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We construct a two parameter family of complete embedded triply periodic minimal surfaces in Nil_3, which are the analogues of the three parameter family of the Schwarz D-surfaces in the Euclidean three space.

1. Introduction

Take a polygon in $\mathbb{R}^3$ consisting of three pairs of oppositely parallel edges of a cube (Figure 1) and solve the Plateau problem with respect to this polygon to get an embedded minimal disk, and then reflect this surface successively with respect to the edges. Continuing this process indefinitely one gets an embedded completely triply periodic minimal surface, which is called the Schwarz D-surface.

In this paper we construct a two parameter family of embedded complete triply periodic minimal surfaces in Nil_3, which are the analogues of this Schwarz D-surfaces in the Euclidean three space. In Nil_3, a reflection with respect to a geodesic is not necessarily an isometry, however, reflections with respect to the horizontal or vertical geodesic are isometries. As in the case in [Shin et al. 2018], we suitably take a geodesic polygon consisting of the segments of horizontal and vertical geodesics only to get an embedded minimal disk. The polygon for the Schwarz D-surface in Figure 1 consists of the traces of the integral curves of the vector fields ∂x, ∂y and ∂z, starting from (0, 0, 0) in the order of ∂x, ∂y, ∂z, ∂x, ∂y and ∂z completing a Hamiltonian circuit. Our geodesic polygon consists of the traces of the integral curves of the vector fields $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, starting from (0, 0, 0) in the order of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ completing a Hamiltonian circuit (see Figure 2). On the other hand, the geodesic polygon for the Schwarz CLP-surface in Nil_3 in our earlier paper [Shin et al. 2019] consists of the traces of the integral curves of the vector fields $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ too, however, in the order of $\boldsymbol{e}_1$, $\boldsymbol{e}_3$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{e}_2$. Then our construction consists of the successive reflections with respect to boundary geodesics. The

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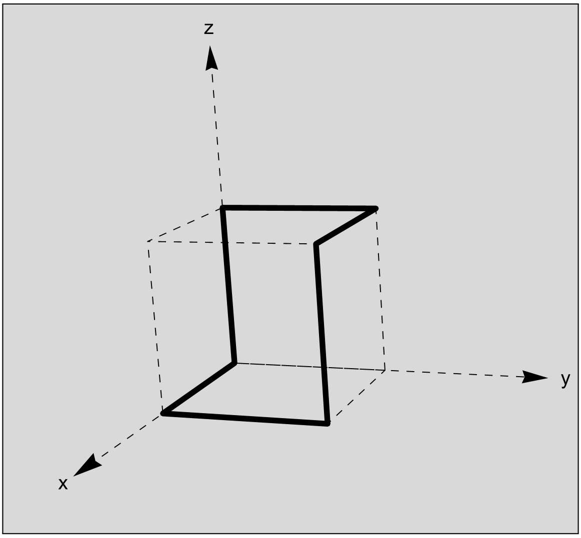

Figure 1. Geodesic polygon for Schwarz D-surface.

different surfaces obtained after reflections are still smooth because the surfaces considered are conformal embeddings of a disk that extends continuously up to the boundary and each piece of boundary is sent to a piece of the geodesic line along which the successive reflections are to be applied. Since the glued surface with the reflected surface is smooth along the geodesic line of the reflection, a crucial issue in our construction lies in how to make the pieces fit together along the boundary geodesic to form a smooth embedded surface.

2. Nil_3

The three-dimensional Heisenberg group is the Lie group $(\mathbb{R}^3, *)$, with $*$ defined as

$$(x, y, z) * (x', y', z') = \big(x + x', y + y', z + z' + \tfrac{1}{2}(xy' - x'y)\big).$$

The identity element is $(0, 0, 0)$ and the inverse element of (x, y, z) is $(-x, -y, -z)$. We denote by Nil_3 the three-dimensional Heisenberg group endowed with the left-invariant metric

$$g = dx^2 + dy^2 + \big(dz + \tfrac{1}{2}(ydx - xdy)\big)^2.$$

It is a Riemannian fibration over the Euclidean plane $\mathbb{R}^2$ with the projection

$$\pi : (x, y, z) \mapsto (x, y).$$

For the left-invariant orthonormal frame field

$$\boldsymbol{e}_1 := \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad \boldsymbol{e}_2 := \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad \boldsymbol{e}_3 := \frac{\partial}{\partial z},$$

