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SINGULAR PERIODIC SOLUTIONS TO A CRITICAL EQUATION IN THE HEISENBERG GROUP

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We construct positive solutions to the equation

$$-\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}}$$

on the Heisenberg group, singular in the origin, similar to the Fowler solutions of the Yamabe equations on $\mathbb{R}^n$. These satisfy the homogeneity property $u \circ \delta_T = T^{-(Q-2)/2} u$ for some T large enough, where $Q = 2n+2$ and δ_T is the natural dilation in $\mathbb{H}^n$. We use the Lyapunov–Schmidt method applied to a family of approximate solutions built by periodization from the global regular solution classified by Jerison and Lee (1988).

1. Introduction

Let $\mathbb{H}^n$ be the Heisenberg group with its standard pseudohermitian structure. The problem of studying constant *Webster curvature* pseudohermitian structures conformal to the standard one, in the spirit of the classical Riemannian case, is equivalent to finding the positive solutions of the equation

$$-\Delta_{\mathbb{H}^n} u = u^{\frac{Q+2}{Q-2}}, \tag{1}$$

where $\Delta_{\mathbb{H}^n}$ is the sublaplacian and $Q = 2n+2$ is the homogeneous dimension (in Section 2 we will recall the preliminary definitions about the Heisenberg group).

The positive solutions of (1) satisfying some integrability hypotheses were classified by Jerison and Lee [1988], and they correspond to conformal factors that transform the standard pseudohermitian structure of $\mathbb{H}^n$ into the push-forward of the pseudohermitian structure of the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with respect to the Cayley transform, up to translations and dilations. This classification plays an important role in the solution of the *CR Yamabe problem*, see [Jerison and Lee 1987; Gamara and Yacoub 2001; Gamara 2001; Cheng et al. 2017].

In the Euclidean space the analogous equation,

$$-\Delta_{\mathbb{H}^n} u = u^{\frac{n+2}{n-2}}, \tag{2}$$

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is well studied, being related to the Yamabe problem, and being analytically interesting due to a lack of compactness.

The Yamabe equation in $\mathbb{R}^n$ also arises when looking for extremals of the critical Sobolev–Gagliardo–Nirenberg inequality. These were classified as “bubble functions” independently by Aubin [1976] and Talenti [1976].

A complete classification for solutions of (2) (without integrability hypotheses) was given by Caffarelli, Gidas and Spruck [Caffarelli et al. 1989]. In this case also the solutions on $\mathbb{R}^n \setminus \{0\}$ were classified. In addition to the regular ones on the whole space, there is a singular solution corresponding geometrically to the cylindrical metric, and a family of *singular solutions*, the Fowler solutions, which correspond to a family of periodic metrics on the cylinder which are isometric to the Delaunay surfaces (see [Mazzeo and Pacard 1999]).

This terminology is in analogy with the structure for axially symmetric constant mean curvature surfaces: in this case Delaunay surfaces bridge the sphere and the cylinder (see [Mazzeo and Pacard 2001]). Furthermore, the Fowler solutions have been used as building blocks (see for example [Mazzeo and Pacard 1999]) for the construction of more general solutions (as well as for the constant mean curvature Delaunay surfaces).

The above classification has been used to study the asymptotic profiles of general singular solutions (see [Korevaar et al. 1999]), and solutions with singular behavior as with Fowler’s ones arise in the study of blow-up limits for the prescribed scalar curvature problem (see [Li 1995; 1996; Chen and Lin 1998]).

The aim of this article is to prove, in analogy with the Euclidean case, the existence of a family of solutions to (1) satisfying a periodicity condition with respect to dilations, that is such that $u \circ \delta_T = T^{\alpha} u$ for some T, α (see Section 2 for the notation). A simple computation shows that it must necessarily hold that $\alpha = -(Q-2)/2$. The main result of the paper is the following.

Theorem 1.1. *There exists T_0 such that for $T \geq T_0$ there exists a positive solution of Equation (1) on $\mathbb{H}^n \setminus \{0\}$ such that*

$$u \circ \delta_T = T^{-\frac{Q-2}{2}} u,$$

and T is the smallest period.

On the Euclidean space the proof of the uniqueness of such solutions relies on a result (in [Caffarelli et al. 1989]), proved by the moving planes method, stating that the positive solutions of (1) are radially symmetric. In this way the construction of solutions and their classification is carried out by a standard ODE analysis. This cannot be done on $\mathbb{H}^n$. We point out that on the Heisenberg group one cannot expect a symmetric solution, because the sublaplacian is not rotationally invariant.

We also point out the recent results in [Guidi et al. 2019], where solutions with singularities at higher-dimensional sets were constructed with different methods.

Theorem 1.1 is proved by writing Equation (1) as the variational equation of the functional

$$\mathscr{J}_T(u) = \int_{\Omega_T} \left(|\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{2^*} |u|^{2^*} \right)$$

on a space of functions satisfying $u \circ \delta_T = T^{-(Q-2)/2} u$ (the integral is with respect to the Haar measure, see Section 2).

In Section 3 we will find an estimate of the Sobolev constant for periodic solution through a Hardy–Littlewood–Sobolev-type theorem for Lorentz spaces. This will be used to carry out the estimates in the subsequent sections.

In Section 4 we will build a family $\mathscr{Z}_T$ of approximate critical points of $\mathscr{J}_T$ by gluing a sequence of suitable dilations of the global regular solution ω_λ. We will show that these solutions are "almost critical" points, in the sense that on $\mathscr{Z}_T$ the differential of the functional $\mathscr{J}_T$ is small.

In Section 5 we will prove that a nondegeneracy condition holding for ω_λ can be carried on Ψ_λ.

In the final section we will prove the existence of the desired solutions through the Lyapunov–Schmidt method, reducing the problem to the orthogonal of the tangent of the curve $\mathscr{Z}_T$, and therein applying the contraction theorem.

We believe that the construction should give perspectives for the study of more general singular solutions in the Heisenberg group, in the spirit of the cited results on the Euclidean space.

2. Preliminaries and notation

In this section we recall some basic definitions and facts on the Heisenberg group, widely present in the literature. See, for example, Chapter 10 of [Chen and Shaw 2001].

Let us consider the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, with the convention on the product

$$(z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\,\mathfrak{Im}(z_1 \cdot \overline{z_2})).$$

Let

$$X_i = T_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = T_{n+i} = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad T = T_0 = \frac{\partial}{\partial t}$$

be the standard basis of left invariant vector fields,

$$\nabla_{\mathbb{H}^n} u = \sum_i X_i(u) X_i + Y_i(u) Y_i$$

the subriemannian gradient,

$$\operatorname{div}\left(\sum_i f_i X_i + g_i Y_i\right) = \sum_i X_i(f_i) + Y_i(g_i)$$

the divergence (which coincides with the divergence with respect to a Haar volume form), and

$$\Delta_{\mathbb{H}^n} = \operatorname{div} \circ \nabla_{\mathbb{H}^n} = \sum_i X_i^2 + Y_i^2$$

the sublaplacian. There exists a constant $C = C(n)$ such that

$$K(x) = \frac{C}{|x|^{Q-2}} \tag{3}$$

is a fundamental solution of the sublaplacian. Let

$$S^1(\mathbb{H}^n) = \{u \in L^2(\mathbb{H}^n) \mid X_i u, Y_i u \in L^2(\mathbb{H}^n)\}.$$

We endow $\mathbb{H}^n$ with the set of dilations

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t)$$

and with the homogeneous norm

$$|(z, t)| = (|z|^4 + t^2)^{1/4}.$$

The Lebesgue measure dx is a biinvariant Haar measure on $\mathbb{H}^n$ satisfying

$$(\delta_\lambda)_\# \, dx = \lambda^Q \, dx;$$

this is essentially the reason why Q takes the place of the topological dimension n in many analytic questions.

Let us set $B_r = \{|x| < r\}$ and $\Omega_T = \overline{B}_R \setminus B_1$. We define the Hilbert space

$$X_T = \{u \in S^1_{\mathrm{loc}}(\mathbb{H}^n) \mid u \circ \delta_T = T^{-(Q-2)/2} u\}$$

with the product

$$\langle u, v\rangle = \int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v.$$

Let $\widetilde{X}_T$ the closed subspace of X_T of the functions of the form $u(|z|, t)$.

It is known that the positive solutions of (1) are

$$\omega_\lambda = \lambda^{(2-Q)/2} \omega \circ \delta_{\lambda^{-1}}$$

and the translates thereof, where

$$\omega(z, t) = c_0 \frac{1}{\big(t^2 + (1 + |z|^2)^2\big)^{(Q-2)/4}}.$$

The problem is variational: the solutions in $S^1(\mathbb{H}^n)$ of the equation are the critical points of the functional

$$\mathscr{J}(u) = \int_{\mathbb{H}^n} \Big(|\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{2^*}|u|^{2^*} \Big).$$

Analogously the solutions of the equation in X_T are the critical points of the functional

$$\mathscr{J}_T(u) = \int_{\Omega_T} \Big(|\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{2^*}|u|^{2^*} \Big).$$

It holds that

$$d\mathscr{J}_T(u)[\varphi] = \int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \varphi - u|u|^{2^*-2}\varphi$$

and that

$$d^2\mathscr{J}_T(u)[\varphi, \psi] = \int_{\Omega_T} \nabla_{\mathbb{H}^n} \varphi \cdot \nabla_{\mathbb{H}^n} \psi - (2^* - 1)|u|^{2^*-2}\varphi\psi.$$

We call $\mathscr{J}_T''$ the operator associated with this bilinear form in the natural way:

$$\langle \mathscr{J}_T''(u)[\varphi], \psi \rangle = d^2\mathscr{J}_T(u)[\varphi, \psi].$$

Let us notice that, if $u \in X_T$ and $E \subseteq \mathbb{H}^n$ then

$$\int_{\delta_r(E)} |u|^{\frac{2Q}{Q-2}} = \int_E |r^{(Q-2)/2} u \circ \delta_r|^{\frac{2Q}{Q-2}} \tag{4}$$

and

$$\int_{\delta_r(E)} |\nabla_{\mathbb{H}^n} u|^2 = \int_E |r^{(Q-2)/2} \nabla_{\mathbb{H}^n}(u \circ \delta_r)|^2. \tag{5}$$

In particular, if $1 \le r \le T$ then

$$\begin{aligned} \int_{\delta_r \Omega_T} |u|^{\frac{2Q}{Q-2}} &= \int_{\Omega_T \setminus \Omega_r} |u|^{\frac{2Q}{Q-2}} + \int_{\Omega_{rT} \setminus \Omega_T} |u|^{\frac{2Q}{Q-2}} \\ &= \int_{\Omega_T \setminus \Omega_r} |u|^{\frac{2Q}{Q-2}} + \int_{\Omega_r} |T^{(Q-2)/2} u \circ \delta_T|^{\frac{2Q}{Q-2}} = \int_{\Omega_T} |u|^{\frac{2Q}{Q-2}}, \end{aligned} \tag{6}$$

and by induction and inversion one can extend this formula to every value of r. Analogously

$$\int_{\delta_r \Omega_T} |\nabla_{\mathbb{H}^n} u|^2 = \int_{\Omega_T} |\nabla_{\mathbb{H}^n} u|^2, \tag{7}$$

and by polarization

$$\int_{\delta_r \Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v = \int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v. \tag{8}$$

The following lemma shows that in integration by parts in X_T boundary terms are null.

Lemma 2.1. *If $u, v \in X_T$ then*

$$\int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v = -\int_{\Omega_T} \Delta_{\mathbb{H}^n} u \cdot v.$$

Proof. Let us write $v = v_1 + v_2$ with $u_1, u_2 \in X_T$, $\operatorname{supp} u_1 \cap \Omega_T \subset B_{7(T+1)/8} \setminus B_{(T+1)/8}$ and $\operatorname{supp} u_2 \cap \delta_{(T+1)/2}\Omega_T \subset B_{(T+1)^2/8} \setminus B_{3(T+1)/4}$ (this can be carried out through a partition of unity).

Then, using formula (8),

$$\begin{aligned}\int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v &= \int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v_1 + \int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v_2 \\ &= -\int_{\Omega_T} \Delta_{\mathbb{H}^n} u \cdot v_1 + \int_{\delta_{(T+2)/2}\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} v_2 \\ &= -\int_{\Omega_T} \Delta_{\mathbb{H}^n} u \cdot v_1 - \int_{\delta_{(T+2)/2}\Omega_T} \Delta_{\mathbb{H}^n} u \cdot v_2 \\ &= -\int_{\Omega_T} \Delta_{\mathbb{H}^n} u \cdot v_1 - \int_{\Omega_T} \Delta_{\mathbb{H}^n} u \cdot v_2 \\ &= -\int_{\Omega_T} \Delta_{\mathbb{H}^n} u \cdot v. \end{aligned}$$ □

We will need to restrict ourselves to solutions in $\widetilde{X}_T$. In order to do this, we observe that, under the identification $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$, the functional $\mathscr{I}_T$ is invariant by the group of transformations of the form $(z, t) \mapsto (Az, t)$ with $A \in O(\mathbb{R}^{2n}) \cap \mathrm{Sp}(\mathbb{R}^{2n})$. In fact it is known that if $A = (a_{ij}) \in \mathrm{Sp}(\mathbb{R}^{2n})$ then this transformation is a group automorphism of $\mathbb{H}^n$ (see [Folland 1989], Chapter 1, Section 2), and so it maps the fields T_i into the fields $\sum_j a_{ij} T_j$. So, using the fact that $A \in O(\mathbb{R}^{2n})$, it is easy to verify that $\mathscr{I}_T$ is invariant by this group.

Furthermore, under the canonical identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$,

$$O(\mathbb{R}^{2n}) \cap \mathrm{Sp}(\mathbb{R}^{2n}) = U(\mathbb{C}^n)$$

[Folland 1989, Proposition 4.6].

Since $U(n)$ acts transitively on the unit sphere of $\mathbb{C}^n$, $\widetilde{X}_T$ is the set of the functions in X_T invariant under the transformations of this form, and so, by the Palais criticality principle [1979], the critical points of the restriction of $\mathscr{I}_T$ to it are critical points in all of X_T.

In the sequel we will need also a particular vector field that plays an important role in $\mathbb{H}^n$ (and more in general in homogeneous groups), the generator of the dilations. It is characterized by the equation

$$\frac{d}{d\lambda}\Big|_{\lambda=1}(u\circ\delta_\lambda)=Zu$$

for every $u\in\mathscr{C}^1(\mathbb{H}^n)$. An explicit expression for it is

$$Z=\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}+y_i\frac{\partial}{\partial y_i}+2t\frac{\partial}{\partial t}.$$

It is easy to verify that

$$\lambda\frac{d}{d\lambda}(u\circ\delta_\lambda)=Z(u\circ\delta_\lambda)=(Zu)\circ\delta_\lambda. \tag{9}$$

Using this formula, it is easy to prove that a function u is homogeneous of degree α if and only if $Zu=\alpha u$ (an extension to $\mathbb{H}^n$ of Euler's theorem).

Furthermore it holds that $[X_i,Z]=X_i$ and $[Y_i,Z]=Y_i$, and so $[\nabla_{\mathbb{H}^n},Z]=\nabla_{\mathbb{H}^n}$.

Lorentz spaces. In Section 3, to overcome the nonintegrability of the functions in X_T in the whole space, will need to use the Lorentz spaces, which we recall briefly.

Given a σ-finite measure space (X,μ) and $1\le p<\infty,\ 1\le q\le\infty$, the Lorentz quasinorm is defined as

$$\|u\|_{L^{p,q}(X)}=p^{1/q}\left\|\lambda\mu\{|u|>\lambda\}^{1/p}\right\|_{L^q(dt/t)}.$$

Furthermore one defines $\|u\|_{L^{\infty,\infty}(X)}=\|u\|_{L^\infty(X)}$. The Lorentz space $L^{p,q}(X)$ is the set of functions such that this quantity is finite. When $p=q$, $\|u\|_{L^{p,p}}=\|u\|_{L^p}$, while when $q=\infty$, $L^{p,\infty}$ coincides with weak L^p.

We will need the following generalization of the Young inequality, which sometimes is referred to in the literature as the Young–O'Neil inequality. It can be deduced applying Theorem 2.6 in [O'Neil 1963] (with the corrections in [Yap 1969]) and Theorem 1.2.12, Remark 1.2.11 in [Grafakos 2008].

Theorem 2.2. *If* $1<p,p_1,p_2<\infty,\ 1\le q,q_1,q_2\le\infty$ *are such that*

$$\frac{1}{p_1}+\frac{1}{p_2}=1+\frac{1}{p}\quad\text{and}\quad\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$$

then there exists C such that for every $f\in L^{p_1,q_1}(\mathbb{H}^n)$, $g\in L^{p_2,q_2}(\mathbb{H}^n)$ it holds that

$$\|fg\|_{L^{p,q}(\mathbb{H}^n)}\le C\|f\|_{L^{p_1,q_1}(\mathbb{H}^n)}\|g\|_{L^{p_2,q_2}(\mathbb{H}^n)}.$$

Basic definitions on CR geometry. For convenience of the reader, we recall the basic definitions about CR manifolds, also if we will not use them. The reader can find more on the topic in [Boggess 1991; Dragomir and Tomassini 2006].

A CR manifold is a real smooth manifold M endowed with a subbundle $\mathscr{H}$ of the complexified tangent bundle of M, $T^{\mathbb{C}}M$, such that $\mathscr{H}\cap\overline{\mathscr{H}}=\{0\}$ and $[\mathscr{H},\mathscr{H}]\subseteq\mathscr{H}$.

We will assume M to be of hypersurface type, that is that $\dim M = 2n+1$ and that $\dim \mathscr{H} = n$. There exists a nonzero real differential form θ that is zero on $\mathfrak{Re}(\mathscr{H} \oplus \overline{\mathscr{H}})$; it is unique up to scalar multiple by a function. Such a form is called pseudohermitian structure. On a pseudohermitian manifold, the Levi form is defined as the 2-form on $\mathscr{H}$ $L_\theta(V, W) = -id\theta(V, \overline{W}) = id\theta([V, \overline{W}])$. A CR manifold is said to be pseudoconvex if it admits a positive definite Levi form (this implies every Levi form to be definite).

The Heisenberg group is the simplest pseudoconvex CR manifold, if endowed with the bundle $\mathscr{H} = \operatorname{span}(Z_1, \dots, Z_n)$ with $Z_j = \frac{1}{2}(X_j - iY_j)$.

On a nondegenerate pseudohermitian manifold one can define a connection, the Tanaka–Webster connection. This allows to define curvature operators in an analogous manner as in Riemannian geometry: the pseudohermitian curvature tensor is the curvature of the Tanaka–Webster connection, the Ricci tensor is

$$\operatorname{Ric}(X, Y) = \operatorname{trace}(Z \mapsto R(Z, X)Y),$$

and the Webster scalar curvature is the trace of the Ricci tensor with respect to the Levi form.

Being a pseudohermitian structure defined only up to a conformal factor on a CR manifold, in CR geometry the Yamabe problem is even more natural than in Riemannian geometry. If $\tilde{\theta} = u^{2/n}\theta$, the transformation law of the Webster curvature is

$$\widetilde{W} = u^{-1-2/n}\Big(\frac{2n+2}{n}\Delta_b u + Wu\Big),$$

where Δ_b is the sublaplacian, which can be defined in a similar way as the Heisenberg group. So the Yamabe problem takes to the equation

$$\frac{2n+2}{n}\Delta_b u + Wu = \lambda u^{1+2/n}.$$

Since the Heisenberg group has zero Webster curvature, and since the pseudohermitian sublaplacian coincides with the sublaplacian defined formerly, the Yamabe problem, up to an inessential constant, is equivalent to finding a positive solution to Equation (1).

The solution of this case plays in the solution in the general case the same role that the solution on $\mathbb{R}^n$ plays in the solution of the general Riemannian case.

3. Estimate of the Sobolev constant on X_T

In order to carry out the estimates in the next sections, we will need an explicit bound on the Sobolev constant on X_T. We will achieve this relating the L^p norm on Ω_T and the $L^{p,\infty}$ norm on the whole space.

Proposition 3.1. *If f is an L^p_{loc} function on $\mathbb{H}^n \setminus \{0\}$ such that $f \circ \delta_T = T^{-\alpha} f$ and $\alpha p = Q$ then*

$$\Big(\frac{T^Q - 1}{T^Q}\Big)^{1/p} \|u\|_{L^{p,\infty}(\mathbb{H}^n)} \leq_2 \|u\|_{L^p(\Omega_T)} \leq Q^{1/p} (\log T)^{1/p} \|u\|_{L^{p,\infty}(\mathbb{H}^n)}.$$

Proof. Let us call $f(\lambda) = \mu\{x \in \Omega_T \mid u(x) > \lambda\}$ and $g(t) = \mu\{x \in \mathbb{H}^n \mid u(x) > \lambda\}$. Then it holds that

$$g(\lambda) = \sum_{k\in\mathbb{Z}} T^{Qk} f(\lambda T^\alpha).$$

Therefore for every $\lambda > 0$

$$\begin{aligned}
\|u\|^p_{L^p(\Omega_T)} &= p\int_0^\infty \xi^{p-1} f(\xi)\, d\xi = p \sum_{k\in\mathbb{Z}} \int_{\lambda T^{\alpha(k-1)}}^{\lambda T^{\alpha k}} \xi^{p-1} f(\xi)\, d\xi \\
&\geq p \sum_{k\in\mathbb{Z}} f(\lambda T^{\alpha k}) \int_{\lambda T^{\alpha(k-1)}}^{\lambda T^{\alpha k}} \xi^{p-1}\, d\xi = \frac{T^Q - 1}{T^Q} \lambda^p \sum_{k\in\mathbb{Z}} T^{Qk} f(\lambda T^{\alpha k}) \\
&= \frac{T^Q - 1}{T^Q} \lambda^p g(\lambda).
\end{aligned}$$

Taking the supremum with respect to λ we get the first inequality.

For the other one, let us pick an integer $N > 0$ and write

$$\begin{aligned}
\|u\|^p_{L^p(\Omega_T)} &= p\int_0^\infty \xi^{p-1} f(\xi)\, d\xi = p \sum_{k\in\mathbb{Z}} \int_{T^{\alpha k/N}}^{T^{\alpha(k+1)/N}} \xi^{p-1} f(\xi)\, d\xi \\
&\leq p \sum_{k\in\mathbb{Z}} f(T^{\alpha k/N}) \int_{T^{\alpha k/N}}^{T^{\alpha(k+1)/N}} \xi^{p-1}\, d\xi \\
&= \sum_{m=1}^N \sum_{j\in\mathbb{Z}} (T^{\alpha p/N} - 1) T^{\alpha p j} T^{\alpha p m/N} f(T^{\alpha j} T^{\alpha m/N}) \\
&= (T^{Q/N} - 1) \sum_{m=1}^N T^{Qm/N} \sum_{j\in\mathbb{Z}} T^{Qj} f(T^{\alpha j} T^{\alpha m/N}) \\
&= (T^{Q/N} - 1) \sum_{m=1}^N T^{Qm/N} g(T^{\alpha m/N}) \leq N(T^{Q/N} - 1) \|u\|^p_{L^{p,\infty}}.
\end{aligned}$$

Taking the limit for $N \to \infty$ we get the second inequality. □

Using Theorem 2.2 we can prove a Sobolev type inequality for weak L^p spaces.

Proposition 3.2. *There exists a constant C such that every function $u \in L^{2,\infty}(\mathbb{H}^n)$ with $\nabla u \in L^{2,\infty}(\mathbb{H}^n)$ verifies*

$$\|u\|_{L^{2Q/(Q-2),\infty}} \leq C \|\nabla u\|_{L^{2,\infty}}.$$

Proof. Let $E = \{u > 1\}$, $E^c = \mathbb{H}^n \setminus E$, $u_1 = u\chi_{E^c} + \chi_E$ and $u_2 = (u-1)\chi_E$, so that $u = u_1 + u_2$. It is standard to prove that u_1 and u_2 have weak subriemannian gradient and that

$$\nabla_{\mathbb{H}} u_1 = (\nabla_{\mathbb{H}} u)\chi_{E^c}, \quad \nabla_{\mathbb{H}} u_2 = (\nabla_{\mathbb{H}} u)\chi_E$$

(the proof is the same as on $\mathbb{R}^n$). It is easy to prove that $u_1 \in S^p(\mathbb{H}^n)$ for $p > 2$ and that $u_2 \in S^q(\mathbb{H}^n)$ for $q < 2$. If $\varphi \in \mathscr{C}_c^\infty(\mathbb{H}^n)$ it holds that

$$\begin{aligned} \varphi(x) &= (\varphi * \delta)(x) = (\varphi * (-\Delta_{\mathbb{H}^n} K))(x) \\ &= \int_{\mathbb{H}^n} (\nabla_{\mathbb{H}^n}\varphi)(xy^{-1}) * (\nabla_{\mathbb{H}^n} K)(y)\, dy := (\nabla_{\mathbb{H}^n}\varphi * \nabla_{\mathbb{H}^n} K)(x) \end{aligned} \tag{10}$$

Formula (3) implies that $\nabla_{\mathbb{H}^n} K \in L^{Q/(Q-1),\infty}$, and so, by Theorem 2.2, the operator $f \mapsto f * \nabla_{\mathbb{H}^n} K$ is bounded from L^p and L^q to some other Lebesgue spaces. Therefore, using the density of $\mathscr{C}_c^\infty$ in $S^p(\mathbb{H}^n)$ for $1 \le p < \infty$, formula (10) holds almost everywhere for functions in these spaces, and so it holds for u_1 and u_2. By summing one obtains that

$$u = \nabla_{\mathbb{H}^n} u * \nabla_{\mathbb{H}^n} K.$$

The thesis follows applying Theorem 2.2 once more. □

We point out that in the proof of the last proposition the splitting of u in two pieces belonging to some L^p space was necessary because $\mathscr{C}_c^\infty$ functions are not dense in the weak L^p spaces.

Combining Propositions 3.1 and 3.2 we get the following Sobolev theorem for X_T spaces with an explicit constant.

Proposition 3.3. *There exist a constant C independent of T such that for every $u \in X_T$*

$$\|u\|_{L^{2Q/(Q-2)}(\Omega_T)} \le C(\log T)^{\frac{Q-2}{2Q}} \left(\frac{T^Q}{T^Q-1}\right)^{1/2} \|u\|_{X_T}.$$

4. Construction of a family of approximate solutions

In order to apply a perturbative method, we find a family of approximate stationary points of $\mathscr{J}_T$ for T big enough.

The family is the following:

$$\Psi_{\lambda,T} = \sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k} = \sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}k} \omega_\lambda \circ \delta_{T^k}$$

(we will hide the dependence on T when convenient). The series converges uniformly on compact sets, because, if $x \in K$,

$$\begin{aligned}\Psi_\lambda(x) &= \sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}k}\omega_\lambda \circ \delta_{T^k} \\ &\le C_{\lambda,K}\sum_{k\ge 0} T^{\frac{Q-2}{2}k}\frac{1}{T^{k(Q-2)}} + C_{\lambda,K}\sum_{k<0} T^{\frac{Q-2}{2}k} \le C_{\lambda,K}.\end{aligned}$$

The subriemannian gradient satisfies

$$\begin{aligned}|\nabla_{\mathbb{H}^n}\Psi_\lambda(x)| &\le \sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}k}T^k|\nabla_{\mathbb{H}^n}\omega_\lambda|\circ\delta_{T^k} \\ &\le C_{\lambda,K}\sum_{k\ge 0} T^{\frac{Q}{2}k}\frac{1}{T^{k(Q-1)}} + C_{\lambda,K}\sum_{k<0} T^{\frac{Q}{2}k} \le C_{\lambda,K}\end{aligned}$$

and so it converges uniformly on compact sets. The same holds for higher order subriemannian derivatives. $\Psi_\lambda \in X_T$ because

$$\Psi_\lambda\circ\delta_T = \sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}k}\omega_\lambda\circ\delta_{T^k}\circ\delta_T = T^{-\frac{Q-2}{2}}\sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}k}\omega_\lambda\circ\delta_{T^k} = T^{-\frac{Q-2}{2}}\Psi_\lambda.$$

It holds that

$$\begin{aligned}\Psi_{T\lambda} &= \sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}k}\omega_{T\lambda}\circ\delta_{T^k} = \sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}k}\frac{1}{(T\lambda)^{(Q-2)/2}}\omega\circ\delta_{1/T\lambda}\circ\delta_{T^k} \\ &= \sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}(k-1)}\frac{1}{\lambda^{(Q-2)/2}}\omega\circ\delta_{1/\lambda}\circ\delta_{T^{k-1}} = \sum_{k\in\mathbb{Z}} T^{\frac{Q-2}{2}(k-1)}\omega_\lambda\circ\delta_{T^{k-1}} = \Psi_\lambda.\end{aligned}$$

Therefore the set $\mathscr{Z}_T = \{\Psi_\lambda \mid \lambda \in (0,\infty)\}$ is a closed curve in X_T.

Moreover, using formula (9), it can be computed that

$$\begin{aligned}\frac{\partial\Psi_\lambda}{\partial\lambda} &= \frac{\partial}{\partial\lambda}\sum_{k\in\mathbb{Z}}\omega_{\lambda/T^k} = \sum_{k\in\mathbb{Z}}\frac{\partial}{\partial\lambda}\big(\lambda^{-(Q-2)/2}\omega_{1/T^k}\circ\delta_{\lambda^{-1}}\big) \\ &= \sum_{k\in\mathbb{Z}}\left(-\frac{Q-2}{2}\frac{1}{\lambda}\omega_{\lambda/T^k} - \lambda^{-\frac{Q-2}{2}}\frac{1}{\lambda^2}\lambda Z(\omega_{1/T^k}\circ\delta_{\lambda^{-1}})\right) \\ &= \sum_{k\in\mathbb{Z}}\left(-\frac{Q-2}{2}\frac{1}{\lambda}\omega_{\lambda/T^k} - \frac{1}{\lambda}Z(\omega_{\lambda/T^k})\right) \\ &= -\frac{Q-2}{2}\frac{1}{\lambda}\Psi_\lambda - \frac{1}{\lambda}Z(\Psi_\lambda).\end{aligned} \tag{11}$$

This implies that the curve $\mathscr{Z}_T$ is immersed for T big enough, because if $\frac{\partial\Psi_\lambda}{\partial\lambda}$ was zero then $Z(\Psi_\lambda) = -\frac{Q-2}{2}\Psi_\lambda$ would be zero, and by the aforementioned Euler's theorem Ψ_λ would be homogeneous of degree $-\frac{Q-2}{2}$; but it is clearly not by construction if T is big enough.

We want to prove the following proposition.

Proposition 4.1. *For every ε there exists T_0, depending only on n, such that if $T \geq T_0$ then $\|\nabla_{\mathbb{H}^n} \mathscr{J}_T\| < \varepsilon$ on $\mathscr{Z}_T$.*

We divide the proof in several lemmas.

First we compute the differential of $\mathscr{J}_T$ in Ψ_λ:

$$
\begin{aligned}
(12)\quad d\mathscr{J}_T(\Psi_\lambda)[u] &= \int_{\Omega_T} \nabla_{\mathbb{H}^n}\Psi_\lambda \cdot \nabla_{\mathbb{H}^n} u - \Psi_\lambda^{2^*-1} u \\
&= \int_{\Omega_T} \sum_{k\in\mathbb{Z}} \nabla_{\mathbb{H}^n}\omega_{\lambda/T^k} \cdot \nabla_{\mathbb{H}^n} u - \Big(\sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k}\Big)^{2^*-1} u \\
&= \sum_{k\in\mathbb{Z}} \Big(\int_{\Omega_T} \nabla_{\mathbb{H}^n}\omega_{\lambda/T^k} \cdot \nabla_{\mathbb{H}^n} u - \omega_{\lambda/T^k}^{\frac{Q+2}{Q-2}} u \Big) \\
&\qquad - \int_{\Omega_T} \Big[\Big(\sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k}\Big)^{\frac{Q+2}{Q-2}} - \sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k}^{\frac{Q+2}{Q-2}} \Big] u \\
&:= A + B.
\end{aligned}
$$

Lemma 4.2. *In the above notation, $A = 0$.*

Proof. We have

$$
\begin{aligned}
A &= \sum_{k\in\mathbb{Z}} \int_{\Omega_T} T^{\frac{Q-2}{2}k} (\nabla_{\mathbb{H}^n}\omega_\lambda)\circ\delta_{T^k} \cdot \nabla_{\mathbb{H}^n} u - (T^{(Q-2)/2k})^{\frac{Q+2}{Q-2}} (\omega_\lambda\circ\delta_{T^k})^{\frac{Q+2}{Q-2}} u \\
&= \sum_{k\in\mathbb{Z}} \int_{\Omega_T} T^{\frac{Q}{2}k} (\nabla_{\mathbb{H}^n}\omega_\lambda)\circ\delta_{T^k} \cdot \nabla_{\mathbb{H}^n} u - T^{\frac{Q+2}{2}k} (\omega_\lambda\circ\delta_{T^k})^{\frac{Q+2}{Q-2}} u \\
&= \sum_{k\in\mathbb{Z}} \int_{\delta_{T^k}(\Omega_T)} T^{-kQ} \Big[T^{\frac{Q}{2}k} \nabla_{\mathbb{H}^n}\omega_\lambda \cdot (\nabla_{\mathbb{H}^n} u)\circ\delta_{T^{-k}} - T^{\frac{Q+2}{2}k} \omega_\lambda^{\frac{Q+2}{Q-2}} u\circ\delta_{T^{-k}} \Big] \\
&= \sum_{k\in\mathbb{Z}} \int_{\delta_{T^k}(\Omega_T)} T^{-\frac{Q}{2}k} T^k \nabla_{\mathbb{H}^n}\omega_\lambda \cdot \nabla_{\mathbb{H}^n}(u\circ\delta_{T^{-k}}) - \omega_\lambda^{\frac{Q+2}{Q-2}} u \\
&= \sum_{k\in\mathbb{Z}} \int_{\delta_{T^k}(\Omega_T)} \nabla_{\mathbb{H}^n}\omega_\lambda \cdot \nabla_{\mathbb{H}^n} u - \omega_\lambda^{\frac{Q+2}{Q-2}} u = \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}\omega_\lambda \cdot \nabla_{\mathbb{H}^n} u - \omega_\lambda^{\frac{Q+2}{Q-2}} u
\end{aligned}
$$

Let us pick a family of smooth functions $\varphi_{\varepsilon,R}$ such that $\varphi_{\varepsilon,R} \equiv 1$ on $B_R \setminus B_{2\varepsilon}$, $\varphi_{\varepsilon,R} \equiv 0$ on B_ε and $\mathbb{H}^n \setminus B_{R+1}$, $|\nabla_{\mathbb{H}^n}\varphi_{\varepsilon,R}| \leq \frac{C}{\varepsilon}$ on $B_{2\varepsilon} \setminus B_\varepsilon$ and $|\nabla_{\mathbb{H}^n}\varphi_{\varepsilon,R}| \leq C$ on

$B_{R+1} \setminus B_R$. Then

$$\begin{aligned}
A &= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\mathbb{H}^n} \big(\nabla_{\mathbb{H}^n}\omega_\lambda \cdot \nabla_{\mathbb{H}^n} u - \omega_\lambda^{(Q+2)/(Q-2)} u\big)\varphi_{\varepsilon,R} \\
&= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\mathbb{H}^n} -\big(\Delta_{\mathbb{H}^n}\omega_\lambda + \omega_\lambda^{(Q+2)/(Q-2)}\big) u\varphi_{\varepsilon,R} - u\nabla_{\mathbb{H}^n}\omega_\lambda \cdot \nabla_{\mathbb{H}^n}\varphi_{\varepsilon,R} \\
&= -\lim_{R\to\infty} \int_{B_{R+1}\setminus B_R} u\nabla_{\mathbb{H}^n}\omega_\lambda \cdot \nabla_{\mathbb{H}^n}\varphi_{\varepsilon,R} - \lim_{\varepsilon\to 0} \int_{B_{2\varepsilon}\setminus B_\varepsilon} u\nabla_{\mathbb{H}^n}\omega_\lambda \cdot \nabla_{\mathbb{H}^n}\varphi_{\varepsilon,R}.
\end{aligned}$$

If $x \to \infty$ then $\nabla_{\mathbb{H}^n}\omega_\lambda \lesssim 1/|x|^{Q-1}$ and $u \lesssim |x|^{-(Q-2)/2}$, and so the first limit is zero. If $x \to 0$ then $\nabla_{\mathbb{H}^n}\omega_\lambda \lesssim 1$ and $u \lesssim |x|^{-(Q-2)/2}$, and so also the second limit is zero. Therefore $A = 0$. $\square$

Now we have to estimate the term B from formula (12).

Lemma 4.3. *In the above notation*

$$|B| \le C(T)\|u\|_{X_T},$$

where $C(T)$ tends to zero uniformly in λ as T tends to infinity.

Proof.

$$\begin{aligned}
|B| &\le \int_{\Omega_T} \bigg[\Big(\sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k}\Big)^{\frac{Q+2}{Q-2}} - \sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k}^{\frac{Q+2}{Q-2}}\bigg]|u| \\
&\le \int_{\Omega_T} \bigg[\Big(\sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k}\Big)^{\frac{Q+2}{Q-2}} - \omega_\lambda^{\frac{Q+2}{Q-2}}\bigg]|u| \\
&\le \bigg\{\int_{\Omega_T} \bigg[\Big(\sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k}\Big)^{\frac{Q+2}{Q-2}} - \omega_\lambda^{\frac{Q+2}{Q-2}}\bigg]^{\frac{2Q}{Q+2}}\bigg\}^{\frac{Q+2}{2Q}} \|u\|_{L^{2Q/(Q-2)}(\Omega_T)} \\
&\le C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \bigg\{\int_{\Omega_T} \bigg[\Big(\sum_{k\in\mathbb{Z}} \omega_{\lambda/T^k}\Big)^{\frac{Q+2}{Q-2}} - \omega_\lambda^{\frac{Q+2}{Q-2}}\bigg]^{\frac{2Q}{Q+2}}\bigg\}^{\frac{Q+2}{2Q}} \\
&= C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \\
&\qquad \cdot \bigg\{\int_{\Omega_T} \bigg[\Big(\sum_{k\in\mathbb{Z}} |x|^{\frac{Q-2}{2}} \omega_{\lambda/T^k}\Big)^{\frac{Q+2}{Q-2}} - (|x|^{(Q-2)/2}\omega_\lambda)^{\frac{Q+2}{Q-2}}\bigg]^{\frac{2Q}{Q+2}} \frac{dx}{|x|^Q}\bigg\}^{\frac{Q+2}{2Q}}
\end{aligned}$$

by Proposition 3.3 (taking $T \ge T_0 > 1$, since we are going to make a limit for $T \to \infty$). Let us define $\eta_\lambda = |x|^{(Q-2)/2}\omega_\lambda$. Then

$$|B| \le C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \bigg\{\int_{\Omega_T} \bigg[\Big(\sum_{k\in\mathbb{Z}} \eta_{\lambda/T^k}\Big)^{\frac{Q+2}{Q-2}} - \eta_\lambda^{\frac{Q+2}{Q-2}}\bigg]^{\frac{2Q}{Q+2}} \frac{dx}{|x|^Q}\bigg\}^{\frac{Q+2}{2Q}}.$$

By periodicity we can suppose that $\frac{|x|}{\lambda} \in \left[1/\sqrt{T}, \sqrt{T}\right]$, with $\lambda = \lambda(x)$. The function η_λ is bounded and tends to zero for $|x| \to 0, \infty$. If $k \geq 0$ and T is large enough then η_{λ/T^k} satisfies estimates

$$|\eta_{\lambda/T^k}(x)| \lesssim \left(\frac{(T^k/\lambda)|x|}{1+(T^k/\lambda)^2|x|^2}\right)^{\frac{Q-2}{2}} \lesssim \left(T^k \frac{|x|}{\lambda}\right)^{-\frac{Q-2}{2}} \leq \left(\frac{1}{T}\right)^{(k-1/2)(Q-2)/2}$$

and

$$|\eta_{\lambda/T^{-k}}(x)| \lesssim \left(\frac{(T^{-k}/\lambda)|x|}{1+(T^{-k}/\lambda)^2|x|^2}\right)^{\frac{Q-2}{2}} \lesssim \left(\frac{1}{T^k}\frac{|x|}{\lambda}\right)^{\frac{Q-2}{2}} \leq \left(\frac{1}{T}\right)^{(k-1/2)(Q-2)/2}$$

uniformly in λ. It is easy to verify that, for α, $\beta \geq 1$ the function

$$\frac{[(x+y)^\alpha - x^\alpha]^\beta}{x^{(\alpha-1)\beta} y^\beta + y^{\alpha\beta}}$$

is bounded on $(0, \infty)^2$, and so there exist C such that

$$[(x+y)^\alpha - x^\alpha]^\beta \leq C(x^{(\alpha-1)\beta} y^\beta + y^{\alpha\beta})$$

for x, $y \geq 0$. Taking

$$x = \eta_\lambda, \quad y = \sum_{k\in\mathbb{Z}\setminus\{0\}} \eta_{\lambda/T^k}, \quad \alpha = \frac{Q+2}{Q-2} \quad \text{and} \quad \beta = \frac{2Q}{Q+2}$$

one gets that

$$|B| \leq C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \cdot \left\{\int_{\Omega_T}\left[\eta_\lambda^{\frac{8Q}{(Q+2)(Q-2)}}\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \eta_{\lambda/T^k}\right)^{\frac{2Q}{Q+2}} + \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \eta_{\lambda/T^k}\right)^{\frac{2Q}{Q-2}}\right]\frac{dx}{|x|^Q}\right\}^{\frac{Q+2}{2Q}}.$$

Let

$$\Omega_T^1 = \{x \in \Omega_T \mid \lambda(x) < 1\} \quad \text{and} \quad \Omega_T^2 = \{x \in \Omega_T \mid \lambda(x) \geq 1\}.$$

Then

$$\begin{aligned}|B| \leq C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} &\left\{\left(\int_{\Omega_T^1} + \int_{\Omega_T^2}\right)\left[\eta_\lambda^{\frac{8Q}{(Q+2)(Q-2)}}\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \eta_{\lambda/T^k}\right)^{\frac{2Q}{Q+2}}\right.\right. \\ &\qquad\left.\left. + \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \eta_{\lambda/T^k}\right)^{\frac{2Q}{Q-2}}\right]\frac{dx}{|x|^Q}\right\}^{\frac{Q+2}{2Q}} \\ \lesssim C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} &\left\{\int_{\Omega_T}\left[\left(\frac{1}{T}\right)^{\frac{Q-2}{4}\cdot\frac{2Q}{Q+2}} + \left(\frac{1}{T}\right)^{\frac{Q-2}{4}\cdot\frac{2Q}{Q-2}}\right]\frac{dx}{|x|^Q}\right\}^{\frac{Q+2}{2Q}}\end{aligned}$$

$$\lesssim C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \left\{ \left(\frac{1}{T}\right)^{\frac{Q(Q-2)}{2(Q+2)}} \int_{\Omega_T} \frac{dx}{|x|^Q} \right\}^{\frac{Q+2}{2Q}}$$

$$\lesssim C(\log T)^{\frac{Q-2}{2Q}} \|u\|_{X_T} \left\{ \left(\frac{1}{T}\right)^{\frac{Q(Q-2)}{2(Q+2)}} \log T \right\}^{\frac{Q+2}{2Q}} \to 0$$

uniformly in λ. □

Proof of Proposition 4.1. It follows from the above lemmas. □

5. Nondegeneracy of the second differential

In order to verify the nondegeneracy of the second differential, we restrict ourselves to the space $\widetilde{X}_T$ defined in Section 2 (which contains $\mathscr{Z}_T$). We recall the following result.

Proposition 5.1 [Malchiodi and Uguzzoni 2002]. *A function $u \in S^1(\mathbb{H}^n)$ is a solution of the following equation*:

$$-\Delta_{\mathbb{H}^n} u = (Q^* - 1)\omega^{Q^*-2} u \tag{13}$$

if and only if there exist coefficients $\mu, \nu_1, \dots, \nu_{2n} \in \mathbb{R}$ such that

$$u = \mu \frac{\partial \omega_\lambda}{\partial \lambda}\Big|_{\lambda=1} + \sum_{i=0}^{2n} \nu_i T_i(\omega_\lambda).$$

For u to solve (13) is equivalent to being in the kernel of $\mathscr{J}''$. Since the operator $\mathscr{J}''$ is the sum of an isomorphism and a compact operator on $S^1(\mathbb{H}^n)$ (see [Malchiodi and Uguzzoni 2002]) and that it only a negative eigenvalue whose one-dimensional eigenspace is spanned by ω_λ (see [Birindelli and Capuzzo Dolcetta 2000]) there exists a constant C such that if $u \in S^1(\mathbb{H}^n)$ and

$$\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \frac{\partial \omega_\lambda}{\partial \lambda} = 0, \quad \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} T_i(\omega_\lambda) = 0, \quad \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \omega_\lambda = 0 \tag{14}$$

then

$$d^2\mathscr{J}(\omega_\lambda)[u,u] \ge C \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2. \tag{15}$$

Furthermore, since $\mathscr{J}''$ is selfadjoint and ω_λ is an eigenfunction,

$$d^2\mathscr{J}(\omega_\lambda)[\omega_\lambda, u] = 0. \tag{16}$$

We want to use this to prove a similar nondegeneracy result for Ψ_λ on Ω_T for T large enough.

In order to do this, we introduce on X_T the norm

$$\|u\|^2_{T,\mathfrak{H}} = \int_{\Omega_T} \Big(|\nabla_{\mathbb{H}^n} u|^2 + \Big|\frac{u}{|x|}\Big|^2 \Big).$$

Thanks to Hardy's inequality in $\mathbb{H}^n$ (see Lemma 2.1 in [Bahouri et al. 2005], or otherwise apply the Hölder inequality for Lorentz spaces), if $u \in S^1(\mathbb{H}^n)$, then, under the aforementioned hypotheses (14),

$$|d^2 \mathscr{J}(\omega_\lambda)[u,u]| \geq C \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 + \Big|\frac{u}{|x|}\Big|^2.$$

Using this we will prove that, if $u \in \widetilde{X}_T$ satisfies

$$\int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \frac{\partial \Psi_\lambda}{\partial \lambda} = 0 \tag{17}$$

and

$$\int_{\Omega_T} \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \Psi_\lambda = 0, \tag{18}$$

then, given $\varepsilon > 0$, for T large

$$d^2 \mathscr{J}_T(\Psi_\lambda)[u,u] \geq C \int_{\Omega_T} |\nabla_{\mathbb{H}^n} u|^2 + \Big|\frac{u}{|x|}\Big|^2,$$

$$\big|d^2 \mathscr{J}_T(\Psi_\lambda)[\Psi_\lambda,\Psi_\lambda]\big| \geq C \int_{\Omega_T} |\nabla_{\mathbb{H}^n} \Psi_\lambda|^2 + \Big|\frac{\Psi_\lambda}{|x|}\Big|^2,$$

$$\text{and} \quad \big|d^2 \mathscr{J}_T(\Psi_\lambda)[\Psi_\lambda,u]\big| < \varepsilon \|\Psi_\lambda\|_{T,\mathfrak{H}} \|u\|_{T,\mathfrak{H}}.$$

This implies that $\mathscr{J}''_T(\Psi_\lambda)$ is invertible orthogonally to $\frac{\partial \Psi_\lambda}{\partial \lambda}$, and that the norm of the inverse is bounded uniformly in T.

Let us take a radial function $\rho = \rho(|x|)$ such that $\rho = 1$ on Ω_T, $\rho = 0$ on $B_{1/2} \cup (\mathbb{H}^n \setminus B_{2T})$, $0 \leq \rho \leq 1$, $|\nabla_{\mathbb{H}^n} \rho| \leq C$ on $B_1 \setminus B_{1/2}$, $|\nabla_{\mathbb{H}^n} \rho| \leq C/T$ on $B_{2T} \setminus B_T$.

By the computations in formula (11) it follows that

$$\frac{\partial \omega_\lambda}{\partial \lambda} = -\frac{Q-2}{2}\frac{1}{\lambda}\omega_\lambda - \frac{1}{\lambda} Z(\omega_\lambda).$$

Thanks to formula (11), it can easily be proved that

$$\left| \nabla_{\mathbb{H}^n} \frac{\partial \Psi_{\lambda,T}}{\partial \lambda} \right| \leq \frac{C}{\lambda} \frac{1}{|x|^{Q/2}}. \tag{19}$$

By periodicity with respect to dilations we can suppose the quantity

$$r^Q \int_{B_2 \setminus B_{1/2}} \Big(|\nabla_{\mathbb{H}^n} (u \circ \delta_r)|^2 + \Big|\frac{u}{|x|} \circ \delta_r\Big|^2 \Big)$$

to be minimal for $r = 1$. Since there are $\sim \log T$ mutually disjoint annuli in Ω_T of the form $\delta_r\left\{\frac{1}{2} \le |x| \le 2\right\}$, by easy computations one gets that

$$\int_{B_2\setminus B_{1/2}} |\nabla_{\mathbb{H}^n} u|^2 + \left|\frac{u}{|x|}\right|^2 \le \frac{C}{\log T}\|u\|^2_{T,\mathfrak{H}},$$

and so, calling $W = (B_{2T} \setminus B_T) \cup (B_1 \setminus B_{1/2})$,

$$\int_W |\nabla_{\mathbb{H}^n} u|^2 + \left|\frac{u}{|x|}\right|^2 \le \frac{C}{\log T}\|u\|^2_{T,\mathfrak{H}}. \tag{20}$$

Lemma 5.2. *If ρ is a cut-off function as above, for every ε there exists T_0 such that for $T \ge T_0$ if (17) holds then*

$$\left|\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\nabla_{\mathbb{H}^n}\frac{\partial \Psi_{T,\lambda}}{\partial\lambda}\right| \le \varepsilon\frac{1}{\lambda}\|u\|_{T,\mathfrak{H}}$$

and

$$\left|\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\nabla_{\mathbb{H}^n}\Psi_{T,\lambda}\right| \le \varepsilon\|u\|_{T,\mathfrak{H}}.$$

Proof.

$$\begin{aligned}\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\nabla_{\mathbb{H}^n}\frac{\partial \Psi_{\lambda}}{\partial\lambda} &= \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\nabla_{\mathbb{H}^n}\frac{\partial \Psi_{\lambda}}{\partial\lambda} - \int_{\Omega_T} \nabla_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}\frac{\partial \Psi_{\lambda}}{\partial\lambda}\\ &= \int_W \left[(\rho\nabla_{\mathbb{H}^n}u + u\nabla_{\mathbb{H}^n}\rho)\nabla_{\mathbb{H}^n}\frac{\partial \Psi_{\lambda}}{\partial\lambda}\right].\end{aligned}$$

Thanks to formulas (19) and (20) the first estimate follows by easy computations. The proof of the second one is identical. □

Lemma 5.3. *For every ε there exists T_0 such that for $T \ge T_0$ if (17) holds then*

$$\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\nabla_{\mathbb{H}^n}\frac{\partial \omega_{\lambda}}{\partial\lambda} \le \varepsilon\frac{1}{\lambda}\|u\|_{T,\mathfrak{H}} \quad \textit{and} \quad \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\nabla_{\mathbb{H}^n}\omega_{\lambda} \le \varepsilon\|u\|_{T,\mathfrak{H}}.$$

Proof. Thanks to Lemma 5.2, we can estimate

$$\begin{aligned}&\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\lambda\nabla_{\mathbb{H}^n}\frac{\partial \Psi_{\lambda}}{\partial\lambda} - \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\lambda\nabla_{\mathbb{H}^n}\frac{\partial \omega_{\lambda}}{\partial\lambda}\\ &\qquad\le C\|u\|_{T,\mathfrak{H}}\left(\int_{\Omega_T\cup W}\left|\lambda\nabla_{\mathbb{H}^n}\frac{\partial \Psi_{\lambda}}{\partial\lambda} - \lambda\nabla_{\mathbb{H}^n}\frac{\partial \omega_{\lambda}}{\partial\lambda}\right|^2\right)^{1/2}.\end{aligned}$$

This quantity can be estimated almost identically as in the proof of Lemma 4.3. The proof of the second inequality estimate is identical. □

Lemma 5.4. *For every* $\varepsilon > 0$ *there exist constants* T_0 *and* C *such that for* $T \geq T_0$ *if* (17) *and* (18) *hold then*

$$\left|d^2\mathscr{J}(\omega_\lambda)[\rho u, \rho u]\right| \geq C\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}(\rho u)|^2 + \left|\frac{\rho u}{|x|}\right|^2,$$

$$\left|d^2\mathscr{J}(\omega_\lambda)[\rho\Psi_\lambda, \rho\Psi_\lambda]\right| \geq C\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}(\rho\Psi_\lambda)|^2 + \left|\frac{\rho\Psi_\lambda}{|x|}\right|^2,$$

$$\text{and} \quad \left|d^2\mathscr{J}(\omega_\lambda)[\rho\Psi_\lambda, \rho u]\right| \leq \varepsilon\|\Psi_\lambda\|_{T,\mathfrak{H}}\|u\|_{T,\mathfrak{H}}.$$

Proof. Since $u \in \widetilde{X}_T,\ u\rho$ is invariant with respect to the symmetry $(x,t) \mapsto (-x,t)$, one has

$$\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}(\rho u)\cdot\nabla_{\mathbb{H}^n}T_i(\omega_\lambda) = 0.$$

The claim follows by Lemma 5.3, by (15) and (16), and elementary linear algebra. □

Lemma 5.5. *For every* $\varepsilon > 0$ *there exist constants* T_0 *and* C *such that for* $T \geq T_0$ *if conditions* (17) *and* (18) *hold, then*

$$\left|d^2\mathscr{J}_T(\Psi_\lambda)[u,u]\right| \geq C\int_{\Omega_T} |\nabla_{\mathbb{H}^n}u|^2,$$

$$\left|d^2\mathscr{J}_T(\Psi_\lambda)[\Psi_\lambda,\Psi_\lambda]\right| \geq C\int_{\Omega_T} |\nabla_{\mathbb{H}^n}\Psi_\lambda|^2$$

$$\text{and} \quad \left|d^2\mathscr{J}_T(\Psi_\lambda)[\Psi_\lambda,u]\right| < \varepsilon\|\Psi_\lambda\|_{X_T}\|u\|_{X_T}.$$

Proof. By direct computation we find

$$\begin{aligned}
&\left|d^2\mathscr{J}(\omega_\lambda)[\rho u,\rho u] - d^2\mathscr{J}_T(\Psi_\lambda)[u,u]\right| \\
&= \left|\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}(\rho u)|^2 - (2^*-1)|\omega_\lambda|^{2^*-2}\rho^2u^2 - \int_{\Omega_T}|\nabla_{\mathbb{H}^n}u|^2 - (2^*-1)|\Psi_\lambda|^{2^*-2}u^2\right| \\
&\leq (2^*-1)\left|\int_{\Omega_T}\left(|\Psi_\lambda|^{2^*-2} - |\omega_\lambda|^{2^*-2}\right)u^2\right| \\
&\qquad + (2^*-1)\left|\left(\int_{B_{2T}\setminus B_T} + \int_{B_1\setminus B_{1/2}}\right)|\omega_\lambda|^{2^*-2}\rho^2u^2\right| \\
&\qquad\qquad + 2\left|\left(\int_{B_{2T}\setminus B_T} + \int_{B_1\setminus B_{1/2}}\right)(u^2|\nabla_{\mathbb{H}^n}\rho|^2 + \rho^2|\nabla_{\mathbb{H}^n}u|^2)\right|.
\end{aligned}$$

The first term can be estimated as in Lemma 4.3, the second in a trivial way, and the third has been essentially already estimated, to prove that for every ε there exists T big enough to ensure that the whole sum is bounded by $\varepsilon\|u\|^2_{X_T}$.

Analogously

$$\left| \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}(\rho u)|^2 - \int_{\Omega_T} |\nabla_{\mathbb{H}^n} u|^2 \right| \le \varepsilon \|u\|^2_{X_T}.$$

This implies the first part of the thesis. The other statements are deduced in an analogous manner. □

Proposition 5.6. *There exist constants T_0 and C such that for $T \ge T_0$ the operator $\mathscr{J}''_T(\Psi_\lambda)$ is invertible on the orthogonal space of $\frac{\partial \Psi_\lambda}{\partial \lambda}$ in X_T, and*

$$\| \mathscr{J}''_T(\Psi_\lambda)^{-1} \|_{\mathscr{L}(X_T)} \le C.$$

Proof. It follows from the preceding lemmas and elementary Hilbert space theory. □

6. Proof of the main Theorem

We have proved that, for T big enough, on the orthogonal in $\widetilde{X}_T$ of the tangent of the curve $\mathscr{Z}_T$ the second differential of $\mathscr{J}_T$ is nondegenerate, with norm bounded independently by λ and T. Let us call W this orthogonal in the point $\Psi_\lambda \in \mathscr{Z}$ and π the orthogonal projection on W. We remember that our aim is to solve $\nabla_{\mathbb{H}^n} \mathscr{J}_T(u) = 0$. Following the standard reasoning in [Ambrosetti and Malchiodi 2006] we note that this is equivalent to solving

$$\pi \nabla_{\mathbb{H}^n} \mathscr{J}_T(\Psi_\lambda + w) = 0$$

(auxiliary equation) and

$$(I - \pi) \nabla_{\mathbb{H}^n} \mathscr{J}_T(\Psi_\lambda + w) = 0$$

(bifurcation equation) with $w \in W$.

Lemma 6.1. *There exists T_0 such that the auxiliary equation has a unique solution $w_T(\lambda)$; furthermore* $\sup_\lambda \|w_T(\lambda)\| \to 0$ *for $T \to \infty$.*

Proof. Write

$$\nabla_{\mathbb{H}^n} \mathscr{J}_T(\Psi_\lambda + w) = \nabla_{\mathbb{H}^n} \mathscr{J}_T(\Psi_\lambda) + \mathscr{J}''_T[w] + R(\Psi_\lambda, w)$$

with $R(\Psi_\lambda, w) = o(\|w\|)$ and $R(\Psi_\lambda, w) - R(\Psi_\lambda, v) = o(\|w - v\|)$, so that the auxiliary equation becomes

$$\pi \nabla_{\mathbb{H}^n} \mathscr{J}_T(\Psi_\lambda) + \pi \mathscr{J}''_T(\Psi_\lambda)[w] + \pi R(\Psi_\lambda, w) = 0,$$

namely

$$w = -(\pi \mathscr{J}''_T(\Psi_\lambda))^{-1} \big[\pi \nabla_{\mathbb{H}^n} \mathscr{J}_T(\Psi_\lambda) + \pi R(\Psi_\lambda, w) \big] := N_\lambda(w).$$

By Propositions 4.1 and 5.6, N is a contraction if T is big enough, and so the auxiliary equation has an unique solution $w = w_T(\lambda)$. Furthermore for every $r > 0$ there exists T big enough such that $B_r(\Psi_\lambda) \cap W$ is mapped into itself by N. So $\sup_\lambda \|w_T(\lambda)\|$ tends to zero for $T \to \infty$. □

Proof of Theorem 1.1. Let us consider the function

$$\Phi(\lambda) = \mathscr{J}_T(\Psi_\lambda + w(\lambda)).$$

It is continuous and periodic, so it has a stationary point λ_0. Following the standard argument of Theorem 2.12 and Remark 2.14 in [Ambrosetti and Malchiodi 2006], with the need for only formal modifications, the fact that

$$\Phi'(\lambda_0) = \mathscr{J}_T'(\Psi_{\lambda_0} + w(\lambda_0)) \cdot \left(\frac{\partial \Psi_{\lambda_0}}{\partial \lambda} + w'(\lambda_0)\right)$$

implies $u = \Psi_{\lambda_0} + w(\lambda_0)$ to solve the bifurcation equation, and so to be a stationary point of $\mathscr{J}_T$.

The smoothness of the solution can be proved with the same method of Appendix B in [Struwe 1996].

Also $\lambda^{(2-Q)/2} u \circ \delta_{\lambda^{-1}}$ is a critical point of $\mathscr{J}_T$, and by the uniqueness in the fixed point theorem it must be equal to $\Psi_{\lambda_0\lambda} + w(\lambda_0\lambda)$, and so the whole curve $\widetilde{\mathscr{Z}}_T = \{\Psi_\lambda + w(\lambda)\}$ consists of critical points of $\mathscr{J}$.

To prove the positivity, let us notice that from the proof of Proposition 5.6 it follows that $\mathscr{J}(\omega_\lambda)$ has Morse index one on $\left\{\lambda \frac{\partial \omega_\lambda}{\partial \lambda}\right\}^\perp$. By continuity, the same holds for the orthogonal to the tangent space to $\widetilde{\mathscr{Z}}_T$. Since $d\mathscr{J}_T$ is zero on $\widetilde{\mathscr{Z}}_T$, the tangent of $\widetilde{\mathscr{Z}}_T$ is in the kernel of $\mathscr{J}_T''$. So the Morse index of $\mathscr{J}_T$ on $\widetilde{X}_T$ is one.

By a slight adaptation of the proof of Proposition 3.2 in [Birindelli and Capuzzo Dolcetta 2000] the set $\{u \neq 0\}$ has at most one connected component modulo δ_T, and so u does not change sign. By construction it is evident that it must be weakly positive (and even if it was not, it would be enough to change sign). The strict positivity follows from Bony's maximum principle [1969].

The last assertion follows by construction. □

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ON A THEOREM OF HEGYVÁRI AND HENNECART

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We study growth rate of product of sets in the Heisenberg group over finite fields and the complex numbers. More precisely, we will give improvements and extensions of recent results due to Hegyvári and Hennecart (2018).

1. Introduction

Let $\mathbb{F}_q$ be an arbitrary finite field with order $q = p^r$ for some positive integer r and an odd prime p. For an integer $n \geq 1$, the Heisenberg group of degree n, denoted by $H_n(\mathbb{F}_q)$, is defined by a set of the following matrices:

$$[\boldsymbol{x}, \boldsymbol{y}, z] := \begin{bmatrix} 1 & \boldsymbol{x} & z \\ \mathbb{0} & I_n & \boldsymbol{y}^t \\ 0 & \mathbb{0} & 1 \end{bmatrix}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_q^n$, $z \in \mathbb{F}_q$, $\boldsymbol{y}^t$ denotes the column vector of $\boldsymbol{y}$, and I_n is the $n \times n$ identity matrix. For $A \subset \mathbb{F}_q$, $E, F \subset \mathbb{F}_q^n$, we define

$$[E, F, A] := \{[\boldsymbol{x}, \boldsymbol{y}, z] : \boldsymbol{x} \in E, \boldsymbol{y} \in F,\ z \in A\},$$

and

$$[E, F, A][E, F, A] := \{[\boldsymbol{x}, \boldsymbol{y}, z] \cdot [\boldsymbol{x}', \boldsymbol{y}', z'] : [\boldsymbol{x}, \boldsymbol{y}, z], [\boldsymbol{x}', \boldsymbol{y}', z'] \in [E, F, A]\},$$

Over recent years, there is an intensive study on growth rate in the Heisenberg group over finite fields and applications. Hegyvári and Hennecart [2013] proved a structure result for *bricks* in Heisenberg groups. The precise statement is as follows.

Theorem 1.1 [Hegyvári and Hennecart 2013]. *For every $\epsilon > 0$, there exists a positive integer $n_0(\epsilon)$ such that for all $n \geq n_0(\epsilon)$ and any sets $X_i, Y_i, Z \subset \mathbb{F}_p$, $i \in [n]$, $X = \prod_{i=1}^n X_i \subset \mathbb{F}_p^n$, $Y = \prod_{i=1}^n Y_i \subset \mathbb{F}_p^n$, if*

$$(1) \qquad |[X, Y, Z]| > |H_n(\mathbb{F}_p)|^{3/4+\epsilon},$$

then $[X, Y, Z][X, Y, Z]$ contains at least $|[X, Y, Z]|/p$ cosets of $[\mathbb{0}, \mathbb{0}, \mathbb{F}_p]$.

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It follows from the proof of Theorem 1.1 in [Hegyvári and Hennecart 2013] that $\epsilon = O(1/n)$. In a very recent work, Shkredov [2020] obtained the following theorem which improves the relation between ϵ and n in the specific case when sizes of sets X_i, Y_i, Z are comparable.

Theorem 1.2 [Shkredov 2020]. *Let $n \geq 2$ be an even number, and $X_i, Y_i, Z \subset \mathbb{F}_p$, $i \in [n]$, $X = \prod_{i=1}^n X_i \subset \mathbb{F}_p^n$, $Y = \prod_{i=1}^n Y_i \subset \mathbb{F}_p^n$ such that X_i, Y_i have comparable sizes. Set $\mathcal{X} = \max_i |X_i|$ and $\mathcal{Y} = \max_i |Y_i|$. If $|Z| \leq \mathcal{X}\mathcal{Y}$, $\mathcal{X} \leq |Z|\mathcal{Y}$, $\mathcal{Y} \leq |Z|\mathcal{X}$ and*

$$\mathcal{X}\mathcal{Y} \gtrsim p^{3/2} \cdot \left(\frac{\mathcal{X}\mathcal{Y}}{p|Z|^{1/2}}\right)^{2^{-n/2}}, \tag{2}$$

then $[X, Y, Z][X, Y, Z]$ contains at least $|[X, Y, Z]|/p$ cosets of $[\mathbb{0}, \mathbb{0}, \mathbb{F}_p]$.

Suppose that $X_i = Y_j = Z$ for all $1 \leq i, j \leq n$. Then it follows from Theorem 1.2 that $[X, Y, Z][X, Y, Z]$ contains at least $|[X, Y, Z]|/p$ cosets of $[\mathbb{0}, \mathbb{0}, \mathbb{F}_p]$ under the condition $|Z| \gtrsim p^{3/4+1/(2^{n/2+4}-12)}$, which improves the threshold $p^{3/4+O(1/n)}$ of Theorem 1.1. Moreover, Shkredov [2020] gives an introduction to representation theory which is good for products of general sets in the affine and in the Heisenberg groups.

Throughout this paper, we use $X \ll Y$ if $X \leq CY$ for some constant $C > 0$ independent of the parameters related to X and Y, and write $X \gg Y$ for $Y \ll X$. The notation $X \sim Y$ means that both $X \ll Y$ and $Y \ll X$ hold. In addition, we use $X \lesssim Y$ to indicate that $X \ll (\log_2 Y)^{C'} Y$ for some constant $C' > 0$.

It is worth noting that there is an interesting application of products of sets in the Heisenberg group to so-called models of *Freiman isomorphisms*; see [Hegyvári and Hennecart 2012]. Moreover, it has been indicated in [Tao and Vu 2006, §5.3] that any set in the Heisenberg group with the doubling constant less than two does not have any good model.

It is well-known that there is a connection between the sum-product phenomenon and growth in the group of affine transformations, for example, see [Rudnev and Shkredov 2018]. Such a connection has been discovered in the setting of Heisenberg group by Hegyvári and Hennecart [2018]. More precisely, in the case $n = 1$, using sum-product estimates, they proved that if $A \subset \mathbb{F}_p$ with $|A| \geq p^{1/2}$, then

$$|[A, A, 0][A, A, 0]| \gg \min\{p^{1/2}|[A, A, 0]|^{5/4},\ p^{-1/2}|[A, A, 0]|^2\}. \tag{3}$$

When the size of A is not too big, the authors obtained the following.

Theorem 1.3 [Hegyvári and Hennecart 2018]. *Let A be a set in $\mathbb{F}_p$. Suppose that $|A| \leq p^{2/3}$, then we have*

$$|[A, A, 0][A, A, 0]| \gg |[A, A, 0]|^{7/4}.$$

Notice that the method in the proof of Theorem 1.3 can be extended to arbitrary finite fields, and as a consequence, we obtain the following.

Theorem 1.4 [Hegyvári and Hennecart 2018]. *Let A be a set in $\mathbb{F}_q$. Suppose that $|A| \geq q^{2/3}$, then we have*

$$|[A, A, 0][A, A, 0]| \gg q|[A, A, 0]|.$$

Note that the lower bound in Theorem 1.4 is stronger than that of (3).

The main purpose of this paper is to give improvements and extensions of Theorems 1.3 and 1.4 in the setting of arbitrary finite fields $\mathbb{F}_q$ and the complex numbers $\mathbb{C}$.

In our first theorem, we will show that Theorem 1.4 can be improved in the case where the additive energy of A is small.

Theorem 1.5. *Let A be a set in $\mathbb{F}_q$. Let $\Lambda^+(A)$ be the number of quadruples $(a, b, c, d) \in A^4$ such that $a+b = c+d$. Suppose that $\Lambda^+(A) \leq |A|^3/K$ for some $K > 0$ and $|A| \geq K^{1/3}q^{2/3}$, then we have*

$$|[A, A, 0][A, A, 0]| \gg Kq|[A, A, 0]|.$$

Our next theorem is an extension of Theorem 1.4 in the setting of $H_n(\mathbb{F}_q)$ for any $n \geq 1$.

Theorem 1.6. *Let E be a set in $\mathbb{F}_q^n$. Suppose that $|E| \gg q^{n/2+1/4}$, then we have*

$$|[E, E, 0][E, E, 0]| \gg q|[E, E, 0]|.$$

Notice that in general the conclusion of Theorem 1.6 is sharp, since E can be a subspace in $\mathbb{F}_q^n$, which implies that $[E, E, 0][E, E, 0] \subset [E, E, \mathbb{F}_q]$. Moreover, the exponent $\frac{1}{2}n + \frac{1}{4}$ can not be decreased to $\frac{1}{2}n$, since, supposing that $q = p^2$, one can take $E = \mathbb{F}_p^n$, which gives us $|[E, E, 0][E, E, 0]| \ll p|[E, E, 0]| = q^{1/2}|[E, E, 0]|$.

In the setting of prime fields, if E is a set in the plane $\mathbb{F}_p^2$ and the size of E is not too big, then we have the following theorem in $H_2(\mathbb{F}_p)$.

Theorem 1.7. *Let $\mathbb{F}_p$ be a prime field with $p \equiv 3 \bmod 4$, and E be a set in $\mathbb{F}_p^2$ with $|E| \ll p^{8/5}$. Then*

$$|[E, E, 0][E, E, 0]| \gg |[E, E, 0]|^{19/15}.$$

When A is a multiplicative subgroup of $\mathbb{F}_p^*$, we are able to show that the exponent $\frac{7}{4}$ in Theorem 1.3 can be improved significantly.

Theorem 1.8. *Let A be a multiplicative subgroup of $\mathbb{F}_p^*$ with $|A| \leq p^{1/2} \log p$. We have*

$$|[A, A, 0][A, A, 0]| \gtrsim |[A, A, 0]|^{151/80}.$$

In the setting of the real numbers, for any $A \subset \mathbb{R}$, using a point-plane incidence bound due to Elekes and Tóth [2005] and an energy variant of the sum-product conjecture due to Rudnev, Shkredov, and Stevens [Rudnev et al. 2020], Hegyvári and Hennecart [2018] proved that

$$|[A, A, 0][A, A, 0]| \gtrsim |[A, A, 0]|^{15/8}.$$

In our next theorem, we employ a point-line incidence bound over the complex numbers due to Tóth [2015] and an energy variant of the sum-product conjecture due to Rudnev, Shkredov, and Stevens [Rudnev et al. 2020] to study an extension in the setting of the complex numbers.

Theorem 1.9. *Let A be a set in $\mathbb{C}$ with $|A| \geq 2$. We have*

$$|[A, A, 0][A, A, 0]| \gtrsim |A|^{29/8} = |[A, A, 0]|^{29/16}.$$

2. Proof of Theorem 1.5

To prove Theorem 1.5, we need to recall a lemma given by the third, fourth, and fifth listed authors in [Koh et al. 2018].

Let X be a multiset in $\mathbb{F}_q^{2n} \times \mathbb{F}_q$. We denote by $\overline{X}$ the set of distinct elements in the multiset X. The cardinality of X, denoted by $|X|$, is $\sum_{x \in \overline{X}} m_X(x)$, where $m_X(x)$ is the multiplicity of $\boldsymbol{x}$ in X. For multisets $\mathcal{A}, \mathcal{B} \subset \mathbb{F}_q^{2n+1}$, let $N(\mathcal{A}, \mathcal{B})$ be the number of pairs $((\boldsymbol{a}, b), (\boldsymbol{c}, d)) \in \mathcal{A} \times \mathcal{B} \subset (\mathbb{F}_q^{2n} \times \mathbb{F}_q)^2$ such that $\boldsymbol{a} \cdot \boldsymbol{c} = b + d$. We have the following lemma on an upper bound of $N(\mathcal{A}, \mathcal{B})$.

Lemma 2.1 [Koh et al. 2018, Lemma 8.1]. *Let $\mathcal{A}, \mathcal{B}$ be multisets in $\mathbb{F}_q^{2n} \times \mathbb{F}_q$. We have*

$$\left| N(\mathcal{A}, \mathcal{B}) - \frac{|\mathcal{A}||\mathcal{B}|}{q} \right| \leq q^n \left(\sum_{(a,b) \in \overline{\mathcal{A}}} m_{\mathcal{A}}((a, b))^2 \sum_{(c,d) \in \overline{\mathcal{B}}} m_{\mathcal{B}}((c, d))^2 \right)^{1/2}.$$

Theorem 1.5 is a direct consequence of the following theorem.

Theorem 2.2. *For $A \subset \mathbb{F}_q$, we have*

$$|[A, A, 0][A, A, 0]| \gg \min\left\{ \frac{|A|^5}{q}, \frac{q|A|^5}{\Lambda^+(A)} \right\}.$$

Proof. Without loss of generality, we assume that $0 \notin A$. Let S be the number of quadruples of matrices (m_1, m_2, m_3, m_4) in $[A, A, 0]^4$ such that $m_1 m_2 = m_3 m_4$. By the Cauchy–Schwarz inequality, we have

$$|[A, A, 0]^2| \geq \frac{|A|^8}{S}.$$

To complete the proof, it will be enough to show that

$$S \ll \frac{|A|^3 \Lambda^+(A)}{q} + q|A|^3.$$

In the next step we are going to prove this. Indeed, from the definition of S, we have that S is equal to the number of tuples $(a, b, c, d, a', b', c', d')$ in A^8 such that

$$a + c = a' + c', \tag{4}$$

$$b + d = b' + d', \tag{5}$$

$$ad = a'd'. \tag{6}$$

It follows from (4) and (5) that $a = a' + c' - c$ and $d' = b + d - b'$. Substituting into (6), we obtain

$$(a' + c' - c) \cdot d = a' \cdot (b + d - b').$$

This implies that

$$d(c' - c) = a'(b - b'). \tag{7}$$

In case $b \neq b'$, we also have $c \neq c'$, since $0 \notin A$. In this case, the above equality is equivalent with

$$a' = \frac{d}{b - b'}(c' - c).$$

It follows from (7) that if $b = b'$ then $c = c'$. We note that the number of tuples $(a', b, b', c, c', d) \in A^6$ with $b = b'$ and $c = c'$ is at most $|A|^4$. We now count the number of tuples with $b \neq b'$ and $c \neq c'$. It follows from (4), (5), and (6) that the number of tuples $(a, b, c, d, a', b', c', d') \in A^8$ satisfying these equalities is at most the number of tuples $(a', c, c', b, b', d) \in A^6$ such that

$$a' = \frac{d}{b - b'}(c' - c) \quad \text{and} \quad b + d - b' \in A.$$

Let X be the number of such tuples.

Define $P = A \times A$. Let L be the multiset of lines of the form $y = d/(b - b')(x - c)$ with $b + d - b' \in A$. It is clear that $|P| = |A|^2$ and $|L| = \Lambda^+(A)|A|$. One can check that X is bounded by the number of incidences between points in P and lines in L.

Let $\mathcal{L}$ be the multiset in $\mathbb{F}_q^2$ containing points of the form $(d/(b - b'), d/(b - b') \cdot c)$ with $b + d - b' \in A$ and $c \in A$. On the other hand, by an elementary calculation, we have $\sum_{l \in \bar{\mathcal{L}}} m_{\mathcal{L}}(l)^2 \leq X|A|$, and $|\mathcal{L}| = |L|$. With this new set $\mathcal{L}$, we have $X = N(P, \mathcal{L})$, where $N(P, \mathcal{L})$ is defined as in Lemma 2.1. Applying Lemma 2.1, we have

$$X \leq \frac{|A|^3 \Lambda^+(A)}{q} + q^{1/2} X^{1/2} |A|^{3/2},$$

which implies that
$$X \le \frac{|A|^3\Lambda^+(A)}{q} + q|A|^3.$$
In other words, we have
$$S \le \frac{|A|^3\Lambda^+(A)}{q} + q|A|^3 + |A|^4 \ll \frac{|A|^3\Lambda^+(A)}{q} + q|A|^3. \qquad \square$$

3. Proof of Theorem 1.6

In order to prove Theorem 1.6, we first prove the following lemma.

Lemma 3.1. *Let E be a set in $\mathbb{F}_q^n$. Let T be the number of triples $(\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{x}') \in E^3$ such that $\boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}') = 0$. Then we have*
$$T \le \frac{|E|^3}{q} + q^n|E|.$$
Before proving Lemma 3.1, we need to review the Fourier transform of functions on $\mathbb{F}_q^n$. Let χ be a nontrivial additive character on $\mathbb{F}_q$. For a function $f : \mathbb{F}_q^n \to \mathbb{C}$, the Fourier transform of f, denoted by $\hat{f}$, is defined by
$$\hat{f}(\boldsymbol{m}) = q^{-n} \sum_{\boldsymbol{x} \in \mathbb{F}_q^n} \chi(-\boldsymbol{x} \cdot \boldsymbol{m}) f(\boldsymbol{x}).$$
The following Fourier inversion theorem can be easily proved by the orthogonality relation of χ:
$$f(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in \mathbb{F}_q^n} \chi(\boldsymbol{x} \cdot \boldsymbol{m}) \hat{f}(\boldsymbol{m}).$$
It follows that
$$\sum_{\boldsymbol{m} \in \mathbb{F}_q^n} |\hat{f}(\boldsymbol{m})|^2 = q^{-n} \sum_{\boldsymbol{x} \in \mathbb{F}_q^n} |f(\boldsymbol{x})|^2,$$
which is referred to as the Plancherel theorem.

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. The number T can be expressed as follows:
$$\begin{aligned} T &= \sum_{\boldsymbol{x}\cdot\boldsymbol{v} - \boldsymbol{x}'\cdot\boldsymbol{v} = 0} E(\boldsymbol{v})E(\boldsymbol{x})E(\boldsymbol{x}') \\ &= \frac{|E|^3}{q} + \frac{1}{q} \sum_{s \neq 0} \sum_{\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{x}' \in \mathbb{F}_q^n} \chi(s\boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}')) E(\boldsymbol{v})E(\boldsymbol{x})E(\boldsymbol{x}') \\ &= \frac{|E|^3}{q} + q^{2n-1} \sum_{s \neq 0} \sum_{\boldsymbol{v} \in \mathbb{F}_q^n} |\widehat{E}(s\boldsymbol{v})|^2 E(\boldsymbol{v}). \end{aligned}$$

Using a change of variables by letting $z = s\boldsymbol{v}$, we have

$$T \leq \frac{|E|^3}{q} + q^{2n} \sum_{z \in \mathbb{F}_q^n} |\widehat{E}(z)|^2 = \frac{|E|^3}{q} + q^n |E|,$$

where we used $\sum_{z\in\mathbb{F}_q^n} |\widehat{E}(z)|^2 = q^{-n}|E|$. □

We are ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let $\boldsymbol{S}$ be the number of quadruples of matrices $(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3, \boldsymbol{m}_4)$ in $[E, E, 0]^4$ such that $\boldsymbol{m}_1\boldsymbol{m}_2 = \boldsymbol{m}_3\boldsymbol{m}_4$. By the Cauchy–Schwarz inequality, we have

$$|[E, E, 0][E, E, 0]| \geq \frac{|E|^8}{\boldsymbol{S}}.$$

In the next step, we are going to show that

$$\boldsymbol{S} \leq \frac{|E|^6}{q} + q^{n-1}|E|^4 + q^{2n}|E|^2.$$

Indeed, as in the proof of Theorem 2.2, we have that $\boldsymbol{S}$ is equal to the number of tuples $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}', \boldsymbol{d}')$ in E^8 such that

$$\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{a}' + \boldsymbol{c}', \tag{8}$$

$$\boldsymbol{b} + \boldsymbol{d} = \boldsymbol{b}' + \boldsymbol{d}', \tag{9}$$

$$\boldsymbol{a} \cdot \boldsymbol{d} = \boldsymbol{a}' \cdot \boldsymbol{d}'. \tag{10}$$

It follows from (8) and (9) that $\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{c}' - \boldsymbol{c}$ and $\boldsymbol{d}' = \boldsymbol{b} + \boldsymbol{d} - \boldsymbol{b}'$. Substituting into (10), we obtain

$$(\boldsymbol{a}' + \boldsymbol{c}' - \boldsymbol{c}) \cdot \boldsymbol{d} = \boldsymbol{a}' \cdot (\boldsymbol{b} + \boldsymbol{d} - \boldsymbol{b}').$$

This implies that

$$\boldsymbol{d} \cdot (\boldsymbol{c}' - \boldsymbol{c}) = \boldsymbol{a}' \cdot (\boldsymbol{b} - \boldsymbol{b}'). \tag{11}$$

For any tuples $(\boldsymbol{c}, \boldsymbol{c}', \boldsymbol{b}, \boldsymbol{b}', \boldsymbol{d}, \boldsymbol{a}')$ satisfying (11), we have $\boldsymbol{a}$ and $\boldsymbol{d}'$ are determined uniquely by (8) and (9).

Let $\mathcal{A}$ and $\mathcal{B}$ be multisets defined as follows:

$$\mathcal{A} = \{(\boldsymbol{d}, -\boldsymbol{b}, \boldsymbol{d} \cdot \boldsymbol{c}) : \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in E\}, \quad \mathcal{B} = \{(\boldsymbol{c}', \boldsymbol{a}', -\boldsymbol{a}' \cdot \boldsymbol{b}') : \boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}' \in E\}.$$

Let $N(\mathcal{A}, \mathcal{B})$ be the number defined as in Lemma 2.1. We have that the number of tuples satisfying (11) is equal to $N(\mathcal{A}, \mathcal{B})$.

To apply Lemma 2.1, we need to estimate $\sum_{\boldsymbol{x}\in\overline{\mathcal{A}}} m_{\mathcal{A}}(\boldsymbol{x})^2$ and $\sum_{\boldsymbol{y}\in\overline{\mathcal{B}}} m_{\mathcal{B}}(\boldsymbol{y})^2$.

We have

$$\sum_{\boldsymbol{x}\in\bar{\mathcal{A}}} m_{\mathcal{A}}(\boldsymbol{x})^2, \ \sum_{\boldsymbol{y}\in\bar{\mathcal{B}}} m_{\mathcal{B}}(\boldsymbol{y})^2 \le |E|T,$$

where T is the number of triples $(\boldsymbol{v}, \boldsymbol{x}, \boldsymbol{x}') \in E^3$ such that $\boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{x}') = 0$.

On the other hand, Lemma 3.1 gives us

$$T \le \frac{|E|^3}{q} + q^n|E|.$$

Therefore, one can apply Lemma 2.1 with $|\mathcal{A}| = |\mathcal{B}| = |E|^3$ to derive

$$\boldsymbol{S} \le \frac{|E|^6}{q} + q^n\left(\frac{|E|^4}{q} + q^n|E|^2\right) \ll \frac{|E|^6}{q},$$

whenever $|E| \gg q^{(2n+1)/4}$. □

4. Proof of Theorem 1.7

To prove Theorem 1.7, we need to use the following lemmas. The first lemma is a consequence of the point-line incidence bound due to Stevens and de Zeeuw [2017]. To see a simple proof, see [Lund and Petridis 2018, Theorem 14].

Lemma 4.1. *Let P be a point set in $\mathbb{F}_p^2$ and L be a set of lines in $\mathbb{F}_p^2$. If $|P| \le p^{8/5}$, then the number of incidences between P and L, denoted by $I(P, L)$, satisfies*

$$I(P, L) \ll |P|^{11/15}|L|^{11/15} + |P| + |L|.$$

Lemma 4.2. *Let E be a set in $\mathbb{F}_p^2$ with $p \equiv 3 \bmod 4$ and define $\Pi(E) := \{\boldsymbol{a}\cdot\boldsymbol{b} : \boldsymbol{a}, \boldsymbol{b} \in E\}$. If $|E| \le p^{8/5}$, then $|\Pi(E)| \gg |E|^{8/15}$.*

Proof. Since $p \equiv 3 \bmod 4$, there is no isotropic line in $\mathbb{F}_p^2$. For each $\boldsymbol{a} \in E$, we denote the set $\{\boldsymbol{a}\cdot\boldsymbol{b} \colon \boldsymbol{b} \in E\}$ by $\Pi_{\boldsymbol{a}}(E)$. Suppose that

$$\max_{\boldsymbol{a}\in E} |\Pi_{\boldsymbol{a}}(E)| = t.$$

It is clear that $|\Pi(E)| \gg \max_{\boldsymbol{a}\in E} |\Pi_{\boldsymbol{a}}(E)|$.

Without loss of generality, we may assume that $0 \notin E$. We now fall into two following cases:

Case 1: If there is a line passing through the origin with at least m points of E, then those m points will contribute at least m distinct values to the set $\Pi(E)$. So, $|\Pi(E)| \gg m$.

Case 2: Suppose that all lines passing through the origin contain at most m points of E. This implies that the number of lines passing through the origin and a point in E is at least $|E|/m$.

Let L_0 be a set of lines passing through the origin and at least one point from E such that $|L_0| \sim |E|/m$. From each line l in L_0, we pick one point in $l \cap E$ arbitrarily, and let P be the set of those points. So $|P| = |L_0|$.

For any point $\boldsymbol{a} = (a_1, a_2) \in E$, let $L_{\boldsymbol{a}}$ be the set of lines defined by the equation $a_1x + a_2y = r$ with $r \in \Pi_{\boldsymbol{a}}(E)$. One can check that the size of $L_{\boldsymbol{a}}$ is the same as the size of $\Pi_{\boldsymbol{a}}(E)$. Moreover, we also have that $L_{\boldsymbol{a}} = L_{\boldsymbol{b}}$ when both a and b lie on a line in L_0, and $L_{\boldsymbol{a}} \cap L_{\boldsymbol{b}} = \varnothing$ when the $\boldsymbol{a}$ and $\boldsymbol{b}$ are distinct elements of P.

Let $L = \bigcup_{\boldsymbol{a}\in P} L_{\boldsymbol{a}}$. Since $|\Pi_{\boldsymbol{a}}(E)| \le t$ for any $\boldsymbol{a} \in E$, we have $|L_{\boldsymbol{a}}| \le t$ for all $\boldsymbol{a} \in E$. Thus $|L| \le |P|t = |L_0|t \sim |E|t/m$.

Let $I(E, L)$ be the number of incidences between E and L. For each $\boldsymbol{a} \in P$, we have $I(E, L_{\boldsymbol{a}}) = |E|$. Thus,

$$I(E, L) \gg |E|^2/m.$$

On the other hand, it follows from Lemma 4.1 that

$$I(E, L) \ll |E|^{11/15}(|E|t/m)^{11/15} + |E| + |E|t/m.$$

Hence, we have

$$|E|^2/m \ll |E|^{11/15}(|E|t/m)^{11/15} + |E| + |E|t/m.$$

Since $|E|^2/m \gg |E| + |E|t/m$, solving this inequality for t, we obtain $t \gg |E|^{8/11}m^{-4/11}$.

Optimizing two cases by choosing $m = |E|^{8/15}$, the lemma follows. □

Proof of Theorem 1.7. We first observe that

$$|[E, E, 0][E, E, 0]| \gg |\Pi(E)||E|^2.$$

It follows from Lemma 4.2 that if $|E| \le p^{8/15}$ then we have

$$|\Pi(E)| \gg |E|^{8/15}.$$

Therefore,

$$|[E, E, 0][E, E, 0]| \gg |\Pi(E)||E|^2 \gg |E|^{38/15},$$

whenever $|E| \ll p^{8/15}$. Since $|[E, E, 0]| = |E|^2$, this completes the proof. □

5. Proof of Theorem 1.8

In the proof of Theorem 1.8, the following results will be used.

Lemma 5.1. *Let A be a multiplicative subgroup of $\mathbb{F}_p^*$ with $|A| \lesssim p^{1/2}$. Let L be a set of lines in $\mathbb{F}_p^2$, and $I(A \times A, L)$ be the number of incidences between $A \times A$ and L. We have*

$$I(A \times A, L) \lesssim |A|^{4/3}|L|^{2/3}.$$

Proof. Let $T(A)$ be the number of collinear triples of points in $A \times A$. It has been shown in [Macourt et al. 2018, Theorem 1.2] that if $|A| \lesssim p^{1/2}$, then we have

$$T(A) \lesssim |A|^4.$$

For any $l \in L$, let $i(l)$ be the number of points of $A \times A$ on l. We have

$$I(A \times A, L) = \sum_{l \in L} i(l) \leq |L|^{2/3} \left(\sum_{l \in L} i(l)^3 \right)^{1/3} \ll |L|^{2/3} T(A)^{1/3} \lesssim |A|^{4/3} |L|^{2/3},$$

where we used the Hölder inequality in the inequality step. □

The following theorem is given in [Murphy et al. 2017, Theorem 3].

Theorem 5.2. *Let A be a multiplicative subgroup of $\mathbb{F}_p^*$. Suppose that $|A| \leq p^{1/2}$, then we have*

$$\Lambda^+(A) \lesssim |A|^{49/20}.$$

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. We first repeat the first paragraph in the proof of Theorem 1.5.

Let S be the number of quadruples of matrices (m_1, m_2, m_3, m_4) in $[A, A, 0]^4$ such that $m_1 m_2 = m_3 m_4$. By the Cauchy–Schwarz inequality, we have

$$[|A, A, 0][A, A, 0]| \geq \frac{|A|^8}{S}.$$

Thus, to complete the proof, we only need to show that

$$S \lesssim |A|^4 + |A|^{169/40}.$$

Let X be the number of incidences between the point set $P = A \times A$ and the multiset L of lines of the form $y = d/(b - b')(x - c)$ with $b + d - b' \in A$. It is clear that $|P| = |A|^2$ and $|L| = \Lambda^+(A)|A|$. As in the proof of Theorem 2.2, we have $S \leq X + |A|^4$. Hence, it is enough to show that $|X| \lesssim |A|^{169/40}$.

For any line $l \in L$, let $m(l)$ be the multiplicity of l. By an elementary calculation,

$$\sum_{l \in \overline{L}} m(l)^2 \leq X|A|,$$

where $\overline{L}$ denotes the set of distinct lines in the multiset L. For $k \geq 1$, let L_k be the set of lines $l \in \overline{L}$ (without multiplicity) with $k \leq m(l) < 2k$. For any $k \geq 1$, we have

$$k|L_k| \leq |L| = \Lambda^+(A)|A|, \quad k^2|L_k| \leq \sum_{l \in \overline{L}} m(l)^2 \leq X|A|.$$

Namely, we obtain

$$|L_k| \leq \min\left\{ \frac{\Lambda^+(A)|A|}{k}, \frac{X|A|}{k^2} \right\}, \tag{12}$$

for every $k \geq 1$.

For any line $l \in L$, let $i(l)$ be the size of $l \cap P$. Using Lemma 5.1 and (12), we have

$$
\begin{aligned}
I(P, L) & \\
&= \sum_{l \in \bar{L}} m(l) i(l) < \sum_i \sum_{l \in \bar{L},\, 2^i \leq m(l) < 2^{i+1}} 2^{i+1} \cdot i(l) \\
&= \sum_i 2^{i+1} \cdot I(P, L_{2^i}) \\
&= \sum_{i, 2^{i+1} \leq X/(\Lambda^+(A))} 2^{i+1} \cdot I(P, L_{2^i}) + \sum_{i,\, 2^{i+1} > X/(\Lambda^+(A))} 2^{i+1} \cdot I(P, L_{2^i}) \\
&\lesssim \sum_{i,\, 2^{i+1} \leq X/(\Lambda^+(A))} 2^{i+1} \cdot |A|^{4/3} \left(\frac{\Lambda^+(A)|A|}{2^i} \right)^{2/3} \\
&\qquad\qquad + \sum_{i,\, 2^{i+1} > X/(\Lambda^+(A))} 2^{i+1} \cdot |A|^{4/3} \left(\frac{X|A|}{2^{2i}} \right)^{2/3} \\
&\lesssim \sum_{i,\ 2^{i+1} \leq X/(\Lambda^+(A))} (2^i)^{1/3} |A|^2 \Lambda^+(A)^{2/3} + \sum_{i,\ 2^{i+1} > X/(\Lambda^+(A))} (2^i)^{-1/3} X^{2/3} |A|^2 \\
&\lesssim \sum_i \left(\frac{X}{\Lambda^+(A)} \right)^{1/3} |A|^2 \Lambda^+(A)^{2/3} + \sum_i \left(\frac{X}{\Lambda^+(A)} \right)^{-1/3} |A|^2 X^{2/3} \\
&\lesssim X^{1/3} |A|^2 \Lambda^+(A)^{1/3},
\end{aligned}
$$

where we have used the fact that each line in L has multiplicity at most $|A|^2$, which implies that $2^i \leq |A|^2$, so i is at most $2 \log_2 |A|$.

Since $X = I(P, L)$, we have proved that

$$X \lesssim |A|^2 X^{1/3} \Lambda^+(A)^{1/3},$$

which implies that $X \lesssim |A|^3 \Lambda^+(A)^{1/2}$. Applying Theorem 5.2, we have $X \lesssim |A|^{169/40}$, whenever $|A| \lesssim p^{1/2}$. This completes the proof of the theorem. □

6. Proof of Theorem 1.9

The proof of Theorem 1.9 is quite similar compared to that of Theorem 1.8. More precisely, we will need the following point-line incidence bound over the complex numbers due to Tóth [2015].

Theorem 6.1 [Tóth 2015]. *Let P be a set of points in $\mathbb{C}^2$ and L be a set of lines in $\mathbb{C}^2$. The number of incidences between P and L, denoted by $I(P, L)$, satisfies*

$$I(P, L) \ll |P|^{2/3} |L|^{2/3} + |P| + |L|.$$

Corollary 6.2 [Tóth 2015]. *Let P be a set of points in $\mathbb{C}^2$. For any integer $t \geq 2$, the number of lines containing at least t points from P is bounded by*

$$O\left(\frac{|P|^2}{t^3} + \frac{|P|}{t}\right).$$

Using these results, we have the following corollary.

Corollary 6.3. *For a set A in $\mathbb{C}$, let $T(A)$ be the number of collinear triples of points in $A \times A$. Then we have*

$$T(A) \lesssim |A|^4.$$

Proof. Let L_k be the set of lines l such that $2^k \leq |l \cap (A \times A)| < 2^{k+1}$. Since $|l \cap (A \times A)| \leq |A|$ for any l, we have $k \leq \log_2 |A|$. Thus, using Corollary 6.2, we have

$$\begin{aligned} T(A) &\leq \sum_{k=0}^{\log_2 |A|} \sum_{l \in L_k} |l \cap (A \times A)|^3 \\ &\leq \sum_{k=0}^{\log_2 |A|} \left(\frac{|A|^4}{2^{3k}} + \frac{|A|^2}{2^k}\right) \cdot 2^{3k+3} \\ &\leq \sum_{k=0}^{\log_2 |A|} 8|A|^4 + \sum_{k=0}^{\log_2 |A|} 8|A|^2 2^{2k} \lesssim |A|^4, \end{aligned}$$

where we have used the fact that $2^k \leq |A|$. □

Lemma 6.4. *Let A be a set in $\mathbb{C}$ with $|A| \geq 2$. Denote by $\Lambda^\times(A)$ the number of quadruples $(a, b, c, d) \in A^4$ such that $ab = cd$. Then the number of tuples $(a, b, c, a', b', c') \in A^6$ such that*

$$a(b-c) = a'(b'-c')$$

is $\lesssim \Lambda^\times(A)^{1/2}|A|^3 + |A|^4 \leq 2\Lambda^\times(A)^{1/2}|A|^3$.

Proof. Since $|A| \geq 2$, without loss of generality, we assume that $0 \notin A$. We first have an observation that the number of desired tuples with $b = c$ or $b' = c'$ is at most $|A|^4 \leq \Lambda^\times(A)^{1/2}|A|^3$ since $\Lambda^\times(A) \geq |A|^2$.

Let M be the number of tuples with $b \neq c$ and $b' \neq c'$. We have M is equal to the number of desired tuples $(a, b, c, a', b', c') \in A^6$ such that

$$\frac{a}{a'} = \frac{b'-c'}{b-c}.$$

Using the Cauchy–Schwarz inequality, we have

$$M \leq \Lambda^\times(A)^{1/2} \cdot \left|\left\{(b_1, c_1, b_2, c_2, b_3, c_3, b_4, c_4) \in A^8 : \frac{b_1 - c_1}{b_2 - c_2} = \frac{b_3 - c_3}{b_4 - c_4}\right\}\right|^{1/2}.$$

Using the Cauchy–Schwarz inequality one more time, we have

$$\begin{aligned}\left|\left\{(b_1, c_1, b_2, c_2, b_3, c_3, b_4, c_4) \in A^8 : \frac{b_1-c_1}{b_2-c_2} = \frac{b_3-c_3}{b_4-c_4}\right\}\right| \\ \le |A|^2 \cdot \left|\left\{(b_1, c_1, b_2, c_2, d_1, d_2) \in A^6 : \frac{b_1-c_1}{b_2-c_2} = \frac{d_1-c_1}{d_2-c_2}\right\}\right| \\ \le |A|^2 \cdot T(A) \lesssim |A|^6,\end{aligned}$$

where we have used Corollary 6.3 in the last inequality. Hence, $M \lesssim \Lambda^\times(A)^{1/2}|A|^3$. This completes the proof of theorem. □

Proof of Theorem 1.9. Without loss of generality, we assume that $0 \notin A$. It has been proved in [Rudnev et al. 2020] that there exist $B, C \subset A$ such that $|B|, |C| \ge |A|/3$ and

$$\Lambda^+(B) \cdot \Lambda^\times(C) \lesssim |A|^{11/2}.$$

This implies that $\Lambda^+(B) \lesssim |A|^{11/4}$ or $\Lambda^\times(C) \lesssim |A|^{11/4}$. If $\Lambda^+(B) \lesssim |A|^{11/4}$ then we replace the set A in the Theorem 1.9 by B, otherwise, we replace the set A by C. Thus, we may assume that either $\Lambda^+(A) \lesssim |A|^{11/4}$ or $\Lambda^\times(A) \lesssim |A|^{11/4}$

The rest of proof of Theorem 1.9 is almost identical with that of Theorem 1.8, and the last step is to estimate X.

