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AO SUN

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COMPACTNESS OF CONSTANT MEAN CURVATURE SURFACES IN A THREE-MANIFOLD WITH POSITIVE RICCI CURVATURE

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We prove a compactness theorem for constant mean curvature surfaces with area and genus bound in a three-manifold with positive Ricci curvature. As an application, we give a lower bound of the first eigenvalue of constant mean curvature surfaces in a three-manifold with positive Ricci curvature.

1. Introduction

Let M be a three-dimensional manifold and $\Sigma \subset M$ be a surface. Let H be the mean curvature of Σ. We say Σ is a constant mean curvature (CMC) surface if H is a constant. In particular, if H is constant 0, Σ is a minimal surface. There are many examples of CMC surfaces in $\mathbb{R}^3$; see [Meeks et al. 2016]. Recently, Zhou and Zhu [2019] proved the existence of embedded CMC hypersurfaces in closed $(n+1)$-dimensional manifolds with $2 \leq n \leq 6$.

We will prove the following compactness theorem for embedded CMC surfaces in a three-dimensional manifold with positive Ricci curvature:

Theorem 1.1. *Let M be a three-dimensional compact manifold with positive Ricci curvature and no boundary. Suppose $\Sigma_i \subset M$ is a sequence of closed embedded CMC surfaces with constant mean curvature H_i, satisfying the following conditions*:

(1) $|H_i| \leq H_0$ *for some constant* H_0.

(2) *The genus of* Σ_i *is uniformly bounded.*

(3) *The area of* Σ_i *is uniformly bounded.*

Then either

(1) *there is a self-touching smoothly immersed CMC surface Σ with finitely many neck pinching points, such that a subsequence of Σ_i converges to Σ in C^k topology for any $k \geq 2$ apart from those neck pinching points*, *or*

(2) *there is an embedded minimal surface Σ such that Σ_i converges to Σ with multiplicity* 2.

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Here we say Σ is *self-touching* if at any nonembedded point $p \in \Sigma$, there is a small r such that $B_r(p) \cap \Sigma$ is a union of two disks D_1, D_2, and D_1 can be written as a graph of function ϕ over D_2 where $\phi \geq 0$ on D_2. Intuitively this means that Σ is immersed but cannot cross itself. *Neck pinching* points are special touching points. We will give the precise definition in Section 4. Intuitively one can imagine that we are pinching a piece of a plasticine into two pieces, and just at the moment they are detached, the point connecting them is a neck pinching point.

Compactness theorem. The compactness theorem for minimal surfaces was first developed by Choi and Schoen [1985]. They proved the compactness theorem of embedded minimal surfaces in a three-dimensional manifold with positive Ricci curvature. Later their result was generalized to many other situations. For example, White [1987] generalized the compactness theorem to surfaces which are stationary for parametric elliptic functionals, and Colding and Minicozzi [2012] generalized the compactness theorem to self-shrinkers in $\mathbb{R}^3$. We will follow the key ideas of these papers.

There are two main ingredients in the proof by Choi and Schoen. The first ingredient is a curvature estimate for minimal surfaces. Then we can get uniform curvature bound on Σ_i apart from finitely many points, so we can find a subsequence of Σ_i converging smoothly to a limit surface Σ apart from finitely many points. Here we need to generalize the curvature estimate to CMC surfaces, and get an uniform curvature estimate only depending on the uniform mean curvature bound H_0.

The second ingredient is showing the multiplicity of the convergence is no more than two. In particular, if the multiplicity is one, then by a result of Allard [1972] the convergence is smooth. There are two methods to show the multiplicity is no more than two. Choi and Schoen argued by constructing a family of functions which contradict the eigenvalue estimate in [Choi and Wang 1983]; another method by White and by Colding and Minicozzi argued that if the multiplicity is more than two, then the linearized operator has a positive Jacobi field, which is impossible if M has positive Ricci curvature. We will follow the second method, because we do not have an eigenvalue estimate for CMC surfaces. In our case, a key observation is that although CMC surfaces and minimal surfaces satisfy different equations, their linearized operators are the same. Hence we may conduct the same argument as the minimal surfaces case.

If the limit surface is minimal, then the CMC surfaces may approach it on both sides with different orientation, and the differential operator is not the same as the differential operator of minimal surfaces. Thus, the convergence may not be multiplicity 1. However, if the multiplicity of the convergence is more than 2, then we can still find two sheets with the same orientation, and so the differential operator is again the same as the differential operator of minimal surfaces. Again we can obtain a positive Jacobi field to argue for contradiction.

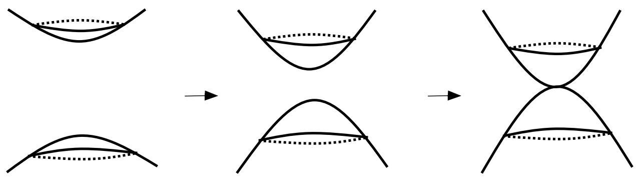

Figure 1. Kissing process.

As an application of the compactness theorem (Theorem 1.1), we obtain a lower bound for the first eigenvalue of CMC surfaces.

Theorem 1.2. *Let M be a three-dimensional manifold with positive Ricci curvature. Suppose there is no embedded minimal surface in M which is the multiplicity 2 limit of a sequence of CMC surfaces. Then for any embedded CMC surface with area bound V, genus bound G and mean curvature bound $|H| \leq H_0$, we have the first eigenvalue lower bound*:

$$\lambda \geq \frac{\min \mathrm{Ric} - HC}{2}, \tag{1-1}$$

where C is a constant depending on M, V, G, H_0.

Touching phenomenon. The touching phenomenon does not occur in minimal surfaces due to the maximum principle, but it may occur in CMC surfaces, especially in a convergence process. The appearance of touching points makes the convergence of CMC surfaces much more complicated.

The touching phenomenon is natural in our physical world. For example, one can observe many soap bubbles touching each other. More complicated examples appear in general three-manifold rather than $\mathbb{R}^3$, and we give some examples in Section 5.

In general touching points in the limit do not influence the smooth convergence in our main theorem (Theorem 1.1) if they are generated when two parts of the surface are kissing each other. One can imagine the convergence is smooth on each piece; see Figure 1.

However, neck pinching points are generated with some topological changes, so smooth convergence cannot cross these points; see Figure 2.

The neck pinching phenomenon is very common in geometric analysis. For example, the neck pinching phenomenon appears in geometric flows, such as mean curvature flow (see [Gang and Sigal 2009]) and Ricci flow (see [Angenent and Knopf 2004]). In order to deal with the nonsmoothness of the flow across these neck pinching points, one needs to do surgery for the geometric flows. For example

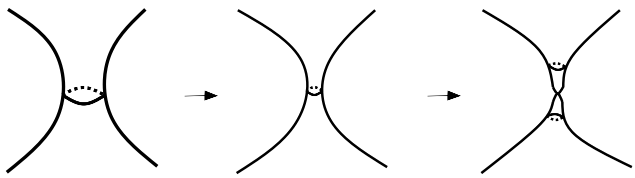

Figure 2. Neck pinching process.

Perelman [2002] studied surgery of Ricci flows, and Brendle and Huisken [2016] studied surgery of mean curvature flows in $\mathbb{R}^3$.