the Koszul formula gives

$$\begin{aligned} &\nabla_{\boldsymbol{e}_1}\boldsymbol{e}_1 = 0, && \nabla_{\boldsymbol{e}_2}\boldsymbol{e}_1 = -\tfrac{1}{2}\boldsymbol{e}_3, && \nabla_{\boldsymbol{e}_3}\boldsymbol{e}_1 = -\tfrac{1}{2}\boldsymbol{e}_2, \\ &\nabla_{\boldsymbol{e}_1}\boldsymbol{e}_2 = \tfrac{1}{2}\boldsymbol{e}_3, && \nabla_{\boldsymbol{e}_2}\boldsymbol{e}_2 = 0, && \nabla_{\boldsymbol{e}_3}\boldsymbol{e}_2 = \tfrac{1}{2}\boldsymbol{e}_1, \\ &\nabla_{\boldsymbol{e}_1}\boldsymbol{e}_3 = -\tfrac{1}{2}\boldsymbol{e}_2, && \nabla_{\boldsymbol{e}_2}\boldsymbol{e}_3 = \tfrac{1}{2}\boldsymbol{e}_1, && \nabla_{\boldsymbol{e}_3}\boldsymbol{e}_3 = 0, \end{aligned}$$

from which one can see that any integral curve of the vector field $\boldsymbol{e}_i$, $i = 1, 2, 3$, is a geodesic.

Let $\gamma^i_{(a,b,c)}(t)$ be the integral curves of $\boldsymbol{e}_i$ with $\gamma^i_{(a,b,c)}(0) = (a, b, c)$, $i = 1, 2, 3$. Then, since each $\boldsymbol{e}_i$ is left invariant, one can see that the traces $\{\gamma^1_{(0,0,0)}\}$ of the integral curve $\gamma^1_{(0,0,0)}(t)$ is the x-axis $\{(t, 0, 0)\}$, the trace $\{\gamma^2_{(0,0,0)}\}$ of the integral curve $\gamma^2_{(0,0,0)}(t)$ is the y-axis $\{(0, t, 0)\}$ and the trace $\{\gamma^3_{(0,0,0)}\}$ of the integral curve $\gamma^3_{(0,0,0)}(t)$ is the z-axis $\{(0, 0, t)\}$. In fact, the trace $\{\gamma^i_{(a,b,c)}\}$ is the fiber of the fibration π through (a, b, c).

Let $\mathcal{R}^i_{(a,b,c)}$ be the reflection with respect to the geodesic $\{\gamma^i_{(a,b,c)}\}$, $i = 1, 2, 3$. The following are proved in [Shin et al. 2018]:

$$\begin{aligned} \mathcal{R}^1_{(0,0,0)}(x, y, z) &= (x, -y, -z), \\ \mathcal{R}^2_{(0,0,0)}(x, y, z) &= (-x, y, -z), \\ \mathcal{R}^3_{(0,0,0)}(x, y, z) &= (-x, -y, z). \end{aligned}$$

Proposition 1.

$$\begin{aligned} \mathcal{R}^1_{(a,b,c)}(x, y, z) &= (x, -y + 2b, -z - bx + ab + 2c), \\ \mathcal{R}^2_{(a,b,c)}(x, y, z) &= (-x + 2a, y, -z + ay - ab + 2c), \\ \mathcal{R}^3_{(a,b,c)}(x, y, z) &= (-x + 2a, -y + 2b, z + bx - ay). \end{aligned}$$

Proof. We give a proof of the first formula. The other formulae follow similarly. Note first that, since $\boldsymbol{e}_1$ is a left invariant vector field, one has

$$(-a, -b, -c) * \{\gamma^1_{(a,b,c)}\} = \{(t, 0, 0)\},$$

the x-axis. Then one has

$$\begin{aligned} \mathcal{R}^1_{(a,b,c)}(x, y, z) &= (a, b, c) * \mathcal{R}^1_{(0,0,0)}((-a, -b, -c) * (x, y, z)) \\ &= (x, 2b - y, 2c - z - bx + ab). \end{aligned}$$ □

Then one can see that every reflection $\mathcal{R}^i_{(a,b,c)}$ is an isometry. Moreover,

$$\mathcal{R}^1_{(a,b,c)} \circ \mathcal{R}^3_{(a,b,c)} = \mathcal{R}^3_{(a,b,c)} \circ \mathcal{R}^1_{(a,b,c)} = \mathcal{R}^2_{(a,b,c)},$$
$$\mathcal{R}^2_{(a,b,c)} \circ \mathcal{R}^3_{(a,b,c)} = \mathcal{R}^3_{(a,b,c)} \circ \mathcal{R}^2_{(a,b,c)} = \mathcal{R}^1_{(a,b,c)},$$
$$\mathcal{R}^1_{(a,b,c)} \circ \mathcal{R}^2_{(a,b,c)} = \mathcal{R}^2_{(a,b,c)} \circ \mathcal{R}^1_{(a,b,c)} = \mathcal{R}^3_{(a,b,c)}$$

and

(1) $d\mathcal{R}^1_{(a,b,c)}(\boldsymbol{e}_1) = \boldsymbol{e}_1$, $d\mathcal{R}^1_{(a,b,c)}(\boldsymbol{e}_2) = -\boldsymbol{e}_2$, $d\mathcal{R}^1_{(a,b,c)}(\boldsymbol{e}_3) = -\boldsymbol{e}_3$,

(2) $d\mathcal{R}^2_{(a,b,c)}(\boldsymbol{e}_1) = -\boldsymbol{e}_1$, $d\mathcal{R}^2_{(a,b,c)}(\boldsymbol{e}_2) = \boldsymbol{e}_2$, $d\mathcal{R}^2_{(a,b,c)}(\boldsymbol{e}_3) = -\boldsymbol{e}_3$,

(3) $d\mathcal{R}^3_{(a,b,c)}(\boldsymbol{e}_1) = -\boldsymbol{e}_1$, $d\mathcal{R}^3_{(a,b,c)}(\boldsymbol{e}_2) = -\boldsymbol{e}_2$, $d\mathcal{R}^3_{(a,b,c)}(\boldsymbol{e}_3) = \boldsymbol{e}_3$.