Using Theorem 6.1 and the same argument as in the proof of Theorem 1.8, we have

$$X \lesssim |A|^3 \Lambda^+(A)^{1/2}.$$

On the other hand, using Lemma 6.4, we have

$$X \lesssim |A|^3 \Lambda^\times(A)^{1/2}.$$

Since either $\Lambda^+(A) \lesssim |A|^{11/4}$ or $\Lambda^\times(A) \lesssim |A|^{11/4}$, we have

$$X \lesssim |A|^{3+11/8}.$$

Therefore, $|[A, A, 0][A, A, 0]| \gtrsim |A|^{5-11/8} = |A|^{29/8} = |[A, A, 0]|^{29/16}$. This completes the proof of the theorem. □

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ON THE EKELAND–HOFER SYMPLECTIC CAPACITIES OF THE REAL BIDISC

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In $\mathbb{C}^2$ with the standard symplectic structure we consider the bidisc $D^2 \times D^2$ constructed as the product of two open real discs of radius 1. We compute explicit values for the first, second and third Ekeland–Hofer symplectic capacity of $D^2 \times D^2$. We discuss some applications to questions of symplectic rigidity.

1. Introduction and main result

The first striking result about a nontrivial obstruction to the existence of a symplectic embedding was obtained by Gromov [1985]. He proved that one can symplectically embed a sphere into a cylinder only if the radius of the sphere is less than or equal to the radius of the cylinder. Since this celebrated nonsqueezing theorem appeared, many similar results of symplectic rigidity have been obtained for a variety of domains. For instance, McDuff [2009; 2011] studied symplectic embeddings of even-dimensional open ellipsoids into one another (see also [Hutchings 2011a; 2011b; McDuff and Schlenk 2012]), and Guth [2008] gave asymptotic results on when a complex polydisc can be symplectically embedded into another one. A useful tool to tackle these questions is given by global symplectic invariants for symplectic manifolds called capacities.

A symplectic capacity is a functor c that assigns to every symplectic manifold (M, ω) of dimension $2n$ a nonnegative (possibly infinite) number $c(M, \omega)$ that satisfies the following conditions:

- (monotonicity) If there exists a symplectic embedding of (M_1, ω_1) into (M_2, ω_2), then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$.
- (conformality) If $\lambda > 0$, then $c(M, \lambda\omega) = \lambda c(M, \omega)$.
- (local nontriviality) For the open unit ball $B \subset \mathbb{R}^{2n}$ we have $c(B, \omega_0) > 0$.
- (nontriviality) For the open cylinder $Z = \{z \in \mathbb{C}^n : |z_1| < 1\}$ we have $c(Z, \omega_0) < \infty$.

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Here ω_0 denotes the standard symplectic structure on $\mathbb{R}^{2n}$. The reader can consult any standard textbook in the subject such as [McDuff and Salamon 2017, Chapter 12] for an extensive treatment of this topic.

The first nontrivial capacity arose in Gromov's proof of the nonsqueezing theorem, and over the years many more such symplectic invariants have been constructed [Ekeland and Hofer 1989; 1990; Hofer and Zehnder 1990; Floer et al. 1990; Viterbo 1992; Gutt and Hutchings 2018].

In this paper we consider the construction of Ekeland and Hofer (which is recalled in Section 2C). For any subset of a symplectic vector space, they define an infinite sequence c_n of symplectic capacities. These quantities are notoriously difficult to compute explicitly, and precise values appear in the literature only for very special classes of domains, such as ellipsoids and polydiscs [Ekeland and Hofer 1990]. The main purpose of this work is to compute some of these capacities for the real bidisc $D^2 \times D^2$, which we now introduce.

In the complex space $\mathbb{C}^2$ with coordinates $z_j = x_j + iy_j$, $j = 1, 2$, endowed with the standard symplectic structure $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, consider the real bidisc

$$D^2 \times D^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid x_1^2 + x_2^2 < 1,\ y_1^2 + y_2^2 < 1\}. \tag{1-1}$$

Our main result is the following (see Theorems 4.2, 4.4 and 4.6).

Main Theorem. *For the unit real bidisc $D^2 \times D^2$ we have*

$$c_1(D^2 \times D^2) = 4, \quad c_2(D^2 \times D^2) = 3\sqrt{3}, \quad c_3(D^2 \times D^2) = 8.$$

The fact that $c_1(D^2 \times D^2) = 4$ is known [Artstein-Avidan and Ostrover 2014]. The referee pointed out to us that the values of $c_2(D^2 \times D^2)$ and $c_3(D^2 \times D^2)$ can also be obtained from the work of Ramos [2017], which uses very different techniques than the ones developed in this paper (see Remarks 4.5 and 4.9).

The Ekeland–Hofer capacities are closely related to the existence of closed Hamiltonian orbits. The key feature of our computation is the use of Hamiltonians modeled on the gauge function of the domain, rather than Hamiltonians that are quadratic at infinity, as in the original definition of Ekeland and Hofer. This same idea already appears in [Berestycki et al. 1985] in the context of smooth domains, and in [Fan 1992] for the case of Lipschitz domains. The different choice of Hamiltonians does not affect the computation of the capacities, provided that no periodic orbits of period 1 are introduced at infinity. This feature is easily achieved by rescaling, since the closed periodic orbits of the Hamiltonians in question can be computed explicitly. The advantage in modeling the Hamiltonians on the gauge function is that we can then exploit the symmetries of the domain to simplify the estimates involved in the computations of the capacities.

We remark that our strategy can be easily adapted to compute the Ekeland–Hofer capacities of other domains. For instance, for the product of two real spheres in $\mathbb{C}^3$

one just needs to repeat the same computations while taking an additional coordinate into account.

We now point out how the results of this paper are related to recent work of Gutt and Hutchings [2018]. They use positive S^1-equivariant symplectic homology to introduce a sequence c_k of symplectic capacities for star-shaped domains in $\mathbb{R}^{2n}$. Their capacities c_k share many properties with the Ekeland–Hofer capacities, and in particular the remarkable "product property" (see Section 5). Combining the results of [Gutt and Hutchings 2018] with Ramos' insight [2017] that the real bidisc is a toric domain, one can see that the Gutt–Hutchings capacities of $D^2 \times D^2$ are the same as the Ekeland–Hofer capacities that we obtain in our Main Theorem. Our computations therefore support the conjecture made by Gutt and Hutchings [2018] that their capacities c_k are always equal to the Ekeland–Hofer capacities.

The organization of this paper is as follows. In Section 2 we recall some background material and set the notation. In Section 3 we show how one can approximate the bidisc $D^2 \times D^2$ with a sequence of smooth convex domains. The main result is then proved in Section 4, while in Section 5 we present some applications to questions of symplectic embedding. It is the authors' hope that further applications to symplectic rigidity of these explicit values of the capacities will be found in the future.

2. Background

2A. ***Basic definitions and notation.*** Let $\mathbb{C}^n$ denote the standard complex vector space of dimension n with variables $z_j = x_j + iy_j$. We endow $\mathbb{C}^n$ with the Euclidean scalar product

$$\langle z, w\rangle := \operatorname{Re}\left(\sum_{j=1}^{n} z_j \overline{w}_j\right)$$

and the standard symplectic form

$$\omega := \frac{1}{2i}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

All symplectic embeddings considered in this paper will be with respect to the standard symplectic form.

Let Ω be a bounded subset of $\mathbb{C}^n$ with smooth boundary $\partial\Omega$. We denote by $T\mathbb{C}^n$ the (real) tangent bundle of $\mathbb{C}^n$ and by $T\partial\Omega$ the tangent bundle of $\partial\Omega$. We write $Y \in T\mathbb{C}^n$ to mean that Y is a local section of $T\mathbb{C}^n$. Consider a smooth defining equation ρ of $\partial\Omega$ such that $|d\rho(z)| \neq 0$ for $z \in \partial\Omega$. The *characteristic vector field* of $\partial\Omega$ is the unique vector field X such that

$$\omega(X, Y) = Y(\rho) \qquad \text{for all } Y \in T\mathbb{C}^n.$$

The characteristic vector field X is tangent to $\partial\Omega$ and its restriction to $\partial\Omega$ does not depend on the choice of ρ. Moreover, X generates the kernel of the restricted form $\omega|_{T\partial\Omega}$.

Let J denote the standard complex structure on $\mathbb{C}^n$. For all $z \in \partial\Omega$ we have $X(z) = J\nabla\rho(z)$, where $\nabla\rho$ is the Euclidean gradient of ρ. Hence an integral curve of X on $\partial\Omega$ is the solution of a system

$$\begin{cases} \dot{z} = J\nabla\rho(z), \\ z(0) = z_0, \end{cases} \tag{2-1}$$

where $z_0 \in \partial\Omega$. These integral curves are called the *characteristics* of $\partial\Omega$.

Of particular interest in symplectic geometry is the study of closed characteristics, that is, the solutions to (2-1) for which there exists a time $t_0 > 0$ such that $z(t_0) = z_0$. Let T be the smallest such t_0. The image $\{z(t),\ 0 \le t \le T\}$ is called an *orbit* and T the *period* of the orbit.

If $\gamma : [0, T] \to \partial\Omega$ is a closed characteristic, the *action* of γ is defined to be

$$\mathcal{A}(\gamma) := -\frac{1}{2}\int_0^T \langle J\dot{\gamma}, \gamma\rangle \, dt.$$

If Ω is a bounded convex subset of $\mathbb{C}^n$ and $\{\gamma_i\}_{i\in I}$ is the set of closed characteristics of $\partial\Omega$, we can define the *action spectrum* of Ω as the set

$$\Sigma(\Omega) := \{|k\mathcal{A}(\gamma_i)|, k \in \mathbb{N}, i \in I\}.$$

Following Ekeland and Hofer [1990], we will see how it is possible to choose some elements of $\Sigma(\Omega)$ called capacities that are symplectic invariants (see Section 2C). We now describe how to adapt the concepts introduced above to nonsmooth domains.

Let $\Omega \subset \mathbb{C}^n$ be a convex bounded domain and let $p \in \partial\Omega$. We say that a unit vector $n(p)$ is *normal* at p for $\partial\Omega$ if

$$\langle n(p), x - p\rangle \le 0 \quad \text{for all } x \in \Omega. \tag{2-2}$$

If p is a smooth boundary point, then $n(p)$ is the usual exterior normal vector. If $\partial\Omega$ is not smooth at p, then there could be more than one choice for a normal vector. In this case, we let $n(p)$ denote the set of all vectors satisfying (2-2).

Definition 2.1. Let Ω be a convex domain in $\mathbb{C}^n$. We say that $z : \mathbb{R} \to \Omega$ is a *characteristic* if $z(t)$ has right and left derivative $\dot{z}^{\pm}(t)$ for all t, and

$$\dot{z}^{\pm}(t) \in Jn(z(t)).$$

Note that, for Ω smooth, this definition coincides with the definition of a characteristic given before.

2B. ***The action spectrum of the real bidisc.*** We now turn our attention to the domain that is the main object of interest of this paper: the real bidisc $D^2 \times D^2$. Recall that $D^2 \times D^2$ was defined in (1-1) as the product of two open real discs of radius 1. The next proposition, which follows from the work of Artstein-Avidan and Ostrover [2014], describes the closed characteristics of $D^2 \times D^2$ and its action spectrum $\Sigma(D^2 \times D^2)$.

Proposition 2.2. *The unitary real bidisc $D^2 \times D^2$ has infinitely many closed characteristics. The action spectrum is given by the set*

$$\Sigma(D^2 \times D^2) = \{2n\cos(\theta_{k,n}) \mid k, n \in \mathbb{N}, \theta_{k,n} \in J_n\} \cup \{2n\pi \mid n \in \mathbb{N}\}, \tag{2-3}$$

where $J_n = \{(2k-1)\pi/2n, 1 \leq k \leq (n-1)/2\}$ if n is odd, and, if n is even, $J_n = \{k\pi/n,\ 0 \leq k \leq n/2 - 1\}$.

Remark 2.3. The elements of $\Sigma(D^2 \times D^2)$ are precisely the lengths of all closed billiard orbits in a circle of radius 1.

Remark 2.4. Note that $\min \Sigma(D^2 \times D^2) = 4$. Moreover, the second smallest element in $\Sigma(D^2 \times D^2)$ is $3\sqrt{3}$.

2C. ***Ekeland–Hofer symplectic capacities.*** Following [Ekeland and Hofer 1990], we recall the definition of these symplectic capacities. We first set up the functional analytical framework. For more details, see also [Ekeland and Hofer 1989, Section II].

Let E be the Hilbert space of all functions $f \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{C}^n)$ such that the Fourier series

$$f(t) = \sum_{k\in\mathbb{Z}} f_k e^{2k\pi i t}, \qquad f_k \in \mathbb{C}^n,$$

satisfies

$$\sum_{k\in\mathbb{Z}} |k||f_k|^2 < \infty.$$

The inner product in E is defined by

$$(f, g) := \langle f_0, g_0\rangle + 2\pi \sum_{k\in\mathbb{Z}} |k|\langle f_k, g_k\rangle.$$

E is the most natural space on which the action functional $\mathcal{A}$ can be defined. It is easy to see that

$$\mathcal{A}(f) = \pi \sum_{k=0}^{+\infty} k(|f_k|^2 - |f_{-k}|^2).$$

Note that there is a natural action $T : \mathbb{S}^1 \to \mathrm{Aut}(E)$ of $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ on E given by the phase shift

$$T_{e^{2\pi i\theta}} f(t) := f(t+\theta).$$

The space E has a natural orthogonal splitting, compatible with the phase shift action, given by

$$E = E^- \oplus E^0 \oplus E^+.$$

Here we have defined

$$\begin{aligned} E^- &= \{f \in E \mid f_k = 0 \text{ for } k \geq 0\}, \\ E^0 &= \{f \in E \mid f_k = 0 \text{ for } k \neq 0\} = \mathbb{C}^n, \\ E^+ &= \{f \in E \mid f_k = 0 \text{ for } k \leq 0\}. \end{aligned}$$

We denote by P^+, P^0 and P^- the corresponding orthogonal projections.

We now need to introduce the notion of the index of a subspace. Let X be a Hilbert space over $\mathbb{C}$, and let $T : S^1 \to \operatorname{Aut}(X)$ be a representation of S^1 on the vector space

$$\operatorname{Aut}(X) := \{f : X \to X \mid f \text{ is a linear isometry}\}.$$

A subset $A \subseteq X$ is called *invariant* if $T(\theta)(A) = A$ for all $\theta \in S^1$. Let Y be another Hilbert space, and $R : S^1 \to \operatorname{Aut}(Y)$ a representation of S^1 on $\operatorname{Aut}(Y)$. A linear map $f : X \to Y$ is called *equivariant* if $f \circ T(\theta) = R(\theta) \circ f$ for all $\theta \in S^1$. Let $A \subseteq X$ be an invariant subset. For every $k \in \mathbb{N}$, we let $\mathcal{F}(A, k)$ denote the collection of functions $f : A \to \mathbb{C}^k \setminus \{0\}$ such that

- f is continuous,
- there exists a positive integer n such that $f(T(\theta)(x)) = e^{2\pi n i \theta} f(x)$ for all $\theta \in S^1$ and for all $x \in A$.

We define the *index* of A as the quantity

$$\alpha(A) := \min\{k \in \mathbb{N} \mid \mathcal{F}(A, k) \neq \varnothing\}. \tag{2-4}$$

If $\mathcal{F}(A, k) = \varnothing$ for every $k \in \mathbb{N}$, we set $\alpha(A) = +\infty$. Moreover, we set $\alpha(\varnothing) = 0$. Observe that if F is the set of fixed points of X for T, that is,

$$F := \{x \in X \mid T(\theta)(x) = x \ \text{ for all } \theta \in S^1\},$$

then $A \cap F \neq \varnothing$ implies $\alpha(A) = \infty$.

In the following paragraph we describe a pseudoindex theory in the sense of Benci relative to E^+. In our exposition we mainly follow [Benci 1982] (see also [Ekeland and Hofer 1990]).

Consider the group of homeomorphisms

$$\Gamma := \{h : E \to E \mid h = e^{\gamma^+} P^+ + e^{\gamma^-} P^- + P^0 + K\} \tag{2-5}$$

such that the following conditions are satisfied:

- K is a compact equivariant operator.

- $\gamma^+, \gamma^- : E \to \mathbb{R}^+$ map bounded sets in precompact sets and are invariant.
- There exists a constant $c > 0$ such that $\mathcal{A}(x) \leq 0$ or $\|x\| > c$ implies that $\gamma^+(x) = \gamma^-(x) = 0$ and $K(x) = 0$.

Let $S := \{x \in E \mid \|x\|_E = 1\}$ and let $\xi \subseteq E$ be an invariant subset of E. We define the *pseudoindex* of ξ as

$$\operatorname{ind}(\xi) := \inf\{\alpha(h(\xi) \cap S \cap E^+) \mid h \in \Gamma\}.$$

For the basic properties of the pseudoindex we refer to [Ekeland and Hofer 1990]. In particular, the following result is often useful.

Proposition 2.5 [Ekeland and Hofer 1990, Proposition 1]. *If $V_k \subseteq E^+$ is a finite-dimensional invariant subspace of E^+ of complex dimension k*, $\operatorname{ind}(V_k \oplus E^0 \oplus E^-) = k$.

We need to introduce one last concept before defining the symplectic capacities. We call a smooth function $H : \mathbb{C}^n \to (0, +\infty)$ an *admissible Hamiltonian* for a bounded domain $\Omega \subset \mathbb{C}^n$ if

- H is 0 on some open neighborhood of $\overline{\Omega}$,
- $H(z) = c|z|^2$ for $|z|$ large enough, where $c > \pi$ and $c \notin \mathbb{Z}\pi$.

We denote by $\mathcal{H}(\Omega)$ the set of the admissible Hamiltonians for Ω. For j a positive integer and $H \in \mathcal{H}(\Omega)$, we define a number $c_{H,j} \in (0, +\infty) \cup \{\infty\}$ by

$$c_{H,j} := \inf\{\sup \mathcal{A}_H(\xi) \mid \xi \subset E \text{ is } S^1\text{-invariant and } \operatorname{ind}(\xi) \geq j\}.$$

Here $\mathcal{A}_H : E \to \mathbb{R}$ is the action functional associated to a Hamiltonian H defined by

$$\mathcal{A}_H(f) := \mathcal{A}(f) - \int_0^1 H(f(t))\, dt.$$

Every number $c_{H,j}$ is nonnegative and, if finite, is a critical value of $\mathcal{A}_H$ [Ekeland and Hofer 1990, page 559].

We can now define the j-th *Ekeland–Hofer symplectic capacity* of Ω as

$$c_j(\Omega) := \inf_{H \in \mathcal{H}(\Omega)} c_{H,j}.$$

Remark 2.6. In our computation of the Ekeland–Hofer symplectic capacities, we will not use Hamiltonians with quadratic behavior at infinity, as in the definition. We will use instead Hamiltonians of the form $H(z) = f(r(z))$, where r is the gauge function of the domain, and f is linear at infinity (see [Berestycki et al. 1985; Fan 1992]). This choice does not affect the values obtained in the computation of the capacities, as one can easily prove by combining the following two facts:

- For Hamiltonians H_1 and H_2 we have

$$H_1 \leq H_2 \Longrightarrow \mathcal{A}_{H_1} \geq \mathcal{A}_{H_2} \Longrightarrow c_{H_1,k} \leq c_{H_2,k}.$$

- For every f linear at infinity there exists $H \in \mathcal{H}(\Omega)$ such that $f(r) \leq H$. Similarly, for any $H \in \mathcal{H}(\Omega)$ there exists f such that $H \leq f(r)$.

Observe also that if we choose f so that the Hamiltonian $f(r)$ has no periodic solutions of period 1 at infinity, then $\mathcal{A}_{f(r)}$ satisfies the Palais–Smale condition. This implies that $c_{f(r),k}$, if it is finite, is a critical value of $\mathcal{A}_{f(r)}$ (see [Aebischer et al. 1994, page 71]).

The next theorem, from [Hofer and Zehnder 1990], characterizes the first symplectic capacity as the infimum of the action spectrum.

Theorem 2.7. *Let Ω be a smoothly bounded convex domain in $\mathbb{C}^n$ and let $\alpha = \min \Sigma(\Omega)$. Then $c_1(\Omega) = \alpha$.*

Note that we cannot apply Theorem 2.7 directly to the real bidisc, since its boundary $\partial(D^2 \times D^2)$ is not smooth. In the next section we show how to overcome this difficulty by appropriately approximating $D^2 \times D^2$ with smooth domains.

3. Approximation with smooth domains

We start by constructing a decreasing sequence of smooth convex domains $\mathcal{D}_n$ converging to $D^2 \times D^2$. Let $g : \mathbb{R} \to [0, +\infty)$ be a convex, increasing function such that $g(1) = 1$ and $g(s) = 0$ for $s < 0$. Consider the following subsets of $\mathbb{C}^2$:

$$\mathcal{D}_n := \{z \in \mathbb{C}^2 \mid g(n(x_1^2 + x_2^2 - 1)) + g(n(y_1^2 + y_2^2 - 1)) \leq 1\}. \tag{3-1}$$

The domains defined in (3-1) are smooth and convex. Moreover, they satisfy the following properties:

- $\mathcal{D}_n \supset \mathcal{D}_{n+1}$ for all $n \in \mathbb{N}$.
- $\bigcap_{n=1}^{+\infty} \mathcal{D}_n = D^2 \times D^2$.
- For all $n \in \mathbb{N}$, we have
$$\frac{1}{\sqrt{1+\frac{1}{n}}}\mathcal{D}_n \subset D^2 \times D^2.$$

In particular, by the properties of the capacities, for any choice of positive integers k and n, the following double inequality holds:

$$\frac{1}{1+\frac{1}{n}} c_k(\mathcal{D}_n) \leq c_k(D^2 \times D^2) \leq c_k(\mathcal{D}_n).$$

Proposition 3.2 shows how the closed characteristics of the real bidisc $D^2 \times D^2$ are approximated by the closed characteristics of the approximating domains $\mathcal{D}_n$. Before stating the result, it is convenient to give the following definition.

Definition 3.1. For all $M > 0$ and $\varepsilon > 0$ we define

$$\Sigma_{M,\varepsilon}(\mathcal{D}_n) := (\Sigma(\mathcal{D}_n) \cap [0, M]) \setminus \bigcup_{k=1}^{\infty} [2k\pi - \varepsilon, 2k\pi + \varepsilon].$$

Here $\Sigma(\mathcal{D}_n)$ denotes the action spectrum of $\mathcal{D}_n$.

Proposition 3.2. *Let $\alpha \in \Sigma(D^2 \times D^2)$ and suppose that $\alpha = \mathcal{A}(\gamma)$, where γ is a nongliding closed characteristic of $D^2 \times D^2$. Then there exists a sequence γ_n, where each γ_n is a closed characteristic of $\mathcal{D}_n$, such that γ_n converges to γ and $\mathcal{A}(\gamma_n)$ converges to α. In particular, for all $M > 0$ and $\varepsilon > 0$, we have*

$$d(\Sigma_{M,\varepsilon}(\mathcal{D}_n), \Sigma_{M,\varepsilon}(D^2 \times D^2)) \to 0 \quad \textit{for } n \to +\infty,$$

where $d(\cdot\,,\cdot)$ is the Hausdorff distance between sets.

Proof. Note that, for every n, we can decompose the boundary $\partial\mathcal{D}_n$ of $\mathcal{D}_n$ into three components

$$\begin{aligned} X_n^3 &:= \left\{(z_1, z_2) \in \mathbb{C}^2,\ x_1^2 + x_2^2 - 1 < 0,\, y_1^2 + y_2^2 = 1 + \frac{1}{n}\right\} \\ Y_n^3 &:= \left\{(z_1, z_2) \in \mathbb{C}^2,\ y_1^2 + y_2^2 - 1 < 0,\, x_1^2 + x_2^2 = 1 + \frac{1}{n}\right\} \end{aligned} \tag{3-2}$$

and

$$T_n^3 := \partial\mathcal{D}_n \setminus (X_n^3 \cup Y_n^3). \tag{3-3}$$

To find the closed characteristics of $\mathcal{D}_n$, we consider the system of differential equations

$$\begin{cases} \dot{x} = -g'(n(|y|^2 - 1))2ny, \\ \dot{y} = g'(n(|x|^2 - 1))2nx, \end{cases} \tag{3-4}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$. We first note that $\det(x, y) = x_1y_2 - x_2y_1$ is constant along the solutions of (3-4). It is convenient to use polar coordinates in the planes defined by the variables x and y, respectively. We recall the notation already introduced in the proof of Proposition 2.2:

$$\begin{aligned} r_1e^{i\varphi_1} &:= r_1(\cos\varphi_1, \sin\varphi_1) = (x_1, x_2), \\ r_2e^{i\varphi_2} &:= r_2(\cos\varphi_2, \sin\varphi_2) = (y_1, y_2). \end{aligned}$$

We can now rewrite the system (3-4) as

$$\begin{cases} (\dot{r_1} + ir_1\dot{\varphi_1})e^{i\varphi_1} = -g'(n(r_2^2 - 1))2nr_2e^{i\varphi_2}, \\ (\dot{r_2} + ir_2\dot{\varphi_2})e^{i\varphi_2} = g'(n(r_1^2 - 1))2nr_1e^{i\varphi_1}, \end{cases} \tag{3-5}$$

from which we obtain the two systems

$$\begin{cases} r_1\dot{\varphi}_1 = g'(n(r_2^2-1))2nr_2\sin(\varphi_1-\varphi_2), \\ r_2\dot{\varphi}_2 = g'(n(r_1^2-1))2nr_2\sin(\varphi_1-\varphi_2), \\ r_1r_2\sin(\varphi_1-\varphi_2) = \det(x,y) = \text{const} \end{cases} \tag{3-6}$$

and

$$\begin{cases} r_1\dot{r}_1 = -g'(n(r_2^2-1))2nr_3, \\ r_2\dot{r}_2 = g'(n(r_1^2-1)2nr_3, \\ \dot{r}_3 = 2n(g'(n(r_1^2-1))r_1^2 - g'(n(r_2^2-1))r_2^2). \end{cases} \tag{3-7}$$

Here $r_3 := x\cdot y = x_1y_1 + x_2y_2 = r_1r_2\cos(\varphi_1-\varphi_2)$.

Clearly the characteristics in $\mathcal{D}_n$ are straight segments when they lie on X_n^3 or Y_n^3. We now want to understand the behavior of the characteristics at their intersections with T_n^3. Let us consider the Cauchy problem (3-4) with initial data $x(0) = (1,0)$ and $y(0) = \sqrt{1+1/n}(\cos\theta_0, \sin\theta_0)$ for $\theta_0 \in (0,\pi/2)$. The corresponding solution of (3-4) enters X_n^3 (this can be seen by inspection of (3-7), since $r_3(0) > 0$ implies that r_1 is decreasing and r_2 is constant). Reasoning as in Proposition 2.2, we see that the solution reaches the point

$$(x_1,x_2) = (\cos(\pi+2\theta_0), \sin(\pi+2\theta_0)), \quad (y_1,y_2) = \sqrt{1+\tfrac{1}{n}}(\cos\theta_0, \sin\theta_0).$$

The solution then enters T_n^3 at a time t_0. From (3-7) we see that $r_3(t_0) < 0$, r_1 increases and r_2 decreases. Let $T > 0$ be the smallest positive real number such that $\dot{r}_3(t_0+T) = 0$. By symmetry, we see that $r_1(t) = r_2(2T+2t_0-t)$, $r_2(t) = r_1(2T+2t_0-t)$ and $r_3(t) = r_3(2T+2t_0-t)$. In particular, this tells us that the solution eventually leaves T_n^3. By the third equation in (3-6), the angle between $x(t_0+2T)$ and $y(t_0+2T)$ is the same as the angle between $x(t_0)$ and $y(t_0)$ but $\varphi_1(t_0) \neq \varphi_1(t_0+2T)$. We want to compute $\Delta\varphi := \varphi_1(t_0+2T) - \varphi_1(t_0) = \varphi_2(t_0+2T) - \varphi_2(t_0)$. By (3-6),

$$\begin{aligned} \Delta\varphi &= \int_0^{2T} \frac{g'(n(r_2^2-1))2nr_2}{r_1} 2\sin(\varphi_1-\varphi_2)\,dt \\ &= \int_0^{2T} \frac{g'(n(r_2^2-1))2n}{r_1^2}\sqrt{1+\tfrac{1}{n}}\sin\theta_0\,dt. \end{aligned} \tag{3-8}$$

Since $r_3 = x\cdot y = r_1r_2\cos(\varphi_1-\varphi_2)$, (3-7) implies

$$\begin{aligned} (\dot{r_1^2}) &= -g'(n(r_2^2-1))2nr_1r_2\cos(\varphi_1-\varphi_2) \\ &= -g'(n(r_2^2-1))2n\sqrt{r_1^2r_2^2 - \left(1+\tfrac{1}{n}\right)\sin^2\theta_0}. \end{aligned} \tag{3-9}$$

Using (3-9) inside (3-8) we obtain

$$\Delta\varphi = -\int_1^{1+\frac{1}{n}} \frac{\sqrt{1+\frac{1}{n}}\sin(\theta_0)}{r_1^2\sqrt{r_1^2r_2^2-\left(1+\frac{1}{n}\right)\sin^2(\theta_0)}}\,d(r_1^2). \tag{3-10}$$

It follows from

$$g(n(r_1^2-1))+g(n(r_2^2-1))=1$$

that

$$r_2^2 = 1+\frac{g^{-1}(1-g(n(r_1^2-1)))}{n}. \tag{3-11}$$

Plugging (3-11) into (3-10) we obtain

$$\Delta\varphi = -\int_1^{1+\frac{1}{n}} \frac{\sqrt{1+\frac{1}{n}}\sin\theta_0}{u\sqrt{u(1+g^{-1}(1-g(n(u-1)))/n)-(1+\frac{1}{n})\sin^2\theta_0}}\,du. \tag{3-12}$$

From (3-12) we see that $\Delta\varphi$ is small for $\theta_0 < \pi/2$ and n large. Following the same reasoning as in Proposition 2.2 we can see that after $2m$ straight sides the characteristic hits the point

$$P_{2m} = \Big((-1)^m e^{i(2m(\theta+\Delta\varphi))}, (-1)^m\sqrt{1+\tfrac{1}{n}}e^{i((2m+1)\theta+2m\Delta\varphi)}\Big).$$

The characteristic is closed if and only if

$$\begin{cases}\theta+\Delta\varphi = \frac{k\pi}{m} \text{ for } k=0,\dots,\frac{m}{2}-1 & \text{if } m \text{ is even,}\\ \theta+\Delta\varphi = \frac{2k-1}{2m}\pi \text{ for } k=1,\dots,\frac{m-1}{2} & \text{if } m \text{ is odd.}\end{cases} \tag{3-13}$$

Since $\Delta\varphi$ depends continuously on θ_0 and is small if n is large, then the equations in (3-13) are solvable. In particular, if $\theta_{k,m} \in J_m$ (see Proposition 2.2), for n big enough there exists θ close to $\theta_{k,m}$ such that (3-13) is satisfied. The corresponding characteristics of $\mathcal{D}_n$ and $D^2\times D^2$ are close to each other and their actions are also close. Note that there could also be some characteristics that are entirely contained in T_n^3. These characteristics are left out by the description above. If n is large, they are close to the gliding trajectories of the bidisc $D^2\times D^2$ and their actions are close to a multiple of 2π. □

4. The Ekeland–Hofer symplectic capacities of the real bidisc

In this section we compute the symplectic capacities of the real bidisc $D^2\times D^2$. The following result will be the main tool for our computations.

Proposition 4.1. *Let Ω be a smooth convex domain in $\mathbb{C}^n$ containing 0, and let r be its gauge function. For all $c > 0$, let $\Psi_c : E \to \mathbb{R}$ be the functional*

$$\Psi_c(\zeta) := \mathcal{A}(\zeta) - c\int_0^1 r(\zeta(t))\,dt. \tag{4-1}$$

If $W \subset E$ is an invariant subset of pseudoindex at least k such that $\Psi_c|_W \le 0$, then $c_k(\Omega) \le c$.

Proof. Let $\varepsilon > 0$ and choose a smooth function $f_\varepsilon : [0,\infty) \to [0,\infty)$ such that

$$f_\varepsilon(s) = \begin{cases} 0 & \text{if } s \le 1, \\ (c+\varepsilon)s & \text{if } s \text{ is large.} \end{cases} \tag{4-2}$$

Moreover, we require that $0 \le f'_\varepsilon(s) \le c + 2\varepsilon$ for all s, and that $f'_\varepsilon(s) \in \Sigma(\Omega)$ only for finitely many numbers $s_1, \dots, s_m$, which we can assume to be arbitrarily close to 1. Let $f'_\varepsilon(s_j) = \alpha_j \in \Sigma(\Omega)$ and define $H_{f_\varepsilon}(z) := f_\varepsilon(r(z))$. The periodic orbits of $\dot{z} = J\nabla H_{f_\varepsilon}(z)$ of period 1 are obtained by scaling. Namely, they are the curves $\sqrt{s_j}\gamma_j(\alpha_j t)$, where γ_j is the closed characteristic in $\partial\Omega$ such that $\mathcal{A}(\gamma_j) = \alpha_j$. The corresponding critical values of $\mathcal{A}_{H_{f_\varepsilon}}$ are

$$\mathcal{A}_{H_{f_\varepsilon}}(\sqrt{s_j}\gamma_j(\alpha_j t)) = s_j\alpha_j - f_\varepsilon(s_j). \tag{4-3}$$

As $\varepsilon \to 0$, these critical values tend to α_j, and each of them is less than c. Note that

$$\begin{aligned} \mathcal{A}_{H_{f_\varepsilon}}(\zeta) &= \Psi_c(\zeta) + \int_0^1 [cr(\zeta(t)) - f_\varepsilon(r(\zeta(t)))]\,dt \\ &\le \Psi_c(\zeta) + \int_0^1 [C - \varepsilon r(\zeta(t))]\,dt \le \Psi_c(\zeta) + D \end{aligned}$$

for some constants C and D. Recalling that $\Psi|_W \le 0$, then

$$\mathcal{A}_{H_{f_\varepsilon}}|_W \le C \tag{4-4}$$

for some new constant C. Equation (4-4) implies $c_{H_{f_\varepsilon},k} < \infty$. Since there are no periodic orbits at infinity of period 1, then $\mathcal{A}_{H_{f_\varepsilon}}$ satisfies the Palais–Smale condition and therefore $c_{H_{f_\varepsilon},k}$ is a critical value of $\mathcal{A}_{H_{f_\varepsilon}}$. Hence $c_{H_{f_\varepsilon},k} \le c$ by (4-3). Since $c_k(\Omega) \le c_{H_{f_\varepsilon},k}$, the conclusion follows. □

In the next theorem we compute the first Ekeland–Hofer capacity of the real bidisc. A different proof of the same result appears in [Artstein-Avidan and Ostrover 2014].

Theorem 4.2. *For the unit real bidisc $D^2 \times D^2$ we have $c_1(D^2 \times D^2) = 4$.*

Proof. Recall that $4 = \min \Sigma(D^2 \times D^2)$ (Remark 2.4). Proposition 3.2 then implies the existence of a sequence of closed characteristics $\gamma_n \subset \partial\mathcal{D}_n$ with $\alpha_n := \mathcal{A}(\gamma_n) \to 4$.

By Theorem 2.7 we have

$$\lim_{n\to+\infty} c_1(\mathcal{D}_n) = 4. \tag{4-5}$$

Now observe, from the construction of the $\mathcal{D}_n$, that

$$\frac{1}{\sqrt{1+\frac{1}{n}}}\mathcal{D}_n \subset D^2 \times D^2 \subset \mathcal{D}_n.$$

The basic properties of the capacities then yield the double inequality

$$\frac{1}{1+\frac{1}{n}} c_1(\mathcal{D}_n) \le c_1(D^2 \times D^2) \le c_1(\mathcal{D}_n),$$

from which we can conclude that $c_1(D^2 \times D^2) = 4$ by applying (4-5). □

For the next theorem we need the following simple lemma.

Lemma 4.3. *Let $f(t) = \sum_{k\in\mathbb{Z}} f_k e^{2k\pi i t}$ be an L^1-convergent series. Then, for every integer n,*

$$\int_0^1 |f(t)|\, dt \ge |f_n|.$$

Proof. For every n we have

$$\begin{aligned}\int_0^1 \Bigl|\sum_{k\in\mathbb{Z}} f_k e^{2k\pi i t}\Bigr|\, dt &= \int_0^1 \Bigl| e^{2n\pi i t} \sum_{k\in\mathbb{Z}} f_k e^{2(k-n)\pi i t}\Bigr| dt \\ &\ge \Bigl|\int_0^1 \sum_{k\in\mathbb{Z}} f_{k+n} e^{2k\pi i t}\, dt\Bigr| = |f_n|.\end{aligned}$$ □

Theorem 4.4. *For the real bidisc $D^2 \times D^2$ we have $c_2(D^2 \times D^2) = 3\sqrt{3}$.*

Proof. Let W be the following invariant subset of E:

$$W := \left\{ (\alpha,\beta) e^{2\pi i t} + \gamma([\alpha:\beta]) \left(\bar{\alpha}\frac{\alpha^3}{|\alpha|^3}, \bar{\beta}\frac{\alpha^3}{|\alpha|^3} \right) e^{4\pi i t} \,\middle|\, \alpha,\beta\in\mathbb{C} \right\} \oplus E^0 \oplus E^-,$$

where $\gamma : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{R}$ is a nonnegative continuous function which is nonzero only in a neighborhood of the two points $[1:i]$ and $[1:-i]$. We will specify later how γ is chosen. We now prove that W has pseudoindex 2. First note that

$$W\cap S\cap E^+ = \left\{ (\alpha,\beta) e^{2\pi i t} + \gamma([\alpha:\beta]) \left(\bar{\alpha}\frac{\alpha^3}{|\alpha|^3}, \bar{\beta}\frac{\alpha^3}{|\alpha|^3} \right) e^{4\pi i t} \,\middle|\, \alpha,\beta\in\mathbb{C},\, (|\alpha|^2+|\beta|^2)(1+\gamma^2)=1 \right\},$$

and therefore $W\cap S\cap E^+$ has index 2. Assume now by contradiction that there exists $h\in\Gamma$ such that $F := h(W)\cap E^+\cap S$ has index strictly smaller than 2. Then, by the properties of the index, there exists an open neighborhood of U of F in E such that $\alpha(U)=\alpha(F)$. Let $E_k=\{f\in E \mid f_j=0 \text{ for } |j|>k\}$ and denote by $Q_k : E\to E_k$ the corresponding orthogonal projection. We claim, for k large, that

$$Q_k(h(W\cap E_k))\cap S\cap E^+\subset U. \tag{4-6}$$

Assume by contradiction that (4-6) is false. Then there exists a sequence of functions $f_k\in E_k\cap W$ such that $Q_k(h(f_k))\in E^+\cap S$ and $Q_k(h(f_k))\notin U$. Recalling the structure of the homeomorphism h (see (2-5)), we have

$$\begin{cases} f_k^0+e^{\gamma^-(f_k)}f_k^-+(P^-+P^0)Q_kK(f_k)=0,\\ \|e^{\gamma^+(f_k)}f_k^++P^+Q_kK(f_k)\|=1. \end{cases} \tag{4-7}$$

Note that f_k must be a bounded sequence, otherwise we have $K(f_k)=0$, which together with (4-7) implies $\|f_k^+\|=1$ and $f_k^0=f_k^-=0$, thus giving a contradiction. We can therefore assume that $K(f_k)$ and $\gamma^\pm(f_k)$ converge. Hence, by (4-7), the sequences f_k^- and f_k^0 also converge. Furthermore, the sequence f_k^+ converges as well, since it lies in a finite-dimensional space. We therefore have that f_k converges to some element $f_\infty\in W$ with $h(f_\infty)\in E^+\cap S$ and $h(f_\infty)\notin U$, which is a contradiction.

In order to apply [Fadell et al. 1982, Proposition 3.3] as done in [Ekeland and Hofer 1990, page 558], we consider the following “truncated” set: for $M>0$ let

$$W_M:=\biggl\{(\alpha,\beta)e^{2\pi it}+\chi\biggl(\frac{|\alpha|^2+|\beta|^2+\|f^0+f^-\|^2}{M}\biggr)\gamma([\alpha:\beta]) \\ \times\biggl(\bar\alpha\frac{\alpha^3}{|\alpha|^3},\bar\beta\frac{\alpha^3}{|\alpha|^3}\biggr)e^{4\pi it}+f^0+f^-\;\bigg|\;\alpha,\beta\in\mathbb{C},\ f^0\in E^0,\ f^-\in E^-\biggr\},$$

where χ is a smooth function such that $\chi(t)=1$ for $t<1$ and $\chi(t)=0$ for $t>\frac{3}{2}$. Note that W_M coincides with W inside the ball of radius M in E and that $h(W_M)\cap E^+\cap S=h(W)\cap E^+\cap S$. Consider now the equivariant map

$$\begin{aligned}\phi:\mathbb{C}^2\oplus E^0\oplus(E^-\cap E_k)&\to W_M,\\ \phi(\alpha,\beta,f^0,f_k^-)&=\chi\biggl(\frac{|\alpha|^2+|\beta|^2+\|f^0+f^-\|^2}{M}\biggr)\gamma([\alpha:\beta])\biggl(\bar\alpha\frac{\alpha^3}{|\alpha|^3},\bar\beta\frac{\alpha^3}{|\alpha|^3}\biggr)e^{4\pi it}\\ &\qquad+(\alpha,\beta)e^{2\pi it}+f^0+f_k^-.\end{aligned} \tag{4-8}$$

By [Fadell et al. 1982, Proposition 3.3] applied to the map $Q_kh(\phi)$, we obtain

$$\alpha(Q_k(W_M)\cap E^+\cap S)\geq 2.$$

The conclusion that $\operatorname{ind}(W)=2$ is achieved by taking M large enough.

Now, let r be the gauge function of $D^2 \times D^2$ and $\Psi_c : E \to \mathbb{R}$ the functional defined in (4-1):

$$\Psi_c(\zeta) := \mathcal{A}(\zeta) - c \int_0^1 r(\zeta(t))\, dt.$$

We recall that

$$r(z_1, z_2) = \frac{|z_1|^2 + |z_2|^2}{2} + \frac{|\mathrm{Re}(z_1^2 + z_2^2)|}{2}.$$

We will prove that $\Psi_c|_W < 0$ for $c = 4\sqrt{2}$.

Let $v = (\zeta_1, \zeta_2) \in W$. Then

$$v = v_1 e^{2\pi i t} + v_2 e^{4\pi i t} + \sum_{k=0}^{+\infty} w_{-k} e^{-2k\pi i t},$$

where $v_1 = (\alpha, \beta)$ and $v_2 = \gamma \alpha^3/|\alpha|^3(\bar{\alpha}, \bar{\beta})$ for some $\alpha, \beta \in \mathbb{C}$. We have

$$\text{(4-9)} \quad \Psi_c(v) = \Big(\pi - \frac{c}{2}\Big)|v_1|^2 + \Big(2\pi - \frac{c}{2}\Big)|v_2|^2 - \sum_{k=1}^{+\infty} k\pi |w_{-k}|^2 - \frac{c}{2}\sum_{k=0}^{+\infty}|w_{-k}|^2 - \frac{c}{2}\int_0^1 \Big|\mathrm{Re}(\zeta_1^2(t) + \zeta_2^2(t))\Big|\, dt.$$

To estimate the integral on the right side of (4-9) we compute the Fourier coefficients of order 4 and 6 of $\mathrm{Re}\big(\zeta_1^2(t) + \zeta_2^2(t)\big)$. We denote them respectively by I_4 and I_6.

$$\text{(4-10)} \quad 2I_4 = v_1^2 + 2(v_2 \cdot w_0) + \overline{(w_{-1})^2 + 2(w_{-2} \cdot w_0) + 2(w_{-3} \cdot v_1) + 2(w_{-4} \cdot v_2)},$$

$$\text{(4-11)} \quad 2I_6 = 2(v_2 \cdot v_1) + \overline{2(w_{-1} \cdot w_{-2}) + 2(w_0 \cdot w_{-3}) + 2(v_1 \cdot w_{-4}) + 2(v_2 \cdot w_{-5})}.$$

By Lemma 4.3 applied to I_6 and (4-11) we get

$$\text{(4-12)} \quad \int_0^1 \Big|\mathrm{Re}\big(\zeta_1^2(t) + \zeta_2^2(t)\big)\Big|\, dt \geq |I_6| \geq |v_2 \cdot v_1| - |w_{-1} \cdot w_{-2}| - |w_0 \cdot w_{-3}| - |v_1 \cdot w_{-4}| - |v_2 \cdot w_{-5}|.$$

Since all the functionals on the right side of (4-12), which we are going to estimate, are homogeneous, it is not restrictive to assume that $|\alpha|^2 + |\beta|^2 = 1$. In particular, $|v_2 \cdot v_1| = |\gamma|$. Applying the Cauchy–Schwarz inequality we obtain the estimates

$$\text{(4-13)} \quad |w_{-1} \cdot w_{-2}| \leq \tfrac{1}{2}(|w_{-1}|^2 + |w_{-2}|^2) \quad \text{and} \quad |w_0 \cdot w_{-3}| \leq \tfrac{1}{2}(|w_0|^2 + |w_{-3}|^2).$$

For the terms $|v_1 \cdot w_{-4}|$ and $|v_2 \cdot w_{-5}|$ we give small and large constants to obtain

$$\text{(4-14)} \quad \begin{aligned} 2|v_1 \cdot w_{-4}| &\leq \frac{\frac{c}{2} + 4\pi}{\frac{c}{4}}|w_{-4}|^2 + \frac{\frac{c}{4}}{\frac{c}{2} + 4\pi}, \\ 2|v_2 \cdot w_{-5}| &\leq \frac{\frac{c}{2} + 5\pi}{\frac{c}{4}}|w_{-5}|^2 + \frac{\frac{c}{4}}{\frac{c}{2} + 5\pi}\gamma^2, \end{aligned}$$

where we have used that $|v_1| = 1$ and $|v_2|^2 = \gamma^2$. Combining the estimate for the integral in (4-12) with (4-13) and (4-14) and plugging into (4-9), we see that

$$\Psi_c(v) < \left(\pi - \frac{c}{2} - \frac{c}{2}\gamma + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2}+4\pi}\right) + \left(2\pi - \frac{c}{2} + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2}+5\pi}\right)\gamma^2 + \cdots, \tag{4-15}$$

where the dots stand for negative terms that arise after absorbing the terms that involve w_{-4} and w_{-5} on the right side of (4-14) with the terms $\left(-4\pi - \frac{c}{2}\right)|w_{-4}|^2$ and $\left(-5\pi - \frac{c}{2}\right)|w_{-5}|^2$ on the right side of (4-9). The right side of (4-15) is negative if

$$\left(2\pi - \frac{c}{2} + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2}+5\pi}\right)\gamma^2 - \frac{c}{2}\gamma + \left(\pi - \frac{c}{2} + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2}+4\pi}\right) < 0. \tag{4-16}$$

For $c = 4\sqrt{2}$ the inequality in (4-16) is satisfied by the solutions to

$$0.44 - 2.82\gamma + 3.56\gamma^2 < 0.$$

In particular, $\Psi_c(v) < 0$ for

$$0.22 \le \gamma \le 0.58. \tag{4-17}$$

Using I_4 in place of I_6, we get that

$$\begin{aligned}\int_0^1 \left|\operatorname{Re}\left(\zeta_1^2(t)+\zeta_2^2(t)\right)\right| dt \\ \ge |I_4| \ge |\alpha^2+\beta^2| - |v_2\cdot w_0| - |w_0\cdot w_{-2}| - |v_1\cdot w_{-3}| - |v_2\cdot w_{-4}| - \frac{|w_{-1}|^2}{2}.\end{aligned} \tag{4-18}$$

We first estimate

$$\begin{aligned}
2|v_2\cdot w_0| &\le \frac{\frac{3}{4}c+4\pi}{\frac{c}{2}+2\pi}|w_0|^2 + \frac{\frac{c}{2}+2\pi}{\frac{3}{4}c+4\pi}\,\gamma^2,\\
2|w_0\cdot w_{-2}| &\le \frac{\frac{c}{2}+2\pi}{\frac{c}{4}}|w_{-2}|^2 + \frac{\frac{c}{4}}{\frac{c}{2}+2\pi}\,|w_0|^2,\\
2|v_1\cdot w_{-3}| &\le \frac{\frac{c}{2}+3\pi}{\frac{c}{4}}|w_{-3}|^2 + \frac{\frac{c}{4}}{\frac{c}{2}+3\pi},\\
2|v_2\cdot w_{-4}| &\le \frac{\frac{c}{2}+4\pi}{\frac{c}{4}}|w_{-4}|^2 + \frac{\frac{c}{4}}{\frac{c}{2}+4\pi}\,\gamma^2.
\end{aligned} \tag{4-19}$$

We have used again that $|v_1|^2 = 1$ and $|v_2|^2 = \gamma^2$. Combining (4-19) with (4-18) and replacing inside (4-9) we obtain

$$\Psi_c(v) \le \left(\pi - \frac{c}{2} - \frac{c}{4}|\alpha^2+\beta^2| + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2}+3\pi}\right) + \left(2\pi - \frac{c}{2} + \frac{c}{4}\left(\frac{\frac{c}{2}+2\pi}{\frac{3}{4}c+4\pi} + \frac{\frac{c}{4}}{\frac{c}{2}+4\pi}\right)\right)\gamma^2 + \cdots$$

Once again, the dots indicate negative terms which come after absorption with the negative terms involving $|w_k|^2$ for $k = 0, -2, -3, -4$. Hence $\Psi_c(v) < 0$ if

$$\left(\pi-\frac{c}{2}+\frac{(\frac{c}{4})^2}{\frac{c}{2}+3\pi}\right)-\frac{c}{4}|\alpha^2+\beta^2|+\left(2\pi-\frac{c}{2}+\frac{c}{4}\left(\frac{\frac{c}{2}+2\pi}{\frac{3}{4}c+4\pi}\right)+\frac{\frac{c}{4}}{\frac{c}{2}+4\pi}\right)\gamma^2<0. \tag{4-20}$$

For $c = 4\sqrt{2}$, the inequality (4-20) is satisfied by the solutions to

$$0.477 - 1.41|\alpha^2 + \beta^2| + 4.352\gamma^2 < 0. \tag{4-21}$$

Letting $\gamma_0 = 0.23$ by (4-17) and solving the corresponding equation to (4-21), we can define $\delta_0 := (0.47 + 4.22 \cdot 0.23^2)/1.41 = 0.496$. We then choose a continuous function γ such that $0 \le \gamma \le \gamma_0$ and

$$\gamma([\alpha:\beta]) = \begin{cases} \gamma_0 & \text{if } |\alpha^2+\beta^2| < \delta_0 = 0.502, \\ 0 & \text{if } |\alpha^2+\beta^2| > 0.6. \end{cases} \tag{4-22}$$

With this choice, we conclude that $\Psi_c|_W < 0$. The conclusion remain valid even if we choose a constant c slightly smaller than $4\sqrt{2}$.

Now let $\mathcal{D}_n$ be the approximating sets for $D^2 \times D^2$ constructed in Proposition 3.2. Note that

$$\frac{\mathcal{D}_n}{\sqrt{1+\frac{1}{n}}} \subseteq D^2 \times D^2 \subseteq \mathcal{D}_n. \tag{4-23}$$

The first inclusion in (4-23) implies

$$\frac{r_n}{1+\frac{1}{n}} \ge r, \tag{4-24}$$

where r_n is the gauge function of the set $\mathcal{D}_n$. For each n, let Ψ_c^n be defined as

$$\Psi_c^n(\zeta) := \mathcal{A}(\zeta) - c\int_0^1 \frac{r_n(\zeta(t))}{1+\frac{1}{n}}\,dt, \qquad \zeta \in E.$$

By (4-24), we have $\Psi_c^n \le \Psi_c \le 0$. Propositions 4.1 and 3.2 then imply

$$\lim_{n\to+\infty} \frac{c_2(\mathcal{D}_n)}{1+\frac{1}{n}} = 3\sqrt{3}.$$

This is because $\Sigma(\mathcal{D}_n \cap [0, 4\sqrt{2}-\varepsilon] = \{\alpha_1^n, \alpha_2^n\}$ for $\varepsilon > 0$ and n large, where α_1^n and α_2^n are two sequences converging to 4 and $3\sqrt{3}$, respectively (Proposition 3.2). Together with the fact that $c_2(\mathcal{D}_n)/(1+\frac{1}{n}) < 4\sqrt{2}$ we have that either $c_2(\mathcal{D}_n) = \alpha_1^n$ or $c_2(\mathcal{D}_n) = \alpha_2^n$. It cannot be $c_2(\mathcal{D}_n) = \alpha_1^n = c_1(\mathcal{D}_n)$, otherwise the set of characteristics of $\mathcal{D}_n$ of action α_1^n would have index 2, and this is not the case. Hence $c_2(\mathcal{D}_n) = \alpha_2^n \to 3\sqrt{3}$, and this gives the conclusion. □

Remark 4.5. One can deduce that $c_2(D^2 \times D^2) = 3\sqrt{3}$ from the work of Ramos [2017]. Let $E(a, b)$ denote the ellipsoid

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \pi\left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\right) \le 1 \right\}. \tag{4-25}$$

It follows from [Ramos 2017, Corollary 9] that $E(4-\varepsilon, 4+\delta)$ embeds symplectically into the bidisc $D^2 \times D^2$ for some $\varepsilon, \delta > 0$. Hence

$$4 + \delta \le c_2(D^2 \times D^2) \le 3\sqrt{3}.$$

Since $3\sqrt{3}$ is the only number in the spectrum $\Sigma(D^2 \times D^2)$ with this property, we conclude that $c_2(D^2 \times D^2) = 3\sqrt{3}$.

Theorem 4.6. *For the unit real bidisc $D^2 \times D^2$ we have $c_3(D^2 \times D^2) = 8$.*

Proof. Let W be the subspace of E defined by

$$W := E^- \oplus E^0 \oplus \langle (e^{2\pi i t}, 0), (0, e^{2\pi i t}), (e^{4\pi i t}, 0) \rangle.$$

By [Ekeland and Hofer 1990, Proposition 1] the pseudoindex of W is equal to 3. We now prove that for some constant c we have $\Psi_c|_W \le 0$, where Ψ_c is the functional defined in (4-1). For an element $(\zeta_1, \zeta_2) = (\alpha e^{2\pi i t} + \gamma e^{4\pi i t}, \beta e^{2\pi i t}) + w^- + w^0 \in W$, with $\alpha, \beta, \gamma \in \mathbb{C}$, $w^- \in E^-$, $w^0 \in E^0$, we have

$$\begin{aligned} &\Psi_c((\alpha e^{2\pi i t} + \gamma e^{4\pi i t}, \beta e^{2\pi i t}) + w^- + w^0) \\ &\quad = \mathcal{A}(\zeta) - c \int_0^1 r(\zeta(t))\, dt \\ &\quad = -\frac{c}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2) - \|w^-\|_E - \frac{c}{2}\|w^0 + w^-\|^2_{L^2} \\ &\qquad\quad - \frac{c}{2}\int_0^1 |\mathrm{Re}(\zeta_1^2(t) + \zeta_2^2(t))|\, dt + \pi(|\alpha|^2 + |\beta|^2 + 2|\gamma|^2). \end{aligned} \tag{4-26}$$

To give an estimate of the last integral in (4-26), we compute the coefficient I_8 of $e^{8\pi i t}$ in the Fourier expansion of $\mathrm{Re}(\zeta_1^2(t) + \zeta_2^2(t))$:

$$I_8 = \gamma^2 + \overline{2(\alpha, \beta) \cdot w_5^- + 2(\gamma, 0) \cdot w_6^- + 2w^0 \cdot w_4^- + 2w_1^- \cdot w_3^- + w_2^- \cdot w_2^-}.$$

We want to find a constant c such that

$$\begin{aligned} &\pi(|\alpha|^2 + |\beta|^2 + 2|\gamma|^2) \\ &\qquad - \frac{c}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2) - \|w^-\|_E - \frac{c}{2}\|w^0 + w^-\|^2_{L^2} - \frac{c}{4}|I_8| < 0. \end{aligned} \tag{4-27}$$

Applying the Cauchy–Schwarz inequality, we get

$$\begin{aligned} |I_8| \ge |\gamma|^2 - a_1|w_5^-|^2 - \frac{1}{a_1}|(\alpha, \beta)|^2 - a_2|w_6^-|^2 - \frac{1}{a_2}|\gamma|^2 \\ - |w^0|^2 - |w_4^-|^2 - |w_1^-|^2 - |w_3^-|^2 - |w_2^-|^2. \end{aligned}$$

Choosing a_1 such that $\frac{c}{4}a_1 = \frac{c}{2} + 5\pi$ and a_2 such that $\frac{c}{4}a_2 = \frac{c}{2} + 6\pi$, (4-27) becomes

$$(|\alpha|^2 + |\beta|^2)\left(\pi - \frac{c}{2} - \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 5\pi}\right) + |\gamma|^2\left(2\pi - \frac{c}{2} - \frac{c}{4} + \frac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 6\pi}\right) + \cdots, \tag{4-28}$$

where the dots indicate other negative terms not involving α, β or γ. The expression in (4-28) is negative if

$$\begin{cases} \pi - \dfrac{c}{2} - \dfrac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 5\pi} < 0, \\ 2\pi - \dfrac{c}{2} - \dfrac{c}{4} + \dfrac{\left(\frac{c}{4}\right)^2}{\frac{c}{2} + 6\pi} < 0. \end{cases} \tag{4-29}$$

Solving the system (4-29), we obtain

$$c > 4\pi \frac{\sqrt{109} - 7}{5}.$$

Considering the approximating domains $\mathcal{D}_n$ and reasoning as in the last part of the proof of Theorem 4.4, we conclude that

$$c_3(D^2 \times D^2) < 4\pi \frac{\sqrt{109} - 7}{5}. \tag{4-30}$$

Moreover, Proposition 4.8 implies

$$c_3(D^2 \times D^2) > c_3(\mathbb{B}^2) = 2\pi. \tag{4-31}$$

Looking at the spectrum $\Sigma(D^2 \times D^2)$ computed in Proposition 2.2, we see that (4-30) and (4-31) imply $c_3(D^2 \times D^2) = 8$. □

Note that in the proof of Theorem 4.6 we have used the fact that the capacity c_3 of the real bidisc is strictly greater than the corresponding capacity for the unit ball $\mathbb{B}^2$. This inequality is proved below in Proposition 4.8 for every capacity c_k. The proof relies on the following lemma.

Lemma 4.7. *Let $D_1 \subset D_2$ be two convex smooth subdomains of $\mathbb{C}^n$ such that for some k we have $c_k(D_1) = c_k(D_2) =: c$. Assume also that c is isolated in $\Sigma(D_1)$ and $\Sigma(D_2)$. Then there exists a closed characteristic $\gamma \subset \partial D_1 \cap \partial D_2$ such that $\mathcal{A}(\gamma) = c$.*

Proof. Let r_1 and r_2 be the gauge functions of D_1 and D_2, respectively. Let f be an increasing positive function such that

$$f(s) = \begin{cases} 0 & \text{if } s \leq 1, \\ Cs & \text{if } s \text{ is large}, \end{cases} \tag{4-32}$$

and $f'(s_0) = c$ only for one $s_0 > 1$. We will specify later how s_0 and C are chosen.

Let $H_i^f := f \circ r_i$ for $i = 1, 2$. Note that

$$c_k(D_1) = \inf_{H_1}\{c^k_{H_1}\} = \inf_{H_2}\{c^k_{H_2}\} = c_k(D_2), \tag{4-33}$$

where each infimum is taken over all possible admissible Hamiltonians H_i for D_i. Equation (4-33) together with the fact that c is isolated in the spectra $\Sigma(D_1)$ and $\Sigma(D_2)$ implies that, choosing a function f with C sufficiently large, we obtain

$$c^k_{H_1^f} = c^k_{H_2^f}.$$

Let $\mathcal{A}_{H_i^f}$ be the corresponding Hamiltonian actions. Since $D_1 \subset D_2$, then $r_1 \geq r_2$ and $\mathcal{A}_{H_1^f} \leq \mathcal{A}_{H_2^f}$. Let

$$W_\varepsilon = \{\zeta \in E \mid \mathcal{A}_{H_2^f}(\zeta) \leq c^k_{H_2^f} + \varepsilon\}.$$

W_ε is a closed equivariant set of pseudoindex at least k. We have

$$c^k_{H_1^f} \leq \sup_{\zeta\in W_\varepsilon}\{\mathcal{A}_{H_1^f}(\zeta)\} \leq \sup_{\zeta\in W_\varepsilon}\{\mathcal{A}_{H_2^f}(\zeta)\} = c^k_{H_2^f} + \varepsilon.$$

We now claim that there exist two sequences $\varepsilon_n \to 0$ and $\zeta_n \in W_{\varepsilon_n}$ such that

$$\nabla_E \mathcal{A}_{H_1^f}(\zeta_n) \to 0 \quad \text{and} \quad \mathcal{A}_{H_1^f}(\zeta_n) \to c^k_{H_1^f}.$$

Indeed, assume that this is not true. Then there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$ and for all $\zeta \in W_\varepsilon$ such that $|\mathcal{A}_{H_1^f}(\zeta) - c^k_{H_1^f}| \leq \varepsilon_0$ we have $\|\nabla\mathcal{A}_{H_1^f}(\zeta)\|_E > \delta_0$. Following [Ekeland and Hofer 1990, Lemma 1] and [Ekeland and Hofer 1989, Proposition 2], we denote by Φ_t the flow at the time t in E of the vector field $-\nabla\mathcal{A}_{H_1^f}$ with a suitable cut-off. Choosing ε small enough we have, for $t > \varepsilon/\delta_0^2$, that

$$\sup_{\zeta\in\Phi_t(W_\varepsilon)} \mathcal{A}_{H_1^f}(\zeta) < c^k_{H_1^f},$$

which gives a contradiction.

We have thus proved that there exists a subsequence of ζ_n converging in E to ζ_0, which is a critical point for $\mathcal{A}_{H_1^f}$ and such that $\mathcal{A}_{H_1^f}(\zeta_0) = c^k_{H_1^f}$. If the image of ζ_0 is not in $\partial D_1 \cap \partial D_2$, then $r_1(\zeta_0(t)) \geq r_2(\zeta_0(t))$ and the inequality is strict in some open interval. This implies that $\mathcal{A}_{H_2^f}(\zeta_0) > c^k_{H_2^f}$ and in turn $\mathcal{A}_{H_2^f}(\zeta_n) > c^k_{H_2^f} + \varepsilon$ for some $\varepsilon > 0$ not depending on n, but this is in contradiction with the definition of W_{ε_n}. □

Proposition 4.8. *For the real bidisc $D^2 \times D^2$ and the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$, we have $c_k(D^2 \times D^2) > c_k(\mathbb{B}^2)$ for every positive integer k.*

Proof. Since the unit ball $\mathbb{B}^2$ centered at 0 is contained in $D^2 \times D^2$ we have, for any k, that $c_k(D^2 \times D^2) \geq c_k(\mathbb{B}^2)$. Choose a smooth domain W containing $\mathbb{B}^2$ and contained in $D^2 \times D^2$ with a discrete action spectrum and having the same intersection at the boundary with $D^2 \times D^2$ and $\mathbb{B}^2$. If $c_k(D^2 \times D^2) = c_k(\mathbb{B}^2)$ for some k, then Lemma 4.7 implies that there exists a characteristic contained in

$$\begin{aligned}\partial W \cap \mathbb{B}^2 &= \partial(D^2 \times D^2) \cap \mathbb{B}^2 \\ &= \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1^2 + x_2^2 = 1\} \cup \{(iy_1, iy_2) \in \mathbb{C}^2 \mid y_1^2 + y_2^2 = 1\}.\end{aligned}$$

Since there are no characteristics in the intersection of the boundaries, then we must have $c_k(D^2 \times D^2) > c_k(\mathbb{B}^2)$. □

Remark 4.9. Let $B(a) = E(a, a)$, that is, $B(a)$ is the Euclidean ball of radius $\sqrt{a/\pi}$. It follows from [Ramos 2017] that the real bidisc $D^2 \times D^2$ can be symplectically embedded into the ellipsoid $E(4, 3\sqrt{3})$ and that the ball $B(4)$ can be symplectically embedded into $D^2 \times D^2$. This implies that $c_3(D^2 \times D^2) = 8$. Proposition 4.8 can also be obtained from [Ramos 2017]. Indeed, $\mathbb{B}^2 = B(\pi)$ has strictly smaller Ekeland–Hofer capacities than $B(4)$, which embeds into $D^2 \times D^2$.

5. Applications

We now exploit our computations of the capacities of $D^2 \times D^2$ to prove some results of symplectic rigidity. First recall that the complex bidisc $\Delta^2 \subset \mathbb{C}^2$ is defined by

$$\Delta^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid \ x_1^2 + y_1^2 < 1, x_2^2 + y_2^2 < 1\}.$$

Sukhov and Tumanov [2012] applied techniques from classical complex analysis to prove that there exists no symplectic embedding of $D^2 \times D^2$ into Δ^2. Using symplectic capacities we can easily show that no symplectic embedding is possible in the other direction.

Corollary 5.1. *There is no symplectic embedding of Δ^2 into $D^2 \times D^2$.*

Proof. Assume by contradiction that there is such an embedding

$$\psi : \Delta^2 \to D^2 \times D^2.$$

By the extension after restriction principle, for any $\varepsilon > 0$ there exists a symplectic map with compact support $\psi_\varepsilon : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\psi_\varepsilon|_{\Delta^2_{1-\varepsilon}} = \psi|_{\Delta^2_{1-\varepsilon}}$. Therefore,

$$3\pi(1-\varepsilon)^2 = c_3(\Delta^2_{1-\varepsilon}) = c_3(\psi_\varepsilon(\Delta^2_{1-\varepsilon})) \leq c_3(D^2 \times D^2) = 8,$$

which gives a contradiction. □

Note that the proof of Corollary 5.1 implies that the complex bidisc cannot be embedded even in a slightly larger real bidisc.

Remark 5.2. Corollary 5.1 can also be obtained without using the Ekeland–Hofer capacities. By [Ramos 2017], the bidisc $D^2 \times D^2$ is a concave toric domain. One can then apply [Gutt and Hutchings 2018, Theorem 1.18] to conclude that the cube capacity $c_\square$ of $D^2 \times D^2$ is equal to 2. By the very definition of the cube capacity [Gutt and Hutchings 2018, Definition 1.17], Corollary 5.1 follows.

To prove the next rigidity result we need to recall a product property of the Ekeland–Hofer capacities: if $A \subset \mathbb{C}^n$ and $B \subset \mathbb{C}^m$, then

$$c_k(A \times B) = \min_{i+j=k} \{c_i(A) + c_j(B)\}.$$

Here we use the convention that the zero-th capacity is equal to 0, that is, $c_0(A) = c_0(B) = 0$. We denote by Δ_R the standard complex disc of radius R.

Corollary 5.3. *The product $D^2 \times D^2 \times \Delta_R$ is not symplectomorphic to $\Delta^2 \times \Delta_R$ for $R > \sqrt{3\sqrt{3}/2\pi}$.*

Proof. The case $R \geq 1$ is known [Wong 2018, Theorem 4.1], hence let $R < 1$. We have

$$c_2(\Delta^2 \times \Delta_R) = 2\pi R^2.$$

On the other hand,

$$c_2(D^2 \times D^2 \times \Delta_R) = \min\{3\sqrt{3}, 4 + \pi R^2, 2\pi R^2\}.$$

The two capacities are different if $R > \sqrt{3\sqrt{3}/2\pi}$. □

Remark 5.4. The bound in Corollary 5.3 can be improved to $\sqrt{2/\pi}$ using Ekeland–Hofer symplectic capacities of higher order. More precisely, one has to show, for each positive integer n, that $c_{2n-1}(D^2 \times D^2) = 4n$. This can be achieved by arguing in a similar way as in Theorem 4.6, where we computed the value of $c_3(D^2 \times D^2)$. Vinicius Gripp Barros Ramos has informed the authors that the bound $\sqrt{2/\pi}$ can also be obtained using the results in [Gutt and Hutchings 2018].

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FREENESS CHARACTERIZATIONS ON FREE CHAOS SPACES

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We deal with characterizing the freeness and asymptotic freeness of free multiple integrals with respect to a free Brownian motion or a free Poisson process. We obtain three characterizations of freeness, in terms of contraction operators, covariance conditions, and free Malliavin gradients. We show how these characterizations can be used in order to obtain limit theorems, transfer principles, and asymptotic properties of converging sequences.

1. Introduction

A classical result in probability theory asserts that one can decompose any functional of a Brownian motion W as an infinite sum of multiple integrals. That is, to any square integrable random variable F measurable with respect to W, one can associate a unique sequence of symmetric and square integrable kernels $\{f_n : n \geq 0\}$ such that

$$F = \sum_{n=0}^{\infty} I_n^W(f_n).$$

The set of all multiple Wiener–Itô integrals of the form $I_n^W(f)$, the so-called *n-th Wiener chaos* of W, thus plays a fundamental role in modern stochastic analysis. Analyzing its many rigid properties (notably those related to independence and normal approximation) has become a subject in its own right, and has grown into a mature and widely applicable mathematical theory.

Among the most striking results about Wiener chaos are the following two theorems, which will play a central role in the present paper. The first one characterizes independence of multiple Wiener–Itô integrals.

Theorem 1.1 [Üstünel and Zakai 1989]. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then $I_n^W(f)$ and $I_m^W(g)$ are independent if and only if, for almost all $x_1, \dots, x_{n-1}, y_1, \dots, y_{m-1} \in \mathbb{R}_+$,*

$$\int_0^{\infty} f(x_1, \dots, x_{n-1}, u) g(y_1, \dots, y_{m-1}, u)\, du = 0.$$

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The second result is nowadays one of the most central tools of analysis on Wiener chaos, as it represents a drastic simplification with respect to the method of moments for the normal approximation of sequences of multiple Wiener–Itô integrals.

Theorem 1.2 [Nualart and Peccati 2005]. *A unit-variance sequence in a Wiener chaos of fixed order converges in law to the standard Gaussian distribution if and only if the corresponding sequence of fourth moments converges to* 3.

Since its introduction by Voiculescu in the eighties in order to solve some longstanding conjectures about von Neumann algebras of free groups, free probability theory has become a vivid and powerful branch of mathematics, with many applications (including signal processing, channel capacity estimation and nuclear physics) and deep connections with other mathematical fields (like operator algebra, theory of random matrices or combinatorics). Free probability has many parallels with the usual probability theory (hence its name), and the study of these links often brings a new point of view which may then enrich the theory of both worlds (classical and free).

Starting from the free independence property, a genuine stochastic calculus with respect to the free Brownian motion (the free analog of the classical Brownian motion) has emerged within the last twenty years, following the route paved by the seminal paper of Biane and Speicher [1998]. In particular, a common property of the classical and free settings is the possibility of expanding the space as a sum of free chaos, giving rise to the so-called *Wigner chaos*. By their very construction, these free chaos play in the free world a similar role as Wiener chaos in the classical setting. It is thus natural to investigate the similarities and differences between these two mathematical objects. For instance, do we have an analog of Theorem 1.2 in the free world? The answer is "yes", and is given by the following theorem.

Theorem 1.3 [Kemp et al. 2012]. *A unit-variance sequence in a Wigner chaos of fixed order converges in law to the semicircular distribution if and only if the corresponding sequence of fourth moments converges to* 2.

Shortly after the publication of [Kemp et al. 2012], many other results in the spirit of Theorem 1.3 have been added to the literature, including the following ones (the list is not exhaustive).

In [Nourdin et al. 2013], it is shown that component-wise convergence to the semicircular distribution is equivalent to joint convergence, thus extending to the free probability setting a seminal result by Peccati and Tudor [2005].

In [Nourdin and Peccati 2013], a noncentral counterpart of Theorem 1.3 is provided. More precisely, it is shown that any adequately rescaled sequence $\{F_n : n \geq 0\}$ of self-adjoint operators living inside a fixed Wigner chaos of even order converges in distribution to a centered free Poisson random variable with rate $\lambda > 0$ if and only if $\varphi(F_n^4) - 2\varphi(F_n^3) \to 2\lambda^2 - \lambda$ (where φ is the relevant tracial state).

In [Nourdin and Poly 2012], convergence in law of any sequence belonging to the second Wigner chaos is characterized by means of the convergence of only a finite number of cumulants.

In [Deya and Nourdin 2012], making use of heavy combinatorics it is shown that any adequately rescaled sequence $\{F_n : n \geq 0\}$ of self-adjoint operators living inside a fixed Wigner chaos converges in distribution to the tetilla law $\mathcal{T}$ if and only if

$$\varphi(F_n^4) \to \varphi(\mathcal{T}^4) \quad \text{and} \quad \varphi(F_n^6) \to \varphi(\mathcal{T}^6)$$

(where φ is the relevant tracial state). Note that this finding is not an extension of a result known in the classical probability theory, as the existence of such a result in the classical setting is still an open problem.

In [Bourguin and Peccati 2014], a class of sufficient conditions, ensuring that a sequence of multiple integrals with respect to a free Poisson measure converges to a semicircular limit, is established, thus providing an analog of Theorem 1.3 in the context of free Poisson chaos.

In [Bourguin 2015], a fourth moment type condition is given, for an element of a free Poisson chaos of arbitrary order to converge to a free centered Poisson distribution.

In [Arizmendi and Jaramillo 2014], an estimate for the Kolmogorov distance between a freely infinitely divisible distribution and the semicircle distribution is given, in terms of the difference between the fourth moment and 2.

In [Bourguin 2016], a multidimensional counterpart of the aforementioned central limit theorem on the free Poisson chaos is given.

In [Bourguin and Campese 2018], a quantitative version of Theorem 1.3 is derived, using free stochastic analysis as well as a new biproduct formula for bi-integrals.

In the present paper, our main goal is to provide characterizations of free independence on the Wigner and free Poisson chaos, as well as investigate the similarities and dissimilarities between classical and free chaos, as far as (possibly asymptotic) independence properties are concerned.

Our first set of investigations yields a characterization of freeness on the Wigner and free Poisson chaos, in terms of contractions, covariances, or free Malliavin gradient, thus providing a suitable extension of Theorem 1.1 (and related results) to the free setting. Most of our results turn out to be similar to the classical setting, with the notable exception of the characterization of freeness in terms of the free Malliavin gradient, this last fact illustrating a fundamental difference between the classical and the free cases.

Our second set of investigations is concerned again with the independence property, but this time in an asymptotic context. Here, the problem is to find what conditions need to be imposed on *limits* of multiple integrals to be free.

The remainder of this paper is organized as follows: Section 2 contains a short introduction to free probability theory, with a special emphasis to the material needed for the rest of the paper. Section 3 is devoted to the characterization of freeness on the Wigner and free Poisson chaos, in terms of contractions, covariances, or free Malliavin gradient. This section also provides several lemmas which will be used to prove our main results in the following sections. In Section 4, we study different characterizations of asymptotic freeness, in several contexts. We devote Section 5 to the study of transfer principles between classical and free chaos. Finally, Section 6 contains auxiliary results that are used throughout the paper.

2. Preliminaries

Elements of free probability. In the following, a short introduction to free probability theory is provided. For a thorough and complete treatment, see [Nica and Speicher 2006; Voiculescu et al. 1992; Hiai and Petz 2000]. Let $(\mathscr{A}, \varphi)$ be a tracial W^*-probability space, that is $\mathscr{A}$ is a von Neumann algebra with involution $*$ and $\varphi : \mathscr{A} \to \mathbb{C}$ is a unital linear functional assumed to be weakly continuous, positive (meaning that $\varphi(X) \geq 0$ whenever X is a nonnegative element of $\mathscr{A}$), faithful (meaning that $\varphi(XX^*) = 0 \Rightarrow X = 0$ for every $X \in \mathscr{A}$) and tracial (meaning that $\varphi(XY) = \varphi(YX)$ for all $X, Y \in \mathscr{A}$). The self-adjoint elements of $\mathscr{A}$ will be referred to as random variables. The noncommutative space $L^2(\mathscr{A}, \varphi)$ denotes the completion of $\mathscr{A}$ with respect to the norm $\|X\|_2 = \sqrt{\varphi(XX^*)}$.

Recall the definition of freeness (see [Nica and Speicher 2006, Definition 5.3] and [Nica and Speicher 2006, Remarks 5.4] or [Tao 2012, Definition 2.5.18]) for a collection of noncommutative random variables living on an appropriate noncommutative probability space $(\mathscr{A}, \varphi)$.

Definition 2.1. A collection of random variables $X_1, \dots, X_n$ on $(\mathscr{A}, \varphi)$ is said to be free if

$$\varphi\big([P_1(X_{i_1}) - \varphi(P_1(X_{i_1}))] \cdots [P_m(X_{i_m}) - \varphi(P_m(X_{i_m}))]\big) = 0$$

whenever $P_1, \dots, P_m$ are polynomials and $i_1, \dots, i_m \in \{1, \dots, n\}$ are indices with no two adjacent i_j equal.

Let $X \in \mathscr{A}$. The p-th moment of X is given by the quantity $\varphi(X^p)$, $p \in \mathbb{N}_0$. Now assume that X is a self-adjoint bounded element of $\mathscr{A}$ (in other words, X is a bounded random variable), and write $\rho(X) = \|X\| \in [0, \infty)$ to indicate the *spectral radius* of X.

Definition 2.2. The *law* (or *spectral measure*) of X is defined as the unique Borel probability measure μ_X on the real line such that $\int_{\mathbb{R}} P(t)\, d\mu_X(t) = \varphi(P(X))$ for every polynomial $P \in \mathbb{R}[X]$. A consequence of this definition is that μ_X has support in $[-\rho(X), \rho(X)]$.

The existence and uniqueness of μ_X in such a general framework are proved, e.g., in [Tao 2012, Theorem 2.5.8] (see also [Nica and Speicher 2006, Proposition 3.13]). Note that, since μ_X has compact support, the measure μ_X is completely determined by the sequence $\{\varphi(X^p) : p \geq 1\}$.

Let $\{X_k : k \geq 1\}$ be a sequence of noncommutative random variables, each possibly belonging to a different noncommutative probability space $(\mathscr{A}_k, \varphi_k)$.

Definition 2.3. The sequence $\{X_k : k \geq 1\}$ is said to converge in distribution to a limiting noncommutative random variable X_∞ (defined on $(\mathscr{A}_\infty, \varphi_\infty)$), if $\varphi_k(P(X_k)) \xrightarrow[k\to+\infty]{} \varphi_\infty(P(X_\infty))$ for every polynomial $P \in \mathbb{R}[X]$.

If X_k, X_∞ are bounded (and therefore the spectral measures $\mu_{X_k}, \mu_{X_\infty}$ are well-defined), this last relation is equivalent to saying that

$$\int_{\mathbb{R}} P(t)\, \mu_{X_k}(dt) \xrightarrow[k\to+\infty]{} \int_{\mathbb{R}} P(t)\, \mu_{X_\infty}(dt).$$

An application of the method of moments yields immediately that, in this case, one has also that μ_{X_k} weakly converges to μ_{X_∞}, that is $\mu_{X_k}(f) \xrightarrow[k\to+\infty]{} \mu_{X_\infty}(f)$, for every $f : \mathbb{R} \to \mathbb{R}$ bounded and continuous (note that no additional uniform boundedness assumption is needed).

In this paper, we will also deal with *joint* convergences in law, for sequences $\{X_k = (X_k^1, \dots, X_k^d) : k \geq 1\}$ of noncommutative random vectors, each possibly belonging to a different noncommutative probability space $(\mathscr{A}_k, \varphi_k)$.

Definition 2.4. The vector-valued sequence $\{X_k = (X_k^1, \dots, X_k^d) : k \geq 1\}$ is said to converge jointly in distribution to a limiting noncommutative random vector $X_\infty = (X_\infty^1, \dots, X_\infty^d)$ (defined on $(\mathscr{A}_\infty, \varphi_\infty)$), if any moment in the variables $X_k^1, \dots, X_k^d$ converges, as $k \to \infty$, to the corresponding moments in $X_\infty^1, \dots, X_\infty^d$; otherwise stated, $(X_k^1, \dots, X_k^d) \xrightarrow{\text{law}} (X_\infty^1, \dots, X_\infty^d)$ if for any $r \in \mathbb{N}$ and positive integers $i_1, \dots, i_r$, as $k \to \infty$, one has

$$\varphi_k[X_k^{i_1} \cdots X_k^{i_r}] \to \varphi_\infty[X_\infty^{i_1} \cdots X_\infty^{i_r}].$$

Let us now define the two main processes we will deal with in this paper, namely the free Brownian motion and the free Poisson process.

Definition 2.5. (1) The centered semicircular distribution with variance $t > 0$, denoted by $\mathcal{S}(0, t)$, is the probability distribution given by

$$\mathcal{S}(0, t)(dx) = (2\pi t)^{-1} \sqrt{4t - x^2}\, \mathbb{1}_{[-2\sqrt{t}, 2\sqrt{t}]}(x)\, dx.$$

(2) A free Brownian motion S consists of

(i) a filtration $\{\mathscr{A}_t : t \geq 0\}$ of von Neumann subalgebras of $\mathscr{A}$ (in particular, $\mathscr{A}_s \subset \mathscr{A}_t$ for $0 \leq s < t$),

(ii) a collection $S = \{S_t : t \geq 0\}$ of self-adjoint operators in $\mathscr{A}$ such that

  (a) $S_0 = 0$ and $S_t \in \mathscr{A}_t$ for all $t \geq 0$,

  (b) for all $t \geq 0$, S_t has a semicircular distribution with mean zero and variance t,

  (c) for all $0 \leq u < t$, the increment $S_t - S_u$ is free with respect to $\mathscr{A}_u$, and has a semicircular distribution with mean zero and variance $t - u$.

Definition 2.6. (1) The free Poisson distribution with rate $\lambda > 0$, denoted by $P(\lambda)$, is the probability distribution defined as follows:

(i) if $\lambda \in (0, 1]$, then $P(\lambda) = (1 - \lambda)\delta_0 + \lambda\widetilde{\nu}$, and

(ii) if $\lambda > 1$, then $P(\lambda) = \widetilde{\nu}$, where δ_0 stands for the Dirac mass at 0. Here,

$$\widetilde{\nu}(dx) = (2\pi x)^{-1}\sqrt{4\lambda - (x - 1 - \lambda)^2}\,\mathbb{1}_{[(1-\sqrt{\lambda})^2,(1+\sqrt{\lambda})^2]}(x)dx.$$

(2) A free Poisson process N consists of

(i) a filtration $\{\mathscr{A}_t : t \geq 0\}$ of von Neumann subalgebras of $\mathscr{A}$ (in particular, $\mathscr{A}_s \subset \mathscr{A}_t$ for $0 \leq s < t$),

(ii) a collection $N = \{N_t : t \geq 0\}$ of self-adjoint operators in $\mathscr{A}_+$ ($\mathscr{A}_+$ denotes the cone of positive operators in $\mathscr{A}$) such that

  (a) $N_0 = 0$ and $N_t \in \mathscr{A}_t$ for all $t \geq 0$,

  (b) for all $t \geq 0$, N_t has a free Poisson distribution with rate t, and

  (c) for all $0 \leq u < t$, the increment $N_t - N_u$ is free with respect to $\mathscr{A}_u$, and has a free Poisson distribution with rate $t - u$. $\hat{N}$ will denote the collection of random variables $\hat{N} = \{\hat{N}_t = N_t - t\mathbf{1} : t \geq 0\}$, where $\mathbf{1}$ stands for the unit of $\mathscr{A}$. $\hat{N}$ will be referred to as a compensated free Poisson process.

Remark 2.7. In the sequel, $\mathfrak{M}$ will stand for either the free Brownian motion S or the compensated free Poisson process $\hat{N}$.

We continue with some definitions that will play a crucial role in the rest of the paper. For every integer $n \geq 1$, the space $L^2(\mathbb{R}_+^n; \mathbb{C}) = L^2(\mathbb{R}_+^n)$ denotes the collection of all complex-valued functions on $\mathbb{R}_+^n$ that are square-integrable with respect to the Lebesgue measure on $\mathbb{R}_+^n$.

Definition 2.8. Let n be a natural number and let f be a function in $L^2(\mathbb{R}_+^n)$.

(1) The adjoint of f is the function $f^*(t_1, \dots, t_n) = \overline{f(t_n, \dots, t_1)}$.

(2) The function f is called mirror-symmetric if $f = f^*$, i.e., if

$$f(t_1, \dots, t_n) = \overline{f(t_n, \dots, t_1)}$$

for almost all $(t_1, \dots, t_n) \in \mathbb{R}_+^n$ with respect to the product Lebesgue measure.

(3) The function f is called (fully) symmetric if it is real-valued and, for any permutation σ in the symmetric group $\mathfrak{S}_n$, it holds that $f(t_1, \dots, t_n) = f(t_{\sigma(1)}, \dots, t_{\sigma(n)})$ for almost all $(t_1, \dots, t_n) \in \mathbb{R}_+^n$ with respect to the product Lebesgue measure.

Definition 2.9. Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$. Let $p \le n \wedge m$ be a natural number. The p-th nested contraction $f \overset{p}{\frown} g$ of f and g is the $L^2(\mathbb{R}_+^{n+m-2p})$ function defined by nested integration of the middle p variables in $f \otimes g$:

$$(f \overset{p}{\frown} g)(t_1, \dots, t_{n+m-2p}) = \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_1, \dots, s_p) \\ \times g(s_p, \dots, s_1, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p.$$

In the case where $p = 0$, the function $f \overset{0}{\frown} g$ is just given by $f \otimes g$.

Similarly, we define the star contraction $f \star_k^j g$ of f and g.

Definition 2.10. Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$. Let $k \in \{1, \dots, n \wedge m\}$ and $j \in \{0, \dots, k\}$ be two natural numbers. We set

$$(f \star_k^j g)(t_1, \dots, t_{n+m-2k+j}) = \int_{\mathbb{R}_+^{k-j}} f(t_1, \dots, t_{n-k+j}, s_{k-j}, \dots, s_1) \\ \times g(s_1, \dots, s_{k-j}, t_{n-k+1}, \dots, t_{n+m-2k+j}) ds_1 \cdots ds_{k-j}.$$

For $f \in L^2(\mathbb{R}_+^n)$, we denote by $I_n^S(f)$ the multiple Wigner integral of f with respect to the free Brownian motion as introduced in [Biane and Speicher 1998]. The space $L^2(\mathcal{S}, \varphi) = \{I_n^S(f) : f \in L^2(\mathbb{R}_+^n), n \ge 0\}$ is a unital $*$-algebra, with product rule given, for any $n, m \ge 1$, $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$, by

$$I_n^S(f) I_m^S(g) = \sum_{p=0}^{n \wedge m} I_{n+m-2p}^S(f \overset{p}{\frown} g) \tag{1}$$

and involution $I_n^S(f)^* = I_n^S(f^*)$. For a proof of (1), see [Biane and Speicher 1998].

Similarly, we can define free Poisson multiple integrals with respect to $\hat{N}$ (these integrals were studied in [Bourguin and Peccati 2014], and we refer to this reference for details). The space $L^2(\mathcal{N}, \varphi) = \{I_n^{\hat{N}}(f) : f \in L^2(\mathbb{R}_+^n), n \ge 0\}$ is a unital $*$-algebra, with product rule given, for any $n, m \ge 1$, $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$, by

$$I_n^{\hat{N}}(f) I_m^{\hat{N}}(g) = \sum_{p=0}^{n \wedge m} I_{n+m-2p}^{\hat{N}}(f \overset{p}{\frown} g) + \sum_{p=1}^{n \wedge m} I_{m+n-2p+1}^{\hat{N}}(f \star_p^{p-1} g) \tag{2}$$

and involution $I_n^{\hat{N}}(f)^* = I_n^{\hat{N}}(f^*)$. For a proof of this formula, see [Bourguin and Peccati 2014].

Furthermore, as is well known, both Wigner and free Poisson multiple integrals of different orders are orthogonal in $L^2(\mathcal{A},\varphi)$, whereas for two integrals of the same order, the Wigner isometry holds:

$$\varphi(I_n^{\mathfrak{M}}(f)I_n^{\mathfrak{M}}(g)^*) = \langle f, g\rangle_{L^2(\mathbb{R}_+^n)}. \tag{3}$$

Remark 2.11. (1) Observe that it follows from the definition of the involution on the algebras $L^2(\mathcal{S},\varphi)$ and $L^2(\mathcal{N},\varphi)$ that operators of the type $I_n^{\mathfrak{M}}(f)$ are self-adjoint if and only if f is mirror-symmetric.

(2) In what follows, we will use the notation I_n^S, $I_n^{\hat{N}}$, I_n^W and $I_n^{\hat{\eta}}$ to denote multiple Wigner integrals, multiple free Poisson integrals, multiple Wiener integrals, and multiple classical Poisson integrals, respectively.

Bi-integrals and free gradient operator. In this particular subsection, we only focus on the Wigner case, as the tools we are about to introduce do not exist in the context of free Poisson processes.

Let $(\mathcal{A},\varphi)$ be a W^*-probability space. An $\mathcal{A}\otimes\mathcal{A}$-valued stochastic process $t\mapsto U_t$ is called a biprocess. For $p\geq 1$, U is an element of $\mathscr{B}_p$, the space of L^p-biprocesses, if its norm

$$\|U\|_{\mathscr{B}_p}^2 = \int_0^\infty \|U_t\|^2_{L^p(\mathcal{A}\otimes\mathcal{A},\varphi\otimes\varphi)}\,dt$$

is finite.

Let n, m be two positive integers and $f = g\otimes h\in L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m)$. Then, the Wigner bi-integral $[I_n^S\otimes I_m^S](f)$ is defined as

$$[I_n^S\otimes I_m^S](f) = I_n^S(g)\otimes I_m^S(h).$$

From the Wigner isometry for multiple integrals, we obtain the so-called Wigner bisometry: for $f\in L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m)$ and $g\in L^2(\mathbb{R}_+^{n'})\otimes L^2(\mathbb{R}_+^{m'})$ having the form of a tensor product,

$$\begin{aligned}\varphi\otimes\varphi([I_n^S\otimes I_m^S](f)[I_{n'}^S\otimes I_{m'}^S](g)^*) \\ = \begin{cases}\langle f, g\rangle_{L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m)} & \text{if } n=n' \text{ and } m=m',\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{4}$$

Formula (4) is then extended linearly to generic elements $f\in L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m)\cong L^2(\mathbb{R}_+^{n+m})$, where the symbol $\cong$ denotes an isomorphic identification.

A crucial tool in the analysis of Wigner integrals is the product formula (1), and a biproduct formula for bi-integrals was recently obtained in [Bourguin and Campese 2018], which will be a crucial tool in the sequel. It makes use of a new type of contraction, referred to in [Bourguin and Campese 2018] as bi-contractions, defined as follows. Let n_1, m_1, nf_2, m_2 be positive integers. Let

$f \in L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1}) \cong L^2(\mathbb{R}_+^{n_1+m_1})$ and $g \in L^2(\mathbb{R}_+^{n_2}) \otimes L^2(\mathbb{R}_+^{m_2}) \cong L^2(\mathbb{R}_+^{n_2+m_2})$ and let $p \le n_1 \wedge n_2,\ r \le m_1 \wedge m_2$ be natural numbers. The (p,r)-bicontraction $f \overset{p,r}{\frown} g$ is the $L^2(\mathbb{R}_+^{n_1+n_2-2p}) \otimes L^2(\mathbb{R}_+^{m_1+m_2-2r}) \cong L^2(\mathbb{R}_+^{n_1+n_2+m_1+m_2-2p-2r})$ function defined by

$$\begin{aligned} f \overset{p,r}{\frown} g&(t_1,\dots,t_{n_1+n_2+m_1+m_2-2p-2r}) \\ &= \int_{\mathbb{R}_+^{p+r}} f\big(t_1,\dots,t_{n_1-p},s_p,\dots,s_1,y_1,\dots,y_r, \\ &\qquad\qquad\qquad\qquad t_{n_1+n_2+m_2-2p-r+1},\dots,t_{n_1+n_2+m_1+m_2-2p-2r}\big) \\ &\quad \times g(s_1,\dots,s_p,t_{n_1-p+1},\dots,t_{n_1+n_2+m_2-2p-r},y_r,\dots,y_1)\,ds_1\cdots ds_p\,dy_1\cdots dy_r. \end{aligned}$$

Remark 2.12. Observe that these bicontractions have the following properties (for a proof, see [Bourguin and Campese 2018]). For $n_1,m_1,n_2,m_2 \in \mathbb{N}$, let $f \in L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1}) \cong L^2(\mathbb{R}_+^{n_1+m_1})$ and $g \in L^2(\mathbb{R}_+^{n_2}) \otimes L^2(\mathbb{R}_+^{m_2}) \cong L^2(\mathbb{R}_+^{n_2+m_2})$ be fully symmetric functions. Furthermore, let $p \le n_1 \wedge n_2$ and $r \le m_1 \wedge m_2$ be natural numbers such that $p+r = p'+r'$. Then, the following holds.

(1) $f \overset{p,r}{\frown} g \cong f \overset{p+r}{\frown} g$.

(2) $f \overset{p,r}{\frown} g = f \overset{p',r'}{\frown} g$.

(3) $\|f \overset{p,r}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n_1+n_2-2p}) \otimes L^2(\mathbb{R}_+^{m_1+m_2-2r})} = \|f \overset{p+r}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n_1+n_2+m_1+m_2-2p-2r})}$.

(4) $f \overset{n_1,m_1}{\frown} f = \|f\|^2_{L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1})} 1 \otimes 1$, which is a constant in $L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1})$.

We introduce $\sharp$ to be the associative action of $\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$ (where $\mathscr{A}^{\mathrm{op}}$ denotes the opposite algebra) on $\mathscr{A} \otimes \mathscr{A}$, as

$$(A \otimes B)\sharp(C \otimes D) = (AC) \otimes (DB). \tag{5}$$

We also write $\sharp$ to denote the action of $\mathscr{A} \otimes L^2(\mathbb{R}_+) \otimes \mathscr{A}^{\mathrm{op}}$ on $\mathscr{A} \otimes L^2(\mathbb{R}_+) \otimes \mathscr{A}$, as

$$(A \otimes f \otimes B)\sharp(C \otimes g \otimes D) = (AC) \otimes fg \otimes (DB).$$

Using the bicontractions definition, the biproduct formula for Wigner bi-integrals proved in [Bourguin and Campese 2018] can be stated as follows.

Proposition. *For $n_1,m_1,n_2,m_2 \in \mathbb{N}$, let $f \in L^2(\mathbb{R}_+^{n_1}) \otimes L^2(\mathbb{R}_+^{m_1}) \cong L^2(\mathbb{R}_+^{n_1+m_1})$ and $g \in L^2(\mathbb{R}_+^{n_2}) \otimes L^2(\mathbb{R}_+^{m_2}) \cong L^2(\mathbb{R}_+^{n_2+m_2})$. Then*

$$[I^S_{n_1} \otimes I^S_{m_1}](f)\sharp[I^S_{n_2} \otimes I^S_{m_2}](g) = \sum_{p=0}^{n_1\wedge n_2}\sum_{r=0}^{m_1\wedge m_2} [I^S_{n_1+n_2-2p} \otimes I^S_{m_1+m_2-2r}](f \overset{p,r}{\frown} g). \tag{6}$$

Finally, the free gradient operator $\nabla : L^2(\mathcal{S}, \varphi) \to \mathscr{B}_2$ is a densely defined and closable operator whose action on Wigner integrals is given by

$$\nabla_t I_n^S(f) = \sum_{k=1}^{n} [I_{k-1}^S \otimes I_{n-k}^S](f_t^{(k)}),$$

where $f_t^{(k)}(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{k-1}, t, x_k, \dots, x_{n-1})$ is viewed as an element of $L^2(\mathbb{R}_+^{k-1}) \otimes L^2(\mathbb{R}_+^{n-k})$. We also define the pairing $\langle \cdot, \cdot \rangle$ between $\mathscr{B}_2 \times \mathscr{B}_2$ and $L^2(\mathscr{A} \otimes \mathscr{A}, \varphi \otimes \varphi)$ to be

$$\langle \cdot, \cdot \rangle : \mathscr{B}_2 \times \mathscr{B}_2 \mapsto L^2(\mathscr{A} \otimes \mathscr{A}, \varphi \otimes \varphi), \quad \langle U, V \rangle = \int_{\mathbb{R}_+} U_s \sharp V_s^* ds. \tag{7}$$

3. Characterizations of freeness

In this section, we are interested in providing several characterizations of freeness between two multiple integrals. We will derive those characterizations in terms of contractions, covariances and free Malliavin gradients respectively.