Organization of the paper. In Section 2 we will prove the Choi–Schoen type curvature estimate for CMC surfaces. We will just follow Choi and Schoen's proof. A similar estimate for CMC surfaces in $\mathbb{R}^3$ appears in [Zhang 2005].

In Section 3 we discuss the linearized equation and the linearized operator.

In Section 4 we prove the main compactness theorem. We follow the idea from [White 1987] and [Colding and Minicozzi 2012].

In Section 5 we present some touching examples of CMC surfaces.

In Section 6, as an application of the compactness theorem, we prove a lower bound for the first eigenvalue of CMC surfaces in a three-manifold.

2. Curvature estimate

In this section we generalize the Choi–Schoen curvature estimate for minimal surfaces to CMC surfaces. We first need some tools.

Tools for curvature estimate. The first lemma is a Simon's type inequality for CMC surfaces. We need to keep track of the mean curvature term.

Lemma 2.1. *Let Σ be a CMC surface with mean curvature H, $|H| \le H_0$. Then*

$$\Delta_\Sigma |A|^2 \ge -C(\delta^2 + |A|^2)^2, \tag{2-1}$$

where C is a universal constant, and δ is quadratic under the rescaling of the M, i.e., suppose we rescale the metric g to $\tilde{g} = \sigma g$, then δ becomes $\tilde{\delta} = \sigma^{-1}\delta$.

Proof. See [Ilias et al. 2012, Theorem 3.1]. □

The next lemma generalizes the monotonicity formula for minimal surfaces to CMC surfaces.

Lemma 2.2. *Let M be a closed three-manifold with sectional curvature bounded by k and injective radius bounded from below by i_0. Let $\Sigma \subset M$ be a CMC surface with mean curvature H, $|H| \leq H_0$. Let f be a function on Σ satisfying $\Delta_\Sigma f \geq -\lambda t^{-2} f$, where λ is a fixed constant and $t < \min\{i_0, 1/\sqrt{k}\}$. Then we have*

$$f(x_0) \leq \frac{e^{\lambda + C(H_0,k)t/2}}{\pi} \int_{B_t(x_0)\cap\Sigma} f. \tag{2-2}$$

Before we prove this lemma, let us state a lemma of the famous Laplacian comparison theorem in three-manifold. See [Colding and Minicozzi 2011, Chapter 7, Lemma 7.1] for a proof.

Lemma 2.3. *Suppose that M is a closed three-manifold with sectional curvature bounded by k and injective radius bounded from below by i_0. Let $x \in M$ be a fixed point, and r be the distance function from x. Then for $r < \min\{i_0, 1/\sqrt{k}\}$ and any vector X with $|X| = 1$,*

$$\left|\mathrm{Hess}_r(X, X) - \frac{1}{r}\langle X - \langle X, Dr\rangle Dr, X - \langle X, Dr\rangle Dr\rangle\right| \leq \sqrt{k}. \tag{2-3}$$

Here D is the gradient on M.

Proof of Lemma 2.2. Let $y \in \Sigma$ be a point with $r(y) < \min\{i_0, 1/\sqrt{k}\}$. We choose a local orthonormal frame E_1, E_2. Then by Laplacian comparison (Lemma 2.3), we have

$$\left|\mathrm{Hess}_r(E_1, E_1) - \frac{1}{r}\langle E_1 - \langle E_1, Dr\rangle Dr, E_1 - \langle E_1, Dr\rangle Dr\rangle\right| \leq \sqrt{k}, \tag{2-4}$$

$$\left|\mathrm{Hess}_r(E_2, E_2) - \frac{1}{r}\langle E_2 - \langle E_2, Dr\rangle Dr, E_2 - \langle E_2, Dr\rangle Dr\rangle\right| \leq \sqrt{k}. \tag{2-5}$$

Adding these two inequalities and noting Σ is a CMC surface, we get (compare to the minimal surfaces case in [Colding and Minicozzi 2011, Chapter 7, (7.2)])

$$|\Delta_\Sigma r^2 - 4 - \langle \nabla^\perp r^2, H\boldsymbol{n}\rangle| \leq 4\sqrt{k}r. \tag{2-6}$$

Noting $|Dr| \leq 1$, we get

$$|\Delta_\Sigma r^2 - 4| \leq (4\sqrt{k} + 2H_0)r = \alpha r. \tag{2-7}$$

where C only depends on k, H_0. Let us define

$$F(s) = \frac{1}{s^2}\int_{B_s(x_0)\cap\Sigma} f.$$

We can differentiate it for almost every $s < t$

$$F'(s) = -\frac{2}{s^3}\int_{B_s(x_0)\cap\Sigma} f + \frac{1}{s^2}\int_{\partial B_s(x_0)\cap\Sigma} \frac{f}{|\nabla_\Sigma r|}. \tag{2-8}$$

Here we use the co-area formula; see [Colding and Minicozzi 2011, p. 24, (1.59)]. Let us estimate the first term on the right-hand side. Using inequality (2-7) and integrating by parts gives

$$
\begin{aligned}
(2\text{-}9)\quad -\frac{2}{s^3}\int_{B_s(x_0)\cap\Sigma} f &= -\frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma} 4f \\
&\geq -\frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma} \Delta_\Sigma(r^2-s^2)f - \frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma} \alpha r f \\
&= -\frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma} (r^2-s^2)\Delta_\Sigma f \\
&\qquad - \frac{1}{2s^3}\int_{\partial B_s(x_0)\cap\Sigma} \nabla_\Sigma(r^2) f - \frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma} \alpha r f.
\end{aligned}
$$

So we get the following inequality

$$
\begin{aligned}
(2\text{-}10)\quad F'(s) &\geq \frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}(s^2-r^2)\Delta_\Sigma f + \frac{1}{s^2}\int_{\partial B_s(x_0)\cap\Sigma} \frac{1-|\nabla_\Sigma r|^2}{|\nabla_\Sigma r|} f - \frac{\alpha}{2}F(s) \\
&\geq \frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}(s^2-r^2)\Delta_\Sigma f - \frac{\alpha}{2}F(s) \\
&\geq -\frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}(s^2-r^2)t^{-2}\lambda f - \frac{\alpha}{2}F(s) \\
&\geq -\frac{\lambda}{t}F(s) - \frac{\alpha}{2}F(s).
\end{aligned}
$$

Hence $e^{(\lambda/t+\alpha/2)s}F(s)$ is monotone nondecreasing. Then we can conclude that

$$
(2\text{-}11)\qquad f(x_0) \leq \frac{e^{C+\alpha t/2}}{\pi}\int_{B_t(x_0)\cap\Sigma} f. \qquad \square
$$

Choi–Schoen type estimate.

