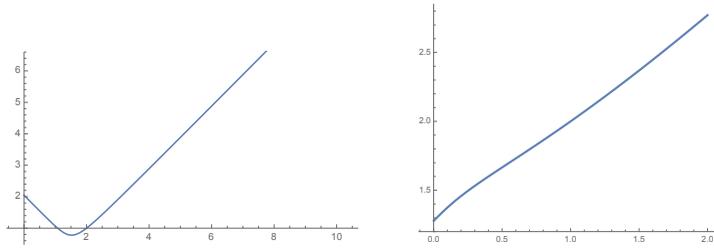


Figure 3. Solutions of (15), case $\alpha > 0$ and $u'_0 \neq 0$. Left: case $u'_0 = -1$. Right: case $u'_0 = 1$.

Proof. In order to find a fixed point of T , we prove that T is a contraction in \mathcal{B} for some $\delta > 0$ to be chosen. The functions f and ϕ^{-1} are locally Lipschitz continuous of constant $L > 1$ in $[u_0 - \epsilon, u_0 + \epsilon] \times [-\epsilon, \epsilon]$ and $[-\epsilon, \epsilon]$ respectively, provided $\epsilon < \{u_0, 1\}$. Since $\phi^{-1}(y) = y/\sqrt{1 + y^2}$, then $L < 1$. Then for all $u, v \in \overline{B(0, \epsilon)}$ and for all $r \in [0, \delta]$,

$$\begin{aligned}
 |(Tu)(r) - (Tv)(r)| &\leq L \int_0^r \left| \int_0^s \frac{t}{s} (f(u, u') - f(v, v')) dt \right| \\
 &\leq L^2 \int_0^r \int_0^s \frac{t}{s} \|u - v\| dt = \frac{L^2}{4} r^2 \|u - v\|. \\
 |(Tu)'(r) - (Tv)'(r)| &\leq \frac{L}{r} \left| \int_0^r t (f(u, u') - f(v, v')) dt \right| \\
 &\leq \frac{L^2}{r} \int_0^r t \|u - v\| dt = \frac{L^2}{2} r \|u - v\|.
 \end{aligned}$$

By choosing $\delta > 0$ small enough, we deduce that T is a contraction in the closed ball $\overline{B(0, \delta)} \subset \mathcal{B}$. Thus the Schauder point fixed theorem proves the existence of one fixed point of T , so the existence of a local solution of the initial value problem (16). This solution belongs to $C^1([0, \delta]) \cap C^2((0, \delta])$. The C^2 -regularity up to 0 is verified directly by using the L'Hôpital rule because (14) leads to

$$\lim_{r \rightarrow 0} u''(r) + \lim_{r \rightarrow 0} \frac{u'(r)}{r} = \frac{\alpha}{u_0},$$

that is,

$$(17) \quad \lim_{r \rightarrow 0} u''(r) = \frac{\alpha}{2u_0}.$$

The continuous dependence of local solutions on the initial data is a consequence of the continuous dependence of the fixed points of T . □

In the following result we describe the geometric properties of the rotational solutions of (15). See Figures 2, 3 and 4.

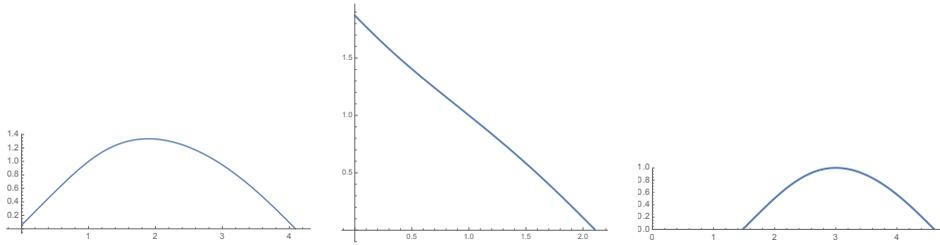


Figure 4. Solutions of (15), case $\alpha < 0$ and $u'_0 \neq 0$. Left: case $u'_0 = 1$. Middle: case $u'_0 = -1$. Right: a solution that does not meet the rotation axis

Theorem 3.3. Let u be a solution of (14) with $u > 0$ and $u'^2 < 1$.

- (1) Case $\alpha > 0$. The maximal domain of u is $(0, \infty)$. Let $u'_0 = \lim_{r \rightarrow 0} u'(r)$. Then we have the following cases: $u'_0 = 0$ and the function u is increasing; $u'_0 = -1$ and u has a unique critical point which is a global minimum; $u'_0 = 1$ and the function is increasing. In all cases,

$$(18) \quad \lim_{r \rightarrow \infty} u(r) = \infty.$$

Also, the function $u(r) = \sqrt{\alpha}r$ is a solution of (14).