Characterization in terms of contractions. Recall the well-known characterization of independence of multiple Wiener–Itô integrals by Üstünel and Zakai [1989] in terms of the first contraction of the associated kernels.

Theorem 3.1 [Üstünel and Zakai 1989]. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then, $I_n^W(f)$ and $I_m^W(g)$ are independent if and only if $f \otimes_1 g = 0$ almost everywhere (for the definition of $\otimes_1$, see the first point of Remark 3.2 below).*

Remark 3.2. • In Theorem 3.1 and throughout the text, the notation $\otimes_r$ stands for the usual r-th contraction operator, defined as follows: if $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ are symmetric and if $r \in \{1, \dots, n \wedge m\}$, we set

$$(f \otimes_r g)(t_1, \dots, t_{n+m-2r}) = \int_{\mathbb{R}_+^r} f(t_1, \dots, t_{n-r}, x_1, \dots, x_r) \times g(t_{n-r+1}, \dots, t_{n+m-2r}, x_1, \dots, x_r) dx_1 \dots dx_r.$$

• In the context of a multiple Wiener–Itô integral $I_n^W(f)$, note that one can always assume without loss of generality that the kernel f is symmetric, as $I_n^W(f) = I_n^W(\tilde{f})$, where $\tilde{f}$ denotes the symmetrization of the function f given by

$$\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

with $\mathfrak{S}_n$ the symmetric group of $\{1, \dots, n\}$.

A natural question is to ask whether or not the characterization of independence of Üstünel and Zakai has a counterpart in the free setting. It turns out that a similar characterization of freeness holds on both the Wigner and the free Poisson space, which is the first result of this paper.

Theorem 3.3. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then*:

(i) *$I_n^S(f)$ and $I_m^S(g)$ are free if and only if $f \overset{1}{\frown} g = 0$ almost everywhere.*

(ii) *$I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free if and only if $f \star_1^0 g = 0$ almost everywhere.*

Proof. First, assume that $I_n^{\mathfrak{M}}(f)$ and $I_m^{\mathfrak{M}}(g)$ are free. Then, by Definition 2.1, it holds that, in particular

$$\begin{aligned}\varphi\big([I_n^{\mathfrak{M}}(f)^2 - \varphi(I_n^{\mathfrak{M}}(f)^2)][I_m^{\mathfrak{M}}(g)^2 - \varphi(I_m^{\mathfrak{M}}(g)^2)]\big)\\ = \varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2) - \varphi(I_n^{\mathfrak{M}}(f)^2)\varphi(I_m^{\mathfrak{M}}(g)^2) = 0.\end{aligned}$$

Observe that

$$\begin{aligned}\varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2) &= \sum_{p=0}^{n}\sum_{r=0}^{m} \varphi(I_{2n-2p}^{\mathfrak{M}}(f \overset{p}{\frown} f) I_{2m-2r}^{\mathfrak{M}}(g \overset{r}{\frown} g)) \\ &\quad + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n}\sum_{r=1}^{m} \varphi(I_{2n-2p+1}^{\mathfrak{M}}(f \star_p^{p-1} f) I_{2m-2r+1}^{\mathfrak{M}}(g \star_r^{r-1} g)) \\ &= \sum_{p=0}^{n}\sum_{r=0}^{m} \varphi(I_{2p}^{\mathfrak{M}}(f \overset{n-p}{\frown} f) I_{2r}^{\mathfrak{M}}(g \overset{m-r}{\frown} g)) \\ &\quad + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=0}^{n-1}\sum_{r=0}^{m-1} \varphi(I_{2p+1}^{\mathfrak{M}}(f \star_{n-p}^{n-p-1} f) I_{2r+1}^{\mathfrak{M}}(g \star_{m-r}^{m-r-1} g)).\end{aligned}$$

Using the isometry property (3), we get

$$\begin{aligned}\varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2) &= \sum_{p=0}^{n\wedge m} \langle f \overset{n-p}{\frown} f, g \overset{m-p}{\frown} g\rangle_{L^2(\mathbb{R}_+^{2p})} \\ &\quad + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=0}^{(n\wedge m)-1} \langle f \star_{n-p}^{n-p-1} f, g \star_{m-p}^{m-p-1} g\rangle_{L^2(\mathbb{R}_+^{2p+1})} \\ &= \sum_{p=0}^{n\wedge m} \| f \overset{p}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p})} + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n\wedge m} \| f \star_p^{p-1} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})} \\ &= \|f\|^2_{L^2(\mathbb{R}_+^n)} \|g\|^2_{L^2(\mathbb{R}_+^m)} + \sum_{p=1}^{n\wedge m} \| f \overset{p}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p})} \\ &\quad + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n\wedge m} \| f \star_p^{p-1} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})}.\end{aligned}$$

Recalling that $\varphi(I_n^{\mathfrak{M}}(f)^2)=\|f\|^2_{L^2(\mathbb{R}_+^n)}$ and $\varphi(I_m^{\mathfrak{M}}(g)^2)=\|g\|^2_{L^2(\mathbb{R}_+^m)}$ yields

$$\begin{aligned}(8)\quad &\varphi(I_n^{\mathfrak{M}}(f)^2 I_m^{\mathfrak{M}}(g)^2)-\varphi(I_n^{\mathfrak{M}}(f)^2)\varphi(I_m^{\mathfrak{M}}(g)^2)\\ &\qquad=\sum_{p=1}^{n\wedge m}\|f\overset{p}{\frown} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p})}+\mathbb{1}_{\{\mathfrak{M}=\hat{N}\}}\sum_{p=1}^{n\wedge m}\|f\star_p^{p-1} g\|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})}.\end{aligned}$$

As the left-hand side of the above equality is zero, the fact that $f\overset{1}{\frown} g=0$ a.e. in the Wigner case and $f\star_1^0 g=0$ a.e. in the free Poisson case follows.

Conversely, assume that $f\overset{1}{\frown} g=0$ a.e. in the Wigner case and that $f\star_1^0 g=0$ a.e. in the free Poisson case. According to Definition 2.1 together with the linearity of the functional φ, we must prove that, for any natural number ℓ and for any natural numbers $k_1,\dots,k_{2\ell}$,

$$\begin{aligned}&\varphi([I_n^{\mathfrak{M}}(f)^{k_1}-\varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2}-\varphi(I_m^{\mathfrak{M}}(g)^{k_2})]\\ &\qquad\cdots[I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}}-\varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g)^{k_{2\ell}}-\varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})])=0.\end{aligned}$$

Remark 3.4. Observe that we only consider an even number of powers k. This comes from the tracial property of the functional φ together with the condition that no two adjacent indices i_j can be equal in Definition 2.1. Indeed, if we consider an odd number of powers k, we would have

$$\begin{aligned}&\varphi([I_n^{\mathfrak{M}}(f)^{k_1}-\varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2}-\varphi(I_m^{\mathfrak{M}}(g)^{k_2})]\\ &\qquad\qquad\cdots[I_n^{\mathfrak{M}}(f)^{k_{2\ell+1}}-\varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell+1}})])\\ &\quad=\varphi([I_n^{\mathfrak{M}}(f)^{k_{2\ell+1}}-\varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell+1}})][I_n^{\mathfrak{M}}(f)^{k_1}-\varphi(I_n^{\mathfrak{M}}(f)^{k_1})]\\ &\qquad\qquad[I_m^{\mathfrak{M}}(g)^{k_2}-\varphi(I_m^{\mathfrak{M}}(g)^{k_2})]\cdots[I_m^{\mathfrak{M}}(g)^{k_{2\ell}}-\varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})]),\end{aligned}$$

where the first two indices would be the same in the framework of Definition 2.1.

Let $q<k$ be two nonnegative integers. For $0\le q\le k-1$, define the multisets $S_q^k=\{1,\dots,1,0,\dots,0\}$ where the element 1 has multiplicity q and the element 0 has multiplicity $k-q-1$. Such a set is sometimes denoted $\{(1,q),(0,k-q-1)\}$. We denote the group of permutations of the multiset S_q^k by $\mathfrak{S}_q^k$ and its cardinality is given by the multinomial coefficient

$$\binom{k-1}{q,\,m-q-1}=\frac{(k-1)!}{q!(k-q-1)!}=\binom{k-1}{q}.$$

Observe that in the definition of the group of permutations of a multiset, each permutation yields a different ordering of the elements of the multiset, which is why the cardinality of $\mathfrak{S}_q^k$ is $\binom{k-1}{q}$ and not $(k-1)!$. Using the Wigner and free Poisson product formulas along with Equation (4.1) in [Nourdin and Peccati 2013] and Lemma 4.1 in [Bourguin 2015], we can write

$$I_n^{\mathfrak{M}}(f)^k = \varphi(I_n^{\mathfrak{M}}(f)^k) + \sum_{r=1}^{kn} I_r^{\mathfrak{M}}(a_r(f)) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{r=1}^{kn} I_r^{\mathfrak{M}}(b_r(f)),$$

where

$$a_r(f) = \sum_{(p_1,\dots,p_{k-1})\in A_r} (\cdots((f \overset{p_1}{\frown} f) \overset{p_2}{\frown} f)\cdots f) \overset{p_{k-1}}{\frown} f$$

with

$$A_r = \left\{(p_1,\dots,p_{k-1}) \in \{0,1,\dots,n\}^{k-1} : kn - 2\sum_i^{k-1} p_i = r\right\}$$

and where (recall Definition 2.10 for the contractions appearing below)

$$b_r(f) = \sum_{q=1}^{k-1} \sum_{\pi\in\mathfrak{S}_q^k} \sum_{(p_1,\dots,p_{k-1})\in B_{r,q}^{\pi}} (\cdots((f \star_{p_1}^{p_1-\pi(1)} f) \star_{p_2}^{p_2-\pi(2)} f)\cdots f) \star_{p_{k-1}}^{p_{k-1}-\pi(k-1)} f$$

with, for each $q = 1,\dots,k-1$ and each $\pi \in \mathfrak{S}_q^k$,

$$B_{r,q}^{\pi} = \left\{(p_1,\dots,p_{k-1}) \in \bigotimes_{s=1}^{k-1} \{\pi(s),\dots,n\} : kn + q - 2\sum_i^{k-1} p_i = r\right\}.$$

We get that

$$\begin{aligned}
&[I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \\
&\qquad \cdots [I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})] \\
&= \sum_{r_1=1}^{k_1 n} \sum_{r_2=1}^{k_2 m} \cdots \sum_{r_{2\ell-1}=1}^{k_{2\ell-1} n} \sum_{r_{2\ell}=1}^{k_{2\ell} m} I_{r_1}^{\mathfrak{M}}(a_{r_1}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_1}(f)) \\
&\qquad \times I_{r_2}^{\mathfrak{M}}(a_{r_2}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_2}(g)) \cdots I_{r_{2\ell-1}}^{\mathfrak{M}}(a_{r_{2\ell-1}}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{2\ell-1}}(f)) \\
&\qquad\qquad \times I_{r_{2\ell}}^{\mathfrak{M}}(a_{r_{2\ell}}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{2\ell}}(g)).
\end{aligned}$$

At this point, observe that the assumptions that $f \overset{1}{\frown} g = 0$ a.e in the Wigner case and $f \star_1^0 g = 0$ a.e in the free Poisson case imply, by Lemmas 6.1 and 6.2 respectively, that for any given $i = 1,\dots,2\ell-1$, the contractions between $(a_{r_i}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_i}(f))$ and $(a_{r_{i+1}}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{i+1}}(g))$ resulting from using the appropriate product formula iteratively will all be zero a.e. except for the ones of order zero corresponding to the tensor product operation (it is the only contraction that can be nonzero under both the Wigner and free Poisson case assumptions).

Remark 3.5. Note that for the above argument to hold, we need to assume that the functions f and g are symmetric in order to be able to freely reorder variables appearing in the contractions of $a_{r_i}(f)$ and $a_{r_j}(g)$ (as well as in the contractions of $b_{r_{i+1}}(f)$ and $b_{r_{j+1}}(g)$) so that the assumptions $f \overset{1}{\frown} g = 0$ a.e. in the Wigner case and $f \star_1^0 g = 0$ a.e. in the free Poisson case can be used to deduce that the resulting contractions will all be zero.

Hence, keeping only the nonzero terms in the above expression yields

$$\begin{aligned}
&[I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \\
&\qquad \cdots [I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})] \\
&= \sum_{r_1=1}^{k_1 n} \sum_{r_2=1}^{k_2 m} \cdots \sum_{r_{2\ell-1}=1}^{k_{2\ell-1} n} \sum_{r_{2\ell}=1}^{k_{2\ell} m} I^{\mathfrak{M}}_{r_1+\cdots+r_{2\ell}}((a_{r_1}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_1}(f)) \\
&\qquad \otimes (a_{r_2}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_2}(g)) \otimes \cdots \otimes (a_{r_{2\ell-1}}(f) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{2\ell-1}}(f)) \\
&\qquad\qquad \otimes (a_{r_{2\ell}}(g) + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} b_{r_{2\ell}}(g))).
\end{aligned}$$

As the quantity $r_1 + \cdots + r_{2\ell}$ is strictly positive, applying φ to the above expression yields

$$\begin{aligned}
&\varphi\big([I_n^{\mathfrak{M}}(f)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f)^{k_1})][I_m^{\mathfrak{M}}(g)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g)^{k_2})] \\
&\qquad \cdots [I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g)^{k_{2\ell}})]\big) = 0,
\end{aligned}$$

which is the desired result. □

Observe that the above characterization of freeness is stated and proven for symmetric kernels only. A natural question is whether or not this characterization continues to hold in the more general case of a mirror-symmetric kernel. We provide a negative answer to this question, proving that our characterization is exhaustive. Concretely, we will exhibit two mirror-symmetric kernels $f, g \in L^2([0,2]^3)$ such that $\| f \overset{1}{\frown} g \|_{L^2([0,2]^3)} = 0$ but $I_3^S(f)$ and $I_3^S(g)$ are not free.

Indeed, consider $f = \mathbb{1}_{[0,1]\times[0,2]\times[0,1]}$ and $g = \mathbb{1}_{[1,2]\times[0,2]\times[1,2]}$. It is readily checked that $f \overset{1}{\frown} g = 0$. On the other hand, using the product formula (1) iteratively, we can write

$$\begin{aligned}
I_3^S(f)^7 &= \sum_{(r_1,\dots,r_6)\in C} I^S_{21-2r_1-\cdots-2r_6}((((((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f) \overset{r_3}{\frown} f) \overset{r_4}{\frown} f) \overset{r_5}{\frown} f) \overset{r_6}{\frown} f) \\
I_3^S(g)^7 &= \sum_{(r_1,\dots,r_6)\in C} I^S_{21-2r_1-\cdots-2r_6}((((((g \overset{r_1}{\frown} g) \overset{r_2}{\frown} g) \overset{r_3}{\frown} g) \overset{r_4}{\frown} g) \overset{r_5}{\frown} g) \overset{r_6}{\frown} g),
\end{aligned}$$

where

$$\begin{aligned}
C = \{(r_1, \dots, r_6) \in \{0,1,2,3\}^6 : r_2 \le 6 - 2r_1,& \\
r_3 \le 9 - 2r_1 - 2r_2, \dots, r_6 \le 18 - 2r_1 - \cdots - 2r_5\}&.
\end{aligned}$$

Using the Wigner isometry (3), we deduce that $\varphi(I_3^S(f)^7) = 0$ and $\varphi(I_3^S(g)^7) = 0$, as well as (the functions f and g being positive)

$$\varphi(I_3^S(f)^7 I_3^S(g)^7) \geq \Big\langle \Big(((((f \overset{2}{\frown} f) \overset{2}{\frown} f) \overset{1}{\frown} f) \overset{1}{\frown} f) \overset{1}{\frown} f\Big) \overset{3}{\frown} f, \\ \Big(((((g \overset{2}{\frown} g) \overset{2}{\frown} g) \overset{1}{\frown} g) \overset{1}{\frown} g) \overset{1}{\frown} g\Big) \overset{3}{\frown} g \Big\rangle_{L^2([0,2])} = 32 \neq 0.$$

Consequently, according to the definition of freeness given in Definition 2.1, $I_3^S(f)$ and $I_3^S(g)$ are not free.

Remark 3.6. The same counterexample would also yield the same conclusion in the free Poisson case (replacing the Wigner integrals by free Poisson ones) as it is also the case that $f \star_1^0 g = 0$ and as the first part of the free Poisson product formula (2) is the same as the Wigner product formula used above.

However, even if establishing a characterization of freeness in terms of contractions in the mirror-symmetric case is not possible, we can still give a sufficient condition for freeness, which is the object of the following result.

Theorem 3.7. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be mirror-symmetric functions.*

(i) *If dealing with Wigner integrals, assume that $f^{(\sigma)} \overset{1}{\frown} g^{(\pi)} = 0$ almost everywhere for all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$, where*

$$f^{(\sigma)}(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad x_1, \dots, x_n \in \mathbb{R}_+,$$

and a similar definition for $g^{(\pi)}$. Then, $I_n^S(f)$ and $I_m^S(g)$ are free.

(ii) *If dealing with free Poisson integrals, assume that $f^{(\sigma)} \star_1^0 g^{(\pi)} = 0$ almost everywhere for all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$. Then, one has that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free.*

Proof. Apply the same strategy as in the proof of Theorem 3.3 with the stronger assumptions. □

Characterization in terms of covariances. The next result is a free analog of [Rosiński and Samorodnitsky 1999, Corollary 5.2], which is itself a consequence of Theorem 3.1 by Üstünel and Zakai.

Corollary 3.8. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then, $I_n^{\mathfrak{M}}(f)$ and $I_m^{\mathfrak{M}}(g)$ are free if and only if their squares are uncorrelated, i.e., if and only if*

$$\mathrm{Cov}(I_n^{\mathfrak{M}}(f)^2, I_m^{\mathfrak{M}}(g)^2) = 0.$$

Proof. First, assume that $I_n^{\mathfrak{M}}(f)$ and $I_m^{\mathfrak{M}}(g)$ are free. Then, by Definition 2.1,

$$\varphi\big([I_n^{\mathfrak{M}}(f)^2-\varphi(I_n^{\mathfrak{M}}(f)^2)][I_m^{\mathfrak{M}}(g)^2-\varphi(I_m^{\mathfrak{M}}(g)^2)]\big) \\ =\varphi(I_n^{\mathfrak{M}}(f)^2I_m^{\mathfrak{M}}(g)^2)-\varphi(I_n^{\mathfrak{M}}(f)^2)\varphi(I_m^{\mathfrak{M}}(g)^2)=0.$$

As $\operatorname{Cov}(I_n^{\mathfrak{M}}(f)^2, I_m^{\mathfrak{M}}(g)^2)=\varphi(I_n^{\mathfrak{M}}(f)^2I_m^{\mathfrak{M}}(g)^2)-\varphi(I_n^{\mathfrak{M}}(f)^2)\varphi(I_m^{\mathfrak{M}}(g)^2)$, the desired conclusion follows.

Conversely, assume that $\operatorname{Cov}(I_n^{\mathfrak{M}}(f)^2, I_m^{\mathfrak{M}}(g)^2)=0$. Using (8), it holds that

$$\operatorname{Cov}(I_n^{\mathfrak{M}}(f)^2, I_m^{\mathfrak{M}}(g)^2) \\ =\sum_{p=1}^{n\wedge m}\|f\overset{p}{\frown}g\|^2_{L^2(\mathbb{R}_+^{n+m-2p})}+\mathbb{1}_{\{\mathfrak{M}=\hat{N}\}}\sum_{p=1}^{n\wedge m}\|f\star_p^{p-1}g\|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})},$$

which implies that all the contraction norms appearing on the right-hand side of the above equality are zero. In particular, in the Wigner case, $\|f\overset{1}{\frown}g\|^2_{L^2(\mathbb{R}_+^{n+m-2})}=0$, and in the free Poisson case, $\|f\star_1^0 g\|^2_{L^2(\mathbb{R}_+^{n+m-1})}=0$, which, by Theorem 3.3 implies that $I_n^{\mathfrak{M}}(f)$ and $I_m^{\mathfrak{M}}(g)$ are free. □

Characterization in terms of free Malliavin gradients. In the context of Wiener integrals, Üstünel and Zakai [1989, Proposition 2] proved that a necessary condition for two Wiener integrals $I_n^W(f)$ and $I_m^W(g)$ to be independent was that the inner product of their Malliavin derivatives was zero almost surely. More precisely, their statement reads as follows.

Theorem 3.9 [Üstünel and Zakai 1989]. *A necessary condition for the independence of $I_n^W(f)$ and $I_m^W(g)$ is*

$$\langle DI_n^W(f), DI_m^W(g)\rangle_{L^2(\mathbb{R}_+)}=0\quad a.s. \tag{9}$$

However, they were also able to show that this condition is not sufficient and hence cannot provide a proper characterization of independence of Wiener integrals. The technical reason for this is that this condition implies that only the symmetrization of the first contraction of f and g be zero almost everywhere, which in turn does not necessarily imply that the first contraction itself be zero almost everywhere. As the latter is an equivalent statement to independence, the sufficiency of (9) fails.

In the free case, a free version of the Malliavin calculus (with respect to the free Brownian motion) has been developed by Biane and Speicher [1998], and it is a natural question to ask whether it can be used to provide a characterization of freeness for Wigner integrals.

Remark 3.10. In this subsection, we only focus on Wigner integrals and not on the free Poisson case. The reason for this is that there is no free Malliavin calculus available for free Poisson random measures, which is what would be needed to explore similar statements in the free Poisson case.

The following result is the main result of this subsection, which is a characterization of freeness in terms of the free gradient operator for Wigner integrals with symmetric kernels. It is worth noting that, as opposed to the case of Wiener integrals studied by Üstünel and Zakai, we are able to provide a positive answer to the question of characterizing freeness in terms of free gradients, which illustrates a fundamental difference between the classical case and the free case.

Theorem 3.11. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Then, $I_n^S(f)$ and $I_m^S(g)$ are free if and only if*

$$\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle = 0 \text{ in } L^2(\mathcal{A}\otimes\mathcal{A}, \varphi\otimes\varphi), \tag{10}$$

where the notation $\langle\cdot,\cdot\rangle$ is defined in (7).

Proof. In the following we will use the shorthand $f_s^{(k)}$ to denote the function given by

$$f_s^{(k)}(x_1,\dots,x_{n-1}) = f(x_1,\dots,x_{k-1},s,x_{k+1},\dots,x_n).$$

Applying the definition of the action of ∇ on Wigner integrals, we get that

$$\begin{aligned}\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle &= \int_{\mathbb{R}_+} (\nabla_s I_n^S(f))\sharp(\nabla_s I_m^S(g))^* \, ds\\ &= \sum_{k=1}^n\sum_{q=1}^m \int_{\mathbb{R}_+} [I_{k-1}^S \otimes I_{n-k}^S](f_s^{(k)})\sharp([I_{q-1}^S\otimes I_{m-q}^S](g_s^{(q)}))^* \, ds\\ &= \sum_{k=1}^n\sum_{q=1}^m \int_{\mathbb{R}_+} [I_{k-1}^S \otimes I_{n-k}^S](f_s^{(k)})\sharp[I_{q-1}^S\otimes I_{m-q}^S](g_s^{(q)}) \, ds,\end{aligned}$$

where the last equality follows from the full symmetry of the function g. The biproduct formula (6) yields

$$\begin{aligned}&\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle\\ &\quad= \sum_{k=1}^n\sum_{q=1}^m \int_{\mathbb{R}_+} \sum_{p=0}^{(k\wedge q)-1}\ \sum_{r=0}^{(n-k)\wedge(m-q)} [I_{k+q-2-2p}^S \otimes I_{n+m-k-q-2r}^S](f_s^{(k)} \overset{p,r}{\frown} g_s^{(q)}) \, ds,\end{aligned}$$

and by using a Fubini argument, it follows that

$$\begin{aligned}&\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle\\ &\quad= \sum_{k=1}^n\sum_{q=1}^m \sum_{p=0}^{(k\wedge q)-1}\ \sum_{r=0}^{(n-k)\wedge(m-q)} [I_{k+q-2-2p}^S \otimes I_{n+m-k-q-2r}^S]\Big(\int_{\mathbb{R}_+} f_s^{(k)} \overset{p,r}{\frown} g_s^{(q)} \, ds\Big).\end{aligned}$$

The full symmetry of f and g implies that $f_s^{(k)} = f_s^{(n)}$ for every $1 \le k \le n$ and $g_s^{(q)} = g_s^{(1)}$ for every $1 \le q \le m$. Hence, using Remark 2.12, we get

$$\int_{\mathbb{R}_+} f_s^{(k)} \overset{p,r}{\frown} g_s^{(q)} ds = f \overset{p+r+1}{\frown} g,$$

so that we finally get

$$(11)\quad \langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle = \sum_{k=1}^{n}\sum_{q=1}^{m}\sum_{p=0}^{(k\wedge q)-1}\sum_{r=0}^{(n-k)\wedge(m-q)} [I^S_{k+q-2-2p} \otimes I^S_{n+m-k-q-2r}](f \overset{p+r+1}{\frown} g).$$

Using the Wigner bisometry (4), we see that the quantity

$$\varphi\otimes\varphi(|\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle|^2)$$

is just a sum with strictly positive coefficients only involving the contractions norms

$$\|f \overset{1}{\frown} g\|^2_{L^2(R_+^{n+m-2})}, \|f \overset{2}{\frown} g\|^2_{L^2(R_+^{n+m-4})}, \dots, \|f \overset{n\wedge m}{\frown} g\|^2_{L^2(R_+^{n+m-2(n\wedge m)})}.$$

Formally, we have an equality of the type

$$(12)\qquad \varphi\otimes\varphi(|\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle|^2) = \sum_{u=1}^{n\wedge m} c_u \|f \overset{u}{\frown} g\|^2_{L^2(R_+^{n+m-2u})},$$

with $c_u > 0$.

Now assume that $I_n^S(f)$ and $I_m^S(g)$ are free. By Theorem 3.3, this is equivalent to $f \overset{1}{\frown} g = 0$ almost everywhere, which by Lemma 6.1 implies that $f \overset{p}{\frown} g = 0$ almost everywhere for all $1 \le p \le n \wedge m$. Using (12), we get (10).

Conversely, assume that

$$\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle = 0.$$

Then,

$$\varphi\otimes\varphi(|\langle \nabla I_n^S(f), \nabla I_m^S(g)\rangle|^2) = 0.$$

This implies that all the norms appearing in the representation (12) are zero, and in particular that $f \overset{1}{\frown} g = 0$ almost everywhere. Using Theorem 3.3 concludes the proof. □

4. Characterizations of asymptotic freeness

In the asymptotic context, the problem of interest is to find necessary and sufficient conditions for the limits in law of multiple integrals to be free. It is a much more general problem than before, as limits in law of multiple integrals need not be multiple integrals themselves.

Characterization in terms of contractions. In the classical case, the following result holds.

Theorem 4.1 [Nourdin and Rosiński 2014, Theorem 3.1]. *Let n, m be natural numbers and let $\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$ be sequences of symmetric functions. Assume that $(I_n^W(f_k), I_m^W(g_k)) \xrightarrow{\text{law}} (F, G)$ as $k \to \infty$, where F, G are square integrable random variables with laws determined by their moments. Then, F and G are independent if and only if $f_k \otimes_p g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2p})$ for all $p = 1, \ldots, n \wedge m$.*

Remark 4.2. The fact that the limiting random variables in the above theorem need to have laws determined by their moments (a condition that we get automatically in the free setting) has been shown in [Nourdin et al. 2016] to be not necessary. On the other hand, observe that the necessary and sufficient condition for asymptotic independence is not

$$f_k \otimes_1 g_k \xrightarrow[k\to+\infty]{} 0 \ \text{ in } \ L^2(\mathbb{R}_+^{n+m-2}),$$

as one could have expected in view of Theorem 3.1. This weaker condition is necessary but not sufficient in the asymptotic case, as pointed out in [Nourdin and Rosiński 2014, Remark 3.2]. In the free case, the same phenomenon happens in the sense that the condition $f_k \overset{1}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2})$ (in the Wigner case) and $f_k \star_1^0 g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2})$ (in the free Poisson case) will prove to be necessary but not sufficient either, for the same reason.

The following result in the free case is hence rather an analog of the stronger results of [Nourdin et al. 2016] instead of those found in [Nourdin and Rosiński 2014]. In Theorem 4.1 or in the forthcoming Theorem 4.3, note that F and G do not need to have the form of a multiple integral. This implies that sequences of multiple integrals can be used in order to prove the freeness of general random variables in $L^2(\varphi)$ (provided these random variables admit approximating sequences of multiple integrals with symmetric kernels).

Theorem 4.3. *Let n, m be natural numbers and let $\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$ be sequences of symmetric functions such that*

$$(I_n^{\mathfrak{M}}(f_k), I_m^{\mathfrak{M}}(g_k)) \xrightarrow{\text{law}} (F, G) \tag{13}$$

as $k \to \infty$, where F, G are random variables in $L^2(\mathscr{A}, \varphi)$. Then,

(i) *If $\mathfrak{M} = S$, then F and G are free if and only if $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2p})$ for all $p = 1, \ldots, n \wedge m$.*

(ii) *If $\mathfrak{M} = \hat{N}$, then F and G are free if and only if $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2p})$ and $f_k \star_p^{p-1} g_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^{n+m-2p+1})$ for all $p = 1, \ldots, n \wedge m$.*

Proof. First, assume that F and G are free. Then, $\mathrm{Cov}(F^2, G^2) = 0$. Using (8) along with assumption (13) yields

$$\mathrm{Cov}(I_n^{\mathfrak{M}}(f_k)^2, I_m^{\mathfrak{M}}(g_k)^2) = \sum_{p=1}^{n\wedge m} \| f_k \overset{p}{\frown} g_k \|^2_{L^2(\mathbb{R}_+^{n+m-2p})} + \mathbb{1}_{\{\mathfrak{M}=\hat{N}\}} \sum_{p=1}^{n\wedge m} \| f_k \star_p^{p-1} g_k \|^2_{L^2(\mathbb{R}_+^{n+m-2p+1})} \xrightarrow[k\to+\infty]{} \mathrm{Cov}(F^2, G^2) = 0,$$

so that for all $p = 1, \dots, n \wedge m$, $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ (in the Wigner case) and for all $p = 1, \dots, n \wedge m$, $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ and $f_k \star_p^{p-1} g_k \xrightarrow[k\to+\infty]{} 0$ (in the free Poisson case).

Conversely, assume that, for all $p = 1, \dots, n \wedge m$, $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ (in the Wigner case) or that, for all $p = 1, \dots, n \wedge m$, $f_k \overset{p}{\frown} g_k \xrightarrow[k\to+\infty]{} 0$ and $f_k \star_p^{p-1} g_k \xrightarrow[k\to+\infty]{} 0$ (in the free Poisson case). As in the proof of Theorem 3.3 (together with assumption (13)), these conditions imply that, for any natural number ℓ and for any natural numbers $k_1, \dots, k_{2\ell}$,

$$\varphi\big([I_n^{\mathfrak{M}}(f_k)^{k_1} - \varphi(I_n^{\mathfrak{M}}(f_k)^{k_1})][I_m^{\mathfrak{M}}(g_k)^{k_2} - \varphi(I_m^{\mathfrak{M}}(g_k)^{k_2})] \cdots [I_n^{\mathfrak{M}}(f_k)^{k_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f_k)^{k_{2\ell-1}})][I_m^{\mathfrak{M}}(g_k)^{k_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g_k)^{k_{2\ell}})]\big) \xrightarrow[k\to+\infty]{} 0,$$

which implies that F and G are free as they are determined by their moments. □

Remark 4.4. Observe that the only difference between the proofs of Theorem 3.3 and Theorem 4.3 is the fact that in the nonasymptotic case, we have one additional step which states that the seemingly weaker condition $f \overset{1}{\frown} g = 0$ a.e. implies that, for all $p = 1, \dots, n \wedge m$, $f \overset{p}{\frown} g = 0$ a.e. (in the Wigner case) and that the condition $f \star_1^0 g = 0$ a.e. implies that, for all $p = 1, \dots, n \wedge m$, $f \overset{p}{\frown} g = 0$ and $f \star_p^{p-1} g = 0$ a.e. (in the free Poisson case). Recall that these implications do not necessarily hold true asymptotically, as pointed out in [Nourdin and Rosiński 2014, Remark 3.2]. For instance, the sequence $\{f_k : n \geq 1\} \subset L^2([0, 1]^2)$ given by

$$f_k = \sqrt{k} \sum_{i=0}^{k-1} \mathbb{1}_{[i/k,(i+1)/k]^2}$$

satisfies $f_k \overset{1}{\frown} f_k \xrightarrow[k\to+\infty]{} 0$ in $L^2(\mathbb{R}_+^2)$, although $f_k \overset{2}{\frown} f_k = 1$ for all k. As we directly assume the asymptotic equivalent of the conclusions of these implications, the same arguments as in the proof of Theorem 3.3 yield the desired conclusion in the proof of Theorem 4.3.

As before with Theorem 3.7, we can give sufficient conditions for the asymptotic freeness of F and G whenever the sequences of multiple integrals have mirror-symmetric kernels instead of symmetric ones.

Theorem 4.5. *Let n, m be natural numbers and let $\{f_k : k \geq 0\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 0\} \subset L^2(\mathbb{R}_+^m)$ be sequences of mirror-symmetric functions. Assume that $(I_n^{\mathfrak{M}}(f_k), I_m^{\mathfrak{M}}(g_k)) \xrightarrow{\text{law}} (U, V)$ and that $f_k^{(\sigma)} \overset{p}{\frown} g_k^{(\pi)} \to 0$ as $k \to \infty$, for all $p = 1, \ldots, n \wedge m$ and all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$, where $f_k^{(\sigma)}$ and $g_k^{(\pi)}$ are defined as in Theorem 3.7. Finally, if dealing with free Poisson integrals, assume moreover that $f_k^{(\sigma)} \star_p^{p-1} g_k^{(\pi)} \to 0$ as $k \to \infty$, for all $p = 1, \ldots, n \wedge m$ and all $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_m$. Then U and V are free.*

Proof. Using the exact same argument as in the proof of Theorem 3.3, we can obtain that, for any natural number ℓ and for any natural numbers $p_1, \ldots, p_{2\ell}$,

$$\varphi\big([I_n^{\mathfrak{M}}(f_k)^{p_1} - \varphi(I_n^{\mathfrak{M}}(f_k)^{p_1})][I_m^{\mathfrak{M}}(g_k)^{p_2} - \varphi(I_m^{\mathfrak{M}}(g_k)^{p_2})] \\ \cdots [I_n^{\mathfrak{M}}(f_k)^{p_{2\ell-1}} - \varphi(I_n^{\mathfrak{M}}(f_k)^{p_{2\ell-1}})][I_m^{\mathfrak{M}}(g_k)^{p_{2\ell}} - \varphi(I_m^{\mathfrak{M}}(g_k)^{p_{2\ell}})]\big) \xrightarrow[k\to+\infty]{} 0.$$

Taking the limit as $k \to \infty$,

$$\varphi\big([U^{p_1} - \varphi(U^{p_1})][V^{p_2} - \varphi(V^{p_2})] \cdots [U^{p_{2\ell-1}} - \varphi(U^{p_{2\ell-1}})][V^{p_{2\ell}} - \varphi(V^{p_{2\ell}})]\big) = 0,$$

which concludes the proof. □

Characterization in terms of covariances. Based on Theorem 4.1, Nourdin and Rosiński [2014, Corollary 3.6] obtained the following result that links component-wise convergence and joint convergence of multiple integrals. As before, note that in the following results, the random variables F and G need not have the form of multiple integrals. This implies that sequences of multiple integrals can be used in order to prove the freeness of general random variables in $L^2(\varphi)$ (provided these random variables admit approximating sequences of multiple integrals with symmetric kernels).

Theorem 4.6. *Let n, m be natural numbers and let $\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$ be sequences of symmetric functions such that $I_n^W(f_k) \xrightarrow{\text{law}} F$ and $I_m^W(g_k) \xrightarrow{\text{law}} G$ as $k \to \infty$, where F, G are square integrable independent random variables with laws determined by their moments. If*

$$\operatorname{Cov}(I_n^W(f_k)^2, I_m^W(g_k)^2) \xrightarrow[k\to+\infty]{} 0,$$

then $(I_n^W(f_k), I_m^W(g_k)) \xrightarrow{\text{law}} (F, G)$, as $k \to \infty$.

In the free case, we obtain the following similar result.

Theorem 4.7. *Let n, m be natural numbers and let*

$$\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n) \quad \text{and} \quad \{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$$

be sequences of symmetric functions such that $(I_n^{\mathfrak{M}}(f_k), I_m^{\mathfrak{M}}(g_k)) \xrightarrow{\text{law}} (F, G)$ as $k \to \infty$. Then, F and G are free if and only if

$$\operatorname{Cov}(I_n^{\mathfrak{M}}(f_k)^2, I_m^{\mathfrak{M}}(g_k)^2) \xrightarrow[k\to+\infty]{} 0.$$

Proof. Combine (8) with Theorem 4.3. □

Characterization in terms of free Malliavin gradients. It is also possible to characterize asymptotic freeness in terms of the free gradient quantity appearing in Theorem 3.11. We offer the following statement.

Theorem 4.8. *Let n, m be natural numbers and let $\{f_k : k \geq 1\} \subset L^2(\mathbb{R}_+^n)$ and $\{g_k : k \geq 1\} \subset L^2(\mathbb{R}_+^m)$ be sequences of symmetric functions such that*

$$(I_n^S(f_k), I_m^S(g_k)) \xrightarrow{\text{law}} (F, G)$$

as $k \to \infty$, where F, G are random variables in $L^2(\mathscr{A}, \varphi)$. Then, F and G are free if and only if

$$\langle \nabla I_n^S(f_k), \nabla I_m^S(g_k) \rangle \xrightarrow[k\to+\infty]{} 0 \text{ in } L^2(\mathscr{A} \otimes \mathscr{A}, \varphi \otimes \varphi),$$

where the notation $\langle \cdot, \cdot \rangle$ is defined in (7).

Proof. Combine the representation (12) with Theorem 4.3. □

5. Transfer principles

Since the characterizations of freeness we have obtained in Section 3 involve quantities which are similar whatever the context (classical or free, Brownian or Poisson), it is natural to study possible transfer principles from one setting to another one. It is the goal of this section to study these aspects.

Theorem 5.1. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Assume that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free. Then, $I_n^S(f)$ and $I_m^S(g)$ are free. However, the fact that $I_n^S(f)$ and $I_m^S(g)$ are free does not necessarily imply that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free, as illustrated by Example 5.2.*

Proof. By Theorem 3.3, if $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free, then it holds that $f \star_1^0 g = 0$ a.e. Lemma 6.2 guarantees that $f \star_1^0 g = 0$ a.e. implies $f \overset{1}{\frown} g = 0$ a.e. Using Theorem 3.3 again concludes the proof. □

Example 5.2. Let T be a positive real number and let $f, g \in L^2(\mathbb{R}_+)$ be functions defined by

$$f(x) = x\mathbb{1}_{[0,T]}(x) \quad \text{and} \quad g(x) = \Big(x^2 - \frac{3T}{4}x\Big)\mathbb{1}_{[0,T]}(x).$$

Note that

$$f \overset{1}{\frown} g = \langle f, g \rangle_{L^2(\mathbb{R}_+)} = \int_0^T x\Big(x^2 - \frac{3T}{4}x\Big)\,dx = \int_0^T \Big(x^3 - \frac{3T}{4}x^2\Big)\,dx = 0$$

whereas

$$f \star_1^0 g(x) = f(x) \cdot g(x) = \Big(x^3 - \frac{3T}{4}x^2\Big)\mathbb{1}_{[0,T]}(x) \neq 0.$$

Hence, by Theorem 3.3, $I_1^S(f)$ and $I_1^S(g)$ are free but $I_1^{\hat{N}}(f)$ and $I_1^{\hat{N}}(g)$ are not.

Based on Theorems 3.1 and 3.3, we can obtain the following transfer principles between the Wiener and Wigner chaos.

Proposition. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. It holds that $I_n^S(f)$ and $I_m^S(g)$ are free if and only if $I_n^W(f)$ and $I_m^W(g)$ are independent.*

Proof. Observe that as f and g are symmetric functions; it holds that $f \otimes_1 g = f \overset{1}{\frown} g$. Using Theorems 3.1 and 3.3 concludes the proof. □

Remark 5.3. In the classical Poisson case, there is no known characterization of independence in terms of the almost sure nullity of a contraction. By using similar techniques to the ones used in the proof of Theorem 3.3 (using the definition of moment independence in place of the definition of freeness), one can prove that the condition $f \star_1^0 g = 0$ a.e. implies moment independence. However, moment independence only implies $\widetilde{f \star_1^0 g} = 0$ a.e., which is weaker than $f \star_1^0 g = 0$ a.e. Summing up, one can prove that the condition $f \star_1^0 g = 0$ a.e. is sufficient but not necessary and that the condition $\widetilde{f \star_1^0 g} = 0$ a.e. is necessary but not sufficient (the fact that it is not sufficient is illustrated by the counterexample provided in [Rosiński and Samorodnitsky 1999, Example 5.3]). Also pointed out therein is the fact that the squares of multiple Poisson integrals being uncorrelated does not imply that these multiple integrals are independent. This makes it difficult to establish any independence correspondence or transfer principles between the classical and free Poisson chaos. However, it can be pointed out that the freeness of free Poisson multiple integrals implies the freeness of the corresponding Wigner integrals and the independence of the corresponding Wiener integrals.

Despite the above remark, we can still provide the following partial transfer result.

Corollary 5.4. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be symmetric functions. Assume that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free. Then, $I_n^{\hat{\eta}}(f)$ and $I_m^{\hat{\eta}}(g)$ are moment independent.*

Proof. Assuming $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free, Theorem 3.3 states that $f \star_1^0 g = 0$ a.e., which, as pointed out in Remark 5.3, is a sufficient condition for $I_n^{\hat{\eta}}(f)$ and $I_m^{\hat{\eta}}(g)$ to be moment independent. Conversely, if $I_n^{\hat{\eta}}(f)$ and $I_m^{\hat{\eta}}(g)$ are moment independent and $f \star_1^0 g = 0$ a.e., Theorem 3.3 ensures that $I_n^{\hat{N}}(f)$ and $I_m^{\hat{N}}(g)$ are free. □

6. Auxiliary results

This last section contains two auxiliary results that have been used along the proof of Theorem 3.3.

Lemma 6.1. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be mirror-symmetric functions. Assume furthermore that $f \overset{1}{\frown} g = 0$ almost everywhere. Then, for all $p = 1, \dots, n \wedge m$, it holds that $f \overset{p}{\frown} g = 0$ almost everywhere.*

Proof. Observe that, for any $p = 1, \dots, n \wedge m$,

$$
\begin{aligned}
&f \overset{p}{\frown} g(t_1, \dots, t_{n+m-2p}) \\
&= \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_p, \dots, s_1) g(s_1, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p \\
&= \int_{\mathbb{R}_+^{p-1}} \biggl(\int_{\mathbb{R}_+} f(t_1, \dots, t_{n-p}, s_p, \dots, s_1) \\
&\qquad\qquad g(s_1, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \biggr) ds_2 \cdots ds_p \\
&= \int_{\mathbb{R}_+^{p-1}} f \overset{1}{\frown} g(t_1, \dots, t_{n-p}, s_p, \dots, s_2, s_2, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_2 \cdots ds_p.
\end{aligned}
$$

Using the assumption that $f \overset{1}{\frown} g = 0$ a.e., we get $f \overset{p}{\frown} g = 0$ a.e., which concludes the proof. □

Lemma 6.2. *Let n, m be natural numbers and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$ be mirror-symmetric functions. Assume furthermore that $f \star_1^0 g = 0$ almost everywhere. Then, for all $p = 1, \dots, n \wedge m$ and all $r = 2, \dots, n \wedge m$, it holds that $f \overset{p}{\frown} g = 0$ and $f \star_r^{r-1} g = 0$ almost everywhere.*

Proof. Observe that, for any $p = 1, \dots, n \wedge m$,

$$
\begin{aligned}
&f \overset{p}{\frown} g(t_1, \dots, t_{n+m-2p}) \\
&\quad= \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_p, \dots, s_1) g(s_1, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p \\
&\quad= \int_{\mathbb{R}_+^p} f \star_1^0 g(t_1, \dots, t_{n-p}, s_p, \dots, s_1, s_2, \dots, s_p, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p.
\end{aligned}
$$

Similarly, it holds that, for any $r = 2, \dots, n \wedge m$,

$$
\begin{aligned}
&f \star_r^{r-1} g(t_1, \dots, t_{n+m-2r+1}) \\
&\qquad= \int_{\mathbb{R}_+^{r-1}} f(t_1, \dots, t_{n-r+1}, s_{r-1}, \dots, s_1) \\
&\qquad\qquad\qquad g(s_1, \dots, s_{r-1}, t_{n-r+1}, \dots, t_{n+m-2r+1})\, ds_1 \cdots ds_{r-1} \\
&\qquad= \int_{\mathbb{R}_+^{r-1}} f \star_1^0 g(t_1, \dots, t_{n-r+1}, s_{r-1}, \dots, s_1, s_2, \dots, s_{r-1}, \\
&\qquad\qquad\qquad t_{n-r+1}, \dots, t_{n+m-2r+1})\, ds_1 \cdots ds_{r-1}.
\end{aligned}
$$

Using the assumption that $f \star_1^0 g = 0$ a.e. concludes the proof. □

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DOMINANCE ORDER AND MONOIDAL CATEGORIFICATION OF CLUSTER ALGEBRAS

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We study a compatibility relationship between Qin's dominance order on a cluster algebra $\mathcal{A}$ and partial orderings arising from classifications of simple objects in a monoidal categorification $\mathcal{C}$ of $\mathcal{A}$. Our motivating example is Hernandez and Leclerc's monoidal categorification using representations of quantum affine algebras. In the framework of Kang, Kashiwara, Kim and Oh's monoidal categorification via representations of quiver Hecke algebras, we focus on the case of the category R-gmod for a symmetric finite type A_n quiver Hecke algebra using Kleshchev and Ram's classification of irreducible finite-dimensional representations.

1. Introduction

Cluster algebras were introduced in [Fomin and Zelevinsky 2002] to study total positivity and canonical bases of quantum groups. Cluster algebras are commutative $\mathbb{Z}$-subalgebras of fields of rational functions over $\mathbb{Q}$ generated by certain distinguished generators, called *cluster variables*, satisfying relations called exchange relations. These cluster variables are grouped into overlapping finite sets of fixed cardinality (the rank of the cluster algebra) called clusters. A monomial in cluster variables of

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the same cluster is called a *cluster monomial*. Applying exchange relations to one variable of a cluster leads to another cluster and this procedure is called *mutation*. Any cluster can be reached from a given initial cluster by a finite sequence of mutations. Moreover, it is shown in [Fomin and Zelevinsky 2007] that once an initial seed $((x_1, \dots, x_n), B)$ has been fixed, any cluster variable of any other seed $((x_1^t, \dots, x_n^t), B^t)$ can be related to the initial cluster variables in the following way:

$$x_l^t = \frac{F^{l,t}(\hat{y}_1, \dots, \hat{y}_n)}{F^{l,t}{}_{|\mathbb{P}}(y_1, \dots, y_n)} x_1^{g_1^{l,t}} \cdots x_n^{g_n^{l,t}},$$

where $(g_1^{l,t}, \dots, g_n^{l,t})$ is an n-tuple of integers called the g-vector of the cluster variable x_l^t and $F^{l,t}$ is a polynomial called the F-polynomial of x_l^t. The variables $y_j, \hat{y}_j$, $1 \le j \le n$, are Laurent monomials in the initial cluster variables $x_1, \dots, x_n$ which do not depend on l and t. Thus the combinatorics of a cluster algebra is entirely contained in the behavior of g-vectors and F-polynomials.

Berenstein and Zelevinsky [2005] defined quantum cluster algebras as quantizations of cluster algebras; these algebras are not commutative: cluster variables belonging to the same cluster q-commute, i.e., satisfy relations of the form $x_i x_j = q^{\lambda_{ij}} x_j x_i$ for some integers λ_{ij}.

The notion of monoidal categorification of a cluster algebra has been introduced in [Hernandez and Leclerc 2010]. The idea is to identify a given cluster algebra $\mathcal{A}$ with the Grothendieck ring of a monoidal category; more specifically, the categories involved in this procedure will be categories of modules over algebras such as quantum affine algebras. Monoidal categorification requires a correspondence between cluster monomials and *real* simple objects in the category (i.e., simple objects S such that $S \otimes_{\mathcal{C}} S$ is again simple) as well as between cluster variables and *prime* real simple objects (real simple objects that cannot be decomposed as tensor products of several nontrivial objects). Given a cluster algebra $\mathcal{A}$, the existence of a monoidal categorification of $\mathcal{A}$ can give some fruitful information about the category itself, such as the existence of decompositions of simple objects into tensor products of prime simple objects. Hernandez and Leclerc [2010] defined a sequence $\{\mathcal{C}_l, l \in \mathbb{N}\}$ of subcategories of modules over quantum affine algebras and conjectured that the category $\mathcal{C}_1$ is a monoidal categorification of a cluster algebra. They proved this conjecture in types A_n and D_4 and exhibited a remarkable link between the g-vector (resp. F-polynomial) of each cluster variable and the dominant monomial (resp. the (truncated) q-character) of the corresponding simple module. In [Hernandez and Leclerc 2013], they introduce other subcategories of finite-dimensional representations of quantum affine algebras and prove several monoidal categorification statements for these categories in types A_n and D_n.

Other examples of monoidal categorification of cluster algebras appeared in various contexts. For instance, in the study of categories of sheaves on the Nakajima varieties: these varieties were constructed by Nakajima [1998; 2001] for the purpose of providing a geometric realization of quantum groups as well as their highest weight representations. In [Nakajima 2011], certain categories of perverse sheaves on these varieties are shown to be monoidal categorifications of cluster algebras. In particular, Nakajima used these constructions to study the category $\mathcal{C}_1$ and proved Hernandez and Leclerc's conjecture in ADE-types. Recently, Cautis and Williams [2019] exhibited a new example of monoidal categorification of cluster algebras using the $\mathbb{G}_m$-equivariant coherent Satake category, i.e., the category of $G(\mathcal{O})$-equivariant perverse coherent sheaves on the affine Grassmannian Gr_G. In the case of the general linear group GL_n, they show that this category is a monoidal categorification of a quantum cluster algebra and construct explicitly an initial seed.

A large proportion of this paper will be devoted to another situation of monoidal categorification of (quantum) cluster algebras that came out of the works of Kang, Kashiwara, Kim and Oh [Kang et al. 2018a; 2015; 2018b], involving categories of representations of quiver Hecke algebras. Introduced in [Khovanov and Lauda 2009] and independently in [Rouquier 2012], quiver Hecke algebras (or KLR algebras) are $\mathbb{Z}$-graded algebras which categorify the negative part $U_q(\mathfrak{n})$ of the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetric Kac–Moody algebra and $\mathfrak{n}$ the nilpotent subalgebra arising from a triangular decomposition. Khovanov and Lauda conjectured that this categorification provides a bijection between the canonical basis of $U_q(\mathfrak{n})$ and the set of indecomposable projective modules over the quiver Hecke algebra corresponding to $\mathfrak{g}$. This was proved in [Rouquier 2012] and independently in [Varagnolo and Vasserot 2011] using a geometric realization of quiver Hecke algebras. The category of finite-dimensional modules over quiver Hecke algebras can be given a monoidal structure using parabolic induction. Kleshchev and Ram [2011] gave a combinatorial classification of simple finite-dimensional modules over quiver Hecke algebras of finite types (i.e., associated with a finite type Lie algebra $\mathfrak{g}$) using Lyndon bases. More precisely, they introduced a certain class of simple modules, called *cuspidal modules*, which are in bijection with the set of positive roots of $\mathfrak{g}$. The simple modules are realized as quotients of products of cuspidal modules, and are parametrized by *dominant words* or *root partitions* (see Definition 3.25).

Fix a total order on the set of vertices of the Dynkin diagram of $\mathfrak{g}$. The corresponding lexicographic order is a total ordering on the set of dominant words. In Section 5, we use this framework for quiver Hecke algebras of finite type A_n and exhibit an easy combinatorial way to compute the highest dominant word appearing in the decomposition of the product of classes of two simples into a sum of classes of simples. For $n \geq 1$, consider $\mathfrak{g}$ a Lie algebra of finite type A_n. Denote by $r_n = n(n+1)/2$ the number of positive roots for the root system associated to $\mathfrak{g}$. Let R-gmod denote the category of finite-dimensional representations of the quiver Hecke algebra arising from $\mathfrak{g}$ and let $\boldsymbol{M}$ be the set of dominant words. For every $\mu \in \boldsymbol{M}$ we let $L(\mu)$ denote the unique (up to isomorphism) simple object in R-gmod corresponding to μ. For any $\mu, \mu' \in \boldsymbol{M}$, we define $\mu \odot \mu'$ as the highest word appearing in the decomposition of the product $[L(\mu)] \cdot [L(\mu')]$ as a sum of classes of simple objects in R-gmod. This is well-defined as $\boldsymbol{M}$ is totally ordered.

Theorem 1.1 (cf. Theorem 5.5). *The law $\odot$ provides $\boldsymbol{M}$ with a monoid structure and there is an isomorphism of monoids*

$$(\boldsymbol{M}, \odot) \simeq (\mathbb{Z}_{\geq 0}^{r_n}, +).$$

Moreover this isomorphism is explicitly constructed.

Kang et al. [2018b] adapted the notion of monoidal categorification to the quantum setting (in particular they defined a notion of quantum monoidal seed) and proved that the category R-gmod gives a monoidal categorification of the quantum cluster algebra structure on $U_q(\mathfrak{n})$. For this purpose, they introduced in [Kang et al. 2018a] some R-matrices for categories of finite-dimensional representations of quiver Hecke algebras, which give rise to exact sequences corresponding to cluster mutations in the Grothendieck ring. The notion of admissible pair is introduced in [Kang et al. 2018b] as a sufficient condition for a quantum monoidal seed to admit mutations in every exchange direction (see Definition 3.12). The main result of [Kang et al. 2018b] consists in proving that given an initial seed coming from an admissible pair, the seeds obtained after any mutation again come from admissible pairs. Hence the existence of an admissible pair implies monoidal categorification statements. The main results of [Kang et al. 2018b] (Theorems 11.2.2 and 11.2.3) consist in constructing admissible pairs for certain subcategories $\mathcal{C}_w$ (for each w in the Weyl group of $\mathfrak{g}$) of finite-dimensional representations of quiver Hecke algebras. These categories thus provide monoidal categorifications of cluster algebra structures on the quantum coordinate rings $\mathcal{A}_q(\mathfrak{n}(w))$ introduced by Geiss, Leclerc and Schröer [Geiss et al. 2013]. We refer to Kashiwara's ICM talk [2018] for a survey on monoidal categorifications of cluster algebras and connections with the theory of crystal bases and in particular global bases of quantum coordinate rings.

In this framework, it is natural to consider the dominant word of the simple module corresponding to a cluster variable x_l^t and try to relate it to the dominant words of the simple modules belonging to an initial monoidal seed, for instance the seed arising from the construction of [Kang et al. 2018b, Theorem 11.2.2]. As mentioned above, the theory of cluster algebras encourages us to look at the variables $\hat{y}_j$ defined in [Fomin and Zelevinsky 2007]. Using the above result on dominant words, we can associate in a natural way analogs of dominant words that we call *generalized parameters* to each of these variables $\hat{y}_j$ (with respect to the initial seed constructed in [Kang et al. 2018b]). It turns out that in the case of the category R-gmod $= \mathcal{C}_{w_0}$ (where w_0 stands for the longest element of the Weyl group of $\mathfrak{g}$), these generalized parameters share some remarkable properties. In particular, it implies some correspondence between the lexicographic ordering on dominant words and the following ordering on Laurent monomials in the initial cluster variables: for any two such Laurent monomials $\boldsymbol{x}^{\boldsymbol{\alpha}} = \prod_i x_i^{\alpha_i}$ and $\boldsymbol{x}^{\boldsymbol{\beta}} = \prod_i x_i^{\beta_i}$, one sets

$$\boldsymbol{x}^{\boldsymbol{\alpha}} \preccurlyeq \boldsymbol{x}^{\boldsymbol{\beta}} \Leftrightarrow \text{ there exists } (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_{\geq 0}^n \text{ such that } \boldsymbol{x}^{\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\alpha}}. \prod_j \hat{y}_j^{\gamma_j}.$$

This ordering can be easily seen to coincide with the *dominance order* for the considered initial seed, introduced in [Qin 2017] as an ordering on multidegrees. As proven in [Hernandez and Leclerc 2010], this dominance order corresponds to the Nakajima order on monomials in the case of the monoidal categorification by the category $\mathcal{C}_1$ (see Proposition 4.14). Geometrically, it is related to the partial ordering on subvarieties of Nakajima varieties, defined as inclusions of subvarieties into the closures of others (see [Nakajima 2011]). Qin uses this order to introduce the notions of pointed elements and pointed sets (see Definition 4.1) and define triangular bases in a (quantum) cluster algebra. As an application, he proves that in the context of monoidal categorification of cluster algebras using representations of quantum affine algebras, cluster monomials correspond to classes of simple modules, which partly proves Hernandez and Leclerc's conjecture [2010].

In this paper we study this relationship between orderings in a more general setting; considering the data of a cluster algebra $\mathcal{A}$ together with a monoidal categorification $\mathcal{C}$ of $\mathcal{A}$, we assume the simple objects in $\mathcal{C}$ are parametrized by elements of a partially ordered set $\boldsymbol{M}$. Given a seed $\mathcal{S}$ in $\mathcal{A}$, we define a notion of compatibility between the ordering on $\boldsymbol{M}$ and the dominance order $\preccurlyeq$ associated to $\mathcal{S}$ and we say that $\mathcal{S}$ is a *compatible seed* if this compatibility holds (see Definition 4.7). Not all seeds are compatible and it even seems that most seeds are not. We conjecture that under some technical assumptions on the category $\mathcal{C}$, there exists a compatible seed (Conjecture 4.10). The existence of such a seed $\mathcal{S}$ implies some combinatorial relationships between the g-vector with respect to $\mathcal{S}$ of any cluster variable on the one hand, and the parameter of the corresponding simple object in $\mathcal{C}$ on the other hand. The results of Hernandez and Leclerc [2010] provide a beautiful example of such a relationship.

We then focus on the case of monoidal categorifications of cluster algebras via representations of quiver Hecke algebras of finite type A_n. More precisely, we consider the category R-gmod of finite-dimensional representations of a type A_n quiver Hecke algebra. One of the main results of this paper consists in the explicit computation of the dominant words of the simple modules of the initial (quantum) monoidal seed constructed in [Kang et al. 2018b] for this category.

Theorem 1.2 (cf. Theorem 6.1). *Let $\mathcal{S}_0^n$ be the initial seed constructed in* [*Kang et al. 2018b*] *for the category R-gmod associated with a Lie algebra of type A_n.*

Then the cluster variables of the seed $\mathcal{S}_0^n$ can be explicitly described in terms of dominant words as follows:

$$\begin{array}{ccccc}
[L(1)] & & & & \\
[L(12)] & [L((2)(1))] & & & \\
[L(123)] & [L((23)(12))] & [L((3)(2)(1))] & & \\
\vdots & \vdots & & & \\
[L(1\dots k)] & [L((2\dots k)(1\dots k-1))] & \cdots & [L((k)\cdots(1))] & \\
\vdots & \vdots & & & \\
[L(1\dots n)] & [L((2\dots n)(1\dots n-1))] & \cdots & \cdots & [L((n)\cdots(1))].
\end{array}$$

The set of frozen variables corresponds to the last line and the set of unfrozen variables consists in the union of lines $1, \dots, n-1$.

Using this description, we can deduce the main result of this paper:

Theorem 1.3 (Theorem 6.2). *The seed $\mathcal{S}_0^n$ is compatible in the sense of Definition 4.7.*

In particular, Conjecture 4.10 holds for the category of finite dimensional representations of a quiver Hecke algebra arising from a Lie algebra of type A_n.

This paper is organized as follows: in Section 2 we recall the definitions and main results of the theory of cluster algebras from [Fomin and Zelevinsky 2007], as well as the notion of monoidal categorification of cluster algebras from [Hernandez and Leclerc 2010]. Section 3 is devoted to the representation theory of quiver Hecke algebras. After some reminders of their definitions and main properties, we recall the constructions of Kang et al. [2015; 2018a; 2018b] of renormalized R-matrices for modules over quiver Hecke algebras as well as their results on monoidal categorification of quantum cluster algebras. We also recall the construction of admissible pairs for the categories $\mathcal{C}_w$ from [Kang et al. 2018b]. We end the section with the results of Kleshchev and Ram [2011] about the classification of irreducible finite-dimensional representations of finite type quiver Hecke algebras. In Section 4, we consider the general situation of monoidal categorifications of cluster algebras. We introduce a partial ordering on Laurent monomials in the cluster variables of a given seed and show that it coincides with the dominance order introduced by Qin [2017]. Then we define the notion of admissible seed and state our main conjecture (Conjecture 4.10). We show that the results of Hernandez and Leclerc [2010] in the context of monoidal categorification of cluster algebras via quantum affine algebras provide an example where this conjecture holds. In Section 5 we focus on the case of monoidal categorifications of cluster algebras via finite type A_n quiver Hecke algebras. Section 5A is devoted to the proof of Theorem 5.5. We use it in the framework of [Kang et al. 2018b] to obtain a combinatorial rule of transformation of dominant words under cluster mutation. Then we study in detail the example of a quiver Hecke algebra of type A_3 and exhibit examples of compatible and noncompatible seeds in the corresponding category R-gmod. In Section 6 we state and prove the two main results of this paper (Theorems 6.1 and 6.2). We conclude with some possible further developments.

2. Cluster algebras and their monoidal categorifications

In this section, we recall the main results of the theory of cluster algebras from [Fomin and Zelevinsky 2002; 2003; 2007], as well as the notion of monoidal categorification from [Hernandez and Leclerc 2010].

2A. ***Cluster algebras.*** Cluster algebras were introduced in [Fomin and Zelevinsky 2002]. They are commutative $\mathbb{Z}$-subalgebras of the field of rational functions over $\mathbb{Q}$ in a finite number of algebraically independent variables. They are defined as follows.

Let $1 \leq n < m$ be two nonnegative integers and let $\mathcal{F}$ be the field of rational functions over $\mathbb{Q}$ in m independent variables. The initial data is a couple $((x_1, \dots, x_m), B)$ called an initial seed and made out of a *cluster*, i.e., m algebraically independent variables $x_1, \dots, x_m$ generating $\mathcal{F}$ and an $m \times n$ matrix $B := (b_{ij})$ called the *exchange matrix* whose principal part (i.e., the square submatrix $(b_{ij})_{1 \leq i,j \leq n}$) is skew-symmetric.

To the exchange matrix B one can associate a quiver, whose index set is $\{1, \dots, m\}$ with b_{ij} arrows from i to j if $b_{ij} \geq 0$, and $-b_{ij}$ arrows from j to i if $b_{ij} \leq 0$.

By construction, one can recover the exchange matrix from the data of a quiver without loops and 2-cycles in the following way:

$$b_{ij} = (\text{number of arrows from } i \text{ to } j) - (\text{number of arrows from } j \text{ to } i).$$

For any $k \in \{1, \dots, n\}$ one defines new variables:

$$x'_j := \begin{cases} \dfrac{1}{x_k}\left(\displaystyle\prod_{b_{lk}>0} x_l^{b_{lk}} + \prod_{b_{lk}<0} x_l^{-b_{lk}} \right) & \text{if } j = k, \\ x_j & \text{if } j \neq k, \end{cases} \tag{1}$$

as well as a new matrix B':

$$\text{for all } i \text{ and } j, \quad (B')_{ij} := \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \text{if } i \neq k \text{ and } j \neq k. \end{cases}$$

Note that the principal part of the matrix B' is again skew-symmetric.

The procedure producing the seed $((x'_1, \dots, x'_m), B')$ out of the initial seed $((x_1, \dots, x_m), B)$ is called the *mutation* in the direction k of the initial seed $((x_1, \dots, x_m), B)$. This procedure is involutive, i.e., the mutation of the seed $(x'_1, \dots, x'_m), B')$ in the same direction k gives back the initial seed $((x_1, \dots, x_m), B)$.

Any seed can give rise to n new seeds, each of them obtained by a mutation in the direction k for $1 \leq k \leq n$. Let $\mathbb{T}$ be the tree whose vertices correspond to the seeds and edges to mutations. There are exactly n edges adjacent to each vertex. This tree can have a finite or infinite number of vertices depending on the initial seed. Let $((x_1, \ldots, x_m), B)$ be a fixed seed and t_0 be the corresponding vertex in $\mathbb{T}$. For any vertex $t \in \mathbb{T}$ one denotes by $((x_1^t, \ldots, x_m^t), B^t)$ the seed corresponding to the vertex t. It is obtained from the initial seed $((x_1, \ldots, x_m), B)$ by applying a sequence of mutations following a path starting at t_0 and ending at t.

Definition 2.1. The cluster algebra generated by the initial seed $((x_1, \ldots, x_m), B)$ is the $\mathbb{Z}[x_{n+1}^{\pm 1}, \ldots, x_m^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by all the variables $x_1^t, \ldots, x_n^t$ for all the vertices $t \in \mathbb{T}$.

For any seed $((x_1, \ldots, x_m), B)$, the variables $x_1, \ldots, x_m$ are called the cluster variables, $x_1, \ldots, x_n$ are the unfrozen variables and $x_{n+1}, \ldots, x_m$ are the frozen variables. These last variables do not mutate and are present in every cluster.

The first main result of the theory of cluster algebras is the *Laurent phenomenon*:

Theorem 2.2 [Fomin and Zelevinsky 2002, Theorem 3.1]. *Let $((x_1, \ldots, x_m), B)$ be a fixed seed in a cluster algebra $\mathcal{A}$. Then for any seed $((x_1^t, \ldots, x_m^t), B^t)$ in $\mathcal{A}$ and any $1 \leq j \leq n$, the cluster variable x_j^t is a Laurent polynomial in the variables $x_1, \ldots, x_m$.*

Let $\mathbb{P}$ be the multiplicative group of all Laurent monomials in the frozen variables $x_{n+1}, \ldots, x_m$. One can endow it with an additional structure given by

$$\prod_i x_i^{\alpha_i} \oplus \prod_i x_i^{\beta_i} := \prod_i x_i^{\min(\alpha_i, \beta_i)}$$

making $\mathbb{P}$ a semifield. Any substraction-free rational expression $F(u_1, \ldots, u_k)$ with integer coefficients in some variables $u_1, \ldots, u_k$ can be specialized on some elements $p_1, \ldots, p_k$ in $\mathbb{P}$. This will be denoted by $F_{|\mathbb{P}}(p_1, \ldots, p_k)$.

The mutation relation (1) can be rewritten as

$$x_k x_k' = p_k^+ \prod_{1 \leq i \leq n} x_i^{[b_{ik}]_+} + p_k^- \prod_{1 \leq i \leq n} x_i^{[-b_{ik}]_+},$$

where

$$p_k^+ := \prod_{n+1 \leq i \leq m} x_i^{[b_{ik}]_+} \quad \text{and} \quad p_k^- := \prod_{n+1 \leq i \leq m} x_i^{[-b_{ik}]_+}$$

belong to the semifield $\mathbb{P}$.

Thus the frozen variables $x_{n+1}, \ldots, x_m$ play the role of coefficients and the cluster algebra $\mathcal{A}$ can be viewed as the $\mathbb{Z}\mathbb{P}$ algebra generated by the (exchange) variables $x_1^t, \ldots, x_n^t$ for all the vertices t of the tree $\mathbb{T}$. Here $\mathbb{Z}\mathbb{P}$ denotes the group ring of the multiplicative group of the semifield $\mathbb{P}$. This group is always torsion-free and hence the ring $\mathbb{Z}\mathbb{P}$ is a domain.

The notion of isomorphism of cluster algebras is introduced in [Fomin and Zelevinsky 2003]: two cluster algebras $\mathcal{A} \subset \mathcal{F}$ and $\mathcal{A}' \subset \mathcal{F}'$ with the same coefficient part $\mathbb{P}$ are said to be isomorphic if there exists a $\mathbb{Z}\mathbb{P}$ algebras isomorphism $\mathcal{F} \to \mathcal{F}'$ sending a seed in $\mathcal{A}$ onto a seed in $\mathcal{A}'$. In particular the set of seeds of $\mathcal{A}$ is in bijection with the set of seeds of $\mathcal{A}'$, and $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic as algebras.

The second important result is the classification of finite type cluster algebras, i.e., the ones with a finite number of seeds.

Theorem 2.3 [Fomin and Zelevinsky 2003, Theorem 1.4]. *There is a canonical bijection between isomorphism classes of cluster algebras of finite type and Cartan matrices of finite type.*

Let $\mathcal{A}$ be a cluster algebra and let us fix $((x_1, \dots, x_m), B)$ an initial seed. Fomin and Zelevinsky [2007] define, for any $1 \le j \le n$,

$$y_j := \prod_{n+1 \le i \le m} x_i^{b_{ij}} \quad \text{and} \quad \hat{y}_j := \prod_{1 \le i \le m} x_i^{b_{ij}}.$$

Theorem 2.4 [Fomin and Zelevinsky 2007, Corollary 6.3]. *Let*

$$((x_1^t, \dots, x_n^t, x_{n+1}, \dots, x_m), B^t)$$

be any seed in $\mathcal{A}$. Then for any $1 \le l \le n$, the cluster variable x_l^t can be expressed in terms of the initial cluster variables $x_1, \dots, x_m$ in the following way:

$$x_l^t = \frac{F^{l,t}(\hat{y}_1, \dots, \hat{y}_n)}{F^{l,t}{}_{|\mathbb{P}}(y_1, \dots, y_n)} x_1^{g_1^{l,t}} \cdots x_n^{g_n^{l,t}}. \tag{2}$$

In this formula, $F^{l,t}$ is a polynomial called the F-polynomial associated to the variable x_l^t, and the $g_i^{l,t}$ are integers. We write for short $\boldsymbol{x}^{\boldsymbol{g}^{l,t}}$ for

$$x_1^{g_1^{l,t}} \cdots x_n^{g_n^{l,t}}$$

and $\boldsymbol{g}^{l,t} = (g_1^{l,t}, \dots, g_n^{l,t})$ is called the g-vector associated to the variable x_l^t.

The F-polynomial associated to any cluster variable satisfies several important and useful properties, which have been conjectured in [Fomin and Zelevinsky 2007] and proved by Derksen, Weyman and Zelevinsky in [Derksen et al. 2010] using the theory of quivers with potentials. We recall here some of these results, which we will use in Section 4 in the study of compatible seeds.

Theorem 2.5 [Derksen et al. 2010, Theorem 1.7]. *Let*

$$((x_1^t, \dots, x_n^t, x_{n+1}, \dots, x_m), B^t)$$

be any seed in $\mathcal{A}$. Let $1 \le l \le n$, and $F^{l,t}$ be the F-polynomial associated to the *cluster variable x_l^t. Then*:

(i) *There is a unique monomial in $F^{l,t}$ that is strictly divisible by any other monomial in $F^{l,t}$. This monomial has coefficient* 1.

(ii) *The polynomial $F^{l,t}$ has constant term* 1.

2B. ***Monoidal categorification of cluster algebras.*** The notion of monoidal categorification of a cluster algebra was introduced in [Hernandez and Leclerc 2010]. Recall that, if $\mathcal{C}$ is a monoidal category, a simple object M in $\mathcal{C}$ is said to be *real* if the tensor product $M \otimes_{\mathcal{C}} M$ is simple. It is said to be *prime* if it is not invertible in $\mathcal{C}$ and cannot be decomposed as $M = M_1 \otimes_{\mathcal{C}} M_2$ with M_1 and M_2 two simple noninvertible modules neither trivial nor equal to M itself. We denote by $K_0(\mathcal{C})$ the Grothendieck ring of the category $\mathcal{C}$. Recall that for any objects A, B, C in $\mathcal{C}$, the relation $[B] = [A] + [C]$ holds in $K_0(\mathcal{C})$ if there is a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$. The ring structure on $K_0(\mathcal{C})$ is directly inherited from the monoidal structure of $\mathcal{C}$: $[M] \cdot [M'] = [M \otimes_{\mathcal{C}} M']$ for any objects M, M' in $\mathcal{C}$.

Remark 2.6. In the category R-gmod that we will mostly be studying in this paper, all the simple objects are noninvertible. However in other categories, simple objects may be invertible. This happens for instance in categories of modules over Borel subalgebras of quantum affine algebras in [Hernandez and Leclerc 2016].

Definition 2.7 (monoidal categorification of a cluster algebra). A monoidal category $\mathcal{C}$ is a monoidal categorification of a cluster algebra $\mathcal{A}$ if the following conditions simultaneously hold:

(i) There is a ring isomorphism
$$K_0(\mathcal{C}) \simeq \mathcal{A}.$$

(ii) Under this isomorphism, classes of simple real objects in $\mathcal{C}$ correspond to cluster monomials in $\mathcal{A}$ and classes of simple real prime objects in $\mathcal{C}$ correspond to cluster variables in $\mathcal{A}$.

Several examples of monoidal categorifications of cluster algebras appeared more recently in various contexts: using categories of (finite-dimensional) representations of quiver Hecke algebras through the works of Kang et al. [2018a; 2015; 2018b], or via the coherent Satake category studied by Cautis and Williams [2019]. Let us point out that these examples use a slightly different notion of monoidal categorification:

Definition 2.8 (monoidal categorification of a cluster algebra in the sense of [Kang et al. 2018b; Cautis and Williams 2019]). A monoidal category $\mathcal{C}$ is a monoidal categorification of a cluster algebra $\mathcal{A}$ if:

(i) There is a ring isomorphism
$$K_0(\mathcal{C}) \simeq \mathcal{A}.$$

(ii) Under this isomorphism, any cluster monomial in $\mathcal{A}$ is the class of a simple real object in $\mathcal{C}$.

See for instance Definition 3.11 for a precise definition in the context of quiver Hecke algebras.

2C. ***Example: representations of quantum affine algebras.*** Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra of type A_n, D_n, or E_n and $\hat{\mathfrak{g}}$ be the corresponding Kac–Moody algebra. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ can be defined as a quantization of the universal enveloping algebra of $\hat{\mathfrak{g}}$ (see [Drinfeld 1987] or [Chari and Pressley 1994] for precise definitions). Consider the category $\mathcal{C}$ of finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules. Chari and Pressley [1994] proved that simple objects in this category are parametrized by their highest weights. More precisely, let I be the set vertices of the Dynkin diagram of $\mathfrak{g}$ and, for each $i \in I$ and $a \in \mathbb{C}^*$, let $Y_{i,a}$ be some indeterminate. The notion of q-character of a finite-dimensional $U_q(\hat{\mathfrak{g}})$-module was introduced by Frenkel and Reshetikhin [1999] as an injective ring homomorphism

$$\chi_q : K_0(\mathcal{C}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^*].$$

Let $\mathcal{M}$ be the set of Laurent monomials in the variables $Y_{i,a}$. For any $i \in I$ and $a \in \mathbb{C}^*$, set

$$A_{i,a} := Y_{i,aq} Y_{i,aq^{-1}} \prod_{j \neq i} Y_{j,a}^{a_{ji}} \in \mathcal{M}.$$

One defines a partial ordering (the Nakajima order) on $\mathcal{M}$ in the following way:

$$\mathfrak{m} \leq \mathfrak{m}' \Leftrightarrow \frac{\mathfrak{m}'}{\mathfrak{m}} \text{ is a monomial in the } A_{i,a}$$

for any monomials $\mathfrak{m}, \mathfrak{m}' \in \mathcal{M}$.

A monomial $\mathfrak{m} \in \mathcal{M}$ is called dominant if it does not contain negative powers of the variables $Y_{i,a}$. Let $\mathcal{M}^+$ denote the subset of $\mathcal{M}$ of all dominant monomials. For any simple object V of $\mathcal{C}$, the set of monomials occurring in the q-character of V has a unique maximal element μ_V for the above order, and this monomial is always dominant. Conversely, it is possible to associate a simple finite-dimensional $U_q(\hat{\mathfrak{g}})$-module to any dominant monomial in the variables $Y_{i,a}$, providing a bijection between the set of simple objects in $\mathcal{C}$ and $\mathcal{M}^+$. For any dominant monomial $\mathfrak{m}$, we let $L(\mathfrak{m})$ denote the unique (up to isomorphism) simple object in $\mathcal{C}$ corresponding to $\mathfrak{m}$ via this bijection. In the case where $\mathfrak{m}$ is reduced to a single variable $Y_{i,a}$ for some $i \in I$ and $a \in \mathbb{C}^*$, the simple module $L(\mathfrak{m}) = L(Y_{i,a})$ is called a *fundamental representation*.

The Dynkin diagram of $\mathfrak{g}$ is a bipartite graph hence its vertex set I can be decomposed as $I = I_0 \sqcup I_1$ such that every edge connects a vertex of I_0 with one of I_1. Then for any $i \in I$, set

$$\xi_i := \begin{cases} 0 & \text{if } i \in I_0, \\ 1 & \text{if } i \in I_1. \end{cases}$$

Hernandez and Leclerc introduced a subcategory $\mathcal{C}_1$ of $\mathcal{C}$ whose Grothendieck ring is generated (as a ring) by the classes of the fundamental representations $L(Y_{i,q^{\xi_i}})$ and $L(Y_{i,q^{\xi_i+2}})$, $i \in I$. One of the main results of [Hernandez and Leclerc 2010] can be stated as follows:

Theorem 2.9 [Hernandez and Leclerc 2010, Conjecture 4.6]. *The category $\mathcal{C}_1$ is a monoidal categorification of a (finite type) cluster algebra of the same Lie type as the Lie algebra $\mathfrak{g}$.*

Hernandez and Leclerc [2010, Sections 10 and 11] prove this conjecture for $\mathfrak{g}$ of types A_n $(n \geq 1)$ and D_4. Nakajima [2011] proved this conjecture in types ADE using geometric methods involving graded quiver varieties. Note that this geometric construction is valid for any orientation of the Dynkin graph of $\mathfrak{g}$. Hernandez and Leclerc [2013, Theorems 4.2 and 5.6] exhibited other examples of monoidal categorifications of cluster algebras via categories of representations of quantum affine algebras in types A_n and D_n.

3. Quiver Hecke algebras

Kang et al. [2018a; 2015; 2018b] provided many examples of monoidal categorifications of cluster algebras arising from certain categories of modules over quiver Hecke algebras. In this section, we recall the main definitions and properties of quiver Hecke algebras; then we recall the constructions of renormalized R-matrices from [Kang et al. 2018a] as well as the statements of monoidal categorification from [Kang et al. 2018b]. We also recall the classification of simple finite-dimensional representations of quiver Hecke algebras of finite type using combinatorics of Lyndon words from [Kleshchev and Ram 2011].

3A. ***Definition and main properties.*** In this subsection we recall the definitions and main properties of quiver Hecke algebras, as defined in [Khovanov and Lauda 2009] and [Rouquier 2012].

Let $\mathfrak{g}$ be a Kac–Moody algebra, P the associated weight lattice and $\Pi = \{\alpha_i, i \in I\}$ the set of simple roots. We also define the coweight lattice as $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$ and we let $\Pi^\vee$ denote the set of simple coroots. We also denote by A the generalized Cartan matrix, W the Weyl group of $\mathfrak{g}$, and $(\cdot\,,\cdot)$ a W-invariant symmetric bilinear form on P. Let $\boldsymbol{k}$ be a base field.

The root lattice is defined as $Q := \bigoplus_i \mathbb{Z}\alpha_i$. We also set $Q_+ := \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$ and $Q_- := \bigoplus_i \mathbb{Z}_{\leq 0}\alpha_i$. For any $\beta \in Q$ which we write as $\sum_i m_i\alpha_i$, its length is defined as $\sum_i |m_i|$. When $\beta \in Q_+$, as in [Khovanov and Lauda 2009] we denote by $\mathrm{Seq}(\beta)$ the set of all finite sequences (called words) of the form $i_1, \ldots, i_n$ (where n is the length of β) with m_i occurrences of the integer i for all i. For the sake of simplicity, we identify letters with simple roots. In particular, for any $i, j \in \{1, \ldots, n\}$, (i, j) stands for (α_i, α_j) and if $\mu = i_1, \ldots, i_n$ and $\nu = j_1, \ldots, j_m$ are two words in $\mathrm{Seq}(\beta)$, (μ, ν) stands for $\sum_{p,q}(i_p, j_q)$.

To define quiver Hecke algebras, we fix a nonnegative integer n and a family $\{Q_{i,j}, 1 \leq i, j \leq n\}$ of two-variables polynomials with coefficients in $\boldsymbol{k}$. These polynomials are required to satisfy certain properties, in particular $Q_{i,j} = 0$ if $i = j$ and $Q_{i,j}(u, v) = Q_{j,i}(v, u)$ for any i, j (see for instance [Kang et al. 2018b, Section 2.1] for more details). In the case of finite type A_n symmetric quiver Hecke algebras (which we will focus on in Sections 5 and 6), the polynomials $Q_{i,j}$ are the following (see [Kleshchev and Ram 2011]):

$$Q_{i,j}(u, v) = \begin{cases} (u - v) & \text{if } j = i + 1, \\ (v - u) & \text{if } j = i - 1, \\ 0 & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 3.1. For any β in Q_+ of length n, the quiver Hecke algebra $R(\beta)$ at β associated to the Kac–Moody algebra $\mathfrak{g}$ and the family $\{Q_{i,j}, 1 \leq i, j \leq n\}$ is the $\boldsymbol{k}$-algebra generated by operators $\{e(\nu)\}_{\nu \in \mathrm{Seq}(\beta)}$, $\{x_i\}_{i \in \{1,\ldots,n\}}$, and $\{\tau_k\}_{k \in \{1,\ldots,n-1\}}$ satisfying the following relations:

$$\begin{gathered} e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu), \\ \sum_{\nu \in \mathrm{Seq}(\beta)} e(\nu) = 1, \\ x_i x_j = x_j x_i, \\ x_i e(\nu) = e(\nu) x_i, \\ \tau_k e(\nu) = e(s_k(\nu)) \tau_k, \\ \tau_k \tau_l = \tau_l \tau_k \quad \text{if } |k - l| > 1, \\ \tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu), \\ (\tau_k x_i - x_{s_k(i)} \tau_k) e(\nu) = \begin{cases} -e(\nu) & \text{if } i = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } i = k+1, \nu_k = \nu_{k+1} \\ 0 & \text{otherwise} \end{cases}, \\ (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) = \begin{cases} \dfrac{Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise,} \end{cases} \end{gathered}$$

where for any $\nu \in \mathrm{Seq}(\beta)$, ν_k stands for the k-th letter of the word ν.

The quiver Hecke algebra $R(\beta)$ is called symmetric when the polynomials $Q_{i,j}$ are polynomials in $u-v$.

The first main property of quiver Hecke algebras is that they naturally come with a $\mathbb{Z}$-grading by setting

$$\deg e(\nu)=0, \quad \deg x_k e(\nu)=2, \quad \deg \tau_i e(\nu)=-(\nu_i,\nu_{i+1}).$$

For any β and γ in Q_+ of respective lengths m and n, let M be an $R(\beta)$ module and N an $R(\gamma)$ module. One defines the convolution product of M and N via parabolic induction (see [Khovanov and Lauda 2009; Kang et al. 2018a]).

Set

$$e(\beta,\gamma):=\sum_{\substack{\nu\in\mathrm{Seq}(\beta)\\ \lambda\in\mathrm{Seq}(\gamma)}} e(\nu\lambda)\in R(\beta+\gamma).$$

It is an idempotent in $R(\beta+\gamma)$. Consider the homomorphism of $\boldsymbol{k}$-algebras

$$R(\beta)\otimes R(\gamma)\to e(\beta,\gamma)R(\beta+\gamma)e(\beta,\gamma)$$

given by

$$\begin{gathered} e(\nu)\otimes e(\lambda)\mapsto e(\nu\lambda), \quad \nu\in\mathrm{Seq}(\beta),\ \lambda\in\mathrm{Seq}(\gamma)\\ x_k\otimes 1\mapsto x_k e(\beta,\gamma), \quad 1\le k\le m, \qquad 1\otimes x_l\mapsto x_{m+l}e(\beta,\gamma), \quad 1\le l\le n\\ \tau_k\otimes 1\mapsto \tau_k e(\beta,\gamma), \quad 1\le k< m, \qquad 1\otimes \tau_l\mapsto \tau_{m+l}e(\beta,\gamma), \quad 1\le l<n. \end{gathered}$$

Then one defines

$$M\circ N:=R(\beta+\gamma)\otimes_{R(\beta)\otimes R(\gamma)} M\otimes N.$$

For any $\beta\in Q_+$, let $R(\beta)$-pmod be the category of (left) graded finite type projective $R(\beta)$ modules, $R(\beta)$-gmod the category of left finite-dimensional graded $R(\beta)$-modules, and also

$$R\text{-pmod}:=\bigoplus_{\beta\in Q_+} R(\beta)\text{-pmod}, \quad R\text{-gmod}:=\bigoplus_{\beta\in Q_+} R(\beta)\text{-gmod}.$$

Convolution product induces a monoidal structure on the categories R-gmod and R-pmod. The grading on quiver Hecke algebras also yields shift functors for these categories: decompose any object M as

$$M=\bigoplus_{n\in\mathbb{Z}} M_n$$

and define qM as

$$qM=\bigoplus_{n\in\mathbb{Z}} M_{n-1}.$$

The natural $\mathbb{Z}[q^{\pm 1}]$ action

$$q\cdot[M]:=[qM]$$

gives rise to $\mathbb{Z}[q^{\pm 1}]$-algebras structures on the Grothendieck rings of the categories R-pmod and R-gmod.

The following definition introduces a notion of graded character for representations of quiver Hecke algebras.

Definition 3.2 [Khovanov and Lauda 2009; Kleshchev and Ram 2011]. Let M be a finite-dimensional graded $R(\beta)$-module. For any $\nu \in \mathrm{Seq}(\beta)$, set $M_\nu := e(\nu) \cdot M$. The module M can be decomposed as

$$M = \bigoplus_{\nu} M_\nu.$$

Define

$$\mathrm{ch}_q(M) := \sum_{\nu} (\dim_q M_\nu).\nu,$$

where for any graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, $\dim_q(V) := \sum_{n\in\mathbb{Z}} q^n \dim V_n$. This is a formal series in words belonging to $\mathrm{Seq}(\beta)$ with coefficients in $\mathbb{Z}[q, q^{-1}]$.

One can put a ring structure on the image set of ch_q by defining a "product" of two words called the *quantum shuffle product*. For any nonnegative integer n, let $\mathfrak{S}_n$ denote the symmetric group of rank n.

Definition 3.3 (quantum shuffle product). Let $\boldsymbol{i} = i_1, \dots, i_r$ and $\boldsymbol{j} = j_1, \dots, j_s$ be two words. We set $i_{r+1} := j_1,\ i_{r+s} := j_s$ so that we can consider the concatenation $\boldsymbol{i}\boldsymbol{j} = i_1 \cdots i_{r+s}$.

Define the quantum shuffle product of $\boldsymbol{i}$ and $\boldsymbol{j}$:

$$\boldsymbol{i} \circ \boldsymbol{j} := \sum_{\sigma \in \mathfrak{S}_{r,s}} q^{-e(\sigma)} (i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(r+s)}),$$

where $\mathfrak{S}_{r,s}$ denotes the subset of $\mathfrak{S}_{r+s}$ defined as

$$\mathfrak{S}_{r,s} := \{\sigma \in \mathfrak{S}_{r+s} \mid \sigma(1) < \cdots < \sigma(r) \text{ and } \sigma(r+1) < \cdots < \sigma(r+s)\}$$

and, for any element $\sigma \in \mathfrak{S}_{r,s}$,

$$e(\sigma) := \sum_{\substack{1 \le k \le r < l \le r+s \\ \sigma(k) > \sigma(l)}} (i_k, i_l).$$

By linearity one can also define quantum shuffle products of two formal series in elements of $\mathrm{Seq}(\beta)$ with coefficients in $\mathbb{Z}[q, q^{-1}]$ (for any $\beta \in Q_+$).

Proposition 3.4 [Khovanov and Lauda 2009, Lemma 2.20]. *For any $\beta, \gamma \in Q_+$, and any $M \in R(\beta)$-gmod and $N \in R(\gamma)$-gmod, we have*

$$ch_q(M \circ N) = ch_q(M) \circ ch_q(N).$$

One can now state the main property of quiver Hecke algebras, which is to categorify the negative part of the quantum group $U_q(\mathfrak{g})$ in a way that induces a correspondence between the basis of indecomposable objects in R-pmod and the canonical basis of $U_q(\mathfrak{n})$. In the following we will mostly consider the category R-gmod; hence we give here the dual statements, involving the category R-gmod and the quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ (the precise definition of which can be found in [Geiss et al. 2013] or [Kang et al. 2018b]). The first theorem was proved by Khovanov and Lauda [2009] and Rouquier [2012]. The second was conjectured by Khovanov and Lauda, and proved by Rouquier [2012] and Varagnolo and Vasserot [2011] using geometric methods.

Theorem 3.5 (Khovanov and Lauda, Rouquier). *The map* ch_q *induces a* $\mathbb{Z}[q, q^{-1}]$*-algebra isomorphism*

$$K_0(R\text{-}gmod) \simeq \mathcal{A}_q(\mathfrak{n}).$$

Theorem 3.6 (Rouquier, Varagnolo and Vasserot). *The map* ch_q *(see Definition 3.2) induces a bijection between the canonical basis of the quantum coordinate ring* $\mathcal{A}_q(\mathfrak{n})$ *and the set of isomorphism classes of self-dual simple modules in the category* R*-gmod.*

3B. ***Renormalized R-matrices for quiver Hecke algebras.*** From Section 3A, recall that the weight lattice associated to the Kac–Moody algebra $\mathfrak{g}$ is given with a symmetric bilinear form $(\cdot\,,\cdot)$. Denoting by A the symmetrizable generalized Cartan matrix of $\mathfrak{g}$, this bilinear form is entirely determined by its values on simple roots, namely, for all i, j,

$$(\alpha_i, \alpha_j) = \boldsymbol{s}_i a_{ij},$$

where the $\boldsymbol{s}_i$ are the entries of a diagonal matrix D such that DA is symmetric.

One also defines another symmetric bilinear form $(\cdot\,,\cdot)_n$ on the root lattice Q as in [Kang et al. 2018a]:

$$(\alpha_i, \alpha_j)_n := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

for all i, j. Let $\beta \in Q_+$ of length m and $1 \leq k < m$; the following operators φ_k are introduced in [Kang et al. 2018a]:

$$\varphi_k e(\nu) := \begin{cases} (\tau_k(x_k - x_{k+1}) + 1)e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \tau_k e(\nu) & \text{otherwise,} \end{cases}$$

for all $\nu \in \mathrm{Seq}(\beta)$. These operators satisfy the braid relation, hence for any permutation σ, $\varphi_\sigma := \varphi_{i_1} \cdots \varphi_{i_l}$ does not depend on the choice of a reduced expression $\sigma = s_{i_1} \cdots s_{i_l}$.

For any $m, n \in \mathbb{Z}_{\geq 0}$, let $w[m, n]$ be the element of $\mathfrak{S}_{m+n}$ sending k on $k + n$ if $1 \leq k \leq m$ and on $k - m$ if $m < k \leq m + n$.

Consider a nonzero $R(\beta)$-module M and a nonzero $R(\gamma)$-module N. The following map is defined in [Kang et al. 2018a]:

$$M \otimes N \to N \circ M, \qquad u \otimes v \mapsto \varphi_{w[n,m]}(v \otimes u).$$

It is $R(\beta) \otimes R(\gamma)$ linear and hence induces a homomorphism of $R(\beta+\gamma)$-modules,

$$R_{M,N} : M \circ N \to N \circ M.$$

The map $R_{M,N}$ satisfies the Yang–Baxter equation (see [Kang et al. 2018a]).

Let z be an indeterminate, homogeneous of degree 2. For any $\beta \in Q_+$ and any nonzero module M in $R(\beta)$-gmod, one defines $M_z := \boldsymbol{k}[z] \otimes M$ with the following $\boldsymbol{k}[z] \otimes R(\beta)$-module structure:

$$\begin{aligned} e(\nu).(P \otimes m) &:= P \otimes (e(\nu)m), \\ x_k.(P \otimes m) &:= (zP) \otimes m + P \otimes (x_k m), \\ \tau_k.(P \otimes m) &:= P \otimes (\tau_k m), \end{aligned}$$

for any $\nu \in \mathrm{Seq}(\beta)$, $P \in \boldsymbol{k}[z]$ and $m \in M$.

It is shown in [Kang et al. 2018a] that for any $\beta, \gamma \in Q_+$ and any nonzero $R(\beta)$-module M and nonzero $R(\gamma)$-module N, the map $R_{M_z,N}$ is polynomial in z and does not vanish. Let s be the largest nonnegative integer such that the image of $R_{M_z,N}$ is contained in $z^s N \circ M_z$. One defines R-matrices in the category R-gmod in the following way:

Definition 3.7. Let $\beta, \gamma \in Q_+$. For any nonzero $R(\beta)$-module M and nonzero $R(\gamma)$-module N, define a homomorphism of $R(\beta+\gamma)$-modules

$$r_{M,N} : M \circ N \to N \circ M$$

by setting

$$r_{M,N} := (z^{-s} R_{M_z,N})_{|z=0},$$

where s is the integer defined above.

Proposition 3.8 [Kang et al. 2018a]. *The homomorphism $r_{M,N}$ does not vanish and satisfies the Yang–Baxter equation.*

Thus the maps $r_{M,N}$ are R-matrices for the category R-gmod. They are called renormalized R-matrices. As in the case of categories of representations of quantum affine algebras, these R-matrices are in general not invertible and thus yield (graded) short exact sequences in the category R-gmod. Consequently this produces some relations in the Grothendieck ring $K_0(R\text{-gmod}) \simeq \mathcal{A}_q(\mathfrak{n})$. In the context of monoidal categorifications of (quantum) cluster algebras (see Section 3C), the exchange relations in $\mathcal{A}_q(\mathfrak{n})$ will be identified with some of these relations.

The corresponding relations in the Grothendieck ring $K_0(R\text{-gmod})$ will be identified with exchange relations. For any nonzero modules M and N, we denote by $\Lambda(M, N)$ the homogeneous degree of the morphism $r_{M,N}$. It is given by

$$\Lambda(M, N) = -(\beta, \gamma) + 2(\beta, \gamma)_n - 2s.$$

The next statement gives a criterion for the renormalized R-matrix $r_{M,N}$ to be an isomorphism. It will be particularly useful for the proof of Theorem 6.1 (see, for instance, Corollary 6.6).

Lemma 3.9 [Kang et al. 2018b, Lemma 3.2.3]. *Let M and N be two simples in the category R-gmod and assume one of them is real. Then the following are equivalent*:

(i) $\Lambda(M, N) + \Lambda(N, M) = 0$.

(ii) *$r_{M,N}$ and $r_{N,M}$ are inverse to each other up to a constant multiple.*

(iii) *$M \circ N$ and $N \circ M$ are isomorphic up to grading shift.*

(iv) *$M \circ N$ is simple in the category R-gmod.*

One says that M and N commute if they satisfy these properties.

3C. ***Monoidal categorification via representations of quiver Hecke algebras.*** In this subsection we focus on the case where $\mathcal{C}$ is a full subcategory of R-gmod stable under convolution products, subquotients, extensions, and grading shifts. $\mathcal{C}$ can be decomposed as

$$\mathcal{C} = \bigoplus_{\beta \in Q_+} \mathcal{C}_\beta$$

with $\mathcal{C}_\beta := \mathcal{C} \cap R(\beta)$-gmod for every $\beta \in Q_+$, so that the tensor product in $\mathcal{C}$ sends $\mathcal{C}_\beta \times \mathcal{C}_\gamma$ onto $\mathcal{C}_{\beta+\gamma}$ for any $\beta, \gamma \in Q_+$.

Kang et al. [2018b] adapted the notion of monoidal categorification to the setting of quantum cluster algebras. In the classical setting, a monoidal seed in $\mathcal{C}$ is defined as a triple $(\{M_i\}_{1 \leq i \leq n}, B, D)$ where $\{M_i\}_{1 \leq i \leq n}$ is a collection of simple objects in $\mathcal{C}$ such that for any $i_1, \dots, i_t$ in $\{1, \dots, n\}$, the object $M_{i_1} \circ \cdots \circ M_{i_t}$ is simple in $\mathcal{C}$, B is an integer-valued matrix with skew-symmetric principal part and D is a diagonal matrix encoding the weights of the modules M_i (i.e., the elements $\beta_i \in Q_+$ such that $M_i \in \mathcal{C}_{\beta_i}$). Cluster mutations correspond to some (ungraded) short exact sequences in the category $\mathcal{C}$. These exact sequences come from the failure of the renormalized R-matrices (see Definition 3.7) to be isomorphisms. The cluster mutations being involutive imposes some relations between the entries of the matrices B and D.

In the framework of [Kang et al. 2018b], one takes into account the natural grading of quiver Hecke algebras defined in Section 3A: objects in $\mathcal{C}$ are graded as well. A quantum monoidal seed is the data of such a triple $(\{M_i\}, B, D)$ with the further assumption that there exist integers λ_{ij} and isomorphisms of graded modules $M_i \otimes M_j \simeq q^{\lambda_{ij}} M_j \otimes M_i$ for any $i, j \in \{1, \dots, n\}$. The matrix $L = (\lambda_{ij})_{1 \leq i, j \leq n}$ is a skew-symmetric matrix and is assumed to satisfy some compatibility relations with the matrix B as in [Berenstein and Zelevinsky 2005]. See [Kang et al. 2018b, Section 6.2.1] for a precise definition.