Theorem 2.4. *Let M be a three-dimensional manifold. Let $p \in M$ and $r > 0$ such that $B_r(p)$ has a compact closure in M. Let Σ be a compact immersed CMC surface with mean curvature H in M such that $B_r(p)\cap\partial\Sigma = \varnothing$. Here $|H| \leq H_0$. Then there exists $\varepsilon_0 > 0$ depending on the geometry of $B_r(p)$ and H_0 such that if*

$$
\int_{\Sigma\cap B_r(p)} |A|^2 \leq \varepsilon_0.
$$

and $r \leq \varepsilon_0$, then

$$
(2\text{-}12)\qquad \max_{0\leq\sigma\leq r}\sigma^2\sup_{B_{r-\sigma}(p)}|A|^2 \leq C = C(H_0, B_r(p)).
$$

Proof. We follow the idea of Choi and Schoen. Since $\sigma^2\sup_{B_{r-\sigma}(p)}|A|^2$ vanishes on ∂B_r, the supremum of $\sigma^2\sup_{B_{r-\sigma}(p)}|A|^2$ must be achieved inside B_r. Let σ_0 be the number such that

$$\sigma_0^2 \sup_{B_{r-\sigma_0}(p)} |A|^2 = \max_{0\le\sigma\le r} \sigma^2 \sup_{B_{r-\sigma}(p)} |A|^2.$$

and let $q \in B_{r-\sigma_0}(p)$ be chosen to satisfy

$$|A|^2(q) = \sup_{B_{r-\sigma_0}(q)} |A|^2.$$

Then

$$\sup_{B_{\frac{1}{2}\sigma_0}(q)} |A|^2 \le 4|A|^2(q). \tag{2-13}$$

If $\sigma_0^2|A|^2(q) \le 4$, then the inequality holds. So we only need to consider the case $\sigma_0^2|A|^2(q) > 4$. Now we rescale the metric ds^2 on M by setting $\tilde{d}s^2 = |A|^2(q)ds^2$, and we denote the balls and the quantities under the rescaled metric with tilde. $\sigma_0^2|A|^2(q) > 4$ implies that $\partial\Sigma \cap \widetilde{B}_1(q) = \varnothing$. Inequality (2-13) implies that

$$\sup_{\widetilde{B}_1(q)} |\widetilde{A}|^2 \le 4.$$

By Simons' inequality (Lemma 2.1),

$$\tilde{\Delta}_\Sigma |\widetilde{A}|^2 \ge -C(\delta^2 + |A|^2)^2.$$

Note here $\delta^2 \le C\sigma_0^2 \le C\varepsilon_0^2$. Together with the inequality $\sup_{\widetilde{B}_1(q)} |\widetilde{A}|^2 \le 4$ we get

$$\tilde{\Delta}_\Sigma u \ge -Cu \text{ on } \widetilde{B}_1(q),$$

where $u = \delta^2 + |\widetilde{A}|^2$ and C is a universal constant. So the monotonicity formula (Lemma 2.2) gives

$$u(x_0) \le \frac{e^{C+\widetilde{\alpha}/2}}{\pi} \int_{\widetilde{B}_1(q)\cap\Sigma} u. \tag{2-14}$$

Noting $\widetilde{\alpha} \le \alpha\sigma_0 \le \alpha\varepsilon_0$, we have

$$|A|^2(q) \le u(q) \le \frac{e^{C+\alpha\varepsilon_0/2}}{\pi} \int_{\widetilde{B}_1(q)\cap\Sigma} (\delta^2 + |A|^2) \le C\varepsilon_0. \tag{2-15}$$

where we use the conformal invariance of the integral of $|A|^2$ and the area bound of CMC surface. If ε_0 is small enough, we will get a contradiction since $|\widetilde{A}|^2(q) = 1$. Thus we finish the proof. □

3. Linearized equation

Let Σ be a CMC surface in M. Let us define a differential operator L such that

$$Lu = \Delta_\Sigma u + \mathrm{Ric}(\boldsymbol{n}, \boldsymbol{n})u + |A|^2 u. \tag{3-1}$$

Here u is a function on Σ. We call L the *linearized operator*. In this section we study some properties of this operator.

Difference of two CMC surfaces. Let M be a three-dimensional manifold. Suppose $\Sigma_1, \Sigma_2 \subset M$ are two CMC surfaces, with mean curvature H_1, H_2 respectively.

Theorem 3.1. *Suppose Σ_2 is a graph over Σ_1, i.e.,*

$$\Sigma_2 = \{x + \varphi \boldsymbol{n} : x \in \Sigma_1\}.$$

Then φ satisfies the second order elliptic equation

$$L\varphi - (H_2 - H_1) = \operatorname{div}(a\nabla\varphi) + b \cdot \nabla\varphi + c\varphi. \tag{3-2}$$

Here a, b, c turns to 0 as $\|\varphi\|_{C^2}$ goes to 0.

This can be viewed as a noninfinitesimal version of the second variational formula. The computations are in the Appendix. Intuitively, one can imagine the second order variational formula gives the second order derivative of minimal surfaces, and the difference formula here gives the Taylor expansion of the minimal surfaces up to the second order. In particular letting $\varphi \to 0$, we will again get the second variational formula.

Stability of linearized operator. Suppose Σ is a CMC surface. We say the linearized operator L of Σ is *stable* if for any function u on Σ,

$$\int_\Sigma uLu \leq 0. \tag{3-3}$$

Otherwise we say L is unstable. Note this definition of stability is not the same as the stability of the CMC surface itself, since when we talk about the stability of a CMC surface Σ, we only consider the variational fields which preserve the (local) volume enclosed by Σ.

For a surface Σ in a three-dimensional manifold M with positive Ricci curvature, letting $u \equiv 1$ we see that

$$\int_\Sigma uLu = \int_\Sigma |A|^2 + \operatorname{Ric}(\boldsymbol{n}, \boldsymbol{n}) > 0.$$

Hence L is always unstable.

Recall a Jacobi field on Σ is a variational field $f\boldsymbol{n}$ such that $Lf = 0$. The following lemma shows that a positive Jacobi field implies the stability of L.

Lemma 3.2. *Suppose there is a positive function u on Σ such that $Lu = 0$. Then L is stable.*

Proof. Let $w = \log u$. Then $\Delta_\Sigma w = -|A|^2 - \operatorname{Ric}(\boldsymbol{n}, \boldsymbol{n}) - |\nabla_\Sigma w|^2$.

Let v be any smooth function on Σ. Multiplying both sides of the above identity by v^2 gives

$$\begin{aligned}\int_\Sigma v^2(|A|^2+\mathrm{Ric}(\boldsymbol{n},\boldsymbol{n}))+\int_\Sigma|\nabla_\Sigma w|^2v^2 \\ =-\int_\Sigma v^2\Delta_\Sigma w=2\int_\Sigma v\langle\nabla_\Sigma v,\nabla_\Sigma w\rangle \\ \le 2\int_\Sigma|v||\nabla_\Sigma w||\nabla_\Sigma v|\le\int_\Sigma v^2|\nabla_\Sigma w|^2+\int_\Sigma|\nabla_\Sigma v|^2.\end{aligned}\tag{3-4}$$

Then integration by parts gives

$$\int_\Sigma vLv\le 0.\tag{3-5}$$

□

4. Compactness theorem

In this section, we will prove the main compactness theorem.

Smooth limit. We first show that there is a reasonable smooth limit under the conditions in Theorem 1.1.

Theorem 4.1. *Let M be a three-dimensional compact manifold with positive Ricci curvature and no boundary. Suppose $\Sigma_i\subset M$ is a sequence of closed embedded CMC surfaces with constant mean curvature H_i, satisfying the following conditions*:

(1) $|H_i|\le H_0$ *for some constant* H_0.