- (2) Case $\alpha < 0$. The maximal domain of u is (a, b) with $0 \leq a < b < \infty$ and

$$\lim_{r \rightarrow b} u(r) = 0, \quad \lim_{r \rightarrow b} u'(r) = -1.$$

If $a > 0$, then u has a global maximum and

$$\lim_{r \rightarrow a} u(r) = 0, \quad \lim_{r \rightarrow a} u'(r) = 1.$$

If $a = 0$, let $u'_0 = \lim_{r \rightarrow 0} u'(r)$. Then we have the following cases: $u'_0 = 0$ and u is a decreasing function; $u'_0 = -1$ and u is a decreasing function; $u'_0 = 1$ and u has a global maximum.

Proof. We observe that if u has a critical point at $r_o \geq 0$, then (15) implies $u''(r_o) = \alpha/u(r_o) \neq 0$, hence all critical points are all maxima or are all minima. Thus there is one critical point at most. In such a case, this point is a global minimum (resp. maximum) if $\alpha > 0$ (resp. $\alpha < 0$).

Claim A. If the graph of u meets the x -axis at $r_* > 0$, then $\alpha < 0$ and $\lim_{r \rightarrow r_*} u'(r)^2 = 1$.

The proof follows by multiplying (15) by $2u'$ and integrating. Then

$$\log(1 - u'(r)^2) + 2\alpha \log u(r) = 2\alpha \int^r \frac{u'(t)^2}{t} dt + \mu, \quad \mu \in \mathbb{R}.$$

In a neighborhood of r_* , the right-hand side of the above equation is finite. Since $\log(u(r)) \rightarrow -\infty$ as $r \rightarrow r_*$, the same occurs with $\lim_{r \rightarrow r_*} \log(1 - u'(r)^2)$, proving that $u'(r)^2 \rightarrow 1$ as $r \rightarrow r_*$. Moreover, the case $\alpha > 0$ is not possible because the left-hand side would be $-\infty$.

Claim B. If the graph of u meets the y-axis, then $\lim_{r \rightarrow 0} u'(r) = 0$ or $\lim_{r \rightarrow 0} u'(r)^2 = 1$.

Denote $u'_0 = \lim_{r \rightarrow 0} u'(r)$. From Theorem 3.2, we know the existence of solutions when $u'_0 = 0$. Suppose now $u'_0 \neq 0$. By contradiction, we assume that $u_0'^2 \neq 1$. For $\delta > 0$ close to 0 and by (16),

$$(19) \quad \frac{ru'(r)}{\sqrt{1-u'(r)^2}} - \frac{\delta u'(\delta)}{\sqrt{1-u'(\delta)^2}} = \int_{\delta}^r \frac{\alpha t}{u\sqrt{1-u^2}} dt.$$

Since $u_0'^2 \neq 1$, letting $r \rightarrow 0$ we have

$$\frac{u'(\delta)}{\sqrt{1-u'(\delta)^2}} = \frac{1}{\delta} \int_0^{\delta} \frac{\alpha t}{u\sqrt{1-u^2}} dt.$$

Letting $\delta \rightarrow 0$ and by the L'Hôpital rule, we deduce

$$\lim_{\delta \rightarrow 0} \frac{u'(\delta)}{\sqrt{1-u'(\delta)^2}} = \lim_{\delta \rightarrow 0} \frac{\alpha \delta}{u\sqrt{1-u^2}} = 0,$$

hence $u'_0 = 0$, a contradiction.

In particular, Claim B implies that it is not possible to find solutions of the initial value problem (16) when $u_0'^2 \in (0, 1)$.

From now, we will denote by $u(a)$ and $u'(a)$ (and similarly for $r = b$), the limit of $u(r)$ and $u'(r)$ at $r = a$.

Claim C. If $a > 0$ (resp. $b < \infty$), then $u(a) =$ (resp. $u(b) = 0$).

Suppose that $a > 0$ (and similarly for $b < \infty$). If $u(a) \neq 0$, then u'' is bounded around $r = a$ by (15). Since u' and u'' are bounded functions, we could extend the solution u beyond $r = a$, a contradiction.

We are in position to prove the theorem.

(1) Case $\alpha > 0$. Suppose that $u' > 0$ in all its domain. Since u' and u'' are bounded functions by (15), then the value of b in I is $b = \infty$. If $a > 0$, this implies that $u(a) = 0$ by Claim C and this is a contradiction by Claim A. This proves that $I = (0, \infty)$.

Suppose that the sign of u' is negative in all its domain. Then (15) implies that u is a concave function and thus $b < \infty$ because u is decreasing. Then $u(b) = 0$, which is not possible by Claim A.

With the above arguments, we have proved that if u' has a constant sign, then $u' > 0$, $a = 0$ and either $u'_0 = 0$ or $u'_0 = 1$. In the case where u' changes sign, then there is a unique critical point at some point $r = r_o > 0$, which is a global minimum.

In this case, $u' > 0$ for $r > r_o$. Since u' and u'' are bounded, then $b = \infty$. The case $a > 0$ is forbidden by Claim C. Thus $a = 0$. Since $u' < 0$ for $r < r_o$, Claim B asserts $u'_0 = -1$.