In the quantum setting, cluster mutations correspond to some *graded* short exact sequences.

Definition 3.10 [Kang et al. 2018b, Definition 6.2.3]. Let $k \in \{1, \dots, r\}$ be fixed. A quantum monoidal seed $\mathcal{S} = (\{M_i\}_{1 \leq i \leq n}, L, B, D)$ admits a mutation in the direction k if there exists a simple object M_k' of $\mathcal{C}$ such that:

(1) $M_k' \in \mathcal{C}_{d_k'}$ with $d_k' := -d_k + \sum_{b_{ik} > 0} b_{ik} d_i$.

(2) One has the following short exact sequences in $\mathcal{C}$:

$$0 \to q M^{\boldsymbol{b}'} \to q^{m_k} M_k \otimes M_k' \to M^{\boldsymbol{b}''} \to 0,$$
$$0 \to q M^{\boldsymbol{b}''} \to q^{m_k'} M_k' \otimes M_k \to M^{\boldsymbol{b}'} \to 0,$$

where m_k and m_k' are some integers.

(3) $\mathcal{S}'^{(k)} := (\{M_i\}_{i \neq k} \cup \{M_k'\}, L'^{(k)}, B'^{(k)}, D'^{(k)})$ is again a quantum monoidal seed in $\mathcal{C}$, where $L'^{(k)}$ and $B'^{(k)}$ are defined as in [Berenstein and Zelevinsky 2005, Definition 3.5] and $D'^{(k)}$ is the diagonal matrix whose entries are the d_i for $i \neq k$ and d_k' for $i = k$.

Definition 3.11. The category $\mathcal{C}$ is a monoidal categorification of a quantum cluster algebra $\mathcal{A}$ if:

(a) There is an isomorphism of graded rings $\mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K_0(\mathcal{C}) \simeq \mathcal{A}$.

(b) There exists a quantum monoidal seed $\mathcal{S} := (\{M_i\}, L, B, D)$ in $\mathcal{C}$ such that $[\mathcal{S}] := (\{q^{-\frac{1}{4}(d_i, d_i)}[M_i]\}, L, B)$ is a quantum seed in $\mathcal{A}$.

(c) The quantum monoidal seed $\mathcal{S}$ admits arbitrary sequences of mutations in all directions.

In this setting, the existence of a monoidal categorification implies that any (quantum) cluster monomial is the class of some real simple object in $\mathcal{C}$. Recall from Section 2B that this notion is slightly different from the notion of monoidal categorification initially defined by Hernandez and Leclerc [2010].

The following definition provides a sufficient condition for producing quantum monoidal seeds.

Definition 3.12. A pair $(\{M_i\}, B)$ is admissible if:

(i) $\{M_i\}_{1 \leq i \leq n}$ is a family of self-dual real simple modules commuting with each other.

(ii) The matrix B is defined as above.

(iii) For each $1 \leq k \leq r$ there exists a self-dual simple module M'_k such that M'_k commutes with the M_i for $i \neq k$ and there is a short exact sequence of graded objects in $\mathcal{C}$,

$$0 \to qM^{b'} \to q^{\widetilde{\Lambda}(M_k, M'_k)} M_k \circ M'_k \to M^{b''} \to 0,$$

where $\widetilde{\Lambda}(M, N)$ is defined as $\frac{1}{2}(\Lambda(M, N) + (\beta, \gamma))$ for $M \in R(\beta)$-gmod and $N \in R(\gamma)$-gmod.

The data of an admissible pair naturally gives rise to a quantum monoidal seed in $\mathcal{C}$. More precisely, if $(\{M_i\}_{1 \leq i \leq n}, B)$ is an admissible pair in $\mathcal{C}$, and M'_k is as in the previous definition, then one defines an $r \times r$ skew-symmetric matrix L and a diagonal matrix D of size n by setting

$$L_{ij} := \Lambda(M_i, M_j) \quad \text{and} \quad D = \mathrm{Diag}(d_1, \dots, d_n),$$

where d_i stands for the weight of the module M_i. Then ([Kang et al. 2018b, Proposition 7.1.2]) the quadruple $\mathcal{S} := (\{M_i\}_{1 \leq i \leq n}, -L, B, D)$ is a quantum monoidal seed in $\mathcal{C}$ which admits mutations in every direction k (for $1 \leq k \leq r$).

The main result of [Kang et al. 2018b] can now be stated as follows:

Let $(\{M_i\}_{1 \leq i \leq n}, B)$ be an admissible pair in $\mathcal{C}$ and

$$\mathcal{S} := (\{M_i\}_{1 \leq i \leq n}, -L, B, D)$$

be the corresponding quantum monoidal seed. Set

$$[\mathcal{S}] := (\{q^{-\frac{1}{4}(wt(M_i), wt(M_i))}[M_i]\}_{1 \leq i \leq n}, -L, B, D).$$

Theorem 3.13 [Kang et al. 2018b, Theorem 7.1.3]. *Assume there is a $\mathbb{Q}(q^{\frac{1}{2}})$-algebras isomorphism*

$$\mathbb{Q}(q^{\frac{1}{2}}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K_0(\mathcal{C}) \simeq \mathbb{Q}(q^{\frac{1}{2}}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathcal{A}_{q^{\frac{1}{2}}}([\mathcal{S}]).$$

Then for each $1 \leq k \leq r$, the pair $(\{M_i\}_{i \neq k} \cup \{M'_k\}, B'^{(k)})$ is again an admissible pair in the category $\mathcal{C}$.

3D. *Quantum monoidal seeds for $\mathcal{C}_w$.* In this subsection we recall from [Kang et al. 2018b] the definition of the subcategories $\mathcal{C}_w$ of R-gmod as well as the construction of admissible pairs for these categories.

For any element w of the Weyl group W associated to $\mathfrak{g}$, Geiss, Leclerc and Schröer [2013, Section 7.2] defined algebras $\mathcal{A}_q(\mathfrak{n}(w))$ as subalgebras of the quantum coordinate rings $\mathcal{A}_q(\mathfrak{n})$. They showed ([2013, Theorem 12.3]) that it is possible to put a quantum cluster algebra structure on $\mathcal{A}_q(\mathfrak{n}(w))$ for every $w \in W$. Kang et al. [2018b] introduced, for each $w \in W$, a subcategory $\mathcal{C}_w$ of R-gmod such that the Grothendieck ring $K_0(\mathcal{C}_w)$ is the preimage of $\mathcal{A}_q(\mathfrak{n}(w))$ under the isomorphism given by Theorem 3.5: $M \in \mathcal{C}_w$ if and only if $\mathrm{ch}_q(M) \in \mathcal{A}_q(\mathfrak{n}(w))$.

Theorem 3.14 [Kang et al. 2018b, Theorem 11.2.3]. *For each element w of the Weyl group W, the category $\mathcal{C}_w$ is a monoidal categorification of the quantum cluster algebra $\mathcal{A}_{q^{1/2}}(\mathfrak{n}(w))$.*

Thus the categories $\mathcal{C}_w$ provide many examples of monoidal categorifications of (quantum) cluster algebras.

Remark 3.15. The category R-gmod coincides with $\mathcal{C}_{w_0}$ where w_0 stands for the longest element of the Weyl group of $\mathfrak{g}$. When w is the square of a well-chosen Coxeter element c in W, the quantum cell $\mathcal{A}_{q^{1/2}}(\mathfrak{n}(w))$ is also categorified by the category $\mathcal{C}_1$ defined in [Hernandez and Leclerc 2010]. The category $\mathcal{C}_{w_0}$ (resp. $\mathcal{C}_{c^2}$) is related to the category $\mathcal{C}_Q$ (resp. $\mathcal{C}_1$) introduced in [Hernandez and Leclerc 2013] (resp. [Hernandez and Leclerc 2010]) via a functor called *generalized quantum affine Schur–Weyl duality* defined in [Kang et al. 2018a]. In the case of $\mathcal{C}_{w_0}$ Fujita [2017] proved that this functor is an equivalence of categories.

Note that Geiss, Leclerc and Schröer defined categories $\widetilde{\mathcal{C}}_w$ which provide additive categorifications of the quantum coordinate rings $\mathcal{A}_q(\mathfrak{n}(w))$ for each $w \in W$ ([Geiss et al. 2013, Theorem 12.3]). The categories $\widetilde{\mathcal{C}}_w$ are defined as subcategories of the preprojective algebra of certain quivers. The categories $\mathcal{C}_w$ as defined in [Kang et al. 2018b] can be seen as monoidal analogs of the categories $\widetilde{\mathcal{C}}_w$ of [Geiss et al. 2013] in terms of representations of quiver Hecke algebras. However, Theorem 3.14 provides a monoidal categorification statement and is thus of different nature than the results of [Geiss et al. 2013].

In order to prove Theorem 3.14, Kang et al. constructed an admissible pair (see Definition 3.12) in the category $\mathcal{C}_w$ for each $w \in W$. We now recall this construction. By the results of [Kang et al. 2018b], the existence of such a pair implies Theorem 3.14.

First one defines *unipotent quantum minors* as some distinguished elements of $\mathcal{A}_q(\mathfrak{n})$: for any dominant weight λ in the weight lattice P and any couple (μ, ζ) of elements of $W\lambda$, the unipotent quantum minor $D(\mu, \zeta)$ is an element of $\mathcal{A}_q(\mathfrak{n})$ which is either a member of the canonical basis of $\mathcal{A}_q(\mathfrak{n})$ or zero ([Kang et al. 2018b, Lemma 9.1.1]). The following statement gives a necessary and sufficient condition so that $D(\mu, \zeta)$ is nonzero. First recall some notation from [Kang et al. 2018b]:

Definition 3.16. Let $\lambda \in P^+,\ \mu, \zeta \in W\lambda$. We write $\mu \precsim \zeta$ if there exists a finite sequence $(\beta_1, \dots, \beta_l)$ such that, setting $\lambda_0 := \zeta,\ \lambda_k = s_{\beta_k}\lambda_{k-1},\ 1 \leq k \leq l$, one has $\lambda_l = \mu$ and for all $1 \leq k \leq l,\ (\beta_k, \lambda_{k-1}) \geq 0$.

Lemma 3.17 [Kang et al. 2018b, Lemma 9.1.4]. *Let $\lambda \in P^+,\ \mu, \zeta \in W\lambda$. Then $D(\mu, \zeta) \neq 0$ if and only if $\mu \precsim \zeta$.*

The following statement is a direct consequence of Theorem 3.6 and Lemma 3.17:

Corollary 3.18. *Let $\lambda \in P^+$, $\mu, \zeta \in W\lambda$ such that $\mu \lesssim \zeta$. There exists a unique self-dual simple module $M(\mu, \zeta) \in R$-gmod whose image under the character map* ch_q *is $D(\mu, \zeta)$. Moreover, $M(\mu, \zeta)$ is real.*

This module is called a *determinantial module* ([Kang et al. 2018b, Definition 10.2.1]). Its weight is equal to $\zeta - \mu$, i.e., $M(\mu, \zeta) \in R(\zeta - \mu)$-gmod.

Remark 3.19. This is one of the key points that we will use to compute the dominant words of the modules corresponding to the frozen variables in R-gmod in Section 6.

One can now construct an admissible seed for the category $\mathcal{C}_w$. Fix some element w in the Weyl group W and a reduced expression $w = s_{i_1} \cdots s_{i_r}$. For $s \in \{1, \dots, r\}$, set

$$s_+ := \min(\{k \mid s < k \leq r,\, i_k = i_s\} \cup \{r+1\})$$
$$s_- := \max(\{k \mid 1 \leq k < s,\, i_k = i_s\} \cup \{0\})$$

For $1 \leq k \leq r$, set

$$\lambda_k := s_{i_1} \cdots s_{i_k} \omega_{i_k}.$$

For $0 \leq t \leq s \leq r$, set

$$D(s,t) := \begin{cases} D(\lambda_s, \lambda_t) & \text{if } 0 < t, \\ D(\lambda_s, \omega_{i_s}) & \text{if } 0 = t < s \leq r, \\ 1 & \text{if } t = s = 0. \end{cases}$$

Definition 3.20 [Kang et al. 2018b]. As in Corollary 3.18, consider $M(s,t)$ the unique simple real module (up to shift and isomorphism) such that $\mathrm{ch}(M(s,t)) = D(s,t)$ for any $0 \leq s \leq t \leq r$.

Set $J = \{1, \dots, r\}$, $J_{fr} = \{k \in J \mid k_+ = r+1\}$ and $J_{ex} = J \setminus J_{fr}$. The initial quiver is set to have $J = \{1, \dots, r\}$ as the set of vertices with the following arrows:

$$\begin{aligned} s \to t \quad & \text{if } 1 \leq s < t < s_+ < t_+ \leq r+1, \\ s \to s_- \quad & \text{if } 1 \leq s_- < s \leq r. \end{aligned}$$

Denoting by B the corresponding exchange matrix, the main result of [Kang et al. 2018b] can be stated in the following way:

Theorem 3.21 [Kang et al. 2018b, Theorem 11.2.2]. *The pair $(\{M(k,0)\}_{1 \leq k \leq r}, B)$ is admissible in the category $\mathcal{C}_w$.*

3E. ***Irreducible representations of quiver Hecke algebras.*** In this subsection we recall from [Kleshchev and Ram 2011] the classification of simple finite-dimensional modules over finite type quiver Hecke algebras. The main result (Theorem 3.31) is that simple objects in the category R-gmod are parametrized in a combinatorial way by *dominant words*, which are analogs of Zelevinsky's multisegments in the classification of simple representations of affine Hecke algebras of type A. As for Lie algebras, simple modules over quiver Hecke algebras are constructed as quotients of tensor products of some distinguished irreducible representations, called *cuspidal modules* in [Kleshchev and Ram 2011].

Choose a labeling of the vertices of the Dynkin diagram of $\mathfrak{g}$ by $I = \{1, \dots, n\}$. A word is a finite set of elements of I. We fix a total order on I by setting $1 < \cdots < n$. The set of all words is a totally ordered set with respect to the lexicographic order induced by $<$.

For $\boldsymbol{i} := (i_1, \dots, i_d)$, set $|\boldsymbol{i}| := \alpha_1 + \cdots + \alpha_d \in Q_+$. Recall from Section 3A that for any $\beta \in Q_+$, $\mathrm{Seq}(\beta) = \{\boldsymbol{i}, |\boldsymbol{i}| = \beta\}$.

Definition 3.22. A word is called Lyndon if it is smaller than all its proper right factors.

Example 3.23. The words 123, 24, 13 are Lyndon. The word 231 is not.

The following statement is well known (see [Lothaire 1997, Theorem 5.1.5]):

Proposition 3.24 (canonical factorization). *Any word μ can be written in a unique way as a concatenation of Lyndon words in the decreasing order*:

$$\mu = (\boldsymbol{i}^{(1)})^{n_1} \cdots (\boldsymbol{i}^{(r)})^{n_r}$$

with $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(r)}$ Lyndon words satisfying $\boldsymbol{i}^{(1)} > \cdots > \boldsymbol{i}^{(r)}$ and $n_1, \dots, n_r$ nonnegative integers.

This is called the *canonical factorization* of the word μ. Recall from Section 3A (Definition 3.2) that for $\beta \in Q_+$, any $R(\beta)$-module M decomposes as a direct sum of vector spaces $M = \bigoplus_{\nu \in \mathrm{Seq}(\beta)} M_\nu$ with $M_\nu := e(\nu)M$.

Definition 3.25. A word μ is dominant if there is an $R(\beta)$-module M such that μ is the highest word among the words ν such that M_ν is not zero: $M = M_\mu \oplus \bigoplus_{\nu < \mu} M_\nu$ and $M_\mu \neq 0$.

Dominant words play the same role as highest weights in the representation theory of finite dimensional semisimple Lie algebras (see [Chari and Pressley 1994]). The next statement provides a very useful combinatorial criterion to determine whether a word is dominant or not. In particular it shows that a dominant word can be seen as a collection (or a sum with positive coefficients) of positive roots, which is why the terminology *root partitions* is sometimes used (see [McNamara 2017]).

Theorem 3.26 [Kleshchev and Ram 2011]. (i) *There is a bijection between the set of dominant Lyndon words and the set Δ_+ of positive roots of $\mathfrak{g}$, given by $\boldsymbol{i} \mapsto |\boldsymbol{i}|$.*

(ii) *A word μ is dominant if and only if all the Lyndon words appearing in the canonical factorization of μ are dominant.*

Example 3.27. In type A_4, 24 is Lyndon but not dominant Lyndon, and 123 is dominant Lyndon. The word 12312 is dominant but the word 3213 is not.

Remark 3.28. Dominant Lyndon words already appear in the work of Leclerc [2004] as *good Lyndon words* in the study of dual canonical bases for quantum groups and quantum coordinate rings.

Proposition 3.29 introduces the notion of *cuspidal modules*. The existence of cuspidal modules follows from results of Varagnolo and Vasserot [2011] in simply laced cases and Rouquier [2012] in the general case. Cuspidal modules can be seen as analogs of fundamental representations for Lie algebras.

Proposition 3.29 [Kleshchev and Ram 2011, Proposition 8.4]. *For any dominant Lyndon word $\boldsymbol{i}$ in* $\mathrm{Seq}(\beta)$, *there is a unique* (*up to isomorphism and shift*) *irreducible $R(\beta)$-module of highest weight $\boldsymbol{i}$. We denote it by $L(\boldsymbol{i})$.*

Kleshchev and Ram [2011] gave explicit constructions of cuspidal modules for each finite type. For instance, in type A_n, the set of positive roots is $\Delta_+ = \{\alpha_i + \cdots + \alpha_j,\ 1 \le i \le j \le n\}$ and the cuspidal module $L(k \ldots l)$ corresponding to the positive root $\alpha_k + \cdots + \alpha_l$ is the one-dimensional vector space spanned by a vector v with action of $R(\alpha_k + \cdots + \alpha_l)$ given by

$$x_i \cdot v = 0, \quad \tau_j \cdot v = 0, \quad e(\nu) \cdot v = \begin{cases} v & \text{if } \nu = k \ldots l, \\ 0 & \text{otherwise.} \end{cases}$$

Recall from Section 3A that the graded character of a finite dimensional $R(\beta)$-module M is a (finite) formal sum of elements of $\mathrm{Seq}(\beta)$ with coefficients in $\mathbb{Z}[q, q^{-1}]$. For any such formal sum $S := \sum_\lambda P_\lambda(q)\lambda$, we let $\max(S)$ denote the greatest word appearing in this sum (for the lexicographic order). In particular, for any finite dimensional $R(\alpha)$-module M, we set $\max(M) := \max(\mathrm{ch}_q(M))$. The word $\max(M)$ is called the *highest weight* of the module M in [Kleshchev and Ram 2011].

The next statement shows that any word (not necessarily dominant) always appears as the highest word in the quantum shuffle product of the Lyndon words appearing in its canonical factorization.

Proposition 3.30 [Kleshchev and Ram 2011, Lemma 5.3]. *Let μ be a word, and $\mu = \boldsymbol{i}^{(1)} \cdots \boldsymbol{i}^{(r)}$ its canonical factorization. Then we have* $\max(\boldsymbol{i}^{(1)} \circ \cdots \circ \boldsymbol{i}^{(r)}) = \mu$.

One can now state the main result of [Kleshchev and Ram 2011]. It shows that finite-dimensional irreducible representations of finite type quiver Hecke algebras are parametrized by dominant words.

Theorem 3.31 [Kleshchev and Ram 2011, Theorem 7.2]. *Let μ be a dominant word, and $\mu = (\boldsymbol{i}^{(1)})^{n_1} \cdots (\boldsymbol{i}^{(r)})^{n_r}$ its canonical factorization. Set*

$$\Delta(\mu) := L(\boldsymbol{i}^{(1)})^{\circ n_1} \circ \cdots \circ L(\boldsymbol{i}^{(r)})^{\circ n_r} \langle s(\mu) \rangle,$$

where $s(\mu) := \sum_{k=1}^{r} (\boldsymbol{i}_k . \boldsymbol{i}_k) n_k (n_k - 1)/4$.

Then:

(i) $\Delta(\mu)$ *has an irreducible head, denoted $L(\mu)$.*

(ii) *The highest weight of $L(\mu)$ is μ*: $\max(L(\mu)) = \mu$.

(iii) *The set $\{L(\mu)\}$ for μ dominant words in* $\mathrm{Seq}(\beta)$ *is a complete and irredundant set of irreducible graded $R(\beta)$-modules up to isomorphism and shift.*

Moreover for μ of the form $\boldsymbol{j}^n$ with $\boldsymbol{j}$ dominant Lyndon, one has $L(\mu) = L(\boldsymbol{j})^{\circ n}$.

Example 3.32. Here are some examples of characters of some simple modules:

$$\mathrm{ch}_q(L(1)) = (1), \quad \mathrm{ch}_q(L(12)) = (12), \quad \mathrm{ch}_q(L(21)) = (21),$$
$$\mathrm{ch}_q(L(312) = (312) + (132), \quad \mathrm{ch}_q(L(11)) = (q + q^{-1})(11).$$

4. Dominance order and compatible seeds

In this section we define a partial ordering on the set of Laurent monomials in the cluster variables of a cluster algebra $\mathcal{A}$. This ordering coincides with the *dominance order* introduced by Qin [2017]. In the context of monoidal categorification of a cluster algebra, we use this dominance order to introduce the notion of *compatible seed* and state the main conjecture of this work (Conjecture 4.10). We end this section with a discussion of monoidal categorifications of cluster algebras via representations of quantum affine algebras following the work of Hernandez and Leclerc [2010], which provides a first example where Conjecture 4.10 holds.

4A. ***Partial ordering on monomials.*** Consider a cluster algebra $\mathcal{A}$ and choose a seed $((x_1, \dots, x_n, x_{n+1}, \dots, x_m), B)$, where $x_1, \dots, x_n$ are the unfrozen variables and $x_{n+1}, \dots, x_m$ are the frozen variables. Let $\mathcal{M}_{\boldsymbol{x}}$ be the monoid of all monomials in the x_i and $\mathcal{G}_{\boldsymbol{x}}$ the abelian group of all Laurent monomials in the x_i. Recall from Section 2A the variables $\hat{y}_j$ defined as

$$\hat{y}_j := \prod_{1 \le i \le m} x_i^{b_{ij}}$$

for any $1 \le j \le n$. In what follows, we write $\boldsymbol{x}^{\alpha}$ for $\prod_i x_i^{\alpha_i}$ for any integers α_i.

One now defines a partial preorder on $\mathcal{G}_{\boldsymbol{x}}$ in the following way: given two Laurent monomials $\boldsymbol{x}^{\boldsymbol{\alpha}} = \prod_i x_i^{\alpha_i}$ and $\boldsymbol{x}^{\boldsymbol{\beta}} = \prod_i x_i^{\beta_i}$ in $\mathcal{G}_{\boldsymbol{x}}$, we set

$$\boldsymbol{x}^{\boldsymbol{\alpha}} \preccurlyeq \boldsymbol{x}^{\boldsymbol{\beta}}$$

if and only if there exist nonnegative integers $\gamma_j,\ 1 \leq j \leq n$, such that

$$\boldsymbol{x}^{\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\alpha}}.\prod_j \hat{y}_j^{\gamma_j}.$$

We denote by $\succcurlyeq$ the opposite preorder.

Assume that the initial exchange matrix B has full rank n. Then the preorder $\preccurlyeq$ becomes an order on $\mathcal{G}_{\boldsymbol{x}}$. This order is the same as the *dominance order* as defined in [Qin 2017, Definition 3.1.1]. Indeed, by definition, the relation $\prod_i x_i^{\alpha_i} \preccurlyeq \prod_i x_i^{\beta_i}$ is equivalent to the existence of nonnegative integers γ_j, $1 \leq j \leq n$ such that, for all i,

$$\beta_i = \alpha_i + \sum_j b_{ij}\gamma_j.$$

In vector notation this can be rewritten as

$$\beta = \alpha + B\gamma$$

and thus the order $\preccurlyeq$ coincides with Qin's dominance order on multi-indices.

Following [Qin 2017], one can use this dominance order to introduce the notions of *pointed elements* and *pointed sets*.

Definition 4.1 [Qin 2017, Definitions 3.1.4 and 3.1.5]. Fix a seed

$$((x_1, \dots, x_n, x_{n+1}, \dots, x_m), B)$$

in $\mathcal{A}$ and assume B has full rank n.

(i) Let P be a Laurent polynomial in the cluster variables $x_1, \dots, x_m$. One says that P is pointed with respect to the seed $((x_1, \dots, x_m), B)$ if among the monomials of P, there is a unique monomial which is a maximal element (for the dominance order $\preccurlyeq$) and has coefficient 1. This monomial is called the leading term of P in [Qin 2017].

(ii) Let L be any set of Laurent polynomials in the cluster variables $x_1, \dots, x_m$. One says that L is pointed with respect to the seed $((x_1, \dots, x_m), B)$ if all the elements of L are pointed and two distinct elements of L have different leading terms.

One can associate a degree to each of the cluster variables x_i (see [Qin 2017]). If P is a pointed element with respect to the seed $((x_1, \dots, x_m), B)$, then the degree of its leading term can be seen as a generalization of the notion of g-vector.

4B. ***Generalized parameters.*** Let us consider an Artinian monoidal category $\mathcal{C}$ and assume we are given a classification of simple objects in $\mathcal{C}$. That is, suppose we are given a poset $(\boldsymbol{M}, \leq)$ together with a bijection ψ between $\boldsymbol{M}$ and the set $\boldsymbol{S} := \{[V], V \text{simple in } \mathcal{C}\}$. Let $L(\mu)$ denote a representative of the isomorphism class in $K_0(\mathcal{C})$ corresponding to $\mu \in \boldsymbol{M}$ via this bijection:

$$\psi : \boldsymbol{S} \to \boldsymbol{M}, \quad [L(\mu)] \mapsto \mu.$$

In what follows, μ will be referred to as the *parameter* of the simple object $L(\mu)$ in $\mathcal{C}$. We will also assume that the identity object is simple in $\mathcal{C}$.

From now on we assume that the category $\mathcal{C}$ satisfies the following property:

Assumption A (decomposition property). Let $\mu, \mu' \in \boldsymbol{M}$ and let $L(\mu)$ and $L(\mu')$ be the corresponding simple objects in $\mathcal{C}$; then the following equality holds in the Grothendieck ring $K_0(\mathcal{C})$:

$$[L(\mu)] \cdot [L(\mu')] = \sum_{\nu \in \boldsymbol{N}_{\mu,\mu'} \subset \boldsymbol{M}} a_\nu [L(\nu)],$$

where $\boldsymbol{N}_{\mu,\mu'}$ is a finite subset of $\boldsymbol{M}$ such that there exists a unique maximal element in $\boldsymbol{N}_{\mu,\mu'}$ and $\{a_\nu,\ \nu \in \boldsymbol{N}_{\mu,\mu'}\}$ is a family of nonzero integers. The maximal element of $\boldsymbol{N}_{\mu,\mu'}$ is denoted by $\mu \odot \mu'$.

Remark 4.2. In various examples of categories satisfying this property (for instance categories of modules over quiver Hecke algebras or quantum affine algebras), the integer $a_{\mu\odot\mu'}$ happens to be equal to 1 for any $\mu, \mu' \in \boldsymbol{M}$, but we will not need this assumption here.

In what follows we will need the additional assumption that the law $\odot$ is compatible with the partial ordering on $\boldsymbol{M}$ in the following sense:

Assumption B. For all $\mu, \nu, \lambda \in \boldsymbol{M}$,

$$\mu \leq \nu \ \Rightarrow\ \lambda \odot \mu \leq \lambda \odot \nu \text{ and } \mu \odot \lambda \leq \nu \odot \lambda.$$

Combining Assumptions A and B leads to the following:

Lemma 4.3. *The law $\odot$ on $\boldsymbol{M}$ is associative.*

Proof. First note that for any $\mu, \mu' \in \boldsymbol{M}$, the set $\boldsymbol{N}_{\mu,\mu'}$ is finite and has a unique maximal element by Assumption A, hence this element (namely $\mu \odot \mu'$) is a greatest element in $\boldsymbol{N}_{\mu,\mu'}$. Now let μ, μ' and μ'' in $\boldsymbol{M}$ and decompose in two different ways the product $[L(\mu)][L(\mu')][L(\mu'')]$. On the one hand, Assumption A gives

$$[L(\mu)][L(\mu')][L(\mu'')] = [L(\mu)] \cdot \sum_{\nu \in \boldsymbol{N}_{\mu',\mu''}} a_\nu [L(\nu)]$$

with $\nu \leq \mu' \odot \mu''$ for every $\nu \in \boldsymbol{N}_{\mu',\mu''}$. For any $\nu \in \boldsymbol{N}_{\mu',\mu''}$, the parameters appearing in the decomposition of $[L(\mu)] \cdot [L(\nu)]$ into classes of simples are all smaller than $\mu \odot \nu$; as $\nu \leq \mu' \odot \mu''$, Assumption B implies $\mu \odot \nu \leq \mu \odot (\mu' \odot \mu'')$. Hence all the parameters involved in the decomposition of $[L(\mu)][L(\mu')][L(\mu'')]$ into classes of simples are smaller than $\mu \odot (\mu' \odot \mu'')$.

On the other hand, one can write

$$[L(\mu)][L(\mu')][L(\mu'')] = \sum_{\nu \in \boldsymbol{N}_{\mu,\mu'}} a_\nu [L(\nu)].[L(\mu'')]$$

and the same arguments show that all the parameters involved in the decomposition of $[L(\mu)][L(\mu')][L(\mu'')]$ into classes of simples are smaller than $(\mu \odot \mu') \odot \mu''$. In particular we get $\mu \odot (\mu' \odot \mu'') \leq (\mu \odot \mu') \odot \mu''$ and $\mu \odot (\mu' \odot \mu'') \geq (\mu \odot \mu') \odot \mu''$ and hence $\mu \odot (\mu' \odot \mu'') = (\mu \odot \mu') \odot \mu''$. □

Thus the operation

$$\boldsymbol{M} \times \boldsymbol{M} \to \boldsymbol{M}, \quad (\mu, \mu') \mapsto \mu \odot \mu'$$

provides $\boldsymbol{M}$ with a monoid structure. The neutral element 1_M is the image via ψ of the class of the identity object of $\mathcal{C}$. By Assumption B, the monoid $(\boldsymbol{M}, \odot)$ is an ordered monoid with respect to $\leq$.

We now assume that $\mathcal{C}$ is a monoidal categorification of a cluster algebra $\mathcal{A}$. Let ϕ be a ring isomorphism

$$\phi : K_0(\mathcal{C}) \xrightarrow{\simeq} \mathcal{A}.$$

As $K_0(\mathcal{C})$ is isomorphic to $\mathcal{A}$, it is in particular commutative which implies that the monoid $(\boldsymbol{M}, \odot)$ is commutative as well. Hence it can be canonically embedded into its Grothendieck group $G(\boldsymbol{M})$, which is defined as follows (see [Bourbaki 1974]):

Definition 4.4 (Grothendieck group of $\boldsymbol{M}$). Elements of $G(\boldsymbol{M})$ are equivalence classes of couples (μ, ν) of elements of $\boldsymbol{M}$ with respect to the equivalence relation

$$(\mu, \nu) \sim (\mu', \nu') \Leftrightarrow \text{there exists } \lambda \in \boldsymbol{M} \text{ such that } \mu \odot \nu' \odot \lambda = \nu \odot \mu' \odot \lambda.$$

The group $G(\boldsymbol{M})$ is an abelian group. We denote it by $\boldsymbol{G}$ in what follows.

The inverse in $\boldsymbol{G}$ of an element $\mu \in \boldsymbol{M}$ will be denoted by $\mu^{\odot -1}$. Similarly $g^{\odot -1}$ stands for the inverse in $\boldsymbol{G}$ of any element g of $\boldsymbol{G}$. We will refer to elements of $\boldsymbol{M}$ as *parameters* and elements of $\boldsymbol{G}$ as *generalized parameters*.

Proposition 4.5. *The ordering $\leq$ on $\boldsymbol{M}$ naturally extends to a partial ordering on $\boldsymbol{G}$ that we also denote by $\leq$.*

Proof. One defines a partial ordering on $\boldsymbol{M} \times \boldsymbol{M}$ by setting

$$(\mu, \nu) \leq (\mu', \nu') \Leftrightarrow \text{there exists } \lambda \in \boldsymbol{M} \text{ such that } \lambda \odot \mu \odot \nu' \leq \lambda \odot \mu' \odot \nu.$$

Using the Assumption B, one can check that if $(\mu, \nu) \sim (\mu', \nu')$ then for any $(\mu'', \nu'') \in \boldsymbol{M} \times \boldsymbol{M}$, one has $(\mu, \nu) \leq (\mu'', \nu'') \Leftrightarrow (\mu', \nu') \leq (\mu'', \nu'')$. Thus $\leq$ naturally gives rise to a well-defined partial ordering on $\boldsymbol{G}$. □

Let us now fix a seed $((x_1, \dots, x_m), B)$ in $\mathcal{A}$ and choose for each $1 \le i \le m$ a representative $L(\mu_i)$ of the isomorphism class $\phi^{-1}(x_i) \in \boldsymbol{S}$ corresponding to the cluster variable x_i. Recall that $\mathcal{M}_{\boldsymbol{x}}$ stands for the monoid of all monomials in the x_i and $\mathcal{G}_{\boldsymbol{x}}$ for the abelian group of all Laurent monomials in the x_i. For any m-tuple of integers $(\alpha_1, \dots, \alpha_m)$, we let $\boldsymbol{\mu}^{\boldsymbol{\alpha}}$ denote the element $\bigodot_{1\le i\le m} {\mu_i}^{\alpha_i}$ of $\boldsymbol{G}$. Of course $\boldsymbol{\mu}^{\boldsymbol{\alpha}}$ belongs to $\boldsymbol{M}$ if all the α_i are nonnegative.

Let us consider the subset $\mathcal{P}$ of $K_0(\mathcal{C})$ consisting of nonzero classes $[M]$ such that there is a unique maximal element among the parameters of the simples appearing in the Jordan–Hölder series of M. Let Ψ be the map

$$\Psi : \mathcal{P} \to \boldsymbol{G}, \quad [M] = a_1[M_1] + \cdots + a_r[M_r] \mapsto \max_{\le}(\psi([M_k]), 1 \le k \le r).$$

Here the a_k are integers and the M_i are the simples of the Jordan–Hölder series of M. Note that $\mathcal{P}$ contains 1, whose image by Ψ is the neutral element of $\boldsymbol{M}$. The set $\boldsymbol{S}$ of classes of simples in $\mathcal{C}$ is a basis of $K_0(\mathcal{C})$ so any element of $\mathcal{P}$ can be written as $a_1[M_1]+\cdots+a_r[M_r]$ in a unique way up to reordering. Thus the map Ψ is well defined.

Let $\widetilde{\mathcal{P}}$ be the subset of $Frac(\mathcal{A})$ defined in the following way:

$$\widetilde{\mathcal{P}} := \{\boldsymbol{x}^{\boldsymbol{\alpha}}\phi(p), \boldsymbol{\alpha} \in \mathbb{Z}^m, p \in \mathcal{P}\}.$$

In particular $\widetilde{\mathcal{P}}$ contains $\mathcal{G}_{\boldsymbol{x}}$. Let $\widetilde{\Psi}$ be the map

$$\widetilde{\Psi} : \widetilde{\mathcal{P}} \to \boldsymbol{G}, \quad \boldsymbol{x}^{\boldsymbol{\alpha}}\phi(p) \mapsto \boldsymbol{\mu}^{\boldsymbol{\alpha}} \odot \Psi(p).$$

Proposition 4.6. *The map $\widetilde{\Psi}$ is well defined and satisfies the following properties*:

(i) $\widetilde{\Psi} \circ \phi$ *coincides with ψ on $\boldsymbol{S}$.*

(ii) $\widetilde{\Psi}$ *defines an abelian group morphism from $\mathcal{G}_{\boldsymbol{x}}$ (for the natural multiplication) to $(\boldsymbol{G}, \odot)$.*

Proof. In order to show that the map $\widetilde{\Psi}$ is well defined, we need to check that if $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are two m-tuples of integers and p, q are two elements of $\mathcal{P}$ such that $\boldsymbol{x}^{\boldsymbol{\alpha}}\phi(p) = \boldsymbol{x}^{\boldsymbol{\beta}}\phi(q)$, then the equality $\boldsymbol{\mu}^{\boldsymbol{\alpha}} \odot \Psi(p) = \boldsymbol{\mu}^{\boldsymbol{\beta}} \odot \Psi(q)$ holds in $\boldsymbol{G}$. Let us write $p = a_1[M_1] + \cdots + a_r[M_r]$ and $q = b_1[N_1] + \cdots + b_s[N_s]$, with $r, s \ge 0$, $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{Z}$ and $[M_1], \dots, [M_r], [N_1], \dots, [N_s] \in \boldsymbol{S}$. Let $\boldsymbol{\gamma}$ be an m-tuple of nonnegative integers such that $\boldsymbol{x}^{\boldsymbol{\gamma}}\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\gamma}}\boldsymbol{x}^{\boldsymbol{\beta}}$ are monomials in the x_i. One can write

$$\begin{aligned}
&\boldsymbol{x}^{\boldsymbol{\alpha}}\phi(p) = \boldsymbol{x}^{\boldsymbol{\beta}}\phi(q)\\
&\quad \Leftrightarrow \boldsymbol{x}^{\boldsymbol{\gamma}}\boldsymbol{x}^{\boldsymbol{\alpha}}\phi(p) = \boldsymbol{x}^{\boldsymbol{\gamma}}\boldsymbol{x}^{\boldsymbol{\beta}}\phi(q) \Leftrightarrow \phi\Big(\prod_i [L(\mu_i)]^{\gamma_i+\alpha_i} p\Big) = \phi\Big(\prod_i [L(\mu_i)]^{\gamma_i+\beta_i} q\Big)\\
&\quad \Leftrightarrow [L(\boldsymbol{\mu}^{\boldsymbol{\gamma}+\boldsymbol{\alpha}})].(a_1[M_1] + \cdots + a_r[M_r]) = [L(\boldsymbol{\mu}^{\boldsymbol{\gamma}+\boldsymbol{\beta}})].(b_1[N_1] + \cdots + b_s[N_s])
\end{aligned}$$

as ϕ is an isomorphism. Let us set $\mu := \boldsymbol{\mu}^{\boldsymbol{\gamma}+\boldsymbol{\alpha}}$ and $\nu := \boldsymbol{\mu}^{\boldsymbol{\gamma}+\boldsymbol{\beta}}$; they are elements of $\boldsymbol{M}$. By Assumption A, for every $1 \leq i \leq r$, the product $[L(\mu)].[M_i]$ decomposes as a sum of classes of simples and $\mu \odot \psi([M_i])$ is maximal among the corresponding parameters. As $p \in \mathcal{P}$, the finite set of parameters $\psi([M_1]), \dots, \psi([M_r])$ has a unique maximal element. For simplicity, let us assume it is $\psi([M_1])$. Then Assumption B implies that $\mu \odot \psi([M_1])$ is maximal among the parameters appearing in the decompositions of $[L(\mu)].[M_i]$, $1 \leq i \leq r$, into classes of simples. The same arguments of course hold for the right-hand side $[L(\nu)].q$. As $\boldsymbol{S}$ is a basis of $K_0(\mathcal{C})$, one gets

$$\mu \odot \psi([M_1]) = \nu \odot \psi([N_1]).$$

Now by definition of Ψ, one has $\psi([M_1]) = \Psi(p)$ and $\psi([N_1]) = \Psi(q)$. Hence we can write in $\boldsymbol{G}$:

$$\begin{aligned}\boldsymbol{\mu}^{\boldsymbol{\alpha}} \odot \Psi(p) = \boldsymbol{\mu}^{-\boldsymbol{\gamma}} \odot \mu \odot \Psi(p) = \boldsymbol{\mu}^{-\boldsymbol{\gamma}} \odot \mu \odot \psi([M_1]) \\ = \boldsymbol{\mu}^{-\boldsymbol{\gamma}} \odot \nu \odot \psi([N_1]) = \boldsymbol{\mu}^{-\boldsymbol{\gamma}} \odot \nu \odot \Psi(q) = \boldsymbol{\mu}^{\boldsymbol{\beta}} \odot \Psi(q)\end{aligned}$$

which is the desired equality. Thus $\widetilde{\Psi}$ is well defined.

For any $\mu \in \boldsymbol{M}$, $\phi([L(\mu)])$ belongs to $\widetilde{\mathcal{P}}$ with $\boldsymbol{\alpha}$ being zero and $p = [L(\mu)]$. Hence by definition one has

$$\widetilde{\Psi}(\phi([L(\mu)])) = \Psi([L(\mu)]) = \mu = \psi([L(\mu)])$$

which proves (i).

Taking elements of $\widetilde{\mathcal{P}}$ for which p is the empty sum, one gets

$$\widetilde{\Psi}(\boldsymbol{x}^{\boldsymbol{\alpha}}) = \boldsymbol{\mu}^{\boldsymbol{\alpha}} = \bigodot_{1 \leq i \leq m} {\mu_i}^{\alpha_i} = \bigodot_{1 \leq i \leq m} \widetilde{\Psi}(\phi([L(\mu_i)]))^{\alpha_i} = \bigodot_{1 \leq i \leq m} \widetilde{\Psi}(x_i)^{\alpha_i},$$

where the third equality is given by (i), which proves (ii). □

4C. ***Compatible seeds.*** In this subsection, we introduce the notion of compatible seed, state our main conjecture (Conjecture 4.10) and explain some consequences.

Definition 4.7 (compatible seed). Let $\mathcal{S} := ((x_1, \dots, x_m), B)$ be a seed in $\mathcal{A}$, let $\mathcal{G}_{\boldsymbol{x}}$ be the group of Laurent monomials in the cluster variables $x_1, \dots, x_m$, and let $\preccurlyeq$ be the corresponding dominance order on $\mathcal{G}_{\boldsymbol{x}}$. Let $\widetilde{\Psi}$ be the map given by Proposition 4.6. For any $1 \leq j \leq n$, set

$$\hat{\mu}_j := \widetilde{\Psi}(\hat{y}_j) = \bigodot_{1 \leq i \leq m} \mu_i^{\odot b_{ij}}.$$

We say that the seed $\mathcal{S}$ is compatible if the restriction $\widetilde{\Psi}_{|\mathcal{G}_{\boldsymbol{x}}} : (\mathcal{G}_{\boldsymbol{x}}, \preccurlyeq) \to (\boldsymbol{G}, \leq)$ is either increasing or decreasing.

Remark 4.8. By construction, the restriction of $\widetilde{\Psi}$ to $\mathcal{G}_{\boldsymbol{x}}$ is increasing if and only if for any Laurent monomials $\prod_i x_i^{\alpha_i}$ and $\prod_i x_i^{\beta_i}$ one has

$$\prod_i x_i^{\alpha_i} \preccurlyeq \prod_i x_i^{\beta_i} \Rightarrow \bigodot_i \mu_i^{\odot\alpha_i} \leq \bigodot_i \mu_i^{\odot\beta_i}.$$

This is equivalent to requiring that for all $\mu \in \boldsymbol{M}$ and for all $1 \leq j \leq n$,

$$\mu \leq \hat{\mu}_j \odot \mu.$$

Similarly, $\widetilde{\Psi}$ is decreasing on $\mathcal{G}_{\boldsymbol{x}}$ if and only if, for all $\mu \in \boldsymbol{M}$ and for all $1 \leq j \leq n$,

$$\mu \geq \hat{\mu}_j \odot \mu.$$

In many examples of monoidal categorifications of cluster algebras, for instance via quantum affine algebras or quiver Hecke algebras, we will check this condition to prove that a seed is compatible.

Remark 4.9. Note that if the seed $\mathcal{S} = ((x_1, \dots, x_m), B)$ is compatible with $\widetilde{\Psi}$ increasing on $\mathcal{G}_{\boldsymbol{x}}$, then $\widetilde{\mathcal{P}}$ contains all the Laurent polynomials in the x_i that are pointed with respect to $\mathcal{S}$, i.e., $\widetilde{\mathcal{P}} \supseteq \mathcal{PT}(0)$ with the notation of [Qin 2017].

One can now state the main conjecture of this paper:

Conjecture 4.10. Let $\mathcal{A}$ be a cluster algebra and $\mathcal{C}$ an Artinian monoidal categorification of $\mathcal{A}$. Assume there exists a poset $(\boldsymbol{M}, \leq)$ as above such that Assumptions A and B hold. Then there exists a compatible seed in $\mathcal{A}$.

The next statements provide some useful consequences of the existence of a compatible seed. In particular we combine it with the results of [Fomin and Zelevinsky 2007] and [Derksen et al. 2010] recalled in Section 2A to relate parameters of simple objects in $\mathcal{C}$ to some cluster algebra invariants, such as g-vectors and F-polynomials.

Let $\mathcal{C}$ be an Artinian monoidal categorification of a cluster algebra $\mathcal{A}$ and assume Conjecture 4.10 holds. Let $((x_1, \dots, x_n, x_{n+1}, \dots, x_m), B)$ be a compatible seed. Consider x_l^t a cluster variable in $\mathcal{A}$ belonging to another cluster and let $F^{l,t}$ be its F-polynomial. Let

$$X_1^{a_1^{l,t}} \cdots X_n^{a_n^{l,t}}$$

be the monomial given by Theorem 2.5(i).

Corollary 4.11. *One has $F^{l,t}(\hat{y}_1, \dots, \hat{y}_n) \in \widetilde{\mathcal{P}}$ and*

$$\widetilde{\Psi}(F^{l,t}(\hat{y}_1, \dots, \hat{y}_n)) = \begin{cases} \bigodot_j \hat{\mu}_j^{\odot a_j^{l,t}} & \text{if } \widetilde{\Psi} \text{ is increasing on } \mathcal{G}_{\boldsymbol{x}}, \\ 1_{\boldsymbol{G}} & \text{otherwise.} \end{cases}$$

Here $1_{\boldsymbol{G}}$ denotes the neutral element of $\boldsymbol{G}$.

Proof. By Theorem 2.4, $F^{l,t}(\hat{y}_1, \dots, \hat{y}_n)$ is the product of a Laurent monomial in the x_i with the cluster variable $x^{l,t}$. Since $\mathcal{C}$ is a monoidal categorification of $\mathcal{A}$, $x^{l,t}$ is the image by ϕ of the class of a simple object in $\mathcal{C}$. Hence $F^{l,t}(\hat{y}_1, \dots, \hat{y}_n) \in \widetilde{\mathcal{P}}$.

By Theorem 2.5 (i) and (ii), any monomial of $F^{l,t}$ can be written as $X_1^{b_1} \cdots X_n^{b_n}$ with $0 \le b_j \le a_j^{l,t}$ for all $1 \le j \le n$. As the seed $((x_1, \dots, x_m), B)$ is compatible, evaluating on the $\hat{y}_j$ and considering the corresponding generalized parameters $\hat{\mu}_j$ yields

$$\bigodot_j \hat{\mu}_j^{a_j^{l,t}} \ge \bigodot_j \hat{\mu}_j^{b_j} \quad \text{if } \widetilde{\Psi} \text{ is increasing on } \mathcal{G}_{\boldsymbol{x}}$$

and

$$\bigodot_j \hat{\mu}_j^{a_j^{l,t}} \le \bigodot_j \hat{\mu}_j^{b_j} \quad \text{if } \widetilde{\Psi} \text{ is decreasing on } \mathcal{G}_{\boldsymbol{x}}.$$

Hence among all the generalized parameters appearing in the term $F^{l,t}(\hat{y}_1, \dots, \hat{y}_n)$, there is one which is greater than the others, namely,

$$\bigodot_j \hat{\mu}_j^{\odot a_j^{l,t}}$$

if $\widetilde{\Psi}$ is increasing on $\mathcal{G}_{\boldsymbol{x}}$, and $1_{\boldsymbol{G}}$ otherwise. □

Corollary 4.12 shows that the existence of a compatible seed in $\mathcal{A}$ implies relations between the g-vector (with respect to this initial (compatible) seed) of any cluster variable in $\mathcal{A}$ and the parameter of the corresponding simple object in $\mathcal{C}$.

Let

$$x_{n+1}^{c_1^{l,t}} \cdots x_m^{c_{m-n}^{l,t}}$$

be the monomial in the frozen variables equal to the denominator $F^{l,t}{}_{|\mathbb{P}}(y_1, \dots, y_n)$ in the right-hand side of (2). Note that, in view of the definition of the semifield $\mathbb{P}$, the $c_i^{l,t}$ are negative integers, as the F-polynomial $F^{l,t}$ has constant term equal to 1 by Theorem 2.5(ii).

Corollary 4.12. *Let $\mu^{l,t}$ be the parameter of the simple module corresponding to the cluster variable $x^{l,t}$, i.e., $x^{l,t} = \phi([L(\mu^{l,t})])$. Then*

$$\mu^{l,t} = \begin{cases} \bigodot_{1\le j\le n} \hat{\mu}_j^{\odot a_j^{l,t}} \odot \bigodot_{1\le i\le m-n} \mu_{n+i}^{\odot(-c_i^{l,t})} \odot \bigodot_{1\le i\le n} \mu_i^{\odot g_i^{l,t}} & \text{if } \widetilde{\Psi} \text{ is increasing on } \mathcal{G}_{\boldsymbol{x}}, \\ \bigodot_{1\le i\le m-n} \mu_{n+i}^{\odot(-c_i^{l,t})} \odot \bigodot_{1\le i\le n} \mu_i^{\odot g_i^{l,t}} & \text{if } \widetilde{\Psi} \text{ is decreasing on } \mathcal{G}_{\boldsymbol{x}}. \end{cases}$$

Proof. We give the proof only in the case where the restriction of $\widetilde{\Psi}$ to $\mathcal{G}_{\boldsymbol{x}}$ is increasing, the other case being analogous.

By Proposition 4.6(i), $\widetilde{\Psi}(x_l^t) = \psi([L(\mu^{l,t})]) = \mu^{l,t}$. On the other hand, one can use the previous corollary and apply $\widetilde{\Psi}$ to both sides of (2):

$$\bigodot_j \hat{\mu}_j^{\odot a_j^{l,t}} = \widetilde{\Psi}(F^{l,t}(\hat{y}_1,\dots,\hat{y}_n)) = \widetilde{\Psi}\left(\frac{x_{n+1}^{c_1^{l,t}}\cdots x_m^{c_{m-n}^{l,t}}}{x_1^{g_1^{l,t}}\cdots x_n^{g_n^{l,t}}}.x^{l,t}\right)$$

$$= \mu^{l,t} \odot \bigodot_{1\le i\le m-n} \mu_{n+i}^{\odot c_i^{l,t}} \odot \bigodot_{1\le i\le n} \mu_i^{\odot(-g_i^{l,t})} \qquad \text{(by Proposition 4.6(ii)).}$$

Finally one can conclude:

$$\mu^{l,t} = \bigodot_{1\le j\le n} \hat{\mu}_j^{\odot a_j^{l,t}} \odot \bigodot_{1\le i\le m-n} \mu_{n+i}^{\odot(-c_i^{l,t})} \odot \bigodot_{1\le i\le n} \mu_i^{\odot g_i^{l,t}}. \qquad \square$$

4D. *First example: the category $\mathcal{C}_1$.* The first example of compatible seed appears in [Hernandez and Leclerc 2010] and is one of the main motivations for this work.

Recall from Section 2C the definition of the category $\mathcal{C}_1$. This category was introduced in [Hernandez and Leclerc 2010] as a (monoidal) subcategory of the category of finite-dimensional representations of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$. For $\mathfrak{g}$ of types ADE, the category $\mathcal{C}_1$ is a monoidal categorification of a cluster algebra of the same cluster type (in the classification of [Fomin and Zelevinsky 2003]) as the Lie type of $\mathfrak{g}$ ([Hernandez and Leclerc 2010; Nakajima 2011]). As explained in Section 2C, the simple finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules are parametrized by dominant monomials. The monoid $\boldsymbol{M}$ parametrizing simple objects in the category $\mathcal{C}_1$ is a submonoid of the set of dominant monomials involving only the variables $Y_{i,a}$, $i \in I$, $a \in q^{\mathbb{Z}}$. The monoid law $\odot$ is simply the natural multiplication of monomials. The ordering $\le$ on $\boldsymbol{M}$ is the restriction of the Nakajima order on dominant monomials (see Section 2C). Assumptions A and B are obviously satisfied for the category $\mathcal{C}_1$.

Hernandez and Leclerc [2010] explicitly gave an initial seed in the category $\mathcal{C}_1$:

Theorem 4.13 [Hernandez and Leclerc 2010]. *Each seed has $n = |I|$ unfrozen variables and n frozen variables. These frozen variables are given by the classes of the modules $L(Y_{i,q^{\xi_i}} Y_{i,q^{\xi_i+2}})$, $i \in I$.*

Moreover, an initial seed in this cluster algebra is given by the classes

$$[L(Y_{i,q^{\xi_i+2}})], \quad [L(Y_{i,q^{\xi_i}} Y_{i,q^{\xi_i+2}})], \qquad i \in I,$$

together with the exchange matrix $B = (b_{ij})$ whose columns are indexed by I and rows by $I \sqcup I' = [1,n] \sqcup [n+1,2n]$, and whose entries are given by

$$b_{ij} := \begin{cases} (-1)^{\xi_j} a_{ij} & \text{if } i, j \in I \text{ and } i \neq j, \\ -1 & \text{if } j \in I \text{ and } i = j+n \in I', \\ -a_{kj} & \text{if } j \in J_0 \text{ and } i = k+n \in I' \text{ with } k \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Here the a_{ij} are the entries of the Cartan matrix associated to $\mathfrak{g}$.

The cluster algebra $\mathcal{A}$ is the cluster algebra (in the classification of [Fomin and Zelevinsky 2003]) generated by the initial seed $((x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}), B)$ where B is the matrix above. Following [Hernandez and Leclerc 2010], we denote by ι the unique ring isomorphism

$$\iota : \mathcal{A} \xrightarrow{\simeq} K_0(\mathcal{C}_1)$$

such that

$$\iota(x_i) = [L(Y_{i,q^{\xi_i+2}})], \quad \iota(x_{n+i}) = [L(Y_{i,q^{\xi_i}} Y_{i,q^{\xi_i+2}})], \qquad 1 \le i \le n.$$

The isomorphism ι is the inverse of the isomorphism ϕ in Section 4B. Using the map $\widetilde{\Psi}$ associated to the seed $((x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}), B)$ above, one can compute the generalized parameters $\hat{\mu}_j$ corresponding to this seed. This is done using the following statement, as a direct consequence of Theorem 4.13:

Proposition 4.14 [Hernandez and Leclerc 2010, Lemma 7.2]. *With the notation of Section 4B, for all* $1 \le j \le n$,

$$\hat{\mu}_j = A^{-1}_{j,q^{\xi_j+1}}.$$

Corollary 4.15. *Conjecture 4.10 holds for the category* $\mathcal{C}_1$.

Proof. By definition of the Nakajima ordering on monomials (see Section 2C), Proposition 4.14 implies that for any dominant monomial $\mathfrak{m}$,

$$\mathfrak{m} \ge \hat{\mu}_j \mathfrak{m}.$$

for all $1 \le j \le n$. By Remark 4.8, this implies that the map $\widetilde{\Psi}$ associated to the seed $((x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}), B)$ is decreasing. Hence this seed is compatible in the sense of Definition 4.7 and Conjecture 4.10 holds. ☐

We conclude this section with an illustration of the relationships between g-vectors and highest weights for quantum affine algebras that occur as a consequence of the initial seed of [Hernandez and Leclerc 2010] being compatible. The cluster structure on $K_0(\mathcal{C}_1)$ is a finite type cluster algebra; thus one can use the results of [Fomin and Zelevinsky 2003] and label the cluster variables by almost positive roots, i.e., positive roots together with the opposite of the simple roots. Let $x[\alpha]$ denote the cluster variable associated to the almost positive root α with respect to the above initial seed.

Following [Fomin and Zelevinsky 2003], one defines piecewise linear involutions τ_ϵ ($\epsilon \in \{-1, 1\}$) of the root lattice Q of $\mathfrak{g}$: for any $\gamma \in Q$,

$$[\tau_\epsilon(\gamma) : \alpha_i] = \begin{cases} -[\gamma : \alpha_i] - \sum_{j \ne i} a_{ij} \max(0, [\gamma, \alpha_j]) & \text{if } \epsilon_i = \epsilon, \\ [\gamma : \alpha_i] & \text{if } \epsilon_i \ne \epsilon, \end{cases}$$

where $[\gamma : \alpha_i]$ stands for the coefficient of α_i in the expansion of γ on simple roots.

Corollary 4.16 [Hernandez and Leclerc 2010, Corollary 7.4]. *Let α be an almost positive root. Set $\beta := \tau_-(\alpha)$. Write $\beta = \sum_i b_i\alpha_i$. The highest weight of the simple module corresponding to the cluster variable $x[\alpha]$ is given by*

$$\prod_{i\in I_0} Y_{i,1}^{b_i} \cdot \prod_{i\in I_1} Y_{i,q^3}^{b_i}.$$

It is known from [Fomin and Zelevinsky 2007] that in the case of a cluster algebra of finite type, the g-vector of the variable $x[\alpha]$ is given by

$$\boldsymbol{g}(\alpha) = E\tau_-(\alpha),$$

where E is the automorphism of the root lattice of $\mathfrak{g}$ which sends the simple root α_i onto $(-1)^{\xi_i+1}\alpha_i$.

Thus the previous corollary can be reformulated in the following way:

Corollary 4.17. *Let α be an almost positive root and let $\boldsymbol{g}(\alpha)$ be the g-vector of the cluster variable $x[\alpha]$ with respect to the above initial seed. The highest weight of the simple module corresponding to $x[\alpha]$ is given by*

$$\prod_{i\in I_0} Y_{i,1}^{-g_i} \cdot \prod_{i\in I_1} Y_{i,q^3}^{g_i}.$$

5. A mutation rule for parameters of simple representations of quiver Hecke algebras

In this section, we consider the category $\mathcal{C} = R$-gmod of finite-dimensional representations of symmetric quiver Hecke algebras of finite type A_n. The set $\boldsymbol{M}$ is the set of *dominant words* (see Section 2) and the order $\leq$ is the natural lexicographic order; it is a total ordering, hence Assumption A obviously holds. Moreover, with the notation of Section 3E, one has $\mu\odot\nu = \max(L(\mu)\circ L(\nu))$ for any dominant words μ and ν. We begin by describing explicitly the monoid operation $\odot$ for dominant words. In particular it can be easily computed using *canonical factorizations* of dominant words (see Proposition 3.24). We apply this in the context of monoidal categorifications of cluster algebras via quiver Hecke algebras following the works of Kang et al. [2018a; 2018b]. We obtain a combinatorial rule for the transformation of dominant words under cluster mutation.

5A. ***Convolution product of simple modules.*** This subsection is devoted to the description of the monoid structure $\odot$ on the monoid $\boldsymbol{M}$ of dominant words in the case of a symmetric quiver Hecke algebra of type A_n. First we restrict to the case where the canonical factorizations of two words μ, μ' are ordered with respect to each other. We show that in this case, Proposition 3.30 implies that the monoidal product $\mu\odot\mu'$ is simply the concatenation of μ and μ' (Corollary 5.3). Then we state the main result of this section (Proposition 5.4) which gives a combinatorial expression for $\mu \odot \mu'$ for any μ, μ'. Our proof involves ideas similar to the ones used in [Kleshchev and Ram 2011] for the proof of Proposition 3.30, but here we use the specific form of dominant Lyndon words (in bijection with positive roots) in type A_n.

Recall (see Proposition 3.24) that any word μ can be written in a unique way as a concatenation of Lyndon words in the decreasing order. This is called the canonical factorization of μ. Moreover, if the word μ is dominant, then all the Lyndon words involved in the canonical factorization of μ are dominant as well (Theorem 3.26(ii)). By Theorem 3.26(i), the canonical factorization of a dominant word μ can be seen as a sum of positive roots in the decreasing order. In particular, in type A_n these positive roots correspond to words of the form $k(k+1)\dots l$ with $k \leq l$.

We begin by recalling a technical result from [Kleshchev and Ram 2011].

Lemma 5.1 [Kleshchev and Ram 2011, Lemma 5.1]. *Let $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(r)}, \boldsymbol{j}^{(1)}, \dots, \boldsymbol{j}^{(r)}$ be words such that for each $k \in \{1, \dots, r\}$, $\boldsymbol{i}^{(k)}$ and $\boldsymbol{j}^{(k)}$ have same length. Assume that $\boldsymbol{i}^{(1)} \geq \boldsymbol{j}^{(1)}, \dots, \boldsymbol{i}^{(r)} \geq \boldsymbol{j}^{(r)}$. Then* $\max(\boldsymbol{i}^{(1)} \circ \dots \circ \boldsymbol{i}^{(r)}) \geq \max(\boldsymbol{j}^{(1)} \circ \dots \circ \boldsymbol{j}^{(r)})$. *Moreover, this inequality is an equality if and only if all the inequalities $\boldsymbol{i}^{(k)} \geq \boldsymbol{j}^{(k)}$ are equalities.*

Proposition 5.2 states that the product $\mu \odot \mu'$ of two dominant words μ and μ' coincides with the highest word in the quantum shuffle product of μ and μ'.

Proposition 5.2. *Let μ, ν be dominant words. Then*

$$\mu \odot \nu = \max(\mu \circ \nu).$$

Proof. By Theorem 3.31, we can write

$$\mathrm{ch}_q(L(\mu)) = P(q).\mu + \sum_{\mu' < \mu} a_{\mu'}(q).\mu' \quad \text{and} \quad \mathrm{ch}_q(L(\nu)) = Q(q).\nu + \sum_{\nu' < \nu} b_\nu(q).\nu',$$

where P, Q, a, b are Laurent polynomials in q (P and Q nonzero). By definition

$$\mu \odot \nu = \max(L(\mu) \circ L(\nu)) = \max(\mathrm{ch}_q(L(\mu) \circ L(\nu))) = \max(\mathrm{ch}_q(L(\mu)) \circ \mathrm{ch}_q(L(\nu))).$$

By Lemma 5.1, for any $\mu' < \mu$, $\max(\mu' \circ \nu) < \max(\mu \circ \nu)$. Similarly, $\max(\mu \circ \nu') < \max(\mu \circ \nu)$ and $\max(\nu' \circ \nu) < \max(\mu \circ \nu)$ for any $\mu' < \mu$ and $\nu' < \nu$. So the highest word of $\mathrm{ch}_q(L(\mu)) \circ \mathrm{ch}_q(L(\nu))$ can only come from the shuffle product $\mu \circ \nu$. Hence

$$\mu \odot \nu = \max(\mu \circ \nu). \qquad \square$$

More generally, the same proof shows that for any finite formal series R and S on W with coefficients in $\mathbb{Z}[q, q^{-1}]$, one has $\max(R \circ S) = \max(\max(R) \circ \max(S))$. The following corollary is then a direct consequence of Proposition 3.30.

Corollary 5.3. *Let μ and μ' be two dominant words. Write their canonical factorizations,*

$$\mu = (\boldsymbol{i}^{(1)})^{n_1} \cdots (\boldsymbol{i}^{(r)})^{n_r} \quad \text{and} \quad \mu' = (\boldsymbol{i}'^{(1)})^{n'_1} \cdots (\boldsymbol{i}'^{(r')})^{n'_{r'}}$$

with $\boldsymbol{i}^{(1)} > \cdots > \boldsymbol{i}^{(r)}$ and $\boldsymbol{i}'^{(1)} > \cdots > \boldsymbol{i}'^{(r')}$. Assume $\boldsymbol{i}^{(r)} \geq \boldsymbol{i}'^{(1)}$.

Then

$$\mu \odot \mu' = (\boldsymbol{i}^{(1)})^{n_1} \cdots (\boldsymbol{i}^{(r)})^{n_r} (\boldsymbol{i}'^{(1)})^{n'_1} \cdots (\boldsymbol{i}'^{(r')})^{n'_{r'}}.$$

Proof. Setting $R := (\boldsymbol{i}^{(1)})^{n_1} \circ \cdots \circ (\boldsymbol{i}^{(r)})^{n_r}$ and $R' := (\boldsymbol{i}'^{(1)})^{n'_1} \circ \cdots \circ (\boldsymbol{i}'^{(r)})^{n'_{r'}}$, Proposition 3.30 implies $\mu = \max(R)$ and $\mu' = \max(R')$. Then by Proposition 5.2, one has:

$$\begin{aligned} \mu \odot \mu' &= \max(\mu \circ \mu') && \text{(by Proposition 5.2)} \\ &= \max(\max(R) \circ \max(R')) \\ &= \max(R \circ R') && \text{(by Proposition 5.2)} \\ &= \max\big((\boldsymbol{i}^{(1)})^{n_1} \circ \cdots \circ (\boldsymbol{i}^{(r)})^{n_r} \circ (\boldsymbol{i}'^{(1)})^{n'_1} \circ \cdots \circ (\boldsymbol{i}'^{(r)})^{n'_{r'}}\big). \end{aligned}$$

The assumption $\boldsymbol{i}^{(r)} \geq \boldsymbol{i}'^{(1)}$ implies that $(\boldsymbol{i}^{(1)})^{n_1} \cdots (\boldsymbol{i}^{(r)})^{n_r} (\boldsymbol{i}'^{(1)})^{n'_1} \cdots (\boldsymbol{i}'^{(r)})^{n'_{r'}}$ is the canonical factorization of the concatenation $\mu\mu'$. Hence by Proposition 3.30, we get

$$\mu \odot \mu' = \mu\mu' = (\boldsymbol{i}^{(1)})^{n_1} \cdots (\boldsymbol{i}^{(r)})^{n_r} (\boldsymbol{i}'^{(1)})^{n'_1} \cdots (\boldsymbol{i}'^{(r')})^{n'_{r'}}. \qquad \square$$

One can now state the main result of this section. It can be seen as a generalization of Corollary 5.3, as we now drop the hypothesis $\boldsymbol{i}^{(r)} \geq \boldsymbol{i}'^{(1)}$.

Proposition 5.4. *Let μ and μ' be two dominant words. Write their canonical factorizations,*

$$\mu = (\boldsymbol{i}^{(1)})^{n_1} \cdots (\boldsymbol{i}^{(r)})^{n_r} \quad \text{and} \quad \mu' = (\boldsymbol{i}'^{(1)})^{n'_1} \cdots (\boldsymbol{i}'^{(r')})^{n'_{r'}}.$$

Let $\{\boldsymbol{j}^{(1)}, \ldots, \boldsymbol{j}^{(s)}\}$ be the set of all the words $\boldsymbol{i}^{(1)}, \ldots, \boldsymbol{i}^{(r)}, \boldsymbol{i}'^{(1)}, \ldots, \boldsymbol{i}'^{(r')}$ ranged in the decreasing order. Let $m_1, \ldots, m_s, m'_1, \ldots, m'_s$ be the positive integers uniquely determined by

$$\mu = (\boldsymbol{j}^{(1)})^{m_1} \cdots (\boldsymbol{j}^{(s)})^{m_s} \quad \text{and} \quad \mu' = (\boldsymbol{j}^{(1)})^{m'_1} \cdots (\boldsymbol{j}^{(s)})^{m'_s}.$$

Then

$$\mu \odot \mu' = (\boldsymbol{j}^{(1)})^{m_1+m'_1} \cdots (\boldsymbol{j}^{(s)})^{m_s+m'_s}.$$

Using Theorem 3.31, one can reformulate this statement in the following way: write the positive roots in the decreasing order

$$\begin{aligned} (3) \quad \alpha_n > \alpha_{n-1} + \alpha_n > \alpha_{n-1} > \cdots > \alpha_i + \alpha_{i+1} + \cdots + \alpha_n > \cdots \\ \cdots > \alpha_i > \cdots > \alpha_1 + \cdots + \alpha_n > \alpha_1 \end{aligned}$$

and let $r_n = n(n+1)/2$ denote the number of these positive roots. Define a map

$$(\boldsymbol{M}, \odot) \to (\mathbb{Z}_{\geq 0}^{r_n}, +), \quad \mu \mapsto \overrightarrow{\mu}, \tag{4}$$

such that the i-th coordinate of the vector $\overrightarrow{\mu}$ is equal to the multiplicity of the Lyndon word corresponding to the i-th positive root (in the above decreasing order) in the canonical factorization of μ.

Theorem 5.5. *The map (4) is an isomorphism of abelian monoids.*

Proof of Proposition 5.4. For simplicity we use a slight change of notation for the proof: we write

$$\mu = \boldsymbol{i}^{(1)} \cdots \boldsymbol{i}^{(r)} \quad \text{and} \quad \mu' = \boldsymbol{i}'^{(1)} \cdots \boldsymbol{i}'^{(r')}$$

with $\boldsymbol{i}^{(1)} \geq \cdots \geq \boldsymbol{i}^{(r)}$ and $\boldsymbol{i}'^{(1)} \geq \cdots \geq \boldsymbol{i}'^{(r')}$ dominant Lyndon words not necessarily distinct. Let n (resp. n') be the length of μ (resp. μ'). The starting point is the word $\mu.\mu'$ which is the concatenation of the words μ and μ' and we consider permutations $\sigma \in \mathfrak{S}_{r,s}$, i.e., those permutations whose restrictions to $[|1; n|]$ and $[|n+1; n+n'|]$ are increasing (see Definition 3.3).

First note that the word $\mu \odot \mu'$ indeed appears in the quantum shuffle product of μ with μ': consider the permutation σ simply defined by rearranging the blocks $(\boldsymbol{i}^{(1)}), \ldots, (\boldsymbol{i}^{(r)}), (\boldsymbol{i}'^{(1)}), \ldots, (\boldsymbol{i}'^{(r')})$ of the concatenation $\mu.\mu'$ and putting them in the decreasing order.

We write $\mu = h_1, \ldots, h_n$ and $\mu' = h'_1, \ldots, h'_{n'}$. The concatenation of μ and μ' is then $\mu.\mu' = h_1, \ldots, h_n, h'_1, \ldots, h'_{n'}$. As in Definition 3.3, we set $h_{n+1} := h'_1, \ldots, h_{n+n'} := h'_{n'}$ and thus $\mu.\mu' = h_1, \ldots, h_{n+n'}$. We set

$$\sigma(\mu.\mu') := h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n+n')}$$

for any permutation $\sigma \in \mathfrak{S}_{r,s}$. From now on we fix $\sigma \in \mathfrak{S}_{r,s}$ and we assume that the word $\sigma(\mu.\mu')$ is greater than or equal to $\mu \odot \mu'$ (for the lexicographic order). We show that under this assumption, one necessarily has $\sigma(\mu.\mu') = \mu \odot \mu'$.

The proof is based on an induction on $r + r'$ or equivalently on the sum of the lengths of μ and μ'.

We first look at the action of σ on the Lyndon words $\boldsymbol{i}^{(1)}$ and $\boldsymbol{i}'^{(1)}$ and show that σ necessarily rearranges these two blocks so that in the word $\sigma(\mu.\mu')$ they will appear in the decreasing order. Then, considering the restriction of the action of σ on the other Lyndon words, we find ourselves considering a shuffle of parameters $\tilde{\mu}$ and $\tilde{\mu'}$, one of them being of length strictly smaller than the corresponding initial parameter.

First case: $\boldsymbol{i}'^{(1)} > \boldsymbol{i}^{(1)}$. Here, the word $\mu \odot \mu'$ begins with $\boldsymbol{i}'^{(1)}$.

The two words $\boldsymbol{i}^{(1)}$ and $\boldsymbol{i}'^{(1)}$ are dominant Lyndon words so we write them $\boldsymbol{i}^{(1)} = (k, k+1, \dots, k+d_1)$ and $\boldsymbol{i}'^{(1)} = (k', k'+1, \dots, d'_1)$ with either $k' > k$ or $k' = k$ and $d'_1 > d_1$.

We first show that the assumption $\sigma(\mu.\mu') \geq \mu \odot \mu'$ implies $\sigma(n+1) = 1$. Indeed, if $\sigma(n+1) \geq d_1 + 1$ then, as the restrictions of σ to $[|1; n|]$ and $[|n+1; n+n'|]$ are increasing, we have $\sigma(1) = 1, \dots, \sigma(d_1) = d_1$. The word $\sigma(\mu.\mu')$ then begins with $(k, \dots, k+d_1, l, \dots)$, where l is equal to k if $\sigma(n+1) > d_1 + 1$ and to k' if $\sigma(n+1) = d_1 + 1$. If $k < k'$ then $k, \dots, k+d_1 < \boldsymbol{i}'^{(1)}$ and hence $\sigma(\mu.\mu') < \mu \odot \mu'$. If $k' = k$ and $d'_1 > d_1$ then

$$\begin{aligned} k, \dots, k+d_1, l \dots &= k', \dots, k'+d_1, k' \\ &< k', \dots, k'+d_1, k'+d_1+1 \\ &\leq k', \dots, k'+d'_1 \\ &= \boldsymbol{i}'^{(1)} \end{aligned}$$

and the conclusion is the same.

If $\sigma(n+1) \in \{2, 3, \dots, d_1\}$, then $\sigma(\mu.\mu')$ begins with $(k, k+1, \dots, k+p, k', \dots)$ where p is some integer such that $0 \leq p < d_1$. If $k < k'$ then $k, k+1, \dots, k+p, k' < \boldsymbol{i}'^{(1)}$ and hence $\sigma(\mu.\mu') < \mu \odot \mu'$. If $k' = k$ and $d'_1 > d_1$ then

$$\begin{aligned} k, k+1, \dots, k+p, k' &= k, k+1, \dots, k+p, k \\ &< k, k+1, \dots, k+p, k+p+1 \\ &\leq k, k+1, \dots, k+d_1 \\ &< k, k+1, \dots, k+p, k+d'_1 \\ &= \boldsymbol{i}'^{(1)} \end{aligned}$$

and the conclusion is the same. Thus $\sigma(n+1) = 1$.

As the restrictions of σ to $[|1; n|]$ and $[|n+1; n+n'|]$ are increasing, $\sigma^{-1}(2)$ is either equal to 1 or to $n+2$; but the first possibility gives a word beginning with $k'k$ which is obviously strictly smaller than $\mu \odot \mu'$. Hence $\sigma(n+2) = 2$. Then by iterating this, we see that necessarily

$$\sigma(n+1) = 1, \dots, \sigma(n+d'_1) = d'_1.$$

In other words σ sends the blocks $\boldsymbol{i}'^{(1)}$ onto the left of the blocks $\boldsymbol{i}^{(1)}$, i.e., to the beginning of the word $\sigma(\mu.\mu')$.

Second case: $\boldsymbol{i}'^{(1)} < \boldsymbol{i}^{(1)}$. Here, $\mu \odot \mu'$ begins with $(\boldsymbol{i}^{(1)})$ and with the previous notation, one has either $k' < k$ or $k' = k$ and $d'_1 < d_1$.

We show that the assumption $\sigma(\mu.\mu') \geq \mu \odot \nu$ implies $\sigma(n+1) > d_1$.

Indeed, if $\sigma(n+1) \in \{2, \ldots, d_1\}$, then $\sigma(\mu.\mu')$ begins with $(k, \ldots, k+p, k')$ where p is some integer such that $0 \leq p < d_1$. But then

$$\begin{aligned} k, k+1, \ldots, k+p, k' &\leq k, k+1, \ldots, k+p, k \\ &< k, k+1, \ldots, k+p, k+p+1 \\ &\leq k, k+1, \ldots, k+d_1 \\ &= \boldsymbol{i}^{(1)} \end{aligned}$$

and hence $\sigma(\mu.\mu') < \mu \odot \mu'$.

If $\sigma(n+1) = 1$, then $\sigma(\mu.\mu') \geq \mu \odot \mu'$ implies $k' = k$ (and hence $d'_1 < d_1$). But then it is easy to see that necessarily $\sigma(n+2) = 2, \ldots, \sigma(n+d'_1) = d'_1$, i.e., $\sigma(\mu.\mu')$ begins with $(k, \ldots, k+d'_1, \ldots)$. The letter coming after $k+d'_1$ is either the first letter of $\boldsymbol{i}^{(1)}$ (if $\sigma(1) = d'_1+1$) or the first letter of $\boldsymbol{i}'^{(2)}$ (if $\sigma(n+d'_1+1) = d'_1+1$); in both cases it is smaller than k and in particular smaller than $k+d'_1+1$ and hence $\sigma(\mu.\mu') < \mu \odot \mu'$. Thus $\sigma(n+1) > d_1$.

In particular, $\sigma(1) = 1, \ldots, \sigma(d_1) = d_1$ (in other words, σ fixes the block $\boldsymbol{i}^{(1)}$, i.e., leaves it at the beginning of the resulting word.

Third case: $\boldsymbol{i}'^{(1)} = \boldsymbol{i}^{(1)}$. Here, the word $\mu \odot \mu'$ begins with $(\boldsymbol{i}^{(1)})^2$.

We show that under the assumption $\sigma(\mu.\mu') \geq \mu \odot \mu'$, one has either $\sigma(n+1) = 1, \ldots, \sigma(n+d_1) = d_1$ (i.e., σ sends the block $\boldsymbol{i}^{(1)}$ coming from μ' to the left of the block $\boldsymbol{i}^{(1)}$ coming from μ) or $\sigma(1) = 1, \ldots, \sigma(d_1) = d_1$ (i.e., σ fixes the block $\boldsymbol{i}^{(1)}$ coming from μ).

Indeed, as the restrictions of σ to $[|1; n|]$ and $[|n+1; n+n'|]$ are increasing, $\sigma^{-1}(1)$ is either equal to 1 or to $n+1$.

If $\sigma(1) = 1$, then $\sigma(n+1)$ is necessarily strictly greater than d_1, otherwise $\sigma(\mu.\mu')$ would begin with $(k, k+1, \ldots, k+p, k, \ldots)$ (where p is some integer such that $0 \leq p < d_1$) and would be strictly smaller than $\mu \odot \mu'$. Hence in this case we get $\sigma(1) = 1, \ldots, \sigma(d_1) = d_1$.

If $\sigma(n+1) = 1$, then the same argument shows that $\sigma(1)$ is necessarily strictly greater than d_1, and hence we get $\sigma(n+1) = 1, \ldots, \sigma(n+d_1) = d_1$.

In conclusion, we have shown that the permutations we are seeking fix the block $\boldsymbol{i}^{(1)}$ if $\boldsymbol{i}^{(1)} > \boldsymbol{i}'^{(1)}$, send the block $\boldsymbol{i}'^{(1)}$ to the left of the block $\boldsymbol{i}^{(1)}$ if $\boldsymbol{i}^{(1)} < \boldsymbol{i}'^{(1)}$, and send either $\boldsymbol{i}^{(1)}$ or $\boldsymbol{i}'^{(1)}$ to the beginning of the resulting word if $\boldsymbol{i}^{(1)} = \boldsymbol{i}'^{(1)}$. The desired result follows by induction on $r+r'$. □

5B. ***A mutation rule for dominant words.*** We now use Theorem 5.5 (or equivalently Proposition 5.4) to obtain a mutation rule on the parameters of simple modules corresponding to cluster variables in the setting of [Kang et al. 2018b]. We express it in a vector setting, i.e., in terms of the images of dominant words under the isomorphism (4). Recall that the image of any dominant word μ under the isomorphism (4) is the vector $\overrightarrow{\mu}$ whose i-th coordinate is equal to the multiplicity of the Lyndon word corresponding to the i-th positive root (in the decreasing order (3)) in the canonical factorization of μ. Such vectors are of size r_n, the number of positive roots in type A_n ($r_n = n(n+1)/2$)).

Example 5.6. In type A_2, there are three positive roots: $\alpha_2 > \alpha_1 + \alpha_2 > \alpha_1$. The word 21 will be represented by the vector ${}^t(1, 0, 1)$.

In type A_3 there are six positive roots: $\alpha_3 > \alpha_2 + \alpha_3 > \alpha_2 > \alpha_1 + \alpha_2 + \alpha_3 > \alpha_1 + \alpha_2 > \alpha_1$. The word 2312 will be represented by the vector ${}^t(0, 1, 0, 0, 1, 0)$ and the word 321 by the vector ${}^t(1, 0, 1, 0, 0, 1)$.

Let us consider a quantum monoidal seed $\mathcal{S} := (\{M_i\}_{i \in I}, B, \Lambda, D)$ in the sense of [Kang et al. 2018b]. Recall that I splits into $I = J_{ex} \cup J_{fr}$ with the $\{[M_i]\}_{i \in J_{ex}}$ corresponding to unfrozen variables and the $\{[M_i]\}_{i \in J_{fr}}$ corresponding to frozen variables. For every $i \in I$, let μ_i be the parameter of the simple module M_i and let $\overrightarrow{\mu_i}$ be the corresponding vector.

Remark 5.7. (1) The abelian monoid isomorphism (4) naturally extends to an abelian group isomorphism between the respective Grothendieck groups of $(\boldsymbol{M}, \odot)$ and $(\mathbb{Z}_{\geq 0}^{r_n}, +)$, namely,

$$(\boldsymbol{G}, \odot) \simeq (\mathbb{Z}^{r_n}, +). \tag{5}$$

Under this isomorphism, the inverse in $\boldsymbol{G}$ of a parameter $\mu \in \boldsymbol{M}$ corresponds to the opposite vector in $\mathbb{Z}^{r_n}$. For instance the vector corresponding to the generalized parameter $\hat{\mu}_j$ is

$$\overrightarrow{\hat{\mu}_j} = \sum_{1 \leq i \leq n+m} b_{ij} \overrightarrow{\mu_i}.$$

(2) The lexicographic order $\leq$ on $\boldsymbol{M}$ and $\boldsymbol{G}$ also turns into a (total) ordering on $\mathbb{Z}^{r_n}$ through the above isomorphism: a vector $\overrightarrow{\mu_1}$ is strictly greater than a vector $\overrightarrow{\mu_2}$ if and only if the first nonzero component of $\overrightarrow{\mu_1} - \overrightarrow{\mu_2}$ is positive.

Let k be fixed in J_{ex} and let us look at the mutation in direction k of the seed $\mathcal{S}$. It leads to a new seed $\mathcal{S}'$ with the same variables, except for M_k which has turned into M_k', such that we have a short exact sequence of graded modules

$$0 \to q \bigodot_{b_{ik}>0} M_i^{\circ b_{ik}} \to q^{\widetilde{\Lambda}(M_k, M_k')} M_k \circ M_k' \to \bigodot_{b_{ik}<0} M_i^{\circ(-b_{ik})} \to 0. \tag{6}$$

The next statement shows that one can deduce the parameter of the simple module M_k' from the knowledge of the parameters μ_i and the exchange matrix B of the seed $\mathcal{S}$.

Proposition 5.8. *Let μ'_k be the parameter of the simple module M'_k and $\overrightarrow{\mu_k}'$ be the corresponding vector. Then we have*

$$\overrightarrow{\mu_k}' = -\overrightarrow{\mu_k} + \max\left(\sum_{b_{ik}>0} b_{ik}\overrightarrow{\mu_i}, \sum_{b_{ik}<0} (-b_{ik})\overrightarrow{\mu_i} \right).$$

Proof. As the real simple modules M_i commute, the modules $\bigodot_{b_{ik}>0} {M_i}^{\circ b_{ik}}$ and $\bigodot_{b_{ik}<0} {M_i}^{\circ(-b_{ik})}$ are simple. Thus they correspond to some dominant words μ_+ and μ_-. Using Theorem 3.31(ii), one can write

$$\mu_+ = \max(L(\mu_+)) = \max\left(\bigodot_{b_{ik}>0} L(\mu_i)^{\circ b_{ik}} \right) = \bigodot_{b_{ik}>0} \mu_i^{\odot b_{ik}}.$$

Under the isomorphism (4) we get

$$\overrightarrow{\mu_+} = \sum_{b_{ik}>0} b_{ik}\overrightarrow{\mu_i}.$$

Now the short exact sequence (6) gives the relation

$$q^{\widetilde{\Lambda}(M_k,M'_k)}[M_k][M'_k] = q \prod_{b_{ik}>0} [M_i]^{b_{ik}} + \prod_{b_{ik}<0} [M_i]^{-b_{ik}}$$

in the Grothendieck ring of the category R-gmod. Taking the characters we get

$$q^{\widetilde{\Lambda}(M_k,M'_k)} \operatorname{ch}_q(M_k) \circ \operatorname{ch}_q(M'_k) = q \operatorname{ch}_q(L(\mu_+)) + \operatorname{ch}_q(L(\mu_-)).$$

Looking at the highest weight on both sides of this equality we get

$$\begin{aligned} \mu_k \odot \mu'_k &= \max(\operatorname{ch}_q(M_k) \circ \operatorname{ch}_q(M'_k)) \\ &= \max\bigl(\max(\operatorname{ch}_q(L(\mu_+))), \max(\operatorname{ch}_q(L(\mu_-)))\bigr). \end{aligned}$$

Applying isomorphism (4), we get

$$\overrightarrow{\mu_k} + \overrightarrow{\mu_k}' = \max(\overrightarrow{\mu_+}, \overrightarrow{\mu_-}) = \max\left(\sum_{b_{ik}>0} b_{ik}\overrightarrow{\mu_i}, \sum_{b_{ik}<0} (-b_{ik})\overrightarrow{\mu_i} \right)$$

which is the desired statement in the image of isomorphism (5). □

5C. ***An example in type A_3.*** In this subsection we apply Proposition 5.8 to the example of the category R-gmod for a Lie algebra of type A_3. This example provides an illustration of Theorem 6.2 which will be proved in general type A_n in Section 6. The category R-gmod corresponds to $\mathcal{C}_w$ with $w = w_0$ the longest element of the Weyl group of $\mathfrak{g}$ (see Section 3D). In type A_3 this element can be written as

$$w_0 = s_1 s_2 s_3 s_1 s_2 s_1.$$

Theorem 3.21 provides an admissible pair (in the sense of Definition 3.12), which gives rise to a quantum monoidal seed for this category. We denote this seed by $\mathcal{S}_0^3$. Firstly, one can see that $J_{ex} = \{1, 2, 3\}$ and $J_{fr} = \{4, 5, 6\}$ (see Section 3D). The simple modules whose classes are the cluster variables of the seed $\mathcal{S}_0^3$ can be computed directly using [Kang et al. 2018b, Proposition 10.2.4].

Lemma 5.9. *The seed $\mathcal{S}_0^3$ for the category R-gmod in type A_3 is given by three unfrozen variables $[L(1)]$, $[L(12)]$, $[L(21)]$ and three frozen variables $[L(123)]$, $[L(2312)]$, $[L(321)]$ together with the following exchange matrix*:

$$B_0 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. By definition of the modules $M(k, 0)$ defining the underlying admissible pair of the seed $\mathcal{S}_0^3$ (see Section 3D), one has

$$\begin{aligned} M(1,0) &= M(s_1\omega_1, \omega_1)), \\ M(2,0) &= M(s_1 s_2 \omega_2, \omega_2), \\ M(3,0) &= M(s_1 s_2 s_1 \omega_1, \omega_1), \\ M(4,0) &= M(s_3 s_1 s_2 s_1 \omega_1, \omega_1), \\ M(5,0) &= M(s_2 s_3 s_1 s_2 s_1 \omega_1, \omega_1), \\ M(6,0) &= M(s_1 s_2 s_3 s_1 s_2 s_1 \omega_1, \omega_1). \end{aligned}$$

Using [Kang et al. 2018b, Proposition 10.2.4], we get $M(1,0) = L(1)$, $M(2,0) = \mathrm{hd}(L(1) \circ L(2)) = L(12)$, $M(3,0) = \mathrm{hd}(L(2) \circ L(1)) = L(21)$. Here $\mathrm{hd}(M)$ stands for the head of a module M in R-gmod. The computations are similar for $M(k, 0)$, $k \in \{4, 5, 6\}$. □

The (ungraded) short exact sequences corresponding to the mutations in each of the three exchange directions can be written as follows:

$$0 \to L(21) \to L(1) \circ L \to L(12) \to 0.$$

$$0 \to L(1) \circ L(2312) \to L(12) \circ M \to L(21) \circ L(123) \to 0.$$

$$0 \to L(12) \circ L(321) \to L(21) \circ N \to L(1) \circ L(2312) \to 0.$$

Let λ (resp. μ, ν) be the parameters of the simple module L (resp. M, N). We can compute these parameters using Proposition 5.8. For instance consider the second of the above exact sequences. Then with the notation of Remark 5.7, a straightforward computation gives the parameter of M as $\overrightarrow{\mu} = {}^t(0,1,0,0,0,1)$. Hence $M = L(231)$.

In the same way one can compute $L = L(2)$ and $N = L(312)$.

Let $\mathcal{S}_1$ be the seed obtained from the seed $\mathcal{S}_0$ by mutation in the first direction. One can now show that $\mathcal{S}_0$ is compatible in the sense of Definition 4.7 and $\mathcal{S}_1$ is not.

First we write the exchange matrix for $\mathcal{S}_1$:

$$B_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then for $\mathcal{S}_0$, the images under isomorphism (5) of the generalized parameters $\hat{\mu}_j$ are:

$$\overrightarrow{\hat{\mu}_1} = {}^t(0,0,1,0,1,1), \quad \overrightarrow{\hat{\mu}_2} = {}^t(0,1,-1,1,1,0), \quad \overrightarrow{\hat{\mu}_3} = {}^t(1,-1,1,0,0,0)$$

and for $\mathcal{S}_1$ we get:

$$\overrightarrow{\hat{\mu}_1} = {}^t(0,0,-1,0,1,-1), \quad \overrightarrow{\hat{\mu}_2} = {}^t(0,1,-1,-1,1,0), \quad \overrightarrow{\hat{\mu}_3} = {}^t(1,-1,2,0,-1,1).$$

Combining Remark 5.7(2) and Remark 4.8, one can see that $\mathcal{S}_0$ is compatible in the sense of Definition 4.7 but this is not the case for $\mathcal{S}_1$.

In a more general sense, given an initial quantum monoidal seed $(\{M_i\}_i, B)$, Proposition 5.8 we can explicitly compute the parameters of the simple modules appearing when mutating the initial seed an arbitrary number of times in any directions.

6. A compatible seed for R-gmod in type A

In this section we compute in type A_n the parameters of the simple modules of the monoidal seed $\mathcal{S}_0^n$ arising from the construction of [Kang et al. 2018b] (see Section 3D) for the category R-gmod.

6A. ***Statements of the main theorems.*** In this subsection we state the two main results of this paper, Theorems 6.1 and 6.2. Recall that $\mathcal{A}_q(\mathfrak{n}) = \mathcal{A}_q(\mathfrak{n}(w_0))$ where w_0 is the longest element of the Weyl group of $\mathfrak{g}$. The category R-gmod coincides with the category $\mathcal{C}_{w_0}$ (see Section 3D). In type A_n we have

$$w_0 = (1\ldots n)(1\ldots(n-1))\cdots(12)(1).$$

Recall that $r_n := n(n+1)/2$ stands for the length of w_0. In the category R-gmod $= \mathcal{C}_{w_0}$ in type A_n, Theorem 3.21 provides an admissible pair (as in Definition 3.12), which gives rise to a quantum monoidal seed for this category. We denote this seed by $\mathcal{S}_0^n$.

Our first main result is the following:

Theorem 6.1. *The cluster variables of the seed $\mathcal{S}_0^n$ can be explicitly described in terms of parameters as follows*:

$$\begin{array}{ccccc}
[L(1)] & & & & \\
[L(12)] & [L((2)(1))] & & & \\
[L(123)] & [L((23)(12))] & [L((3)(2)(1))] & & \\
\vdots & \vdots & & & \\
[L(1\dots k)] & [L((2\dots k)(1\dots k{-}1))] & \cdots & [L((k)\cdots(1))] & \\
\vdots & \vdots & & & \\
[L(1\dots n)] & [L((2\dots n)(1\dots n{-}1))] & \cdots & \cdots & [L((n)\cdots(1))].
\end{array}$$

The set of frozen variables corresponds to the last line and the set of unfrozen variables consists in the union of lines $1, \dots, n-1$.

The three following sections are devoted to some intermediate steps for the proof of Theorem 6.1. Recall from Section 3D that J_{fr} denotes the index set of the frozen variables of the monoidal seed $\mathcal{S}_0^n$. In Section 6B, we prove that $|J_{fr}| = n$. We show that the knowledge of the dominant words of the M_j, $j \in J_{fr}$, is sufficient to recover the whole seed $\mathcal{S}_0^n$. Section 6C is devoted to the computation of the weights of the M_j, $j \in J_{fr}$. These weights are determined by the construction of [Kang et al. 2018b] (see Section 3D). In Section 6D, we use the fact that for any $j \in J_{fr}$, the module M_j necessarily commutes with any other simple module in R-gmod. This strongly constrains the form of the corresponding dominant word. Together with the weights obtained in Section 6C, we find at most n possible dominant words, which is exactly the number of frozen variables computed in Section 6B. Hence we get a bijection between these parameters and the set of modules $\{M_j, j \in J_{fr}\}$.

We complete the proof in Section 6E by determining which parameter corresponds to every simple module, which is more precise than just a global bijection. The key argument is provided by Proposition 5.8.

Theorem 6.2 is our second main result. We deduce it from Theorem 6.1.

Theorem 6.2. *The seed $\mathcal{S}_0^n$ is compatible in the sense of Definition 4.7.*

In particular Conjecture 4.10 holds for the category R-gmod in type A_n.

6B. ***Initial seed for R-gmod.*** For $n \geq 1$, we consider a Lie algebra $\mathfrak{g}$ of type A_n. We let $\{\alpha_1, \dots, \alpha_n\}$ denote the simple roots and $Q_+^n := \bigoplus_{i=1,\dots,n} \mathbb{N}\alpha_i$. We also denote by Δ_+^n the set of positive roots, R-gmod^n the category of (graded) finite-dimensional representation of the quiver Hecke algebras associated with $\mathfrak{g}$, and $\boldsymbol{M}^n$ the set of dominant words in bijection with the set of simple objects in R-gmod^n (up to isomorphism). There is a canonical embedding ι_n^m of $\boldsymbol{M}^n$ into $\boldsymbol{M}^m$ for any $m \geq n$.

In particular the set of simple objects in R-gmod^n is naturally included into the set of simple objects in R-gmod^m. We again denote this inclusion by ι_n^m.

Let J_{ex}^n (resp. J_{fr}^n) denote the index set of the unfrozen variables (resp. frozen variables) of the seed $\mathcal{S}_0^n$ for every $n \geq 1$. We also set $J^n := J_{ex}^n \cup J_{fr}^n$. In order to prove Theorem 6.1, we begin by determining the sets J_{ex}^n and J_{fr}^n. We point out an inductive property of this seed: the set $\{M_i, i \in J_{ex}^n\}$ coincides with the set $\{\iota_{n-1}^n(M_i),\ i \in J^{n-1}\}$.

Proposition 6.3. *The cluster variables of the seed $\mathcal{S}_0^n$ split into the following exchange and frozen sets*:

1. *There are n frozen variables in $\mathcal{S}_0^n$, which correspond to the classes of the last n modules $M(r_{n-1}+1, 0), \ldots, M(r_n, 0)$.*
2. *The set $\{M_i, i \in J_{ex}^n\}$ coincides with the union of the sets $\{\iota_k^n(M_i),\ i \in J_{fr}^k\}$, $1 \leq k \leq n-1$.*

Proof. For the first statement, write

$$w_0 = (s_1 \ldots s_n)(s_1 \ldots s_{n-1}) \cdots (s_1 s_2)(s_1) = s_{r_n} \cdots s_1.$$

Then for any $l \in \{1, \ldots, r_{n-1}\}$, the letter s_l is in $\{1, \ldots, n-1\}$ and this letter obviously appears again in the word w_0 as $s_{l'}$ for some $l' > l$. In other words, $l_+ \leq r_n$ and thus $l \in J_{ex}^n$. Conversely, if $l \in \{r_{n-1}+1, r_n\}$, then all the letters $s_{l'}$, $l' > l$, are distinct from s_l and thus $l_+ = r+1$. Hence one has

$$J_{ex}^n = \{1, \ldots, r_{n-1}\} \quad \text{and} \quad J_{fr}^n = \{r_{n-1}+1, \ldots, r_n\}.$$

In particular the modules corresponding to the frozen variables of the seed $\mathcal{S}_0^n$ can be written as

$$\begin{aligned} M(r_{n-1}+1, 0) &= M(s_1(s_2 s_1) \cdots (s_{n-1} \ldots s_1) s_n.\omega_n, \omega_n) \\ &\vdots \\ M(r_n, 0) &= M(s_1(s_2 s_1) \cdots (s_{n-1} \ldots s_1)(s_n \ldots s_1).\omega_1, \omega_1). \end{aligned}$$

The second statement follows from the first statement applied to the seeds $\mathcal{S}_0^k$, $1 \leq k \leq n-1$. Indeed, in the same way, one can write the modules $M_i,\ i \in J_{fr}^{n-1}$, as

$$\begin{gathered} M(s_1(s_2 s_1) \cdots (s_{n-2} \ldots s_1) s_{n-1}.\omega_{n-1}, \omega_{n-1}) \\ \vdots \\ M(s_1(s_2 s_1) \cdots (s_{n-2} \ldots s_1)(s_{n-1} \ldots s_1).\omega_1, \omega_1). \end{gathered}$$

The images via ι_{n-1}^n of these modules respectively coincide with the modules $M(r_{n-2}+1, 0), \ldots, M(r_{n-1}, 0)$, whose classes are exactly the last $n-1$ unfrozen variables of the seed $\mathcal{S}_0^n$. Iterating this, we conclude that the set $\{M_l, l \in J_{ex}^n\}$ is the union of the sets $\{\iota_k^n(M_l), l \in J_{fr}^k\}$, $1 \leq k \leq n-1$. ☐

As a direct consequence of the previous proposition, it suffices to compute the parameters of the modules corresponding to the frozen variables of the seed $\mathcal{S}_0^n$. This is what we focus on in the next two subsections.