Furthermore, since an isometry is uniquely defined by its differential at one point (for the proof of this fact, see for example [Petersen 1998, p. 137]) one has:

Proposition 2. *Let I be an isometry in Nil_3 satisfying* (i), $i = 1, 2, 3$ *of the above conditions with $I(a, b, c) = (a, b, c)$. Then $I = \mathcal{R}^i_{(a,b,c)}$.* □

One can also see that the translation along the fiber

$$T_a(x, y, z) := (x, y, z + a)$$

is also an isometry. Then one has the following relations:

$$\mathcal{R}^i_{(0,0,c)} \circ T_a = T_{-a+2c} \circ \mathcal{R}^i_{(0,0,0)}, \qquad i = 1, 2,$$
$$\mathcal{R}^3_{(0,0,0)} \circ T_a = T_a \circ \mathcal{R}^3_{(0,0,0)}.$$

Finally, note that for any constant a, the Euclidean planes

$$\{(a, y, z)\}, \quad \{(x, a, z)\}, \quad \{(x, y, a)\}$$

are minimal surfaces in Nil_3.

3. A construction

Our construction follows the procedure: Partition the whole space into cylindrical regions

$$C_{n,m} := \{(x, y, z) \mid (2n-1)p \le x \le (2n+1)p,\ (2m-1)q \le y \le (2m+1)q\}.$$

Then construct a smooth minimal surface $D_{0,0}$ embedded in $C_{0,0}$ with $\partial D_{0,0} \subset \partial C_{0,0}$. Since $\mathcal{R}^3_{(np,mq,0)}(C_{0,0}) = C_{n,m}$, there exists a smooth minimal surface $D_{n,m} := \mathcal{R}^3_{(np,mq,0)}(D_{0,0})$ embedded in $C_{n,m}$. Then we show that the surfaces $D_{n,m}$ are joined smoothly along the boundary segments to get the complete smooth minimal surface $D := \bigcup_{n,m\in\mathbb{Z}} D_{n,m}$.

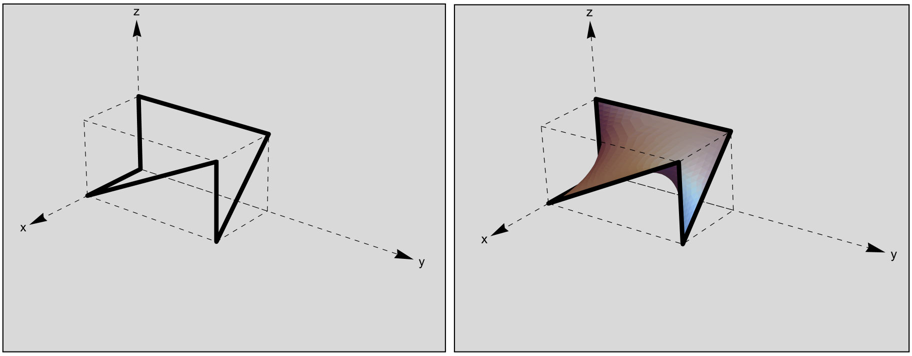

Figure 2. Geodesic polygon $\Gamma_{p,q}$ and minimal disk D_0.

Let us now begin the construction. We first construct a minimal surface embedded in the cylindrical region

$$C_{0,0} := \{(x, y, z) : -p \le x \le p, -q \le y \le q\}.$$

For constants $p, q > 0$, let $\Gamma_{p,q}$ be the geodesic polygon consisting of the six geodesic segments

$$\begin{aligned}
\gamma^1_{(0,0,0)}(s) &= (s, 0, 0), && 0 \le s \le p,\\
\gamma^2_{(p,0,0)}(s) &= \left(p, s, \tfrac{1}{2}ps\right), && 0 \le s \le q,\\
\gamma^3_{(p,q,0)}(s) &= (p, q, s), && 0 \le s \le \tfrac{1}{2}pq,\\
\gamma^1_{(0,q,\frac{1}{2}pq)}(s) &= \left(s, q, \tfrac{1}{2}(p-s)q\right), && 0 \le s \le p,\\
\gamma^2_{(0,0,\frac{1}{2}pq)}(s) &= \left(0, s, \tfrac{1}{2}pq\right), && 0 \le s \le q,\\
\gamma^3_{(0,0,0)}(s) &= (0, 0, s), && 0 \le s \le \tfrac{1}{2}pq.
\end{aligned}$$

which is contained in the boundary of the parallelepiped

$$C_0 := \left\{(x, y, z) : 0 \le x \le p,\ 0 \le y \le q,\ 0 \le z \le \tfrac{1}{2}pq\right\},$$

which is mean-convex since it is bounded by six minimal surfaces

$$\{(0, y, z)\},\ \{(p, y, z)\},\ \{(x, 0, z)\},\ \{(x, q, z)\},\ \{(x, y, 0)\},\ \left\{\left(x, y, \tfrac{1}{2}pq\right)\right\}.$$