(2) *The genus of Σ_i is uniformly bounded.*

(3) *The area of Σ_i is uniformly bounded.*

Then there is a self-touching smoothly immersed CMC surface Σ such that a subsequence of Σ_i converges to Σ in C^k topology for any $k\ge 2$ apart from a finite singular set $\mathcal{S}$.

Proof. We follow [Choi and Schoen 1985] and [Colding and Minicozzi 2012]. First of all, since the mean curvature, the area and the genus of Σ_i is uniformly bounded, by the Gauss–Bonnet theorem the total curvature of Σ_i is also uniformly bounded by a constant C. For each positive integer m, take a finite covering $\{B_{r_m}(y_j)\}$ of M such that each point of M is covered at most h times by balls in this covering, and $\{B_{r_m/2}(y_j)\}$ is still a covering of M. Here we set $r_m=2^{-m}\varepsilon_0$ and h only depends on M. Then we have

$$\sum_j\int_{\Sigma_i\cap B_{r_m}(y_j)}|A|^2\le hC$$

Therefore for each i there are at most hC/ε_0 number of balls such that

$$\int_{\Sigma_i\cap B_{r_m}(y_j)} |A|^2 \geq \varepsilon_0$$

By passing to a subsequence of Σ_i we can always assume that all the Σ_i have the same balls with total curvature $\geq \varepsilon_0$. Call the center of these balls $\{x_{1,m}, \dots, x_{l,m}\}$, where l is an integer at most hC/ε_0. Then on the balls other than $B_{x_{k,m}(r_m)}$, by Theorem 2.4 we have a uniformly point-wise curvature bound for Σ_i. Passing to a subsequence we may assume that the Σ_i converge smoothly on a half size of those balls to Σ. Since the Σ_i are embedded, the limit Σ is self-touching in the balls other than $B_{x_{k,m}(r_m)}$.

We can continue this process as m increases. Finally by a diagonal argument we can get a subsequence $\{\Sigma_i\}$, converging smoothly everywhere to Σ apart from those points $x_1, \dots, x_l$ which are the limit of those $\{x_{1,m}\}, \dots, \{x_{l,m}\}$. Moreover, since there is no maximum principle for CMC surfaces, the limit is only immersed. However if we consider the compactness for each connected components in any fixed ball, we can see the limit is self-touching away from $x_1, \dots, x_l$. □

Next we will show that Σ is actually smooth everywhere. We will follow White to prove that the singularities are removable. The main ingredient is a more delicate curvature estimate near the singularities.

Lemma 4.2. *Suppose Σ is a properly self-touching CMC surface in $B_R(x_0) \setminus \{x_0\}$ with mean curvature $|H| \leq H_0$. Then there exists $\varepsilon = \varepsilon(H_0, R, x_0) > 0$ such that if $\int_\Sigma |A|^2 \leq \varepsilon$, there exists C such that*

$$|A(x)|(\mathrm{dist}(x, x_0)) \leq C. \tag{4-1}$$

Proof. We show this by contradiction. If the criterion is not true, we can find a sequence of points $x_n \in ((B_R(x_0) \setminus B_{1/n}(x_0)) \cap \Sigma)$ such that

$$|A(x_n)|^2\Big(\mathrm{dist}(x, x_0) - \frac{1}{n}\Big) \to +\infty.$$

Otherwise we will have uniform bound for $|A(x)|^2\big(\mathrm{dist}(x, x_0) - \frac{1}{n}\big)$ for a sequence of $n \to \infty$, then passing to limit we will have a uniform bound for $|A(x)|^2 \,\mathrm{dist}(x, x_0)$.

We can choose $z_n \in ((B_R(x_0)\setminus B_{1/n}(x_0))\cap\Sigma)$ such that $|A(z_n)|^2\big(\mathrm{dist}(z_n, x_0) - \frac{1}{n}\big)$ achieves its maximum. Note that $|A(x)|^2\big(\mathrm{dist}(x, x_0) - \frac{1}{n}\big)$ equals 0 on $\partial B_{1/n}(x_0)\cap\Sigma$, so $d_n := \mathrm{dist}(z_n, x_0) - \frac{1}{n} > 0$.

We rescale $B_{d_n/2}(z_n)$ by $|A(z_n)|$, and denote the set $\{x \in \Sigma : \mathrm{dist}(x, z_n) \leq d_n/2\}$ after rescaling by $\widetilde{\Sigma}_n$. We will use tilde to denote the quantities on this new surface. Moreover, since $|A(z_n)| \to \infty$, the limit of the rescaling of $B_{d_n/2}(z_n)$ will converge to $\mathbb{R}^3$, so we can assume n is sufficiently large such that $\widetilde{\Sigma}_n$ actually lives in $\mathbb{R}^3$ with a metric which is perturbed from the standard Euclidean metric.

$\widetilde{\Sigma}_n$ satisfies the following properties:

(i) $|\widetilde{A}(0)| = 1$.

(ii) Since

$$|A(z_n)|^2 d_n \to +\infty,$$

we know that, for any fixed $R > 0$, $\widetilde{\Sigma}_n \cap \partial B_R(0) \neq \varnothing$ in $\mathbb{R}^3$ if n is large enough, and $\partial \widetilde{\Sigma}_n \cap B_R(0) = \varnothing$ if n is large enough.

(iii) For any $x' = |A(z)|x \in \widetilde{\Sigma}_n$, we have

$$|A(x)|\Big(\text{dist}(x - x_0)\ \text{dist} - \frac{1}{n}\Big) \leq |A(z)| d_n.$$

Since $\text{dist}(x, z) \leq d_n/2$, we have $\text{dist}(x, x_0) - \frac{1}{n} \geq d_n/2$, thus $|A(x)| \leq 2|A(z)|$, $|\widetilde{A}(x')| \leq 2$.

By the uniform curvature bound of $\widetilde{\Sigma}_n$, for each $R > 0$, there exists a subsequence (still denoted by $\widetilde{\Sigma}_n$) converging smoothly on $B_R(0)$ to a complete surface $\widetilde{\Sigma}$. By checking the equation after rescaling, we see that the limit $\widetilde{\Sigma}$ must be a minimal surface, i.e., $\widetilde{H} = 0$.

Since the rescaling does not change the integral of the squared curvature, we have

$$\int_{B_R(0)\cap\widetilde{\Sigma}} |A|^2 \leq \varepsilon.$$

Thus $\widetilde{\Sigma}$ has to be a plane if ε is small enough; see [White 1987, p. 249]. This is a contradiction to the condition that $|\widetilde{A}(0)| = 1$. □

Theorem 4.3. *The limit surface in Theorem 4.1 is smoothly immersed. Moreover, for $y \in \mathcal{S}$ a nonembedded point, in a small neighborhood of y, Σ is a union of two disks which are touching at y.*

Proof. We only need to prove that Σ is smooth around the singular set $\mathcal{S}$. Suppose $y \in \mathcal{S}$ is a singularity. We may assume r small enough such that $\int_{B_r(y)\cap\Sigma} |A|^2 \leq \varepsilon$ (Note $\Sigma\backslash\mathcal{S}$ has finite total curvature since the Σ_i's have uniformly bounded total curvature).