6C. *Weights of the simple modules $M(r_{n-1}+k,0)$, $1 \le k \le n$.* From now on, the integer n is fixed. We write J_{fr} for J_{fr}^n. By Proposition 6.3, the simple modules corresponding to the frozen variables of the seed $\mathcal{S}_0^n$ are the $M(r_{n-1}+k,0)$, $1 \le k \le n$. For simplicity we set $M_k := M(r_{n-1}+k,0)$ for any $1 \le k \le n$. This subsection is devoted to the computation of the weights of the simple modules M_k, i.e., the elements β_k such that $M_k \in R(\beta_k)$-gmod for every $1 \le k \le n$. Our main tool is the definition of the modules $M(l,0)$ from [Kang et al. 2018b] (see Definition 3.20).

Proposition 6.4. *For each $1 \le k \le n/2$, the two modules M_k and M_{n-k+1} both belong to the subcategory*

$$R(\alpha_n+2\alpha_{n-1}+\cdots+k\alpha_{n-k+1}+\cdots+k\alpha_k+\cdots+2\alpha_2+\alpha_1)\text{-}mod.$$

Proof. For $1 \le l \le n$, we have

$$\begin{aligned} M_l &= M(r_{n-1}+l,0) \\ &= M\big(s_1(s_2s_1)\cdots(s_{n-1}\ldots s_1)(s_n\ldots s_k)\omega_k,\omega_k\big), \text{ where } k := n-l+1. \end{aligned}$$

One computes

$$\zeta_k := s_1(s_2s_1)\cdots(s_{n-1}\ldots s_1)(s_n\ldots s_k)\omega_k.$$

The weight of M_l is given by $\omega_k-\zeta_k$ (see Corollary 3.18):

$$\begin{aligned} \zeta_k &= s_1(s_2s_1)\cdots(s_{n-1}\ldots s_1)(\omega_k-(\alpha_n+\cdots+\alpha_k)) \\ &= s_1(s_2s_1)\cdots(s_{n-2}\ldots s_1)(\omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+2\alpha_k+\alpha_{k-1})) \\ &= s_1(s_2s_1)\cdots(s_{n-3}\ldots s_1)(\omega_k-(\alpha_n+2\alpha_{n-1}+3\alpha_{n-2}+\cdots+3\alpha_k+2\alpha_{k-1}+\alpha_{k-2})). \end{aligned}$$

If $2k \le n$ then by iterating we get

$$\begin{aligned} \zeta_k = s_1(s_2s_1)&\cdots(s_{n-k}\ldots s_1) \\ &\times(\omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+k\alpha_{n-k+1}+\cdots+k\alpha_k+\cdots+2\alpha_2+\alpha_1)) \end{aligned}$$

but

$$\omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+k\alpha_{n-k+1}+\cdots+k\alpha_k+\cdots+2\alpha_2+\alpha_1)$$

is invariant under the action of $s_1,\ldots,s_{n-k}$. Hence

$$\zeta_k = \omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+k\alpha_{n-k+1}+\cdots+k\alpha_k+\cdots+2\alpha_2+\alpha_1).$$

If $2k > n$ then by iterating we get

$$\begin{aligned}\zeta_k &= s_1(s_2s_1)\cdots(s_k\dots s_1)(\omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+(n-k)\alpha_{k+1}\\ &\qquad +(n-k)\alpha_k+\cdots+2\alpha_2+\alpha_{2k-n+1}))\\ &= s_1(s_2s_1)\cdots(s_{k-1}\dots s_1).(\omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+(n-k)\alpha_{k+1}\\ &\qquad +(n-k+1)\alpha_k+(n-k)\alpha_{k-1}\cdots+2\alpha_2+\alpha_{2k-n}))\\ &\ \vdots\\ &= s_1(s_2s_1)\cdots(s_{n-k}\dots s_1).(\omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+(n-k)\alpha_{k+2}\\ &\qquad +(n-k+1)\alpha_{k+1}+\cdots+(n-k+1)\alpha_{n-k+1}+\cdots+2\alpha_2+\alpha_1))\end{aligned}$$

and $\omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+(n-k+1)\alpha_{k+1}+\cdots(n-k+1)\alpha_{n-k+1}+\cdots+2\alpha_2+\alpha_1)$ is invariant under the action of $s_1,\dots,s_{n-k}$. Hence we get

$$\zeta_k = \begin{cases}\omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+k\alpha_{n-k+1}+\cdots+k\alpha_k+\cdots+2\alpha_2+\alpha_1) & \text{if } 2k\le n,\\ \omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+(n-k+1)\alpha_k+\cdots+(n-k+1)\alpha_{n-k+1}+\cdots+2\alpha_2+\alpha_1) & \text{if } 2k>n,\\ \omega_k-(\alpha_n+2\alpha_{n-1}+\cdots+k\alpha_l+\cdots+k\alpha_k+\cdots+2\alpha_2+\alpha_1) & \text{if } k<l,\\ \omega_k-(\alpha_n+2\alpha_{n-1}+\cdots l\alpha_k+\cdots+l\alpha_l+\cdots+2\alpha_2+\alpha_1) & \text{if } k\ge l.\end{cases}$$

Hence for each $1\le k\le n/2$, the two modules M_k and M_{n-k+1} both belong to the subcategory $R(\alpha_n+2\alpha_{n-1}+\cdots+k\alpha_{n-k+1}+\cdots+k\alpha_k+\cdots+2\alpha_2+\alpha_1)$-mod. □

Let us now fix $1\le k\le n$ and consider the parameter μ_k of the simple module M_k. Let m_k be the length of μ_k. Since k and $n-k+1$ play symmetric roles, we assume from now on that $k\le n/2$.

The following statement is a direct consequence of the previous proposition.

Corollary 6.5. *For any* $1\le i\le n$,

$$(\mu_k, i) = \begin{cases}1 & \textit{if } i=k \textit{ or } i=n-k+1,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. For $1\le i\le k-1$ or $n-k+2\le i\le n$, by Proposition 6.4 there are $i-1$ (resp. $i, i+1$) occurrences of the letter $i-1$ (resp. $i, i+1$) in the word μ_k and hence $(\mu_k, i)=2i-(i-1)-(i+1)=0$.

If $k+1\le i\le n-k$ then by Proposition 6.4 each of the letters $i-1,\ i,\ i+1$ appears k times and hence $(\mu_k, i)=2i-i-i=0$.

Finally if $i=k$ then by Proposition 6.4 there are k occurrences of the letters k and $k+1$, and $k-1$ occurrences of the letter $k-1$ which gives $(\mu_k, i)=2k-k-(k-1)=1$. If $i=n-k+1$ then there are k occurrences of the letters $i-1$ and i, and $k-1$ occurrences of the letter $i+1$ and thus $(\mu_k, i)=2k-k-(k-1)=1$. □

In particular, one can compute the quantities $\Lambda(M_k, L(i))$ for any $1\le i\le n$:

Corollary 6.6. *For any $1 \le i \le n$, let N_i be the number of occurrences of the letter i in the word μ_k. Let s_i and s_i' be the integers such that (see Remark 6.7)*

$$\Lambda(L(i), M_k) = -(\mu_k, i) + 2N_i - 2s_i$$

and

$$\Lambda(M_k, L(i)) = -(\mu_k, i) + 2N_i - 2s_i'.$$

Then one has $s_i = s_i' = N_i$ if $i \notin \{k, n-k+1\}$ and either $s_i = N_i$ and $s_i' = N_i - 1$ or $s_i = N_i - 1$ and $s_i' = N_i$ if $i \in \{k, n-k+1\}$.

Proof. By Corollary 6.5, the quantity $\Lambda(L(i), M_k)$ can be written as

$$\Lambda(L(i), M_k) = \begin{cases} 2N_i - 1 - 2s_i & \text{if } i = k \text{ or } i = n-k+1, \\ 2N_i - 2s_i & \text{otherwise,} \end{cases}$$

and similarly for $\Lambda(M_k, L(i))$ with s_i'. As M_k commutes with $L(i)$, by Lemma 3.9, one has

$$s_i + s_i' = \begin{cases} 2N_i - 1 & \text{if } i = k \text{ or } i = n-k+1, \\ 2N_i & \text{otherwise.} \end{cases}$$

As the integers s_i and s_i' are always smaller than N_i, one gets the desired result. □

Remark 6.7. In the following we will make several computations of some $\Lambda(M, N)$ for various simple real objects M, N in R-gmod in order to check commutation between these modules. For any $\beta \in Q_+$, any simple (left) $R(\beta)$-module M is cyclic, i.e., is isomorphic to $R(\beta).u$ for some $u \in M$. We will refer to any such vector u in M as a generating vector in M. Now let $\beta, \gamma \in Q_+$, let M be a simple $R(\beta)$-module, and N be a simple $R(\gamma)$-module. As the morphism

$$M \otimes N \to N \circ M, \quad u \otimes v \mapsto \varphi_{w[n,m]}(v \otimes u)$$

is $R(\beta) \otimes R(\gamma)$-linear, computing the map $R_{M,N}$ is equivalent to computing the action of $\varphi_{w[n,m]} \in R(\beta+\gamma)$ on the tensor product of generating vectors u and v for M and N.

Now let $u_z := 1 \otimes u \in M_z$ and let $\tilde{s}$ be the valuation of the polynomial in z given by $\varphi_{w[n,m]}.(v \otimes u_z)$. As the actions of the generators x_i, τ_k and $e(\nu)$ can only make the degree in z increase, the image of the map $R_{M_z,N}$ is contained in $z^{\tilde{s}} N \circ M_z$. Moreover, by definition of $\tilde{s}$, $\varphi_{w[n,m]}.(v \otimes u_z)$ contains a nonzero term of degree $\tilde{s}$ hence it does not belong to $z^k N \circ M_z$ for any $k > \tilde{s}$. Hence $\tilde{s}$ coincides with s in Definition 3.7. Thus in what follows, for any simple $R(\beta)$-module M and any simple $R(\gamma)$-module N, we will always fix some choices of generating vectors $u \in M$ and $v \in N$ and write

$$\Lambda(M, N) = -(\beta, \gamma) + 2(\beta, \gamma)_n - 2s$$

with s being the valuation of the polynomial in z given by $\varphi_{w[n,m]}.(v \otimes u_z)$.

6D. ***Dominant words associated to frozen variables in R-gmod.*** In this subsection, we compute the dominant words associated to the frozen variables for the category R-gmod in type A_n. As in the previous subsection, we fix k such that $1 \leq k \leq n/2$ and we consider $M_k = M(r_{n-1}+k, 0)$ the simple module whose isomorphism class is the k-th frozen variable in the seed $\mathcal{S}_0^n$ constructed in [Kang et al. 2018b]. We use the fact that M_k commutes with all the simple modules in R-gmod. In particular, it commutes with all the cuspidal modules $L(i)$, $1 \leq i \leq n$. Together with the form of the weight of M_k given by Proposition 6.4 and Corollary 6.6, this leads to only n possible dominant words. As there are exactly n frozen variables in the seed $\mathcal{S}_0^n$ (see Proposition 6.3), we get a bijection between the possible parameters and the frozen variables for R-gmod.

For every $1 \leq i \leq n$, the algebra $R(\alpha_i)$ is generated by one generator x_i and one generator $e(i)$ commuting with each other (with the notation of Section 3A, the set $\mathrm{Seq}(\alpha_i)$ is a singleton consisting in the word reduced to a single letter i). Recall from Section 3E that for every $1 \leq i \leq n$, the cuspidal module $L(i)$ is a one-dimensional vector space spanned by a generating vector v_i with action of $R(\alpha_i)$ given by

$$x_i \cdot v_i = 0, \quad e(i) \cdot v_i = v_i.$$

It is the only simple object in the category $R(\alpha_i)$-mod.

As above we let μ_k denote the parameter of the simple module M_k and m_k the length of μ_k. In what follows we will write the word μ_k as

$$\mu_k = h_1, \ldots, h_{m_k}.$$

Note that in this setting the h_j are the *letters* of the word μ_k, whereas we use bold letters $\boldsymbol{i}_l$ to refer to *Lyndon words* in the canonical factorization of μ_k (see after Remark 6.11).

As the module M_k is simple and real, Lemma 3.9 shows that checking its commutation with any other simple module L is equivalent to computing the quantities $\Lambda(L, M_k)$ and $\Lambda(M_k, L)$. When $L = L(i)$ for some $i \in \{1, \ldots, n\}$, these quantities are given by Corollary 6.6. Thus as explained in Remark 6.7 above, once we have fixed a generating vector u for M_k, we will compute the valuations s_i (resp. s'_i) of the polynomial functions

$$\varphi_{w[m_k,1]}(u \otimes (v_i)_z) = \varphi_1 \cdots \varphi_{m_k}(u \otimes (v_i)_z)$$

(resp. $\varphi_{w[1,m_k]}(v_i \otimes u_z) = \varphi_{m_k} \cdots \varphi_1(v_i \otimes u_z)$) for various choices of $i \in \{1, \ldots, n\}$. We fix once and for all a generating vector u (resp. v_i, $1 \leq i \leq n$) for M_k (resp. $L(i)$, $1 \leq i \leq n$).

We begin by showing that there are only two possibilities for the first letter of μ_k.

Lemma 6.8. *Let $p = h_1$ denote the first letter of μ_k. The letter p is equal either to k or $n-k+1$.*

Proof. With the same notation as in Corollary 6.6, we show that $s_p \leq N_p - 1$.

$$\begin{aligned}\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) &= \varphi_1 \cdots \varphi_{m_k} e(h_1, \dots, h_{m_k}, p).(u \otimes (v_p)_z) \\ &= \varphi_1 e(h_1, p, h_2, \dots, h_{m_k}) \varphi_2 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) \\ &= (\tau_1(x_1 - x_2) + 1)\varphi_2 \cdots \varphi_{m_k}.(u \otimes (v_p)_z).\end{aligned}$$

The operator x_1 commutes with $\varphi_2, \dots, \varphi_{m_k}$ and acts trivially on u. Moreover, $x_2 \varphi_2 \cdots \varphi_{m_k} = \varphi_2 \cdots \varphi_{m_k} x_{m_k+1}$ (see for example [Kang et al. 2018a, Lemma 1.3.1]) and $x_{m_k+1} \cdot (u \otimes (v_p)_z) = z(u \otimes (v_p)_z)$. Hence we get

$$\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) = -z.\tau_1.\varphi_2 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) + \varphi_2 \cdots \varphi_{m_k}.(u \otimes (v_p)_z).$$

The operator $\varphi_2 \cdots \varphi_{m_k}$ acts nontrivially on $(u \otimes (v_p)_z)$ (as the renormalized R-matrix $r_{L(p),M_k}$ never vanishes). Thus $\varphi_2 \cdots \varphi_{m_k}.u$ is a nonzero polynomial function, and the above equality implies that s_p is equal to its valuation. This polynomial function has degree less than $N_p - 1$, as the only operators φ_j that can make the degree rise are the ones corresponding to an occurrence of p in μ_k.

This implies $s_p \leq N_p - 1$. Hence by Corollary 6.6, $p \in \{k, n-k+1\}$. □

Remark 6.9. With the same proof, one can show that the last letter of the word μ_k is either k or $n-k+1$ as well.

Lemma 6.10. (i) *For each $1 \leq k' \leq k$, there is exactly one Lyndon word ending with $n-k'+1$ in the canonical factorization of μ_k.*

(ii) *Moreover, denoting by $\boldsymbol{j}_{n-k'+1}$ the unique Lyndon word ending with $n-k'+1$ (for each $1 \leq k' \leq k$), one has $\boldsymbol{j}_n > \cdots > \boldsymbol{j}_{n-k+1}$.*

Proof. We prove the first statement by induction on k'. We know that the letter n appears exactly once; thus there is a unique Lyndon word $\boldsymbol{j}_n$ containing (and thus ending with) n, which proves the statement for $k' = 1$. Suppose $k \geq 2$ and (i) holds for $1 \leq k' < k$. Denote by $\boldsymbol{j}_n, \dots, \boldsymbol{j}_{n-k'+1}$ the Lyndon words respectively ending with $n, \dots, n-k'+1$. Their first letters are all smaller than p and in particular smaller than $n-k+1$ by Lemma 6.8. Hence they all contain the letter $n-k'$, which makes k' occurrences of this letter. By Proposition 6.4, $n-k'$ has to appear $k'+1$ times in the word μ_k. Hence there is a unique Lyndon word $\boldsymbol{j}_{n-k'}$ containing $n-k'$ but none of the letters $n, \dots, n-k'+1$, which means that this Lyndon word ends with $n-k'$. Thus the first statement holds by induction.

For the second statement, let $m \in \{n-k+1, \dots, n\}$ such that $\boldsymbol{j}_m$ is the smallest of the $\boldsymbol{j}_l$; this is equivalent to saying that it is the last (among the $\boldsymbol{j}_l$) to appear in the canonical factorization of μ_k.

Note the first statement implies that, for each $1 \leq k' \leq k$, the letter $n-k'+1$ appears k' times among the Lyndon words $\boldsymbol{j}_n, \dots, \boldsymbol{j}_{n-k+1}$: once in each of $\boldsymbol{j}_n, \dots, \boldsymbol{j}_{n-k'+1}$, and never in the others. Together with Proposition 6.4, this implies that the letters $n-k'+1,\ 1 \le k' \le k$, do not appear in any other Lyndon word of the canonical factorization of μ_k. Hence denoting by i the position of the last letter of $\boldsymbol{j}_m$ in the word μ_k, one has $h_i = m$ and $h_{i+1}, \dots, h_{m_k} < m$. Thus one has

$$\begin{aligned}(7)\quad & \varphi_{m_k} \cdots \varphi_1.(v_m \otimes u_z) \\ &= \tau_{m_k} \cdots \tau_{i+1}(\tau_i(x_i - x_{i+1}) + 1)\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \\ &= \tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \pm z.\tau_{m_k} \cdots \tau_{i+1}\tau_i\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z).\end{aligned}$$

We denote by $Q(z)$ the first term

$$\tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z).$$

We show that if $m \ge n-k+2$, then $Q(z)$ is nonzero. As the renormalized R-matrix $r_{M_k,L(m)}$ does not vanish, the action of the operator $\varphi_{i-1} \cdots \varphi_1$ on $(v_m \otimes u_z)$ is a nonzero polynomial function in z of degree less than $N_m - 1$. Consider now the action of τ_{m_k} on $Q(z)$:

$$\begin{aligned}&\tau_{m_k}.Q(z) \\ &\quad= \tau_{m_k}.\tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \\ &\quad= \tau_{m_k}.\tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.e(m\mu_k).(v_m \otimes u_z) \\ &\quad= \tau_{m_k}.\tau_{m_k} e(s_{m_k-1} \cdots s_{i+1}s_{i-1} \cdots s_1.m\mu_k).\tau_{m_k-1} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \\ &\quad= \tau_{m_k}^2.e(h_1, \dots, h_{m_k-1}, m, h_{m_k}).\tau_{m_k-1} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \\ &\quad= e(h_1, \dots, h_{m_k-1}, m, h_{m_k}).\tau_{m_k-1} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z) \qquad (\text{as } h_{m_k} \le m-2) \\ &\quad= \tau_{m_k-1} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z)\end{aligned}$$

Similarly, all the letters in position $i+1, \dots, m_k$ are less than $n-k$ and in particular they are less than $m-2$. The same argument can be applied to $\tau_{m_k-1}, \dots, \tau_{i+1}$ and thus we get

$$\begin{aligned}\tau_{i+1} \cdots \tau_{m_k}.Q(z) &= \tau_{i+1} \cdots \tau_{m_k}.(\tau_{m_k} \cdots \tau_{i+1}\varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z)) \\ &= \varphi_{i-1} \cdots \varphi_1.(v_m \otimes u_z)\end{aligned}$$

which is not zero. A fortiori $Q(z)$ itself is nonzero. It is thus a nonzero polynomial function of degree less than $N_m - 1$ and the equality (7) above shows that s'_m is necessarily equal to its valuation.

This implies $s'_m \le N_m - 1$, and in particular $m \in \{k, n-k+1\}$ by Corollary 6.6. This contradicts the hypothesis $m \ge n-k+2$. Thus we have shown $m = n-k+1$.

By iterating this we conclude that the Lyndon words $\boldsymbol{j}_n, \dots, \boldsymbol{j}_{n-k+1}$ appear in this order in the canonical factorization of μ_k, which is the desired statement. □

From now on we write the canonical factorization of μ_k as $\mu_k = \boldsymbol{i}_0 \cdots \boldsymbol{i}_r$ with $\boldsymbol{i}_0 \geq \cdots \geq \boldsymbol{i}_r$. For each $0 \leq j \leq r$ we denote by p_j the first letter of the Lyndon word $\boldsymbol{i}_j$.The sequence $(p_j)_{0 \leq j \leq r}$ is decreasing, with $p_0 = p$ and $p_r = 1$ (the letter 1 appears once, necessarily in the smallest of the $\boldsymbol{i}_j$). We also denote by a_j the position of the letter p_j in the word μ_k.

Remark 6.11. Note that as an immediate consequence of the previous lemma, one has $r \geq k - 1$.

With these notations we can make the following observation, as a straightforward consequence of Lemma 6.10.

Corollary 6.12. *The Lyndon word $\boldsymbol{i}_0$ ends with the letter n. In other words, $\boldsymbol{i}_0 = \boldsymbol{j}_n$.*

Proof. As $\boldsymbol{i}_0$ is greater than any other Lyndon word appearing in the canonical factorization of μ_k, in particular it is greater than $\boldsymbol{j}_n$. Hence by Lemma 6.10(ii), one can write

$$\boldsymbol{i}_0 \geq \boldsymbol{j}_n > \cdots > \boldsymbol{j}_{n-k+1}. \tag{8}$$

Thus all of the Lyndon words $\boldsymbol{j}_n, \ldots, \boldsymbol{j}_{n-k+1}$ begin with a letter smaller than $p_0 = p$. By definition they end with letters greater than or equal to $n-k+1$ which is greater than p by Lemma 6.8 (recall that we assumed $k \leq n-k+1$ at the beginning of this section). Hence each of the Lyndon words $\boldsymbol{j}_n, \ldots, \boldsymbol{j}_{n-k+1}$ contains the letter p which makes k occurrences of p. From Lemma 6.8 and Proposition 6.4, we conclude that no other Lyndon word contains p. Thus the Lyndon word $\boldsymbol{i}_0$ has to be one of the $\boldsymbol{j}_l$ and the inequalities (8) impose $\boldsymbol{i}_0 = \boldsymbol{j}_n$. □

Lemma 6.13. *For any $1 \leq j \leq r$, if $p_j \neq k$, then $p_{j-1} - p_j \leq 1$.*

Proof. Assume there exists $j \in \{1, \ldots, r\}$ such that $p_j \neq k$ and $p_{j-1} - p_j \geq 2$. We set $q := p_j$ and show that $s_q \leq N_q - 1$. Let i denote the position of q in the word μ_k. Then $h_i = q$; on the other hand all the letters in position $1, \ldots, i-1$ are greater than $q+2$ (as they are greater than p_{j-1}), hence

$$\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_q)_z) = \tau_1 \cdots \tau_{i-1}(\tau_i(x_i - x_{i+1}) + 1)\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_q)_z).$$

By similar arguments as in the proof of Lemma 6.10(ii), this implies $s_q \leq N_q - 1$. Hence by Corollary 6.6, $q \in \{k, n-k+1\}$. Now by hypothesis $q \leq p_{j-1} - 2 \leq p_0 - 2 < p_0$ and $p_0 \leq n-k+1$ by Lemma 6.8. In particular $q < n-k+1$. Since, by assumption, $q = p_j \neq k$, we get the desired contradiction. □

Proposition 6.14. *With the previous notation, one has*:

(i) $p_1 < p_0$.

(ii) *For all $j \geq 1$, if $p_j \neq k$ then $p_{j+1} < p_j$.*

Proof. Assume $p_1 = p_0 = p$ and let i denote the position of the letter p_1 in the word μ_k, i.e., $h_i = p_1$. First note that this implies $2 \leq p \leq n-1$ (as the letters 1 and n appear only once in the word μ_k). In particular $\boldsymbol{i}_0$ is of length strictly greater than 2, as $\boldsymbol{i}_0 = (p \dots n)$ by Corollary 6.12. In other words $i > 2$. One has

$$\begin{aligned} (9) \quad & \varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_p)_z) \\ & = (\tau_1(x_1 - x_2) + 1)\tau_2 \cdots \tau_{i-1}(\tau_i(x_i - x_{i+1}) + 1)\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z). \end{aligned}$$

As the renormalized R-matrix $r_{L(p),M_k}$ does not vanish, the operator $\varphi_{i+1} \cdots \varphi_{m_k}$ acts as a nonzero polynomial function on $(u \otimes (v_p)_z)$. We set

$$Q(z) := \varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z).$$

Note that $\deg(Q) \leq N_p - 2$. Let $P(z)$ denote the polynomial function given by the term

$$\tau_2 \cdots \tau_{i-1}\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z) = \tau_2 \cdots \tau_{i-1}.Q(z)$$

from the above equality and let us consider the action of the operator $\tau_{i-1} \cdots \tau_1$ on $P(z)$. Recall that $i > 2$. We first write

$$\tau_2\tau_1.P(z) = \tau_2\tau_1.(\tau_2 \cdots \tau_{i-1})Q(z) = (\tau_1\tau_2\tau_1 + 1).\tau_3 \cdots \tau_{i-1}\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z)$$

using the braid relation. The operator τ_1 commutes with $\tau_3 \cdots \tau_{i-1}$ as well as with $\varphi_{i+1}, \dots, \varphi_{m_k}$. As $\boldsymbol{i}_0$ is of length greater than 2, the action of τ_1 on $(u \otimes (v_p)_z)$ is the same as its action on the cuspidal module $L(\boldsymbol{i}_0)$ which is trivial. Hence we get

$$\tau_2\tau_1.P(z) = \tau_3 \cdots \tau_{i-1}\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z).$$

The letters $h_3, \dots, h_{i-1}$ are greater than $p+2$; hence by arguments similar to those in the proof of Lemma 6.10 (ii), we get

$$\tau_{i-1} \cdots \tau_3.(\tau_3 \cdots \tau_{i-1}).Q(z) = Q(z).$$

Finally we get

$$\tau_{i-1} \cdots \tau_1.P(z) = \varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z) = Q(z)$$

which is not zero. A fortiori $P(z)$ itself is a nonzero polynomial function. All the other terms in (9) are either zero, or of valuation strictly greater than the valuation of $Q(z)$. This implies that s_p is equal to the valuation of $Q(z)$. In particular $s_p \leq N_p - 2$. This contradicts Corollary 6.6. Hence $p_1 < p_0$.

For the second statement assume we have $j \geq 1$ such that $q := p_{j+1} = p_j \neq k$ and consider j minimal for this property. For the sake of simplicity, we only deal with the case where only the two Lyndon words $\boldsymbol{i}_j$ and $\boldsymbol{i}_{j+1}$ begin with the letter q i.e., $p_{j-1} > q, \; p_j = p_{j+1} = q$ and $p_{j+2} \leq q-1$. The proof is analogous if there are several words $\boldsymbol{i}_{j'}$ beginning with q.

Since, by hypothesis, $q \neq k$, Lemma 6.13 implies $p_{j-1} \in \{q, q+1\}$ and hence $p_{j-1} = q+1$ as $p_{j-1} > q$. Moreover by minimality of j, $q+1$ appears exactly once in the subsequence $(p_i)_{i<j}$.

Set $a := a_{j-1}$, $b := a_j$ and $c := a_{j+1}$. As above, N_q denotes the number of occurrences of q in the word w_k. We write

$$\begin{aligned}(10)\quad & \varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_q)_z) \\ &\qquad = \tau_1 \cdots \tau_{b-1}(\tau_b(x_b - x_{b+1}) + 1)\tau_{b+1} \cdots \tau_{c-1}(\tau_c(x_c - x_{c+1}) + 1)Q(z),\end{aligned}$$

where $Q(z) := \varphi_{c+1} \cdots \varphi_{m_k}.(u \otimes (v_q)_z)$. As the renormalized R-matrix $r_{L(q),M_k}$ does not vanish, Q is a nonzero polynomial function. Its degree is equal to $N_q - 2$.

Now we prove that $Q(z)$ is in fact a monomial in z. Indeed, any occurrence of q in a position $i \in \{c+1, \dots, m_k\}$ appears inside a Lyndon word $\boldsymbol{i}_{j'}$ (with $j' > j+1$) beginning with a letter strictly smaller than q. Then the operator τ_{i-1} commutes with any φ_h, $h > i$, and acts by zero on $(u \otimes (v_q)_z)$. Thus

$$\begin{aligned}\tau_{i-1}\varphi_i\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_q)_z) &= \tau_{i-1}(\tau_i(x_i - x_{i+1}) + 1)\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_q)_z) \\ &= \tau_{i-1}\tau_i(x_i - x_{i+1})\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_q)_z) \\ &= z.\tau_{i-1}\tau_i\varphi_{i+1} \cdots \varphi_{m_k}.(u \otimes (v_q)_z)\end{aligned}$$

up to some sign. This is valid for any occurrence of q between the positions $c+1$ and m_k and hence

$$Q(z) = \varphi_{c+1} \cdots \varphi_{m_k}.(u \otimes (v_p)_z) = z^{N_q-2}\tau_{c+1} \cdots \tau_{m_k}.(u \otimes (v_p)_z)$$

is a monomial in z of degree $N_q - 2$.

The term $\tau_1 \cdots \tau_{b-1}\tau_{b+1} \cdots \tau_{c-1}Q(z)$ coming from (10) is necessarily zero: if it was not, then (10) implies that s_q would be equal to $N_q - 2$, which contradicts Corollary 6.6. There are two terms of degree $N_q - 1$ in (10): $\tau_1 \cdots \tau_{c-1}Q(z)$ and $\tau_1 \cdots \tau_{b-1}\tau_{b+1} \cdots \tau_c Q(z)$. Denote them, respectively, by $A(z)$ and $B(z)$.

We show that the operator $\tau_{c-1} \cdots \tau_1$ acts nontrivially on $A(z)$ and trivially on $B(z)$. This implies that $A(z) + B(z)$ cannot be zero and therefore there is a nonzero term of degree $N_q - 1$ in (10).

Action of $\tau_{c-1} \cdots \tau_1$ on $A(z)$. Let us first look at the action of the operators $\tau_1, \cdots, \tau_{a-1}$ on $A(z)$; if $j = 1$ this is of course not necessary as $a = a_0 = 1$. Otherwise one has $j \geq 2$ and this action is easy to compute: for instance for τ_1,

$$\begin{aligned}&\tau_1.A(z) \\ &\qquad = {\tau_1}^2\tau_2 \cdots \tau_{c-1}Q(z) \\ &\qquad = {\tau_1}^2 e(h_1, h_c, h_2, \dots, h_{c-1}, q, h_{c+1}, \dots, h_{m_k})\tau_2 \cdots \tau_{c-1}Q(z) \\ &\qquad = \tau_2 \cdots \tau_{c-1}Q(z) \quad \text{(as } h_c = q \text{ and } h_1 = p_0 > p_1 > q \text{ by minimality of } j \geq 2.)\end{aligned}$$

Similarly $h_2, \dots, h_{a-1} \geq q+2$ and thus one gets

$$\tau_{a-1} \cdots \tau_1 . A(z) = \tau_a \cdots \tau_{c-1} . Q(z).$$

Let us now look at the action of τ_a:

$$\begin{aligned}
&\tau_a.(\tau_a \cdots \tau_{c-1}.Q(z)) \\
&\qquad = {\tau_a}^2 e(h_1, \dots, h_a, h_c, h_{a+1}, \dots, h_{c-1}, q, h_{c+1}, \dots, h_{m_k}) \tau_{a+1} \cdots \tau_{c-1} Q(z) \\
&\qquad = (x_a - x_{a+1}) \tau_{a+1} \cdots \tau_{c-1} Q(z) \text{ as } h_a = p_{j-1} = q+1 \text{ and } h_c = p_{j+1} = q.
\end{aligned}$$

The operator x_a commutes with $\tau_{a+1}, \cdots \tau_{c-1}$ and acts trivially on the generating vector, hence one gets

$$\begin{aligned}
&\tau_a.(\tau_a \cdots \tau_{c-1}.Q(z)) \\
&\qquad = -x_{a+1} \tau_{a+1} \cdots \tau_{c-1}.Q(z) \\
&\qquad = -x_{a+1} \tau_{a+1} \cdots \tau_b e(h_1, \dots, h_b, h_c, h_{b+1}, \dots, h_{c-1}, \dots) \tau_{b+1} \cdots \tau_{c-1} Q(z) \\
&\qquad = -\tau_{a+1} \cdots \tau_{b-1} x_b \tau_b e(h_1, \dots, h_b, h_c, h_{b+1}, \dots) \tau_{b+1} \cdots \tau_{c-1}.Q(z) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (h_c = q,\ h_{a+1}, \dots, h_{b-1} \geq q+2) \\
&\qquad = -\tau_{a+1} \cdots \tau_{b-1} (\tau_b x_{b+1} + 1) \tau_{b+1} \cdots \tau_{c-1}.Q(z) \qquad\qquad (h_b = h_c = q) \\
&\qquad = -\tau_{a+1} \cdots \tau_{b-1} \tau_b \tau_{b+1} \cdots \tau_{c-1} x_c.Q(z) - \tau_{a+1} \cdots \tau_{b-1} \tau_{b+1} \cdots \tau_{c-1}.Q(z).
\end{aligned}$$

The operator x_c acts trivially on $(u \otimes (v_q)_z)$, hence the first term of the right-hand side in the last equality is zero. Now as $h_{a+1}, \dots, h_{b-1} \geq q+2$ and $h_b = q$, the action of the operator $\tau_{b-1} \cdots \tau_{a+1}$ on the surviving term is similar to the action of $\tau_{a-1} \cdots \tau_1$ computed above. Hence we get

$$\tau_{b-1} \cdots \tau_a.(\tau_a \cdots \tau_{c-1}.Q(z)) = -\tau_{b+1} \cdots \tau_{c-1}.Q(z).$$

The situation is now similar to (i): using the braid relation, one can see that the action of $\tau_{b+1}\tau_b$ on $\tau_{b+1} \cdots \tau_{c-1}.Q(z)$ will give two terms, the only nontrivial one being $\tau_{b+2} \cdots \tau_{c-1}.Q(z)$. The letters $h_{b+2}, \dots, h_{c-1}$ are all greater than $q+2$ and hence one concludes as before that

$$\tau_{c-1} \cdots \tau_{b+2}.(\tau_{b+2} \cdots \tau_{c-1}.Q(z)) = Q(z).$$

Finally we have shown that $\tau_{c-1} \cdots \tau_1$ acts by identity on $A(z)$ (up to some sign).

Action of $\tau_{c-1} \cdots \tau_1$ *on* $B(z)$. One again has

$$\begin{aligned}
\tau_{a-1} \cdots \tau_1.B(z) &= \tau_{a-1} \cdots \tau_1.(\tau_1 \cdots \tau_{b-1} \tau_{b+1} \cdots \tau_c Q(z)) \\
&= \tau_a \cdots \tau_{b-1} \tau_{b+1} \cdots \tau_c Q(z).
\end{aligned}$$

But then

$$\begin{aligned}
\tau_a.(\tau_a \cdots \tau_{b-1} \tau_{b+1} \cdots \tau_c Q(z)) &= (x_a - x_{a+1}) \tau_{a+1} \cdots \tau_{b-1} \tau_{b+1} \cdots \tau_c Q(z) \\
&= \tau_{a+1} \cdots \tau_{b-1} x_b \tau_{b+1} \cdots \tau_c Q(z)
\end{aligned}$$

up to some sign, and then the operator x_b commutes with $\tau_{b+1}, \dots, \tau_c, \varphi_{c+1}, \dots, \varphi_{m_k}$ and acts by zero on $(u \otimes (v_q)_z)$.

Thus we have shown that the operator $\tau_1 \cdots \tau_{c-1}$ acts nontrivially on $A(z)+B(z)$ and in particular $A(z)+B(z) \neq 0$. Therefore $s_q = N_q - 1$. Now, $q < p_0 \leq n-k+1$ by Lemma 6.8 and by assumption $q \neq k$, hence $q \notin \{k, n-k+1\}$. By Corollary 6.6, this contradicts the inequality $s_q \leq N_q - 1$.

In conclusion, (ii) holds. □

Corollary 6.15. *The sequence (p_j) takes every value $1, \dots, k-1$ exactly once and the value k at least once.*

Proof. The last term of the sequence (p_j) is $p_r = 1$. Recall that $r \geq k-1$ (Remark 6.11). By (finite) induction on $t \in \{0, \dots, k-1\}$ one shows that $p_{r-t} = t+1$. Indeed, if $k=1$ there is nothing to prove. If $k \geq 2$, assume $p_r = 1, \dots, p_{r-t} = t+1$ with $t < k-1$; then $p_{r-t} \leq k-1$ and Lemma 6.13 implies $p_{r-t-1} \leq p_{r-t} + 1$. If $p_{r-t-1} \neq k$ then Proposition 6.14 (ii) implies $p_{r-t-1} > p_{r-t}$ and thus $p_{r-t-1} = p_{r-t} + 1$ which gives $p_{r-(t+1)} = t+1$. If $p_{r-t-1} = k$ then since $p_{r-t} \leq k-1$, one necessarily has $p_{r-t} = t = k-1$ and $p_{r-t-1} = k$ which again gives $p_{r-(t+1)} = t+1$. This implies that the sequence (p_j) takes each value $1, \dots, k-1$ exactly once (and the value k at least once). □

Corollary 6.16. *In the case $p=k$, the parameter μ_k of M_k is given by*

$$\mu_k = (k \dots n)(k-1 \dots n-1) \cdots (1 \dots n-k+1).$$

Proof. By Proposition 6.14(i), the sequence (p_j) takes the value k exactly once. Together with Corollary 6.15, we deduce that the word μ_k has the form

$$\mu_k = (k \dots)(k-1 \dots) \cdots (1 \dots).$$

Combining this with Lemma 6.10 (i) and (ii), we get the desired statement. □

One can now focus on the case $p_0 = n-k+1$.

Proposition 6.17. *The sequence $(p_j)_{0 \leq j \leq r}$ takes every value between $n-k+1$ and 1 exactly once. In other words, $r = n-k+1$ and $p_j = n-k+1-j$ for all $1 \leq j \leq n-k+1$.*

Proof. By Corollary 6.15, we already know that the values $1, \dots, k-1$ are taken exactly once and the value k at least once.

Values $k+1, \dots, n-k+1$. Let $i := \max\{j, p_j > k\}$ (it exists since $p_0 = n-k+1 > k$) and $m := p_i$.

Assume $m \geq k+2$. Then the commuting of M_k with $L(k)$ implies that $\boldsymbol{i}_{i+1}$ is the only Lyndon word beginning with k (equivalently, $p_{i+1} = k, p_{i+2} = k-1, \dots, p_r = 1$).

Indeed, all the letters in position strictly smaller than a_{i+1} are greater than $k+2$, hence $\varphi_{i'} = \tau_{i'}$ for all $i' < a_{i+1}$ and

$$\tau_{a_{i+1}-1} \cdots \tau_1.(\varphi_1 \cdots \varphi_{m_k}.(u \otimes (v_k)_z)) = \varphi_{a_{i+1}} \cdots \varphi_{m_k}.(u \otimes (v_k)_z).$$

Then the same proof as for Proposition 6.14(i) shows that $k = p_{i+1} > p_{i+2}$.

Thus there is exactly one Lyndon word $\boldsymbol{i}_j$ beginning with each letter $1, \dots, k$. The letter $m-1$ does not appear in any of the words $\boldsymbol{i}_j$ for $j \le i$ (all these words begin with letters greater than m) and appears exactly k times in the word μ_k (as $n-k \ge m-1 \ge k+1$), hence it appears exactly once in each of the words $\boldsymbol{i}_{i+1}, \dots, \boldsymbol{i}_r$. This implies that the last letters of all of these words are greater than $m-1$ and in particular so is k (the last letter of $\boldsymbol{i}_r$), i.e., $m \le k+1$, which contradicts the hypothesis.

Hence $m = k+1$, i.e., the sequence (p_j) takes all the values $n-k+1, \dots, 1$. By Proposition 6.14(ii), the values $n-k+1, \dots, k+1$ appear exactly once in the sequence (p_j).

Value k. If there are more than two Lyndon words $\boldsymbol{i}_j$ beginning with the letter k then the same proof as for Proposition 6.14(ii) (it can be applied since $m = k+1$) implies $s_k \le N_k - 1$. But since the last letter of the word μ_k is k, the same proof as for Lemma 6.8 shows that s'_k is also smaller than $N_k - 1$. Hence both s_k and s'_k are less than $N_k - 1$, which contradicts Corollary 6.6.

Therefore the sequence (p_j) takes every value $n-k+1, \dots, 1$ exactly once. □

Corollary 6.18. *For any* $0 \le j \le n-k$, *the Lyndon word* $\boldsymbol{i}_j$ *is* $(n-k+1-j \dots n-j)$.

Proof. We show this by induction on j. In fact we prove the following properties:

(i) For every $0 \le j \le n-k$ there is exactly one Lyndon word ending with each of the letters $n-j, \dots, n-k+1-j$.

(ii) The Lyndon word ending with $n-j$ begins with the letter $n-k+1-j$.

For $j = 0$ this follows from Corollary 6.12.

Assume (i) and (ii) hold until the rank j. By hypothesis, the Lyndon words ending with the letters $n, n-1 \dots, n-j$, respectively, begin with the letters $n-k+1, \dots, n-k+1-j$, and in particular do not contain $n-k+j$. Since by Proposition 6.17 there is exactly one Lyndon word beginning with each of the letters $n-k+1, \dots, 1$, the Lyndon words ending with letters $n-1-j, \dots, n-k+1-j$ begin with letters less than $n-k-j$ and hence contain $n-k-j$. This gives $k-1$ Lyndon words containing the letter $n-k-j$. As this letter appears exactly k times in the word μ_k, there exists a Lyndon word that contains $n-k-j$ but is not one of the previous words, i.e., does not end with any of the letters $n, \dots, n-j$. Hence it does not contain $n-k-j+1$ (since $n-k-j+1$ appears k times, once in each of the k words ending with $n-j,\dots,n-k-j+1$). This means there is a unique Lyndon word ending with the letter $n-k-j$, which proves (i) at the rank $j+1$.

Now (ii) at rank j and (i) at rank $j+1$ together with Proposition 6.17 easily imply (ii) at rank $j+1$. □

From Corollary 6.16 and Corollary 6.18, one concludes that one of the two simple modules M_k and M_{n-k+1} has a parameter whose first letter is k, namely $(k\dots n)(k-1\dots n-1)\cdots(1\dots n-k+1)$, and the other has a parameter whose first letter is $n-k+1$, namely $(n-k+1\dots n)\cdots(1\dots k)$.

6E. ***Proofs of main theorems.*** At this stage, one only has bijections between pairs of modules and pairs of dominant words: for each $1\le k\le n/2$, the set of modules $\{M_k, M_{n-k+1}\}$ is in one-to-one correspondence with the set

$$\{(k\dots n)(k-1\dots n-1)\cdots(1\dots n-k+1),(n-k+1\dots n)\cdots(1\dots k)\}.$$

A priori this yields two possibilities for each k. To complete the proof of Theorem 6.1, we need to show that for every $1\le k\le n/2$, one has

$$M_k = L((k\dots n)(k-1\dots n-1)\cdots(1\dots n-k+1))$$

and

$$M_{n-k+1} = L((n-k+1\dots n)(n-k\dots n-1)\cdots(1\dots k)).$$

The key argument is the mutation rule for dominant words given by Proposition 5.8.

Proof of Theorem 6.1. We prove by induction on $k\in\{1,\dots,n\}$ that

$$\begin{aligned} M_{r_{k-1}+1} &= L(1\dots k)\\ M_{r_{k-1}+2} &= L((2\dots k)(1\dots k-1))\\ &\;\vdots\\ M_{r_k} &= L(k\cdots 1). \end{aligned}$$

The result already holds for $k=1$ and $k=2$. Consider $1\le k\le n$ and assume the result holds at the rank k.

Let $j\in\{r_{k-1}+2,\dots,r_k-1\}$ and let us write the (ungraded) short exact sequence corresponding to the mutation in direction j:

$$0\to M_{j_+}\circ M_{j-1}\circ M_{j_-+1}\to M_j\circ M_j{}'\to M_{j_-}\circ M_{j+1}\circ M_{j_+-1}\to 0.$$

Let $p:=j-r_{k-1}$.

By the induction hypothesis, one has

$$\begin{aligned} M_j &= L((p\dots k)\cdots(1\dots k-p+1)),\\ M_{j-1} &= L((p-1\dots k)\cdots(1\dots k-p+2)),\\ M_{j+1} &= L((p+1\dots k)\cdots(1\dots k-p)). \end{aligned}$$

The Lyndon word $(p \dots k)$ appears in the parameter of M_j, and hence in the parameter of $M_j \circ M'_j$. Hence by Proposition 5.8, it necessarily appears either in $\mu_{j_+} \odot \mu_{j-1} \odot \mu_{j_-+1}$ or in $\mu_{j_-} \odot \mu_{j+1} \odot \mu_{j_+-1}$. Obviously, it does not appear in μ_{j-1} nor in μ_{j+1}. Moreover, μ_{j_-} and μ_{j_-+1} do not contain the letter k, hence $(p \dots k)$ does not appear in the canonical factorizations of these parameters either.

Now by Proposition 6.4, μ_{j_+} is either

$$(p+1 \dots k+1) \cdots (1 \dots k-p) \quad \text{or} \quad (k-p+1 \dots k+1) \cdots (1 \dots p+1)$$

and μ_{j_+-1} is either

$$(p \dots k+1) \cdots (1 \dots k-p+2) \quad \text{or} \quad (k-p+2 \dots k+1) \cdots (1 \dots p).$$

The only one of these words in which the Lyndon $(p \dots k)$ appears is

$$(p+1 \dots k+1) \cdots (1 \dots k-p)$$

and thus $\mu_{j_+} = \mu_{r_k+p+1} = (p+1 \dots k+1) \cdots (1 \dots k-p)$.

One can do this for any $j \in \{r_{k-1}+2, \dots, r_k-1\}$, and the same arguments hold for $j = r_{k-1}+1$ and $j = r_k$. Thus the desired result holds at rank $k+1$. $\square$

One can now prove Theorem 6.2.

Proof of Theorem 6.2. We begin by describing the exchange matrix corresponding to the quiver given in [Kang et al. 2018b, Definition 11.1.1]. For any $1 \leq k \leq n-1$, define the matrices

$$A_k := \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ -1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}, \quad B_k := \begin{pmatrix} -1 & 0 & \cdots & \cdots & 0 \\ 1 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & \cdots & \cdots 0 & 1 \end{pmatrix}, \quad C_k := -\,{}^tB_{k-1}$$

of respective sizes $k \times k$, $k+1 \times k$, and $k-1 \times k$.

Now the whole exchange matrix can be written by blocks as

$$\begin{pmatrix} A_1 & C_2 & 0 & \cdots & \cdots & 0 \\ B_1 & A_2 & C_3 & \ddots & & \vdots \\ 0 & B_2 & A_3 & C_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & \ddots & \ddots & C_{n-1} \\ 0 & \cdots & \cdots & 0 & B_{n-2} & A_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & B_{n-1} \end{pmatrix}.$$

Recall that for any parameter $\mu \in \boldsymbol{M}$, $\mu^{\odot -1}$ denotes the inverse of μ in the Grothendieck group $\boldsymbol{G}$ of $\boldsymbol{M}$. We can now compute the parameters $\hat{\mu}_j$ associated to the $\hat{y}_j$ as in Definition 4.7. For instance, for any $2 \leq j \leq n-2$,

$$\begin{aligned}
&\hat{\mu}_{r_{n-2}+j} \\
&\quad = ((j-1 \dots n-2) \cdots (1 \dots n-j))^{\odot -1} \odot ((j \dots n-2) \cdots (1 \dots n-j-1)) \\
&\qquad \odot ((j-1 \dots n-1) \cdots (1 \dots n-j+1)) \odot ((j+1 \dots n-1) \cdots (1 \dots n-j-1))^{\odot -1} \\
&\qquad \odot ((j \dots n) \cdots (1 \dots n-j+1))^{\odot -1} \odot ((j+1 \dots n) \cdots (1 \dots n-j))
\end{aligned}$$

which simplifies as

$$\hat{\mu}_{r_{n-2}+j} = ((j+1 \dots n)(j \dots n-1)) \odot ((j+1 \dots n-1)(j \dots n))^{\odot -1}.$$

Hence for any parameter $\mu \in \boldsymbol{M}$, one has

$$\begin{aligned}
((j+1 \dots n-1)(j \dots n)) \odot \hat{\mu}_{r_{n-2}+j} \odot \mu &= ((j+1 \dots n)(j \dots n-1)) \odot \mu \\
&> ((j+1 \dots n-1)(j \dots n)) \odot \mu.
\end{aligned}$$

This exactly means $\hat{\mu}_{r_{n-2}+j} \odot \mu > \mu$ in $\boldsymbol{G}$ for any $\mu \in \boldsymbol{M}$. The computations for any other index $s \in \{1, \dots, r_{n-1}\}$ are similar.

Using Remark 4.8 we conclude that the seed $\mathcal{S}_0^n$ is compatible. □

7. Possible further developments

In this section we mention a couple of situations where interesting consequences may arise from the study of compatible seeds in various contexts of monoidal categorifications of cluster algebras.

7A. ***Dominant words and g-vectors.*** By Theorem 6.2, the seed $\mathcal{S}_0^n$ for the category R-gmod in type A_n is compatible in the sense of Definition 4.7. As explained in Section 4C, this yields some interesting combinatorial relationships between dominant words and g-vectors.

More precisely, consider as in Section 4C x_l^t, any cluster variable in $\mathcal{A}$, M_l^t the simple module in R-gmod such that $[M_l^t] = x_l^t$ and μ_l^t the dominant word associated to M_l^t. For simplicity, we will write x (resp. M, μ) for x_l^t (resp. M_l^t, μ_l^t) without ambiguity as we will focus here on this module. For any dominant Lyndon word (i.e., any positive root in type A_n) $(k \dots l)$, we let $m_{(k \dots l)}$ denote the multiplicity of the Lyndon word $(k \dots l)$ in the canonical factorization of μ. As in Section 2A, we consider F and $\boldsymbol{g} = (g_1, \dots, g_{r_{n-1}})$ the F-polynomial and the g-vector associated to x. We also let $a_1, \dots, a_{r_{n-1}}$ denote the exponents of the unique monomial of maximal degree of F (see Theorem 2.5(i)), and $c_1, \dots, c_n$ the (negative) integers such that $F_{|\mathbb{P}}(y_1, \dots, y_n) = x_{r_{n-1}+1}^{c_1} \cdots x_{r_n}^{c_n}$ (see Section 4C).

First consider the positive roots ending with the letter n. It follows from Theorem 6.1 that these Lyndon words do not appear in the dominant words associated to the unfrozen variables of the seed $\mathcal{S}_0^n$. Hence the g-vectors will not be involved. Moreover, for any $1 \leq j \leq n$, the Lyndon word $(j \dots n)$ appears in exactly one of the dominant words corresponding to the frozen variables of $\mathcal{S}_0^n$, namely, $(j \dots n) \cdots (1 \dots n-j+1)$. If $2 \leq j \leq n-1$, then the Lyndon word $(j \dots n)$ appears in exactly two of the $\hat{\mu}_j$, namely, $\hat{\mu}_{r_{n-1}+j-1}$ and $\hat{\mu}_{r_{n-1}+j}$. The relations are

$$m_{(j \dots n)} = a_{r_{n-2}+j-1} - a_{r_{n-2}+j} - c_j.$$

For positive roots of the form $(j \dots n-1)$ with $2 \leq j \leq n-2$, similar arguments show that

$$m_{(j \dots n-1)} = a_{r_{n-2}+j} - a_{r_{n-2}+j+1} - c_{j+1} - a_{r_{n-2}+j-1} + a_{r_{n-2}+j+1} + a_{r_{n-3}+j-1} - a_{r_{n-3}+j} + g_{r_{n-2}+j}$$

which simplifies as

$$m_{(j \dots n-1)} = a_{r_{n-2}+j} - c_{j+1} - a_{r_{n-2}+j-1} + a_{r_{n-3}+j-1} - a_{r_{n-3}+j} + g_{r_{n-2}+j}.$$

7B. ***The coherent Satake category.*** Recently, Cautis and Williams [2019] exhibited a new example of monoidal categorification of cluster algebras, using the coherent Satake category. In this subsection we focus on the case of the general linear group GL_n. We begin by checking that Assumptions A and B hold in the framework of [Cautis and Williams 2019].

The simple objects in the coherent Satake category are parametrized (up to $\mathbb{G}_m$-equivariant shift) by couples of a coweight and a weight, modulo action of the Weyl group. Equivalently they can be parametrized by *dominant pairs*, i.e., couples of a dominant coweight $\lambda^\vee$ together with a weight μ dominant for the Levi factor of $P_{\lambda^\vee}$. Denote by $\mathcal{P}_{\lambda^\vee,\mu}$ the simple perverse coherent sheaf corresponding to a dominant pair $(\lambda^\vee, \mu) \in P^\vee \times P$. Then the following statement shows that Assumption A holds:

Proposition 7.1 [Cautis and Williams 2019, Proposition 2.5]. *Let $\mathcal{P}_{\lambda_1^\vee,\mu_1}$ and $\mathcal{P}_{\lambda_2^\vee,\mu_2}$ be two simple objects in the coherent Satake category. Then in its Grothendieck ring $K^{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_G)$,*

$$[\mathcal{P}_{\lambda_1^\vee,\mu_1} * \mathcal{P}_{\lambda_2^\vee,\mu_2}] = q^s[\mathcal{P}_{\lambda_1^\vee+\lambda_2^\vee,\mu_1+\mu_2}] + \sum_{(\lambda^\vee,\mu)\in S} p_{\lambda^\vee,\mu}[\mathcal{P}_{\lambda^\vee,\mu}],$$

where s is some integer depending on $\lambda_1, \mu_1, \lambda_2, \mu_2$, $p_{\lambda^\vee,\mu} \in \mathbb{Z}[q^{\pm 1/2}]$, and S is a finite collection of dominant pairs such that for every $(\lambda^\vee, \mu) \in S$, one has either $\lambda^\vee < \lambda_1^\vee + \lambda_2^\vee$, or $\lambda^\vee = \lambda_1^\vee + \lambda_2^\vee$ and $\|\mu\|^2 \leq \|\mu_1\|^2 + \|\mu_2\|^2$ for any W-invariant quadratic form $\|\cdot\|^2$.

Taking the lexicographic order on (dominant) pairs $(\lambda^\vee, \mu) \in P^\vee \times P$, the monoid structure on the set of dominant pairs can be simply taken as

$$(\lambda_1^\vee, \mu_1) \odot (\lambda_2^\vee, \mu_2) = (\lambda_1^\vee + \lambda_2^\vee, \mu_1 + \mu_2).$$

It is then clear that Assumption B also holds.

In the case of the general linear group GL_n, Cautis and Williams explicitly describe a monoidal seed in the coherent Satake category. However, this seed is not compatible in the sense of Definition 4.7. For example, for GL_2, it can be written as

$$(([\mathcal{P}_{1,0}], [\mathcal{P}_{1,1}], [\mathcal{P}_{2,0}], [\mathcal{P}_{2,1}]), B),$$

where the first two classes are the unfrozen variables and the last two are the frozen variables, and the exchange matrix B is given by

$$B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Recall from [Cautis and Williams 2019, Section 2.2] that $\mathcal{P}_{k,l}$ stands for $\mathcal{P}_{\omega_k^\vee, l\omega_k}$ for any $1 \le k \le 2$ and any $l \in \{0, 1\}$.

One can now compute the generalized parameters $\hat{\mu}_1$ and $\hat{\mu}_2$ for this seed. A straightforward computation gives $\hat{\mu}_1 = (2\omega_1^\vee - \omega_2^\vee, 2\omega_1 - \omega_2)$ and $\hat{\mu}_2 = (\omega_2^\vee - 2\omega_1^\vee, 0)$. The coweight $2\omega_1^\vee - \omega_2^\vee$ is exactly the coroot $\alpha_1^\vee$, and hence for any dominant pair $(\lambda^\vee, \mu)$, one has $\hat{\mu}_1 \odot (\lambda^\vee, \mu) \geq (\lambda^\vee, \mu)$. However, the coweight part of $\hat{\mu}_2$ is obviously the opposite of $\alpha_1^\vee$ and thus $\hat{\mu}_2 \odot (\lambda^\vee, \mu) \leq (\lambda^\vee, \mu)$ for any dominant pair $(\lambda^\vee, \mu)$. We conclude that this seed is not compatible.

It would be interesting to see if Conjecture 4.10 holds in the coherent Satake category of the general linear group. Note that as the ordering on dominant pairs is partial, it is not clear that one can formulate mutation rules for parameters as in Section 5. Indeed, we crucially used the fact that the ordering on dominant words parametrizing simple modules over quiver Hecke algebras is total. This mutation rule allows us to compute explicitly as many seeds as we want from the data of an initial seed. In the case of a partial ordering, we cannot do so a priori.

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ON THE FINE EXPANSION OF THE UNIPOTENT CONTRIBUTION OF THE GUO–JACQUET TRACE FORMULA

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For a useful class of functions (containing functions whose one finite component is essentially a matrix coefficient of a supercuspidal representation), we establish three results about the unipotent contribution of the Guo–Jacquet relative trace formula for the pair $(\mathrm{GL}_n(D), \mathrm{GL}_n(E))$. First we get a fine expansion in terms of global nilpotent integrals. Second we express these nilpotent integrals in terms of zeta integrals. Finally we prove that they satisfy certain homogeneity properties. The proof is based on a new kind of truncation introduced in a previous article.

1. Introduction

1.1. *The statement.*

1.1.1. Let E/F be a quadratic extension of number fields. Let $\varepsilon \in E^\times$ such that $\mathrm{trace}_{E/F}(\varepsilon) = 0$. Let $n \geqslant 1$ be an integer. Let $G = \mathrm{GL}(2n, F)$ and $H = \mathrm{GL}(n, E)$ viewed as an algebraic group over F. The F-basis $(1, \varepsilon)$ identifies E^n with F^{2n} and gives an embedding $H \subset G$. We identify ε with a scalar matrix in $\mathrm{GL}(n, E)$. Then H is the centralizer of ε in G. Moreover, the quotient G/H is a symmetric space which is easily identified with the subvariety $S \subset G$ of $g \in G$ such that $\varepsilon^{-1} g \varepsilon = g^{-1}$. The group H acts on S by conjugation.

1.1.2. Let $\mathbb{A}$ be the ring of adèles of F and $|\cdot|$ be the product of local normalized absolute values. We identify ε with a central element in $\mathrm{GL}(n, E)$. For any cuspidal automorphic form φ on $G(\mathbb{A})$, one can define its H-period as

$$\mathcal{P}_H(\varphi) = \int_{H(F)\backslash H(\mathbb{A})^1} \varphi(h)\, dh,$$

where $H(\mathbb{A})^1 \subset H(\mathbb{A})$ is the kernel of the morphism $h \mapsto |\det(h)|$ (see Section 2.1.4) and dh is an invariant measure. A cuspidal automorphic representation is said to

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be H-distinguished if $\mathcal{P}_H$ induces a nonzero linear form on its underlying space. A fundamental question is to understand the interplay between the H-distinction, the Jacquet–Langlands functoriality and distinction by some other related subgroups: a beautiful answer is given by the so-called Guo–Jacquet conjecture (see the conjecture in [Guo 1996] extrapolating on results of Waldspurger and Jacquet).

1.1.3. A promising tool to study this problem is the so-called Guo–Jacquet trace formula (a specific example of a relative trace formula) based on the seminal work of Jacquet [1986]. A simple form of it has already been used successfully (see [Feigon et al. 2018]). It consists in expressing geometrically and spectrally the integral

$$\int_{H(F)\backslash H(\mathbb{A})^1} \sum_{\gamma\in S(F)} \Phi(h^{-1}\gamma h)\,dh, \tag{1-1-3-1}$$

where $\Phi \in C_c^\infty(S(\mathbb{A}))$. In general, (1-1-3-1) is not convergent. To remedy this problem, the simple trace formula introduces severe restrictions at two places on the function Φ so that on the spectral side only the cuspidal spectrum appears and on the geometric side only regular semisimple γ's contribute to the rational sum in (1-1-3-1). However, for the purpose of convergence, it suffices to impose a certain local condition at one place on Φ. When it is satisfied, we shall say that Φ is very cuspidal. In the paper, this is the local condition (3-2-2-1) below. Here it suffices to say that the condition is satisfied if the local component of Φ at a finite place v is obtained by integration of a matrix coefficient of a supercuspidal representation of $G(F_v)/Z(F_v)$ (see Remark 3.2.2.1, Z is the center of G).

For a very cuspidal Φ, as a part of (1-1-3-1), we have the (convergent) unipotent contribution defined by

$$\int_{H(F)\backslash H(\mathbb{A})^1} \sum_{X\in\mathcal{N}(F)} \Phi(h^{-1}\exp(X)h)\,dh, \tag{1-1-3-2}$$

where $\mathcal{N}(F)$ is the cone of nilpotent matrices in the tangent space $\mathfrak{s}$ of S at identity (given by matrices X of size $2n$ such that $\varepsilon X + X\varepsilon = 0$) and exp is the usual exponential. We emphasize that besides regular semisimple terms the unipotent contribution is certainly the most important contribution of the geometric side of the Guo–Jacquet trace formula to understand (for general terms there should be some kind of Jordan decomposition).

1.1.4. The goal of the paper is to prove the following properties of the unipotent contribution for a very cuspidal Φ.

Theorem 1.1.4.1 (see Theorem 3.2.4.1 for a more precise and general statement). *Let $\Phi \in C_c^\infty(G(\mathbb{A}))$ be a very cuspidal function. Let $\mathcal{O} \in \mathcal{N}(F)/H(F)$ be a nilpotent orbit and let v be a place of F.*

(1) *For any $t \in F_v^\times$, the integral*

$$J_{\mathcal{O}}^t(\Phi) = \int_{[H]^1} \sum_{X \in \mathcal{O}} \Phi(h^{-1} \exp(t^{-1} X) h)\, dh$$

is absolutely convergent. In particular we get $J_{\mathcal{O}}^1(\Phi)$ for $t = 1$.

(2) *We have*

$$\int_{H(F)\backslash H(\mathbb{A})^1} \sum_{X \in \mathcal{N}(F)} \Phi(h^{-1} \exp(X) h)\, dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} J_{\mathcal{O}}^1(\Phi).$$

(3) *There exists a bound $\eta > 0$ such that for any t in $F_v^\times$ such that $|t|_v < \eta$, we have*

$$J_{\mathcal{O}}^t(\Phi) = \lim_{s \to 0^+} s\theta_{\mathcal{O}}(s) Z_{\mathcal{O}}(\Phi, s),$$

where

- *$\theta_{\mathcal{O}}(s)$ is some holomorphic function which is defined for $s \in \mathbb{C}$ such that its real part satisfies $\Re(s) > 0$ and which does not depend on Φ,*
- *the function $Z_{\mathcal{O}}(\Phi, s)$ is a zeta function attached to Φ of the variable $s \in \mathbb{C}$, holomorphic for $\Re(s) > 0$.*

(4) *Let η be the above bound. For any t_0, t in $F_v^\times$ such that $|t_0|_v < \eta$ and $|t|_v \leqslant 1$ we have*

$$J_{\mathcal{O}}^{tt_0}(\Phi) = |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}^{t_0}(\Phi).$$

Remark 1.1.4.2. Assertions (1) and (2) provide a "fine expansion" of the unipotent contribution that is an expansion according conjugacy classes. Assertion (3) gives a way to compute each nilpotent contribution; moreover the zeta function $Z_{\mathcal{O}}(\Phi, s)$ admits a Eulerian product so it is possible to give an expression of each nilpotent contribution in terms of local objects.

Remark 1.1.4.3. In the paper, the theorem is in fact stated and proved in the broader situation where $G = \mathrm{GL}_n(D)$, where D is a quaternion algebra containing E. It should also hold in the situation where G is the multiplicative group of an F-simple central algebra containing E and H is the centralizer of E in G. The methods of the article should also apply in this context but we have not written the details.

We have the following corollary.

Corollary 1.1.4.4 (see Corollary 3.2.4.2). *Let $\Phi \in C_c^\infty(G(\mathbb{A}))$ be a very cuspidal function. Let v be a place of F. When $t \in F_v^\times$ goes to 0, the expression*

$$\int_{[H]^1} \sum_{X \in \mathcal{N}(F)} \Phi(h^{-1} \exp(t^{-1} X) h)\, dh$$

is equivalent to

$$\mathrm{vol}([H]^1)\Phi(1).$$

Remark 1.1.4.5. One of the motivations for this result is that it plays a role in Xue's approach of local distinction problems. For example, Corollary 1.1.4.4 is used to prove the existence of H-distinguished cuspidal automorphic representations of G with local supercuspidal components at some places (see Corollary 6.2 of [Xue 2019]). Another potential application of Theorem 1.1.4.1 and its corollary is the establishment of some kind of (relative) Weyl's law.

1.2. *The methods.*

1.2.1. The Guo–Jacquet trace formula in our context should share many properties with the usual Arthur–Selberg trace formula. Recall that the unipotent contribution of the Arthur–Selberg trace formula has a fine expansion in terms of local unipotent integrals (although some global constants remain unknown; for some progress on these questions see [Chaudouard 2017; 2018a]). As a consequence it does satisfy general homogeneity properties (see [Arthur 1985]). We could have tried to prove analogs of several results of Arthur in our context (mainly the results of [1978; 1988; 1985]). In this paper we choose to offer a rather simple proof of Theorem 1.1.4.1 based on ideas developed in [Chaudouard 2018b]. The payoff is that we get a (in principle) computable expression (rather than unknown global coefficients).

1.2.2. First one reduces the problem to the case of the infinitesimal situation where H acts on the tangent space $\mathfrak{s}$. This reduction is the content of the final section, Section 3. We introduce in Section 2.3.3 the set of weakly cuspidal functions $f \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$ that satisfy some vanishing conditions. For such functions, we have

$$\text{(1-2-2-1)} \quad \int_{[H]^1} \sum_{X \in \mathcal{N}(F)} f(\mathrm{Ad}(h^{-1})X)\, dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} \int_{[H]^1} \sum_{X \in \mathcal{O}} f(\mathrm{Ad}(h^{-1})X)\, dh;$$

on the right-hand side we sum terms that satisfy the nontrivial convergence result (see Theorem 2.3.4.1)

$$\int_{[H]^1} \left| \sum_{X \in \mathcal{O}} f(\mathrm{Ad}(h^{-1})X) \right| dh < \infty.$$

However these terms are difficult to compute since one cannot permute the sum over X and the integral. This is where enters the truncation of [Chaudouard 2018b]. It enables to recover each term as a limit

$$\lim_{s \to 0} s J_{\mathcal{O}}(f, s),$$

where $J_{\mathcal{O}}(f, s)$ is a holomorphic function for $s \in \mathbb{C}$ of real part $\Re(s) > 0$ (see Theorem 2.5.2.1). The point is that $J_{\mathcal{O}}(f, s)$ can be expressed in terms of a zeta integral under a mild assumption on the support of f (see Theorem 2.6.4.1). For this zeta integral, the homogeneity property is easy to check (see Lemma 2.6.3.1).

2. Infinitesimal situation

2.1. *Notation.*

2.1.1. Let F be a number field and $\tau \in F$ such that $E = F[\sqrt{\tau}]$ is a quadratic extension. Let σ be the generator of the Galois group $\mathrm{Gal}(E/F)$. For each place v of F, let $|\cdot|_v$ be the normalized absolute value.

Let $\mathbb{A}$ be the ring of adèles of F. Let $\mathbb{A}^\times$ be the multiplicative group of $\mathbb{A}$. For any $x = (x_v)_v \in \mathbb{A}^\times$, we shall denote by $|x|$ the product $\prod_v |x|_v$ over all places of F.

For any algebraic group G over F, we shall denote the quotient $G(F)\backslash G(\mathbb{A})$ by $[G]$.

2.1.2. Let V_F be of a vector space of dimension n over F. Let $V = V_F \otimes_F E$. By abuse of notation, we denote by σ the F-automorphism of V given by $\mathrm{Id}_{V_F} \otimes_F \sigma$. Let $\mathfrak{s} \subset \mathrm{End}_F(V)$ be the F-subspace of σ-linear endomorphisms of V, namely the space of maps X of V into itself such that $X \circ \sigma$ is E-linear.

2.1.3. Let H be the algebraic group of automorphisms of the E-vector space V. By restriction of scalars, we will view H as an F-group. The group H is provided with a Galois F-automorphism still denoted by σ. The group H acts by conjugation on $\mathrm{End}_F(V)$: we denote by Ad the restriction of this action on $\mathfrak{s}$. This action is defined over F. By differentiation, we get an action $\mathfrak{h} \times \mathfrak{s} \to \mathfrak{s}$ denoted by $(X, Y) \mapsto [X, Y]$. For any $X \in \mathfrak{s}$, let H_X be the centralizer of X in H.

2.1.4. By abuse of notation, we simply denote by det the F-morphism $H \to \mathbb{G}_{m,F}$ given by $\mathrm{Norm}_{E/F} \circ \det$. Let $H(\mathbb{A})^1$ be the kernel of $h \mapsto |\det(h)|$. We shall denote by $[H]^1$ the quotient $H(F)\backslash H(\mathbb{A})^1$.

2.1.5. *Parabolic decomposition.* We fix an ordered F-basis $(e_1, \ldots, e_n)$ of V_F. By a (standard) parabolic subgroup P, we mean a subgroup which stabilizes an incomplete standard flag of E-subspaces

$$V_0 = (0) \subsetneq V_1 \subsetneq V_1 \oplus V_2 \subsetneq \cdots \subsetneq V_1 \oplus \cdots \oplus V_r,$$

where for $1 \leqslant i \leqslant r$ the E-vector space V_i is generated by vectors

$$e_{d_i+1}, e_{d_i+2}, \ldots, e_{d_i+\dim(V_i)}, \quad \text{where } d_i = \textstyle\sum_{j=1}^{i-1} \dim(V_j).$$

Then we have a standard Levi decomposition $P = MN$, where N is the unipotent radical of P and M is the common stabilizer of the subspaces V_i. We define $\mathfrak{s}_M$ and $\mathfrak{s}_N$ to be the subspaces of $X \in \mathfrak{s}$ such that $XV_i \subset V_i$ and $X(V_1 \oplus \cdots \oplus V_i) \subset$

$V_1 \oplus \cdots \oplus V_{i-1}$, respectively, for all $1 \leqslant i \leqslant r$. Let $\mathfrak{s}_P = \mathfrak{s}_M \oplus \mathfrak{s}_N$. The groups P, M and N act respectively on $\mathfrak{s}_P$, $\mathfrak{s}_M$ and $\mathfrak{s}_N$.

The stabilizer of the complete standard flag is denoted by B. We have a standard Levi decomposition $B = TN_B$.

2.1.6. *Maximal compact subgroup.* Thanks to the basis of Section 2.1.5, we identify $H(\mathbb{A})$ with $\mathrm{GL}(n, \mathbb{A}_E)$. Let $K \subset H(\mathbb{A})$ be the maximal compact subgroup corresponding to the standard maximal compact subgroup of $\mathrm{GL}(n, \mathbb{A}_E)$. For parabolic subgroups $P = MN$, we have the Iwasawa decomposition $H(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})K$.

2.1.7. *Haar measures.* We fix a Haar measure on $\mathbb{A}_E^\times$ the multiplicative group of the adèles of E. For any $n \geqslant 1$, we take on $(\mathbb{A}_E^\times)^n$ the product of the Haar measure. On any unipotent group N over F, we take the Haar measure on $N(\mathbb{A})$ such that the quotient measure on $[N]$ gives the total volume 1. The Haar measure on the standard compact maximal of $\mathrm{GL}(n, \mathbb{A}_E)$ is such that the total volume is 1. Finally we require that the Haar measure on $\mathrm{GL}(n, \mathbb{A}_E)$ is compatible with the Iwasawa decomposition, which normalizes the measure if we view $(\mathbb{A}_E^\times)^n$ as a minimal Levi subgroup. In this way, we get a normalization of the Haar measure on $H(\mathbb{A})$. The Haar measure on $H(\mathbb{A})^1$ is such that the quotient measure on $H(\mathbb{A})/H(\mathbb{A})^1 \simeq \mathbb{R}_+^\times$ is the usual Haar measure. Then we get quotient measures on $[H]$ and $[H]^1$.

2.1.8. *Categorical quotient.* Let $C : \mathfrak{s} \to \mathfrak{c} = \mathfrak{s}//H$ be the categorical quotient. The quotient $\mathfrak{c}$ can be identified with the standard n-dimensional affine space over F in such a way that c is the map that associates to any $X \in \mathfrak{s}$ the coefficients of the characteristic polynomial of $X^2 \in \mathrm{End}_E(V)$ (the coefficients are in fact defined over F). For any $c \in \mathfrak{c}$, let $\mathfrak{s}_c$ be the fiber of C above c. In particular, if c corresponds to the polynomial t^n, we denote by $\mathcal{N}$ the fiber $\mathfrak{s}_c$. Then an element $X \in \mathfrak{s}$ belongs to $\mathcal{N}$ if and only if the E-endomorphism X^2 of V is nilpotent in the usual sense. By abuse, in the following, $\mathcal{N}$ will be called the nilpotent cone and the elements of $\mathcal{N}$ will be called the nilpotent elements of $\mathfrak{s}$.

2.2. *Induction of nilpotent orbits.*

2.2.1. *Classification of nilpotent orbits.* It is given by the following lemma.

Lemma 2.2.1.1. *For any $X \in \mathcal{N}$, there exists an E-basis of V such that the matrix of X is in the Jordan normal form.*

There are finitely many orbits of H on $\mathcal{N}$ and they are classified by their Jordan normal form.

Proof. One can find a proof in [Guo 1997, Lemma 2.3]. Alternatively, one can observe that V has a $E[t]$-module structure of t^{2n}-torsion given by $(P, v) \mapsto P(X)v$ which boils down to the classification of torsion modules over a principal ideal domain. □

2.2.2. *Induction à la Lusztig–Spaltenstein.* We state and prove in our context some results about induction of nilpotent orbits analogous to those of [Lusztig and Spaltenstein 1979].

Let $P = MN$ be a parabolic subgroup of H as above. Let $X \in \mathfrak{s}_M$ be a nilpotent element. Let $\mathcal{O}_X^M$ be the M-orbit of X. The variety $\mathcal{O}_X^M \oplus \mathfrak{s}_N$ is irreducible. Since there are finitely many nilpotent orbits, there is a unique nilpotent H-orbit $\mathcal{O}$ such that

$$\mathcal{O} \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)$$

is a Zariski open dense subset of $\mathcal{O}_X^M \oplus \mathfrak{s}_N$. We denote $\mathcal{O}$ by $I_P(X)$ and we shall call it *the induced orbit*.

Proposition 2.2.2.1. *The intersection $I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)$ is a single P-orbit.*

Proof. Let $Z \in \mathfrak{s}_N$ such that the element $Y = X + Z$ belongs to $I_P(X) \cap (\mathcal{O}_X^M \oplus \mathfrak{s}_N)$. Let $\mathcal{B}$ be the variety of complete flags of E-subspaces of V. As usual, we view $\mathcal{B}$ as an F-variety. Let $\mathcal{B}_Y$ be the subvariety of complete flags $V_\bullet$ such that $YV \subset V$.

It is not difficult to compute the dimensions of $\mathcal{B}_Y$ and H_Y in terms of the Jordan decomposition of Y (one can use the method of [Spaltenstein 1982]). Then one can check the equality

$$\dim(H_Y) = 2\dim(\mathcal{B}_Y) + \dim(T). \tag{2-2-2-1}$$

By using the group B as a base point, we get an equivariant map $\pi_1 : H \to \mathcal{B}$. Let π_2 be the map from H to the H-orbit $\mathcal{O}_Y$ of Y given by $h \mapsto h^{-1}Yh$. We have $\pi_2^{-1}(\mathcal{O}_Y \cap \mathfrak{s}_B) = \pi_1^{-1}(\mathcal{B}_Y)$. The dimension of the fibers of π_1 and π_2 are respectively $\dim(B)$ and $\dim(H_Y)$. We deduce that

$$\dim(\mathcal{O}_Y \cap \mathfrak{s}_B) + \dim(H_Y) = \dim(\mathcal{B}_Y) + \dim(B). \tag{2-2-2-2}$$

Combining (2-2-2-1) and (2-2-2-2), we get

$$\dim(\mathcal{B}_Y) + \dim(\mathcal{O}_Y \cap \mathfrak{s}_B) = \dim(N_B). \tag{2-2-2-3}$$

We can also define a variety $\mathcal{B}_X^M$ relative to M and X. We have an identification

$$\mathcal{B}_X^M \simeq \mathcal{B}_Y^P = \mathcal{B}_Y \cap \mathcal{B}^P,$$

where $\mathcal{B}^P$ is the variety of the complete flags that refine the flag associated to P. In particular, $\dim(\mathcal{B}_X^M) \leqslant \dim(\mathcal{B}_Y)$. As before, one proves:

$$\dim(\mathcal{B}_X^M) + \dim(\mathcal{O}_X^M \cap \mathfrak{s}_{M\cap N_B}) = \dim(N_B \cap M).$$

Following the proof of theorem 1.3 in [Lusztig and Spaltenstein 1979], we have

$$\begin{aligned}\dim(\mathcal{B}_Y)+\dim(\mathcal{O}_Y\cap\mathfrak{s}_B) &\geqslant \dim(\mathcal{B}_X^M)+\dim((\mathcal{O}_X^M\oplus\mathfrak{s}_N)\cap\mathfrak{s}_B)\\ &=\dim(\mathcal{B}_X^M)+\dim(\mathcal{O}_X^M\cap\mathfrak{s}_{M\cap N_B})+\dim(\mathfrak{s}_N)\\ &=\dim(N_B\cap M)+\dim(\mathfrak{s}_N)\\ &=\dim(N_B).\end{aligned}$$

By (2-2-2-3), all the inequalities are in fact equalities. In particular, we deduce $\dim(\mathcal{B}_Y)=\dim(\mathcal{B}_X^M)$ and thus $\dim(H_Y)=\dim(M_X)$ where M_X is the centralizer of X in M.

Let $\mathcal{O}_Y^P$ be the P-orbit of Y and P_Y be its centralizer in P. We have

$$\begin{aligned}\dim(I_P(X)\cap(\mathcal{O}_X^M\oplus\mathfrak{s}_N)) &\geqslant \dim(\mathcal{O}_Y^P)=\dim(P)-\dim(P_Y)\\ &\geqslant \dim(P)-\dim(H_Y)\\ &=\dim(P)-\dim(M_X)\\ &=\dim(\mathcal{O}_X^M\oplus\mathfrak{s}_N)\\ &=\dim\big(I_P(X)\cap(\mathcal{O}_X^M\oplus\mathfrak{s}_N)\big).\end{aligned}$$

Thus we have $\dim(\mathcal{O}_Y^P)=\dim\big(I_P(X)\cap(\mathcal{O}_X^M\oplus\mathfrak{s}_N)\big)$. But one gets the same result for any $Y'\in I_P(X)\cap(\mathcal{O}_X^M\oplus\mathfrak{s}_N)$. Thus the orbits $\mathcal{O}_Y^P$ and $\mathcal{O}_{Y'}^P$ must intersect by irreducibility of $I_P(X)\cap(\mathcal{O}_X^M\oplus\mathfrak{s}_N)$. □

2.2.3. The following lemma is a variant of [Chaudouard 2018a, lemme 2.9.1].

Lemma 2.2.3.1. *There exists a finite family of polynomial maps $(\Phi_i)_{i\in I}$ on $\mathfrak{s}_M\oplus\mathfrak{s}_N$ such that for any nilpotent M-orbit $\mathcal{O}$ in $\mathfrak{s}_M$, there exists $I_{\mathcal{O}}\subset I$ that satisfies the following two properties*:

(1) *For any $X\in\mathcal{O}$ and $Y\in\mathfrak{s}_N$, one has $X+Y\in I_P(X)$ if and only if there exists $i\in I_{\mathcal{O}}$ such that $P_i(X,Y)\neq 0$.*

(2) *For any $X\in\mathcal{O}$, there exists $i\in I_{\mathcal{O}}$ such that the restriction of $P_i(X,\cdot)$ to $\mathfrak{s}_N$ is nontrivial.*

Proof. Let $\mathcal{O}$ be an nilpotent M-orbit in $\mathfrak{s}_M$. Let $X\in\mathcal{O}$ and $Y\in\mathfrak{s}_N$. Let $\mathcal{O}'$ be the P-orbit of $X+Y$. By Proposition 2.2.2.1, we have $X+Y\in I_P(X)$ if and only if $\mathcal{O}'$ is a dense open subset of $\mathcal{O}\oplus\mathfrak{s}_N$. The latter condition holds if and only if we have the following equality between tangent spaces:

$$[\mathfrak{p},X+Y]=[\mathfrak{m}_P,X]\oplus\mathfrak{s}_N. \tag{2-2-3-4}$$

One always has the inclusion $[\mathfrak{p},X+Y]\subset[\mathfrak{m}_P,X]\oplus\mathfrak{s}_N$. The dimension $d_{\mathcal{O}}=\dim([\mathfrak{m}_P,X]\oplus\mathfrak{s}_N)$ does depend only on $\mathcal{O}$. Thus the condition (2-2-3-4) holds if and only if the rank of the F-map $\mathfrak{p}\to\mathfrak{s}_P$ given by $Z\mapsto[Z,X+Y]$ is at least $d_{\mathcal{O}}$.

This condition defines a dense Zariski open subset of $\mathfrak{s}_P$ which does intersect nontrivially $X+\mathfrak{s}_N$ since, for any nilpotent $X\in\mathfrak{s}_M$, there exists $Y\in\mathfrak{s}_N$ such that $X+Y\in I_P(X)$. The statement is then clear. □

2.3. *Nilpotent expansion and the homogeneity property.*

2.3.1. For any F-vector space W, we denote by $\mathcal{S}(W(\mathbb{A}))$ the space of complex Schwartz–Bruhat functions on $W(\mathbb{A})$ and $C_c^\infty(W(\mathbb{A}))\subset\mathcal{S}(W(\mathbb{A}))$ the subspace of smooth compactly supported functions.

2.3.2. Let $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$. For any parabolic subgroup $P=MN$ of H (see Section 2.1.5) and any $x\in H(\mathbb{A})$, we define the constant term of f by

$$f_{P,x}(X)=\int_{\mathfrak{s}_N(\mathbb{A})} f(\mathrm{Ad}(x)(X+U))\,dU,$$

where the Haar measure on $\mathfrak{s}_N(\mathbb{A})$ is normalized in such a way that the quotient $\mathfrak{s}_N(\mathbb{A})/\mathfrak{s}_N(F)$ equipped with the quotient measure by the counting measure is of volume 1. This formula defines a function $f_{P,x}$ in the Schwartz–Bruhat space $\mathcal{S}(\mathfrak{s}_M(\mathbb{A}))$.

2.3.3. We say that f is *weakly cuspidal* if $f_{P,x}$ vanishes on the subset $\mathcal{N}(\mathbb{A})\cap\mathfrak{s}_M(\mathbb{A})$ for any proper parabolic subgroup $P\subsetneq H$ and any $x\in H(\mathbb{A})$.

2.3.4. *A convergence result.* Let $\mathcal{N}(F)/H(F)$ be the finite set of $H(F)$-orbits on $\mathcal{N}(F)$. Let $\mathcal{O}\in\mathcal{N}(F)/H(F)$ be a nilpotent orbit. For any $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $h\in H(\mathbb{A})$, we define

$$k_{\mathcal{O}}(f,h)=\sum_{X\in\mathcal{O}} f(\mathrm{Ad}(h^{-1})X).$$

Theorem 2.3.4.1. *Let $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ be a weakly cuspidal function. The integral*

$$J_{\mathcal{O}}(f)=\int_{[H]^1} k_{\mathcal{O}}(f,h)\,dh$$

is absolutely convergent.

The proof will be given in Section 2.4.4 below. We have the following corollary which is also a simple consequence of a much more general result of Li [2020, Theorem 1.1].

Corollary 2.3.4.2. *Let $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ be a weakly cuspidal function. The integral*

$$\int_{[H]^1}\Bigg|\sum_{X\in\mathcal{N}(F)} f(\mathrm{Ad}(h^{-1})X)\Bigg|\,dh$$

is convergent.

2.3.5. *Homogeneity of nilpotent integrals.* Let v be a place of F. For any $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and any $t \in F_v^\times$, let f_t be the function in $\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ defined by $f_t(X) = f(t^{-1}X)$.

Theorem 2.3.5.1. *Let $f \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$ be a weakly cuspidal compactly supported function. Let v be a place of F. There exists a bound $\eta > 0$ such for any t_0, t in $F_v^\times$ such that $|t_0|_v < \eta$ and $|t|_v \leqslant 1$ we have*

$$J_{\mathcal{O}}(f_{tt_0}) = |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}(f_{t_0}).$$

Remark 2.3.5.2. The proof is given in Section 2.7 but it is built upon several preliminaries results. Indeed to get Theorem 2.3.5.1, the convergence of Theorem 2.3.4.1 is not sufficient. The main difficulty is that we cannot a priori invert the integral and the sum over $\mathcal{O}$: this would in general make appear an infinite volume related to a centralizer. We shall rather use a roundabout method inspired by [Chaudouard 2018b]. As a byproduct, we shall obtain an expression of $J_{\mathcal{O}}(f_{t_0})$ in terms of an explicit zeta integral depending on f (see Theorem 2.6.4.1). Here we will need a mild assumption on the support of f_{t_0} hence the bound on t_0.

For future reference, let's state a simple corollary.

Corollary 2.3.5.3. *Under the hypothesis of Theorem 2.3.5.1, for any $t \in F_v^\times$ such that $|t|_v \leqslant 1$ we have*

$$\int_{[H]^1} \sum_{X \in \mathcal{N}(F)} f_{tt_0}(\mathrm{Ad}(h^{-1})X)\, dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}(f_{t_0}).$$

In particular, when $t \to 0$ the expression above is equivalent to

$$J_{(0)}(f_{t_0}) = \mathrm{vol}([H]^1) f(0),$$

where (0) is the orbit of 0.

2.4. *Refined convergence results.*

2.4.1. Let $\mathcal{O} \in \mathcal{N}(F)/H(F)$ be a nilpotent orbit. Let $P = MN$ be standard parabolic subgroup of H (see Section 2.1.5). We borrow notations from [Chaudouard 2018b]; to do this, we identify H with $\mathrm{GL}(n, E)$ and P with a standard parabolic subgroup of $\mathrm{GL}(n, E)$ by the choice of the basis of Section 2.1.5. We follow [Chaudouard 2018b, §§1.6 and 2.7] (relative to the base field E). We have the Harish-Chandra map from H_P from $H(\mathbb{A})$ to some real vector space a_P. Restricted to $P(\mathbb{A})$ this is a morphism and we denote by $M(\mathbb{A})^1$ the intersection of $M(\mathbb{A})$ with the kernel of H_P. Let τ_P be the characteristic function of the acute Weyl chamber in a_P associated to P. Let Z_P be the maximal central F-split torus in M and let A_P be the neutral component of the group of $\mathbb{R}$-points of the split component of $\mathrm{Res}_{F/\mathbb{Q}}(Z_P)$. We have $M(\mathbb{A}) = M(\mathbb{A})^1 A_P$. The function F^P is the characteristic function of some compact subset of $A_P M(F)N(\mathbb{A})\backslash H(\mathbb{A})$. Note that both F^P and H_P depend on the choice of the compact subgroup K of Section 2.1.6.

Theorem 2.4.1.1. *Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ be a weakly cuspidal function. The integral*

$$\int_{P(F)\backslash H(\mathbb{A})^1} F^P(h)\tau_P(H_P(h))k_{\mathcal{O}}(f,h)\,dh$$

is absolutely convergent.

The proof will be given in Section 2.4.2 below.

Remark 2.4.1.2. Theorem 2.4.1.1 is a simple analog of [Chaudouard 2018a, proposition 3.5.1, corollaire 3.2.2].

2.4.2. *Proof of Theorem 2.4.1.1.* It follows the lines of the proof of [Chaudouard 2018a, théorème 3.2.1]. For the reader's convenience, we will sketch the main steps and the simplifications in our case. The case where $P = H$ is obvious since then $h \mapsto F^P(h)\tau_P(H_P(h)) = F^H(h)$ is compactly supported on $[H]^1$. So we assume $P \subsetneq G$. We denote by Δ_P the set of simple roots of Z_P in N and ρ_P the half-sum of roots of Z_P on N. It is easy to see that Theorem 2.4.1.1 is a direct consequence of the following majorization.

Lemma 2.4.2.1. *Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ be a weakly cuspidal function. Let Ω be a compact subset of $N(\mathbb{A})M(\mathbb{A})^1K$. There exist $\varepsilon > 0$ and $c_0 > 0$ such that*

$$\exp\bigl(-\langle 2\rho_P, H_P(a)\rangle\bigr)|k_{\mathcal{O}}(f,ah)| \leqslant c_0 \cdot \prod_{\alpha\in\Delta_P} \alpha(a)^{-\varepsilon}$$

for all $h \in \Omega$ and all $a \in A_P$ such that $\tau_P(H_P(a)) = 1$.

Lemma 2.4.2.1 itself is a straightforward consequence of

Lemma 2.4.2.2. *Let f and Ω be as in Lemma 2.4.2.1. Let $\alpha \in \Delta_P$. There exists $c_0 > 0$ such that*

$$\exp\bigl(-\langle 2\rho_P, H_P(a)\rangle\bigr)|k_{\mathcal{O}}(f,ah)| \leqslant c_0 \cdot \alpha(a)^{-1}$$

for all $h \in \Omega$ and all $a \in A_P$ such that $\tau_P(H(a)) = 1$.

2.4.3. *Proof of Lemma 2.4.2.2.* The simple root $\alpha \in \Delta_P$ defines a maximal parabolic subgroup Q that contains P. We denote $Q = LR$ be the standard Levi decomposition of Q where R is the unipotent radical of Q. Let $\bar{R}$ be the unipotent radical of the opposite parabolic subgroup.

Let's denote by $Y \mapsto Y_{\bar{R}}$ the projection of $\mathfrak{s}$ on $\mathfrak{s}_{\bar{R}}$ according to the decomposition $\mathfrak{s} = \mathfrak{s}_R \oplus \mathfrak{s}_L \oplus \mathfrak{s}_{\bar{R}}$. To prove Lemma 2.4.2.2, we split the sum $\sum_{X\in\mathcal{O}} f(\mathrm{Ad}(ah)^{-1}X)$ into the following three contributions:

$$\sum_{X\in\mathcal{O},\, X_{\bar{R}}\neq 0} f(\mathrm{Ad}(ah)^{-1}X)\,; \tag{2-4-3-1}$$

$$\sum_{X\in\mathcal{O}\cap\mathfrak{s}_Q(F)} f(\mathrm{Ad}(ah)^{-1}X)-\sum_{\substack{X\in\mathfrak{s}_L(F)\\ I_Q(X)=\mathcal{O}}}\sum_{Y\in\mathfrak{s}_R(F)} f(\mathrm{Ad}(ah)^{-1}(X+Y))\,; \tag{2-4-3-2}$$

$$\sum_{\substack{X\in\mathfrak{s}_L(F)\\ I_Q(X)=\mathcal{O}}}\sum_{Y\in\mathfrak{s}_R(F)} f(\mathrm{Ad}(ah)^{-1}(X+Y)). \tag{2-4-3-3}$$

For the contribution (2-4-3-1) we have a better majorization (see proof of [Chaudouard 2018a, lemme 3.8.2]): for any integer $k\geqslant 1$, there exists $c_0>0$ such that

$$\exp(-\langle 2\rho_P, H_P(a)\rangle)\Bigg|\sum_{X\in\mathcal{O},X_{\bar{R}}\neq 0} f(\mathrm{Ad}(ah)^{-1}X)\Bigg|\leqslant c_0\cdot\alpha(a)^{-k}$$

for all $h\in\Omega$ and all $a\in A_P$ such that $\tau_P(H(a))=1$. For the contribution (2-4-3-3), we have the same kind of majorization. But to see it, we need to introduce a nontrivial continuous additive character $F\backslash\mathbb{A}\to\mathbb{C}^\times$ and a nondegenerate bilinear form on $\mathfrak{s}$ given by $\langle X,Y\rangle=\mathrm{trace}(XY)$ (this is the trace of an F-endomorphism of V_E). Then, using the Poisson summation formula for the sum over $\mathfrak{s}_R(F)$ and the fact that f is weakly cuspidal, one gets

$$\begin{aligned}\sum_{\substack{X\in\mathfrak{s}_L(F)\\ I_Q(X)=\mathcal{O}}}\sum_{Y\in\mathfrak{s}_R(F)} & f(\mathrm{Ad}(ah)^{-1}(X+Y))\\ &=\sum_{\substack{X\in\mathfrak{s}_L(F)\\ I_Q(X)=\mathcal{O}}}\sum_{\substack{Y\in\mathfrak{s}_{\bar{R}}(F)\\ Y\neq 0}}\int_{\mathfrak{s}_R(\mathbb{A})} f(\mathrm{Ad}(ah)^{-1}(X+U))\psi(\langle Y,U\rangle)\,dU.\end{aligned}$$

At this point we can conclude as in the proof of [Chaudouard 2018a, lemme 3.8.3]. The most difficult contribution is thus (2-4-3-2) which can be written as

$$\sum_{\substack{X\in\mathfrak{s}_L(F)\\ I_Q(X)\neq\mathcal{O}}}\sum_{\substack{Y\in\mathfrak{s}_R(F)\\ X+Y\in\mathcal{O}}} f(\mathrm{Ad}(ah)^{-1}X)-\sum_{\substack{X\in\mathfrak{s}_L(F)\\ I_Q(X)=\mathcal{O}}}\sum_{\substack{Y\in\mathfrak{s}_R(F)\\ X+Y\notin\mathcal{O}}} f(\mathrm{Ad}(ah)^{-1}(X+Y)).$$

But then the argument is that used in [Chaudouard 2018a, §3.12] with Lemma 2.2.3.1 above playing the role of [Chaudouard 2018a, lemme 2.9.1].

2.4.4. *Proof of Theorem 2.3.4.1*. Theorem 2.3.4.1 is a straightforward consequence of Theorem 2.4.1.1 and the equality for any $h\in H(\mathbb{A})$

$$\sum_{P}\sum_{\delta\in P(F)\backslash H(F)} F^P(\delta h)\tau_P(H_P(\delta h))=1, \tag{2-4-4-4}$$

where the sum is over all standard parabolic subgroups P of H (see [Chaudouard 2018b, proposition 2.5.1]).

2.5. *A limit formula for nilpotent contributions.*

2.5.1. In this section, we will get an expression for $J_{\mathcal{O}}(f)$ as the residue at $s=0$ of a function $J_{\mathcal{O}}(f,s)$ when f is a weakly cuspidal function. In the next section, under a mild condition on f, the function $J_{\mathcal{O}}(f,s)$ is expressed in terms of a zeta integral. In our context, this is a simple analog of results in [Chaudouard 2018b]. The homogeneity of $J_{\mathcal{O}}(f)$ is then an easy consequence of this result. Once again, we borrow notations from [Chaudouard 2018b]: we will denote by E^H the function $E^{\mathrm{GL}_E(n)}$ defined in [Chaudouard 2018b, §3.2 équation (3.2.1)]: it is a characteristic function on $[H]$.

2.5.2. Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $\mathcal{O} \in \mathcal{N}(F)/H(F)$ be a nilpotent orbit.

Theorem 2.5.2.1. *Assume that f is weakly cuspidal. Then the integral*

$$J_{\mathcal{O}}(f,s) = \int_{[H]} E^H(h) k_{\mathcal{O}}(f,h) |\det(h)|^s \, dh$$

is absolutely convergent for $s \in \mathbb{C}$ with $\Re(s) > 0$. Moreover,

$$\lim_{s \to 0^+} s J_{\mathcal{O}}(f,s) = J_{\mathcal{O}}(f),$$

where $\lim_{s\to 0^+}$ means that the limit is taken over complex numbers s such that $\Re(s) > 0$.

Proof. It is a variation on the proof of théorème 4.6.1 of [Chaudouard 2018b]. In fact, the hypothesis "weakly cuspidal" makes the situation even simpler.

One shows that for a standard parabolic subgroup $P \subset H$ the integral

$$J_{\mathcal{O}}^P(f,s) = \int_{P(F)\backslash H(\mathbb{A})} E^H(h) F^P(h) \tau_P(H_P(h)) k_{\mathcal{O}}(f,h) |\det(h)|^s \, dh$$

is absolutely convergent for $s \in \mathbb{C}$ with $\Re(s) > 0$. The convergence is a straightforward consequence of Theorem 2.4.1.1 and the fact that the function $E^H(h)$ has a simple expression when

$$F^P(h)\tau_P(H_P(h)) = 1,$$

see [ibid., (3.2.1)]. Moreover (as in the proof of [ibid., proposition 4.5.1]), one has

$$\lim_{s\to 0^+} s J_{\mathcal{O}}^P(f,s) = \int_{P(F)\backslash H(\mathbb{A})^1} F^P(h) \tau_P(H_P(h)) k_{\mathcal{O}}(f,h) \, dh.$$

One gets the theorem by adding the contributions of the various parabolic subgroups P, see (2-4-4-4). □

2.6. *Computation of a nilpotent integral.*

2.6.1. Let $X \in \mathcal{N}(F)$. Using the proof of Lemma 2.2.1.1, one can show that there exist

- an integer $r \geqslant 1$;
- a decomposition
$$V_E = \bigoplus_{1 \leqslant i \leqslant j \leqslant r} V_j^i,$$
$d_j = \dim(V_j^i)$ does not depend on i;
- a basis $(e_{k,j}^i)_{1 \leqslant k \leqslant d_j}$ of V_j^i;
- in the basis $(e_{k,j}^i)_{1 \leqslant i \leqslant j \leqslant r, 1 \leqslant k \leqslant d_j}$ we have
$$X e_{k,j}^i = \begin{cases} e_{k,j}^{i-1} & \text{if } i > 1, \\ 0 & \text{if } i = 1. \end{cases}$$

We may order the basis by $e_{k,j}^i < e_{k',j'}^{i'}$ if and only if one the following conditions are satisfied:

- $i < i'$;
- $i = i'$ and $j > j'$;
- $i = i'$, $j = j'$ and $k < k'$.