Then $\Gamma_{p,q}$ spans an embedded minimal disk D_0 lying inside of C_0 (see Figure 2). Let

$$D_i = \mathcal{R}^i_{(0,0,0)}(D_0), \quad i = 1, 2, 3.$$

Then one has

$$\begin{aligned}
&D_0 \cap D_1 = \{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid 0 \le s \le p\},\\
&D_0 \cap D_2 = \{(0, 0, 0)\},\\
&D_0 \cap D_3 = \left\{\gamma^3_{(0,0,0)}(s) = (0, 0, s) \mid 0 \le s \le \tfrac{1}{2}pq\right\},\\
&D_1 \cap D_2 = \left\{\gamma^3_{(0,0,0)}(s) = (0, 0, s) \mid -\tfrac{1}{2}pq \le s \le 0\right\},\\
&D_1 \cap D_3 = \{(0, 0, 0)\},\\
&D_2 \cap D_3 = \{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid -p \le s \le 0\}.
\end{aligned}$$

Let

$$D := D_0 \cup D_1 \cup D_2 \cup D_3$$

which is smooth along $\{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid 0 \le s \le p\}$ since

$$D_1 = \mathcal{R}^1_{(0,0,0)}(D_0),$$

smooth along $\left\{\gamma^3_{(0,0,0)}(s) = (0, 0, s) \mid 0 \le s \le \tfrac{1}{2}pq\right\}$ since

$$D_3 = \mathcal{R}^3_{(0,0,0)}(D_0),$$

smooth along $\left\{\gamma^3_{(0,0,0)}(s) = (0, 0, s) \mid -\tfrac{1}{2}pq \le s \le 0\right\}$ since

$$D_2 = \mathcal{R}^2_{(0,0,0)}(D_0) = \mathcal{R}^3_{(0,0,0)}(\mathcal{R}^1_{(0,0,0)}(D_0)) = \mathcal{R}^3_{(0,0,0)}(D_1)$$

and smooth along $\{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid -p \le s \le 0\}$ since

$$D_2 = \mathcal{R}^2_{(0,0,0)}(D_0) = \mathcal{R}^1_{(0,0,0)}(\mathcal{R}^3_{(0,0,0)}(D_0)) = \mathcal{R}^1_{(0,0,0)}(D_3).$$

Since

$$D_1 \cup D_2 = \mathcal{R}^1_{(0,0,0)}(D_0 \cup D_3),$$

we can see that D is also smooth along the geodesic

$$\{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid -p \le s \le p\}$$

including the corner point $(0, 0, 0)$ of D_0. Hence the surface D is a smooth minimal surface. Moreover, one can see that D is embedded in the region

$$\left\{(x, y, z) : -p \le x \le p,\ -q \le y \le q,\ -\tfrac{1}{2}pq \le z \le \tfrac{1}{2}pq\right\}$$

and that

$$\mathcal{R}^1_{(0,0,0)}(D) = \mathcal{R}^2_{(0,0,0)}(D) = \mathcal{R}^3_{(0,0,0)}(D) = D.$$