By Lemma 4.2, there is a constant C such that, for any $x \in B_r(y) \cap \Sigma$,

$$|A(x)|\ \text{dist}(x, y) \leq C.$$

Now we choose a sequence $r_i \to 0$ and rescale $B_r(y)$ and Σ_i by $1/r_i$ and denote it by $\widetilde{\Sigma}_i$. Note the curvature bound

$$|A(x)|\ \text{dist}(x, y) \leq C$$

is invariant under rescaling, so this uniform curvature bound indicates that $\widetilde{\Sigma}_i$ smoothly converges to a complete surface $\widetilde{\Sigma}$ in $\mathbb{R}^3 \setminus \{0\}$; see [White 1987].

Now for K a compact subset of $\mathbb{R}^3 \setminus \{0\}$,

$$\int_{\widetilde{\Sigma}_i \cap K} |A|^2 = \int_{\Sigma_i \cap r_i K} |A|^2 \to 0 \quad \text{as } r_i \to 0.$$

This implies that $\widetilde{\Sigma}$ is a union of planes. Thus $\Sigma \cap B_r(0)$ is actually a union of disks and punctured disks.

Now let Σ denote one of its connected components which is a punctured disk. Since $\widetilde{\Sigma}_i$ converges to the plane in $\mathbb{R}^3 \setminus \{0\}$, we can assume for some i that $\widetilde{\Sigma}_i$ can be written as a graph φ_i of that plane. Without loss of generality, let the plane be the xy plane in $\mathbb{R}^3$. By the computations in the Appendix, in B_1, φ_i satisfies an elliptic equation over the tangent plane:

$$L\varphi_i - (H_2 - H_1) = \operatorname{div}(a\nabla\varphi_i) + b \cdot \nabla\varphi_i + c\varphi_i. \tag{4-2}$$

Here all terms are defined on $\mathbb{R}^2 \cap B_1(0)$. Again, when i is large, each term on the right-hand side goes to 0. Then by the implicit function theorem, if we fixed the normal direction to point upwards, we can solve $\varphi_{i,t}$ for boundary data

$$\varphi_{i,t} = \varphi_i + t$$

on $\partial B_1(0)$. Then the graphs of $\varphi_{i,t}$ foliate a region of $(B_1(0) \cap \mathbb{R}^2) \times \mathbb{R}$. Since we fixed the direction of normal vectors, we can apply the maximal principle, which indicates that the leaf such that $\varphi_{i,t}(0) = 0$ lies on one side of $\widetilde{\Sigma}_i$. As a result, any sequence of dilations of Σ must converge to the same limit plane, which is just the tangent plane of that leaf at 0.

Thus $\Sigma \cup \{0\}$ is a C^1 graph of a function v in a neighborhood of 0. Since v is a $C^{2,\alpha}$ solution to an elliptic equation except 0, v is actually $C^{2,\alpha}$ everywhere. Hence $\Sigma \cup \{0\}$ is a smooth disk.

We have already shown that $\Sigma \cup \{0\}$ is a union of smooth disks. So Σ is an immersed surface, with locally finitely many curvature concentration points. By the maximal principle, at each touching point Σ consists of two disks which are touching at that point. So Σ is a smoothly self-touching immersed surface. □

Smooth convergence. In this subsection we first assume that the limit surface is not minimal, and discuss the situation when the limit surface is minimal at the end.

We will show the convergence is smooth apart from neck pinching points. Note we have already shown the convergence is smooth apart from $\mathcal{S}$, so we only need to show smooth convergence across points in $\mathcal{S}$ with density 1 (i.e., locally Σ is one disk) and points in $\mathcal{S}$ which are not neck pinching.

We first define neck pinching points. From Theorem 4.3 we know that for any points $y \in \mathcal{S}$ with density more than one, locally Σ is the union of two disks D_1, D_2, and Σ_i can be written as graphs G_i^1, G_i^2 of functions φ_i^1, φ_i^2 over $D_1 \setminus \{y\}$ and $D_2 \setminus \{y\}$ respectively.

Definition 4.4. We say y is a *neck pinching* point if there exists $r_0 > 0$ such that for $0 < r < r_0$, G_i^1 and G_i^2 do not lie in the same connected components of $\Sigma_i \cap B_r$ for at most finitely many Σ_i's.

Now we prove the convergence is smooth apart from these neck pinching points. The main ingredient is to show the convergence has multiplicity one. Then by the regularity theorem of Allard [1972] (see also [Choi and Schoen 1985] and [Colding and Minicozzi 2012]), we can show the convergence is smooth across those singularities with density 1. Finally we show that even for a singularity with density greater than 1, if it is not a neck pinching point we can still argue that the convergence is smooth across it.

Theorem 4.5. *The multiplicity of the convergence in Theorem 4.1 is one when the limit surface is not minimal.*

We follow the idea in [Colding and Minicozzi 2012]. The key ingredient is to show that if the convergence has multiplicity greater than 1, there exists a positive Jacobi field on Σ, which is a contradiction.

Proof. We argue as in [Choi and Schoen 1985] that we only need to consider the case that M is simply connected, and self-touching Σ is two sided (note although in [Choi and Schoen 1985] this argument is for closed embedded surfaces, it can be adapted to self-touching surfaces). If the convergence has multiplicity more than 1, then Σ_i's can be decomposed into several sheets of graphs on $\Sigma \setminus \mathcal{S}$. Since Σ is two-sided, we can label the graphs by height, and let the highest sheet of Σ_i be written as the graph of w_i^+, let the lowest sheet of Σ_i be written as the graph of w_i^-, and let $w_i = w_i^+ - w_i^-$. Fix a point p not in $\mathcal{S}$, and let $u(x) = w(x)/w(p)$. Then $u(p) = 1$ and $u > 0$ on $\Sigma \setminus \mathcal{S}$. Moreover, although w_i^- and w_i^+ do not satisfy a linear elliptic equation, their difference does. Hence u_i satisfies a linear elliptic equation. Then Harnack inequality implies C^α bound for u_i's and then standard elliptic theory gives $C^{2,\alpha}$ bound. Then by the Arezela–Ascoli theorem, a subsequence (still denoted by u_i) converges uniformly in C^2 on a compact subset of $\Sigma \setminus \mathcal{S}$ to a nonnegative function u on $\Sigma \setminus \mathcal{S}$ such that

$$Lu = 0, \quad u(p) = 1. \tag{4-3}$$

Next we show u can be extended smoothly across $\mathcal{S}$ to a solution of $Lu = 0$. Again we follow the idea in [White 1987] and [Colding and Minicozzi 2012]. We only need to show u is bounded around each singularity y, then by the standard elliptic theory u extends smoothly. Suppose u_i satisfies the linearized equation

$$L(u_i) = \operatorname{div}(a_i \cdot \nabla u_i) + b_i \cdot \nabla u_i + c_i u_i.$$

Then choose exponential normal coordinates over $B_\varepsilon(y) \subset \Sigma$ and a cylinder N over $B_\varepsilon(y) \cap \Sigma$; when ε is small, the implicit function theorem gives a foliation of graphs v_t over $B_\varepsilon(y)\cap\Sigma$ in N so that

$$v_0(x)=0 \text{ for all } x\in B_\varepsilon(y) \quad\text{and}\quad v_t(x)=t \text{ for all } x\in\partial B_\varepsilon(y).$$

By the Harnack inequality, $t/C_i\le v_t\le C_i t$ for some $C_i>0$. Since the right-hand side of the linearized equation turns to 0 as $i\to\infty$, C_i actually has uniform bound. Then by the maximum principle, u_i is bounded on $B_\varepsilon(y)$ by a multiple of its supremum on $B_\varepsilon(y)\setminus B_{\varepsilon/2}(y)$. Hence u has a removable of singularity at p.