We prove (18). Since u is increasing close to ∞ , let $c = \lim_{r \rightarrow \infty} u(r)$. If $c < \infty$, then $u'(r) \rightarrow 0$ as $r \rightarrow \infty$ and using (15), $\lim_{r \rightarrow \infty} u''(r) = \alpha/c > 0$, a contradiction. Thus $c = \infty$.

Finally, by a direct computation, we observe that $u(r) = \sqrt{\alpha}r$ is a solution of (14).

(2) Case $\alpha < 0$. Suppose that $u' < 0$ in all its domain. Let $c = \lim_{r \rightarrow b} u(r) \geq 0$. If $b = \infty$, then $u'(r) \rightarrow 0$ and (15) would imply that $\lim_{r \rightarrow \infty} u''(r)$ is either α/c if $c > 0$ or ∞ if $c = 0$, a contradiction. Thus $b < \infty$, hence $u(b) = 0$. By Claim A, $u'(b) = -1$. If $a > 0$, then $u(a) > 0$ because u is decreasing; by Claim C, this is a contradiction. Thus $a = 0$. By Claim B and because u is decreasing, we have two possibilities, namely, $u'_0 = 0$ and $u'_0 = -1$.

Suppose now that $u' > 0$ in all its domain. Then u is a concave function by (15). By Claim C, we have $b = \infty$. In the other end of the interval I , namely $r = a$, we have $a = 0$, $u'_0 = 1$ or $a > 0$, $u(a) = 0$ and $u'(a) = 1$. In both cases, as $u'' < 0$, we find $\lim_{r \rightarrow \infty} u'(r) = \lim_{r \rightarrow \infty} u''(r) = 0$ and $\lim_{r \rightarrow \infty} u(r) = \infty$. By (19)

$$\frac{u'(r)}{\sqrt{1 - u'(r)^2}} = \frac{1}{r} \int_{\delta}^r \frac{\alpha t}{u \sqrt{1 - u'^2}} dt + \mu.$$

Letting $r \rightarrow \infty$, the left-hand side is 0. However, and applying twice the L'Hôpital rule, the limit of the right-hand side is

$$\lim_{r \rightarrow \infty} \frac{\alpha r}{u(r)} + \mu = \lim_{r \rightarrow \infty} \frac{\alpha}{u'(r)} + \mu = \infty,$$

giving a contradiction.

So, if $u' > 0$ at some point, there is a critical point r_o of u , which will be the global maximum of u . Then $a \geq 0$ with $u'(a) = 1$ because u is increasing in (a, r_o) . \square

Remark 3.4. If $\alpha < 0$, there exist solutions that do not meet the rotation axis; see Figure 4 (right). This case appears if $0 < a < b < \infty$, where the function u has a global maximum and $u'(a) = 1 = -u'(b)$. This extends the same property of the solution of (4), where the part of the function $u = u(x)$ given in (5) that lies over the x -axis is formed by successive bounded intervals.

4. The Dirichlet problem

The Dirichlet problem of the singular maximal surface equation asks if given a positive function $\varphi : \partial\Omega \rightarrow \mathbb{R}$ defined in a bounded domain $\Omega \subset \mathbb{R}^2$, there exists a smooth positive function $u : \bar{\Omega} \rightarrow \mathbb{R}$ such that (9) holds in Ω , $u = \varphi$ on $\partial\Omega$

and $|Du| < 1$ on $\bar{\Omega}$. Since any curve in a spacelike surface must be spacelike, the graph Γ of φ is spacelike. The problem is to determine the type of function φ and the boundary $\partial\Omega$ for the solvability of the Dirichlet problem. It is expected that the sign α in (9) plays an important role because we have seen in Sections 2 and 3 the contrast of the behavior of the invariant solutions of (7) depending on whether α is positive or negative.

Following similar ideas in [Jenkins and Serrin 1968; Serrin 1969], we will solve the Dirichlet problem if the domain Ω is mean convex. In fact, we will establish the Dirichlet problem in the n -dimensional case, or equivalently, we will find singular maximal hypersurfaces in the $(n+1)$ -dimensional Lorentz–Minkowski space \mathbb{L}^{n+1} with prescribed boundary data.

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is said to be mean convex if $\partial\Omega$ has nonnegative mean curvature $H_{\partial\Omega}$ with respect to the inward orientation. In the case $n = 2$, the mean convexity property is equivalent to the convexity of Ω , but in arbitrary dimensions, the mean convexity is less restrictive than convexity.

The Dirichlet problem is now formulated as follows. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $\alpha \neq 0$ a given constant. Let $\varphi : \partial\Omega \rightarrow \mathbb{R}$ be a positive spacelike smooth function. The problem is finding a classical solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $u > 0$ in $\bar{\Omega}$, of

$$(20) \quad \begin{cases} \operatorname{div} \left(\frac{Du}{\sqrt{1-|Du|^2}} \right) = \frac{\alpha}{u\sqrt{1-|Du|^2}} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ |Du| < 1 & \text{in } \bar{\Omega}. \end{cases}$$

We solve the Dirichlet problem when the boundary data φ has a spacelike extension in $\bar{\Omega}$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded mean convex domain with smooth boundary $\partial\Omega$. Assume that $\alpha < 0$. If $\varphi \in C^2(\bar{\Omega})$ is a positive function with $\max_{\bar{\Omega}} |D\varphi| < 1$, then there is a unique positive solution u of (20).*

The proof of Theorem 4.1 is accomplished by using the Schauder theory of a priori global estimates, the method of continuity and the Leray–Schauder fixed point theorem. Applying these techniques, we find all elements for proving Theorem 4.1. As usual, we will utilize the distance function d to $\partial\Omega$ to construct a barrier function ([Gilbarg and Trudinger 1983; Jenkins and Serrin 1968; Ladyzhenskaya and Uraltseva 1961]).