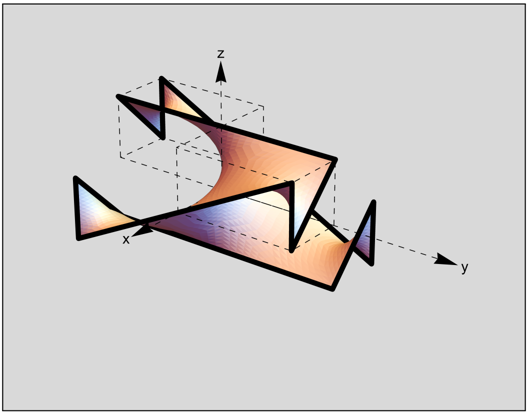

Figure 3. $D_0 \cup D_1 \cup D_2 \cup D_3$

The boundary of D is the Jordan curve consisting of the geodesic segments

$$\begin{aligned}
&\gamma^2_{(0,0,-\frac{1}{2}pq)}(s), && -q \le s \le q, && \gamma^1_{(0,q,-\frac{1}{2}pq)}(s), && -p \le s \le 0,\\
&\gamma^3_{(-p,q,0)}(s), && -\tfrac{1}{2}pq \le s \le 0, && \gamma^2_{(-p,0,0)}(s), && -q \le s \le q,\\
&\gamma^3_{(-p,-q,0)}(s), && 0 \le s \le \tfrac{1}{2}pq, && \gamma^1_{(0,-q,\frac{1}{2}pq)}(s), && -p \le s \le 0,\\
&\gamma^2_{(0,0,\frac{1}{2}pq)}(s), && -q \le s \le q, && \gamma^1_{(0,q,\frac{1}{2}pq)}(s), && 0 \le s \le p,\\
&\gamma^3_{(p,q,0)}(s), && 0 \le s \le \tfrac{1}{2}pq, && \gamma^2_{(p,0,0)}(s), && -q \le s \le q,\\
&\gamma^3_{(p,-q,0)}(s), && -\tfrac{1}{2}pq \le s \le 0, && \gamma^1_{(0,-q,-\frac{1}{2}pq)}(s), && 0 \le s \le p.
\end{aligned}$$

Now we let

$$D_{0,0} := \bigcup_{k\in\mathbb{Z}} T_{kpq}(D).$$

For each $k \in \mathbb{Z}$, the surface $T_{kpq}(D)$ intersects $T_{(k+1)pq}(D)$ on the common boundary

$$\bigl\{\gamma^2_{(0,0,(k+\frac{1}{2})pq)}(s) = \bigl(0,\, s,\, \bigl(k+\tfrac{1}{2}\bigr)pq\bigr) \mid -q \le s \le q\bigr\}.$$

Since $\mathcal{R}^2_{(0,0,(k+\frac{1}{2})pq)} \circ T_{kpq} = T_{(k+1)pq} \circ \mathcal{R}^2_{(0,0,0)}$, we have

$$\mathcal{R}^2_{(0,0,(k+\frac{1}{2})pq)}(T_{kpq}(D)) = T_{(k+1)pq}(\mathcal{R}^2_{(0,0,0)}(D)) = T_{(k+1)pq}(D),$$

that is, $T_{(k+1)pq}(D)$ is the reflection of $T_{kpq}(D)$ with respect to the common boundary. Therefore each $T_{(k+1)pq}(D)$ is smoothly joined to $T_{kpq}(D)$ and $D_{0,0}$ is a smooth embedded minimal surface in the cylinder $C_{0,0}$.

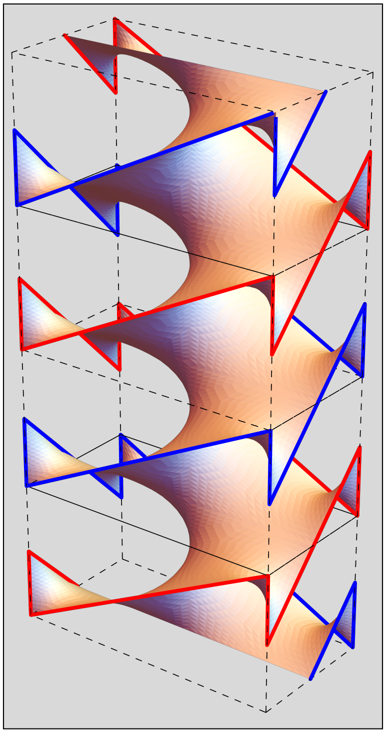

Figure 4. $D_{0,0}$

The boundary $\partial D_{0,0} = D_{0,0} \cap \partial C_{0,0}$ of $D_{0,0}$ consists of the following geodesic segments:

$$\begin{aligned} &\gamma^1_{(0,\pm q,kpq+\frac{1}{2}pq)}(t), && -p \le t \le p,\\ &\gamma^2_{(\pm p,0,kpq)}(t) && -q \le t \le q,\\ &\gamma^3_{(\pm p,\pm q,kpq)}(t), && 0 \le t \le \tfrac{1}{2}pq,\\ &\gamma^3_{(\pm p,\mp q,kpq)}(t), && -\tfrac{1}{2}pq \le t \le 0 \end{aligned}$$

for $k \in \mathbb{Z}$. It is clear that $T_{lpq}(D_{0,0}) = D_{0,0},\ l \in \mathbb{Z}$. We also note that

$$\mathcal{R}^i_{(0,0,0)}(D_{0,0}) = D_{0,0}, \quad i = 1, 2, 3,$$

since $\mathcal{R}^i_{(0,0,0)} \circ T_{kpq} = T_{-kpq} \circ \mathcal{R}^i_{(0,0,0)}$ for $i = 1, 2$, and $\mathcal{R}^3_{(0,0,0)} \circ T_{kpq} = T_{kpq} \circ \mathcal{R}^3_{(0,0,0)}$.