So there exists a nonnegative solution u of the linearized operator $Lu=0$. By $u(p)=1$, Harnack inequality implies that u is positive everywhere. Then by Lemma 3.2, Σ is stable. However, plugging in a test function constant 1 implies that no immersed CMC surface in positive Ricci three-manifold can be stable, which is a contradiction. Then we conclude that the convergence has multiplicity one. □

By [Allard 1972], this theorem implies smooth convergence across those density 1 points. It remains to show smooth convergence across those touching singularities which are not neck pinching singularities.

Theorem 4.6. *The convergence is smooth apart from those neck pinching singularities.*

Proof. Let $y\in\mathcal{S}$ be a singularity with density greater than 1; then by Theorem 4.3 locally Σ is the union of two disks D_1, D_2. Then by the definition of pinching points, we know that if y is not a pinching point, locally $\Sigma_i=G_i^1\cup G_i^2$ is the union of two graphs over $D_1\setminus\{y\}$, $D_2\setminus\{y\}$ respectively. Thus we only need to apply previous analysis to each graph G_i^j to get smooth convergence across y. □

Finally we discuss the situation that the limit Σ is an embedded minimal surface. Now multiplicity 2 convergence may happen because the CMC surfaces can converge to Σ from both sides with different orientation. However, if the convergence is of multiplicity larger than 2, there are at least two graphs that have the same orientation. Repeating the argument for these graphs, we again get a positive Jacobi field, which is a contradiction. Therefore, the convergence has at most multiplicity 2.

Combining all the ingredients in this section we conclude the main theorem (Theorem 1.1).

5. Touching examples

In this section we give some examples of touching points of CMC surfaces in three-dimensional manifolds.

Example 5.1 (kissing itself). Let us consider a sphere $\mathbb{S}_R$ with radius R in $\mathbb{R}^3$. By quotient a $\mathbb{Z}^3$ action of $\mathbb{R}^3$, we get a torus $\mathbb{T}^3$, and the image of $\mathbb{S}_R$ in $\mathbb{T}^3$ is an embedded CMC surface when R sufficiently small. Now we increase the radius of $\mathbb{S}_R$. Then for some specific R_0, in $\mathbb{T}^3$, $\mathbb{S}_{R_0}$ will kiss itself thus form a touching point. This is not a neck pinching point.

The touching set may be very large. For example, we can consider a cylinder $\mathcal{C}_R$ with radius R in $\mathbb{R}^3$. Using the same construction, we can see for some radius R_0, $\mathcal{C}_{R_0}$ kisses itself at a straight line, which is a one-dimensional curve.

Example 5.2 (unduloid neck pinching). An unduloid is a one periodic CMC surface in $\mathbb{R}^3$. See [Hadzhilazova et al. 2007] for a detailed discussion of unduloids.

The unduloid has two parameters a, c to determine its shape; see [Hadzhilazova et al. 2007, Theorem 3.1]. When $a \to 0, c \to 1/H$, we can see the family of unduloids will smoothly converge to the union of spheres apart from the touching points of spheres. This is an example of neck pinching singularity. One can see that the smooth convergence cannot cross these neck pinching points because the topology changes in the limit.

Of course, we can quotient $\mathbb{R}^3$ by some $\mathbb{Z}^3$ actions to make this example be an example in a closed three-manifold.

The reader may notice that these examples do not lie in a Ricci positive three-manifold. It is not known whether the touching behavior of CMC surfaces in positive Ricci three-manifolds is simpler or not. So we suggest the following conjectures:

Conjecture 5.3. A self-touching CMC surface in a three-dimensional manifold with positive Ricci curvature cannot carry infinitely many touching points.

Conjecture 5.4. For a CMC surface in a three-dimensional manifold with a one-dimensional touching set, the touching set must be a geodesic of the ambient space.

Another interesting observation is that a touching point of a self-touching CMC surface can be generated by both kissing and neck pinching process. For example, in $\mathbb{T}^3$, a sphere kissing itself can be generated by both the first example and the second example above. So a very natural question is whether any touching can be generated by both process? Some observations suggest the answer is probably “no”:

Example 5.5. Consider two spheres in $\mathbb{R}^3$ kissing at a single point p. Aleksandrov [1958] proved that any embedded CMC surface in $\mathbb{R}^3$ must be a standard sphere. They cannot be the limit of a sequence of embedded CMC surfaces; hence p cannot be a neck pinching point of a sequence of embedded CMC surfaces.

We suggest the following conjecture.

Conjecture 5.6. Suppose M is a compact three-manifold with positive Ricci curvature. Assume that $S_1 \cup S_2$ is the union of two embedded CMC spheres in M kissing at p. Then p cannot be a neck pinching point.

6. Eigenvalue estimate of CMC surfaces with small $|H|$

In this section we discuss an application of our main theorem. We will give a lower bound of the first eigenvalue of CMC surfaces in a positive Ricci three-manifold with small $|H|$.

The main idea is a method by Choi and Wang [1983]. They used an identity by Reilly to estimate the first eigenvalue of minimal surface in three-manifold. The main issue for generalizing their method to CMC surfaces is that we may not be able to control the term involving mean curvature (in the minimal surface case, this term vanishes). So we need a more delicate estimate for each term in Reilly's identity.

We first recall the proof by Choi and Wang [1983]. They used a formula by Reilly. For u a smooth function defined on a bounded domain Ω we have

$$\int_\Omega \big(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - (\Delta u)^2\big) = \int_{\partial\Omega} \big(A((\nabla u)^\top, (\nabla u)^\top) - 2u_n \Delta_{\partial\Omega} u + H u_n^2\big), \tag{6-1}$$

where u_n is the normal derivative and H is the mean curvature on $\partial\Omega$. Then they applied this formula when $\partial\Omega$ is minimal, where u is the harmonic function solving

$$\Delta_\Omega u = 0 \quad \text{and} \quad u|_{\partial\Omega} = f,$$

where f an eigenfunction of the first eigenvalue on $\partial\Omega$ such that $\int_{\partial\Omega} f^2 = 1$. Then they could get a first eigenvalue estimate for $\partial\Omega$, i.e., the minimal surface, in a simply connected three-dimensional manifold with positive Ricci curvature. Later, Choi and Schoen [1985] used a covering argument to extend the estimate to all closed three-manifolds with positive Ricci curvature.