The C^0 estimates will be obtained by comparing the solution of (20) with the rotational examples studied in Section 3: here the hypothesis $\alpha < 0$ will be essential because if $\alpha > 0$ it is not possible to prevent that $|u| \rightarrow 0$ for a solution u . For the C^1 estimates, we need to prove that $|Du|$ is bounded away from 1 which will be

deduced by using barrier functions. Finally, the hypothesis $\alpha < 0$ will be also used when we apply the implicit function theorem for the existence of the linearized problem associated to (20).

The maximum principle for elliptic equations of divergence type implies the following result.

Proposition 4.2 (touching principle). *Let Σ_1 and Σ_2 be two α -singular maximal surfaces. If Σ_1 and Σ_2 have a common tangent interior point and Σ_1 lies above Σ_2 around p , then Σ_1 and Σ_2 coincide at an open set around p .*

We also need to formulate the comparison principle in the context of α -singular maximal surfaces. We write the equation of (20) in classical notation. Define the operator

$$(21) \quad \begin{aligned} Q[u] &= (1 - |Du|^2)\Delta u + u_i u_j u_{ij} - \frac{\alpha(1 - |Du|^2)}{u} \\ &= a_{ij}(Du)u_{ij} + \mathbf{b}(u, Du), \end{aligned}$$

where

$$a_{ij} = (1 - |Du|^2)\delta_{ij} + u_i u_j, \quad \mathbf{b} = -\frac{\alpha(1 - |Du|^2)}{u}.$$

Here $u_i = \partial u / \partial x_i$, $1 \leq i \leq n$, and we assume the summation convention of repeated indices. It is immediate that u is a solution of (20) if and only if $Q[u] = 0$. The ellipticity of the operator Q is clear because if $A = (a_{ij})$ and $\xi \in \mathbb{R}^n$, then

$$(22) \quad (1 - |p|^2)|\xi|^2 \leq \xi^t A \xi = (1 - |p|^2)|\xi|^2 + \langle p, \xi \rangle^2 \leq |\xi|^2.$$

Moreover, this shows that Q is not uniformly elliptic. We recall the comparison principle ([Gilbarg and Trudinger 1983, Theorem 10.1]).

Proposition 4.3 (comparison principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $Q[u] \geq Q[v]$ and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω .*

Notice that if $\alpha < 0$, the classical theory implies the uniqueness of solutions of the Dirichlet problem.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha < 0$. The solution of (20), if it exists, is unique.*

In arbitrary dimension, any horizontal translation and any dilation from a point of $\mathbb{R}^n \times \{0\}$ preserve (20). Similarly, Theorem 3.3 holds where now (16) is

$$\left(r \frac{u'}{\sqrt{1 - u'^2}} \right)' = r^{n-1} \frac{\alpha}{u \sqrt{1 - u'^2}}.$$

We establish the solvability of (20) in the particular case that Ω is a ball of \mathbb{R}^n and φ is a positive constant.

Proposition 4.5. *Let $\alpha < 0$ and $B_R \subset \mathbb{R}^n$ be a round ball of radius $R > 0$. If $c > 0$, then there is a unique radial solution u of (20) in B_R with $u = c$ on ∂B_R .*

Proof. After a horizontal translation, we suppose that the origin $O \in \mathbb{R}^n$ is the center of B_R . By Theorem 3.2, let $v = v(r)$ be the solution of (16) with $v(0) = 1$. Recall that Theorem 3.3 asserts that the maximal domain of v is a ball B_b for some $b > 0$ with $v(b) = 0$. In the (r, v) -plane, consider the line $x_{n+1} = cr/R$. Since v is a decreasing function, the graph of v meets this line at one point $r = r_o$, $u(r_o) = cr_o/R$. If $\lambda = R/r_o$, then $u_\lambda(r) = \lambda u(r/\lambda)$ is a solution of (20) with $u_\lambda(R) = c$. □

Following a standard scheme, we start by finding C^0 estimates by using the rotational solutions of (7). In the following result, we do not require the mean convexity of Ω .

Proposition 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha < 0$. If u is a positive solution of (20), there exists a constant $C_1 = C_1(\alpha, \Omega, \varphi) > 0$ such that*

$$(23) \quad \min_{\partial\Omega} \varphi \leq u \leq C_1 \quad \text{in } \Omega.$$

Proof. Since the right-hand side of (20) is negative, then $\inf_\Omega u = \min_{\partial\Omega} \varphi$ by the maximum principle. This proves the left inequality of (23).