We now reflect the surface $D_{0,0}$ with respect to the vertical geodesic $\gamma^3_{(np,mq,0)}$ and let

$$D_{n,m} := \mathcal{R}^3_{(np,mq,0)}(D_{0,0}), \quad m, n \in \mathbb{Z}.$$

Then $D_{n,m}$ is an embedded minimal surface in the cylinder

$$\begin{aligned} C_{n,m} &:= \mathcal{R}^3_{(np,mq,0)}(C_{0,0}) \\ &= \{(x,y,z) \mid (2n-1)p \le x \le (2n+1)p,\ (2m-1)q \le y \le (2m+1)q\} \end{aligned}$$

with the boundary $\partial D_{n,m} = D_{n,m} \cap \partial C_{n,m}$. Since

$$\begin{aligned} \mathcal{R}^3_{(np,mq,0)}\big(0, \pm q, kpq + \tfrac{1}{2}pq\big) &= \big(2np, (2m \mp 1)q, (k \mp n)pq + \tfrac{1}{2}pq\big), \\ \mathcal{R}^3_{(np,mq,0)}(\pm p, 0, kpq) &= ((2n \mp 1)p, 2mq, (k \pm m)pq), \\ \mathcal{R}^3_{(np,mq,0)}(\pm p, \pm q, kpq) &= ((2n \mp 1)p, (2m \mp 1)q, (k \mp n \pm m)pg), \\ \mathcal{R}^3_{(np,mq,0)}(\pm p, \mp q, kpq) &= ((2n \mp 1)p, (2m \pm 1)q, (k \pm n \pm m)pg) \end{aligned}$$

and

$$d\mathcal{R}^3_{(np,mq,0)}(\boldsymbol{e}_1) = -\boldsymbol{e}_1, \quad d\mathcal{R}^3_{(np,mq,0)}(\boldsymbol{e}_2) = -\boldsymbol{e}_2, \quad d\mathcal{R}^3_{(np,mq,0)}(\boldsymbol{e}_3) = \boldsymbol{e}_3$$

we can see that the boundary $\partial D_{n,m}$ of $D_{n,m}$ is the union of the geodesic segments

$$\begin{aligned} &\gamma^1_{(2np,(2m\pm1)q,kpq+\frac{1}{2}pq)}(t), && -p \le t \le p, \\ &\gamma^2_{((2n\pm1)p,2mq,kpq)}(t), && -q \le t \le q, \\ &\gamma^3_{((2n\pm1)p,(2m\pm1)q,kpq)}(t), && \begin{cases} 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is even,} \\ -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is odd,} \end{cases} \\ &\gamma^3_{((2n\pm1)p,(2m\mp1)q,kpq)}(t), && \begin{cases} -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is even,} \\ 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is odd.} \end{cases} \end{aligned}$$

Now, we show that any boundary geodesic segment in $\partial D_{n,m}$ is the common boundary segments of exactly two such surfaces and these surfaces are joined smoothly across the common boundary segments. We will consider each type of boundary segments listed above separately.

(i) The boundary segment

$$\gamma^1_{(2np,(2m+1)q,kpq+\frac{1}{2}pq)}(t), \quad -p \le t \le p$$

of $D_{n,m}$ is also a boundary segment of $D_{n,m+1}$ only. Moreover, since ${\mathcal{R}^3_{(np,mq,0)}}^{-1} = \mathcal{R}^3_{(np,mq,0)}$ and $D_{0,0}$ is invariant under the translations T_{lpq} and the reflections $\mathcal{R}^i_{(0,0,0)}$,

$$\begin{aligned} D_{n,m+1} &= \mathcal{R}^3_{(np,(m+1)q,0)}(D_{0,0}) \\ &= \mathcal{R}^3_{(np,(m+1)q,0)}\big(T_{(2k+1)pq}(\mathcal{R}^1_{(0,0,0)}(D_{0,0}))\big) \\ &= \big(\mathcal{R}^3_{(np,(m+1)q,0)} \circ T_{(2k+1)pq} \circ \mathcal{R}^1_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}\big)(D_{n,m}). \end{aligned}$$

For the isometry $\psi = \mathcal{R}^3_{(np,(m+1)q,0)} \circ T_{(2k+1)pq} \circ \mathcal{R}^1_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}$, a direct computation shows that

$$\psi\big(2np, (2m+1)q, kpq + \tfrac{1}{2}pq\big) = \big(2np, (2m+1)q, kpq + \tfrac{1}{2}pq\big)$$

and

$$d\psi(\boldsymbol{e}_1) = \boldsymbol{e}_1, \quad d\psi(\boldsymbol{e}_2) = -\boldsymbol{e}_2, \quad d\psi(\boldsymbol{e}_3) = -\boldsymbol{e}_3.$$

Therefore

$$\psi = \mathcal{R}^1_{(2np,(2m+1)q,kpq+\frac{1}{2}pq)} \quad \text{and} \quad D_{n,m+1} = \mathcal{R}^1_{(2np,(2m+1)q,kpq+\frac{1}{2}pq)}(D_{0,0}).$$

Hence $D_{n,m+1}$ and $D_{n,m}$ are joined smoothly across the common boundary segments

$$\gamma^1_{(2np,(2m+1)q,kpq+\frac{1}{2}pq)}(t), \quad -p \le t \le p.$$

The boundary segment

$$\gamma^1_{(2np,(2m-1)q,kpq+\frac{1}{2}pq)}(t), \quad -p \le t \le p$$

of $D_{n,m}$ is the common boundary segment of $D_{n,m-1}$ and $D_{n,m}$ and they are joined smoothly across these segments by the same argument taking $m-1$ instead of m.