Let us naively follow their method to deal with CMC surfaces. Suppose $\partial\Omega$ is a CMC surface with constant mean curvature H and the first eigenvalue of $\partial\Omega$ is λ. We will get the following inequality (see [Colding and Minicozzi 2011, p. 244]):

$$2\lambda \int_\Omega |\nabla u|^2 \geq (\min \mathrm{Ric}) \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla^2 u|^2 - \int_{\partial\Omega} A((\nabla u)^\top, (\nabla u)^\top) - H \int_{\partial\Omega} u_n^2. \tag{6-2}$$

Since A changes sign if we replace Ω by its complement, we can always assume

$$\int_{\partial\Omega} A((\nabla u)^\top, (\nabla u)^\top)$$

is nonnegative and get

$$2\lambda \int_\Omega |\nabla u|^2 \geq (\min \mathrm{Ric}) \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla^2 u|^2 - H \int_{\partial\Omega} u_n^2. \tag{6-3}$$

So our goal is to control $\int_\Omega |\nabla^2 u|^2 - H\int_{\partial\Omega} u_n^2$.

Trace theorem. In this subsection, we will transform the problem of controlling $\int_\Omega |\nabla^2 u|^2 - H\int_{\partial\Omega} u_n^2$ to the problem of getting an uniform tubular neighborhood of CMC surfaces. We need a trace theorem in three-manifolds. The idea of the proof is based on the proof in [Evans 2010].

Theorem 6.1. *Let $\Sigma = \partial\Omega$ be an embedded surface in a three manifold M. Suppose there is a constant δ such that $\exp_x(t\boldsymbol{n}) : \Sigma \times [-\delta, \delta] \to M$ is a diffeomorphism from $\Sigma \times [-\delta, \delta]$ to its image, and there is a constant A_0 such that $|A| \le A_0$ on Σ. Then there is a constant C only depending on M, A_0 and δ such that*

$$\int_{\partial\Omega} (u_n)^2 \le C \int_\Omega (|\nabla u|^2 + |\nabla^2 u|^2). \tag{6-4}$$

Proof. Note $(u_n)^2 \le |\nabla u|^2$, so we only need to prove a standard trace theorem

$$\int_{\partial\Omega} f^2 \le C \int_\Omega (f^2 + |\nabla f|^2).$$

By the conditions, we can pull back the metric of M to $\Sigma \times [-\delta, \delta]$, and by the uniform curvature bound, the pull back metric is uniformly closed to the standard production metric. In particular, we only need to prove the trace theorem on $\Sigma \times [-\delta, \delta]$ with product metric. Let us choose a cut-off function ζ such that $\zeta = 1$ on $\Sigma \times [-\delta/2, \delta/2]$ and is supported on $\Sigma \times [-\delta, \delta]$. Moreover we may assume that its gradient is bounded by C/δ for some constant C. Then

$$\begin{aligned}\int_{\partial\Omega} f^2\,dx' = \int_\Sigma f^2\zeta\,dx' &= -\int_{\Sigma\times[-\delta,0]} (f^2\zeta)_{x_n}\,dx \\ &= -\int_{\Sigma\times[-\delta,0]} |f|^2\zeta_{x_n} + 2ff_{x_n}\zeta\,dx \\ &\le C\int_{\Sigma\times[-\delta,0]} |f|^2 + |\nabla f|^2\,dx.\end{aligned} \tag{6-5}$$

Here x_n is the normal direction (i.e., the direction on $[-\delta, \delta]$), dx' is the measure on Σ, and dx is the measure of the production metric. In the last inequality we use Young's inequality. Translating this back to M gives the desired trace theorem. □

If H is sufficiently close to 0, we can apply this trace theorem in the inequality (6-3) to get the eigenvalue lower bound

$$\lambda \ge \frac{\min \mathrm{Ric} - HC}{2}, \tag{6-6}$$

where C is a constant depending on M, A_0, δ.

***Uniform bound for CMC surfaces with H close to* 0.** It remains to prove the pointwise curvature bound and the existence of δ in Theorem 6.1. We will use the compactness theorem to get these bounds for CMC surfaces with H small.

Theorem 6.2. *Suppose there is no embedded minimal surface in M which is the multiplicity 2 limit of a sequence of CMC surfaces. There exists $H_0 > 0$ such that an embedded CMC surface Σ with mean curvature $|H| \le H_0$, area less than V and genus less than G has curvature $|A| \le C(H_0, V, G)$*

Proof. We argue by contradiction. Suppose such H_0 does not exist. Then we can find a family of CMC surfaces Σ_i, with mean curvature $H_i \to 0$ such that a point p_i on Σ_i has curvature $|A(p_i)| \to \infty$ as $i \to \infty$. By compactness of M we may assume $p_i \to p$ for a point $p \in M$. Now, by the main theorem (Theorem 1.1), Σ_i converges to a minimal surface Σ. Since Σ is minimal, by the maximum principle there is no touching point. So the convergence is everywhere smooth. However $|A(p)|$ is finite since Σ is a smoothly embedded surface, which is a contradiction. Thus H_0 exists. □

Theorem 6.3. *Suppose there is no embedded minimal surface in M which is the multiplicity 2 limit of a sequence of CMC surfaces. There exists $H_0 > 0$ and $\delta_0 > 0$ such that for an embedded CMC surface Σ with mean curvature $|H| \le H_0$, area less than V and genus less than G, $\exp_x(t\boldsymbol{n}) : \Sigma \times [-\delta, \delta] \to M$ is a diffeomorphism.*

Proof. We argue by contradiction. Suppose such H_0, δ_0 does not exist. Then we can find a family of CMC surfaces Σ_i, with mean curvature $H_i \to 0$ and $\delta_i \to 0$ such that there is a point $p \in M$ such that $p = \exp_{x_i^j}(t_i^j \boldsymbol{n})$, $j = 1, 2$ for $x_i^j \in \Sigma_i$ and $t_i^j \in [-\delta_i, \delta_i]$. Since we have already obtained an uniform curvature bound for Σ_i, we know $\mathrm{dist}_\Sigma(x_i^1, x_i^2) \ge d$ for some constant d when i is large enough.

Again, a subsequence of Σ_i smoothly converges to a smooth embedded minimal surface Σ. By passing to a subsequence we can find two points $x^1, x^2 \in \Sigma$, with intrinsic distance $\mathrm{dist}_\Sigma(x^1, x^2) \ge d$ but extrinsic distance $\mathrm{dist}_M(x^1, x^2) = 0$. This is a contradiction by the maximum principle of minimal surfaces. □

Combining all the ingredients in this section we get the following lower bound for the first eigenvalue of CMC surfaces:

Theorem 6.4 (Theorem 1.2). *Let M be a three-manifold with positive Ricci curvature. Suppose there is no embedded minimal surface in M which is the multiplicity 2 limit of a sequence of CMC surfaces. Then for any embedded CMC surface with area bound V, genus bound G and mean curvature bound $|H| \le H_0$, we have the first eigenvalue lower bound*

$$\lambda \ge \frac{\min \mathrm{Ric} - HC}{2}, \tag{6-7}$$

where C is a constant depending on M, V, G, H_0.

Remark 6.5. An interesting question is: can we directly get the first eigenvalue lower bound for CMC surfaces? If we can, then we can prove the compactness theorem for CMC surfaces without area bound.

Appendix: Difference of two surfaces in three manifold

Here we will present some computations of the difference of two surfaces in a three-manifold. These kinds of computation have already appeared in [Kapouleas 1990] and [Colding and Minicozzi 2011] in three-dimensional Euclidean space.