For the upper estimate of (23), we consider the radial solution v of (16) with $v(0) = 1$ and let $\{v_\lambda : \lambda > 0\}$ where $v_\lambda(r) = \lambda v(r/\lambda)$. Denote B_R the maximal domain of v , with $v(R) = 0$ and let Σ_λ denote the graph of v_λ . Take $\lambda > 0$ sufficiently big so the graph S of u is included in the domain of the halfspace $x_{n+1} > 0$ bounded by $\Sigma_\lambda \cup B_{\lambda R}$. Let λ decrease to 0 until the first time λ_0 that Σ_λ meets Σ_u . By the maximum principle, the first contact must occur at some boundary point of S . Then this point is a point of $\partial\Omega$ or a point of ∂S . Since ∂S is the graph of φ , this value λ_0 depends on Ω and φ . Consequently, $u \leq v_{\lambda_0} \leq \sup_\Omega v_{\lambda_0}$. The proof finishes by letting $C_1 = \sup_\Omega v_{\lambda_0}$, which depends only on α, Ω and φ . □

The next step to prove Theorem 4.1 is the derivation of estimates for $|Du|$. This is done by first proving the supremum of $|Du|$ is attained at some boundary point. In the next result, the assumption that α is negative is essential.

Proposition 4.7 (interior gradient estimates). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\alpha < 0$. If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a positive solution of (20), then*

$$\max_{\bar{\Omega}} |Du| = \max_{\partial\Omega} |Du|.$$

Proof. Let $v^k = u_k, 1 \leq k \leq n$. By differentiating $Q[u] = 0$ with respect to x_k , we find for each k ,

$$(24) \quad ((1-|Du|^2)\delta_{ij} + u_i u_j) v_{ij}^k + 2 \left(u_i \Delta u + u_j u_{ij} - \frac{\alpha u_i}{u} \right) v_i^k + \frac{\alpha(1-|Du|^2)}{u^2} v^k = 0.$$

The equation (24) is a linear elliptic equation in v^k . Because $\alpha < 0$, the coefficient for v^k is negative and the maximum principle ([Gilbarg and Trudinger 1983, Theorem 3.7]) implies that $|v^k|$, and hence $|Du|$, does not have an interior maximum. In particular, the maximum of $|Du|$ on $\bar{\Omega}$ is attained at some boundary point, proving the result. \square

As a consequence of Proposition 4.7, the problem of finding a priori estimates of $|Du|$ reduces to get these estimates along $\partial\Omega$. With this purpose, we prove that u admits barriers from above and from below along $\partial\Omega$. We now use the assumption of the mean convexity of Ω .

Proposition 4.8 (boundary gradient estimates). *Let $\Omega \subset \mathbb{R}^n$ be a bounded mean convex domain and $\alpha < 0$. If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a positive solution of (20), then there is a constant*

$$C_2 = C_2(\alpha, \Omega, C_1, \|\varphi\|_{1;\bar{\Omega}}, \|\varphi\|_{2;\bar{\Omega}}) < 1$$

such that

$$\max_{\partial\Omega} |Du| \leq C_2.$$

Proof. We consider the operator $Q[u]$ defined in (21). For a lower barrier for u , we take the solution v^0 of the Dirichlet problem of the maximal surface equation in Ω with the same boundary φ . The function v^0 is the solution of (20) for $\alpha = 0$ whose existence is assured ([Bartnik and Simon 1982/83, Theorem 4.1]). Then

$$Q[v^0] = -\frac{\alpha(1 - |Dv^0|^2)}{v^0} > 0 = Q[u].$$

Since $v^0 = u$ on $\partial\Omega$, we conclude $v^0 < u$ in Ω by the comparison principle.

We now construct an upper barrier for u by means of the distance function in a small tubular neighborhood of $\partial\Omega$ in Ω .

Consider the distance function $d(x) = \text{dist}(x, \partial\Omega)$ and let $\epsilon > 0$ sufficiently small so $\mathcal{N}_\epsilon = \{x \in \bar{\Omega} : d(x) < \epsilon\}$ is a tubular neighborhood of $\partial\Omega$. We parametrize \mathcal{N}_ϵ using normal coordinates $x \equiv (t, \pi(x)) \in \mathcal{N}_\epsilon$, where $x \equiv \pi(x) + t\nu(\pi(x))$ for some $t \in [0, \epsilon)$, $\pi : \mathcal{N}_\epsilon \rightarrow \partial\Omega$ is the orthogonal projection and ν is the unit normal vector to $\partial\Omega$ pointing to Ω . A straightforward computation gives that d is C^2 , $|Dd|(x) = 1$, and $\Delta d(x) \leq -(n-1)H_{\partial\Omega}(\pi(x))$ for all $x \in \mathcal{N}_\epsilon$. Because Ω is mean convex, then $\Delta d(x) \leq 0$.