In this basis, the matrix of X is given by

$$\begin{pmatrix}
0_{d_1+\cdots+d_r} & I_{d_2+\cdots+d_r} & 0 & \cdots & 0 & 0 \\
 & 0\{\vdots & \vdots & & \vdots & \vdots \\
 & 0_{d_2+\cdots+d_r} & & & 0 & 0 \\
 & & & \ddots & \vdots & \vdots \\
 & & \ddots & & I_{d_{r-1}+d_r} & 0 \\
 & & & & 0\{\vdots & \vdots \\
 & & & & 0_{d_{r-1}+d_r} & I_{d_r} \\
 & & & & & 0\{\vdots \\
 & & & & & 0_{d_r}
\end{pmatrix}.$$

In the following we may replace X by a conjugate. So we can and we shall assume that the ordered basis $(e_{k,j}^i)$ is the basis chosen in Section 2.1.5. Thanks to this basis, we shall also identify the group $H(F)$ with $\mathrm{GL}(n, E)$ and the F-space $\mathfrak{s}$ with the space $\mathfrak{gl}(n, E)$ of $n \times n$ matrices with coefficients in E. Note that through these identifications, the action Ad of the group $H(F)$ on $\mathfrak{s}$ is the σ-conjugation of $\mathrm{GL}(n, E)$ on $\mathfrak{gl}(n, E)$ (given by $(y, Y) \mapsto yY\sigma(y)^{-1}$).

2.6.2. Let

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n},$$

where

- $\mathfrak{m} = \bigoplus_{1 \leqslant i \leqslant j \leqslant r} \mathrm{Hom}_E(V_j^i, V_j^i)$;
- $\mathfrak{n} = \bigoplus \mathrm{Hom}_E(V_j^i, V_{j'}^{i'})$, where the sum is over $1 \leqslant i \leqslant j \leqslant r$ and $1 \leqslant i' \leqslant j' \leqslant r$ such that $i > i'$ or $i = i'$ and $j < j'$.

Let M and N be the F-subgroups of H of Lie algebras $\mathfrak{m}$ and $\mathfrak{n}$. We get a parabolic subgroup $P = MN$ with a Levi decomposition. Recall that H_X denotes the centralizer of X in H. Let $M_X = M \cap H_X$ and $N_X = N \cap H_X$. One has $H_X \subset P$ and $H_X = M_X N_X$ is a Levi decomposition. The groups M_X and M can be respectively identified to $\prod_{1 \leqslant j \leqslant r} \mathrm{GL}_E(V_j^j)$ and $\prod_{1 \leqslant i \leqslant j \leqslant r} \mathrm{GL}_E(V_j^i)$. The inclusion $M_X \subset M$ is given by the diagonal embedding of $\mathrm{GL}_E(V_j^j)$ in $\prod_{1 \leqslant i \leqslant j} \mathrm{GL}_E(V_j^i)$: by "diagonal", we mean that we identify $\mathrm{GL}_E(V_j^j)$ to $\mathrm{GL}_E(V_j^i)$ via the F-isomorphism $V_j^j \simeq V_j^i$ given by X^{j-i}. Let

$$\mathfrak{u}_X = \mathfrak{s} \cap \Big(\bigoplus \mathrm{Hom}_F(V_j^i, V_{j'}^{i'})\Big),$$

where the sum is over $1 < i \leqslant j \leqslant r$ and $1 \leqslant i' \leqslant j' \leqslant r$ such that $i - 1 > i'$ or $i = i' + 1$ and $j < j'$.

Lemma 2.6.2.1. *The map $n \mapsto nXn^{-1} - X$ induces an isomorphism from $N_X \backslash N$ onto $\mathfrak{u}_X$.*

Proof. It is similar to the proof of [Chaudouard 2017, proposition 4.5.1]. □

2.6.3. *Zeta integral.* Let $L = \prod_{1 \leqslant i < j \leqslant r} \mathrm{GL}_E(V_j^i)$. An element A of L is written $(A_{i,j})_{1 \leqslant i < j \leqslant r}$ with $A_{i,j} \in \mathrm{GL}_E(V_j^i)$. For any $A \in L(\mathbb{A})$, let

$$\Delta_A = \left(\begin{array}{cccccc}
0_{d_1+\cdots+d_r} & \Delta_1(A) & 0 & \cdots & 0 & 0 \\
 & 0\{\vdots & \vdots & & \vdots & \vdots \\
 & 0_{d_2+\cdots+d_r} & & \ddots & 0 & 0 \\
 & & \ddots & & \Delta_{r-2}(A) & 0 \\
 & & & & 0\{\vdots & \vdots \\
 & & & & 0_{d_{r-1}+d_r} & \Delta_{r-1}(A) \\
 & & & & & 0\{\vdots \\
 & & & & & 0_{d_r}
\end{array}\right),$$

where

$$\Delta_i(A) = \begin{pmatrix} A_{i,r} & & & \\ & A_{i,r-1} & & \\ & & \ddots & \\ & & & A_{i,i+1} \end{pmatrix}$$

for any $1 \leqslant i \leqslant r-1$.

For any $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ let's define

(2-6-3-1) $$f_X^K(A) = \int_{\mathfrak{u}_X(\mathbb{A})} \int_K f(k^{-1}(U+\Delta_A)\sigma(k))\, dU\, dk$$

and for $s \in \mathbb{C}$ such that $\Re(s) > 0$ the zeta function

(2-6-3-2) $$Z_X(f,s) = \int_{L(\mathbb{A})} f_X^K(A)\delta(A,s)\, dA$$

with

(2-6-3-3) $$\delta(A,s) = \prod_{1\leqslant i<j\leqslant r} |\det(A_{i,j})|^{d_i+\cdots+d_j+(j-i)s}$$

and dA is a Haar measure on $L(\mathbb{A})$ (normalized as in Section 2.1.7). Recall that det is a shortcut for $N_{E/F} \circ \det$ (see Section 2.1.4).

As in [Chaudouard 2018b, §8.3], the integral is convergent and defines a holomorphic function on the domain $\Re(s) > 0$.

Lemma 2.6.3.1. *Let $\mathcal{O}$ be the H-orbit of X. Let v be a place of F. For any $t \in F_v^\times$, we have*

$$Z_X(f_t,s) = |t|_v^{\dim(\mathcal{O})/2+cs} Z_X(f,s),$$

where $c = \sum_{1\leqslant j\leqslant r} j(j-1)d_j$.

Remark 2.6.3.2. As usual in this paper, $\dim(\mathcal{O})$ is the dimension of $\mathcal{O}$ over F.

Proof. To begin with, we have the homogeneity property with the exponent:

$$2\sum_{1\leqslant i<j\leqslant r} d_j(d_i+\cdots+d_j+(j-i)s)+\dim(\mathfrak{u}_X).$$

Indeed, by the change of variables $U \mapsto t^{-1}U$ in the integral (2-6-3-1), we have

$$(f_t)_X^K(A) = |t|_v^{\dim(\mathfrak{u}_X)} f_X^K(t^{-1}A).$$

Then we have by the change of variables $A \mapsto t^{-1}A$ in the integral (2-6-3-2),

$$Z_X(f_t,s) = |t|_v^{\dim(\mathfrak{u}_X)+e} Z_X(f,s),$$

where e satisfies $\delta(tA,s) = |t|^e\delta(A,s)$. By the definition (2-6-3-3) of $\delta(A,s)$ we can compute

$$e = 2\sum_{1\leqslant i<j\leqslant r} d_j(d_i+\cdots+d_j+(j-i)s).$$

The factor 2 is due to our definition of det (see Section 2.1.4).

Since we have

$$2\sum_{1\leqslant i<j\leqslant r} d_j(j-i)s=\sum_{1<j\leqslant r} j(j-1)d_js,$$

it suffices to show that

$$2\sum_{1\leqslant i<j\leqslant r} d_j(d_i+\cdots+d_j)+\dim(\mathfrak{u}_X)=\dim(\mathcal{O})/2.$$

One can compute $\dim(H_X)$ as follows

$$\begin{aligned}\dim(H_X)/2&=\sum_{1\leqslant j,j'\leqslant r} d_jd_{j'}\min(j,j')\\&=2\sum_{1\leqslant i<j\leqslant r} id_id_j+\sum_{1\leqslant j\leqslant r} jd_j^2\\&=2\sum_{1\leqslant i<j\leqslant r} d_j(d_i+\cdots+d_j)-2\sum_{1\leqslant j\leqslant r}(j-1)d_j^2+\sum_{1\leqslant j\leqslant r} jd_j^2\\&=2\sum_{1\leqslant i<j\leqslant r} d_j(d_i+\cdots+d_j)-\sum_{1\leqslant j\leqslant r} jd_j^2+2\sum_{1\leqslant j\leqslant r} d_j^2\\&=2\sum_{1\leqslant i<j\leqslant r} d_j(d_i+\cdots+d_j)-\dim(M)/2+\dim(M_X).\end{aligned}$$

Using the equality $\dim(\mathfrak{u}_X)=\dim(N/N_X)$ (see Lemma 2.6.2.1), we get

$$\begin{aligned}&2\sum_{1\leqslant i<j\leqslant r} d_j(d_i+\cdots+d_j)+\dim(\mathfrak{u}_X)\\&\qquad=\dim(H_X)/2+\dim(M)/2-\dim(M_X)+\dim(N)-\dim(N_X)\\&\qquad=(\dim(M)+2\dim(N))/2-\dim(H_X)/2\\&\qquad=(\dim(H)-\dim(H_X))/2=\dim(\mathcal{O})/2.\end{aligned}$$

□

2.6.4. *Computation of the nilpotent integral.* We will denote by $\theta_X(s)$ the function defined in [Chaudouard 2018b, §7.2] relative to the field E and the datum $(d_1,\ldots,d_r)$ (see Section 2.6.1). The main property of $\theta_X(s)$ we retain is that it is holomorphic for $s\in\mathbb{C}$ such that $\Re(s)>0$.

Theorem 2.6.4.1. *Let $f\in C_c^\infty(\mathfrak{s}(\mathbb{A}))$ and v a place of F. There exists a bound $\eta>0$ such that for any t in $F_v^\times$ such that $|t|_v<\eta$ we have*

$$J_{\mathcal{O}}(f_t,s)=\theta_X(s)\cdot Z_X(f_t,s).$$

Proof. It is analogous to the proof of theorem 9.1.1 of [Chaudouard 2018b]. By the Iwasawa decomposition $H(\mathbb{A})=P(\mathbb{A})K$, we can write

$$g=mnk,\quad \text{with } m=(m_j^i)\in M\simeq\prod\nolimits_{1\leqslant i\leqslant j\leqslant r}\mathrm{GL}_E(V_j^i),\ n\in N(\mathbb{A}),\ k\in K.$$

Let $R = \mathrm{GL}_E(d_1 + \cdots + d_r)$ and $r(g) \in R(\mathbb{A})$ be the matrix extracted from the first $d_1 + \cdots + d_r$ rows and columns of mn. As in [Chaudouard 2018b, p. 115], one shows that $E^G(g) = 1$ if and only if $E^R(r(g)) = 1$. The main new ingredient in our context is the observation that $E^G(g) = E^G(\sigma(g))$. Then the same kind of computations as those of [Chaudouard 2018b, p. 116] leads to the statement (see also [ibid., remarque 9.1.2]). □

2.7. ***Proof of Theorem 2.3.5.1.*** Let $f \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$ be a weakly cuspidal function. Recall that we have defined integrals $J_{\mathcal{O}}(f, s)$ (see Theorem 2.5.2.1). Let η be the bound given by Theorem 2.6.4.1. Let t_0, t in $F_v^\times$ such that $|t_0|_v < \eta$ and $|t|_v \leqslant 1$. By Theorem 2.6.4.1, $J_{\mathcal{O}}(f_{tt_0}, s)$ and $J_{\mathcal{O}}(f_{t_0}, s)$. can be expressed in terms of a zeta function for which we have a homogeneity property (see Lemma 2.6.3.1). We deduce that we have for the constant c of Lemma 2.6.3.1

$$J_{\mathcal{O}}(f_{tt_0}, s) = |t|_v^{\dim(\mathcal{O})/2+cs} J_{\mathcal{O}}(f_{t_0}, s).$$

Taking the product with s and then the limit on $s \to 0^+$ given by Theorem 2.5.2.1, we get the result.

3. The unipotent contribution

3.1. *Algebraic situation.*

3.1.1. We follow the notation of Section 2.1. Let D be a quaternion algebra over F equipped with a fixed embedding $E \hookrightarrow D$. Note that D may be split. Then $V_D = V \otimes_E D$ is a right D-module. Let $G = \mathrm{Aut}_D(V_D)$ viewed as an F-group. Let $\varepsilon \in G$ given by the left multiplication by $\sqrt{\tau}$. Let θ the involution of $\mathrm{End}_F(V_D)$ given by $\theta(X) = \varepsilon X \varepsilon^{-1}$. Let $H' \subset G$ be the subgroup fixed by θ. Let $S' \subset G$ be the F-variety of automorphism $g \in \mathrm{Aut}_D(V_D)$ such that $g\varepsilon = \varepsilon g^{-1}$. The map

$$\rho : g \mapsto g\theta(g)^{-1} \tag{3-1-1-1}$$

induces an isomorphism from G/H' onto S' (see [Guo 1997]). The action of G by left translations on G/H' gives an action of G by θ-conjugation for which ρ is equivariant: we have $\rho(g_1 g_2) = g_1 \rho(g_2)\theta(g_1)^{-1}$. This action induces an action by conjugation of H' on S'.

3.1.2. Let $\mathcal{U} \subset G$ be the unipotent variety. We define $\mathcal{U}_{S'} = \mathcal{U} \cap S'$ and $\mathcal{U}_G = \rho^{-1}(\mathcal{U}_{S'})$. One knows [Guo 1997, Lemma 3.2] that $\mathcal{U}_G(F) = H'(F)\mathcal{U}_{S'}(F)H'(F)$.

3.1.3. The tangent space of S' at Id_{V_D} is denoted by $\mathfrak{s}'$: it is the space of $X \in \mathrm{End}_D(V_D)$ such that $X\varepsilon + \varepsilon X = 0$. Let $\mathcal{N}_V$ be the cone of nilpotent elements in $\mathrm{End}_D(V_D)$. Let $\mathcal{N}' = \mathcal{N}_V \cap \mathfrak{s}'$. The usual exponential map denoted by exp induces an isomorphism from $\mathcal{N}_V$ to $\mathcal{U}$ and also from $\mathcal{N}'$ to $\mathcal{U}_{S'}$.

3.1.4. The map $\mathrm{End}_E(V) \to \mathrm{End}_D(V_D)$ given by $\varphi \mapsto \varphi \otimes \mathrm{Id}_D$ gives an identification of H with H' and $\mathfrak{s}$ with $\mathfrak{s}'$ and $\mathcal{N}$ with $\mathcal{N}'$. Hence we can freely use all the results of Section 2 for the action of H' on $\mathfrak{s}'$.

To simplify the notation, we will suppress the superscript $'$ and we will not distinguish between H and H', S and S', etc. For v a place of F, the measures used on the groups of F_v-points are Haar measures (we do not need any normalization).

3.2. *Main results.*

3.2.1. Let v be a place of F. For any $\Phi \in C_c^\infty(G(F_v))$, there is a unique function, denoted by $\Phi_S \in C_c^\infty(S(F_v))$ such that

$$\Phi_S(\rho(x)) = \int_{H(F_v)} \Phi(xh)\, dh.$$

The map $\Phi \mapsto \Phi_S$ is a surjection from $C_c^\infty(G(F_v))$ onto $C_c^\infty(S(F_v))$.

3.2.2. *Very cuspidal test functions.* We shall say that $\Phi \in C_c^\infty(G(F_v))$ is *very cuspidal* if one has

$$\int_{N(F_v)} \Phi(xny)\, dn = 0$$

for any parabolic subgroup $P \subsetneq G$ and any $x, y \in G(F_v)$. Here N is the unipotent radical of P.

Remark 3.2.2.1. Assume v is finite. Let $\widetilde{\Phi}$ be a matrix coefficient of a supercuspidal representation of $G(F_v)/Z(F_v)$ where Z is the center of G. Let $\Phi \in C_c^\infty(G(F_v))$ such that

$$\int_{Z(F_v)} \Phi(gz)\, dz = \widetilde{\Phi}(g)$$

for any $g \in G(F_v)/Z(F_v)$. Then Φ is very cuspidal.

Note that any very cuspidal function $\Phi \in C_c^\infty(G(F_v))$ is such that

$$\int_{N(F_v)/N_H(F_v)} \Phi_S(\rho(xn))\, dn = 0, \tag{3-2-2-1}$$

where $N_H = N \cap H$ and dn is the quotient of the Haar measures on $N(F_v)$ and $N_H(F_v)$.

3.2.3. *Global setting.* We also have a surjective map $\Phi \mapsto \Phi_S$ from $C_c^\infty(G(\mathbb{A}))$ onto $C_c^\infty(S(\mathbb{A}))$ given by

$$\Phi_S(\rho(x)) = \int_{H(\mathbb{A})} \Phi(xh)\, dh$$

for any $x \in G(\mathbb{A})$.

We shall say that $\Phi \in C_c^\infty(G(\mathbb{A}))$ is *very cuspidal* if there is a place v such that if one writes $\mathbb{A} = F_v \times \mathbb{A}^v$, one has $\Phi = \Phi_v \otimes \Phi^v$ with $\Phi^v \in C_c^\infty(G(\mathbb{A}^v))$ and $\Phi_v \in C_c^\infty(G(F_v))$ is very cuspidal.

3.2.4. Let's define for $\Phi \in C_c^\infty(G(\mathbb{A}))$ and $x, y \in G(\mathbb{A})$

$$K_{\mathcal{U}_G}(\Phi, x, y) = \sum_{\gamma \in \mathcal{U}_G(F)} \Phi(x^{-1}\gamma y)$$

and for $h \in H(\mathbb{A})$

$$K_{\mathcal{U}_S}(\Phi, h) = \sum_{\gamma \in \mathcal{U}_S(F)} \Phi_S(h^{-1}\gamma h).$$

We have the simple relation for $h \in H(\mathbb{A})$

$$\int_{[H]} K_{\mathcal{U}_G}(\Phi, x, h)\, dx = K_{\mathcal{U}_S}(\Phi, h). \tag{3-2-4-2}$$

Theorem 3.2.4.1. *Let $\Phi \in C_c^\infty(G(\mathbb{A}))$ be a very cuspidal function. Let $\mathcal{O} \in \mathcal{N}(F)/H(F)$ and let v be a place of F.*

(1) *For any $t \in F_v^\times$, the integral*

$$J_{\mathcal{O}}^t(\Phi) = \int_{[H]^1} \sum_{X \in \mathcal{O}} \Phi_S(h^{-1}\exp(t^{-1}X)h)\, dh$$

is absolutely convergent. For $t = 1$, $J_{\mathcal{O}}^1(\Phi)$ is denoted by $J_{\mathcal{O}}(\Phi)$.

(2) (*Fine expansion*) *We have*

$$\int_{[H]^1} K_{\mathcal{U}_S}(\Phi, h)\, dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} J_{\mathcal{O}}(\Phi),$$

where the left-hand side is absolutely convergent.

(3) *There exists a bound $\eta > 0$ such that for any t in $F_v^\times$ such that $|t|_v < \eta$, we have*

$$J_{\mathcal{O}}^t(\Phi) = \lim_{s \to 0^+} s\theta_X(s) Z_X(f_t, s),$$

where $X \in \mathcal{O}$ is the element considered in Section 2.6.1, $\theta_X(s)$ (defined in Section 2.6.4) *is holomorphic for $\Re(s) > 0$ and does not depend on Φ, the zeta function $Z_X(f_t, s)$ is defined in Section 2.6.3 relative to any function $f_t \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$ such that $f_t(Y) = \Phi_S(\exp(t^{-1}Y))$.*

(4) *Let η be the bound of (3). For any t_0, t in $F_v^\times$ such that $|t_0|_v < \eta$ and $|t|_v \leqslant 1$ we have*

$$J_{\mathcal{O}}^{tt_0}(\Phi) = |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}^{t_0}(\Phi).$$

Proof. Theorem 3.2.4.1 is deduced from similar results on $\mathfrak{s}$ and the (standard) descent procedure to $\mathfrak{s}$ that is explained in Section 3.2.5. More precisely, assertion (1) results from Theorem 2.3.4.1. Assertion (2) is a consequence of assertion (1) and the finiteness of nilpotent orbits. Assertion (3) is a combination of Theorem 2.5.2.1 and 2.6.4.1. Finally, assertion (4) is a consequence of assertion (3) (see Section 2.7). □

Let's state a corollary which is a straightforward consequence of Theorem 3.2.4.1.

Corollary 3.2.4.2. (We use notation of Theorem 3.2.4.1.) *For any t_0, t in $F_v^\times$ such that $|t_0|_v < \eta$ and $|t|_v \leqslant 1$ we have*

$$\int_{[H]^1} \Big| \sum_{X \in \mathcal{N}(F)} \Phi_S\big(h^{-1} \exp((tt_0)^{-1} X) h\big) \Big| \, dh < \infty$$

and

$$\int_{[H]^1} \sum_{X \in \mathcal{N}(F)} \Phi_S(h^{-1} \exp((tt_0)^{-1} X) h) \, dh = \sum_{\mathcal{O} \in \mathcal{N}(F)/H(F)} |t|_v^{\dim(\mathcal{O})/2} J_{\mathcal{O}}^{t_0}(\Phi).$$

In particular, when $t \in F_v^\times$ goes to 0, the expression

$$\int_{[H]^1} \sum_{X \in \mathcal{N}(F)} \Phi_S(h^{-1} \exp(t^{-1} X) h) \, dh$$

is equivalent to

$$\mathrm{vol}([H]^1) \int_{H(\mathbb{A})} \Phi(h) \, dh.$$

Remark 3.2.4.3. For Φ very cuspidal, we have the equality

$$\int_{[H]^1} \int_{[H]} K_{\mathcal{U}_G}(\Phi, x, y) \, dx dy = \int_{[H]^1} K_{\mathcal{U}_S}(\Phi, h) \, dh,$$

where the left-hand side is at least conditionally convergent. One can prove in fact that it is absolutely convergent using mixed truncation operators (in the sense of the seminal paper [Jacquet et al. 1999]) built upon the combinatorics of [Li 2020].

3.2.5. *Descent to the tangent space.* Let $\mathcal{V}_0$ be a finite set of places of F containing the archimedean places and a fixed place denoted by v_0. Let $A \subset F$ the ring of integers outside $\mathcal{V}_0$. We assume that $\mathcal{V}_0$ is large enough such that all objects G, H, S, $\mathcal{U}_S$, $\mathcal{N}$ come naturally from A-schemes by base change. We assume also that the exponential (denoted by exp) induces an isomorphism of A-scheme from $\mathcal{N}$ to $\mathcal{U}_S$. For $v \notin \mathcal{V}_0$, let $\mathcal{O}_v \subset F_v$ be the ring of integers.

Let

$$\Phi = \Phi_0 \otimes \Phi_1 \otimes \Phi_2,$$

where $\Phi_0 \in C_c^\infty(G(F_0))$ (with $F_0 = F_{v_0}$) is a very cuspidal function, Φ_1 is a test function on

$$F_1 = \prod_{v \in \mathcal{V}_0 \setminus \{v_0\}} F_v$$

and Φ_2 is the characteristic function of $\prod_{v \notin \mathcal{V}_0} G(\mathcal{O}_v)$.

Let $c : \mathfrak{s} \to \mathfrak{c} = \mathfrak{s}//H$ be the categorical quotient. For all $v \in \mathcal{V}_0$, we fix an open subset $\omega_v^\flat \subset \mathfrak{c}(F_v)$ containing $c(0)$ such that the exponential map exp is well-defined and induces an analytic diffeomorphism from $\omega_v = c^{-1}(\omega_v^\flat)$ onto an open subset $\Omega_v \subset S(F_v)$. Let ζ_v be a smooth function on $\mathfrak{c}(F_v)$ with compact support included in $\omega_v^\flat$ and with value 1 in a neighborhood of $c(0)$.

Let's define $\zeta_0 = \zeta_{v_0}$ and $\omega_0 = \omega_{v_0}$ and also $\zeta_1 = \prod_{v \in \mathcal{V}_0 \setminus \{v_0\}} \zeta_v$ and $\omega_1 = \prod_{v \in \mathcal{V}_0 \setminus \{v_0\}} \omega_v$. We define functions $f_0 \in C_c^\infty(\mathfrak{s}(F_0))$ and $f_1 \in C_c^\infty(\mathfrak{s}(F_1))$ by

$$f_i(X) = \begin{cases} \zeta_i(X)\Phi_{i,S}(\exp(X)) & \text{if } X \in \omega_i; \\ 0 & \text{otherwise;} \end{cases}$$

for any $i \in \{0, 1\}$ and $X \in \mathfrak{s}(F_i)$. Let $f = f_0 \otimes f_1 \otimes f_2 \in C_c^\infty(\mathfrak{s}(\mathbb{A}))$, where f_2 is the characteristic function of $\prod_{v \notin \mathcal{V}_0} \mathfrak{s}(\mathcal{O}_v)$.

We have for all $X \in \mathcal{N}(\mathbb{A})$

$$\Phi_S(h^{-1} \exp(X) h) = f(\mathrm{Ad}(h^{-1})(X)).$$

To complete the proof of Theorem 3.2.4.1 it suffices to check the next lemma.

Lemma 3.2.5.1. *The function f is weakly cuspidal* (in the sense of Section 2.3.3).

Proof. Let $X \in \mathfrak{s}_M(F_0) \cap \mathcal{N}(F_0)$. Clearly, it suffices to prove that

$$\int_{\mathfrak{s}_N(F_0)} f_0(\mathrm{Ad}(x)(X+U))\, dU = 0$$

for any $x \in H(F_v)$. But for any $U \in \mathfrak{s}_N(F_0)$ we have

$$f_0(\mathrm{Ad}(x)(X+U)) = \Phi_{0,S}(x \exp(X+U) x^{-1}).$$

Let's write

$$\exp(X+U) = \exp(X/2)\big(\exp(-X/2)\exp(X+U)\exp(-X/2)\big)\theta(\exp(X/2))^{-1}.$$

We observe that $U \mapsto \exp(-X/2)\exp(X+U)\exp(-X/2)$ induces an isomorphism from $\mathfrak{s}_N$ onto $S \cap N$. But the map ρ of (3-1-1-1) induces an isomorphism from N/N_H onto $S \cap N$. We get a bijection from $N(F_0)/N_H(F_0)$ onto $\mathfrak{s}_N(F_0)$. By a change of variables, we get (up to a constant $c \neq 0$)

$$\int_{\mathfrak{s}_N(F_0)} f_0(\mathrm{Ad}(x)(X+U))\, dU = c \int_{N(F_0)/N_H(F_0)} \Phi_{0,S}(\rho(x \exp(X/2) n))\, dn = 0$$

by the vanishing condition (3-2-2-1). ☐

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STRONGLY ALGEBRAIC REALIZATION OF DIHEDRAL GROUP ACTIONS

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Let D_{2q} be the dihedral group with $2q$ elements and suppose that q is not divisible by 4. Let M be a closed smooth D_{2q}-manifold. Then there exists a nonsingular real algebraic D_{2q}-variety X which is equivariantly diffeomorphic to M and all D_{2q}-vector bundles over X are strongly algebraic.

1. Introduction

Suppose G is a compact Lie group and Ω is an orthogonal representation of G, also called a real G-module. A *real algebraic G-variety X*, see Definition 3.1 or [Dovermann and Masuda 1995], is a G-invariant common set of zeros of a finite collection of polynomials. The action on X is given as the restriction of the action on Ω. We use the term *nonsingular* with its classical meaning, see [Whitney 1957] or [Bochnak et al. 1987, Section 3.3]. If M is a closed smooth G-manifold and X is a nonsingular real algebraic G-variety that is equivariantly diffeomorphic to M, then we say that M is *algebraically realized* and that X is an *algebraic model* of M. We call X a *strongly algebraic* model of M if, in addition, all G-vector bundles over X are strongly algebraic. This means that the bundles are classified, up to equivariant homotopy, by equivariant entire rational maps to equivariant Grassmannians with their canonical algebraic structure, see Section 3C. Existing results motivate:

Conjecture 1.1 [Dovermann et al. 1994, p. 32]. *Let G be a compact Lie group. Then every closed smooth G-manifold has a strongly algebraic model.*

Our main result verifies the conjecture in a special case:

Theorem 1.2. *Every closed smooth D_{2q}-manifold, q not divisible by 4, has a strongly algebraic model.*

Nash [1952] had asked whether every closed smooth manifold has an algebraic model, and this was confirmed by Tognoli [1973]. Benedetti and Tognoli [1980] showed that every closed smooth manifold has a strongly algebraic model. See also [Akbulut and King 1981; 1992].

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We have confirmed Conjecture 1.1 in special cases. They include the case where G is the product of an odd order group and a 2-torus [Dovermann et al. 1994, Theorem B] and the case where G is cyclic; see [Dovermann and Wasserman 2008; 2019; Dovermann et al. 2017]. The new difficulty that we face in this paper is that Sylow 2 subgroups of dihedral groups are not central, and being able to deal with this adds credence to the conjecture. With the goal of proving Conjecture 1.1 in greater generality, we are developing ideas and tools to overcome difficulties in its proof.

To prove Proposition 2.3 we combine results from representation theory with extensions of the process of simplifying isotropy structures via blow-ups as applied in [Dovermann and Wasserman 2008; Wasserman 1997]. To prove Theorem 2.4, and its bordism theoretic reformulation as Theorem 3.7, we make creative use of the existing literature.

Algebraic realization problems translate to unoriented bordism problems. One expects 2-Sylow subgroups to play a crucial role. If $G = D_{2q}$ and q is not divisible by 4, then any of its 2-Sylow subgroups G_2 is at most of order 4, and for such groups Conjecture 1.1 has been verified. If 4 divides q, then G_2 is dihedral and has at least 8 elements. The proof of Conjecture 1.1 for this group may require extensive equivariant bordism computations. The concept of being iso-special (see Definition 2.1) will be inadequate. Locally we will have to accept three isotropy groups.

2. Outline of Proof

Throughout, until Section 7, we assume that $G = D_{2q}$ and q is odd. In Section 7 we deduce Theorem 1.2 for $G = D_{2q}$ when 2 divides q, but not 4, from the case when q is odd.

Definition 2.1. A smooth G-manifold is said to be *iso-special* if locally the action has at most two isotropy groups. If there are two isotropy groups, say H and K with $K \subset H$, then we assume that $[H : K] = 2$ and the codimension of the H-fixed point set in the K-fixed point set is 1.

Blow-ups (see [Hirzebruch 1966, p. 175f.; Stong 1970, p. 41]) were used in [Wasserman 1997] to simplify the isotropy structure of a manifold, at least in case of abelian group actions. Our term iso-special corresponds to the term *nonsingular* in [Wasserman 1997, Definition 18]. We will review the process of a blow-up in Section 4. If M is a smooth G-manifold and N is a G-invariant submanifold, then we denote the blow-up of M along N by $B(M, N)$. Previously we have shown:

Proposition 2.2 [Dovermann and Wasserman 2008, Section 4]. *If N and $B(M, N)$ have strongly algebraic models, then so does M.*

In this paper we will show the following two assertions:

Proposition 2.3. *Let M be a closed smooth D_{2q}-manifold, q odd. Then there exists a finite sequence of equivariant blow-ups*

$$M_0 = M, \quad M_1 = B(M_0, A_0), \quad \dots, \quad M_k = B(M_{k-1}, A_{k-1}) \tag{2-1}$$

so that M_k and each A_i, $0 \le i \le k-1$, are iso-special.

Theorem 2.4. *Every iso-special closed smooth D_{2q}-manifold, q odd, has a strongly algebraic model.*

The definition of being iso-special is designed so that in combination with the blow-up procedure the proof of Conjecture 1.1 reduces to the special case of iso-special manifolds. Proposition 2.3 is proved in Section 4. In Section 3E we deduce Theorem 2.4 from a bordism theoretic assertion, Theorem 3.7. The proof of Theorem 3.7 occupies the later sections of the paper.

Proof of Theorem 1.2, q odd. Suppose M has a blow-up sequence as in (2-1). Theorem 2.4 tells us that M_k and A_{k-1} have strongly algebraic models. Proposition 2.2 tells us that M_{k-1} has a strongly algebraic model. Proceeding inductively, we conclude that M has a strongly algebraic model. □

3. Notation, definitions, and background material

The dihedral group D_{2q} is generated by two elements that we call a and b, subject to the relations $a^2 = b^q = e$ and $aba = b^{-1}$. We write T for the subgroup generated by a and $\mathbb{Z}_q$ for the subgroup generated by b. If $q = rq'$ then a and b^r generate a subgroup of D_{2q} that we denote by $D_{2q'}$. It has the subgroup T generated by a and a subgroup $\mathbb{Z}_{q'}$ generated by b^r.

3A. ***Representations of the dihedral groups.*** The dihedral group is ambivalent: its elements are conjugate to their inverses. For finite groups being ambivalent is equivalent to all characters being real [Isaacs 1976, p. 31]. The number of real, as well as complex, irreducible representations is equal to the number of conjugacy classes of elements of the group. The complex irreducible representations are complexifications of real irreducible representations. This, and more, follows from the Frobenius–Schur indicator; see [Serre 1977, p. 90ff.].

Specifically, if q is odd, then there are $\frac{q+3}{2}$ conjugacy classes of elements and irreducible representations, of which 2 are of dimension 1 and $\frac{q-1}{2}$ are of dimension 2. If q is even, then there are $\frac{q+6}{2}$ irreducible representations, of which 4 are of dimension 1 and $\frac{q-2}{2}$ are of dimension 2. They are described in [Serre 1977, p. 37f.].

Suppose $q > 1$ is odd. The trivial representation, denoted by $\mathbb{R}$, is one of the real irreducible representations of dimension 1. We denote the other one by $\mathbb{R}_-$. The element $a \in D_{2q}$ acts by multiplication with -1, while b acts trivially. The remaining real irreducible representations are of dimension 2. The generators act by multiplication with the matrices

$$\theta(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \theta(b) = \begin{pmatrix} \cos(2\pi j/q) & \sin(2\pi j/q) \\ -\sin(2\pi j/q) & \cos(2\pi j/q) \end{pmatrix} \tag{3-1}$$

for $1 \le j \le (q-1)/2$.

3B. ***Real algebraic varieties and entire rational maps.*** Let G be a compact Lie group and Ω an orthogonal representation of G. We think of an orthogonal representation as an underlying Euclidean space $\mathbb{R}^n$ together with an action of G via orthogonal maps.

Definition 3.1. A *real algebraic G-variety* is a G-invariant, common set of zeros of a finite set of polynomials $p_1, \dots, p_m : \Omega \to \mathbb{R}$:

$$V = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\}.$$

The action of G on Ω restricts to an action on V. We use the Euclidean topology on varieties and the term *nonsingular* with its standard meaning [Whitney 1957].

Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ be real algebraic varieties. A map $f : V \to W$ is said to be *regular* if it extends to a map $F : \mathbb{R}^n \to \mathbb{R}^m$ such that each of its coordinates F_i (i.e., $F_i = \delta_i \circ F : \mathbb{R}^n \to \mathbb{R}$, where $\delta_i : \mathbb{R}^m \to \mathbb{R}$ is the projection on the i-th coordinate) is a polynomial. We say that f is *entire rational* if there are regular maps $p : \mathbb{R}^n \to \mathbb{R}^m$ and $q : \mathbb{R}^n \to \mathbb{R}$, such that $f = p/q$ on V and q does not vanish anywhere on V.

These concepts generalize naturally to the equivariant setting.

3C. ***Grassmannians and classification of vector bundles.*** A good reference is [Bochnak et al. 1987, §3.4]. Let Λ stand for $\mathbb{R}$ or $\mathbb{C}$. Let Ξ be a representation of G over Λ. Its underlying space is Λ^n for some n. We assume that the action of G preserves the standard bilinear form on Λ^n. Let $\operatorname{End}_\Lambda(\Xi)$ denote the set of endomorphisms of Ξ over Λ. It is a representation of G with the action given by

$$G \times \operatorname{End}_\Lambda(\Xi) \to \operatorname{End}_\Lambda(\Xi) \quad \text{with} \quad (g, L) \mapsto gLg^{-1}.$$

Let d be a natural number. We set

$$G_\Lambda(\Xi, d) = \{L \in \operatorname{End}_\Lambda(\Xi) \mid L^2 = L,\ L^* = L,\ \operatorname{trace} L = d\} \tag{3-2}$$

$$E_\Lambda(\Xi, d) = \{(L, u) \in \operatorname{End}_\Lambda(\Xi) \times \Xi \mid L \in G_\Lambda(\Xi, d),\ Lu = u\} \tag{3-3}$$

$$\gamma_\Lambda(\Xi, d) = (p : E_\Lambda(\Xi, d) \to G_\Lambda(\Xi, d)). \tag{3-4}$$

Here L^* denotes the adjoint of L. If one chooses an orthogonal (or unitary) basis of Ξ, then $\operatorname{End}_\Lambda(\Xi)$ is canonically identified with the set of $n \times n$ matrices, and L^* is obtained by transposing L and conjugating its entries. This description specifies $G_\Lambda(\Xi, d)$ and $E_\Lambda(\Xi, d)$ as real algebraic G-varieties. These varieties are nonsingular. The map in (3-4) is projection on the first factor, and $\gamma_\Lambda(\Xi, d)$ is an equivariant vector bundle. Its base and total space are nonsingular real algebraic varieties, and the projection map is regular, hence entire rational.

Proposition 3.2. *The variety $G_\Lambda(\Xi, d)$ is the Grassmannian consisting of real* (*resp. complex*) *subspaces of Ξ of real* (*resp. complex*) *dimension d.*

Proof. There is a bijection between subspaces of Ξ and orthogonal (resp. unitary) projections. To a projection one associates its image. □

We may take larger and larger representations Ξ of G and form a direct limit. At the same time, we can take direct limits of $G_\Lambda(\Xi, d)$, $E_\Lambda(\Xi, d)$ and $\gamma_\Lambda(\Xi, d)$. We call Ξ a universe if it contains each irreducible representation of G an infinite number of times. If Ξ is a universe, then $G_\Lambda(\Xi, d)$ is a classifying space for G-vector bundles of dimension d over nice space, like finite G-CW complexes. There is a one-to-one correspondence between isomorphism classes of G-vector bundles of dimension d over a G-CW complex X and equivariant homotopy classes from X to $G_\mathbb{R}(\Xi, d)$. The nonequivariant proof in [Milnor and Stasheff 1974, §5] generalizes easily. See also [Segal 1968, §2; Wasserman 1969].

In the context of our discussion of strongly algebraic vector bundles, we like $G_\mathbb{R}(\Xi, d)$ to be a variety, which is the case as long as Ξ is of finite dimension. Depending on the bundle classification problem, one may get away using a finite-dimensional representation Ξ. For instance, for any G-CW complex X of dimension k, $G_\mathbb{R}(\Xi, d)$ classifies G-vector bundles of dimension d over X if each irreducible representation of G occurs with multiplicity at least $k + d + 1$ in Ξ.

3D. ***Strongly algebraic vector bundles.*** In our setting the preferred concept of a vector bundle is the one of a strongly algebraic vector bundle. See also [Bochnak et al. 1987, §12.1]. One has this notion with real, $\Lambda = \mathbb{R}$, as well as complex, $\Lambda = \mathbb{C}$, coefficients.

Definition 3.3. A *strongly algebraic G-vector bundle* over a real algebraic G-variety is a bundle whose classifying map to $G_\Lambda(\Xi, d)$ is equivariantly homotopic to an equivariant entire rational map.

Occasionally, we think of G-vector bundles as equivariant maps to a Grassmannian $G_\Lambda(\Xi, d)$. Then we need to allow stabilization of Ξ.

3E. ***Results from the literature.*** We will use:

Proposition 3.4 [Dovermann et al. 1994, Proposition 2.13]. *Let G be a compact Lie group and M a closed smooth G-manifold. Suppose that for every finite collection of G-vector bundles over M there is an algebraic model X, such that each bundle in* *this collection, pulled back over X, is strongly algebraic. Then M has an algebraic model over which all G-vector bundles are strongly algebraic.*

Suppose Y is a nonsingular real algebraic G-variety. It is convenient to call $\mu : X \to Y$ an *algebraic map* if X is a nonsingular real algebraic G-variety and μ is equivariant and entire rational. Suppose M is a closed smooth G-manifold and $f : M \to Y$ is equivariant. We call an algebraic map $\mu : X \to Y$ an *algebraic model* of $f : M \to Y$ if there is an equivariant diffeomorphism $\Phi : M \to X$ so that f is equivariantly homotopic to $\mu \circ \Phi$.

Theorem 3.5 [Dovermann et al. 1994, Theorem C]. *Let G be a compact Lie group. An equivariant map from a closed smooth G-manifold to a nonsingular real algebraic G-variety has an algebraic model if and only if its cobordism class has an algebraic representative.*

3F. ***Bordism formulation.*** Consider a finite product of Grassmannians:

$$\mathfrak{G} = G_{\mathbb{R}}(\Xi, d_1) \times \cdots \times G_{\mathbb{R}}(\Xi, d_k), \tag{3-5}$$

where Ξ is a sufficiently large representation of G and $d_1, \ldots, d_k$ is a sequence of natural numbers. Such a space is used as a classifying space for a collection of k bundles.

Let $\mathcal{S}(G)$ be the set of all subgroups of G and $\mathcal{H} \subseteq \mathcal{S}(G)$. We say that a G-manifold M is of *type* $\mathcal{H}$ if the isotropy groups G_x belong to $\mathcal{H}$ for all $x \in M$. Recall that $G_{gx} = gG_xg^{-1}$, so we should assume that $\mathcal{H}$ is invariant under conjugation. To avoid listing all elements in a conjugacy class:

Notation 3.6. We write $\mathcal{K}^\bullet$ for the closure of $\mathcal{K} \subseteq \mathcal{S}(G)$ under conjugation.

We adopt the notation used in [Stong 1970, §2; 1977]. We use $\mathcal{N}_k^G(Y)$ to denote G equivariant unoriented bordism classes of equivariant maps $f : M \to Y$ from closed G-manifolds of dimension k to a G space Y. Given a family $\mathcal{F}$ of subgroups of G we write $\mathcal{N}_k^G[\mathcal{F}](Y)$ to indicate that the isotropy groups for the domain M of the map are assumed to be in $\mathcal{F}$. The same restriction on the isotropy groups of the domain applies to a bordism between two maps. In the iso-special case we add a subscript c and write $\mathcal{N}_{r,c}^G[\{H, K\}^\bullet](Z)$ to indicate that the codimension of the H-fixed point set in the K-fixed point is one. Eventually we will prove:

Theorem 3.7. *Let $G = D_{2q}$ be the dihedral group, where q is odd, and $\mathfrak{G}$ is as in (3-5).*

(1) *If H is a subgroup of G, then all classes in $\mathcal{N}_*^G[\{H\}^\bullet](\mathfrak{G})$ have algebraic representatives.*

(2) *If H and K are two subgroups of G and $[H : K] = 2$, then all classes in $\mathcal{N}_{r,c}^G[\{H, K\}^\bullet](\mathfrak{G})$ have algebraic representatives.*

Deduce Theorem 2.4 from Theorem 3.7. Let M be a closed, smooth, and iso-special D_{2q}-manifold. Consider a finite collection $\xi_1, \dots, \xi_k$ of D_{2q}-vector bundles over M. Classify it by a map χ into a product of Grassmannians $\mathfrak{G}$ as in (3-5). Then $\chi = \chi_1 \times \cdots \times \chi_k$ where the individual χ_j classify the bundles ξ_j. According to Theorem 3.7, the bordism class of $\chi : M \to \mathfrak{G}$ has an algebraic representative. According to Theorem 3.5, there is an algebraic model $\widehat{\chi} : X \to \mathfrak{G}$ for $\chi : M \to \mathfrak{G}$. Then $\widehat{\chi} = \widehat{\chi}_1 \times \cdots \times \widehat{\chi}_k$. The $\widehat{\chi}_j$ are entire rational, up to equivariant homotopy, and they classify strongly algebraic bundles. Being able to do this for every collection $\xi_1, \dots, \xi_k$ of D_{2q}-vector bundles over M implies, according to Proposition 3.4, that M has a strongly algebraic model. □

4. Blow-ups

In this section we recall the definition of blow-ups and study their effect on the isotropy structure of a G-manifold in the special case where $G = D_{2q}$ and q is odd.

Construction 4.1. Let M be a closed smooth G-manifold with a collection $\xi_1, \dots, \xi_k$ of G-vector bundles over it. Let N be a G-invariant submanifold of M with normal bundle ν. Denote the trivial representation by $\mathbb{R}$ and the product bundle with fibre $\mathbb{R}$ by $\underline{\mathbb{R}}$. We may restrict $\xi_1, \dots \xi_k$ over N and then use the projection $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}) \to N$ to pull the bundles back over $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}})$. The resulting bundles are called $\overline{\xi_1}, \dots, \overline{\xi_k}$.

We may identify $(M, \xi_1, \dots, \xi_k)$ and $(\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}), \overline{\xi_1}, \dots, \overline{\xi_k})$ along a neighbourhood of N that is contained in M and $\mathbb{R}P(\nu \oplus \underline{\mathbb{R}})$. The result is commonly called the *blow-up* of $(M, \xi_1, \dots, \xi_k)$ along N. It is denoted by $B((M, \xi_1, \dots, \xi_k), N)$. By construction,

$$B((M, \xi_1, \dots, \xi_k), N) \sim (M, \xi_1, \dots, \xi_k) \sqcup (\mathbb{R}P(\nu \oplus \underline{\mathbb{R}}), \overline{\xi_1}, \dots, \overline{\xi_k}), \tag{4-1}$$

where $\sim$ indicates an equivariant cobordism that incorporates bundle data.

Proof of Proposition 2.3. The argument is inductive. We induct over the partial order on the divisors of q. First we blow up components of the $\mathbb{Z}_p$ fixed point set that do not contain points that are fixed under D_{2p}. Secondly we blow up components of the $\mathbb{Z}_p$ fixed point set that contain points that are fixed under D_{2p}.

(i) Let A_0 be the union of those components of $M^{\mathbb{Z}_q}$ that contain only points of isotropy type $\mathbb{Z}_q$. Clearly A_0 is a D_{2q}-invariant submanifold of M, and having only one isotropy type it is iso-special. Certainly, A_0 can have components of different dimensions. Let $A_0^{=0}$ be the part of A_0 that is of codimension 0 in M. In particular, A_0 consists of a D_{2q}-invariant collection of components of M. For this part of M the assertion of Proposition 2.3 holds. We can exclude it from further consideration. Notationally it is easier to set it aside.

Let $A_0^{>0}$ be the union of those components of A_0 that are of positive codimension in M. Blow up $M \setminus A_0^{=0}$, the remaining part of M, along $A_0^{>0}$. Set

$M_1 = B(M \setminus A_0^{=0}, A_0^{>0})$. Let ν_x stand for the normal slice at a point $x \in A_0^{>0}$. Because $\mathbb{Z}_q$ is of odd order and ν_x does not have the trivial representation $\mathbb{R}$ as a summand, there is no real $\mathbb{Z}_q$-invariant line in ν_x. Hence $\mathbb{R}P(\nu_x \oplus \mathbb{R})$ has exactly one $\mathbb{Z}_q$ fixed point. After the identification of $M \setminus A_0^{=0}$ and $\mathbb{R}P(\nu \oplus \mathbb{R})$ along a common neighbourhood of $A_0^{>0}$ the common $\mathbb{Z}_q$ fixed set $A_0^{>0}$ has been eliminated. Our blow-up removes $A_0^{>0}$ from the $\mathbb{Z}_q$ fixed point set. Any remaining points of isotropy type $\mathbb{Z}_q$ belong to components that contain D_{2q}-fixed points.

Let q' be a maximal nontrivial proper divisor of q. We repeat the above process with $\mathbb{Z}_q$ replaced by $\mathbb{Z}_{q'}$. Let A_1 be the union of the components of the $\mathbb{Z}_{q'}$ fixed point set, all of whose points are of isotropy type $\mathbb{Z}_{q'}$. Note that A_1 is iso-special. As before, set the codimension 0 components $A_1^{=0}$ aside. Blow up $M_1 \setminus A_1^{=0}$ along $A_1^{>0}$, the components of A_1 of positive codimension in M_1. The blow-up removes $A_1^{>0}$ from the $\mathbb{Z}_{q'}$ fixed point set. Any remaining points of isotropy type $\mathbb{Z}_{q'}$ belong to components that also contain $D_{2q'}$-fixed points.

We continue this process for all nontrivial divisors of q, partially ordered by divisibility, and end up with a manifold M_k, for some k. For some $1 \neq r \mid q$ we may have set aside a manifold B_r all of whose points have isotropy type $\mathbb{Z}_r$. Denote their union by B. We have a blow-up sequence that starts with M and ends with $M_k \sqcup B$, and all blow-ups are along iso-special submanifolds.

(ii) We create a second blow-up sequence, starting with $\overline{M} = M_k$. Suppose that $\overline{M}^{\mathbb{Z}_q} \neq \varnothing$. Then $\overline{M}^{D_{2q}} \neq \varnothing$. Set $\bar{A}_0 = \overline{M}^{D_{2q}}$. This manifold is D_{2q}-invariant and all points have isotropy type D_{2q}, hence it is iso-special. Blow up along $\bar{A}_0$. We will show that $B(\overline{M}, \bar{A}_0)^{\mathbb{Z}_q}$ is iso-special. Locally we have isotropy groups D_{2q} and $\mathbb{Z}_q$, and one is of index two in the other. We have to show that the codimension of the D_{2q} fixed set in the $\mathbb{Z}_q$ fixed set is 1.

The normal slice ν_x to $\bar{A}_0$ in $\overline{M}$ at $x \in \bar{A}_0$ is a representation of D_{2q}. It has no trivial irreducible representation as summand. It is of the form $\alpha\mathbb{R}_- \oplus \Omega$. As in Section 3A, $\mathbb{R}_-$ is the nontrivial 1-dimensional irreducible representation of D_{2q}. We denote its multiplicity in ν_x by α. The two dimensional irreducible representations of D_{2q} were described in (3-1). Various values for j, reflecting different angles of rotation, may occur. We gather those that occur as part of ν_x, with their multiplicities, in the summand Ω. Note, as q is odd there are no D_{2q}-invariant real lines in Ω.

A D_{2q}-invariant real line in $\alpha\mathbb{R}_- \oplus \Omega \oplus \mathbb{R}$ is a line in $\alpha\mathbb{R}_-$ or it is the line $\mathbb{R}$. Use the symbol $\approx$ to denote a diffeomorphism. Then

$$\mathbb{R}P(\nu_x \oplus \mathbb{R})^{D_{2q}} = \mathbb{R}P(\alpha\mathbb{R}_- \oplus 0 \oplus 0) \sqcup \mathbb{R}P(0 \oplus 0 \oplus \mathbb{R}) \approx \mathbb{R}P^{\alpha-1} \sqcup \mathbb{R}P^0.$$

Any real line in $\alpha\mathbb{R}_- \oplus 0 \oplus \mathbb{R}$ is fixed under the action of $\mathbb{Z}_q$. Hence

$$\mathbb{R}P(\nu_x \oplus \mathbb{R})^{\mathbb{Z}_q} = \mathbb{R}P(\alpha\mathbb{R}_- \oplus 0 \oplus \mathbb{R}) \approx \mathbb{R}P^{\alpha}.$$

We can compute codimensions (cd) in the manifold by looking at codimensions in normal slices:

$$\mathrm{cd}(B(\overline{M}, \overline{A}_0)^{D_{2q}}, B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_q}) = \mathrm{cd}(\mathbb{R}P(\alpha\mathbb{R}_- \oplus 0 \oplus 0), \mathbb{R}P(\nu_x \oplus \mathbb{R})^{\mathbb{Z}_q}) = 1.$$

Having checked the codimension conditions, we deduce that $B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_q}$ is iso-special.

Blow up along $B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_q}$. We obtain

$$\overline{M}_1 = B(B(\overline{M}, \overline{A}_0), B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_p}).$$

The normal slice ν_x to $B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_p}$ in $B(\overline{M}, \overline{A}_0)$ at a point $x \in B(\overline{M}, \overline{A}_0)^{\mathbb{Z}_p}$ is a sum of irreducible representations as in (3-1). There is a single $\mathbb{Z}_q$-invariant line in $\nu_x \oplus \mathbb{R}$ and $\mathbb{R}P(\nu_x \oplus \mathbb{R})$ has a single $\mathbb{Z}_q$ fixed point. The latter disappears in the blow-up.

In summary, this pair of two blow-ups, each along an iso-special submanifold, removes the $\mathbb{Z}_q$ fixed points from $\overline{M}$.

As before, we perform this step for all nontrivial divisors of q, and we proceed inductively following the partial order on the set of divisors of q. Eventually, after repeated blow-ups along iso-special submanifolds, we end up with a manifold $\overline{M}_s$ whose isotropy types are the trivial group and/or the order 2 subgroup T of D_{2q}.

If 1 and T are the only isotropy types of the action on $\overline{M}_s$, we may still have to arrange the codimension 1 condition for $\overline{M}_s$ to be an iso-special manifold. To achieve this we blow up $\overline{M}_s$ along the iso-special submanifold $\overline{M}_s^T$. This is a special case of the first blow-up in (ii) with $q = 1$ and $\Omega = 0$. The normal fibre ν_x is a representation of T. As we computed earlier, $\mathrm{cd}(B(\overline{M}_s, \overline{M}_s^T)^T, B(\overline{M}_s, \overline{M}_s^T)) = 1$, so that $B(\overline{M}_s, \overline{M}_s^T)$ is iso-special.

Combined with the first sequence of blow-ups, we have a sequence of blow-ups along iso-special submanifolds that starts with M and terminates with the iso-special manifold $\overline{M}_s \sqcup B$. The proposition asserted that this is possible, and we verified it. □

5. Proof of Theorem 3.7 (1): the one isotropy type case

The assertion of Theorem 3.7 is that iso-special $G = D_{2q}$ manifolds (q odd) have strongly algebraic models, and in this section we prove the assertion if the manifold has only one isotropy type.

Some cases are easy to dispose of. If the single isotropy group is the trivial group, then D_{2q} acts freely and the assertion has been shown as Theorem B (2) in [Dovermann et al. 1994]. The same reference covers the case where D_{2q} acts trivially.

Next, suppose that the closed smooth D_{2q}-manifold N has the single isotropy type $(D_{2q'})$. The index of $D_{2q'}$ in its normalizer is q/q', which is odd. Theorem C in [Suh 1996] tells us that N together with the set of all D_{2q}-vector bundles over it can be algebraically realized. Our expression is that the D_{2q}-manifold N has a strongly algebraic model. Hence the assertion of the theorem is proved in this case as well. Setting $q' = 1$ this includes the case when $D_{2q'} = T$.

In our final case, suppose that M is a closed D_{2q}-manifold and that $\mathbb{Z}_{q'}$ is its only isotropy group. Necessarily $1 \neq q'$ divides q. We combine ideas from [Cho et al. 2001, Section 2] and [Dovermann and Masuda 1995]. Applying Proposition 3.4 we need to show: Given any finite collection $\{\xi_1, \dots, \xi_m\}$ of G-vector bundles over M, there is an algebraic model X of M so that these G-vector bundles pull back to strongly algebraic G-vector bundles over X.

Let $\mathcal{E}$ be an indexing set for the irreducible representations of $\mathbb{Z}_{q'}$. The irreducible representation associated with $\epsilon \in \mathcal{E}$ is denoted by α_ϵ. Consider one bundle ξ in the collection. For each $\epsilon \in \mathcal{E}$ there is a unique largest D_{2q} subbundle $\xi(\alpha_\epsilon)$ of ξ whose fibre is a multiple of α_ϵ. This uses the fact that irreducible representations of $\mathbb{Z}_{q'}$ are restrictions of representations of D_{2q}. Following [Cho et al. 2001] there should be one subbundle $\xi(\alpha_\epsilon)$ for a each conjugacy class of irreducible representations. But, the conjugation action of D_{2q} on the set of real irreducible D_{2q} representations is trivial. Thus, in our context, a bundle is α_ϵ-isotypical in the sense of [Cho et al. 2001] if the fibre of the bundle is a multiple of α_ϵ.

There is a direct sum decomposition $\xi = \bigoplus_{\epsilon \in \mathcal{E}} \xi(\alpha_\epsilon)$. The direct sum of strongly algebraic bundles is strongly algebraic; see [Dovermann et al. 1994, Proposition 2.11]. That means, to prove Theorem 3.7 in our special case, we may assume that each of the bundles ξ_i in our collection is isotypical.

Let $\mathrm{Vect}_G(M)$ stand for the semigroup of G-vector bundles over M and let $\mathrm{Vect}_G(M, \alpha_\epsilon)$ stand for the subsemigroup of α_ϵ-isotypical bundles. There is a G-vector bundle L over M whose fibre is α_ϵ. Suppose now that ξ is α_ϵ-isotypical. The assignment that sends ξ to $\mathrm{Hom}_{\mathbb{Z}_{q'}}(L, \xi)$ defines an isomorphism $\mathrm{Vect}_G(M, \alpha_\epsilon) \to \mathrm{Vect}_{G/\mathbb{Z}_{q'}}(M)$; see [Cho et al. 2001, Lemma 2.2]. Its inverse sends a bundle $\eta \in \mathrm{Vect}_{G/\mathbb{Z}_{q'}}(M)$ to $L \otimes \eta$. Depending on the type of α_ϵ, the tensor product will be over $\mathbb{R}$ or $\mathbb{C}$. The tensor product of strongly algebraic bundles is strongly algebraic; see [Dovermann et al. 1994, Proposition 2.11].

The group $G/\mathbb{Z}_{q'}$ acts freely on M and the bundles $\mathrm{Hom}_{\mathbb{Z}_{q'}}(L, \xi)$ and η in the previous paragraph are $G/\mathbb{Z}_{q'}$ bundles over M. Our reduction says that we only need to prove the assertion of Theorem 3.7 in case the group acts freely on M. The latter holds according to [Dovermann et al. 1994, Theorem B (2)].

Thus we have proved Theorem 3.7 if the action of D_{2q} has a single isotropy type.

6. Proof of Theorem 3.7 (2): the two isotropy type case

In the following we will make use of the exact Conner–Floyd sequences. They were established in [Conner and Floyd 1966, §5]. Earlier, in the setup for Theorem 3.7, we recalled basic bordism theoretic notation. We need a little more. Given two families of subgroups, $\mathcal{F}$ and $\mathcal{F}'$ with $\mathcal{F}' \subseteq \mathcal{F}$, there is a relative bordism group $\mathcal{N}_k^G[\mathcal{F}, \mathcal{F}'](Y)$

[Stong 1970]. In this setting we allow the domain M to be a compact manifold with boundary and the isotropy groups of ∂M are assumed to belong to $\mathcal{F}'$. Two maps are bordant if the $\mathcal{F}$ fixed points together with their normal data are bordant.

Suppose that there are two isotropy types. As before we denote them by H and K. It is assumed that $[H:K]=2$. The order of G is twice an odd number, so it follows that K is of odd order and normal in $G=D_{2q}$. We abbreviate $\widetilde{G} := G/K$. We have the following commutative diagram of Conner–Floyd sequences. All bordism groups in the diagram should have $\mathfrak{G}$ as codomain. Due to the restrictions on the isotropy groups we may replace $\mathfrak{G}$ by $\mathfrak{G}^K$. For reasons of space, we suppress this codomain altogether.

$$\begin{tikzcd}
\mathcal{N}^G_*[\{K\}] \arrow[r] \arrow[d, "\cong"'] & \mathcal{N}^G_{*c}[\{H,K\}^\bullet] \arrow[r, "j_G"] \arrow[d, "\cong"'] & \mathcal{N}^G_{*c}[\{H,K\}^\bullet,\{K\}] \arrow[r, "\partial_G"] \arrow[d, "\cong"'] & \mathcal{N}^G_{*-1}[\{K\}] \arrow[d, "\cong"'] \\
\mathcal{N}^{\widetilde{G}}_*[\{1\}] \arrow[r] & \mathcal{N}^{\widetilde{G}}_{*c}[\{T,1\}^\bullet] \arrow[r, "j_\sim"] & \mathcal{N}^{\widetilde{G}}_{*c}[\{T,1\}^\bullet,\{1\}] \arrow[r, "\partial_\sim"] & \mathcal{N}^{\widetilde{G}}_{*-1}[\{1\}] \\
\mathcal{N}^T_*[\{1\}] \arrow[r] \arrow[u, "\mathrm{Ind}'"] & \mathcal{N}^T_{*c}[\{T,1\}] \arrow[r, "j_T"] \arrow[u, "\mathrm{Ind}''"] & \mathcal{N}^T_{*c}[\{T,1\},\{1\}] \arrow[r, "\partial_T"] \arrow[u, "\mathrm{Ind}", "\cong"'] & \mathcal{N}^T_{*-1}[\{1\}] \arrow[u, "\mathrm{Ind}'"]
\end{tikzcd}$$

In the transition from the first to the second row we divide out K, the ineffective part of the action, and see that $H/K \cong T$. The vertical maps are natural isomorphisms.

In the transition from the third to the second row we apply induction. If M is a T-manifold, then $\mathrm{Ind}_T^{\widetilde{G}} M = \widetilde{G} \times_T M$ is a $\widetilde{G}$-manifold. It consists of equivalence classes of pairs $(g,x) \in \widetilde{G} \times M$, where $(gt,x) \sim (g,tx)$ when $t \in T$. If Y is a $\widetilde{G}$ space and $f : M \to Y$ is T equivariant, then $\mathrm{Ind}_T^{\widetilde{G}} f : \mathrm{Ind}_T^{\widetilde{G}} M \to Y$ is defined by setting $(\mathrm{Ind}_T^{\widetilde{G}} f)[g,x] = gf(x)$. Functoriality implies that the squares commute.

Restricting $\widetilde{G}$ actions and $\widetilde{G}$ equivariance to T actions and T equivariance defines the map

$$\mathrm{Res}^T_{\widetilde{G}} : \mathcal{N}^{\widetilde{G}}_{*c}[\{T,1\}^\bullet,\{1\}](\mathfrak{G}^K) \to \mathcal{N}^T_{*c}[\{T,1\},\{1\}](\mathfrak{G}^K).$$

We study the outcome. Set $T^g = gTg^{-1}$. According to the definition of the bordism group, a representative of a class in $\mathcal{N}^{\widetilde{G}}_{*c}[\{T,1\}^\bullet,\{1\}](\mathfrak{G}^K)$ is completely determined by its restriction to a neighbourhood of the T^g fixed point sets for all $g \in \widetilde{G}$. In $\mathcal{N}^T_{*c}[\{T,1\},\{1\}](\mathfrak{G}^K)$, once we restrict the action of $\widetilde{G}$ to one of T, the class is completely determined by its restriction to a neighbourhood of the T fixed point set.

The induction map $\mathrm{Ind}_T^{\widetilde{G}} = \widetilde{G} \times_T$ restores the neighbourboods of all the T^g fixed point sets. By construction, $\mathrm{Res}^T_{\widetilde{G}}$ and $\mathrm{Ind}_T^{\widetilde{G}}$ are inverses of each other.

In [Dovermann et al. 1994, Proposition 5.2] we proved that

$$\mathrm{Ind}' : \mathcal{N}^T_*[\{1\}](\mathfrak{G}^K) \to \mathcal{N}^{\widetilde{G}}_*[\{1\}](\mathfrak{G}^K)$$

is onto. The five lemma implies that Ind'' is onto.

Classes in $\mathcal{N}_{*c}^{T}[\{T,1\}](\mathfrak{G}^K)$ have algebraic representatives; see [Dovermann et al. 1994, Proposition F]. If we apply $\mathrm{Ind}_T^{\widetilde{G}}$ we obtain algebraic representatives of the classes in $\mathcal{N}_{*c}^{\widetilde{G}}[\{T,1\}^\bullet](\mathfrak{G}^K)$. This tells us that all classes in $\mathcal{N}_{*c}^{\widetilde{G}}[\{T,1\}^\bullet](\mathfrak{G}^K)$ and in $\mathcal{N}_{*c}^{G}[\{H,K\}](\mathfrak{G}^K)$ have algebraic representatives. We have completed the proof of Theorem 3.7 also in this second case.

7. Proof of Theorem 1.2, $q = 2r$ and r odd

Let $G = D_{2q}$ where $q = 2r$ and r is odd. Let $a, b \in G$ be as in the beginning of Section 3. The element $\tau = b^r$ is central in G and of order 2. Set $G' = G/\langle\tau\rangle = D_{2r}$. In fact, D_{2q} is a direct product of $\langle\tau\rangle$ and $G' = D_{2r}$.

Let M be a closed smooth G-manifold. According to [Dovermann et al. 1994, Proposition F] M has a strongly algebraic model if and only if the G-manifold $N = M^\tau$ has such a model. We will show the latter.

As τ acts trivially on N, there is an induced action of G' on N. We have seen that, as a G'-manifold, N has a strongly algebraic model. Call it X. At the same time X is a G equivariant algebraic model of N.

Let ξ be a G-vector bundle over X. The action of τ on ξ induces one on the fibres ξ_x of the bundle, $x \in N$. Each fibre, as well as the bundle, decomposes as a direct sum of the fixed point set and its orthogonal complement, on which τ acts by multiplication with -1. We write $\xi = \xi^+ \oplus \xi^-$.

As τ acts trivially on ξ^+ this bundle is actually a G' bundle and strongly algebraic, also as a G-vector bundle.

We have a real 1-dimensional representation σ of G. The element a acts trivially, while b and τ act by multiplication with -1. Clearly $\sigma \otimes_{\mathbb{R}} \sigma = \mathbb{R}$. Let $\underline{\sigma}$ be the product bundle with fibre σ. This bundle is classified by a constant map, which is entire rational. Hence the bundle is strongly algebraic. The action of τ on $\xi^- \otimes_{\mathbb{R}} \underline{\sigma}$ is trivial and this G' bundle is strongly algebraic, also as a G-vector bundle. The tensor product of strongly algebraic bundles is strongly algebraic, and so is $\xi^- \otimes_{\mathbb{R}} \underline{\sigma} \otimes_{\mathbb{R}} \underline{\sigma} = \xi^-$.

The direct sum of strongly algebraic bundles is strongly algebraic, and so is $\xi = \xi^+ \oplus \xi^-$. Hence X is a strongly algebraic model of N as a G-manifold. This is what we needed to show.

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ON COMMUTING BILLIARDS IN HIGHER-DIMENSIONAL SPACES OF CONSTANT CURVATURE

ALEXEY GLUTSYUK

We consider two nested billiards in $\mathbb{R}^d$, $d \geq 3$, with C^2-smooth strictly convex boundaries. We prove that if the corresponding actions by reflections on the space of oriented lines commute, then the billiards are confocal ellipsoids. This together with the previous analogous result of the author in two dimensions solves completely the commuting billiard conjecture due to Sergei Tabachnikov. The main result is deduced from the classical theorem due to Marcel Berger which says that in higher dimensions only quadrics may have caustics. We also prove versions of Berger's theorem and the main result for billiards in spaces of constant curvature (space forms).

1. Introduction

1A. ***Main result.*** Let $\Omega_a \Subset \Omega_b \subset \mathbb{R}^d$ be two nested bounded domains with smooth strictly convex boundaries $a = \partial\Omega_a$ and $b = \partial\Omega_b$. Consider the corresponding billiard transformations σ_a, σ_b acting on the space of oriented lines in space by reflection as follows. Each σ_f, $f = a, b$, acts as identity on the lines disjoint from f. For each oriented line l intersecting f we take its last intersection point x with f in the sense of orientation: the orienting arrow of the line l at x is directed outside Ω_f. The image $\sigma_f(l)$ is the line obtained by reflection of the line l from the hyperplane $T_x f$: the angle of incidence equals the angle of reflection. The line $\sigma_f(l)$ is oriented by a tangent vector at x directed inside Ω_f. This is a continuous mapping that is smooth on the space of lines intersecting f transversely.

Remark 1.1. The above action can be defined for a convex billiard in any Riemannian manifold; the billiard reflection acts on the space of oriented geodesics.

Recall, see, e.g., [Berger 1995; Tabachnikov 2005], that a pencil of *confocal quadrics* in a Euclidean space $\mathbb{R}^d$ is a one-dimensional family of quadrics defined

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in some orthogonal coordinates $(x_1, \dots, x_d)$ by equations

$$\sum_{j=1}^{d} \frac{x_j^2}{a_j^2+\lambda} = 1; \quad a_j \in \mathbb{R} \text{ are fixed; } \lambda \in \mathbb{R} \text{ is the parameter.}$$

It is known that *any two confocal elliptic or ellipsoidal billiards* commute [Tabachnikov 2005, p. 59, Corollary 4.6; 1994, p. 58]. Sergei Tabachnikov [1994, p. 58] stated the conjecture affirming the converse: any two commuting nested convex billiards are confocal ellipses (ellipsoids). In two dimensions this conjecture was proved in [Glutsyuk 2017a, Theorem 5.21, p. 231] for piecewise C^4-smooth boundaries. Here we prove it in higher dimensions in $\mathbb{R}^d$ and in spaces of constant curvature (space forms).

Theorem 1.2. *Let two nested strictly convex C^2-smooth closed hypersurfaces in $\mathbb{R}^d$, $d \geq 3$, be such that the corresponding billiard transformations commute. Then they are confocal ellipsoids.*

To extend Theorem 1.2 to spaces of constant curvature, let us recall the notions of space forms and (confocal) quadrics in them.

Definition 1.3. A *space form* is a complete connected Riemannian manifold of constant curvature.

Remark 1.4. We will deal only with simply connected space forms. It is well-known that they are the Euclidean space $\mathbb{R}^d$, the unit sphere $S^d \subset \mathbb{R}^{d+1}$ in the Euclidean space and the hyperbolic space $\mathbb{H}^d$ (up to normalization of the metric by constant scalar factor, which changes neither geodesics, nor reflections). It is known that the hyperbolic space $\mathbb{H}^d$ admits a standard model in the Minkowski space $\mathbb{R}^{d+1}$. Finally, each space form Σ we will be dealing with is realized as an appropriate hypersurface in the space $\mathbb{R}^{d+1}$ with coordinates $x = (x_0, \dots, x_d)$ equipped with a suitable quadratic form

$$\langle Gx, x\rangle, \quad G \text{ is a symmetric } (d+1)\times(d+1)\text{-matrix.}$$

Here $\langle x, x\rangle := \sum_j x_j^2$.

Euclidean case: $G = \operatorname{diag}(0, 1, \dots, 1)$, $\Sigma = \mathbb{R}^d = \{x_0 = 1\}$.

Spherical case: $G = \operatorname{Id}$, $\Sigma = S^d = \{\langle Gx, x\rangle = 1\}$, $\langle Gx, x\rangle = \sum_j x_j^2$.

Hyperbolic case: $G = \operatorname{diag}(-1, 1, \dots, 1)$, $\Sigma = \mathbb{H}^d = \{\langle Gx, x\rangle = -1\} \cap \{x_0 > 0\}$.

The metric on each hypersurface Σ is the restriction to $T\Sigma$ of the quadratic form $\langle Gx, x\rangle$ on the ambient space. It is well-known that the geodesics on Σ are its intersections with two-dimensional vector subspaces in $\mathbb{R}^{d+1}$. Completely geodesic k-dimensional submanifolds in Σ are its intersections with $(k+1)$-dimensional vector subspaces in $\mathbb{R}^{d+1}$.

Definition 1.5 [Veselov 1990, p. 84]. A *quadric* in Σ is a hypersurface

$$S = \Sigma \cap \{\langle Qx, x\rangle = 0\}, \quad Q \text{ is a symmetric matrix.}$$

The *pencil of confocal quadrics* associated to a symmetric matrix Q is the family of quadrics

$$S_\lambda = \Sigma \cap \{\langle Q_\lambda x, x\rangle = 0\}, \quad Q_\lambda = (Q - \lambda G)^{-1}, \quad \lambda \in \mathbb{R}.$$

Definition 1.6. A germ of C^2-smooth hypersurface S in a space form Σ at a point p is *strictly convex*, if it has quadratic tangency with its tangent completely geodesic hypersurface Γ_p, that is, there exists a constant $C > 0$ such that for every $q \in S$ close to p one has

$$\operatorname{dist}(q, \Gamma_p) > C\|q - p\|^2; \text{ here } \|q - p\| = \operatorname{dist}(q, p).$$

Theorem 1.7. *Let $d \geq 3$, and let Σ be a simply connected d-dimensional space form: either $\mathbb{R}^d$, or the unit sphere, or the hyperbolic space. Let two nested strictly convex C^2-smooth closed hypersurfaces in Σ be such that the corresponding billiard transformations commute. Then they are confocal quadrics.*

Theorem 1.2 follows from Theorem 1.7.

Theorem 1.2 can be deduced from a classical theorem due to Marcel Berger [1995] concerning billiards in $\mathbb{R}^d$, $d \geq 3$, which states that only billiards bounded by quadrics may have caustics (see Definition 1.8 for the notion of caustic), and the caustics are their confocal quadrics. To prove Theorem 1.7 in full generality, we extend Berger's theorem to the case of billiards in space forms (Theorem 1.10 stated in Section 1C and proved in Section 2) and then deduce Theorem 1.7 in Section 3. A local version of Theorem 1.7 is proved in Section 4. In Section 5 we present some open problems.

1B. ***Historical remarks.*** Commuting billiards are closely related to problems of classification of integrable billiards, see [Tabachnikov 1994]. It is known that elliptic and ellipsoidal billiards are integrable, see [Veselov 1988, Proposition 4; Tabachnikov 2005, Chapter 4], and this also holds for non-Euclidean ellipsoids in spheres and in the Lobachevsky (hyperbolic) space of any dimension, see [Veselov 1990, the corollary on p. 95]. The famous Birkhoff conjecture states that in two dimensions the converse is true. Namely, it deals with the so-called *Birkhoff caustic-integrable* convex planar billiards with smooth boundary, that is, billiards for which there exists a foliation by closed caustics (a one-parameter family of nested closed caustics Γ_p, $p > 0$) in an interior neighborhood of the boundary, and the boundary itself is the leaf Γ_0 of this foliation. The Birkhoff conjecture states that the only Birkhoff caustic-integrable billiards are ellipses. The Birkhoff conjecture was first stated in print in [Poritsky 1950], where it was proved under the additional assumption that for any two nested caustics in the above family Γ_p the smaller one is a caustic for the billiard in the bigger one. Poritsky's assumption implies that *the initial billiard map* in Γ_0 (being restricted to the set of those lines that are disjoint from some given caustic Γ_p with $p > 0$) *commutes with the billiard in every caustic* Γ_q. This follows by the arguments presented in [Tabachnikov 2005, Section 4, pp. 58–59].

The set of lines intersecting the given convex billiard is a topological cylinder called the *phase cylinder.* One of the most famous results on the Birkhoff conjecture is a theorem of M. Bialy [1993], who proved that if the phase cylinder of the billiard map is foliated (almost everywhere) by noncontractible closed curves which are invariant under the billiard map, then the boundary is a circle. In [Bialy 2013] he proved the same result for billiards on surfaces of nonzero constant curvature. A local version of the Birkhoff conjecture for integrable deformations of ellipses was recently solved in [Avila et al. 2016; Kaloshin and Sorrentino 2018b]. The recent solution of its polynomial version (stated and partially studied in [Bolotin 1992]) is a result of [Bialy and Mironov 2017a; 2017b; Glutsyuk 2017b; 2018]. For a historical survey of the Birkhoff conjecture see [Tabachnikov 2005, Section 5, p. 95], the recent surveys [Bialy and Mironov 2018; Kaloshin and Sorrentino 2018a] and the papers [Kaloshin and Sorrentino 2018b; Glutsyuk 2017b]. Dynamics in billiards in two and higher dimensions with piecewise smooth boundaries consisting of confocal quadrics was studied in [Dragovich and Radnovich 2010].

1C. *Berger's theorem and its extension to billiards in space forms.*

Definition 1.8. Let a, b be two nested strictly convex closed hypersurfaces in a Riemannian manifold E: the hypersurface b bounds a relatively compact domain in E whose interior contains a. We say that a is a *caustic* for the hypersurface b, if the image of each oriented geodesic tangent to a by the reflection σ_b from b is again a geodesic tangent to a.

Remark 1.9. It is well-known that if a, b are two confocal ellipses (ellipsoids) in Euclidean space, then the smaller one is a caustic for the bigger one. In the plane this is the classical Proclus–Poncelet theorem. In higher dimensions this theorem is due to Jacobi; see [Staude 1914, p. 80]. Similar statement holds in any space form; see, e.g., [Veselov 1990, Theorem 3].

We will deduce Theorem 1.7 from the following theorem, which implies that *in every space form only quadrics have caustics, and the caustics of each quadric S are exactly the quadrics confocal to S.*

Theorem 1.10. *Let $d \geq 3$, and let Σ be a d-dimensional simply connected space form. Let $S, U \subset \Sigma$ be germs of C^2-smooth hypersurfaces at points B and $A \neq B$ respectively with nondegenerate second fundamental forms. Let the geodesic AB be tangent to U at A and transversal to S at B. Let $C \in \Sigma \setminus \{B\}$, and let a vector* *tangent to the geodesic AB at B be reflected from the hyperplane $T_B S$ to a tangent vector to the geodesic BC. Let there exist a germ of C^2-smooth hypersurface V at C tangent to BC at C such that each geodesic close to AB and tangent to U be reflected from the hypersurface S to a geodesic tangent to V. Then S is a piece of a quadric b, and U, V are pieces of one and the same quadric confocal to b.*

Remark 1.11. In the case when $\Sigma = \mathbb{R}^d$, Theorem 1.10 was proved by Berger [1995].

2. Caustics of hypersurfaces in space forms: Proof of Theorem 1.10

The proof of Theorem 1.10 for space forms essentially follows Berger's proof [1995] for the Euclidean case. In Section 2A we first prove that the hypersurfaces U and V are pieces of the same quadric denoted by U. Then in Section 2B we show that S is a quadric confocal to U, using the fact that it is an integral hypersurface of a finite-valued hyperplane distribution: the field of symmetry hyperplanes in $T_x\Sigma$, $x \in \Sigma$, for the geodesic cones K_x circumscribed about the quadric U with vertex at x.

2A. *The hypersurfaces U and V and circumscribed cones.*

Theorem 2.1. *In the conditions of Theorem 1.10 the hypersurfaces U and V are pieces of one and the same quadric.*

Theorem 2.1 is proved below following [Berger 1995]. As in [loc. cit.], we first prove that for every $y \in S$ the geodesic cone with vertex y tangent to U is a quadratic cone tangent to both U and V (Lemma 2.4). Afterwards we apply a result from [Berger 1995] (proved via arguments using projective duality, and stated below as Lemma 2.12), showing that if the latter statement holds, then U and V lie in the same quadric.

Let $\pi : \Sigma \to \mathbb{RP}^d$ denote the restriction to Σ of the tautological projection $\mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{RP}^d$. It is a diffeomorphism onto the image $\pi(\Sigma)$ in nonspherical cases and a degree two covering over $\mathbb{RP}^d$ in the spherical case. Let g denote the metric on $\pi(\Sigma)$ that is the (well-defined) pushforward of the space form metric. Note that the geodesics (completely geodesic subspaces) for the metric g are the intersections of projective lines (respectively, projective subspaces) with $\pi(\Sigma)$. In order to reduce the proof to the Euclidean case treated in [Berger 1995], we use the following property of the metric g.

Proposition 2.2. *For every point $y \in \pi(\Sigma)$ there exist an affine chart $\mathbb{R}^d \subset \mathbb{RP}^d$ centered at y and a Euclidean metric on $\mathbb{R}^d$ (compatible with the affine structure) that has the same 1-jet at y, as the metric g.*

Proof. Without loss of generality we assume that $y = (1 : 0 : \cdots : 0)$, that is, the isometry group of the space form Σ acts transitively, and the projection π conjugates its action on Σ with its action on $\mathbb{RP}^d$ by projective transformations (since the isometry group is a subgroup in $\mathrm{GL}_{d+1}(\mathbb{R})$). Thus, in the standard affine chart $\mathbb{R}^d = \{x_0 = 1\}$ the point y is the origin. The metric g is invariant under the orthogonal transformations of the chart $\mathbb{R}^d$, since the metric of the space form is invariant under the rotations around the x_0-axis. The metric g on $T_y\mathbb{R}^d$ coincides with the standard Euclidean metric of the chart $\mathbb{R}^d$, by definition. The last two statements together imply that the 1-jets of both metrics at $y = 0$ coincide. This proves the proposition. $\square$

Corollary 2.3. *Let $U \subset \Sigma$ be a germ of hypersurface with nondegenerate second fundamental form. Then its projection $\pi(U)$ has nondegenerate second fundamental form in any affine chart $\mathbb{R}^d$ with respect to the standard Euclidean metric.*

Proof. The corollary follows from Proposition 2.2 and invariance of the property of having nondegenerate second fundamental form under projective transformations. Indeed, each germ of projective hypersurface is tangent to some quadric with order 3 (which is not unique). The 2-jet of a quadric determines completely whether it is regular or not. Nondegeneracy of the second fundamental form is equivalent to regularity of the tangent quadric. The space of regular quadrics is invariant under projective transformations. This proves the corollary. $\square$

In what follows in the present subsection we identify the hypersurfaces S, U, V and their points with their projection images: for simplicity the projection images $\pi(S)$, $\pi(U)$, $\pi(B)$ etc. will be denoted by the symbols S, U, B, ...

Lemma 2.4. *Let $S, U, V \subset \mathbb{RP}^d$ be the tautological projection images of the same hypersurfaces in Σ, as in Theorem 1.10 (see the above paragraph). For every $y \in S$ there exists a quadratic cone $K_y \subset \mathbb{RP}^d$ (i.e., given by the zero locus of a homogeneous quadratic polynomial) with vertex at y that is tangent to both hypersurfaces U and V.*

The proof of Lemma 2.4 given below follows [Berger 1995, Section 2].

Let $\sigma_g : (T\mathbb{RP}^d)|_S \to (T\mathbb{RP}^d)|_S$ denote the involution acting as the symmetry of each space $T_y\mathbb{RP}^d$, $y \in S$, with respect to the hyperplane T_yS in the metric g. Its action on the projectivized tangent spaces $\mathbb{RP}^{d-1}_y = \mathbb{P}(T_y\mathbb{RP}^d)$ induces its action on the space of projective lines in $\mathbb{RP}^d \supset S$ intersecting S transversely and so that the intersection point is unique: if ℓ intersects S at a point y, then

$$\hat{\ell} := \sigma_g(\ell)$$

is the line through y that is symmetric to ℓ in the above sense.

For every $y \in \Sigma$ set

$$M_y := \text{ the space of projective lines through } y \text{ that are tangent to } U.$$

It suffices to prove the statement of Lemma 2.4 for an arbitrary point $y \in S$ satisfying the following statements:

Proposition 2.5 (stated in [Berger 1995, pp. 110–111]). *There exists an open and dense subset of points $y \in S \subset \mathbb{RP}^d$ for which there exists an open and dense subset $M_y^0 \subset M_y$ of lines ℓ satisfying the following statements*:

(i) *The line ℓ is quadratically tangent to U, (i.e., ℓ is not an asymptotic direction of the hypersurface U at the tangency point). The projective hyperplane containing ℓ and tangent to U at the latter point is not orthogonal to T_yS.*

(ii) *The line $\hat{\ell} = \sigma_g(\ell)$ is quadratically tangent to V at a point, where the second fundamental form of the hypersurface V is nondegenerate.*

(iii) *The lines ℓ and $\hat{\ell}$ are transversal to T_yS and their above tangency points with U and V are distinct from the point y.*

Proof. Statement (i) holds for an open and dense subset of lines $\ell \in M_y$, since the second fundamental form of the hypersurface U is nondegenerate (by assumptions and Corollary 2.3). Statement (iii) also holds for a generic $\ell \in M_y$, whenever $y \notin U \cup V$. Let us show that statement (ii) also holds generically.

Let $y \in S$, $y \notin U \cup V$, and let ℓ be a line through y satisfying assumption (i). Then the cone K_y with vertex y containing ℓ and circumscribed about U is tangent to U along a $(n-2)$-dimensional submanifold $\mathcal{T}_U \subset U$.

The correspondence sending a point $p \in \mathcal{T}_U$ to the projective hyperplane tangent to U at p (i.e., to the projective hyperplane tangent to the cone along the line yp) is a local immersion to the space of hyperplanes through y. Or equivalently, the correspondence sending a line $L \subset K_y$ through y to the projective hyperplane tangent to K_y along L is a local immersion. This follows from nondegeneracy of the second fundamental form of the hypersurface U. This implies a similar statement for the symmetric cone $\widehat{K_y} = \sigma_g(K_y)$ circumscribed about V: the correspondence sending each line $\widehat{L} \subset \widehat{K_y}$ through y to the hyperplane tangent to $\widehat{K_y}$ along the line $\widehat{L}$ is a local immersion to the space of hyperplanes through y.

Suppose now that a line $\widehat{L} \subset \widehat{K_y}$ through y is quadratically tangent to V at a point q. Then the above immersivity statement for the symmetric cone together with quadraticity of tangency imply nondegeneracy of the second fundamental form of the hypersurface V at the point q. It is clear that for a generic choice of the point $y \in S$ and a line $L \subset K_y$ through y the corresponding symmetric line $\widehat{L} = \sigma_g(L)$ is quadratically tangent to V. This proves the proposition. $\square$

Convention 2.6. In the proof of Lemma 2.4 without loss of generality we assume that $y = B$, and there exists a line ℓ through B that is transversal to S and satisfies statements (i)–(iii) of Proposition 2.5. Without loss of generality we assume that A is the tangency point of the line ℓ with U, and C is the tangency point of the symmetric line $\hat{\ell} = \sigma_g(\ell)$ with V, where $A, C \neq B$. Fix an affine chart $\mathbb{R}^d \subset \mathbb{RP}^d$ centered at B and equipped with an Euclidean metric whose 1-jet at B coincides with the 1-jet of the metric g (Proposition 2.2).

Consider a smooth deformation $x(t) \in S$ of the point B, $x(0) = B$, and a smooth deformation $p(t) \in U$ of the point A, $p(0) = A$, such that the line $\ell(t) = x(t)p(t)$ is tangent to U at $p(t)$, $t \in [0, 1)$. Then the line $\hat{\ell}(t) = \sigma_g(\ell(t))$ symmetric to $\ell(t)$ in the metric g is tangent to the hypersurface V at some point $q(t)$, $q(0) = C$, that depends smoothly on the parameter t (assumptions (i)–(iii)). We will show that the property that every deformation $x(t)$ extends to a pair of deformations $p(t)$ and $q(t)$ as above implies that the cone K_y tangent to both U and V is quadratic. To do this, consider the projective hyperplanes $\mathcal{U}$ and $\mathcal{V}$ through B containing the lines $\ell(0) = BA$ and $\hat{\ell}(0) = BC$ respectively: $\mathcal{U}$ is tangent to U at A, and $\mathcal{V}$ is tangent to V at C.

Remark 2.7. Let $\mathcal{U}$ and $\mathcal{V}$ be as above. The tangent subspaces $T_B\mathcal{U}$, $T_B\mathcal{V} \subset T_B\mathbb{RP}^d$ are σ_g-symmetric. Indeed, consider the germs of the cones circumsribed about the hypersurfaces U and V with vertex B and containing the lines $l(0)$ and $\hat{l}(0)$ respectively: we take the germs of the above cones at the latter lines. The σ_g-symmetry permutes the cones, by statement (ii) of Proposition 2.5, which holds for an open and dense set of lines through B. The hyperplanes $\mathcal{U}$ and $\mathcal{V}$ are tangent to the cones along the lines $l(0)$ and $\hat{l}(0)$ respectively, by construction. Hence they are also σ_g-symmetric, as are the cones, and so are their tangent spaces $T_B\mathcal{U}$ and $T_B\mathcal{V}$.

The latter tangent spaces intersect on a codimension 2 subspace $H \subset T_B\mathbb{R}^d$ lying in T_BS, by symmetry and statement (i):

$$H = T_B\mathcal{U} \cap T_BS = T_B\mathcal{V} \cap T_BS. \tag{2-1}$$

For every deformation $x(t)$, $p(t)$, $q(t)$ as above one has

$$u = x'(0) \in T_BS, \qquad v = p'(0) \in T_A\mathcal{U}, \qquad w = q'(0) \in T_C\mathcal{V}. \tag{2-2}$$

This motivates the following definition:

Definition 2.8. Let S be a germ of hypersurface at a point $B \in \mathbb{R}^d \subset \mathbb{RP}^d$. Let g be a positive definite scalar product on the bundle $T\mathbb{R}^d|_S$. Let ℓ be a projective line through B that is transversal to T_BS, and let $H \subset T_BS$ be a vector subspace of codimension one (codimension two in $T_B\mathbb{R}^d$). Let $A \in \ell,\ C \in \hat{\ell} = \sigma_g(\ell),\ A, C \neq B$. Let $\mathcal{U}$ and $\mathcal{V}$ denote the projective hyperplanes through B that are tangent to H and such that $\ell \subset \mathcal{U},\ \hat{\ell} \subset \mathcal{V}$. Let

$$u \in T_BS,\ u \neq 0, \quad v \in T_A\mathcal{U}, \quad w \in T_C\mathcal{V}.$$

We say that $(B, \ell, H, u, A, v, C, w)$ is a *Berger tuple* with base point B, if there exist germs of C^1-smooth curves of points $x(t) \in S,\ p(t), q(t) \in \mathbb{RP}^d,\ x(0) = B,$ $p(0) = A,\ q(0) = C$, such that statements (2-2) hold and for every small t the lines $x(t)p(t)$ and $x(t)q(t)$ are σ_g-symmetric.

Proposition 2.9. *The property of being a Berger tuple depends only on the* 1*-jet of the metric* g. *Namely, let* S *be a germ of hypersurface at a point* $B \in \mathbb{R}^d \subset \mathbb{RP}^d$. *Let* g_1 *and* g_2 *be two positive definite scalar products on the bundle* $(T\mathbb{R}^d)|_S$ *that have the same* 1*-jet at* B. *Then any Berger tuple for the metric* g_1 *with base point* B *is a Berger tuple for the metric* g_2 *and vice versa.*

Proof. The proposition follows from definition and smoothness of the dependence of the reflection σ_g on the parameters of the metric g: if two metrics have the same 1-jets at B, then the corresponding reflections acting in $T_y\mathbb{R}^d$ differ by a quantity $o(y - B)$. □

Theorem 2.10 [Berger 1995, Section 2]. *Let* S *be a germ of hypersurface at a point* $B \in \mathbb{R}^d \subset \mathbb{RP}^d$. *Consider the standard Euclidean metric on the affine chart* $\mathbb{R}^d$, *and let* S *have nondegenerate second fundamental form. Let* ℓ *be a line through* B *transversal to* T_BS. *Then there exist only a finite number* $k \le d-1$ *of codimension one vector subspaces* $H = H_1(\ell), \dots, H_k(\ell) \subset T_BS$ *such that for every* $u \in T_BS$, $u \neq 0$ *the triple* (ℓ, H, u) *extends to a Berger tuple* $(B, \ell, H, u, A, v, C, w)$ *for the Euclidean metric. The number* k *depends only on the second fundamental form of the hypersurface* S *at* B. *The subspaces* $H_j(\ell)$ *are uniquely determined by the line* ℓ *and the second fundamental form.*

Proposition 2.11 [Berger 1995, p. 114]. *In the conditions of Theorem 2.10 consider the tautological projection* $\pi_B : \mathbb{R}^d \setminus \{B\} \to \mathbb{RP}^{d-1}$ *to the space of lines through* B. *For every line* ℓ *through* B *the corresponding projection* $\pi_B(\ell \setminus \{B\}) \in \mathbb{RP}^{d-1}$ *will be denoted by* $[\ell]$. *For every* ℓ *transversal to* S *let* $\Delta_j(\ell) \subset T_B\mathbb{RP}^d = \mathbb{R}^d$ *denote the codimension* 1 *vector subspace spanned by* $H_j(\ell)$ *and* ℓ. *Let* $\widetilde{\Delta}_j([\ell]) = \pi_B(\Delta_j(\ell) \setminus \{0\}) \subset \mathbb{RP}^{d-1}$ *denote its tautological projection, which is a projective hyperplane through* $[\ell]$. *Set*

$$\mathcal{D}_j([\ell]) := T_{[\ell]}\widetilde{\Delta}_j([\ell]) \subset T_{[\ell]}\mathbb{RP}^{d-1}.$$

The subspaces $\mathcal{D}_1([\ell]), \dots, \mathcal{D}_k([\ell]) \subset T_{[\ell]}\mathbb{RP}^{d-1}$ *form a* k*-valued hyperplane distribution* $\mathcal{D}$ *on* $\mathbb{RP}^{d-1}$, *whose all integral surfaces are quadrics. Moreover, let* $\widetilde{S}$ *be a quadric tangent to* S *at* B *with order* 3: *having the same second fundamental form at* B. *The* π_B*-preimages of the above quadrics in* $\mathbb{RP}^{d-1}$ *(i.e., the preimages of the integral hypersurfaces) are cones with vertex at* B *that are tangent to the quadrics confocal to* $\widetilde{S}$.

Proof of Lemma 2.4. Let K be the cone with vertex at $y = B$ circumscribed about the hypersurface U. Let A be a point of tangency of the cone K with U. Set $\ell = BA$, $\hat{\ell} = \sigma_g(\ell)$. Let C denote the point of tangency of the line $\hat{\ell}$ with V. (We suppose that the assumptions of Convention 2.6 hold.) Then for every germ of smooth curve $x(t) \subset S$, $x(0) = B$, there exist curves $p(t) \subset U$ and $q(t) \subset V$, $p(0) = A$, $q(0) = C$, such that the lines $x(t)p(t)$ and $x(t)q(t)$ are tangent to U and V at $p(t)$ and $q(t)$ respectively and σ_g-symmetric.

Let $\mathcal{U}, \mathcal{V} \subset \mathbb{RP}^d$ be the previously defined projective hyperplanes through B tangent to U and V at A and C respectively, and $H = T_B\mathcal{U} \cap T_B\mathcal{V} \subset T_B S$; see (2-1). The tuple $(B, \ell, H, x'(0), A, p'(0), C, q'(0))$ is a Berger tuple for the metric g, by definition. Therefore, it is also a Berger tuple for the Euclidean metric as well (Proposition 2.9). This together with Theorem 2.10 implies that $H = H_j(\ell)$ for some j.

The cone K is tangent along the line ℓ to the hyperplane generated by ℓ and $H = H_j$, by definition. Therefore, at $[\ell] = \pi_B(\ell \setminus \{B\})$, the tautological projection $\widetilde{K} = \pi_B(K \setminus \{B\}) \subset \mathbb{RP}^{d-1}$ is tangent to the corresponding hyperplane $\mathcal{D}_j([\ell])$ from Proposition 2.11. Finally, $\widetilde{K}$ is an integral hypersurface of the multivalued hyperplane distribution $\mathcal{D}$ from Proposition 2.11, and hence, lies in a quadric $\Gamma(U)$. The preimage $\pi_B^{-1}(\Gamma(U))$ is a quadratic cone K_B with vertex B that contains K and is σ_g-symmetric, being a cone tangent to a quadric confocal to $\widetilde{S}$; see Proposition 2.11. (Recall that for any given quadric $\widetilde{S}$ and $B \in \widetilde{S}$ a cone with vertex B circumscribed about a quadric confocal to $\widetilde{S}$ is symmetric with respect to the hyperplane tangent to $\widetilde{S}$ at B.) Similarly, the punctured cone $\sigma_g(K) \setminus \{B\}$ tangent to V is projected to a quadric $\Gamma(V)$, and $\sigma_g(K)$ lies in a quadratic cone. The latter quadratic cone coincides with K_B, by symmetry. This proves Lemma 2.4. □

Lemma 2.12 [Berger 1995, Section 3]. *Let U, V, S be C^2-smooth germs of hypersurfaces in $\mathbb{RP}^d$ with nondegenerate second fundamental forms. Let for every $x \in S$ there exist a quadratic cone K_x with vertex at x that is tangent to both U and V. Then U and V are pieces of one and the same quadric.*

Proof of Theorem 2.1. For every $y \in S$ close enough to B there exists a quadratic cone K_y with vertex at y circumscribed about both U and V (Lemma 2.4). Applying this statement to an open and dense subset of points $y \in S$ satisfying genericity assumptions from Convention 2.6 together with Lemma 2.12 yield that U and V are pieces of one and the same quadric. Theorem 2.1 is proved. □

2B. ***Symmetry hyperplanes of circumscribed cones and confocal quadrics.*** Here we prove the following lemma and then deduce Theorem 1.10 from it.