(ii) The boundary segment

$$\gamma^2_{((2n+1)p,2mq,kpq)}(t), \quad -q \le t \le q$$

of $D_{n,m}$ is the common boundary segments of $D_{n+1,m}$ and $D_{n,m}$. Moreover,

$$\begin{aligned} D_{n+1,m} &= \mathcal{R}^3_{((n+1)p,mq,0)}(D_{0,0}) \\ &= \mathcal{R}^3_{((n+1)p,mq,0)}(T_{2kpq}(\mathcal{R}^2_{(0,0,0)}(D_{0,0}))) \\ &= (\mathcal{R}^3_{((n+1)p,mq,0)} \circ T_{2kpq} \circ \mathcal{R}^2_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)})(D_{n,m}). \end{aligned}$$

For the isometry $\psi = \mathcal{R}^3_{((n+1)p,mq,0)} \circ T_{2kpq} \circ \mathcal{R}^2_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}$,

$$\psi((2n+1)p, 2mq, kpq) = ((2n+1)p, 2mq, kpq)$$

and

$$d\psi(\boldsymbol{e}_1) = -\boldsymbol{e}_1, \quad d\psi(\boldsymbol{e}_2) = \boldsymbol{e}_2, \quad d\psi(\boldsymbol{e}_3) = -\boldsymbol{e}_3.$$

Therefore $\psi = \mathcal{R}^2_{((2n+1)p,2mq,kpq)}$ and $D_{n+1,m} = \mathcal{R}^2_{((2n+1)p,2mq,kpq)}(D_{n,m})$. Thus $D_{n+1,m}$ and $D_{n,m}$ are joined smoothly across the common boundary segments

$$\gamma^2_{((2n+1)p,2mq,kpq)}(t), \quad -q \le t \le q.$$

The boundary segment

$$\gamma^2_{((2n-1)p,2mq,kpq)}(t), \quad -q \le t \le q$$

of $D_{n,m}$ is the common boundary segment of $D_{n-1,m}$ and $D_{n,m}$ and they are joined smoothly across these segments by the same argument taking $n-1$ instead of n.

(iii) The boundary segment

$$\gamma^3_{((2n+1)p,(2m+1)q,kpq)}(t), \qquad \begin{cases} 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is even,} \\ -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is odd,} \end{cases}$$

of $D_{n,m}$ is the common boundary segment of $D_{n+1,m+1}$ and $D_{n,m}$. Moreover,

$$\begin{aligned} D_{n+1,m+1} &= \mathcal{R}^3_{((n+1)p,(m+1)q,0)}(D_{0,0}) \\ &= \mathcal{R}^3_{((n+1)p,(m+1)q,0)}(T_{2(n-m)pq}(\mathcal{R}^3_{(0,0,0)}(D_{0,0}))) \\ &= (\mathcal{R}^3_{((n+1)p,(m+1)q,0)} \circ T_{2(n-m)pq} \circ \mathcal{R}^3_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)})(D_{n,m}). \end{aligned}$$

For the isometry $\psi = \mathcal{R}^3_{((n+1)p,(m+1)q,0)} \circ T_{2(n-m)pq} \circ \mathcal{R}^3_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}$,

$$\psi((2n+1)p, (2m+1)q, kpq) = ((2n+1)p, (2m+1)q, kpq)$$

and

$$d\psi(\boldsymbol{e}_1) = -\boldsymbol{e}_1, \quad d\psi(\boldsymbol{e}_2) = -\boldsymbol{e}_2, \quad d\psi(\boldsymbol{e}_3) = \boldsymbol{e}_3.$$

Therefore $\psi = \mathcal{R}^3_{((2n+1)p,(2m+1)q,kpq)}$ and $D_{n+1,m+1} = \mathcal{R}^3_{((2n+1)p,(2m+1)q,kpq)}(D_{n,m})$. This implies that $D_{n+1,m+1}$ and $D_{n,m}$ are joined smoothly across the common boundary segments

$$\gamma^3_{((2n+1)p,(2m+1)q,kpq)}(t), \qquad \begin{cases} 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is even,} \\ -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is odd.} \end{cases}$$

The boundary segment

$$\gamma^3_{((2n-1)p,(2m-1)q,kpq)}(t), \qquad \begin{cases} 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is even,} \\ -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is odd} \end{cases}$$

of $D_{n,m}$ is the common boundary segment of $D_{n-1,m-1}$ and $D_{n,m}$ and they are joined smoothly across these segments by the same argument taking $n-1$, $m-1$ instead of n, m.