Define in \mathcal{N}_ϵ a function $w = h \circ d + \varphi$, where we use the same symbol φ for a spacelike extension of φ into Ω . The function h is defined as

$$(25) \quad h(t) = a \log(1 + kb^2t), \quad b, k > 0, \quad a = \frac{C_1}{\log(1 + b)},$$

where C_1 is the constant that appears in (23) and b and k will be chosen later. Let us observe that $w > 0$ and that we require that $|Dw| < 1$. The computation of $Q[w]$

leads to

$$Q[w] = a_{ij}(h''d_i d_j + h'd_{ij} + \varphi_{ij}) - \frac{\alpha}{w}(1 - |Dw|^2).$$

From $|Dd| = 1$, it follows that $\langle D(Dd)_x \xi, Dd(x) \rangle = 0$ for all $\xi \in \mathbb{R}^n$. If $\{e_i\}_i$ is the canonical basis of \mathbb{R}^n and $\xi = e_i$, we find $d_{ij}d_j = 0$. Thus

$$\begin{aligned} w_i w_j d_{ij} &= (h'd_i + \varphi_i)(h'd_j + \varphi_j)d_{ij} = (h'^2 d_i + 2h'\varphi_i)d_j d_{ij} + \varphi_i \varphi_j d_{ij} \\ &= \varphi_i \varphi_j d_{ij} \leq \varphi_i^2 \lambda_i^d \leq 0, \end{aligned}$$

where λ_i^d are the eigenvalues of D^2d , which all are not positive because D^2d is negative semidefinite. Using this inequality, the definition of a_{ij} in (21) and (22), it follows that

$$a_{ij}d_{ij} = (1 - |Dw|^2)\Delta d + w_i w_j d_{ij} \leq (1 - |Dw|^2)\Delta d \leq 0.$$

Again (22) implies $a_{ij}d_i d_j \geq 1 - |Dw|^2$ and $a_{ij}\varphi_{ij} \leq |D^2\varphi|$, where $|D^2\varphi| = \sum_{ij} \sup_{\bar{\Omega}} |\varphi_{ij}|$. Since $h' > 0$ and $\Delta d \leq 0$, we find

$$\begin{aligned} (26) \quad Q[w] &\leq h''(1 - |Dw|^2) + h'\Delta d(1 - |Dw|^2) - \frac{\alpha}{w}(1 - |Dw|^2) + a_{ij}\varphi_{ij} \\ &\leq \left(h'' - \frac{\alpha}{w}\right)(1 - |Dw|^2) + |D^2\varphi|. \end{aligned}$$

We now study the spacelike condition $|Dw| < 1$. The computation of $|Dw|$ and the Cauchy–Schwarz inequality gives

$$|Dw|^2 = h'^2 + |D\varphi|^2 + 2h'\langle Dd, D\varphi \rangle \leq (h' + |D\varphi|)^2.$$

Because $h' > 0$ and h' is decreasing on t , we deduce

$$(27) \quad |Dw| \leq h' + |D\varphi| \leq h'(0) + |D\varphi| \leq akb^2 + \mu \quad \text{in } \bar{\Omega},$$

where $\mu = \|D\varphi\|_{0;\bar{\Omega}} < 1$. Fix a constant δ with the property $\mu < \delta < 1$. Then it is possible to choose k sufficiently small in (27) so $|Dw| \leq akb^2 + \mu < \delta$. Let $\beta = 1 - \delta^2$. If $h'' - \alpha/w < 0$, then (26) implies

$$(28) \quad Q[w] \leq \beta \left(h'' - \frac{\alpha}{w}\right) + \|D^2\varphi\|_{0;\bar{\Omega}}.$$

The right-hand side in (28) is a function defined in $\partial\Omega \times [0, \epsilon]$. Let $\varphi_0 = \min_{\bar{\Omega}} \varphi > 0$ and we evaluate this function at $t = 0$, obtaining

$$\beta \left(-ak^2b^4 - \frac{\alpha}{\varphi}\right) + \|D^2\varphi\|_{0;\bar{\Omega}} \leq \beta \left(-\frac{(\delta - \mu)^2}{a} - \frac{\alpha}{\varphi_0}\right) + \|D^2\varphi\|_{0;\bar{\Omega}}.$$

If b is sufficiently big, then $a \rightarrow 0$, hence the right-hand side in this inequality is negative. By compactness of $\partial\Omega \times [0, \epsilon]$ and by continuity, let us take b sufficiently large in (28) so $Q[w] < 0$. Even more, we require b so large that $1/(kb) < \epsilon$. We

now change the tubular neighborhood \mathcal{N}_ϵ by replacing ϵ by $\epsilon = 1/(kb)$ and we denote this by \mathcal{N}_ϵ again.