Let Σ_1, Σ_2 be two surfaces in three-manifold M, and let H_1, H_2 be their mean curvatures respectively. Moreover, we assume Σ_2 can be viewed as a graph over Σ_1, i.e.,

$$\Sigma_2 = \{\exp_x(\varphi \boldsymbol{n}) : x \in \Sigma_1\},$$

where φ is a C^2 function on Σ_1.

Theorem A.1. *Suppose $\|\varphi\|_{C^2}$ is small enough, then φ satisfies a second order elliptic equation.*

Proof. Since this assertion is a local assertion, we only need to check this in a small neighborhood U of $p \in \Sigma_1$. Let us choose the Fermi coordinate x_1, x_2, x_3 in U (so we can view U as an open subset of $\mathbb{R}^3$ with non-Euclidean metric), such that

$$\Sigma_1 = \{(x_1, x_2, x_3) : x_3 = 0\}.$$

Moreover, the metric g under this coordinate satisfies $g_{i3} = 0$, $i = 1, 2$, and $(0, 0, 1)$ is the unit normal vector at each point in Σ_1. We will use $\partial_1, \partial_2, \partial_3$ to denote the vector fields defined on M with respect to the differential under this coordinate.

Σ_2 is a graph,

$$\Sigma_2 = \{(x_1, x_2, x_3) : x_3 = \varphi(x_1, x_2)\}.$$

From now on we will use tilde over quantity to denote the quantity of Σ_2. We use x_1, x_2 to parametrize Σ_2, then we have

$$\tilde{\partial}_i = \tilde{\partial}_{x_i} = \partial_i + \varphi_i \partial_3, \ i = 1, 2. \tag{A-1}$$

Then the metric on Σ_2 satisfies

$$\tilde{g}_{ij} = g_{ij} + g_{i3}\varphi_j + g_{j3}\varphi_i + g_{33}\varphi_i\varphi_j. \tag{A-2}$$

Now we compute the unit normal vector fields on Σ_2. We observe that Σ_2 can be viewed as the 0-level set of the function $\varphi(x_1, x_2) - x_3$. So we can find a normal vector field $\boldsymbol{m}$ on Σ_2:

$$\boldsymbol{m} = -\nabla^M(\varphi(x_1, x_2) - x_3) = g^{ij}(\varphi_k \delta_i^k - \delta_{3i})\partial_j. \tag{A-3}$$

Note

$$\langle \boldsymbol{m}, \boldsymbol{m} \rangle = g^{ij}(\varphi_k \delta_i^k - \delta_{3i}) g^{pq}(\varphi_k \delta_p^k - \delta_{3p}) g_{qj}$$
$$= g^{ij}\varphi_i\varphi_j - 2g^{3k}\varphi_k + g^{33}.$$

So

$$\boldsymbol{n} = -(g^{pq}\varphi_p\varphi_q - 2g^{3l}\varphi_l + g^{33})^{-1/2} g^{ij}(\varphi_k\delta_i^k - \delta_{3i})\partial_j. \tag{A-4}$$

Now let us calculate the mean curvatures. From now on we will slightly abuse the notation when we use Einstein notation. When we use i, j in the summation we will assume they are in $\{1, 2\}$. On Σ_1, $\boldsymbol{n} = \partial_3$, so we have

$$H_1 = g^{ij}\langle \nabla_{\partial_i}\partial_j, \partial_3\rangle = g^{ij}\Gamma_{ij}^k g_{k3}. \tag{A-5}$$

On Σ_2, first we note the covariant derivative is

$$\begin{aligned} \nabla_{\tilde{\partial}_i}\tilde{\partial}_j &= \nabla_{\partial_i + \varphi_i\partial_3}(\partial_j + \varphi_j\partial_3) \\ &= \nabla_{\partial_i}\partial_j + \varphi_i\nabla_{\partial_3}\partial_j + \varphi_{ij}\partial_3 + \varphi_i\varphi_j\nabla_{\partial_3}\partial_3 \\ &= \nabla_{\partial_i}\partial_j + \varphi_i\nabla_{\partial_3}\partial_j + \varphi_{ij}\partial_3. \end{aligned} \tag{A-6}$$

Here we note that ∂_3 is the direction of the geodesic starting from Σ_1; hence $\nabla_{\partial_3}\partial_3 = 0$. Then the mean curvature of Σ_2 is

$$\begin{aligned} H_2 &= \widetilde{g^{ij}}\langle \nabla_{\tilde{\partial}_i}\tilde{\partial}_j, \boldsymbol{n}\rangle \\ &= \widetilde{g^{ij}}\big\langle \nabla_{\partial_i}\partial_j + \varphi_i\nabla_{\partial_3}\partial_j + \varphi_{ij}\partial_3, \\ &\qquad -(g^{pq}\varphi_p\varphi_q - 2g^{3l}\varphi_l + g^{33})^{-1/2} g^{rs}(\varphi_k\delta_r^k - \delta_{3r})\partial_s\big\rangle \\ &= -\widetilde{g^{ij}}(g^{pq}\varphi_p\varphi_q - 2g^{3l}\varphi_l + g^{33})^{-1/2} \\ &\qquad \times g^{rs}(\varphi_k\delta_r^k - \delta_{3r})(\Gamma_{ij}^m g_{ms} + \varphi_i\Gamma_{3j}^m g_{ms} + \varphi_{ij}g_{3s}) \end{aligned} \tag{A-7}$$

In conclusion, H_2 is a function of φ, $\nabla\varphi$, $\nabla^2\varphi$ and the coordinate in ambient manifold. We define a function $H(x_1, x_2, x_3, v_i, w_{ij})$ where $H(x, y, \varphi, \nabla\varphi, \nabla^2\varphi) = H_2$ as above, where (x_1, x_2, x_3) is the local coordinate. Also note $H(x, y, 0, 0, 0, 0) = H_1$. Then we have

$$\begin{aligned} H_2 - H_1 &= \varphi\int_0^1 \frac{\partial H(x_1, x_2, t\varphi, t\nabla\varphi, t\nabla^2\varphi^2)}{\partial x_3}\,dt \\ &\quad + \varphi_i\int_0^1 \frac{\partial H(x_1, x_2, t\varphi, t\nabla\varphi, t\nabla^2\varphi^2)}{\partial v_i}\,dt \\ &\quad + \varphi_{jk}\int_0^1 \frac{\partial H(x_1, x_2, t\varphi, t\nabla\varphi, t\nabla^2\varphi^2)}{\partial w_{jk}}\,dt. \end{aligned} \tag{A-8}$$

Let the coefficients of φ, $\nabla\varphi$, $\nabla^2\varphi$ on the right-hand side of the above identity be a function depending on φ, $\nabla\varphi$, $\nabla^2\varphi$. Then letting $\|\varphi\|_{C^2}$ go to 0 we can see the right-hand side terms will just be Lu by the second variational formula. Thus we have

$$Lu-(H_2-H_1)=\operatorname{div}(a\nabla\varphi)+b\cdot\nabla\varphi+c\varphi, \tag{A-9}$$

where a, b, c turns to 0 as $\|\varphi\|_{C^2}$ goes to 0. □

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AO SUN
DEPARTMENT OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MA
UNITED STATES
aosun@mit.edu

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