Lemma 2.13 (A generalization of an analogous statement in [Berger 1995, p. 109]). *Let Σ be a simply connected space form of dimension at least three. Let $U \subset \Sigma$ be a quadric with nondegenerate second fundamental form. For every $y \in \Sigma \setminus U$ let K_y denote the geodesic cone circumscribed about the quadric U with the vertex at y (i.e., the union of geodesics through y that are tangent to U). We identify the cone K_y with the cone $\widetilde{K}_y \subset T_y\Sigma$ of vectors tangent to the above geodesics via the exponential mapping* $\exp : T_y\Sigma \to \Sigma$. *Let $S \subset \Sigma$ be a germ of hypersurface at a* *point $B \notin U$ with nondegenerate second fundamental form such that for every $y \in S$ the cone $\widetilde{K}_y$ is symmetric with respect to the hyperplane T_yS. Then S is a quadric confocal to U.*

In the proof of Lemma 2.13 we use the following lemma. To state it, let us recall that the orthogonal polarity in $\mathbb{R}^{d+1}$ is the correspondence sending each vector subspace to its orthogonal complement with respect to the standard Euclidean scalar product. The orthogonal polarity in codimension one, which sends codimension one vector subspaces to their orthogonal lines, induces a projective duality $\mathbb{RP}^{d*} \to \mathbb{RP}^d$ sending hyperplanes to points. It sends each hypersurface $S \subset \mathbb{RP}^d$ to its dual S^*: the family of points dual to the hyperplanes tangent to S.

Definition 2.14. Consider a scalar product $\langle Gx, x\rangle$ on $\mathbb{R}^{d+1}$ defining a space form. Orthogonality with respect to the latter scalar product will be called *G-orthogonality.* Let $V \subset \mathbb{R}^{d+1}$ be a subspace that is *not isotropic:* this means that the restriction to V of the scalar product $\langle Gx, x\rangle$ is a nondegenerate quadratic form (or equivalently, that V is not tangent to the light cone $\{\langle Gx, x\rangle = 0\}$). The *pseudosymmetry* with respect to V is the linear involution $I_V : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ that preserves the above scalar product on $\mathbb{R}^{d+1}$ and whose fixed point set coincides with V: it acts trivially on V and as a central symmetry in the G-orthogonal subspace.

Lemma 2.15. *Let $V \subset \mathbb{R}^{d+1}$ be a nonisotropic vector subspace. Let $k < d+1$. Consider the action $I_{V,k} : G(k, d+1) \to G(k, d+1)$ of the pseudosymmetry with respect to V on the Grassmannian of k-subspaces. The orthogonal polarity $L \mapsto L^\perp$ conjugates the actions $I_{V,k}$ and $I_{V^\perp, d+1-k}$.*[1]

Proof. This lemma seems to be well-known to specialists. In three dimensions it follows from [Bolotin 1992, formula (15), p. 23; Kozlov and Treshchëv 1991, formula (3.12), p. 140]. Let us present its proof for completeness of presentation. As it is shown below, Lemma 2.15 is implied by the two following propositions.

Proposition 2.16. *Let G be a real symmetric $(d+1)\times(d+1)$-matrix such that $G^3 = G$. Let two nonisotropic subspaces $V, W \subset \mathbb{R}^{d+1}$ of complementary dimensions be G-orthogonal. Then their* ***Euclidean*** *orthogonal complements $V^\perp$ and $W^\perp$ are also nonisotropic and G-orthogonal.*

Proof. The condition of the proposition implies that the restrictions of the linear operator G to V and W have zero kernels and

$$GV = W^\perp, \quad GW = V^\perp. \tag{2-3}$$

Thus, to prove G-orthogonality of the latter subspaces, it suffices to show that

$$\langle G^2 v, Gw\rangle = \langle G^3 v, w\rangle = 0 \quad \text{for every } v \in V \text{ and } w \in W.$$

[1] Everywhere below the orthogonality sign $\perp$ means orthogonality with respect to the standard Euclidean scalar product.

The first equality follows from symmetry of the matrix G. The second one follows from G-orthogonality of the subspaces V and W and the equality $G^3 = G$. The subspaces (2-3) are nonisotropic, since the restrictions to them of the scalar product $\langle Gx, x\rangle$ are isomorphic to its restrictions to V and W via the operator G: for every $v_1, v_2 \in V$ one has $\langle G(Gv_1), Gv_2\rangle = \langle Gv_1, v_2\rangle$, since $G^3 = G$. Proposition 2.16 is proved. □

Proposition 2.17. *Let $\langle Gx, x\rangle$ be a scalar product on $\mathbb{R}^{d+1}$ defining a space form. Let $k \in \{1, \dots, d\}$, $V \subset \mathbb{R}^{d+1}$ be a nonisotropic subspace, and let $W \subset \mathbb{R}^{d+1}$ be its G-orthogonal complement.*[2] *Let $N_k(V) \subset G(k, d+1)$ denote the subset of those vector k-subspaces in $\mathbb{R}^{d+1}$ that are direct sums of some subspaces $\ell_1 \subset V$ and $\ell_2 \subset W$. The pseudosymmetry I_V induces a nontrivial projective involution $\mathbb{RP}^d \to \mathbb{RP}^d$ and acts trivially on $N_k(V)$. Vice versa, every nontrivial projective involution acting trivially on $N_k(V)$ is the projectivization of the pseudosymmetry I_V.*

Proof. The first statement of the proposition is obvious. Let us prove the second one. Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be a linear transformation whose projectivization is a nontrivial involution acting trivially on $N_k(V)$. Without loss of generality we assume that $F^2 = \pm\,\mathrm{Id}$. For every vector subspace $L \subset V$ of dimension between 1 and k the transformation F preserves the subset in $N_k(V)$ consisting of the k-subspaces containing L. Their intersection being equal to L, F preserves L. The same statement holds for $L \subset W$. Therefore, the restriction of the transformation F to any of the subspaces V and W is a homothety. The coefficients of the homotheties on V and W are equal to ± 1, since $F^2 = Id$ up to sign. The signs of the latter coefficients are opposite, since the projectivization of the transformation F is nontrivial. Hence, $F = \pm I_V$. This proves the proposition. □

Let us now return to the proof of Lemma 2.15. The action of a linear automorphism $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ on all the vector subspaces of all the dimensions is conjugated via the orthogonal polarity to the similar action of the inverse $(F^*)^{-1}$ to the conjugate operator F^* (with respect to the Euclidean scalar product). In the case, when F is an involution, so is $F^* = (F^*)^{-1}$. Let W be the G-orthogonal complement of the subspace V.

Claim. *The conjugate operator $F = I_V^*$ acts trivially on $N_{d+1-k}(V^\perp)$.*

Proof. The orthogonal polarity sends each k-subspace $\Pi = \ell_1 \oplus \ell_2 \in N_k(V)$, $\ell_1 \subset V$, $\ell_2 \subset W$, to the intersection of two subspaces $L_j = L_j(\Pi) = \ell_j^\perp$:

$$\begin{gathered} L_1 \supset V^\perp, \quad L_2 \supset W^\perp, \quad \Pi^\perp = L_1 \cap L_2, \\ \dim(\Pi^\perp) = \dim L_1 + \dim L_2 - (d+1) = d+1-k. \end{gathered} \tag{2-4}$$

[2] The G-orthogonal complement W to a nonisotropic subspace V is always a vector subspace complementary to V. In the non-Euclidean cases W is automatically nonisotropic. In the Euclidean case, when the matrix G is degenerate, W contains the kernel of the matrix G: the x_0-axis.

The transformation F fixes $\Pi^\perp$, by construction and since the pseudosymmetry I_V fixes Π (Proposition 2.17). The intersection $\Pi^\perp$ is the direct sum of the subspaces $L_1 \cap W^\perp$ and $L_2 \cap V^\perp$, which follows from the inclusions (2-4) and the fact that $W^\perp$ and $V^\perp$ are complementary subspaces, as are V and W. Hence, $\Pi^\perp$ lies in $N_{d+1-k}(V^\perp)$. Vice versa, each point in $N_{d+1-k}(V^\perp)$ can be represented as the intersection $\Pi^\perp$ of some subspaces L_1 and L_2 containing $V^\perp$ and $W^\perp$ respectively. Therefore, F acts trivially on all of $N_{d+1-k}(V^\perp)$. The claim is proved. □

The operator $F = I_V^*$ is a projectively nontrivial involution, as is I_V. It coincides with $I_{V^\perp}$ up to sign, by the claim and Proposition 2.17. This together with the discussion preceding the claim implies the statement of Lemma 2.15. □

Proof of Lemma 2.13. Consider the tautological projection $\pi : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{RP}^d$, the images $\pi(S), \pi(U) \subset \mathbb{RP}^d$ and the hypersurfaces in $\mathbb{RP}^d$ projective-dual to them with respect to the orthogonal polarity. For simplicity the latter projective-dual hypersurfaces will be denoted by S^* and U^* respectively. Let $\widetilde{S}, \widetilde{U}, \widetilde{S}^*, \widetilde{U}^* \subset \mathbb{R}^{d+1}$ denote the complete π-preimages in $\mathbb{R}^{d+1}$ of the hypersurfaces $\pi(S)$, $\pi(U)$, S^* and U^* respectively: the cones in $\mathbb{R}^{d+1} \setminus \{0\}$ defined by the latter hypersurfaces. Recall that $\pi(U)$ and U^* are dual quadrics; thus one can write

$$U^* = \{\langle Qx, x\rangle = 0\}, \quad Q \text{ is a real symmetric } (d+1)\times(d+1)\text{-matrix.}$$

For every $y \in S$ let $\mathcal{T}_y S \subset \mathbb{RP}^d$ denote the projective hyperplane tangent to $\pi(S)$ at $\pi(y)$. Define the following vector subspaces in $\mathbb{R}^{d+1}$:

$$\Pi_y := \pi^{-1}(\mathcal{T}_y S) \cup \{0\} \subset \mathbb{R}^{d+1}, \;\; L_y := \Pi_y^\perp,$$
$$V_y := \text{ the one-dimensional subspace } \pi^{-1}(\pi(y)) \cup \{0\} \subset \Pi_y, \quad W_y := V_y^\perp.$$

The subspaces L_y and W_y are nonisotropic. Indeed, in the case when Σ is non-Euclidean, this follows from obvious nonisotropicity of their orthogonal subspaces Π_y and V_y and the fact that in the non-Euclidean case the orthogonal complement to a nonisotropic subspace is also nonisotropic. The latter statement follows from footnote 2 and Proposition 2.16. In the case when Σ is Euclidean, if, to the contrary, either L_y, or W_y contained the x_0-axis, this would imply that either Π_y or V_y lies in the coordinate $(x_1, \dots, x_d)$-subspace, and hence, is disjoint from Σ. This is obviously impossible.

Claim 1. *The quadric U^* is regular, i.e., the matrix Q is nondegenerate. The hyperplane section $\widetilde{U}^* \cap W_y$ is invariant under the pseudosymmetry with respect to the one-dimensional vector subspace $L_y \subset W_y$.*

Proof. The first statement (nondegeneracy) follows from nondegeneracy of the second fundamental form of the quadric U. The inclusion $L_y \subset W_y$ follows from definition. Recall that the cone $\widetilde{K}_y$ is symmetric with respect to the hyperplane $T_y S$, i.e., the preimage $\pi^{-1}(K_y)$ is pseudosymmetric with respect to Π_y, by assumption.

The latter statement is equivalent to the second statement of the claim, by duality and Lemma 2.15. □

The restriction to the d-dimensional vector subspace W_y of the scalar product $\langle Gx, x\rangle$ is nondegenerate (nonisotropicity), and there exist d values

$$\lambda = \lambda_1(y), \dots, \lambda_d(y)$$

(taken with multiplicity, some of them may coincide) such that the restriction to W_y of the scalar product $\langle (Q - \lambda G)x, x\rangle$ is degenerate. Thus, the d-dimensional vector subspace W_y is the G-orthogonal direct sum of kernels of the scalar products $\langle (Q - \lambda_j(y)G)x, x\rangle|_{W_y}$.

Claim 2. *For every $y \in S$ the pseudosymmetry line L_y lies in the kernel of some of the scalar products $\langle (Q - \lambda_j(y)G)x, x\rangle|_{W_y}$.*

Proof. The scalar product $\langle Qx, x\rangle|_{W_y}$ is invariant under the pseudosymmetry with respect to the line L_y. Indeed, the latter pseudosymmetry is an involution preserving the zero locus (light cone) $\widetilde{U}^* \cap W_y = \{\langle Qx, x\rangle = 0\} \cap W_y$ (Claim 1), and hence, it preserves the above scalar product up to sign. Let us show that the sign is also preserved. For an open and dense subset of points $y \in S$ one has $\langle Qx, x\rangle \neq 0$ on $L_y \setminus \{0\}$: equivalently (via duality), the tangent hyperplane T_yS is not tangent to U. Indeed, the latter statement holds for an open and dense subset of points $y \in S$, since $S \cap U = \varnothing$ and a (germ of) hypersurface is uniquely defined by the family of its tangent hyperplanes (well-definedness of the dual hypersurface). Thus, for the above y the pseudosymmetry fixes the nonzero quadratic form $\langle Qx, x\rangle|_{L_y}$, since the points of the line L_y are fixed. This together with the above discussion implies that the above-mentioned sign, and hence the scalar product $\langle Qx, x\rangle|_{W_y}$ are preserved for all $y \in S$.

For every $\lambda_j(y)$ the kernel of the form $\langle (Q - \lambda_j(y)G)x, x\rangle|_{W_y}$ is invariant under the above pseudosymmetry, by invariance of the scalar products $\langle Qx, x\rangle$ and $\langle Gx, x\rangle$. This is possible only in the case, when the pseudosymmetry line L_y lies in some of the kernels, which form an orthogonal direct sum decomposition of the subspace W_y. This proves Claim 2. □

Remark 2.18. The subspace W_y and hence, the corresponding kernels from Claim 2 depend only on y and are well-defined for all $y \in \Sigma$.

Due to Claim 2, the following two cases are possible.

Case 1: For an open and dense subset S_0 of points $y \in S$ the line L_y coincides with a one-dimensional kernel corresponding to a simple eigenvalue $\lambda_j(y)$. Let us show that in this case S lies in a quadric confocal to U. Indeed then there exist a neighborhood $Y = Y(B) \subset \Sigma$ of the base point B of the hypersurface S and an open and dense subset $Y_0 \subset Y$ containing S_0 such that the correspondence $y \mapsto L_y$ extends to a family of lines depending analytically on $y \in Y_0$: these lines are some of the kernels mentioned in the above remark. This implies that the corresponding hyperplanes $\Pi_y := L_y^{\perp}$ also depend analytically on y and thus, induce a field of hyperplanes $T = T(y) = \Pi_y \cap T_y\Sigma$ on Y_0. The hypersurface S_0 is its integral hypersurface.

Subcase 1.1: U is a generic quadric. Then for a generic point $y \in \Sigma$ (here "generic" means "outside an algebraic subset")

- there are exactly d quadrics through y confocal to U, and any two of them are orthogonal at y;
- the corresponding eigenvalues $\lambda_j(y)$ are simple and the corresponding d kernels in W_y are one-dimensional.

Recall that the tangent hyperplanes at y of the above confocal quadrics are symmetry hyperplanes for the cone K_y, since U is a caustic for its confocal quadrics. Therefore, the orthogonal polarity $\Pi_y \mapsto L_y$ induces a one-to-one correspondence between the above tangent hyperplanes and kernels. This implies that for a generic $y \in Y_0$ the integral hypersurface of the hyperplane field T through y is a confocal quadric to U. Passing to limit, as y tends to a point of the integral hypersurface S, we get that S is a confocal quadric as well.

Subcase 1.2: U is a general regular quadric. Then it is a limit of generic quadrics U_n in the above sense. For each U_n the integral hypersurfaces of the corresponding above hyperplane field T_n are quadrics confocal to U_n. Passing to limit, as $n \to \infty$, we get the same statement for the hyperplane field T associated to U. Hence, S is a quadric confocal to U.

Case 2: There exists an open subset of points $y \in S$ for which L_y lies in at least two-dimensional kernel of the form $\langle (Q - \lambda_j(y))x, x\rangle|_{W_y}$ corresponding to a multiple eigenvalue $\lambda_j(y)$. In this case the latter kernel contains at least two linearly independent vectors $w_1, w_2 \in W_y$, and by definition, both of them are orthogonal to the hyperplane W_y with respect to the scalar product $\langle (Q-\lambda G)x, x\rangle$, $\lambda = \lambda_j(y)$. Hence, their appropriate nonzero linear combination $w = a_1 w_1 + a_2 w_2$ is orthogonal to the whole ambient space $\mathbb{R}^{d+1}$ with respect to the same scalar product. Therefore, w lies in the kernel of the same scalar product taken on all of $\mathbb{R}^{d+1}$, and thus, λ is such that the matrix $Q - \lambda G$ is degenerate: then we'll call such a λ a *global eigenvalue*. The number of global eigenvalues λ is at most $d+1$, and all of them are independent on y.

Finally, there exist a global eigenvalue λ and an open subset $S_0 \subset S$ such that for every $y \in S_0$ one has $\langle (Q - \lambda)x, x\rangle \equiv 0$ on L_y, since L_y lies in the kernel of the restriction to W_y of the scalar product $\langle (Q - \lambda)x, x\rangle$.

Thus, for $y \in S_0$ the projections $p(y) = \pi(L_y \setminus \{0\}) \in \mathbb{RP}^d$ lie in a degenerate quadric $\Gamma \subset \mathbb{RP}^d$ defined by the equation $\langle (Q - \lambda)x, x\rangle = 0$. The points $p(y)$ form the dual hypersurface S_0^*, by definition. Hence, S_0^* lies in a degenerate quadric Γ. This contradicts nondegeneracy of the second fundamental form of the hypersurface S. Hence, the case under consideration is impossible. Lemma 2.13 is proved. □

Proof of Theorem 1.10. The hypersurfaces U and V lie in the same quadric in Σ, which will be now denoted by U (Theorem 2.1). The quadric U is a caustic for the hypersurface S: for every $y \in S$ the cone of geodesics through y that are tangent to U is symmetric with respect to the hyperplane tangent to $T_y S$. Therefore, S is a quadric confocal to U, by Lemma 2.13. This proves Theorem 1.10. □

3. Commuting billiards and caustics: Proof of Theorem 1.7

Proposition 3.1. *Let Σ be a space form of constant curvature of dimension $d \geq 2$. Let two nested strictly convex C^2-smooth closed hypersurfaces $a, b \subset \Sigma$, $a \Subset \Omega_b$ (see the notations at the beginning of the paper) be such that the corresponding billiard transformations σ_a and σ_b commute. Then a is a caustic for the hypersurface b.*

Proof. Let Π_a denote the open subset of geodesics in Σ that are disjoint from the hypersurface a. Its boundary $\partial\Pi_a$ consists of those geodesics that are tangent to a. A geodesic L is fixed by σ_a, if and only if $L \in \overline{\Pi}_a$, i.e., L is either disjoint from a, or tangent to a. In this case $\sigma_b\sigma_a(L) = \sigma_b(L) = \sigma_a\sigma_b(L)$, and thus, $\sigma_b(L)$ is a fixed point of the transformation σ_a. This implies that $\sigma_b(\overline{\Pi}_a) \subset \overline{\Pi}_a$. The subset $\overline{\Pi}_a$ is invariant under two transformations acting on oriented geodesics: the reflection σ_b and the transformation J of the orientation change. The transformations J and $J \circ \sigma_b$ are involutions. Hence, they are homeomorphisms of the whole space of oriented geodesics in Σ. Their restrictions to the common invariant subset $\overline{\Pi}_a$ should be also a homeomorphism: an involution acting on a set is obviously always bijective. Therefore, each of them sends the boundary $\partial\Pi_a$ onto itself homeomorphically, and the same is true for their composition $\sigma_b = J \circ (J \circ \sigma_b)$:

$$\sigma_b(\partial\Pi_a) = \partial\Pi_a.$$

The latter equality means exactly that a is a caustic for the hypersurface b. The proposition is proved. □

Proof of Theorems 1.7 and 1.2. Let $a, b \subset \Sigma$ be two nested strictly convex C^2-smooth closed hypersurfaces in a space form Σ with commuting billiard transformations, $a \Subset \Omega_b$, $\dim \Sigma \geq 3$. Then a is a caustic for the hypersurface b, by Proposition 3.1. This means that, for every point $B \in b$ and $A \in a$ such that the line AB is tangent to a at A, the image $\sigma_b(AB)$ of the line AB (oriented from A to B) is a line through B tangent to a. Recall that a and b are strictly convex, which implies that their second fundamental forms are sign-definite and thus, nondegenerate. Therefore, for every A and B as above the germs at A and B of the hypersurfaces $U = a$ and $S = b$ respectively satisfy the conditions of Theorem 1.10, with V being the germ of the hypersurface a at its point D of tangency with the line $\sigma_b(AB)$. Hence, for every A and B as above the germ (S, B) lies in a quadric, and the germs (U, A), (V, D) lie in one and the same quadric confocal to S. This implies that b is a quadric, and a is a quadric confocal to b. Theorems 1.7 and 1.2 are proved. □

4. A tangential local version of Theorem 1.7

Theorem 4.1. *Let $d \geq 3$. Let (U, A), (S, B), (V, D) be germs of C^2-smooth hypersurfaces in a d-dimensional space form Σ at points A, B and D. Let $B \neq A, D$, and let U and S have nondegenerate second fundamental forms. For every $Z = U, S, V$ consider the action of the reflection σ_Z on the oriented geodesics that intersect Z, defined as at the beginning of the paper: we reflect the geodesic at its last intersection point with the hypersurface Z. Let L_0 be a geodesic through B transversal to S and quadratically tangent to U at A (we orient it from A to B), and let its image $\sigma_S(L_0)$ be quadratically tangent to V at D. Let W be a small neighborhood of the geodesic L_0 in the space of oriented geodesics; in particular, each point in W represents a geodesic intersecting S transversally. Let $\Pi_W \subset W$ denote the subset of those geodesics that intersect U. For every $L \in \Pi_W$ let the image $\sigma_S(L)$ intersect V; more precisely, we suppose that the compositions $\sigma_S \circ \sigma_U$ and $\sigma_V \circ \sigma_S$ are well-defined on Π_W. Let the latter compositions be identically equal on Π_W. Then S lies in a quadric b, and U, V lie in one and the same quadric confocal to b.*

Proof. Every geodesic L tangent to U and close enough to L_0 lies in Π_W. Its image $\sigma_U(L)$ coincides with L (by definition), and hence, $\sigma_S \circ \sigma_U(L) = \sigma_S(L) = \sigma_V \circ \sigma_S(L)$. Thus, the geodesic $\sigma_S(L)$, which should intersect V by assumption, is fixed by σ_V. Hence it is tangent to V (at the last point of its intersection with V). Finally, the germs of hypersurfaces U, S and V satisfy the conditions of Theorem 1.10. Therefore, S lies in a quadric b, and U, V lie in one and the same quadric confocal to b, by Theorem 1.10. This proves Theorem 4.1. □

5. Open problems

The billiards in space forms are particular cases of the projective billiards introduced in [Tabachnikov 1997]. The main results of the present paper (Theorem 1.10 extending Berger's result on caustics [1995], Theorem 1.7 on commuting billiards) are proved for billiards in space forms. It would interesting to extend them to projective billiards.

Problem 1 (appeared as a result of our discussion with Sergei Tabachnikov). Let $S \subset \mathbb{R}^d$, $d \geq 3$ be a germ of hypersurface at a point B equipped with a field Λ of one-dimensional subspaces $\Lambda_y \subset T_y\mathbb{R}^d$, $y \in S$, transversal to S. Consider the family of linear involutions $\sigma_y : T_y\mathbb{R}^d \to T_y\mathbb{R}^d$, $y \in S$, that fix each point of the hyperplane T_yS and have Λ_y as an eigenline with eigenvalue -1. Let there exist two germs of hypersurfaces U and V at points $A, C \neq B$ respectively such that the lines AC, BC are tangent to U and V at points A and C respectively and for every $y \in S$ each line through y that is tangent to U is reflected by σ_y to a line tangent to V. (The defined action of the reflections σ_y on oriented lines transversal to S is called the *projective billiard transformation*, and the pair (S, Λ) is called a *projective billiard*; see [Tabachnikov 1997].) Is it true that then U and V lie in one and the same quadric?

Problem 2 (Tabachnikov). Classify commuting nested pairs of projective billiards in $\mathbb{R}^d$, $d \geq 2$.

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ON THE ARITHMETIC OF A FAMILY OF TWISTED CONSTANT ELLIPTIC CURVES

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Let $\mathbb{F}_r$ be a finite field of characteristic $p > 3$. For any power q of p, consider the elliptic curve $E = E_{q,r}$ defined by $y^2 = x^3 + t^q - t$ over $K = \mathbb{F}_r(t)$. We describe several arithmetic invariants of E such as the rank of its Mordell–Weil group $E(K)$, the size of its Néron–Tate regulator $\mathrm{Reg}(E)$, and the order of its Tate–Shafarevich group $\text{Ш}(E)$ (which we prove is finite). These invariants have radically different behaviors depending on the congruence class of p modulo 6. For instance $\text{Ш}(E)$ either has trivial p-part or is a p-group. On the other hand, we show that the product $|\text{Ш}(E)|\,\mathrm{Reg}(E)$ has size comparable to $r^{q/6}$ as $q \to \infty$, regardless of $p \pmod 6$. Our approach relies on the BSD conjecture, an explicit expression for the L-function of E, and a geometric analysis of the Néron model of E.

1. Introduction

For a prime $p > 3$, and powers q and r of p, we study the elliptic curve

$$E: \quad y^2 = x^3 + t^q - t$$

over the rational function field $K = \mathbb{F}_r(t)$. We are interested in the Mordell–Weil group $E(K)$, its regulator $\mathrm{Reg}(E)$, and the Tate–Shafarevich group $\text{Ш}(E)$ of E. By old results of Tate [1966] and Milne [1975], $\text{Ш}(E)$ is finite and the conjecture of Birch and Swinnerton-Dyer holds for E.

One of our main results says that $\mathrm{Reg}(E)\,|\text{Ш}(E)|$ is an integer comparable in archimedean size to $r^{q/6}$ when r is fixed and q tends to ∞. (See Theorem 11.1 for the precise statement.) On the other hand, we will show that if $p \equiv 1 \pmod 6$, then $E(K) = 0$, $\mathrm{Reg}(E) = 1$, and $|\text{Ш}(E)|$ is a p-adic unit; and that if $p \equiv -1 \pmod 6$ and $\mathbb{F}_r$ is sufficiently large, then $E(K)$ has rank $2(q-1)$, $\mathrm{Reg}(E)\,|\text{Ш}(E)|$ is a power of p, and $\text{Ш}(E)$ is a p-group (Propositions 8.3.1 and 8.4.1, and Corollary 9.2). These results show in particular that the archimedean and p-adic sizes of $\mathrm{Reg}(E)\,|\text{Ш}(E)|$ are independent — in our examples, $\mathrm{Reg}(E)\,|\text{Ш}(E)|$ is

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	$p \equiv 1 \pmod 6$	$p \equiv -1 \pmod 6$
$E(K)_{\mathrm{tors}}$	$\cong \{0\}$ (Proposition 2.4(2))	
BSD conjecture	holds for E (Theorem 8.2)	
Rank $E(K)$	$= 0$ (Proposition 8.3.1(3))	$= 2(q-1)$ for $\mathbb{F}_r$ large enough (Proposition 8.4.1(3))
$\mathrm{Reg}(E)$	$= 1$ (Proposition 8.3.1(4))	is a power of p for $\mathbb{F}_r$ large enough (Corollary 9.2(3))
$\text{Ш}(E)$	has trivial p-part (Proposition 10.1(1))	is a p-group (Corollary 9.2(3))
$\dim \text{Ш}(E)$	$= 0$ (Corollary 9.3(1))	$= \lfloor q/6 \rfloor$ (Corollary 9.3(2))
$\lim_{q\to\infty} \mathrm{BS}(E)$	$= 1$ (Theorem 11.1)	
$\lvert\text{Ш}(E)\rvert \, \mathrm{Reg}(E)$	$\geq r^{\lfloor q/6 \rfloor (1+o(1))}$ as $q \to \infty$ (Corollary 11.9)	$= r^{\lfloor q/6 \rfloor}$ for $\mathbb{F}_r$ large enough (Corollary 9.2(3))

Table 1. A summary of the main results of the paper.

large in the archimedean metric, whereas it may be a p-adic unit or divisible by a large power of p.

To prove these results, we combine an analytic analysis of the special value $L^*(E)$, the Birch and Swinnerton-Dyer (BSD) formula, and an algebraic analysis of $\text{Ш}(E)$. We are able to deduce the BSD formula and analyze $\text{Ш}(E)$ by using the fact that the Néron model $\mathcal{E} \to \mathbb{P}^1$ of E is birational to the quotient of a product of curves by a finite group. In fact, $\mathcal{E}$ has three distinct such presentations, and each is convenient for some aspect of our study.

The plan of the paper is as follows: In the next section, we gather the basic definitions and present a few preliminary results about E. In Section 3, we recall standard results about Gauss and Jacobi sums and use them in Section 4 to give an elementary calculation of the Hasse–Weil L-function of E. In Section 5, we prove results about the geometry and cohomology of certain curves over $\mathbb{F}_r$ which are used in Section 6 to show that the Néron model of E is dominated by a product of curves (in multiple ways). In Section 7, we use these dominations to give alternate calculations of the L-function. In Section 8, we apply the BSD conjecture to study the rank of $E(K)$, and in Section 9 we study the p-adic size of the special value and the order of $\text{Ш}(E)$ using the BSD formula. Section 10 reproves our results about $\text{Ш}(E)$ by a direct, algebraic approach, i.e., independently of the BSD formula. In Section 11, we study the archimedean size of the special value and the "Brauer–Siegel ratio" of Hindry.

Table 1 summarizes our main results. There, "for $\mathbb{F}_r$ large enough" means that there is a finite extension $\mathbb{F}_{r_0}$ of $\mathbb{F}_p$ such that the statement holds for all finite extensions $\mathbb{F}_r$ of $\mathbb{F}_{r_0}$ (see Proposition 8.4.1(3) for an explicit definition of r_0).

2. First results

2.1. ***Definitions and notation.*** Notation from this section will be in force throughout the paper. We refer to [Ulmer 2011] for a review of what is known about elliptic curves over function fields, in particular with regard to the conjecture of Birch and Swinnerton-Dyer.

Let $p > 3$ be a prime number, let $\mathbb{F}_p$ be the field of p elements, and fix an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Let $\mathbb{F}_r \subset \overline{\mathbb{F}}_p$ be the finite extension of $\mathbb{F}_p$ of cardinality $r = p^\nu$, and let $K = \mathbb{F}_r(t)$ be the rational function field over $\mathbb{F}_r$. We write v for a place of K, K_v for the completion of K at v, $\deg(v)$ for the degree of v, $\mathbb{F}_v$ for the residue field at v, and $r_v = r^{\deg(v)}$ for the cardinality of $\mathbb{F}_v$. We identify places of K with closed points of the projective line $\mathbb{P}^1_{\mathbb{F}_r}$ over $\mathbb{F}_r$, and we note that finite places of K are in bijection with monic irreducible polynomials in $\mathbb{F}_r[t]$.

Let $q = p^f$ be a power of p, and let E be the elliptic curve over K defined by

$$E = E_{q,r} : \; y^2 = x^3 + t^q - t. \tag{2-1}$$

Write $E(K)$ for the group of K-rational points on E. By the Lang–Néron theorem, this is a finitely generated abelian group.

Let $\mathcal{E} \to \mathbb{P}^1_{\mathbb{F}_r}$ be the Néron model of E. We write c_v for the number of connected components in the special fiber of $\mathcal{E}$ over v. One also calls c_v the local Tamagawa number of E at v.

We denote the (differential) height of E, as defined in [Ulmer 2011, Lecture 3, §2], by $\deg(\omega_E)$. It follows from [Ulmer 2011, Lecture 3, Exercise 2.2] that for E,

$$\deg(\omega_E) = \lceil q/6 \rceil = \begin{cases} \frac{q+5}{6} & \text{if } q \equiv 1 \pmod 6, \\ \frac{q+1}{6} & \text{if } q \equiv -1 \pmod 6. \end{cases}$$

2.2. ***Reduction types.*** From the Weierstrass equation (2-1), one easily computes

$$\Delta = -2^4 3^3 (t^q - t)^2 \quad \text{and} \quad j(E) = 0.$$

Applying Tate's algorithm (see [Silverman 1994, Chapter IV, §9]), one obtains the following further facts:

- At a finite place dividing $t^q - t$, the curve E has additive reduction of type II.
- At $t = \infty$, the curve E has additive reduction of type II* if $q \equiv 1 \pmod 6$ and of type II if $q \equiv 5 \pmod 6$.
- The curve E has good reduction at all other places of K.

From this collection of local information, one deduces that the conductor $\mathcal{N}_E$ of E has degree $\deg \mathcal{N}_E = 2(q+1)$. One can also recover the fact that $\deg(\omega_E) = \lceil q/6 \rceil$ from this computation.

2.3. ***Isotriviality.*** Consider the finite extension $L = K[u]/(u^6 = t^q - t)$ of K, and let E_0 be the elliptic curve over $\mathbb{F}_r$ defined by

$$E_0 : w^2 = z^3 + 1.$$

Then $E \times_K L$ is isomorphic to the constant curve $E_0 \times_{\mathbb{F}_r} L$ via the substitution $(x, y) = (u^2 z, u^3 w)$. In other words, E is the sextic twist of E_0 (or rather of $E_0 \times_{\mathbb{F}_r} K$) by $t^q - t$.

We record two consequences for later use. Recall that the local Tamagawa number c_v is the number of components in the special fiber of the Néron model at v. Its values in terms of the local reduction type are tabulated in [Silverman 1994, p. 365].

Proposition 2.4. (1) *For every place v of K, the local Tamagawa number c_v is* 1.

(2) $E(K)_{\text{tors}} = 0$.

Proof. Part (1) is immediate from the table cited above. For part (2), suppose that $P \in E(K)$ is a nontrivial torsion point. Let $Q = (\alpha, \beta) \in E_0(L)$ be the image of P under the above isomorphism $E \times_K L \cong E_0 \times_{\mathbb{F}_r} L$. Then Q is again a torsion point, and it is known (e.g., [Ulmer 2011, Proposition I.6.1]) that torsion points on a constant curve have constant coordinates. That is, we have $\alpha, \beta \in \mathbb{F}_r$. The original point P thus has coordinates $(\alpha u^2, \beta u^3)$. However, if $\alpha \in \mathbb{F}_r$, then $\alpha u^2 \in K$ only if $\alpha = 0$, and if $\beta \in \mathbb{F}_r$, then $\beta u^3 \in K$ only if $\beta = 0$. Since $(0, 0) \notin E(K)$, there is no nontrivial torsion point $P \in E(K)$. □

3. Preliminaries on exponential sums

3.1. ***Finite fields.*** Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and a prime ideal $\mathfrak{P}$ above p in the ring of algebraic integers $\overline{\mathbb{Z}} \subset \overline{\mathbb{Q}}$. The quotient $\overline{\mathbb{Z}}/\mathfrak{P}$ is then an algebraic closure of $\mathbb{F}_p$ which we denote by $\overline{\mathbb{F}}_p$. All finite fields in this paper will be viewed as subfields of this $\overline{\mathbb{F}}_p$.

3.2. ***Multiplicative characters.*** Reduction modulo $\mathfrak{P}$ induces an isomorphism between the roots of unity of order prime to p in $\overline{\mathbb{Z}}$ and $\overline{\mathbb{F}}_p^\times$. We let $\boldsymbol{t} : \overline{\mathbb{F}}_p^\times \to \overline{\mathbb{Q}}^\times$ denote the inverse of this isomorphism. The same letter $\boldsymbol{t}$ will be used to denote the restriction of $\boldsymbol{t}$ to the multiplicative group of any finite extension $\mathbb{F}$ of $\mathbb{F}_p$ ($\mathbb{F}$ being viewed as a subextension of $\overline{\mathbb{F}}_p$).

If $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$ and n is a divisor of $|\mathbb{F}^\times|$, define

$$\chi_{\mathbb{F},n} := \boldsymbol{t}^{|\mathbb{F}^\times|/n}.$$

This is a character of $\mathbb{F}^\times$ of order exactly n. In particular, if $n = |\mathbb{F}^\times|$, the character $\chi_{\mathbb{F},n}$ is a generator of the group of multiplicative characters of $\mathbb{F}$.

If $\mathbb{F} \subset \mathbb{F}'$ are finite extensions of $\mathbb{F}_p$, if n divides the order of $\mathbb{F}^\times$, and if $\mathrm{N}_{\mathbb{F}'/\mathbb{F}}$ denotes the norm from $\mathbb{F}'$ to $\mathbb{F}$, then an easy calculation shows that $\chi_{\mathbb{F}',n} = \chi_{\mathbb{F},n} \circ \mathrm{N}_{\mathbb{F}'/\mathbb{F}}$.

3.3. ***Additive characters.*** Fix once and for all a nontrivial additive character

$$\psi_p : \mathbb{F}_p \to \mathbb{Q}(\mu_p)^\times \subset \overline{\mathbb{Q}}^\times.$$

If $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$, if $\mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p}$ denotes the trace from $\mathbb{F}$ to $\mathbb{F}_p$, and if $\alpha \in \mathbb{F}^\times$, then the map $x \mapsto \psi_\alpha(x)$ defined by

$$\psi_\alpha(x) = \psi_p(\mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p}(\alpha x))$$

for all $x \in \mathbb{F}$ is a nontrivial additive character of $\mathbb{F}$. Moreover, any nontrivial additive character of $\mathbb{F}$ is of the form ψ_α for a unique $\alpha \in \mathbb{F}^\times$. When we need to make the underlying field precise, we write $\psi_{\mathbb{F},\alpha}$ instead of ψ_α.

3.4. ***Gauss sums.*** If $\mathbb{F}$ is a finite extension of $\mathbb{F}_p$, χ is a nontrivial character of $\mathbb{F}^\times$, and ψ is a nontrivial additive character of $\mathbb{F}$, define the Gauss sum $G_{\mathbb{F}}(\chi, \psi)$ by

$$G_{\mathbb{F}}(\chi, \psi) = -\sum_{x \in \mathbb{F}^\times} \chi(x)\psi(x).$$

We recall a few well-known properties of these Gauss sums:

(1) If χ has order n, the sum $G_{\mathbb{F}}(\chi, \psi)$ is an algebraic integer in $\mathbb{Q}(\mu_{np})$.

(2) For any nontrivial characters χ and ψ, one has $|G_{\mathbb{F}}(\chi, \psi)| = |\mathbb{F}|^{1/2}$ in any complex embedding of $\overline{\mathbb{Q}}$.

(3) For all nontrivial multiplicative characters χ on $\mathbb{F}^\times$ and all $\alpha \in \mathbb{F}^\times$, one has
$$G_{\mathbb{F}}(\chi, \psi_\alpha) = \chi^{-1}(\alpha) G_{\mathbb{F}}(\chi, \psi_1).$$

(4) (Hasse–Davenport relation) Let χ be a nontrivial multiplicative character on $\mathbb{F}^\times$ and ψ be a nontrivial additive character on $\mathbb{F}$. Then for any finite extension $\mathbb{F}'/\mathbb{F}$, one has
$$G_{\mathbb{F}'}(\chi \circ \mathrm{N}_{\mathbb{F}'/\mathbb{F}}, \psi \circ \mathrm{Tr}_{\mathbb{F}'/\mathbb{F}}) = G_{\mathbb{F}}(\chi, \psi)^{[\mathbb{F}':\mathbb{F}]}.$$

(5) (Stickelberger's theorem) Let ord be the p-adic valuation of $\overline{\mathbb{Q}}$ associated to $\mathfrak{P}$, normalized so that $\mathrm{ord}(p) = 1$. If $\mathbb{F}$ has cardinality p^μ and $0 < s < p^\mu - 1$ has p-adic expansion
$$s = s_0 + s_1 p + \cdots + s_{\mu-1} p^{\mu-1}$$
with $0 \le s_i < p$, then
$$\mathrm{ord}(G_{\mathbb{F}}(\chi_{\mathbb{F},|\mathbb{F}^\times|}^{-s}, \psi)) = \frac{1}{p-1} \sum_{i=0}^{\mu-1} s_i.$$

These results are classical, and the reader may find proofs of them (and the claims in the next two subsections) in [Washington 1997, Chapter VI, §1–§2] for instance.

3.5. ***Explicit Gauss sums.*** Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$, and write $|\mathbb{F}| = p^\mu$. An elementary calculation shows that, for any nontrivial additive character ψ of $\mathbb{F}$,

$$G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi)^2 = ((-1)^{(p-1)/2} p)^\mu. \tag{3-1}$$

In particular, $\operatorname{ord} G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi) = \mu/2$. Here, as above, ord denotes the p-adic valuation on $\overline{\mathbb{Q}}$ associated to $\mathfrak{P}$, normalized to that $\operatorname{ord}(p) = 1$.

If $p \equiv 1 \pmod 3$, then Stickelberger's theorem (see (5) above) shows that, for any nontrivial additive character ψ of $\mathbb{F}$,

$$\operatorname{ord} G_{\mathbb{F}}(\chi_{\mathbb{F},3}, \psi) = \tfrac{2}{3}\mu \quad \text{and} \quad \operatorname{ord} G_{\mathbb{F}}(\chi_{\mathbb{F},3}^{-1}, \psi) = \tfrac{1}{3}\mu. \tag{3-2}$$

On the other hand, if $p \equiv 2 \pmod 3$, then 3 divides $|\mathbb{F}^\times|$ if and only if $\mu = [\mathbb{F} : \mathbb{F}_p]$ is even. If this is the case (i.e., if $|\mathbb{F}| = p^\mu \equiv 1 \pmod 3$), an old result of Tate and Shafarevich (see [Ulmer 2002, Lemma 8.2]) and the Hasse–Davenport relation yield that

$$G_{\mathbb{F}}(\chi_{\mathbb{F},3}, \psi_1) = G_{\mathbb{F}}(\chi_{\mathbb{F},3}^{-1}, \psi_1) = (-p)^{\mu/2},$$

and therefore (see (3) in the previous subsection)

$$G_{\mathbb{F}}(\chi_{\mathbb{F},3}, \psi_\alpha) = \chi_{\mathbb{F},3}^{-1}(\alpha)(-p)^{\mu/2} \quad \text{and} \quad G_{\mathbb{F}}(\chi_{\mathbb{F},3}^{-1}, \psi_\alpha) = \chi_{\mathbb{F},3}(\alpha)(-p)^{\mu/2}. \tag{3-3}$$

In particular, $\operatorname{ord} G_{\mathbb{F}}(\chi_{\mathbb{F},3}^{\pm 1}, \psi_\alpha) = \mu/2$ in this case.

3.6. ***Jacobi sums.*** We require only the simplest case: Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ and let χ_1 and χ_2 be two nontrivial characters of $\mathbb{F}^\times$ such that $\chi_1\chi_2$ is also nontrivial. Define

$$J_{\mathbb{F}}(\chi_1, \chi_2) = -\sum_{x \in \mathbb{F}} \chi_1(x)\chi_2(1-x).$$

An elementary calculation (see [Washington 1997, Chapter VI]) shows that

$$J_{\mathbb{F}}(\chi_1, \chi_2) = \frac{G_{\mathbb{F}}(\chi_1, \psi) G_{\mathbb{F}}(\chi_2, \psi)}{G_{\mathbb{F}}(\chi_1\chi_2, \psi)} \tag{3-4}$$

for any nontrivial additive character ψ of $\mathbb{F}$. One may then deduce the archimedean and p-adic sizes of $J(\chi_1, \chi_2)$ from the results quoted in Section 3.4.

3.7. ***Orbits.*** Recall that $p > 3$ is a prime. Given an integer $n \geq 1$ prime to p, let

$$S = S_{n,q} = (\mathbb{Z}/n\mathbb{Z} \smallsetminus \{0\}) \times \mathbb{F}_q^\times \quad \text{and} \quad S^\times = S_{n,q}^\times = (\mathbb{Z}/n\mathbb{Z})^\times \times \mathbb{F}_q^\times.$$

Let $r = p^\nu$ for some positive integer ν. Write $\langle r \rangle$ for the subgroup of $\mathbb{Q}^\times$ generated by r, and consider the action of $\langle r \rangle$ on S and $S^\times$ given by the rule

$$r(i, \alpha) := (ri, \alpha^{1/r}) \quad \text{for all } (i, \alpha) \in S.$$

In other words, r acts on $\mathbb{Z}/n\mathbb{Z}$ by multiplication, and on $\mathbb{F}_q^\times$ by the inverse of the r-power Frobenius. Let $O_{r,n,q}$ be the set of orbits of $\langle r\rangle$ on S and $O_{r,n,q}^\times$ the set of orbits on $S^\times$.

If $n = 1$, then $O_{r,n,q}^\times$ is just the set of orbits of $\langle r\rangle$ on $\mathbb{F}_q^\times$, which we denote by $O_{r,q}$. Note that if $o \in O_{r,q}$ is the orbit through α, then the cardinality $|o|$ of o is equal to the degree $[\mathbb{F}_r(\alpha) : \mathbb{F}_r]$ of the field extension $\mathbb{F}_r(\alpha)$ over $\mathbb{F}_r$.

For a general n, if $o \in O_{r,n,q}^\times$ is the orbit through (i, α), then

$$|o| = \operatorname{lcm}(\operatorname{ord}^\times(r \bmod n), [\mathbb{F}_r(\alpha) : \mathbb{F}_r]), \tag{3-5}$$

where $\operatorname{ord}^\times(r \bmod n)$ denotes the order of r in $(\mathbb{Z}/n\mathbb{Z})^\times$. Note that, for any $\alpha \in \mathbb{F}_q$, one has $[\mathbb{F}_r(\alpha) : \mathbb{F}_r] = \operatorname{lcm}(\nu, [\mathbb{F}_p(\alpha) : \mathbb{F}_p])/[\mathbb{F}_p(\alpha) : \mathbb{F}_p]$, and $[\mathbb{F}_p(\alpha) : \mathbb{F}_p]$ divides $f = [\mathbb{F}_q : \mathbb{F}_p]$. It is then clear that $|o|$ divides $\operatorname{lcm}(\operatorname{ord}^\times(r \bmod n), \operatorname{lcm}(f, \nu)/f)$ for any orbit $o \in O_{r,n,q}^\times$.

In what follows, we will only need the cases where n divides 6. If $r \equiv 1 \pmod 6$, then $\langle r\rangle$ acts trivially on $\mathbb{Z}/6\mathbb{Z}$ and the orbits $o \in O_{r,6,q}$ are "vertical" in the sense that they are of the form $o = \{(i, \alpha)\}$ where i is fixed and α runs through an orbit of $\langle r\rangle$ on $\mathbb{F}_q^\times$. In particular, $|o| = [\mathbb{F}_r(\alpha) : \mathbb{F}_r]$.

On the other hand, if $r \equiv 5 \equiv -1 \pmod 6$, then orbits $o \in O_{r,6,q}$ "bounce left and right" in the sense that an orbit o contains elements (i, α) and $r(i, \alpha) = (-i, \alpha^{1/r})$. In this case, if o is the orbit through (i, α), then $|o| = \operatorname{lcm}(2, [\mathbb{F}_r(\alpha) : \mathbb{F}_r])$.

In both cases (that is to say, for $r \equiv \pm 1 \pmod 6$), note that $\nu|o|$ is even for all orbits $o \in O_{r,6,q}^\times$.

For $n \in \{2, 3\}$, the natural projection $(\mathbb{Z}/6\mathbb{Z})^\times \to (\mathbb{Z}/n\mathbb{Z})^\times$ induces a map $\pi_n : O_{r,6,q}^\times \to O_{r,n,q}^\times$. We record a few elementary observations about π_n:

- The map π_3 is a bijection, because $(\mathbb{Z}/6\mathbb{Z})^\times \to (\mathbb{Z}/3\mathbb{Z})^\times$ is a bijection.
- If $r \equiv 1 \pmod 6$, then π_2 is two-to-one. (This is essentially the same point as the "vertical" remark above.)
- If $r \equiv -1 \pmod 6$ and if $o' \in O_{r,2,q}^\times$ has $|o'|$ even, then there are two orbits $o \in O_{r,6,q}^\times$ with $\pi_2(o) = o'$. Finally, if $r \equiv -1 \pmod 6$ and if $o' \in O_{r,2,q}^\times$ has $|o'|$ odd, then there is a unique orbit $o \in O_{r,6,q}^\times$ with $\pi_2(o) = o'$ and the underlying map of sets $o \to o'$ is two-to-one.

Motivated by this last remark, for any $o \in O_{r,6,q}^\times$, we define

$$m_2(o) = \frac{|o|}{|\pi_2(o)|}.$$

Thus $m_2(o) = 1$ unless $r \equiv -1 \pmod 6$ and $|\pi_2(o)|$ is odd, in which case $m_2(o) = 2$.

3.8. ***Gauss sums associated to orbits.*** Fix data p, r, q, and n as above, and let $o \in O_{r,n,q}$ be the orbit of $\langle r\rangle$ through $(i, \alpha) \in S_{n,q} = (\mathbb{Z}/n\mathbb{Z} \smallsetminus \{0\}) \times \mathbb{F}_q^\times$. Let $\mathbb{F} = \mathbb{F}_{r^{|o|}}$, i.e., $\mathbb{F}$ is the extension of $\mathbb{F}_r$ of degree $|o|$. By formula (3-5) for $|o|$, $\mathbb{F}$ can be interpreted as the smallest extension of $\mathbb{F}_r$ which admits a multiplicative character of order n and contains α. To the orbit o we then associate the Gauss sum

$$G(o) = G_{\mathbb{F}}(\chi_{\mathbb{F},n}^{i}, \psi_\alpha), \tag{3-6}$$

where $\chi_{\mathbb{F},n}$ and ψ_α are the characters on $\mathbb{F}$ defined in Sections 3.2 and 3.3. An elementary computation, as in [Cohen 2007, Lemma 2.5.8], shows that

$$G_{\mathbb{F}}(\chi, \psi_\alpha) = G_{\mathbb{F}}(\chi^p, \psi_{\alpha^{1/p}}),$$

so that $G(o)$ is indeed well defined independently of the choice of element $(i, \alpha) \in o$.

We next record the valuations of Gauss sums associated to orbits for $n = 2$ and 3. These claims follow immediately from the results of Section 3.5.

When $n = 2$, we have $\operatorname{ord}(G(o)) = \frac{1}{2}\nu|o|$ for all orbits $o \in O_{r,2,q}^{\times}$.

When $n = 3$, $p \equiv 1 \pmod 3$, and $o \in O_{r,3,q}^{\times}$, then

$$\operatorname{ord}(G(o)) = \begin{cases} \frac{2}{3}\nu|o| & \text{if } o \text{ contains an element } (1, \alpha), \\ \frac{1}{3}\nu|o| & \text{if } o \text{ contains an element } (-1, \alpha). \end{cases}$$

When $n = 3$ and $p \equiv -1 \pmod 3$, then $\operatorname{ord}(G(o)) = \frac{1}{2}\nu|o|$ for all $o \in O_{r,3,q}^{\times}$.

The following shows that the Gauss sums $G(o)$ "decompose" as roots of unity times powers of Gauss sums of small weight. This will play a key role in our estimation of the archimedean size of $\operatorname{Reg}(E)\,|Ш(E)|$ in Section 11.

Proposition 3.9. *Let $n \geq 1$ be an integer coprime to p, and write*

$$c := \operatorname{ord}^{\times}(p \bmod n)$$

for the order of p modulo n. Then for all $o \in O_{r,n,q}$, one has

$$G(o) = \zeta g^{|o|\nu/c},$$

where ζ is an n-th root of unity, and $g \in \mathbb{Q}(\mu_{np})$ is a Weil integer of size $p^{c/2}$.

Recall that an algebraic number $z \in \overline{\mathbb{Q}}$ is called *a Weil integer of size p^a* (with $a \in \frac{1}{2}\mathbb{Z}_{\geq 0}$) if z is an algebraic integer such that $|z| = p^a$ in any complex embedding $\mathbb{Q}(z) \hookrightarrow \mathbb{C}$. (These numbers are also sometimes called p-Weil integers of weight $2a$.)

Proof. Note that $\mathbb{F}_{p^c}^{\times}$ admits characters of order exactly n. By definition, for any choice of representative $(i, \alpha) \in o$, we have

$$G(o) = G_{\mathbb{F}}(\chi_{\mathbb{F},n}^{i}, \psi_{\mathbb{F},\alpha}),$$

where $\mathbb{F}$ is the extension of $\mathbb{F}_r$ of degree $|o|$, i.e., $|\mathbb{F}| = p^{|o|\nu}$. By construction, c divides $\nu|o|$, so $\mathbb{F}$ is an extension of $\mathbb{F}_{p^c}$. Then the following holds:

$$\begin{aligned} G(o) = G_{\mathbb{F}}(\chi_{\mathbb{F},n}^{i}, \psi_{\mathbb{F},\alpha}) &= \chi_{\mathbb{F},n}^{-i}(\alpha) G_{\mathbb{F}}(\chi_{\mathbb{F},n}^{i}, \psi_{\mathbb{F},1}) && \text{(by (3) in Section 3.4)} \\ &= \chi_{\mathbb{F},n}^{-i}(\alpha) G_{\mathbb{F}_{p^c}}(\chi_{\mathbb{F}_{p^c},n}^{i}, \psi_{\mathbb{F}_{p^c},1})^{|o|\nu/c} && \text{(by the Hasse–Davenport relation).} \end{aligned}$$

We now let $\zeta := \chi_{\mathbb{F},n}^{-i}(\alpha)$ and $g = G_{\mathbb{F}_{p^c}}(\chi_{\mathbb{F}_{p^c},n}, \psi_{\mathbb{F}_{p^c},1})$. Since $\chi_{\mathbb{F},n}$ has order n, ζ is an n-th root of unity. By (1) and (2) in Section 3.4, g is a Weil integer in $\mathbb{Q}(\mu_{np})$ of size $p^{c/2}$. □

3.10. ***Jacobi sums associated to orbits.*** With data p and r as usual, let $\langle r\rangle$ act on $(\mathbb{Z}/6\mathbb{Z})^\times$ by multiplication, and let $N = N_{r,6}$ be the set of orbits of $\langle r\rangle$ on $(\mathbb{Z}/6\mathbb{Z})^\times$. Thus, if $r \equiv 1 \pmod 6$, there are two orbits, both singletons, and if $r \equiv -1 \pmod 6$, there is a unique orbit, $o = \{1, -1\}$. (This is a somewhat trivial situation, but we introduce it for consistency with our treatment of Gauss sums.) Given $o \in N_{r,6}$, write $\mathbb{F} = \mathbb{F}_{r^{|o|}}$ and associate to o the Jacobi sum

$$J(o) := J_{\mathbb{F}}(\chi_{\mathbb{F},2}^{-i}, \chi_{\mathbb{F},3}^{-i}) = J_{\mathbb{F}}(\chi_{\mathbb{F},6}^{-3i}, \chi_{\mathbb{F},6}^{-2i}) \tag{3-7}$$

for any $i \in o$. As a straightforward calculation shows, $J_{\mathbb{F}}(\chi_1^p, \chi_2^p) = J_{\mathbb{F}}(\chi_1, \chi_2)$, so the sum $J(o)$ is well defined independently of the choice of $i \in o$.

We next record the valuations of $J(o)$ for $o \in N_{r,6}$. These claims follow easily from the expression of Jacobi sums in terms of Gauss sums and Stickelberger's theorem (see Sections 3.4 and 3.6). If $p \equiv -1 \pmod 6$, then

$$\operatorname{ord}(J(o)) = \tfrac{1}{2}\nu|o|$$

for all $o \in N_{r,6}$. On the other hand, if $p \equiv 1 \pmod 6$, then

$$\operatorname{ord}(J(\{1\})) = 0 \quad \text{and} \quad \operatorname{ord}(J(\{-1\})) = \nu.$$

Finally, we introduce the map $\rho_6 : O_{r,6,q}^\times \to N_{r,6}$ induced by the projection

$$(\mathbb{Z}/6\mathbb{Z})^\times \times \mathbb{F}_q^{\times} \to (\mathbb{Z}/6\mathbb{Z})^\times.$$

This will play a role in our geometric calculation of the L-function in Section 7.

4. Elementary calculation of the L-function

Recall that we have fixed a prime number $p > 3$, a finite field $\mathbb{F}_r$ of characteristic p, a power q of p, and that we have defined $E = E_{q,r}$ as the elliptic curve

$$E : y^2 = x^3 + t^q - t$$

over $K = \mathbb{F}_r(t)$. In this section, we give an elementary calculation of the L-function of E over K. The Hasse–Weil L-function of E is defined as the Euler product

$$L(E, T) = \prod_{\text{good } v} (1 - a_v T^{\deg(v)} + r_v T^{2\deg(v)})^{-1} \prod_{\text{bad } v} (1 - a_v T^{\deg(v)})^{-1},$$

where the products are over places v of K. Here "good v" refers to the places where E has good reduction, "bad v" refers to the places of bad reduction, and for any place v, $\mathbb{F}_v$ is the residue field at v, r_v is its cardinality, and a_v is the integer such that the number of points on the plane cubic model of E over $\mathbb{F}_v$ is equal to $r_v - a_v + 1$. Note that, since E has additive reduction at all bad places (Section 2.2), the local factors at such places are all 1, so

$$L(E,T) = \prod_{\text{good } v} (1 - a_v T^{\deg(v)} + r_v T^{2\deg(v)})^{-1}. \tag{4-1}$$

One also considers $L(E,s) = L(E,T)$ with $T = r^{-s}$. Since the curve E is nonconstant, it is known (e.g., [Ulmer 2011, Lecture 1, Theorem 9.3]) that $L(E,s)$ is a polynomial in $T = r^{-s}$ and that it satisfies a functional equation relating $L(E,s)$ and $L(E, 2-s)$.

Recall from Section 3.7 that $O^\times_{r,n,q}$ denotes the set of orbits of $\langle r\rangle$ acting on $(\mathbb{Z}/n\mathbb{Z})^\times \times \mathbb{F}_q^\times$, that $\pi_n : O^\times_{r,6,q} \to O^\times_{r,n,q}$ (for $n = 2, 3$) denotes the map induced by the natural projection $(\mathbb{Z}/6\mathbb{Z})^\times \to (\mathbb{Z}/n\mathbb{Z})^\times$, and that $m_2(o) = |o|/|\pi_2(o)|$. As in Section 3.8, we attach a Gauss sum $G(o)$ to any orbit $o \in O^\times_{r,n,q}$.

The main result of this section is the following.

Theorem 4.1. *In the above setting, we have*

$$L(E,s) = \prod_{o \in O^\times_{r,6,q}} (1 - G(\pi_2(o))^{m_2(o)} G(\pi_3(o)) r^{-s|o|}).$$

We remark that, as a polynomial in r^{-s}, the L-function has degree $\sum_{o\in O^\times_{r,6,q}} |o| = |S^\times_{6,r,q}| = 2(q-1)$. This is consistent with what the Grothendieck–Ogg–Shafarevich formula predicts; i.e., that the L-function has degree $\deg(\mathcal{N}_E) - 4$ where $\mathcal{N}_E$ is the conductor of E (recall from Section 2.2 that $\deg \mathcal{N}_E = 2(q+1)$).

The first, elementary, proof of Theorem 4.1 will be given at the end of this section, after proving several lemmas in the next few subsections. In Section 7, we will provide two more conceptual proofs of this statement (see Theorems 7.2 and 7.4, as well as Section 7.5).

Lemma 4.2. *Let $\mathbb{F}$ be a finite field of characteristic p, and let ψ be a nontrivial additive character of $\mathbb{F}$.*

(1) *For any $u \in \mathbb{F}$ and any power q of p, one has*

$$|\{t \in \mathbb{F} : t^q - t = u\}| = \sum_{\alpha \in \mathbb{F}\cap\mathbb{F}_q} \psi(\alpha u).$$

(2) *Denote the nontrivial quadratic character of $\mathbb{F}^\times$ by $\lambda = \chi_{\mathbb{F},2}$. Consider the sum*

$$S_{\mathbb{F}}(\lambda,\psi) = \sum_{x,z\in\mathbb{F}} \lambda(x^3+z)\psi(z). \tag{4-2}$$

Then

$$S_{\mathbb{F}}(\lambda,\psi) = \begin{cases} 0 & \text{if } |\mathbb{F}| \equiv 2 \pmod 3, \\ G_{\mathbb{F}}(\lambda,\psi)\sum_{i\in\{1,2\}} G_{\mathbb{F}}(\chi^i_{\mathbb{F},3},\psi) & \text{if } |\mathbb{F}| = 1 \pmod 3. \end{cases}$$

Proof. Part (1) is straightforward when $\mathbb{F}$ is an extension of $\mathbb{F}_q$, and the general case is proven in [Griffon 2019, Lemma 4.3]. (The key point is that the kernel and the image of the map $\mathbb{F} \to \mathbb{F},\ t \mapsto t^q - t$ are orthogonal complements with respect to the $\mathbb{F}_p$-bilinear form $\langle \alpha, \beta \rangle = \mathrm{Tr}_{\mathbb{F}/\mathbb{F}_p}(\alpha\beta)$.) We now turn to the proof of (2). For any nontrivial additive character ψ on $\mathbb{F}$, consider

$$S_{\mathbb{F}}(\lambda, \psi) = \sum_{x,z \in \mathbb{F}} \lambda(x^3 + z)\psi(z).$$

Let $\mathbf{1}$ denote the trivial multiplicative character of $\mathbb{F}^\times$. It is classical that for any $y \in \mathbb{F}$,

$$|\{x \in \mathbb{F} : y = x^3\}| = \sum_{\theta^3 = \mathbf{1}} \theta(y),$$

where the sum runs over characters on $\mathbb{F}^\times$ whose order divides 3 (see [Cohen 2007, Lemma 2.5.21]). This allows us to rewrite the sum $S_{\mathbb{F}}(\lambda, \psi)$ as

$$\begin{aligned}
S_{\mathbb{F}}(\lambda, \psi) &= \sum_{y \in \mathbb{F}} \sum_{z \in \mathbb{F}} \Big(\sum_{\theta^3 = \mathbf{1}} \theta(y) \Big) \lambda(y+z)\psi(z) \\
&= \sum_{\theta^3 = \mathbf{1}} \sum_{y \in \mathbb{F}} \theta(y) \Big(\sum_{z \in \mathbb{F}} \lambda(y+z)\psi(z) \Big) \\
&= \sum_{\theta^3 = \mathbf{1}} \sum_{y \in \mathbb{F}} \theta(y) \Big(\sum_{u \in \mathbb{F}} \lambda(u)\psi(u-y) \Big) && \text{(by setting } u = z + y\text{)} \\
&= \Big(\sum_{\theta^3 = \mathbf{1}} \sum_{y \in \mathbb{F}} \theta(y)\psi(-y) \Big) \Big(\sum_{u \in \mathbb{F}} \lambda(u)\psi(u) \Big) \\
&= \Big(\sum_{u \in \mathbb{F}} \lambda(u)\psi(u) \Big) \Big(\sum_{\theta^3 = \mathbf{1}} \theta(-1) \sum_{v \in \mathbb{F}} \theta(v)\psi(v) \Big) && \text{(by setting } v = -y\text{).}
\end{aligned}$$

The first sum equals $-G_{\mathbb{F}}(\lambda, \psi)$ and, for a character θ such that $\theta^3 = \mathbf{1}$, the sum over $v \in \mathbb{F}$ equals $-G_{\mathbb{F}}(\theta, \psi)$. Moreover, $\theta(-1) = 1$ for all θ such that $\theta^3 = \mathbf{1}$, and $G_{\mathbb{F}}(\mathbf{1}, \psi) = 0$, so we have

$$S_{\mathbb{F}}(\lambda, \psi) = G_{\mathbb{F}}(\lambda, \psi) \sum_{\substack{\theta^3 = \mathbf{1} \\ \theta \neq \mathbf{1}}} G_{\mathbb{F}}(\theta, \psi).$$

To conclude the proof, it remains to note that if $|\mathbb{F}| \equiv 2 \pmod 3$, then there are no nontrivial characters of order 3, so that the right-hand side vanishes, while if $|\mathbb{F}| \equiv 1 \pmod 3$, the two nontrivial characters of order 3 are $\chi_{\mathbb{F},3}^i$, $i \in \{1, 2\}$. $\square$

To ease notation, for the rest of this section we write $\mathbb{F}_n$ for $\mathbb{F}_{r^n}$, i.e., $\mathbb{F}_n$ is the extension of $\mathbb{F}_r$ of degree n. Fix a nontrivial additive character $\psi_{\mathbb{F}_n}$ of $\mathbb{F}_n$ and for any $\alpha \in \mathbb{F}_n$, let $\psi_{\mathbb{F}_n,\alpha}$ denote the additive character on $\mathbb{F}_n$ defined by $z \in \mathbb{F}_n \mapsto \psi_{\mathbb{F}_n}(\alpha z)$.

Lemma 4.3. *As Taylor series in T,*

$$-\log L(E,T)=\sum_{n\geq 1}\frac{T^n}{n}\sum_{\alpha\in\mathbb{F}_n\cap\mathbb{F}_q} S_{\mathbb{F}_n}(\lambda_{\mathbb{F}_n},\psi_{\mathbb{F}_n,\alpha})$$

where $\lambda_{\mathbb{F}_n}=\chi_{\mathbb{F}_n,2}$ is the nontrivial quadratic character of $\mathbb{F}_n^\times$ and $S_{\mathbb{F}_n}(\lambda_{\mathbb{F}_n},\psi_{\mathbb{F}_n,\alpha})$ is the sum defined by (4-2).

Proof. In the definition of $L(E,T)$, write the Euler factor at a good place v as

$$(1-a_vT^{\deg(v)}+r_vT^{2\deg(v)})=(1-\alpha_vT^{\deg(v)})(1-\beta_vT^{\deg(v)}).$$

Taking the logarithm of the Euler product (4-1) and reordering terms yields that

$$\log L(E,T)=\sum_{n\geq 1}\frac{T^n}{n}\sum_{\substack{\text{good } v\\ \deg(v)\mid n}}\deg(v)(\alpha_v^{n/\deg(v)}+\beta_v^{n/\deg(v)}).$$

To obtain this expression, we have used the standard identity between Taylor series:

$$\log(1-\alpha T)=-\sum_{n\geq 1}\frac{(\alpha T)^n}{n}. \tag{4-3}$$

If $t\in\mathbb{F}_n$, define $A_E(t,n)$ to be the integer such that $r^n+1-A_E(t,n)$ is the number of $\mathbb{F}_n$-rational points on the reduction of E at t. That

$$\alpha_v^{n/\deg(v)}+\beta_v^{n/\deg(v)}=A_E(t,n)$$

for any $t\in\mathbb{F}_n$ lying over v follows from [Silverman 2009, V.2.3.1]. Thus,

$$L(E,T)=\sum_{n\geq 1}\frac{T^n}{n}\sum_{\substack{\text{good } t\\ t\in\mathbb{F}_n}}A_E(t,n).$$

Denote the nontrivial quadratic character of $\mathbb{F}_n^\times$ by $\lambda_{\mathbb{F}_n}$. Then [Silverman 2009, V.1.3] asserts that

$$A_E(t,n)=-\sum_{x\in\mathbb{F}_n}\lambda_{\mathbb{F}_n}(x^3+t^q-t).$$

Note that if $t\in\mathbb{F}_q$, then $t^q-t=0$, and the sum on the right-hand side vanishes, so we may drop the restriction "good t" in the last expression for $L(E,T)$, i.e.,

$$-\log L(E,T)=\sum_{n\geq 1}\frac{T^n}{n}\sum_{t\in\mathbb{F}_n}\sum_{x\in\mathbb{F}_n}\lambda_{\mathbb{F}_n}(x^3+t^q-t).$$

Now applying Lemma 4.2 part (1), we get that

$$\sum_{t\in\mathbb{F}_n}\sum_{x\in\mathbb{F}_n}\lambda_{\mathbb{F}_n}(x^3+t^q-t)=\sum_{x\in\mathbb{F}_n}\sum_{u\in\mathbb{F}_n}\sum_{\alpha\in\mathbb{F}_n\cap\mathbb{F}_q}\psi(\alpha u)\lambda_{\mathbb{F}_n}(x^3+u)$$
$$=\sum_{\alpha\in\mathbb{F}_n\cap\mathbb{F}_q}S_{\mathbb{F}_n}(\lambda_{\mathbb{F}_n},\psi_{\mathbb{F}_n,\alpha}).$$

Therefore, we have proved, as desired, that

$$-\log L(E,T)=\sum_{n\geq 1}\frac{T^n}{n}\sum_{\alpha\in\mathbb{F}_n\cap\mathbb{F}_q}S_{\mathbb{F}_n}(\lambda_{\mathbb{F}_n},\psi_{\mathbb{F}_n,\alpha}).$$ □

Lemma 4.4. *As Taylor series in T,*

$$-\log\prod_{o\in O^{\times}_{r,6,q}}(1-G(\pi_2(o))^{m_2(o)}G(\pi_3(o))T^{|o|})$$
$$=\sum_{\substack{n\geq 1\\ r^n\equiv 1\ (\mathrm{mod}\ 6)}}\frac{T^n}{n}\sum_{\alpha\in\mathbb{F}_n\cap\mathbb{F}_q}\sum_{i\in\{1,2\}}G_{\mathbb{F}_n}(\chi_{\mathbb{F}_n,2},\psi_{\mathbb{F}_n,\alpha})G_{\mathbb{F}_n}(\chi^i_{\mathbb{F}_n,3},\psi_{\mathbb{F}_n,\alpha}).$$

Proof. To lighten the notation, we write $\omega(o):=G(\pi_2(o))^{m_2(o)}G(\pi_3(o))$ for any $o\in O^{\times}_{r,6,q}$. By identity (4-3), we have

$$-\log\prod_{o\in O^{\times}_{r,6,q}}(1-\omega(o)T^{|o|})=\sum_{n\geq 1}\frac{T^n}{n}\sum_{\substack{o\in O^{\times}_{r,6,q}\\ |o|\text{ divides }n}}|o|\omega(o)^{n/|o|}.$$

Write $\mathbb{F}_o$ for $\mathbb{F}_{r^{|o|}}$, the extension of $\mathbb{F}_r$ of degree $|o|$. Pick a representative $(i,\alpha)\in o$. By definition, we have $G(\pi_3(o))=G_{\mathbb{F}_o}(\chi^i_{\mathbb{F}_o,3},\psi_{\mathbb{F}_o,\alpha})$ and the Hasse–Davenport relation (Section 3.4) yields that

$$G(\pi_3(o))^{n/|o|}=G_{\mathbb{F}_n}(\chi^i_{\mathbb{F}_n,3},\psi_{\mathbb{F}_n,\alpha}).$$

Similarly, using the definition and the Hasse–Davenport relation, we have

$$G(\pi_2(o))^{m_2(o)n/|o|}=G_{\mathbb{F}_n}(\chi_{\mathbb{F}_n,2},\psi_{\mathbb{F}_n,\alpha}).$$

Note that $|o|$ divides n if and only if $r^n\equiv 1\pmod 6$ and $\alpha\in\mathbb{F}_n$. Thus,

$$-\log\prod_{o\in O^{\times}_{r,6,q}}(1-\omega(o)T^{|o|})$$
$$=\sum_{\substack{n\geq 1\\ r^n\equiv 1\ (\mathrm{mod}\ 6)}}\frac{T^n}{n}\sum_{\alpha\in\mathbb{F}_n\cap\mathbb{F}_q}\sum_{i\in\{1,2\}}G_{\mathbb{F}_n}(\chi_{\mathbb{F}_n,2},\psi_{\mathbb{F}_n,\alpha})G_{\mathbb{F}_n}(\chi^i_{\mathbb{F}_n,3},\psi_{\mathbb{F}_n,\alpha}).$$

This completes the proof of the lemma. □

Proof of Theorem 4.1. According to Lemma 4.3,

$$-\log L(E,T)=\sum_{n\geq 1}\frac{T^n}{n}\sum_{\alpha\in\mathbb{F}_n\cap\mathbb{F}_q} S_{\mathbb{F}_n}(\lambda_{\mathbb{F}_n},\psi_{\mathbb{F}_n,\alpha}),$$

and part (2) of Lemma 4.2 says that

$$S_{\mathbb{F}_n}(\lambda_{\mathbb{F}_n},\psi_{\mathbb{F}_n,\alpha}) = \begin{cases} 0 & \text{if } |\mathbb{F}_n|=r^n\equiv 2 \pmod 3,\\ \sum_{i\in\{1,2\}} G_{\mathbb{F}_n}(\chi_{\mathbb{F}_n,2},\psi_{\mathbb{F}_n,\alpha})G_{\mathbb{F}_n}(\chi^i_{\mathbb{F}_n,3},\psi_{\mathbb{F}_n,\alpha}) & \text{if } |\mathbb{F}_n|=r^n\equiv 1 \pmod 3.\end{cases}$$

Noting that $r^n\equiv 1 \pmod 3$ if and only if $r^n\equiv 1 \pmod 6$, we have

$$-\log L(E,T)=\sum_{\substack{n\geq 1\\ r^n\equiv 1 \,(\mathrm{mod}\, 6)}}\frac{T^n}{n}\sum_{\alpha\in\mathbb{F}_n\cap\mathbb{F}_q}\sum_{i\in\{1,2\}} G_{\mathbb{F}_n}(\chi_{\mathbb{F}_n,2},\psi_{\mathbb{F}_n,\alpha})G_{\mathbb{F}_n}(\chi^i_{\mathbb{F}_n,3},\psi_{\mathbb{F}_n,\alpha}).$$

By Lemma 4.4, the expression on the right-hand side is

$$-\log\prod_{o\in O^\times_{r,6,q}}(1-G(\pi_2(o))^{m_2(o)}G(\pi_3(o))T^{|o|}). \qquad \square$$

5. Auxiliary curves

In this section, we record some well-known facts about the geometry of certain curves to be used in the sequel.

5.1. ***Cohomology.*** Throughout this section and the next, we denote by $H^n(-)$ any rational Weil cohomology theory (with coefficients in an algebraically closed field) for varieties over $\mathbb{F}_r$, for example ℓ-adic cohomology $H^n(-\times_{\mathbb{F}_r}\overline{\mathbb{F}}_r,\overline{\mathbb{Q}}_\ell)$ or crystalline cohomology $H^n(-/W)\otimes_{W(\mathbb{F}_r)}\overline{\mathbb{Q}}_p$. (See, for example, [Kleiman 1968].) Among other things, these groups admit a functorial action of the geometric Frobenius Fr_r.

Here is a well-known lemma about characteristic polynomials in induced representations. See [Gordon 1979, Lemma 1.1] or [Ulmer 2007, Lemma 2.2] for a proof.

Lemma 5.1.1. *Let V be a finite-dimensional vector space with subspaces W_i indexed by $i\in\mathbb{Z}/m\mathbb{Z}$ such that $V=\oplus_{i\in\mathbb{Z}/m\mathbb{Z}}W_i$. Let $\phi:V\to V$ be a linear transformation such that $\phi(W_i)\subset W_{i+1}$ for all $i\in\mathbb{Z}/m\mathbb{Z}$. Then*

$$\det(1-\phi T\,|\,V)=\det(1-\phi^m T^m\,|\,W_0).$$

5.2. ***An elliptic curve.*** We have already introduced the elliptic curve

$$E_0: w^2=z^3+1$$

over $\mathbb{F}_r$. The displayed equation defines a smooth affine curve, and there is a unique point at infinity on E_0 which we denote by $O\in E_0$.

The curve E_0 carries an action of μ_6 via $\zeta(z,w)=(\zeta^2 z,\zeta^3 w)$. The character group of μ_6 is $\mathbb{Z}/6\mathbb{Z}$. It is well known that $H^1(E_0)$ has dimension 2, and that under the action of μ_6, it decomposes as the direct sum of two lines corresponding to the subspaces where $\zeta\in\mu_6$ acts by ζ and ζ^{-1} (i.e., corresponding to the characters indexed by $\pm1\in\mathbb{Z}/6\mathbb{Z}$):

$$H^1(E_0)=H^1(E_0)^{(1)}\oplus H^1(E_0)^{(-1)}. \tag{5-1}$$

Also, powers of Fr_r act on the two subspaces as $\langle r\rangle$ acts on $\{\pm1\}=(\mathbb{Z}/6\mathbb{Z})^\times\subset\mathbb{Z}/6\mathbb{Z}$.

More explicitly, if $r\equiv 1 \pmod 6$, so that $\langle r\rangle$ has two orbits on $(\mathbb{Z}/6\mathbb{Z})^\times$, then Fr_r preserves the two subspaces, and the corresponding eigenvalues are

$$J(\{1\})=J_{\mathbb{F}_r}(\chi_{\mathbb{F}_r,6}^{-3},\chi_{\mathbb{F}_r,6}^{-2})\quad\text{and}\quad J(\{-1\})=J_{\mathbb{F}_r}(\chi_{\mathbb{F}_r,6}^{3},\chi_{\mathbb{F}_r,6}^{2}),$$

where the Jacobi sums are as defined in (3-7).

If $r\equiv 5 \pmod 6$, so that $\langle r\rangle$ has a unique orbit on $(\mathbb{Z}/6\mathbb{Z})^\times$, then Fr_r exchanges the two subspaces, and the eigenvalues of Fr_r^2 are both

$$J(\{1,-1\})=J_{\mathbb{F}_{r^2}}(\chi_{\mathbb{F}_{r^2},6}^{-3},\chi_{\mathbb{F}_{r^2},6}^{-2})=J_{\mathbb{F}_{r^2}}(\chi_{\mathbb{F}_{r^2},6}^{3},\chi_{\mathbb{F}_{r^2},6}^{2}).$$

Finally, applying Lemma 5.1.1, we find that

$$\det(1-T\,\mathrm{Fr}_r\mid H^1(E_0))=\prod_{o\in N_{r,6}}(1-J(o)T^{|o|}).$$

We remark that this result and the values of $\mathrm{ord}(J(o))$ recorded in Section 3.10 are compatible with the well-known fact that E_0 is ordinary if $p\equiv 1 \pmod 6$ and supersingular if $p\equiv -1 \pmod 6$.

5.3. *Artin–Schreier curves.* For a positive integer n relatively prime to p, let $C_{n,q}$ be the smooth projective curve over $\mathbb{F}_r$ defined by the equation

$$C_{n,q}: u^n=t^q-t.$$

(We also use the equation $w^n=z^q-z$ when more than one instance of $C_{n,q}$ is under discussion. Only $n=2,3,6$ will be used later in this paper.) The displayed equation defines a smooth affine curve, and there is a unique point at infinity on $C_{n,q}$ which we denote by $\infty\in C_{n,q}$.

The curve $C_{n,q}$ carries natural actions of μ_n via $\zeta(t,u)=(t,\zeta u)$, and of $\mathbb{F}_q$ via $\alpha(t,u)=(t+\alpha,u)$. (In fact, it carries an action of the larger group $\mathbb{F}_q\rtimes\mu_{n(q-1)}$, where $\zeta\in\mu_{n(q-1)}$ acts via $\zeta(t,u)=(\zeta^n t,\zeta u)$. In this section and the next, we will only need the action of the subgroup $\mu_n\times\mathbb{F}_q$. The action of the larger group will be useful in Section 10.) The character group of $\mu_n\times\mathbb{F}_q$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}\times\mathbb{F}_q$.

The cohomology group $H^1(C_{n,q})$ has dimension $(q-1)(n-1)$, and under the action of $\mu_n\times\mathbb{F}_q$, it decomposes into lines where μ_n and $\mathbb{F}_q$ act through their nontrivial characters. (This is proven for $q = p$ in [Katz 1981, Corollary 2.2], and the arguments there generalize straightforwardly to the case $q = p^f$.) In particular, the subspace of $H^1(C_{n,q})$ where μ_n acts via a given nontrivial character has dimension $q-1$, and the subspace where $\mathbb{F}_q$ acts via a given nontrivial character has dimension $n-1$.

Recall from Section 3.7 that $S = S_{n,q} := (\mathbb{Z}/n\mathbb{Z} \smallsetminus \{0\}) \times \mathbb{F}_q^\times$ and that $O_{r,n,q}$ denotes the set of orbits of the action of $\langle r\rangle$ on S. We index the characters of $\mu_n \times \mathbb{F}_q$ (with values in the coefficient field of our cohomology theory) that are nontrivial on both factors by S. The subspace of $H^1(C_{n,q})$ where $\mu_n \times \mathbb{F}_q$ acts via the character indexed by (i, α) will be denoted by $H^1(C_{n,q})^{(i,\alpha)}$. We thus obtain a direct sum decomposition of $H^1(C_{n,q})$ into lines as follows:

$$H^1(C_{n,q}) = \bigoplus_{(i,\alpha)\in S_{n,q}} H^1(C_{n,q})^{(i,\alpha)}. \tag{5-2}$$

Katz [1981, Corollary 2.2] further gave a description of the action of Frobenius on the cohomology $H^1(C_{n,q})$: the Frobenius Fr_r sends the subspace indexed by (i,α) to the subspace indexed by $(ri, \alpha^{1/r})$. If $o \in O_{r,n,q}$ is the orbit through (i,α), then the $|o|$-th iterate $\mathrm{Fr}_r^{|o|}$ stabilizes the subspace $H^1(C_{n,q})^{(i,\alpha)}$ (which is a line) and the eigenvalue of $\mathrm{Fr}_r^{|o|}$ on $H^1(C_{n,q})^{(i,\alpha)}$ is the Gauss sum

$$G(o) := G_{\mathbb{F}}(\chi^i_{\mathbb{F},n}, \psi_\alpha),$$

where $\mathbb{F} = \mathbb{F}_{r^{|o|}}$. (Again, Katz treated the case $q = p$, but the generalization is straightforward.)

Applying Lemma 5.1.1, we have

$$\det(1 - T\,\mathrm{Fr}_r \mid H^1(C_{n,q})) = \prod_{o\in O_{r,n,q}} (1 - G(o)T^{|o|}).$$

We remark that this result together with the values of $\mathrm{ord}(G(o))$ recorded in Section 3.8 are compatible with the well-known fact that $C_{2,q}$ is supersingular, and they show that $C_{3,q}$ is supersingular when $p \equiv -1 \pmod 6$ and neither supersingular nor ordinary if $p \equiv 1 \pmod 6$. (In this last case, the slopes are $\frac{1}{3}$ and $\frac{2}{3}$, both with multiplicity $q-1$, cf. [Pries and Ulmer 2016, §8.3].)

5.4. ***Fermat curves.*** For a positive integer d prime to p, let F_d be the Fermat curve of degree d over $\mathbb{F}_r$. This is by definition the smooth, projective curve in $\mathbb{P}^2$ given by the homogeneous equation

$$F_d : X_0^d + X_1^d + X_2^d = 0.$$

The genus of F_d is $(d-1)(d-2)/2$, so $H^1(F_d)$ has dimension $(d-1)(d-2)$. The curve F_d carries an action of $(\mu_d)^3/\mu_d$ where the three copies of μ_d in the numerator act by multiplication on the three coordinates, and the diagonally embedded μ_d acts trivially. Under the action of this group, $H^1(F_d)$ decomposes into lines on which each of the factors μ_d acts nontrivially and the diagonally embedded μ_d acts trivially. There are $(d-1)(d-2)$ such characters. The action of Frobenius on $H^1(F_d)$ is given by Jacobi sums. Since we will not need the cohomology of F_d later in the paper, we omit the details.

6. Domination by a product of curves

In this section we define the Weierstrass and Néron models $\mathcal{W}$ and $\mathcal{E}$ of E and relate them to products of curves. Throughout, unless explicitly indicated otherwise by the notation, products of varieties are over $\mathbb{F}_r$ (i.e., $\times$ means $\times_{\mathbb{F}_r}$).

Our ultimate aim is to compute the relevant part of the cohomology of a model $\mathcal{E}$ of E by showing that $\mathcal{E}$ is birational to the quotient of a product of curves by a finite group.

6.1. ***Models.*** Let $\mathcal{W} \to \mathbb{P}^1_{\mathbb{F}_r}$ be the Weierstrass model of E over K, i.e., the surface fibered over $\mathbb{P}^1$ whose fibers are the plane cubic reductions of E at the places of K. More precisely, let

$$d = \deg(\omega_E) = \lceil q/6 \rceil = \begin{cases} \frac{q+5}{6} & \text{if } q \equiv 1 \pmod 6, \\ \frac{q+1}{6} & \text{if } q \equiv 5 \pmod 6, \end{cases}$$

and define $\mathcal{W}$ by gluing the surfaces

$$y^2 z = x^3 + (t^q - t)z^3 \subset \mathbb{P}^2_{x,y,z} \times \mathbb{A}^1_t$$

and

$$y'^2 z' = x'^3 + (t'^{6d-q} - t'^{6d-1})z'^3 \subset \mathbb{P}^2_{x',y',z'} \times \mathbb{A}^1_{t'}$$

via the map $([x', y', z'], t') = ([x/t^{2d}, y/t^{3d}, z], 1/t)$. Then $\mathcal{W}$ is an irreducible, normal, projective surface, and projection onto the t and t' coordinates defines a morphism $\mathcal{W} \to \mathbb{P}^1$ whose generic fiber is E.

When $q \equiv 5 \pmod 6$, $\mathcal{W}$ is a regular surface (i.e., is smooth over $\mathbb{F}_r$), and we define $\mathcal{E} = \mathcal{W}$. When $q \equiv 1 \pmod 6$, $\mathcal{W}$ has a singularity at the point $([x', y', z'], t') = ([0, 0, 1], 0)$ and is regular elsewhere. In this case, we define $\mathcal{E}$ as the minimal desingularization of $\mathcal{W}$. (The desingularization introduces eight new components.)

The reduction types of $\mathcal{E}$ at closed points of $\mathbb{P}^1$ (i.e., at places of K) were recorded in Section 2.2.

6.2. ***Sextic twists.*** We saw above that E becomes isomorphic to a constant curve after extension of K to $L = K[u]/(u^6 = t^q - t)$. Geometrically, this means that $\mathcal{E}$

is birational to a quotient of $E_0 \times C_{6,q}$. In this subsection, we make this statement more explicit and deduce a cohomological consequence.

Let μ_6 act on $E_0 \times C_{6,q}$ "antidiagonally," i.e., via

$$\zeta(z, w, t, u) = (\zeta^2 z, \zeta^3 w, t, \zeta^{-1} u).$$

Define a rational map $E_0 \times C_{6,q} \dashrightarrow \mathcal{W}$ by

$$(z, w, t, u) \mapsto ([x, y, z], t) = ([zu^2, wu^3, 1], t).$$

It is obvious that this map factors through the quotient $\mathcal{S} := (E_0 \times C_{6,q})/\mu_6$ and so we have a commutative diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, dashed] \arrow[d] & \mathcal{W} \arrow[d] \\ C_{6,q}/\mu_6 \arrow[r, equal] & \mathbb{P}^1_t \end{tikzcd}$$

where the bottom horizontal arrow is the canonical isomorphism $C_{6,q}/\mu_6 \cong \mathbb{P}^1_t$ and the left vertical arrow is induced by the projection onto $C_{6,q}$.

Now let $\widetilde{\mathcal{S}} \to \mathcal{S}$ be a blow-up so that $\widetilde{\mathcal{S}}$ is smooth and $\mathcal{S} \dashrightarrow \mathcal{W}$ induces a morphism $\widetilde{\mathcal{S}} \to \mathcal{E}$. (This can be made completely explicit in terms of the fixed points of the action of μ_6 and the formula for the rational map $E_0 \times C_{6,q} \dashrightarrow \mathcal{W}$, but the details will not be important for our analysis.) The diagram above then extends to

$$\begin{tikzcd} \widetilde{\mathcal{S}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{S} \arrow[r, dashed] \arrow[d] & \mathcal{W} \arrow[d] \\ C_{6,q}/\mu_6 \arrow[r, equal] & \mathbb{P}^1_t \end{tikzcd}$$

The following encapsulates all we need to know about the geometry of $\widetilde{\mathcal{S}} \to \mathcal{E}$.

Proposition 6.2.1. (1) *The strict transform of $(O \times C_{6,q})/\mu_6$ in $\widetilde{\mathcal{S}}$ maps to the zero section of $\mathcal{E}$.*

(2) *The strict transform of $(E_0 \times \infty)/\mu_6$ in $\widetilde{\mathcal{S}}$ maps to a fiber of $\mathcal{E} \to \mathbb{P}^1$.*

(3) *Every component of the exceptional divisor of $\widetilde{\mathcal{S}} \to \mathcal{S}$ maps into a fiber of $\mathcal{E} \to \mathbb{P}^1$.*

Proof. The first two points are obvious from the formula defining $E_0 \times C_{6,q} \dashrightarrow \mathcal{W}$. The third point follows by examining the outer rectangle of the last displayed diagram. Indeed, if D is a component of the exceptional divisor of $\widetilde{\mathcal{S}} \to \mathcal{S}$, then D lies over a single point of $C_{6,q}/\mu_6 \cong \mathbb{P}^1_t$ and thus maps to a fiber of $\mathcal{E} \to \mathbb{P}^1_t$. $\square$

Let $T \subset H^2(\mathcal{E})$ be the subspace spanned by the classes of the zero section and components of fibers of $\mathcal{E} \to \mathbb{P}^1$. This is the subspace Shioda [1992] calls the "trivial lattice".

Corollary 6.2.2. *There is a canonical isomorphism*

$$H^2(\mathcal{E})/T \cong (H^1(E_0) \otimes H^1(C_{6,q}))^{\mu_6}.$$

Here the exponent μ_6 indicates the subspace invariant under the antidiagonal action of μ_6.

Proof. The dominant morphism $\widetilde{\mathcal{S}} \to \mathcal{E}$ induces a surjection $H^2(\widetilde{\mathcal{S}}) \to H^2(\mathcal{E})$. Using the Künneth formula, taking invariants, and using the blow-up formula, we obtain a canonical isomorphism

$$\begin{aligned} H^2(\widetilde{\mathcal{S}}) &\cong H^2(\mathcal{S}) \oplus B \cong H^2(E_0 \times C_{6,q}/\mu_6) \oplus B \cong H^2(E_0 \times C_{6,q})^{\mu_6} \oplus B \\ &\cong \bigl(H^1(E_0) \otimes H^1(C_{6,q})\bigr)^{\mu_6} \oplus \bigl(H^0(E_0) \otimes H^2(C_{6,q})\bigr) \oplus \bigl(H^2(E_0) \otimes H^0(C_{6,q})\bigr) \oplus B \end{aligned}$$

where B denotes the subspace spanned by the classes of components of the exceptional divisor of $\widetilde{\mathcal{S}} \to \mathcal{S}$.

The proposition shows that $H^0(E_0) \otimes H^2(C_{6,q})$, $H^2(E_0) \otimes H^0(C_{6,q})$, and B all map to T. Thus we have a well-defined and canonical surjection

$$(H^1(E_0) \otimes H^1(C_{6,q}))^{\mu_6} \to H^2(\mathcal{E})/T.$$

To finish, we compare dimensions. We recalled in Section 5 that μ_6 acts on $H^1(E_0)$ through the characters $\zeta \mapsto \zeta^{\pm 1}$, each with multiplicity one (see (5-1)). Similarly, μ_6 acts on $H^1(C_{6,q})$ through characters $\zeta \mapsto \zeta^i$ with $i \not\equiv 0 \pmod 6$, each with multiplicity $q-1$ (see (5-2)). Thus

$$\dim(H^1(E_0) \otimes H^1(C_{6,q}))^{\mu_6} = 2(q-1).$$

On the other hand, the Grothendieck–Ogg–Shafarevich formula says that the quotient $H^2(\mathcal{E})/T$ has dimension $\deg(\mathcal{N}_E) - 4$ where $\mathcal{N}_E$ denotes the conductor of E. We noted above that $\deg(\mathcal{N}_E) = 2(q+1)$, so $H^2(\mathcal{E})/T$ has dimension $2(q-1)$. Therefore the surjection

$$(H^1(E_0) \otimes H^1(C_{6,q}))^{\mu_6} \to H^2(\mathcal{E})/T$$

is in fact a bijection. □

6.3. ***Artin–Schreier quotients.*** In this subsection, we show that $\mathcal{E}$ is birational to a quotient of a product of Artin–Schreier curves, in the style of [Pries and Ulmer 2016]. Let

$$\mathcal{C} = C_{2,q} : w_1^2 = z_1^q - z_1 \quad \text{and} \quad \mathcal{D} = C_{3,q} : w_2^3 = z_2^q - z_2.$$

Write $\infty_{\mathcal{C}}$ and $\infty_{\mathcal{D}}$ for the points at infinity on $\mathcal{C}$ and $\mathcal{D}$, respectively. Let $\mathbb{F}_q$ act on $\mathcal{C}\times\mathcal{D}$ "diagonally," i.e., via $\alpha(z_1, w_1, z_2, w_2) = (z_1+\alpha, w_1, z_2+\alpha, w_2)$. It is easily seen that the sole fixed point of this action is $(\infty_{\mathcal{C}}, \infty_{\mathcal{D}})$.

Define a rational map $\mathcal{C}\times\mathcal{D}\dashrightarrow\mathbb{P}^1_t$ by $(z_1, w_1, z_2, w_2)\mapsto t = z_1 - z_2$, and a rational map $\mathcal{C}\times\mathcal{D}\dashrightarrow\mathcal{W}$ by

$$(z_1, w_1, z_2, w_2)\mapsto ([x, y, z], t) = ([w_2, w_1, 1], z_1 - z_2).$$

Both of these maps are morphisms away from $(\infty_{\mathcal{C}}, \infty_{\mathcal{D}})$, and they clearly factor through the quotient $(\mathcal{C}\times\mathcal{D})/\mathbb{F}_q$.

Proposition 6.3.1. *There is a proper birational morphism $\mathcal{S}'\to\mathcal{C}\times\mathcal{D}$ resolving the indeterminacy of $\mathcal{C}\times\mathcal{D}\dashrightarrow\mathcal{W}$ such that the components of the exceptional divisor of $\mathcal{S}'\to\mathcal{C}\times\mathcal{D}$ map either to the fiber of $\mathcal{W}$ over $t=\infty$ or to the zero section of $\mathcal{W}$.*

Proof. The proof of [Pries and Ulmer 2016, Proposition 3.1.5] gives an explicit recipe for a morphism $\mathcal{S}'\to\mathcal{C}\times\mathcal{D}$ resolving the indeterminacy of $\mathcal{C}\times\mathcal{D}\dashrightarrow\mathbb{P}^1_t$. It is a sequence of four blow-ups of closed points. Straightforward calculation, which we omit, shows that the induced map $\mathcal{S}'\to\mathcal{C}\times\mathcal{D}\dashrightarrow\mathcal{W}$ is in fact a morphism, and that it behaves as stated in the proposition on the components of the exceptional divisor. Indeed, the first three blow-ups map to the fiber over $t=\infty$ and the last maps to the zero section. □

The diagonal action of $\mathbb{F}_q$ on $\mathcal{C}\times\mathcal{D}$ lifts uniquely to $\mathcal{S}'$ and fixes the exceptional divisor pointwise. It is clear that the morphism $\mathcal{S}'\to\mathcal{W}$ factors through the quotient $\mathcal{S}'/\mathbb{F}_q$, so we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{S}'/\mathbb{F}_q \arrow[r] \arrow[d] & \mathcal{W} \arrow[d] \\
\mathbb{P}^1_t \arrow[r, equal] & \mathbb{P}^1_t
\end{tikzcd}$$

Now let $\widetilde{\mathcal{S}}\to\mathcal{S}'/\mathbb{F}_q$ be a proper birational morphism so that $\widetilde{\mathcal{S}}$ is a smooth projective surface and the induced rational map $\widetilde{\mathcal{S}}\dashrightarrow\mathcal{E}$ is a morphism. The diagram above then extends to

$$\begin{tikzcd}
\widetilde{\mathcal{S}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{S}'/\mathbb{F}_q \arrow[r] \arrow[d] & \mathcal{W} \arrow[d] \\
\mathbb{P}^1_t \arrow[r, equal] & \mathbb{P}^1_t
\end{tikzcd}$$

The following summarizes the relevant aspects of the geometry of this picture.

Proposition 6.3.2. (1) *The strict transforms of $\infty_{\mathcal{C}} \times \mathcal{D}$ and $\mathcal{C} \times \infty_{\mathcal{D}}$ in $\widetilde{\mathcal{S}}$ map to the fiber of $\mathcal{E} \to \mathbb{P}^1$ over $t = \infty$.*

(2) *The strict transforms in $\widetilde{\mathcal{S}}$ of the images in $\mathcal{S}'/\mathbb{F}_q$ of the components of the exceptional fiber of $\mathcal{S}' \to \mathcal{C} \times \mathcal{D}$ map to the fiber of $\mathcal{E} \to \mathbb{P}^1$ over $t = \infty$ or to the zero section of $\mathcal{E}$.*

(3) *Every component of the exceptional divisor of $\widetilde{\mathcal{S}} \to \mathcal{S}'/\mathbb{F}_q$ maps to a fiber of $\mathcal{E} \to \mathbb{P}^1$.*

Proof. The first point is obvious from the formula defining $\mathcal{C} \times \mathcal{D} \dashrightarrow \mathcal{W}$. The second point follows from the previous proposition. The third point follows by examining the last displayed diagram. Indeed, if D is a component of the exceptional divisor of $\widetilde{\mathcal{S}} \to \mathcal{S}/\mathbb{F}_q$, then D lies over a single point of $\mathbb{P}^1_t$ and so maps to a fiber of $\mathcal{E} \to \mathbb{P}^1_t$. □

Corollary 6.3.3. *Let $T \subset H^2(\mathcal{E})$ be the trivial lattice, i.e., the subspace spanned by the classes of the zero section and components of fibers of $\mathcal{E} \to \mathbb{P}^1$. There is a canonical isomorphism*

$$H^2(\mathcal{E})/T \cong (H^1(\mathcal{C}) \otimes H^1(\mathcal{D}))^{\mathbb{F}_q}.$$

The exponent $\mathbb{F}_q$ indicates the subspace invariant under the diagonal action of $\mathbb{F}_q$.