In order to assure that w is a local upper barrier in \mathcal{N}_ϵ for the Dirichlet problem (20), we need to have

$$(29) \quad u \leq w \quad \text{in } \partial\mathcal{N}_\epsilon.$$

In $\partial\mathcal{N}_\epsilon \cap \partial\Omega$, the distance function is $d = 0$, so $w = \varphi = u$. On the other hand, in $\partial\mathcal{N}_\epsilon \setminus \partial\Omega$, and because $\epsilon = 1/(kb)$, we find

$$w = h(\epsilon) + \varphi = \frac{C_1}{\log(1+b)} \log(1+kb^2\epsilon) + \varphi = C_1 + \varphi.$$

By Proposition 4.6, we have $u \leq C_1$ and we deduce $u < w$ in $\mathcal{N}_\epsilon \setminus \partial\Omega$. Definitively, we find $Q[w] < 0 = Q[u]$ and $u \leq w$ in $\partial\mathcal{N}_\epsilon$, concluding that $u \leq w$ in \mathcal{N}_ϵ by the comparison principle.

Consequently, we have proved the existence of lower and upper barriers for u in \mathcal{N}_ϵ , namely, $v^0 \leq u \leq w$ in \mathcal{N}_ϵ . Hence we deduce

$$\max_{\partial\Omega} |Du| \leq C_2 := \max\{\|Dw\|_{0;\bar{\Omega}}, \|Dv^0\|_{0;\bar{\Omega}}\}$$

and both values $\|Dw\|_{0;\bar{\Omega}}, \|Dv^0\|_{0;\bar{\Omega}}$ depend only on the initial data of the Dirichlet problem. This completes the proof of proposition. \square

With all of the above ingredients, we are in position to prove Theorem 4.1.

Proof of Theorem 4.1. We establish the solvability of the Dirichlet problem (20) by the method of continuity (see [Gilbarg and Trudinger 1983, Section 17.2]). Define the family of Dirichlet problems parametrized by $t \in [0, 1]$

$$\mathcal{P}_t : \begin{cases} Q_t[u] = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where

$$Q_t[u] = (1 - |Du|^2)\Delta u + u_i u_j u_{ij} - \frac{\alpha t(1 - |Du|^2)}{u}.$$

The graph Σ_{u_t} of a solution of u_t is a $(t\alpha)$ -singular maximal surface. Notice that if $t = 0$, $Q_0[u] = 0$ is the maximal surface equation and the solution of \mathcal{P}_0 is the function v^0 that appeared in Proposition 4.8. As usual, let

$$\mathcal{A} = \{t \in [0, 1] : \text{there exists } u_t \in C^{2,\gamma}(\bar{\Omega}), u_t > 0, Q_t[u_t] = 0, u_t|_{\partial\Omega} = \varphi\}.$$

The proof consists of showing that $1 \in \mathcal{A}$. For this, we prove that \mathcal{A} is a nonempty open and closed subset of $[0, 1]$.

(1) The set \mathcal{A} is not empty. This is because $0 \in \mathcal{A}$ since v^0 is the solution of \mathcal{P}_0 .

(2) The set \mathcal{A} is open in $[0, 1]$. Given $t_0 \in \mathcal{A}$, we need to prove that there is an $\epsilon > 0$ such that $(t_0 - \epsilon, t_0 + \epsilon) \cap [0, 1] \subset \mathcal{A}$. Define the map $T(t, u) = Q_t[u]$ for $t \in \mathbb{R}$ and $u \in C^{2,\gamma}(\bar{\Omega})$. Then $t_0 \in \mathcal{A}$ if and only if $T(t_0, u_{t_0}) = 0$. If we show that the derivative of Q_t with respect to u , say $(DQ_t)_u$, at the point u_{t_0} is an isomorphism, then from the implicit function theorem we have the existence of an open set $\mathcal{V} \subset C^{2,\gamma}(\bar{\Omega})$, with $u_{t_0} \in \mathcal{V}$, and a C^1 function $\xi : (t_0 - \epsilon, t_0 + \epsilon) \rightarrow \mathcal{V}$ for some $\epsilon > 0$, such that $\xi(t_0) = u_{t_0} > 0$ and $T(t, \xi(t)) = 0$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$; this guarantees that \mathcal{A} is an open set of $[0, 1]$.

The proof that $(DQ_t)_u$ is one-to-one is equivalent to proving that for any $f \in C^\gamma(\bar{\Omega})$, there is a unique solution $v \in C^{2,\gamma}(\bar{\Omega})$ of the linear equation $Lv := (DQ_t)_u(v) = f$ in Ω and $v = \varphi$ on $\partial\Omega$. The computation of L was done in Proposition 4.7, obtaining

$$Lv = (DQ_t)_u v = a_{ij}(Du)v_{ij} + \mathbf{b}_i(u, Du, D^2u)v_i + \mathbf{c}(u, Du)v,$$

where a_{ij} are defined in (21) and

$$\mathbf{b}_i = 2\left(\Delta u - \frac{\alpha t}{u}\right)u_i + 2u_j u_{ij}, \quad \mathbf{c} = \frac{\alpha t(1 - |Du|^2)}{u^2}.$$

Since $\alpha < 0$, $\mathbf{c} \leq 0$ and the existence and uniqueness is assured by standard theory ([Gilbarg and Trudinger 1983, Theorem 6.14]).