(iv) The boundary segment

$$\gamma^3_{((2n+1)p,(2m-1)q,kpq)}(t), \qquad \begin{cases} -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is even,} \\ 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is odd} \end{cases}$$

of $D_{n,m}$ is the common boundary segments of $D_{n+1,m-1}$ and $D_{n,m}$. Moreover,

$$\begin{aligned} D_{n+1,m-1} &= \mathcal{R}^3_{((n+1)p,(m-1)q,0)}(D_{0,0}) \\ &= \mathcal{R}^3_{((n+1)p,(m-1)q,0)}(T_{-2(n+m)pq}(\mathcal{R}^3_{(0,0,0)}(D_{0,0}))) \\ &= (\mathcal{R}^3_{((n+1)p,(m-1)q,0)} \circ T_{-2(n+m)pq} \circ \mathcal{R}^3_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)})(D_{n,m}). \end{aligned}$$

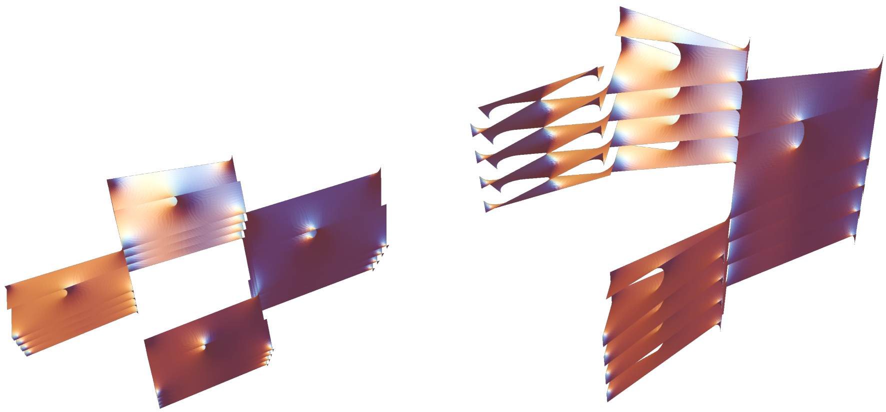

Figure 5. Two views of a part of M.

For the isometry $\psi = \mathcal{R}^3_{((n+1)p,(m-1)q,0)} \circ T_{-2(n+m)pq} \circ \mathcal{R}^3_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}$,

$$\psi((2n+1)p, (2m-1)q, kpq) = ((2n+1)p, (2m-1)q, kpq)$$

and

$$d\psi(\boldsymbol{e}_1) = -\boldsymbol{e}_1, \quad d\psi(\boldsymbol{e}_2) = -\boldsymbol{e}_2, \quad d\psi(\boldsymbol{e}_3) = \boldsymbol{e}_3.$$

Therefore $\psi = \mathcal{R}^3_{((2n+1)p,(2m-1)q,kpq)}$ and $D_{n+1,m-1} = \mathcal{R}^3_{((2n+1)p,(2m-1)q,kpq)}(D_{n,m})$. This implies that $D_{n+1,m-1}$ and $D_{n,m}$ are joined smoothly across the common boundary segments

$$\gamma^3_{((2n+1)p,(2m-1)q,kpq)}(t), \qquad \begin{cases} -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is even,} \\ 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is odd.} \end{cases}$$

The boundary segment

$$\gamma^3_{((2n-1)p,(2m+1)q,kpq)}(t), \qquad \begin{cases} -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is even,} \\ 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is odd} \end{cases}$$

of $D_{n,m}$ is the common boundary segment of $D_{n-1,m+1}$ and $D_{n,m}$ and they are joined smoothly across this segments by the same argument taking $n-1$, $m+1$ instead of n, m.

We have shown that each boundary segments of the minimal surface $D_{n,m}$ are joined smoothly to the neighboring surfaces. For any n, m, above argument shows that $D_{n,m} \cup D_{(n+1),m}$ becomes a smooth minimal surface which contains the vertical geodesic $\gamma^3_{((2n+1)p,(2m\pm1)q,0)}$ in its boundary. Moreover, since

$$D_{n,(m+1)} \cup D_{(n+1),(m+1)} = \mathcal{R}^3_{((2n+1)p,(2m+1)q,0)}(D_{(n+1),m} \cup D_{n,m}),$$

$D_{n,m} \cup D_{(n+1),m} \cup D_{n,(m+1)} \cup D_{(n+1),(m+1)}$ is smooth along the vertical geodesic $\gamma^3_{((2n+1)p,(2m+1)q,0)}$ which includes the corner points of the boundary of $D_{n,m}$.

Since the surfaces $D_{n,m}$ for $n, m \in \mathbb{Z}$ do not intersect with others at interior points, we can conclude that

$$M := \bigcup_{n,m\in\mathbb{Z}} D_{n,m}$$

is a smooth embedded complete minimal surface in Nil_3.

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