The set \mathcal{A} is closed in $[0, 1]$. Let $\{t_k\} \subset \mathcal{A}$ with $t_k \rightarrow t \in [0, 1]$. For each $k \in \mathbb{N}$, there exists $u_{t_k} \in C^{2,\gamma}(\bar{\Omega})$, $u_{t_k} > 0$, such that $Q_{t_k}[u_{t_k}] = 0$ in Ω and $u_{t_k} = \varphi$ in $\partial\Omega$. Define the set

$$\mathcal{S} = \{u \in C^{2,\gamma}(\bar{\Omega}) : \text{there exists } t \in [0, 1] \text{ such that } Q_t[u] = 0 \text{ in } \Omega, u|_{\partial\Omega} = \varphi\}.$$

Then $\{u_{t_k}\} \subset \mathcal{S}$. If we prove that the set \mathcal{S} is bounded in $C^{1,\beta}(\bar{\Omega})$ for some $\beta \in [0, \gamma]$, and since $a_{ij} = a_{ij}(Du)$ in (21), then Schauder theory proves that \mathcal{S} is bounded in $C^{2,\beta}(\bar{\Omega})$, in particular, \mathcal{S} is precompact in $C^2(\bar{\Omega})$ (see Theorem 6.6 and Lemma 6.36 in [Gilbarg and Trudinger 1983]). Thus there exists a subsequence $\{u_{k_l}\} \subset \{u_{t_k}\}$ converging in $C^2(\bar{\Omega})$ to some $u \in C^2(\bar{\Omega})$. Since $T : [0, 1] \times C^2(\bar{\Omega}) \rightarrow C^0(\bar{\Omega})$ is continuous, it follows that $Q_t[u] = T(t, u) = \lim_{l \rightarrow \infty} T(t_{k_l}, u_{k_l}) = 0$ in Ω . Moreover, $u|_{\partial\Omega} = \lim_{l \rightarrow \infty} u_{k_l}|_{\partial\Omega} = \varphi$ on $\partial\Omega$, so $u \in C^{2,\gamma}(\bar{\Omega})$ and consequently, $t \in \mathcal{A}$.

The above reasoning asserts that \mathcal{A} is closed in $[0, 1]$ provided we find a constant M independent of $t \in \mathcal{A}$, such that

$$\|u_t\|_{C^1(\bar{\Omega})} = \sup_{\Omega} |u_t| + \sup_{\Omega} |Du_t| \leq M.$$

However the C^0 and C^1 estimates for the function u_1 , that is, when the parameter t is $t = 1$, are enough as we now see.

The C^0 estimates for u_t follow with the comparison principle. Indeed, let $t_1 < t_2$, $t_i \in [0, 1]$, $i = 1, 2$. Then $Q_{t_1}[u_{t_1}] = 0$ and

$$Q_{t_1}[u_{t_2}] = -\frac{(t_1 - t_2)\alpha(1 - |Du_{t_2}|^2)}{u_{t_2}} < 0 = Q_{t_1}[u_{t_1}]$$

because $\alpha < 0$. Since $u_{t_1} = \varphi = u_{t_2}$ on $\partial\Omega$, the comparison principle yields $u_{t_1} < u_{t_2}$ in Ω . This proves that the solutions u_{t_i} are ordered in an increasing sense according to the parameter t . By (23), we find

$$(30) \quad \sup_{\Omega} u_t \leq \sup_{\Omega} u_1 \leq C_1.$$

In order to derive the gradient estimates for the solution u_t , the same computations obtained in Proposition 4.8 conclude that $\sup_{\partial\Omega} |Du_t|$ is bounded by a constant depending on α , Ω , φ and $\|u_t\|_{0;\bar{\Omega}}$. Now (30) implies that the value $\|u_t\|_{0;\bar{\Omega}}$ is bounded by C_1 , which depends only on α , φ and Ω , but not on t .

The above three steps prove the existence part in Theorem 4.1. The uniqueness is consequence of Proposition 4.4 and this completes the proof of theorem. \square

A consequence of Theorem 4.1 is the solvability of the Plateau problem if $\alpha < 0$ in the following situation.

Corollary 4.9. *Let Γ be a spacelike $(n-1)$ -submanifold of \mathbb{L}^{n+1} with a one-to-one orthogonal projection C on the hyperplane of equation $x_{n+1} = 0$ such that C is the boundary of a mean convex simply-connected domain Ω . Let $\alpha < 0$. If Γ has a spacelike extension to a graph on Ω , then there exists a unique α -singular maximal hypersurface S spanning Γ . Moreover, S is a graph on Ω .*

Proof. Theorem 4.1 asserts the existence of an α -singular maximal hypersurface S whose boundary is Γ and S is a graph on Ω . Assume that M is another such hypersurface. The property that M is spacelike implies that the orthogonal projection $p : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n = \mathbb{R}^n \times \{0\}$, $p(x) = (x_1, \dots, x_n)$ is a local diffeomorphism between M and Ω . In particular, $p : M \rightarrow \Omega$ is a covering map and since Ω is simply connected, the map p is a diffeomorphism, in particular, M is a graph on Ω . Finally, the uniqueness of (7) when α is negative concludes that $M = S$. \square

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