Proof. The proof is completely parallel to that of Corollary 6.2.2, so we just sketch the argument. The dominant morphism $\widetilde{\mathcal{S}} \to \mathcal{E}$ induces a surjection $H^2(\widetilde{\mathcal{S}}) \to H^2(\mathcal{E})$. Using the Künneth formula, taking invariants, using the blow-up formula, and applying the proposition, we obtain a canonical surjection

$$(H^1(\mathcal{C}) \otimes H^1(\mathcal{D}))^{\mathbb{F}_q} \to H^2(\mathcal{E})/T.$$

We conclude by using Section 5 and the proof of Corollary 6.2.2 to check that $H^2(\mathcal{E})/T$ and $(H^1(\mathcal{C}) \otimes H^1(\mathcal{D}))^{\mathbb{F}_q}$ both have dimension $2(q-1)$. Thus the displayed surjection is a bijection. □

6.4. *Fermat quotients.* The surfaces $\mathcal{W}$ and $\mathcal{E}$ have affine open subsets defined by an equation with four monomials in three variables, namely,

$$y^2 = x^3 + t^q - t.$$

In Shioda's terminology, these are "Delsarte surfaces." This allows one to show that (over a sufficiently large ground field) $\mathcal{E}$ is birational to a quotient of a Fermat surface by a finite group. The Fermat surface is itself birational to the quotient of a product of two Fermat curves by a finite group. Thus we arrive at a birational presentation of $\mathcal{E}$ as a quotient of a product of Fermat curves. It turns out that this presentation factors through the sextic twist presentation given in Section 6.2, in a sense to be explained below. Thus, the Fermat quotient presentation does not give essential new information, and we will only sketch the main points, omitting most details.

Let $d = 6q - 6$. Applying the method of Shioda (see [Shioda 1986] and [Ulmer 2007, §6] or [Ulmer 2011, Lecture 2, §10]) yields a dominant rational map from F_d^2 to $\mathcal{E}$. Explicitly, take two copies of F_d with homogeneous coordinates $[X_0, X_1, X_2]$ and $[Y_0, Y_1, Y_2]$, and assume that $\mathbb{F}_r$ is large enough to contain a primitive $2d$-th root of unity ϵ. Consider the rational map $\phi : F_d^2 \dashrightarrow \mathcal{E}$ given by

$$\begin{aligned}([X_0, X_1, X_2], [Y_0, Y_1, Y_2]) &\mapsto (x, y, t) \\ &= \left(\epsilon^2 \frac{X_1^{2q-2}}{X_2^{2q-2}} \frac{Y_0^{2q-2} Y_1^2}{Y_2^{2q}}, \epsilon^{3q} \frac{X_0^{3q-3}}{X_2^{3q-3}} \frac{Y_0^{3q-3} Y_1^3}{Y_2^{3q}}, \epsilon^6 \frac{Y_1^6}{Y_2^6}\right).\end{aligned}$$

Then it is not hard to check that ϕ is dominant of generic degree d^3 and that it induces a birational isomorphism $F_d^2/G \dashrightarrow \mathcal{E}$ where $G \subset (\mu_d^3/\mu_d)^2$ is the group generated by

$$([1, 1, \zeta], [\zeta, 1, 1]), \quad ([\zeta^2, \zeta^3, 1], [1, 1, 1]), \quad \text{and} \quad ([\zeta, \zeta^2, 1], [1, \zeta^{q-1}, 1]),$$

where $\zeta = \epsilon^2$ is a primitive d-th root of unity in $\mathbb{F}_r$.

Analyzing the geometry of ϕ would allow us to show that $H^2(\mathcal{E})/T$ is isomorphic to a certain subspace of $H^2(F_d^2)$. We omit the details, because, as we explain next, ϕ factors through the rational map $E_0 \times C_{6,q} \dashrightarrow \mathcal{W}$ given in Section 6.2.

Indeed, consider the morphism $\tau_1 : F_d \to E_0$ given by

$$[X_0, X_1, X_2] \mapsto (z, w) = \left(\left(\frac{X_1}{X_2}\right)^{2q-2}, \left(\frac{\epsilon X_0}{X_2}\right)^{3q-3}\right)$$

and the morphism $\tau_2 : F_d \to C_{6,q}$ given by

$$[Y_0, Y_1, Y_2] \mapsto (t, u) = \left(\left(\frac{\epsilon Y_1}{Y_2}\right)^6, \frac{\epsilon Y_0^{q-1} Y_1}{Y_2^q}\right).$$

Then it is straightforward to check that the diagram

$$\begin{tikzcd} F_d^2 \arrow[rr, dashed, "\phi"] \arrow[dr, "\tau_1 \times \tau_2"'] & & \mathcal{E} \\ & E_0 \times C_{6,q} \arrow[ur, dashed] & \end{tikzcd}$$

commutes, where the right diagonal rational map is that given in Section 6.2. This implies that $H^2(\mathcal{E})/T$ already appears in the cohomology of $E_0 \times C_{6,q}$, and that, moreover, the relevant map is defined without requiring an extension of $\mathbb{F}_r$. We will thus omit any further consideration of Fermat curves.

7. Geometric calculation of the L-function

In this section, we use the presentation of $\mathcal{E}$ as a quotient of a product of curves to give another calculation of $L(E, s)$ via the cohomological formula for it proved in [Shioda 1992]. As in the previous section, let $T \subset H^2(\mathcal{E})$ be the subspace spanned by the classes of the zero section and all components of all fibers of $\mathcal{E} \to \mathbb{P}^1$. Shioda proved that

$$L(E, s) = \det(1 - \mathrm{Fr}_r\, r^{-s} \mid H^2(\mathcal{E})/T).$$

7.1. ***Via sextic twists.*** Recall from Section 3.7 that $\langle r\rangle$ acts on $S^\times = (\mathbb{Z}/6\mathbb{Z})^\times \times \mathbb{F}_q^\times$, the set of orbits being denoted $O^\times_{r,6,q}$. As in Section 3.10, let $N_{r,6}$ denote the set of orbits of $\langle r\rangle$ on $(\mathbb{Z}/6\mathbb{Z})^\times$, and let $\rho_6 : O^\times_{r,6,q} \to N_{r,6}$ be the map induced by the projection $(\mathbb{Z}/6\mathbb{Z})^\times \times \mathbb{F}_q^\times \to (\mathbb{Z}/6\mathbb{Z})^\times$. Define

$$n_6(o) = \frac{|o|}{|\rho_6(o)|}.$$

Note that $n_6(o)$ is either $|o|$ (if $r \equiv 1 \pmod 6$) or $|o|/2$ (if $r \equiv -1 \pmod 6$). To each orbit $o \in O^\times_{r,6,q}$ we attach the Jacobi sum $J(\rho_6(o))$ (see (3-7)) and the Gauss sum $G(o)$ (see (3-6)).

Theorem 7.2. $$L(E, s) = \prod_{o \in O^\times_{r,6,q}} (1 - J(\rho_6(o))^{n_6(o)} G(o) r^{-s|o|}).$$

Proof. By Corollary 6.2.2, we know that

$$H^2(\mathcal{E})/T \cong (H^1(E_0) \otimes H^1(C_{6,q}))^{\mu_6},$$

where μ_6 acts antidiagonally. Combining (5-1) and (5-2), the right-hand side decomposes as the direct sum

$$\bigoplus_{(i,\alpha)\in S^\times} H^1(E_0)^{(i)} \otimes H^1(C_{6,q})^{(i,\alpha)},$$

where the summands are one-dimensional. If $o \in O^\times_{r,6,q}$, then the subspace

$$\bigoplus_{(i,\alpha)\in o} H^1(E_0)^{(i)} \otimes H^1(C_{6,q})^{(i,\alpha)}$$

is preserved by the r-power Frobenius Fr_r, and by what was recalled in Sections 5.2 and 5.3, the eigenvalue of $\mathrm{Fr}_r^{|o|}$ on $H^1(E_0)^{(i)} \otimes H^1(C_{6,q})^{(i,\alpha)}$ is $J(\rho_6(o))^{n_6(o)} G(o)$. By Lemma 5.1.1, the characteristic polynomial of $\mathrm{Fr}_r\, r^{-s|o|}$ on the displayed subspace is $(1 - J(\rho_6(o))^{n_6(o)} G(o) r^{-s|o|})$. Taking the product over all orbits yields the theorem. $\square$

7.3. ***Via Artin–Schreier quotients.*** Let $\langle r\rangle$ act on $S^\times=(\mathbb{Z}/n\mathbb{Z})^\times\times\mathbb{F}_q^\times$ with orbits $O^\times_{r,n,q}$, as in Section 3.7. For $n=2,3$, the natural projection $(\mathbb{Z}/6\mathbb{Z})^\times\to(\mathbb{Z}/n\mathbb{Z})^\times$ induces a map $\pi_n:O^\times_{r,6,q}\to O^\times_{r,n,q}$. Recall that we write

$$m_2(o)=\frac{|o|}{|\pi_2(o)|}.$$

(There is no need for an analogous $m_3(o)$ since $|\pi_3(o)|=|o|$ for all $o\in O^\times_{r,6,q}$.) To each orbit $o\in O^\times_{r,6,q}$ we associate Gauss sums $G(\pi_2(o))$ and $G(\pi_3(o))$ (see Section 3.8).

Theorem 7.4.
$$L(E,s)=\prod_{o\in O^\times_{r,6,q}}\bigl(1-G(\pi_2(o))^{m_2(o)}G(\pi_3(o))r^{-s|o|}\bigr).$$

Proof. By Corollary 6.3.3, we have

$$H^2(\mathcal{E})/T\cong(H^1(C_{2,q})\otimes H^1(C_{3,q}))^{\mathbb{F}_q},$$

where $\mathbb{F}_q$ acts diagonally. Using (5-2) twice, we get a direct sum decomposition of the right-hand side:

$$\bigoplus_{(i,\alpha)\in S^\times}H^1(C_{2,q})^{(i\bmod 2,\alpha)}\otimes H^1(C_{3,q})^{(i\bmod 3,-\alpha)},$$

where all the summands are one-dimensional. For any orbit $o\in O^\times_{r,6,q}$, the subspace

$$\bigoplus_{(i,\alpha)\in o}H^1(C_{2,q})^{(i\bmod 2,\alpha)}\otimes H^1(C_{3,q})^{(i\bmod 3,-\alpha)}$$

is preserved by the r-power Frobenius. The results recalled in Section 5.3 show that the eigenvalue of $\mathrm{Fr}_r^{|o|}$ acting on the line $H^1(C_{2,q})^{(i\bmod 2,\alpha)}\otimes H^1(C_{3,q})^{(i\bmod 3,-\alpha)}$ is $G(\pi_2(o))^{m_2(o)}G(\pi_3(o))$. (Here we use that $G_{\mathbb{F}}(\chi^i_{\mathbb{F},3},\psi_{-\alpha})=G_{\mathbb{F}}(\chi^i_{\mathbb{F},3},\psi_\alpha)$, a consequence of the fact that -1 is a cube in any finite field $\mathbb{F}$.) Lemma 5.1.1 now implies that the characteristic polynomial of $\mathrm{Fr}_r\,r^{-s|o|}$ on the displayed subspace is $(1-G(\pi_2(o))^{m_2(o)}G(\pi_3(o))r^{-s|o|})$. Taking the product over orbits then yields the theorem. □

7.5. ***Comparison of L-functions.*** As a check, we verify that the three expressions for $L(E,s)$ are in fact equal.

The "Artin–Schreier" expression for the L-function in Theorem 7.4 is visibly equal to the "elementary" expression in Theorem 4.1.

The index sets for the products in the "Artin–Schreier" and "sextic twist" expressions for the L-function (Theorems 7.4 and 7.2, respectively) are the same, namely, $O^\times_{r,6,q}$. If $o\in O^\times_{r,6,q}$ is the orbit through (i,α), let o' be the orbit through $(-i,\alpha)$. The map $o\mapsto o'$ gives a bijection $O^\times_{r,6,q}\to O^\times_{r,6,q}$ with $n_6(o)=n_6(o')$.

Let $o \in O^{\times}_{r,6,q}$ and choose $(i,\alpha)\in o$. Write $\mathbb{F}=\mathbb{F}_{r^{|o|}}$, $\mathbb{F}'=\mathbb{F}_{r^{|\pi_2(o)|}}$, and $\mathbb{F}''=\mathbb{F}_{r^{|\rho_6(o)|}}$, so that $\mathbb{F}/\mathbb{F}'$ is an extension of degree $m_2(o)$, and $\mathbb{F}/\mathbb{F}''$ is an extension of degree $n_6(o)$. Then

$$
\begin{aligned}
&G(\pi_2(o))^{m_2(o)}G(\pi_3(o))\\
&\qquad = G_{\mathbb{F}'}(\chi^i_{\mathbb{F}',2},\psi_\alpha)^{m_2(o)}G_{\mathbb{F}}(\chi^i_{\mathbb{F},3},\psi_\alpha) && \text{(definition of } G(\pi_n(o)))\\
&\qquad = G_{\mathbb{F}}(\chi^i_{\mathbb{F},2},\psi_\alpha)G_{\mathbb{F}}(\chi^i_{\mathbb{F},3},\psi_\alpha) && \text{(Hasse–Davenport relation)}\\
&\qquad = J_{\mathbb{F}}(\chi^i_{\mathbb{F},2},\chi^i_{\mathbb{F},3})G_{\mathbb{F}}(\chi^i_{\mathbb{F},2}\chi^i_{\mathbb{F},3},\psi_\alpha) && \text{(equation (3-4))}\\
&\qquad = J_{\mathbb{F}''}(\chi^i_{\mathbb{F}'',2},\chi^i_{\mathbb{F}'',3})^{n_6(o)}G_{\mathbb{F}}(\chi^i_{\mathbb{F},2}\chi^i_{\mathbb{F},3},\psi_\alpha) && \text{(Hasse–Davenport relation)}\\
&\qquad = J(\rho_6(o'))^{n_6(o')}G_{\mathbb{F}}(\chi^i_{\mathbb{F},2}\chi^i_{\mathbb{F},3},\psi_\alpha) && \big(\text{definition of } J(\rho_6(o'))\\
& && \quad \text{and } n_6(o)=n_6(o')\big)\\
&\qquad = J(\rho_6(o'))^{n_6(o')}G_{\mathbb{F}}(\chi^{-i}_{\mathbb{F},6},\psi_\alpha) && (2+3=-1 \pmod 6)\\
&\qquad = J(\rho_6(o'))^{n_6(o')}G(o') && \text{(definition of } G(o')).
\end{aligned}
$$

Thus the o factor in the "Artin–Schreier" product for $L(E,s)$ equals the o' factor in the "sextic twist" product for $L(E,s)$.

8. First application of the BSD conjecture

In this section, we show that the conjecture of Birch and Swinnerton-Dyer (BSD) holds for E, and we deduce consequences for the Mordell–Weil group $E(K)$.

8.1. ***Notation and definitions.*** We recall the remaining definitions needed to state our BSD result. There is a canonical $\mathbb{Z}$-bilinear pairing

$$\langle\cdot,\cdot\rangle : E(K)\times E(K)\to\mathbb{Q}$$

which is nondegenerate modulo torsion. (This is the canonical Néron–Tate height pairing divided by $\log r$. See [Néron 1965] for the definition and [Hindry and Silverman 2000, B.5] for a friendly introduction.) Choosing a $\mathbb{Z}$-basis $P_1,\dots,P_R$ for $E(K)$ modulo torsion, we define the *regulator* of E as

$$\operatorname{Reg}(E) := |\det\langle P_i,P_j\rangle_{1\le i,j\le R}|.$$

The regulator is a positive rational number, well defined independently of the choice of bases, and by convention, it is 1 when the rank of $E(K)$ is 0.

We write $H^1(K,E)$ for the étale cohomology of K with coefficients in E and similarly for $H^1(K_v,E)$ for any place v of K. The *Tate–Shafarevich group* of E is defined as

$$Ш(E) := \ker\Big(H^1(K, E) \to \prod_v H^1(K_v, E)\Big),$$

where the product is over the places of K and the map is the product of the restriction maps.

The leading coefficient of the L-function (also called its *special value* at $s = 1$ or $T = r^{-1}$) is defined by

$$L^*(E) := \frac{1}{\rho!}\left(\frac{d}{dT}\right)^{\rho} L(E, T)\bigg|_{T=r^{-1}} = \frac{1}{(\log r)^{\rho}}\frac{1}{\rho!}\left(\frac{d}{ds}\right)^{\rho} L(E, s)\bigg|_{s=1}$$

where ρ is the order of vanishing $\rho := \operatorname{ord}_{s=1} L(E, s)$. The point of the normalization by $(\log r)^{-\rho}$ is to ensure that $L^*(E)$ is a rational number (recall indeed that $L(E, s)$ is a polynomial with integral coefficients in $T = r^{-s}$). Note that the above definition directly implies the two relations

$$L^*(E) = \frac{L(E, T)}{(1 - rT)^{\rho}}\bigg|_{T=r^{-1}} \quad \text{and} \quad L^*(E) = \lim_{s\to 1} \frac{L(E, s)}{(1 - r^{1-s})^{\rho}}.$$

We refer to Section 2.1 for the definition of the local Tamagawa numbers c_v.

Here is our main result connecting all these invariants.

Theorem 8.2. *The BSD conjecture holds for E. More precisely*,

(1) $\operatorname{ord}_{s=1} L(E, s) = \operatorname{Rank} E(K)$,

(2) $Ш(E)$ *is finite*,

(3) *we have an equality*

$$L^*(E) = \frac{\operatorname{Reg}(E)\, |Ш(E)| \prod_v c_v}{r^{\deg(\omega_E)-1} |E(K)_{\text{tors}}|^2}.$$

Proof. This follows from the fact (see Sections 6.2 and 6.3) that the Néron model of E is dominated by a product of curves, and earlier work of Tate [1966] and Milne [1975]. See [Ulmer 2011, Theorem 9.1] for more details. □

As we have shown, the L-function $L(E, s)$ is a polynomial of degree $2(q - 1)$ in r^{-s}. In particular, $\rho = \operatorname{ord}_{s=1} L(E, s)$ cannot exceed $2(q - 1)$. By part (1) of the BSD result, this proves that $0 \le \operatorname{Rank} E(K) \le 2(q - 1)$. In what follows, we will describe more precisely the value of $\operatorname{Rank} E(K)$, depending on $p \bmod 6$.

We proved in Proposition 2.4 that $|E(K)_{\text{tors}}| = 1$ and that $\prod_v c_v = 1$, and we noted in Section 2.2 that $\deg(\omega_E) = \lceil q/6 \rceil$. Thus the BSD formula simplifies to

$$L^*(E) = \frac{\operatorname{Reg}(E)\, |Ш(E)|}{r^{\lfloor q/6 \rfloor}}. \tag{8-1}$$

In the rest of this section, we will deduce consequences from part (1) of the theorem, and in the following section we will use parts (2) and (3).

8.3. *Explicit L-function for $p \equiv 1 \pmod 6$.* Recall that we have shown that

$$L(E,T) = \prod_{o \in O^{\times}_{r,6,q}} (1 - G(\pi_2(o))^{m_2(o)} G(\pi_3(o)) T^{|o|}),$$

where we substitute T for r^{-s}. We will make this more explicit using results from Section 3.5.

First, note that when $p \equiv 1 \pmod 6$, the action of $\langle r \rangle$ on $(\mathbb{Z}/6\mathbb{Z})^{\times}$ is trivial, so an orbit $o \in O^{\times}_{r,6,q}$ consists of pairs (i, α) where $i \in (\mathbb{Z}/6\mathbb{Z})^{\times}$ is constant and $\alpha \in \mathbb{F}_q^{\times}$ runs through an orbit $\bar{o} \in O_{r,q}$ (recall that $O_{r,q}$ denotes the set of orbits of the action of $\langle r \rangle$ on $\mathbb{F}_q^{\times}$). In particular, we have $|\pi_2(o)| = |o|$ so that $m_2(o) = 1$.

For a given orbit $\bar{o} \in O_{r,q}$, let us consider the two orbits in $O^{\times}_{r,6,q}$,

$$o = \{(1, \alpha) : \alpha \in \bar{o}\} \quad \text{and} \quad o' = \{(-1, \alpha) : \alpha \in \bar{o}\},$$

"lying over $\bar{o}$" and the two corresponding factors in the product for the L-function. Set $\mathbb{F} = \mathbb{F}_r(\alpha)$ and note that $\mathbb{F}$ is an extension of $\mathbb{F}_r = \mathbb{F}_{p^\nu}$ of degree $|o| = |o'| = |\bar{o}|$. By definition we have

$$\begin{aligned}(8\text{-}2)\quad &\big(1 - G(\pi_2(o))G(\pi_3(o))T^{|o|}\big)\big(1 - G(\pi_2(o'))G(\pi_3(o'))T^{|o'|}\big)\\ &= \big(1 - G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi_\alpha) G_{\mathbb{F}}(\chi_{\mathbb{F},3}, \psi_\alpha) T^{|o|}\big)\big(1 - G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi_\alpha) G_{\mathbb{F}}(\chi^{-1}_{\mathbb{F},3}, \psi_\alpha) T^{|o|}\big)\\ &=: L_{\bar{o}}(T).\end{aligned}$$

Since $|\mathbb{F}| = p^{\nu|o|}$, it follows from (3-1) that

$$\operatorname{ord}(G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi_\alpha)) = \tfrac{1}{2}\nu|o|.$$

On the other hand, (3-2) yields that

$$\operatorname{ord}(G_{\mathbb{F}}(\chi_{\mathbb{F},3}, \psi_\alpha)) = \tfrac{2}{3}\nu|o| \quad \text{and} \quad \operatorname{ord}(G_{\mathbb{F}}(\chi^{-1}_{\mathbb{F},3}, \psi_\alpha)) = \tfrac{1}{3}\nu|o|.$$

In particular, the inverse roots of the product $L_{\bar{o}}(T)$ have valuation $\frac{7}{6}\nu$ and $\frac{5}{6}\nu$. We deduce that $T = r^{-1}$, which satisfies $\operatorname{ord}(r^{-1}) = -\nu$, cannot be a root of $L_{\bar{o}}(T)$.

Since this holds for any orbit $\bar{o} \in O_{r,q}$ and since $L(E,T) = \prod_{\bar{o} \in O_{r,q}} L_{\bar{o}}(T)$, we obtain that $L(E,T)$ does not vanish at $T = r^{-1}$. This establishes the first two points of the following result.

Proposition 8.3.1. *Assume that $p \equiv 1 \pmod 6$.*

(1) *The inverse roots on the right-hand side of (8-2) have valuations $\frac{7}{6}\nu$ and $\frac{5}{6}\nu$.*

(2) $\operatorname{ord}_{s=1} L(E,s) = 0$.

(3) $E(K) = 0$.

(4) $\operatorname{Reg}(E) = 1$.

Proof. Points (1) and (2) follow immediately from the above discussion. It then follows from our BSD result (Theorem 8.2) that $\operatorname{Rank} E(K) = 0$ so that $E(K)$ is torsion. But we showed in Proposition 2.4 that $E(K)_{\text{tors}} = 0$, so $E(K) = 0$. Finally, since $E(K)$ has rank 0, the regulator is 1. □

We remark that point (1) of the proposition leads to another proof of BSD in this case. Indeed, the inequality $0 \le \operatorname{Rank} E(K) \le \operatorname{ord}_{s=1} L(E, s)$ is known in general (see [Tate 1966]), so if $\operatorname{ord}_{s=1} L(E, s) = 0$, then $\operatorname{Rank} E(K) = \operatorname{ord}_{s=1} L(E, s) = 0$, and this equality between algebraic and analytic ranks implies the rest of the BSD conjecture (by the main result of [Kato and Trihan 2003]).

8.4. ***Explicit L-function for $p \equiv -1 \pmod 6$.*** As in the preceding subsection, we start from the expression

$$L(E, T) = \prod_{o \in O^{\times}_{r,6,q}} (1 - G(\pi_2(o))^{m_2(o)} G(\pi_3(o)) T^{|o|}),$$

which we make more explicit, in the case when $p \equiv -1 \pmod 6$, using results from Section 3.5.

Let $o \in O^{\times}_{r,6,q}$ be an orbit, pick $(i, \alpha) \in o$ and write $\mathbb{F} = \mathbb{F}_{r^{|o|}}$. If $m_2(o) = 1$ then, by definition of the Gauss sums, we have

$$\big(1 - G(\pi_2(o))^{m_2(o)} G(\pi_3(o)) T^{|o|}\big) = \big(1 - G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi_\alpha) G_{\mathbb{F}}(\chi^i_{\mathbb{F},3}, \psi_\alpha) T^{|o|}\big).$$

On the other hand, if $m_2(o) = 2$ (i.e., if $|o| = 2|\pi_2(o)|$), setting $\mathbb{F}' = \mathbb{F}_r(\alpha) = \mathbb{F}_{r^{|\pi_2(o)|}}$ (which is a quadratic extension of $\mathbb{F}$), the Hasse–Davenport relation yields

$$G(\pi_2(o))^{m_2(o)} = G_{\mathbb{F}'}(\chi_{\mathbb{F}',2}, \psi_\alpha)^2 = G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi_\alpha).$$

Thus, in both cases, we can rewrite the factor of $L(E, T)$ indexed by $o \in O^{\times}_{r,6,q}$ as

$$\big(1 - G(\pi_2(o))^{m_2(o)} G(\pi_3(o)) T^{|o|}\big) = \big(1 - G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi_\alpha) G_{\mathbb{F}}(\chi^i_{\mathbb{F},3}, \psi_\alpha) T^{|o|}\big),$$

where $\mathbb{F} = \mathbb{F}_{r^{|o|}}$ and $(i, \alpha) \in o$. Now using (3-1) and (3-3) and recalling that $\mathbb{F}_r = \mathbb{F}_{p^\nu}$, we remark that

$$G_{\mathbb{F}}(\chi_{\mathbb{F},2}, \psi_\alpha) G_{\mathbb{F}}(\chi^i_{\mathbb{F},3}, \psi_\alpha) = p^{*\nu|o|/2} \chi^{-i}_{\mathbb{F},3}(\alpha)(-p)^{\nu|o|/2} = \epsilon_o r^{|o|},$$

where ϵ_o is a sixth root of unity, namely,

$$\epsilon_o = (-1)^{(p+1)\nu|o|/4} \chi^{-i}_{\mathbb{F},3}(\alpha). \tag{8-3}$$

Note that, p being odd and $\nu|o|$ being even, the exponent $(p+1)\nu|o|/4$ of -1 is an integer. Therefore, for any orbit $o \in O^{\times}_{r,6,q}$, the factor of $L(E, T)$ indexed by o can be rewritten as

$$(1 - G(\pi_2(o))^{m_2(o)} G(\pi_3(o)) T^{|o|}) = (1 - \epsilon_o r^{|o|} T^{|o|}). \tag{8-4}$$

We can now prove the following result, analogous to Proposition 8.3.1.

Proposition 8.4.1. *Assume that* $p \equiv -1 \pmod 6$. *Let*

$$\rho = \rho_{r,q} := \left|\left\{o \in O^{\times}_{r,6,q} : (p+1)\nu|o| \equiv 0 \pmod 8 \text{ and } \alpha \text{ is a cube in } \mathbb{F}^{\times}_{r^{|o|}} \text{ for any } (i,\alpha) \in o\right\}\right|.$$

Then

(1) $\operatorname{ord}_{s=1} L(E,s) = \rho$.

(2) *$E(K)$ is free abelian of rank ρ.*

(3) *For a given q, $\operatorname{Rank} E(K) = 2(q-1)$ for $\mathbb{F}_r$ sufficiently large. More precisely, if $r = p^{\nu}$ is a power of q, $(p+1)\nu \equiv 0 \pmod 8$, and $3(q-1) \mid (r-1)$, then*
$$\operatorname{Rank} E(K) = 2(q-1).$$

(4) *For a given r, $\operatorname{Rank} E(K)$ is unbounded as q varies. Indeed, for every $\epsilon > 0$, if $q = p^f$ and f is a sufficiently large multiple of 4, then*
$$\operatorname{Rank} E(K) > 2(1-\epsilon)p^f/f.$$

Proof. By our formula for $L(E,s)$ and (8-4), the order of vanishing of $L(E,s)$ at $s=1$ equals the number of orbits $o \in O_{r,6,q}$ such that $G(\pi_2(o))^{m_2(o)}G(\pi_3(o)) = r^{|o|}$, i.e., the number of orbits such that $\epsilon_o = 1$. Part (1) then follows easily from (8-3). For (2), it follows from the BSD theorem (Theorem 8.2) that $\operatorname{Rank} E(K) = \rho$, and we showed in Proposition 2.4 that $E(K)_{\text{tors}} = 0$, so that $E(K)$ is indeed free abelian of rank ρ. The conditions in (3) guarantee that all orbits o have size 1 and satisfy $\epsilon_o = 1$. In this case, there are $2(q-1)$ orbits, all contributing to ρ, and this yields the claim. (Under these assumptions, the L-function of E therefore admits a very simple expression: $L(E,s) = (1-r^{1-s})^{2(q-1)}$).

To prove (4), we first note that it suffices to treat the case $r = p$, i.e., $\nu = 1$. Next, we note that "most" elements $\alpha \in \mathbb{F}_{p^f}$ satisfy $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^f}$. Indeed, it is elementary that the number of elements in $\mathbb{F}_{p^f}$ that do not lie in a smaller field is at least $p^f - (\log_2 f)p^{f/2}$. It follows that for every $\epsilon > 0$, there is a constant f_0 such that

$$|\{\alpha \in \mathbb{F}_{p^f} \mid \mathbb{F}_p(\alpha) = \mathbb{F}_{p^f}\}| \geq (1-\epsilon)p^f$$

for all $f > f_0$. On the other hand, at least $\frac{1}{3}(p^f - 1)$ elements of $\mathbb{F}^{\times}_{p^f}$ are cubes. Thus, if $\epsilon < \frac{1}{3}$, then for all sufficiently large f, the number of elements of $\mathbb{F}_{p^f}$ that are cubes and that generate $\mathbb{F}_{p^f}$ is at least $\left(\frac{1}{3} - \epsilon\right)p^f$. If f is even and α has these properties, then the orbit through (i,α) has size f, and if f is a multiple of 4, then these orbits all contribute to ρ. This shows that for f divisible by 4 and sufficiently large, ρ is bounded below by $2(1-\epsilon)p^f/f$, and this completes the proof of part (4). □

We note that although the rank is always unbounded for varying q, it does not go to infinity with $q = p^f$, i.e., the rank of $E(K)$ may be small even when f is large. For example, when $p \equiv 5 \pmod{12}$ and $\nu = 1$, it follows from part (1) of the proposition that the rank is 0 for all odd f.

9. p-adic size of $L^*(E)$ and Ш(E)

The special value $L^*(E)$ was defined in the previous section. Since $L(E,T)$ is a polynomial in T with integer coefficients, $L^*(E)$ actually lies in $\mathbb{Z}[1/p]$. In this section, we use the explicit presentation of the L-function in terms of exponential sums to estimate the p-adic valuation of $L^*(E)$, and then use the BSD formula to deduce consequences for $\mathrm{Reg}(E)\,|Ш(E)|$.

Recall from Section 3.1 that we fixed a prime ideal $\mathfrak{P}$ of $\overline{\mathbb{Z}}$ that lies over p. As before, we denote by ord the p-adic valuation of $\overline{\mathbb{Q}}$ associated to $\mathfrak{P}$ normalized so that $\mathrm{ord}(p)=1$.

Proposition 9.1. *Given data p, q and $r=p^\nu$ as before, we have*:

(1) *If* $p\equiv 1 \pmod 6$,
$$\mathrm{ord}(L^*(E)) = -\frac{q-1}{6}\nu.$$

(2) *If* $p\equiv -1 \pmod 6$, *then $L^*(E)$ is an integer, so* $\mathrm{ord}(L^*(E))\geq 0$.

(3) *If* $p\equiv -1 \pmod 6$ *and r is sufficiently large* (*in the sense of part* (3) *of Proposition 8.4.1*), $L^*(E)=1$.

Proof. First assume that $p\equiv 1 \pmod 6$. As we saw in Section 8.3, $L^*(E)$ is simply the value of $L(E,T)$ at $T=r^{-1}$. We further showed that $L(E,T)$ is the product over orbits $\bar{o}$ of $\langle r\rangle$ acting on $\mathbb{F}_q^\times$ of factors of the form
$$(1-\gamma_1 T^{|\bar{o}|})(1-\gamma_2 T^{|\bar{o}|}),$$
where $\mathrm{ord}(\gamma_1)=\frac{5}{6}\nu|\bar{o}|$ and $\mathrm{ord}(\gamma_2)=\frac{7}{6}\nu|\bar{o}|$. (See Proposition 8.3.1 (1) and the discussion above that result.) Substituting $T=r^{-1}=p^{-\nu}$, we see that the contribution to $\mathrm{ord}\,L^*(E)$ from the pair of factors associated to $\bar{o}$ has valuation $-\frac{1}{6}\nu|\bar{o}|$. Taking the product over all orbits shows that
$$\mathrm{ord}(L^*(E)) = \sum_{\bar{o}\in O_{r,q}} -\frac{\nu|\bar{o}|}{6} = -\frac{\nu}{6}\sum_{\bar{o}\in O_{r,q}}|\bar{o}| = -\frac{(q-1)\nu}{6},$$
and this establishes part (1) of the proposition.

Now assume that $p\equiv -1 \pmod 6$. In Section 8.4, we showed that $L(E,T)$ is the product over orbits $o\in O_{r,6,q}^\times$ of factors of the form $(1-\epsilon_o r^{|o|}T^{|o|})$ where ϵ_o is a sixth root of unity. If $\epsilon_o\neq 1$, then the contribution of this factor to the special value is $(1-\epsilon_o)$, an algebraic integer. If $\epsilon_o=1$, then the contribution is
$$\left.\frac{(1-r^{|o|}T^{|o|})}{1-rT}\right|_{T=r^{-1}} = \left.(1+rT+\cdots+(rT)^{|o|-1})\right|_{T=r^{-1}} = |o|,$$
an integer. This shows that $L^*(E)$ is an algebraic integer, and since it also lies in

$\mathbb{Z}[1/p] \subset \mathbb{Q}$, $L^*(E)$ is an integer. This establishes part (2) of the proposition. For part (3), we note that if r is sufficiently large, all orbits o are singletons and all the ϵ_o are 1 (see Proposition 8.4.1(3)). The analysis above shows that $L^*(E) = 1$. □

Now we apply the BSD formula, as simplified in (8-1):

$$L^*(E) = \frac{\mathrm{Reg}(E)\,|\text{Ш}(E)|}{r^{\lfloor q/6 \rfloor}}.$$

Corollary 9.2. (1) *If* $p \equiv 1 \pmod 6$, *then*

$$\mathrm{Reg}(E) = 1 \quad \textit{and} \quad \mathrm{ord}(|\text{Ш}(E)|) = 0.$$

In particular, the p-primary part of Ш(E) *is trivial.*

(2) *If* $p \equiv -1 \pmod 6$, *then*

$$\mathrm{ord}(\mathrm{Reg}(E)\,|\text{Ш}(E)|) \geq \lfloor q/6 \rfloor \nu.$$

(3) *If* $p \equiv -1 \pmod 6$ *and* r *is sufficiently large* (*in the sense of part* (4) *of Proposition 8.3.1*), *then*

$$\mathrm{Reg}(E)\,|\text{Ш}(E)| = r^{\lfloor q/6 \rfloor} = p^{\nu \lfloor q/6 \rfloor}.$$

In particular, Ш(E) *is a p-group.*

Proof. If $p \equiv 1 \pmod 6$, then combining Proposition 9.1 with the BSD formula (8-1) yields that

$$\mathrm{ord}(\mathrm{Reg}(E)\,|\text{Ш}(E)|) = 0.$$

We showed in Proposition 8.3.1 that $\mathrm{Reg}(E) = 1$, so that $\mathrm{ord}(|\text{Ш}(E)|) = 0$. This proves part (1).

If $p \equiv -1 \pmod 6$, then Proposition 9.1 says that $L^*(E)$ is an integer, and it follows immediately from (8-1) that $\mathrm{ord}(\mathrm{Reg}(E)\,|\text{Ш}(E)|) \geq \lfloor q/6 \rfloor \nu$. This yields part (2).

For part (3), we know from Proposition 9.1 that $L^*(E) = 1$, so (8-1) implies that $\mathrm{Reg}(E)\,|\text{Ш}(E)| = r^{\lfloor q/6 \rfloor}$. By [Ulmer 2019, Proposition 3.1.1], $\mathrm{Reg}(E)$ is an integer, so both it and $|\text{Ш}(E)|$ are powers of p. This establishes part (3). □

Following [Ulmer 2019, §4], let us consider the limit

$$\dim \text{Ш}(E) := \lim_{n \to \infty} \frac{\log |\text{Ш}(E \times \mathbb{F}_{r^n}(t))[p^\infty]|}{\log(r^n)},$$

where Ш$(-)[p^\infty]$ denotes the p-primary part of Ш$(-)$. As is shown in [loc. cit.], the limit exists and is a nonnegative integer, called the "dimension of Ш" of E. The value of $\dim$ Ш(E) is expressed in terms of the valuations of the inverse roots of $L(E, T)$ in [Ulmer 2019, Proposition 4.2].

In the situation at hand, the mentioned expression and the results of Sections 8.3 and 8.4 directly yield the following values for $\dim Ш(E)$:

Corollary 9.3. (1) *If* $p \equiv 1 \pmod 6$, *then* $\dim Ш(E) = 0$.

(2) *If* $p \equiv -1 \pmod 6$, *then* $\dim Ш(E) = \lfloor q/6 \rfloor$.

10. Algebraic analysis of $Ш(E)[p^\infty]$

In this section we recover the results of Corollaries 9.2 and 9.3 regarding the p-torsion in $Ш(E)$ by algebraic means, more specifically via crystalline cohomology. Here is the statement.

Proposition 10.1. (1) *If* $p \equiv 1 \pmod 6$, *then* $Ш(E)[p] = 0$.

(2) *If* $p \equiv -1 \pmod 6$, *then* $\dim Ш(E) = \lfloor q/6 \rfloor$.

The proof will use that the Néron model $\mathcal{E}$ is dominated by the product of curves $E_0 \times C_{6,q}$, knowledge of the crystalline cohomology of the curves, and p-adic semilinear algebra, as in [Ulmer 2019, §6–8]. We collect the needed background results in the next subsection and treat the cases $p \equiv 1 \pmod 6$ and $p \equiv -1 \pmod 6$ separately in the following two subsections.

10.2. ***Preliminaries.*** Let $W = W(\mathbb{F}_r)$ denote the ring of Witt vectors over $\mathbb{F}_r$ and σ denote its Frobenius morphism. We denote the Dieudonné ring by $A = W\{F, V\}$: this is the noncommutative polynomial ring over W with indeterminates F, V modulo the relations $FV = VF = p \in W$, $Fw = \sigma(w)F$, and $\sigma(w)V = Vw$ for all $w \in W$.

Throughout this section, we write $H^1(C)$ for the integral crystalline cohomology $H^1_{\mathrm{crys}}(C/W)$ of a curve C over $\mathbb{F}_r$. The space $H^1(C)$ is a finitely generated, free $W = W(\mathbb{F}_r)$-module equipped with semilinear actions of F and V such that $FV = VF =$ multiplication by p. In other words, $H^1(C)$ is a module over the Dieudonné ring A. We will apply this for $C = E_0$ and $C = C_{6,q}$ and make it much more explicit below.

We saw in Section 6.2 that the Néron model $\mathcal{E}$ of E, is birational to the quotient of $E_0 \times C_{6,q}$ by the antidiagonal action of μ_6. Then [Ulmer 2019, Proposition 6.2] says that

$$\begin{aligned} (10\text{-}1) \qquad Ш(E)[p^\infty] &\cong \mathrm{Br}(\mathcal{E})[p^\infty] \\ &\cong \mathrm{Br}((E_0 \times C_{6,q})/\mu_6)[p^\infty] \cong \mathrm{Br}(E_0 \times C_{6,q})[p^\infty]^{\mu_6} \end{aligned}$$

where the exponent indicates the invariant subgroup. Moreover, by [Ulmer 2019, Proposition 6.4], for all $n \geq 1$ we have

$$(10\text{-}2) \qquad \mathrm{Br}(E_0 \times C_{6,q})[p^n] \cong \frac{\mathrm{Hom}_A(H^1(E_0)/p^n, H^1(C_{6,q})/p^n)}{\mathrm{Hom}_A(H^1(E_0), H^1(C_{6,q}))/p^n}$$

compatibly with the action of μ_6.

To prove part (1) of the proposition, we will show that the μ_6-invariant part of the numerator in the last expression is 0 whenever $p \equiv 1 \pmod 6$. For part (2), we will recall from [Ulmer 2019, §8] that the growth of $Ш(E \times \mathbb{F}_{r^m}(t))[p^\infty]$ as a function of m is controlled by the numerator in the previous display, and this is in turn computable in terms of the action of $\langle p\rangle$ on a finite set indexing the cohomology of E_0 and $C_{6,q}$.

10.3. ***Explicit A-module structure of $H^1(E_0)$ and $H^1(C_{6,q})$.*** We now make explicit the results on the cohomology groups $H^1(E_0)$ and $H^1(C_{6,q})$ (viewed as A-modules) that will be needed below. All results stated in this subsection follow from well-known results about Fermat curves and their quotients, as recalled in [Ulmer 2019, §7] and in [Katz 1981].

Let $I = \{\pm 1\} \subset (\mathbb{Z}/6\mathbb{Z})^\times = I_0 \cup I_1$ where $I_0 = \{1\}$ and $I_1 = \{-1\}$. As a W-module, $H^1(E_0)$ has rank 2 and is generated by classes e_i with $i \in I$, where e_{-1} is the class of the regular differential dx/y and e_1 is associated to the meromorphic differential $x\,dx/y$. (This can be taken to mean that the restriction of e_1 to $E_0 \setminus \{O\}$ is the class of the regular differential $x\,dx/y$.) The indexing is motivated by the fact that over an extension of $\mathbb{F}_r$ large enough to contain the sixth roots of unity, one has

$$\zeta^*(e_1) = \zeta e_1 \quad \text{and} \quad \zeta^*(e_{-1}) = \zeta^{-1} e_{-1}$$

for all $\zeta \in \mu_6$, where the ζs on the left of each equation are in the finite field $\mathbb{F}_r$ and those on the right are their Teichmüller lifts to the Witt vectors W. The action of A satisfies $F(e_i) = c_i e_{pi}$ for some $c_i \in W$ with

$$\operatorname{ord}(c_i) = \begin{cases} 0 & \text{if } i \in I_0, \\ 1 & \text{if } i \in I_1. \end{cases} \tag{10-3}$$

Since $FV = p$, we deduce that $V(e_i) = p/\sigma^{-1}(c_{i/p}) e_{i/p}$.

Let $J \subset \mathbb{Z}/6(q-1)\mathbb{Z}$ be the set of classes that are nonzero modulo 6. Given $j \in J$, there is a unique pair of integers (a, b) with $1 \le a \le q-1$, $1 \le b \le 5$, and $j \equiv 6a - b \pmod{6(q-1)}$. Then $H^1(C_{6,q})$ is a free W-module of rank $5(q-1)$ with basis elements f_j, $j \in J$, where f_j is associated to the differential $t^{a-1}dt/u^b$ in the following sense: Let $J_1 \subset J$ be the set of classes j whose associated (a, b) satisfy $a < qb/6$. For these j, the differential $t^{a-1}dt/u^b$ is everywhere regular on $C_{6,q}$ and f_j is its class. Let $J_0 = J \setminus J_1$. If $j \in J_0$, the differential $t^{a-1}dt/u^b$ is regular on $C_{6,q} \setminus \{\infty\}$, and the restriction of f_j to the open curve is the class of this differential. Over an extension of $\mathbb{F}_r$ large enough to contain the roots of unity of order $6(q-1)$, we have $\zeta^* f_j = \zeta^j f_j$ for all $\zeta \in \mu_{6(q-1)}$ (with the same convention as before). The action of A on $H^1(C_{6,q})$ is given by $F(f_j) = d_j f_{pj}$, for some $d_j \in W$ satisfying

$$\operatorname{ord}(d_j) = \begin{cases} 0 & \text{if } j \in J_0, \\ 1 & \text{if } j \in J_1. \end{cases}$$

Since $FV = p$, we obtain that $V(f_j) = p/\sigma^{-1}(c_{j/p}) f_{j/p}$.

Fix $j \in J$ with $j \not\equiv 0 \pmod 3$. Let $\mathbb{F} = \mathbb{F}_r(\mu_{6(q-1)})$ and let $m = [\mathbb{F} : \mathbb{F}_p]$, so that $p^m j \equiv j \pmod{6(q-1)}$. Then the m-th power F^m of the Frobenius acts on f_j by multiplication by a Gauss sum. More precisely, let $\chi = \chi_{\mathbb{F},6(q-1)}$ be the character defined in Section 3.2, viewed as a W-valued character. Then $F^m f_j = G_j f_j$ where $G_j = G_{\mathbb{F}}(\chi^j, \psi_1)$. When $p \equiv 1 \pmod 6$, it follows from Stickelberger's theorem that

$$\operatorname{ord}(G_j) = \begin{cases} \frac{2}{3}m & \text{if } j \equiv 1 \pmod 3, \\ \frac{1}{3}m & \text{if } j \equiv 2 \pmod 3. \end{cases} \tag{10-4}$$

(This is essentially the same calculation as that in Section 3.5.)

When $p \equiv 1 \pmod 6$, we will calculate $\operatorname{Hom}_A(H^1(E_0)/p, H^1(C_{6,q})/p)$ explicitly in the next subsection and see that it vanishes. In the following subsection, we will assume $p \equiv -1 \pmod 6$ and use the action of $\langle p \rangle$ on $I \times J$ to compute $\dim Ш(E)$ as in [Ulmer 2019, §8].

10.4. ***Proof of Proposition 10.1(1).*** In light of the isomorphisms (10-1) and (10-2), we remark that it suffices to show that

$$\operatorname{Hom}_A(H^1(E_0)/p, H^1(C_{6,q})/p)^{\mu_6} = 0,$$

to show that $Ш(E)[p] = 0$ in the case when $p \equiv 1 \pmod 6$. To that end, let $\varphi \in \operatorname{Hom}_A(H^1(E_0)/p, H^1(C_{6,q})/p)^{\mu_6}$. Since φ is, in particular, a W-linear map, we can write

$$\varphi(e_i) = \sum_j \alpha_{i,j} f_j$$

for all $i \in I = (\mathbb{Z}/6\mathbb{Z})^\times$, where the sum runs over $j \in J \subset \mathbb{Z}/6(q-1)\mathbb{Z}$, and where $\alpha_{i,j} \in W/p = \mathbb{F}_r$. For φ to commute with the antidiagonal μ_6 action, it is necessary that $\alpha_{i,j} = 0$ unless $i \equiv -j \pmod 6$. Further, φ being an A-module homomorphism means that $\varphi F = F\varphi$ and $\varphi V = V\varphi$. Let us now write down what these conditions mean in terms of the "matrix" $(\alpha_{i,j})_{i,j}$ of φ. Let $m = [\mathbb{F}_r(\mu_{6(q-1)}) : \mathbb{F}_p]$, so that $p^m i \equiv i \pmod 6$ and $p^m j \equiv j \pmod{6(q-1)}$ for all $i \in I$ and $j \in J$. Then, by the results in the previous subsection, we have

$$F^m\varphi(e_1) = F^m\Bigg(\sum_{j\equiv -1 \pmod 6} \alpha_{1,j} f_j\Bigg) = \sum_{j\equiv -1 \pmod 6} \sigma^m(\alpha_{1,j}) G_j f_j$$

and

$$\varphi F^m(e_1) = \varphi(ue_1) = u \sum_{j\equiv -1 \pmod 6} \alpha_{1,j} f_j$$

for a certain $u \in W^\times$ (by (10-3)). Equating coefficients of f_j then yields that $u\alpha_{1,j} = \sigma^m(\alpha_{1,j}) G_j$. However, we know from (10-4) that $\operatorname{ord}(G_j) = \frac{1}{3}m > 0$.

Hence $\alpha_{1,j}=0$ for all $j\in J$. Similarly, we have

$$V^m\varphi(e_{-1})=V^m\Bigl(\sum_{j\equiv 1 \pmod 6}\alpha_{-1,j}f_j\Bigr)=\sum_{j\equiv 1 \pmod 6}\sigma^{-m}(\alpha_{-1,j})(p^m/G_j)f_j$$

and

$$\varphi V^m(e_{-1})=\varphi(ve_{-1})=v\sum_{j\equiv 1 \pmod 6}\alpha_{-1,j}f_j$$

for some $v\in W^\times$ (by (10-3)). Equating coefficients of f_j then shows that

$$v\alpha_{-1,j}=\sigma^{-m}(\alpha_{-1,j})(p^m/G_j).$$

On the other hand, (10-4) tells us that $\operatorname{ord}(p^m/G_j)=\frac{1}{3}m>0$. This implies that $\alpha_{-1,j}=0$ for all $j\in J$.

Thus every $\varphi\in\operatorname{Hom}_A(H^1(E_0)/p,H^1(C_{6,q})/p)^{\mu_6}$ satisfies $\varphi(e_1)=\varphi(e_{-1})=0$. This proves that $\operatorname{Hom}_A(H^1(E_0)/p,H^1(C_{6,q})/p)^{\mu_6}=0$ which completes the proof of part (1) of the proposition. □

10.5. ***Proof of Proposition 10.1 (2).*** We now turn to part (2) of the proposition and assume that $p\equiv -1 \pmod 6$. For any $n\geq 1$, the set $I\times J$ indexes the eigenspaces of $\mu_6\times\mu_{6(q-1)}$ acting on $\operatorname{Hom}(H^1(E_0)/p^n,H^1(C_{6,q})/p^n)$, and the subset (which we denote by $(I\times J)^{\mu_6}$) indexing invariants under the antidiagonal action of μ_6 consists of pairs (i,j) with $i\equiv -j \pmod 6$.

Define a bijection

$$(I\times J)^{\mu_6}\to S:=\{1,5\}\times\{1,\dots,q-1\} \tag{10-5}$$

by $(i,j)\mapsto(b,a)$ where $6a-b\equiv j \pmod{6(q-1)}$ (so that $b\equiv i \pmod 6$). Under this bijection, $(I_0\times J_1)^{\mu_6}$ corresponds to pairs $(1,a)$ where $0<a<q/6$, and $(I_1\times J_0)^{\mu_6}$ corresponds to pairs $(5,a)$ where $5q/6<a<q$. (See the definitions of I_0, I_1, J_0, and J_1 in Section 10.2.) We thus define

$$S_0=\{(1,a):0<a<q/6\}$$

and

$$S_1=\{(5,a):5q/6<a<q\}.$$

The action of $\langle p\rangle$ on $I\times J$ preserves $(I\times J)^{\mu_6}$ and so, by transport of structure, we get a (nonstandard) action on S which we will make explicit below. Let O be the set of orbits of $\langle p\rangle$ on S. Given an orbit $o\in O$, define

$$d(o):=\min(|o\cap S_0|,|o\cap S_1|).$$

Part (2) of the proposition will be a consequence of the following "equidistribution" result.

Proposition 10.6. *For every $o\in O$, $|o\cap S_0|=|o\cap S_1|$.*

Indeed, this proposition implies that

$$\sum_{o\in O} d(o) = \sum_{o\in O} |o\cap S_0| = |S_0| = \lfloor q/6\rfloor.$$

On the other hand, by (10-1), (10-2), and [Ulmer 2019, Theorem 8.3], recall that

$$\dim Ш(E) = \sum_{o\in O} d(o).$$

Hence we have $\dim Ш(E) = \lfloor q/6\rfloor$, so that proving Proposition 10.6 will complete the proof of part (2) of Proposition 10.1.

Proof of Proposition 10.6. We begin the proof by making the action of $\langle p\rangle$ on S more explicit. Suppose that $(i, j) \in (I\times J)^{\mu_6}$ corresponds to $(b, a) \in S$ through the bijection (10-5) and that $p\cdot(i, j) = (pi, pj)$ corresponds to (b', a'). Then $b' = 6-b$ and $6a' - b' \equiv p(6a - b) \pmod{6(q-1)}$, so that

$$a' \equiv pa - \frac{p+1}{6}b + 1 \pmod{q-1} \equiv \begin{cases} pa - \frac{p-5}{6} \pmod{q-1} & \text{if } b = 1,\\ pa - \frac{5p-1}{6} \pmod{q-1} & \text{if } b = 5.\end{cases}$$

We now divide the proof into two cases according to $q \pmod 6$. Suppose first that $q \equiv 1 \pmod 6$, so that $q = p^f$ with f even. Then using the last displayed formula, one finds that q acts on S by $(b, a) \mapsto (b', a')$ where $b' = b$ and

$$a' \equiv \begin{cases} a - \frac{q-1}{6} \pmod{q-1} & \text{if } b = 1,\\ a - \frac{5p-5}{6} \pmod{q-1} & \text{if } b = 5.\end{cases}$$

It follows that the orbits of $\langle q\rangle$ have size exactly 6, all elements of an orbit have the same value of b, and each orbit meets either S_0 or S_1 in exactly one point and does not meet the other. (If the constant value of b is 1, the orbit meets S_0 and if it is 5, the orbit meets S_1.) The orbits of $\langle p\rangle$ are unions of an even number of orbits of $\langle q\rangle$, half of them meeting S_0 and half of them meeting S_1. It follows that $|o\cap S_0| = |o\cap S_1|$ for all orbits o of $\langle p\rangle$. This completes the proof in the case when $q \equiv 1 \pmod 6$.

Now assume that $q \equiv -1 \pmod 6$, so that $q = p^f$ with f odd. In this case, q acts on S by $(b, a) \mapsto (b', a')$ where $b' = 6 - b$ and

$$a' \equiv \begin{cases} a - \frac{q-5}{6} \pmod{q-1} & \text{if } b = 1,\\ a - \frac{5q-1}{6} \pmod{q-1} & \text{if } b = 5.\end{cases}$$

Note that q interchanges the subsets S_0 and S_1, so every orbit of $\langle q\rangle$ on S meets S_0 and S_1 in the same number of points. Since the orbits of $\langle p\rangle$ are unions of orbits of $\langle q\rangle$, it follows that the orbits o of $\langle p\rangle$ satisfy $|o\cap S_0| = |o\cap S_1|$. This completes the proof in the case $q \equiv -1 \pmod 6$, and thus in general. $\square$

11. Archimedean size of $L^*(E)$ and the Brauer–Siegel ratio

Define the exponential differential height of $E = E_{q,r}$ by $H(E) := r^{\deg(\omega_E)}$. As we have seen in Section 2.1, one has $H(E) = r^{\lceil q/6 \rceil}$. Following Hindry and Pacheco [2016], consider the Brauer–Siegel ratio $\mathrm{BS}(E)$ of E:

$$\mathrm{BS}(E) := \frac{\log(\mathrm{Reg}(E)\,|Ш(E)|)}{\log H(E)}.$$

(By Theorem 8.2, $Ш(E)$ is finite so this quantity makes sense). Our goal in this section is to estimate the size of the Brauer–Siegel ratio of $E_{q,r}$ for a fixed r as $q \to \infty$. Here is the statement.

Theorem 11.1. *For a fixed r, as $q \to \infty$ runs through powers of p, one has*

$$\lim_{q\to\infty} \mathrm{BS}(E_{q,r}) = 1.$$

We will actually prove a slightly more precise estimate; namely,

$$\frac{\log(\mathrm{Reg}(E)\,|Ш(E)|)}{\log r} = \frac{q}{6}\left(1 + O\left(\frac{\log\log q}{\log q}\right)\right).$$

Thus for large q, the product $\mathrm{Reg}(E)\,|Ш(E)|$ is of size comparable to $r^{q/6}$. In the case when $p \equiv -1 \pmod 6$ we already know this fact, at least for large enough r (see Corollary 9.2(3)). On the other hand, in the case when $p \equiv 1 \pmod 6$, we know from Proposition 8.3.1(4) that $\mathrm{Reg}(E) = 1$, so we deduce that $|Ш(E)|$ is "large" (of size comparable to $r^{q/6}$).

We saw in (8-1) that

$$L^*(E) = \frac{\mathrm{Reg}(E)\,|Ш(E)|}{H(E) r^{-1}} = \frac{\mathrm{Reg}(E)\,|Ш(E)|}{r^{\lfloor q/6 \rfloor}},$$

so, given the definition of $\mathrm{BS}(E)$, the above theorem will be an immediate consequence of the following one, which is the main result of this section.

Theorem 11.2. *For a fixed r, as $q \to \infty$ runs through powers of p, one has*

$$\lim_{q\to\infty} \frac{\log L^*(E_{q,r})}{q} = 0.$$

To prove this we estimate $\log L^*(E_{q,r})$ from above and from below. While the upper bound is relatively easy to show, proving the required lower bound is more demanding. Before we prove the theorem at the end of this section, we first collect various intermediate results in the next few subsections.

11.3. ***Explicit special value.*** Recall from Theorem 4.1 that

$$L(E,s)=\prod_{o\in O^{\times}_{r,6,q}}\left(1-G(\pi_2(o))^{m_2(o)}G(\pi_3(o))r^{-s|o|}\right),$$

where $O^{\times}_{r,6,q}$ denotes the set of orbits of $\langle r\rangle$ acting on $(\mathbb{Z}/6\mathbb{Z})^{\times}\times\mathbb{F}_q{}^{\times}$. To lighten notation we write

$$\omega(o):=G(\pi_2(o))^{m_2(o)}G(\pi_3(o))$$

for the remainder of the article. Note that $\omega(o)$ is a Weil integer of size $p^{\nu|o|}=r^{|o|}$, where a "Weil integer of size p^c" is an algebraic integer whose absolute value in every complex embedding is p^c.

We partition $O^{\times}:=O^{\times}_{r,6,q}$ as $O^{\times}=O_1^{\times}\cup O_2^{\times}$ where $O_1^{\times}$ consists of those orbits o such that $\omega(o)=r^{|o|}$. Thus the orbits in $O_1^{\times}$ are the ones contributing zeroes at $T=r^{-1}$ to the L-function. In particular, we have $\left|O_1^{\times}\right|=\operatorname{Rank}E(K)$ by our BSD result (Theorem 8.2). From the definition of special value (see Section 8.1), it is a simple exercise to see that

$$L^*(E)=\prod_{o\in O_1^{\times}}|o|\prod_{o\in O_2^{\times}}\left(1-\frac{\omega(o)}{r^{|o|}}\right). \tag{11-1}$$

11.4. ***Estimates for orbits.*** Let us gather here a few estimates to be used below. Although we only need the case $n=6$ in this paper, we work in more generality for future use.

Lemma 11.4.1. *Let p be a prime number, let q and r be powers of p, and let n be an integer prime to p. Let $S^{\times}=(\mathbb{Z}/n\mathbb{Z})^{\times}\times\mathbb{F}_q^{\times}$ and let $O^{\times}$ denote the set of orbits of $\langle r\rangle$ on $S^{\times}$. Then*

(1) $\sum_{o\in O^{\times}}|o|=\left|S^{\times}\right|=\phi(n)(q-1)$,

(2) $\sum_{o\in O^{\times}}1=\left|O^{\times}\right|\ll q/\log q$,

(3) $\sum_{o\in O^{\times}}\log|o|\ll q\log\log q/\log q$.

The implied constants depend only on r and n.

Proof. By general properties of group actions, $S^{\times}$ decomposes as the disjoint union of orbits $o\in O^{\times}$; this yields (1). To prove (2), we study "long" orbits and "short" orbits separately. Let $x\geq 1$ be a parameter to be chosen later. Then

$$\left|\{o\in O^{\times}:|o|>x\}\right|=\sum_{\substack{o\in O^{\times}\\|o|>x}}1\leq\sum_{\substack{o\in O^{\times}\\|o|>x}}\frac{|o|}{x}\leq\frac{1}{x}\sum_{o\in O^{\times}}|o|=\frac{\left|S^{\times}\right|}{x}.$$

Let $o\in O^{\times}$ be the orbit through (i,α). As was noted in Section 3.7, $|o|\geq[\mathbb{F}_r(\alpha):\mathbb{F}_r]$. In particular, $\left|\{o\in O^{\times}:|o|\leq x\}\right|$ is at most $\left|\{\alpha\in\overline{\mathbb{F}}_p:[\mathbb{F}_r(\alpha):\mathbb{F}_r]\leq x\}\right|$. An element $\alpha\in\overline{\mathbb{F}}_p$ has degree $\leq x$ over $\mathbb{F}_r$ if and only if its monic minimal polynomial has degree $\leq x$. The prime number theorem for $\mathbb{F}_r[t]$ implies that there are at most $c_r r^x/x$ monic irreducible polynomials of degree $\leq x$ in $\mathbb{F}_r[t]$ (see [Rosen 2002, Theorem 2.2]) for some constant $c_r > 0$ depending at most on r. This argument yields that $|\{o \in O^\times : |o| \leq x\}| \leq c_r r^x/x$. Adding the two contributions, and choosing $x = \log q/\log r$, we find that $|O^\times| \leq c' q/\log q$ where c' depends only on r and n.

Let us finally turn to the proof of (3): given a parameter $y \geq 1$, we have

$$\sum_{o \in O^\times} \log|o| = \sum_{|o| \leq y} \log|o| + \sum_{|o|>y} \log|o| \leq \log y \sum_{|o|\leq y} 1 + \sum_{|o|>y} \frac{\log|o|}{|o|}|o|$$
$$\leq \log y \sum_{o\in O^\times} 1 + \frac{\log y}{y} \sum_{|o|>y} |o| \leq \log y |O^\times| + \frac{\log y}{y}|S^\times|,$$

because $x \mapsto (\log x)/x$ is decreasing on (e, ∞). Upon using (2) and choosing $y = \log q$, one finds that $\sum_{o\in O^\times} \log|o| \leq c'' q \log\log q/\log q$, where c'' depends only on r and n. This is the desired estimate. $\square$

11.5. *Linear forms in logarithms.* For the convenience of the reader, we quote a special case of the main result of [Baker and Wüstholz 1993] about $\mathbb{Z}$-linear forms in logarithms of algebraic numbers. Choose once and for all an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and fix the branch of the complex logarithm $\log : \mathbb{C} \to \mathbb{C}$ with the imaginary part of $\log z$ in $(-\pi, \pi]$ for all $z \in \mathbb{C}$. In particular, if $|z| = 1$, then $|\log(z)| \leq \pi$ and $\log(-1) = i\pi$. Define the modified height ht'_F as follows: For a number field F and $\alpha \in F$, put

$$ht'_F(\alpha) := \frac{1}{[F:\mathbb{Q}]} \max\{ht_F(\alpha), |\log\alpha|, 1\},$$

where $ht_F(\alpha)$ denotes the usual logarithmic Weil height of α (relative to F); see [Hindry and Silverman 2000, B.2].

Let α_1, α_2 be two algebraic numbers (not 0 or 1) and denote by $\log\alpha_1$, $\log\alpha_2$ their logarithms. Let $F \subset \overline{\mathbb{Q}}$ be the number field generated by α_1, α_2 over $\mathbb{Q}$, and let $d := [F : \mathbb{Q}]$. Let $B = (b_1, b_2)$ with $b_1, b_2 \in \mathbb{Z}$ not both zero and set $ht'(B) := \max\{ht_{\mathbb{Q}}(b_1 : b_2), 1\}$, where $ht_{\mathbb{Q}}$ here denotes the logarithmic Weil height on $\mathbb{P}^1_{\mathbb{Q}}$ (relative to $\mathbb{Q}$). Note that $ht'(B) \leq \log\max\{|b_1|, |b_2|, e\}$.

With notation as above, let $\Lambda := b_1 \log\alpha_1 + b_2 \log\alpha_2 \in \mathbb{C}$. Then the Baker–Wüstholz theorem states that either $\Lambda = 0$ or

$$\log|\Lambda| > -c_d\, ht'_F(\alpha_1)\, ht'_F(\alpha_2)\, ht'(B), \tag{11-2}$$

where $c_d > 0$ is an explicit constant depending only on d.

We make use of the Baker–Wüstholz theorem to prove the following:

Theorem 11.6. *Let p be an odd prime number. Let $z \in \overline{\mathbb{Q}}$ be a Weil integer of size p^a, and let $\zeta \in \overline{\mathbb{Q}}$ be a root of unity. For any integer $L \neq 0$, either $\zeta(zp^{-a})^L = 1$ or*

$$\log\left|1-\zeta(zp^{-a})^L\right| \geq -c_0 - c_1 \log|L|, \tag{11-3}$$

for some effective constants $c_0, c_1 > 0$ depending at most on p, a, the degree of z over $\mathbb{Q}$, and the order of ζ.

Proof. Let $F := \mathbb{Q}(\zeta, z)$ be the number field generated by ζ and z (viewed as a subfield of $\overline{\mathbb{Q}}$), and d be its degree over $\mathbb{Q}$. We begin by estimating the modified height of zp^{-a}. By assumption z is a Weil integer of size p^a. Straightforward estimates imply that the absolute logarithmic Weil height of zp^{-a} is at most $\log p^a$. Therefore,

$$ht'_F(zp^{-a}) \leq \max\left\{\log p^a, \frac{|\log(zp^{-a})|}{d}, \frac{1}{d}\right\} \leq \max\left\{\log p^a, \frac{\pi}{d}\right\},$$

We have used here that $|zp^{-a}| = 1$ in the chosen complex embedding.

For all $|x| \leq \pi/2$, we have $|\sin x| \geq \frac{2}{\pi}|x|$ and thus, for all $|\theta| \leq \pi$, we have

$$|1-e^{i\theta}| = 2\left|\sin\frac{\theta}{2}\right| \geq \frac{2}{\pi}|\theta|.$$

If $0 < |\theta| < \pi$, this leads to $\log|1-e^{i\theta}| \geq \log(2/\pi) + \log|\theta|$.

In the given complex embedding $F \subset \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, one can write $\zeta = e^{2\pi ik/n}$ for some $n \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \ldots, n-1\}$ coprime to n (so that ζ is a primitive n-th root of unity). There is also a unique angle $\phi \in (-\pi, \pi]$ such that $zp^{-a} = e^{i\phi}$. Let $L \neq 0$ be an integer. To prove the theorem, we may assume that $\zeta(zp^{-a})^L \neq 1$. Write

$$\zeta(zp^{-a})^L = e^{i(2\pi k/n + L\phi)} = e^{i\tilde{\theta}},$$

where $\tilde{\theta} \in (-\pi, \pi]$, and let m be the integer such that $2\pi k/n + L\phi = 2\pi m + \tilde{\theta}$. Note that $|m| \leq (|L|+3)/2$. The trigonometric considerations above show that

$$\begin{aligned}
\log\left|1-\zeta(zp^{-a})^L\right| &= \log\left|1-e^{i\tilde{\theta}}\right| \\
&\geq \log(2/\pi) + \log|\tilde{\theta}| \\
&= \log(2/\pi) + \log|2\pi k/n + L\phi - 2\pi m| \\
&= \log(2/(n\pi)) + \log|2\pi(k-nm) + Ln\phi|.
\end{aligned}$$

Let us now consider the $\mathbb{Z}$-linear combination of logarithms of algebraic numbers

$$\Lambda := b_1\log(-1) + b_2\log(zp^{-a}),$$

where $B = (b_1, b_2) := (2(k-mn), nL) \neq (0,0)$. Note that $\log(-1) = i\pi$ and $\log(zp^{-a}) = i\phi$, so that $\Lambda = i(2\pi(k-nm) + Ln\phi)$. By assumption, $\Lambda \neq 0$ so the

Baker–Wüstholz theorem (11-2) yields that

$$\log|\Lambda| \geq -c_d ht'_F(-1) ht'_F(zp^{-a}) ht'(B).$$

As was shown above,

$$ht'_F(zp^{-a}) \leq \max\{\log p^a, \pi/d\},$$

and one can easily see that $ht'_F(-1) = \pi/d$. Also, $ht'(B) \leq \log\max\{|b_1|, |b_2|, e\}$, where

$$|b_1| = |2(k-mn)| \leq 2n(1+|m|) \leq (3+|L|) \leq 3n|L|,$$

and $|b_2| = n|L|$, so that $ht'(B) \leq \log(3n|L|)$.

Putting these estimates together, we arrive at

$$\begin{aligned}\log\left|1-\zeta(zp^{-a})^L\right| &\geq \log\frac{2}{n\pi} - c_d\frac{\pi}{d}\max\left\{\log p^a, \frac{\pi}{d}\right\}\log(3n|L|)\\ &\geq -c_0 - c_1\log|L|,\end{aligned}$$

where c_0 and c_1 are certain positive constants depending only on p, a, n, and d. This completes the proof of the theorem. □

We now apply this result to the situation at hand. For any orbit $o \in O^\times_{r,6,q}$, we deduce from Proposition 3.9 that we can write $G(\pi_2(o)) = \zeta_2 g_2^{|\pi_2(o)|\nu}$ where $\zeta_2 = \pm 1$, and $g_2 \in \mathbb{Q}(\mu_{2p})$ is a Weil integer of size $p^{1/2}$. Similarly, letting c be the order of p modulo 3, Proposition 3.9 implies that $G(\pi_3(o)) = \zeta_3 g_3^{|\pi_3(o)|\nu/c}$ where ζ_3 is a third root of unity and $g_3 \in \mathbb{Q}(\mu_{3p})$ is a Weil integer of size $p^{c/2}$. Since $m_2(o)|\pi_2(o)| = |o|$ and $|\pi_3(o)| = |o|$, and since $c \in \{1, 2\}$, we find that

$$\omega(o) = \zeta_2^{m_2(o)}\zeta_3(g_2^2 g_3^{2/c})^{|o|\nu/2}.$$

For any orbit $o \in O^\times$, it follows that $\omega(o)$ is of the form $\omega(o) = \zeta_o g_o^{|o|\nu/2}$ where $\zeta_o = \zeta_2^{m_2(o)}\zeta_3$ is a sixth root of unity and $g_o = g_2^2 g_3^{2/c} \in \mathbb{Q}(\mu_{6p})$ is a Weil integer of size p^2.

Using the previous theorem for $\zeta = \zeta_o$, $z = g_o$ (with $a = 2$) and $L = |o|\nu/2$, and setting $c_2 = c_0 + c_1\log(\nu/2)$, one obtains the following corollary:

Corollary 11.7. *For any orbit $o \in O^\times_{r,6,q}$, either $\omega(o)/r^{|o|} = 1$ (i.e., $o \in O^\times_1$) or*

$$\log\left|1 - \frac{\omega(o)}{r^{|o|}}\right| \geq -c_2 - c_1\log|o|.$$

11.8. ***Proof of Theorem 11.2.*** Recall that the theorem asserts that

$$\lim_{q\to\infty}\frac{\log L^*(E_{q,r})}{q} = 0.$$

We saw in (11-1) that

$$L^*(E_{q,r}) = \prod_{o\in O_1^\times} |o| \prod_{o\in O_2^\times}\left(1-\frac{\omega(o)}{r^{|o|}}\right),$$

where $O_1^\times \subset O_{r,6,q}^\times$ consists of those orbits o such that $\omega(o) = r^{|o|}$ and $O_2^\times = O_{r,6,q}^\times \smallsetminus O_1^\times$.

It is clear that $|1-\omega(o)/r^{|o|}| \le 2$ for all $o \in O^\times$. We can thus bound $\log L^*(E)$ from above as follows:

$$\begin{aligned}\log L^*(E_{q,r}) = \log\biggl(\prod_{o\in O_1^\times} |o| \prod_{o\in O_2^\times}\left(1-\frac{\omega(o)}{r^{|o|}}\right)\biggr) &\le \sum_{o\in O_1^\times} \log|o| + \sum_{o\in O_2^\times} \log 2 \\ &\ll \frac{q\log\log q}{\log q} + \frac{q}{\log q}\log 2 \ll \frac{q\log\log q}{\log q},\end{aligned}$$

where we made use of Lemma 11.4.1 in the last step. Thus

$$\limsup_{q\to\infty} \frac{\log L^*(E_{q,r})}{q} \ll \limsup_{q\to\infty}\left(\frac{\log\log q}{\log q}\right) = 0.$$

We now turn to a lower bound. We obtain from Corollary 11.7 that

$$\begin{aligned}\log L^*(E_{q,r}) &= \log\biggl(\prod_{o\in O_1^\times} |o| \prod_{o\in O_2^\times}\left(1-\frac{\omega(o)}{r^{|o|}}\right)\biggr) \\ &\ge \sum_{o\in O_1^\times} \log|o| + \sum_{o\in O_2^\times} (-c_2 - c_1\log|o|) \\ &\gg -\frac{q}{\log q} - \frac{q\log\log q}{\log q} \gg -\frac{q\log\log q}{\log q},\end{aligned}$$

using Lemma 11.4.1 again for the penultimate inequality. Therefore

$$\liminf_{q\to\infty} \frac{\log L^*(E_{q,r})}{q} \gg \liminf_{q\to\infty}\left(-\frac{\log\log q}{\log q}\right) = 0.$$

Combining the upper and lower bounds, we finally obtain that

$$\lim_{q\to\infty} \frac{\log L^*(E_{q,r})}{q} = 0,$$

and this completes the proof of Theorem 11.2. □

As a direct consequence of Corollary 9.2(1) and Theorem 11.1, we obtain the following.

Corollary 11.9. *Assume that $p \equiv 1 \pmod 6$. As $q\to\infty$, we have $|Ш(E)[p^\infty]| = 1$ and*

$$|Ш(E)| \ge H(E)^{1+o(1)} = r^{\frac{q}{6}(1+o(1))}.$$

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ON THE NONEXISTENCE OF S^6 TYPE COMPLEX THREEFOLDS IN ANY COMPACT HOMOGENEOUS COMPLEX MANIFOLDS WITH THE COMPACT LIE GROUP G_2 AS THE BASE MANIFOLD

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In a recent preprint Professor Etesi asked a question: could one find a complex three dimensional submanifold S in a compact complex seven dimensional homogeneous space with the compact real 14 dimensional Lie group G_2 as the base manifold, such that S is diffeomorphic to the six dimensional sphere S^6? We apply a result of Tits on compact complex homogeneous space, or of H. C. Wang and Hano–Kobayashi on the classification of compact complex homogeneous manifolds with a compact reductive Lie group to give an answer to his question. In particular, we show that one could not obtain a complex structure of S^6 in his way.

1. Introduction

Let M be a complex manifold, h be an Hermitian metric. For a compact complex manifold, there is always some Hermitian metric h by the partition of the unity argument. If h is an Hermitian metric and G is a compact Lie group acting on M biholomorphically, then by taking average on G, we can always assume that h is invariant under G.

A compact complex homogeneous space with an invariant Hermitian structure was classified by H. C. Wang [1954], see also [Hano and Kobayashi 1960]. In fact, they classified the compact complex homogeneous spaces with compact Lie groups. In particular, an Hermitian manifold is a Riemannian manifold. The identity component of the Riemannian isometric group for a compact Riemannian manifold is a compact Lie group. So is the identity component of the Hermitian isometric group for a compact Hermitian manifold.

Therefore, we have:

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Lemma 1. *If $M = G/H$ is a compact homogeneous Riemannian manifold with G connected, then G is a subgroup of a compact Lie group. In particular, both G and H are reductive with compact semisimple parts.*

We then have (see [Hano and Kobayashi 1960, Theorem B]):

Lemma 2. *Any compact Hermitian homogeneous manifold is a complex torus bundle over a rational (therefore simply connected) projective homogeneous space.*

One could also see [Guan 1994, page 66, Remark] for a detailed understanding of this fibration.

There is also a similar fibration [Tits 1971] closely related to Lemma 2 for any general compact complex homogeneous space:

Proposition 3. *Let $M = G/H$ be a compact complex homogeneous space such that G is a complex Lie group, with H a complex Lie subgroup and $M = G/H$ is the complex quotient. Then there is a complex fibration $G/H \to G/N$ such that $N = \mathrm{Norm}_G(H^0)$ and G/N is a rational projective homogeneous space.*

Here, H^0 is the identity component of H and

$$\mathrm{Norm}_G(H^0) = \{g \in G \mid gH^0g^{-1} \subset H^0\}$$

is the normalizer of H^0 in G.

Let $G = G_2$ be the compact real 14 dimensional Lie group of type G_2, $G^{\mathbb{C}}$ be the complex simple Lie group such that the Lie algebra of $G^{\mathbb{C}}$ is the complexification of the Lie algebra of G. Let B be the Borel subgroup such that B is connected and the Lie algebra of B contains all the negative root vector spaces and the given Cartan subalgebra. Then $G^{\mathbb{C}}/B$ has a complex dimension 6. We notice that the Cartan subalgebra has a complex dimension 2. So, let U be the maximal nilpotent subgroup of B, which is generated by the all the negative root spaces; then B/U is a complex two dimensional algebraic torus $(\mathbb{C}^*)^2$. Let $\pi : B \to B/U$ be the quotient map. Given any complex one dimensional line subspace $l = \mathbb{C}$ in B/U, $S_l = \pi^{-1}(l)$ is a complex codimension one normal subgroup of B. Then $G^{\mathbb{C}}/S_l$ is diffeomorphic to G_2. Therefore, it gives a complex structure on G_2.

On the other hand, if $G = G_2$ is a complex homogeneous space itself, then its complexification $G^{\mathbb{C}}$ acts on G also. And $G = G^{\mathbb{C}}/H$. By Lemma 2, the Hano–Kobayashi fiber bundle is $G^{\mathbb{C}}/H \to G^{\mathbb{C}}/B$. The torus is just the Cartan subgroup of G. By earlier works, e.g., [Guan 2002], we know that the Tits fibration and the HK fibration are the same and therefore, $B = \mathrm{Norm}_{G^{\mathbb{C}}}(H^0)$. But $\dim_{\mathbb{C}} B/H = 1$ and H^0 is normal in B. We see that H contains all the subgroups generated by the negative root spaces. That is, $U \subset H$.

Proposition 4. *Let G be a complex manifold with the base manifold being G_2 such that $G = G_2$ acts on itself biholomorphically. Then there is a holomorphic* *fibration from G into a projective manifold $F : G \to G^{\mathbb{C}}/B$ such that each fiber is a complex one dimensional complex torus. If S is a complex three-dimensional compact submanifold of G, then either $F(S)$ is complex three dimensional or $F(S)$ is complex two dimensional and $S = F^{-1}(F(S))$.*

Therefore, if S is diffeomorphic to S^6, by the result in [Campana et al. 1998; Campana et al. 2020], the algebraic dimension of a complex structure on S is zero. This implies that S can not be a complex submanifold of G.

However, in this paper, we would like to give an argument which is mildly independent of the result from Campana et al.

We shall prove:

Main Theorem. *The conjugate orbit described in both [Etesi 2015] and [Chaves and Rigas 1991] can not be a complex submanifold of G_2.*

Recall that $S^6 = G_2/\operatorname{SU}(3)$. The Cartan subgroup of SU(3) is also a Cartan subgroup of G_2. The Cartan subgroup of SU(3) has elements $\operatorname{diag}(a, b, c)$ with $abc = 1$. The center of SU(3) has elements $aI = \operatorname{diag}(a, a, a)$ with $a^3 = 1$. This implies the center of SU(3) has the structure $C = \mathbb{Z}/(3\mathbb{Z}) = \mathbb{Z}_3$. Let A be one of the generator of $C = \mathbb{Z}_3$. Then $S = \{gAg^{-1} \mid g \in G_2\}$. Since A is in the center of SU(3), it is not difficult to see that SU(3) fixes A.

A description of the Lie algebra of G_2 can be also found in the Section 4 of [Guan 2006], in which the Lie algebra of SU(3) is simply generated by the root spaces with the short roots $e_i - e_j$ with $\{i, j\} \in \{1, 2, 3\}$.

2. Proofs

Proof of Proposition 4. If $F(S)$ in Proposition 4 has two complex dimensions, let $p \in F(S)$ be a regular point for F; then $F^{-1}(p) \cap S$ is a complex one-dimensional manifold. As a closed complex one-dimensional submanifold of $F^{-1}(p)$, it can only be the whole complex torus. Since the fiber bundle is locally free and S a complex submanifold, the regular points for F are a dense set, by continuity we have the last part of Proposition 4 □

Proof of Main Theorem. Assume that S is a complex submanifold, we shall get a contradiction.

If $F(S)$ is complex three dimensional, by $F(G)$ being projective, the pullback of the Kähler form ω is a nonzero H^2 class since it introduces a nonzero measure ω^3 on S. A contradiction.

If $F(S)$ is of complex dimension two, the conjugate orbit of Λ in [Etesi 2015] or [Chaves and Rigas 1991] is S. Here, Λ is one of the two nonidentity elements in the center of SU(3) and therefore is in the given Cartan subgroup of SU(3), which is exactly the Cartan subgroup T of G_2. But $S = F^{-1}(F(S))$, we have that the whole Cartan subgroup of G_2 is $F^{-1}(F(\Lambda))$. Therefore, S could be described as $S_t = \{gtg^{-1} \mid g \in G_2\}$ for any $t \in T$. But $S_t = G/G_t$ with G_t the centralizer $\{g \in G \mid gt = tg\}$. However, in general, $G_t = T$. That is S_t is real 12 dimensional. A contradiction. □

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THE SU(3) TODA SYSTEM WITH MULTIPLE SINGULAR SOURCES

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We consider the singular SU(3) Toda system with multiple singular sources:

$$-\Delta w_1 = 2e^{2w_1} - e^{2w_2} + 2\pi \sum_{\ell=1}^{m} \beta_{1,\ell}\delta_{P_\ell} \quad \text{in } \mathbb{R}^2, \tag{0-1}$$

$$-\Delta w_2 = 2e^{2w_2} - e^{2w_1} + 2\pi \sum_{\ell=1}^{m} \beta_{2,\ell}\delta_{P_\ell} \quad \text{in } \mathbb{R}^2,$$

$$w_i(x) = -2\log|x| + O(1) \quad \text{as } |x| \to \infty;\ i = 1, 2, \tag{0-2}$$

with $m \geq 3$ and $\beta_{i,\ell} \in [0, 1)$. We prove existence and nonexistence results under suitable assumptions on $\beta_{i,\ell}$. This generalizes Luo–Tian's (1992) result for a singular Liouville equation in $\mathbb{R}^2$. We also study existence results for a higher order singular Liouville equation in $\mathbb{R}^n$.

1. Introduction

We consider the following singular SU(3) Toda system with multiple singular sources:

$$\begin{aligned} -\Delta w_1 &= 2e^{2w_1} - e^{2w_2} + 2\pi \sum_{\ell=1}^{m} \beta_{1,\ell}\delta_{P_\ell} \quad \text{in } \mathbb{R}^2, \\ -\Delta w_2 &= 2e^{2w_2} - e^{2w_1} + 2\pi \sum_{\ell=1}^{m} \beta_{2,\ell}\delta_{P_\ell} \quad \text{in } \mathbb{R}^2, \end{aligned} \tag{1-1}$$

where $P_1, \dots, P_m$ are distinct points in $\mathbb{R}^2$, $\beta_{i,\ell} \in [0, 1)$ and δ_P denotes the Dirac measure at P (notice that source terms are written with a plus sign). When $w_1 = w_2$, $\beta_{1,l} = \beta_{2,l} = \beta_l$, the above system reduces to the singular Liouville equation:

$$-\Delta w = e^{2w} + 2\pi \sum_{\ell=1}^{m} \beta_\ell \delta_{P_\ell} \quad \text{in } \mathbb{R}^2. \tag{1-2}$$

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The Toda system (1-1) and the Liouville equation (1-2) have been widely studied in the literature due to its important role in geometry and mathematical physics. For instance, (1-2) is related to the problem of prescribing Gaussian curvature on surfaces with conical singularities, and abelian gauge in Chern–Simons theory [Bartolucci and Tarantello 2002b; Tarantello 1996; Troyanov 1991]. The Toda system (1-1) appears in the description of holomorphic curves in $\mathbb{CP}^3$ [Bolton and Woodward 1997; Calabi 1953; Chern and Wolfson 1987; Doliwa 1997], and in the nonabelian Chern–Simons theory [Dunne 1995; Nolasco and Tarantello 2000; Yang 1997]. For classification, blow-up analysis and existence results for the (singular) Liouville equation and the SU(n) Toda system we refer the reader to [Battaglia et al. 2015; Bartolucci et al. 2019b; Bartolucci and Tarantello 2002a; 2002b; Battaglia and Malchiodi 2016; Brezis and Merle 1991; Chen and Li 1991; Carlotto and Malchiodi 2011; Hyder 2019; Hyder et al. 2019; Jost et al. 2006; Jost and Wang 2002; Lin et al. 2018b; Lin 1998; Lucia and Nolasco 2002; Martinazzi 2009; Prajapat and Tarantello 2001; Lin et al. 2012; 2015; Lin et al. 2018a; Bartolucci et al. 2011; Malchiodi and Ruiz 2011; Bartolucci et al. 2019a; Bartolucci and Malchiodi 2013; D'Aprile et al. 2015; Jevnikar et al. 2015].

Luo–Tian [1992] gave a necessary and sufficient condition for the existence of singular metric with three or more conical singularities on the 2-sphere, whose equivalent statement on $\mathbb{R}^2$ is the following theorem:

Theorem A [Luo and Tian 1992]. *Let $m \geq 3$. Let $P_1, \dots, P_m$ be m distinct points in $\mathbb{R}^2$. Then there exist continuous functions h_ℓ around P_ℓ for $\ell = 1, \dots, m$, a bounded continuous function h_{m+1} outside a compact set, and a solution w to*

$$\begin{aligned} -\Delta w &= e^{2w} \quad \text{in } \mathbb{R}^2 \setminus \{P_1, P_2, \dots, P_m\}, \\ w(x) &= -\beta_\ell \log|x - P_\ell| + h_\ell(x) \text{ around each } P_\ell, \\ w(x) &= -2\log|x| + h_{m+1}(x) \text{ as } |x| \to \infty, \\ &\beta_\ell \in (0,1), \quad \ell = 1, \dots, m \end{aligned} \tag{1-3}$$

if and only if

$$\sum_{\ell=1}^{m} \beta_\ell < 2 \quad \text{and} \sum_{\ell \neq j} \beta_\ell > \beta_j \quad \text{for every } j = 1, 2, \dots, m. \tag{1-4}$$

Moreover, the solution is unique.

Troyanov [1989] studied singular metrics with 2 singularities (i.e., $m = 2$) and constant curvature 1 on the 2-sphere, and showed that the order of both singularities are equal (i.e., $\beta_1 = \beta_2 < 1$). A necessary and sufficient condition on $\{\beta_1, \beta_2, \beta_3\} \subset (-\infty, 1)$ for the existence of singular metric on the 2-sphere has been given in [Eremenko 2004; Umehara and Yamada 2000].

In this paper we study Problem (1-3) in the context of SU(3) Toda system. More precisely, we prove existence and nonexistence of solutions (w_1, w_2) to (1-1) satisfying

$$
\begin{aligned}
&w_i(x) = -\beta_{i,\ell} \log|x - P_\ell| + h_{i,\ell} \text{ around each point } P_\ell,\\
&w_i(x) = -2\log|x| + h_{i,m+1} \text{ as } |x| \to \infty,\\
&h_{i,\ell} \text{ is continuous in a neighborhood of } P_\ell,
\end{aligned}
\tag{1-5}
$$

for $i = 1, 2$ and $\ell = 1, \dots, m$, and $h_{i,m+1}$ is bounded outside a compact set. We write

$$u_i(x) = w_i(x) + \sum_{\ell=1}^{m} \beta_{i,\ell} \log|x - P_\ell|, \quad i = 1, 2.$$

Then w_i solves (1-1) if and only if u_i solves

$$
\begin{aligned}
-\Delta u_1 &= 2K_1 e^{2u_1} - K_2 e^{2u_2} \quad \text{in } \mathbb{R}^2\\
-\Delta u_2 &= 2K_2 e^{2u_2} - K_1 e^{2u_1} \quad \text{in } \mathbb{R}^2\\
K_i(x) &:= \prod_{\ell=1}^{m} \frac{1}{|x - P_\ell|^{2\beta_{i,\ell}}} \quad i = 1, 2.
\end{aligned}
\tag{1-6}
$$

The condition (1-5) in terms of u_i is

$$
\begin{aligned}
&u_i(x) = -\beta_i \log|x| + \text{ a bounded continuous function} \quad \text{on } B_1^c\\
&\beta_i := 2 - \sum_{\ell=1}^{m} \beta_{i,\ell}, \quad i = 1, 2,
\end{aligned}
\tag{1-7}
$$

provided u_i is continuous.

For Toda system with singular sources, the only complete result is [Lin et al. 2012] in which the case of single source, i.e., $m = 1$ is completely solved by PDE and integrable system theory. In [Lin et al. 2018a], some special cases of $m = 2$ are classified using higher order hypergeometric equations. The following theorem gives the *first* existence result when $m \geq 3$:

Theorem 1.1. *Let $m \geq 3$. Let $\{\beta_{i,\ell} : i = 1, 2,\ \ell = 1, 2, \dots m\} \subset [0, 1)$ be such that*

$$
\begin{aligned}
&3(1 + \beta_{i,j}) < 2\sum_{\ell=1}^{m} \beta_{i,\ell} + \sum_{\ell=1}^{m} \beta_{3-i,\ell},\\
&\sum_{\ell=1}^{m} \beta_{i,\ell} < 2 \quad \textit{for } j = 1, 2, \dots, m,\ i = 1, 2.
\end{aligned}
\tag{1-8}
$$

Then given m distinct points $\{P_\ell\}_{\ell=1}^m \subset \mathbb{R}^2$ there exists continuous solution (u_1, u_2) to (1-6) such that (1-7) holds.

Note that if $\sum_\ell \beta_{1,\ell} = \sum_\ell \beta_{2,\ell}$, then the first condition of (1-8) reduces to

$$\sum_{\ell=1}^{m} \beta_{1,\ell} > 1 + \beta_{i,j} \quad \text{for every } i = 1, 2,\ j = 1, \ldots, m,$$

which is stronger than (1-4). We shall show that an equivalent condition of (1-4) for the Toda system, namely a condition of the form

$$\sum_{\ell=1,\ell\neq j}^{m} \beta_{i,\ell} > \max\{\beta_{1,j}, \beta_{2,j}\} \quad \text{for every } j = 1, \ldots, m,\ i = 1, 2, \tag{1-9}$$

is not sufficient for the existence of solutions to (1-6) satisfying the asymptotic behavior (1-7). See Lemma 3.2.

In [Luo and Tian 1992], the existence of a solution to (1-2) is proved by a variational argument. Here, we propose a new proof on the existence via fixed point theory. The crucial step in which we need condition (1-8) is Proposition 2.1 below, a compactness result which follows from the blow-up analysis of sequences of solutions (see Lemma 5.2). This compactness is used to prove the a priori bounds necessary to run the fixed point argument of [Aviles 1986; Hyder et al. 2019; Wei and Ye 2008]. Let us point out that condition (1-9) is sufficient to rule-out a "full blow-up" phenomena (that is, after a suitable rescaling, the limiting profile is a SU(3) Toda system in $\mathbb{R}^2$) for a sequence of solutions to (1-6)–(1-7) (for "half blow-up" and "full blow-up" phenomena; see, e.g., [Ao and Wang 2014; D'Aprile et al. 2016; Musso et al. 2016]). In particular, condition (1-9) is sufficient to prove the a priori estimate when $\beta_{1,\ell} = \beta_{2,\ell} = \beta_\ell$ and $u_1 = u_2$, that is, a priori estimate for the singular Liouville problem (1-3). Moreover, the same method also works for a higher order generalization of it.

Theorem 1.2. *Let $m \geq 3$ and $n \geq 2$. For $\ell = 1, 2, \ldots, m$ let $\beta_\ell \in (0, 1)$ be such that (1-4) holds. Then given m distinct points $\{P_\ell\}_{\ell=1}^m \subset \mathbb{R}^n$ there exists a solution $w \in C^0(\mathbb{R}^n \setminus \{P_1, \ldots, P_m\})$ to*

$$(-\Delta)^{n/2} w = e^{nw} + \gamma_n \sum_{\ell=1}^{m} \beta_\ell \delta_{P_\ell} \quad \text{in } \mathbb{R}^n$$

satisfying the asymptotic behavior

$$w(x) = -2\log|x| + O(1) \text{ as } |x| \to \infty.$$

Here $\gamma_n := \frac{1}{2}(n-1)!|S^n|$ is such that

$$\frac{1}{\gamma_n}(-\Delta)^{n/2} \log\frac{1}{|x|} = \delta_0.$$

2. Proof of Theorem 1.1

It is well-know that if (u_1, u_2) is a solution to (1-6) with $\beta_{i,\ell} < 1$ and u_i, $K_i e^{2u_i} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, then u_i is continuous. On the other hand, if (u_1, u_2) is a continuous solution to (1-6)–(1-7) with $\beta_{i,\ell} < 1$, then $K_i e^{2u_i} = O(|x|^{-4})$ as $|x| \to \infty$. In particular, $\log|\cdot| K_i e^{2u_i} \in L^1(\mathbb{R}^2)$, and u_i satisfies the integral equation

$$(2\text{-}1)\quad u_i(x) := \frac{1}{2\pi}\sum_{j=1}^{2} a_{i,j}\int_{\mathbb{R}^2}\log\left(\frac{1}{|x-y|}\right)K_j(y)e^{2(u_j(y))}\,dy + c_i, \quad i = 1, 2,$$

for some $c_i \in \mathbb{R}$, where $(a_{i,j})$ is the SU(3) Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Moreover, the asymptotic behavior (1-7) implies that

$$\sum_{j=1}^{2} a_{i,j}\int_{\mathbb{R}^2} K_j e^{2u_j}\,dx = 2\pi\beta_i, \quad i = 1, 2,$$

that is,

$$(2\text{-}2)\qquad \int_{\mathbb{R}^2} K_i e^{2u_i}\,dx = 2\pi\bar{\beta}_i, \quad \bar{\beta}_i := \tfrac{1}{3}(2\beta_i + \beta_{3-i}), \quad i = 1, 2.$$

Thus, Theorem 1.1 is equivalent to the existence of solution (u_1, u_2) to (2-1)–(2-2). Moreover, (1-8) in terms of $\bar{\beta}_i$ is

$$(2\text{-}3)\qquad \bar{\beta}_i > 0, \quad \bar{\beta}_i < 1 - \beta_{i,\ell} \quad \text{for every } i = 1, 2;\ \ell = 1, \dots, m.$$

In order to prove existence of solutions to (2-1)–(2-2), we use a fixed point argument on the space

$$X := C_0(\mathbb{R}^2) \times C_0(\mathbb{R}^2),$$
$$\|\mathbf{v}\| := \max\{\|v_1\|_{L^\infty(\mathbb{R}^2)}, \|v_2\|_{L^\infty(\mathbb{R}^2)}\} \text{ for } \mathbf{v} = (v_1, v_2) \in X,$$

where $C_0(\mathbb{R}^2)$ denotes the space of continuous functions vanishing at infinity. We fix $u_0 \in C^\infty(\mathbb{R}^2)$ such that

$$u_0(x) = -\log|x| \text{ on } B_1^c.$$

For $v \in C_0(\mathbb{R}^2)$ let $c_{i,v} \in \mathbb{R}$ be the unique number so that

$$(2\text{-}4)\qquad \int_{\mathbb{R}^2} \overline{K}_i e^{2(v + c_{i,v})}\,dx = 2\pi\bar{\beta}_i, \quad \overline{K}_i := K_i e^{2\beta_i u_0}, \quad i = 1, 2,$$

where $\bar{\beta}_i$ is as in (2-2). Now we define $T : X \to X$, $(v_1, v_2) \mapsto (\bar{v}_1, \bar{v}_2)$, where we have set

$$\bar{v}_i(x) := \frac{1}{2\pi}\sum_{j=1}^{2} a_{i,j}\int_{\mathbb{R}^2}\log\left(\frac{1}{|x-y|}\right)\overline{K}_j(y)e^{2(v_j(y)+c_{j,v_j})}\,dy - \beta_i u_0(x), \quad i=1,2. \tag{2-5}$$

As $\beta_i = 2\bar{\beta}_i - \bar{\beta}_{3-i}$, for $x \in B_1^c$ this can be written as

$$\bar{v}_i(x) := \frac{1}{2\pi}\sum_{j=1}^{2} a_{i,j}\int_{\mathbb{R}^2}\log\left(\frac{|x|}{|x-y|}\right)\overline{K}_j(y)e^{2(v_j(y)+c_{j,v_j})}\,dy, \quad i=1,2.$$

Using that $\overline{K}_i = O(|x|^{-4})$ for $|x|$ large, one can show that $(\bar{v}_1, \bar{v}_2) \in X$. Moreover, the operator T is compact (see, e.g., the proof of [Hyder et al. 2019, Lemma 4.1]).

The following proposition is crucial in proving existence of fixed point of T.

Proposition 2.1. *There exists $C > 0$ such that*

$$\|\mathbf{v}\|_X \le C \textit{ for every } (\mathbf{v}, t) \in X \times [0,1] \textit{ satisfying } \mathbf{v} = tT(\mathbf{v}).$$

Proof. We assume by contradiction that the proposition is false. Then there exists $\mathbf{v}^k = (v_1^k, v_2^k)$ and $t^k \in (0,1]$ with $\mathbf{v}^k = t^k T(\mathbf{v}^k)$ such that $\|\mathbf{v}^k\| \to \infty$. We set

$$\psi_i^k := v_i^k + c_i^k, \quad c_i^k := c_{i,v_i^k} + \tfrac{1}{2}\log t^k.$$

Then we have

$$\begin{aligned} \psi_1^k(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\frac{1}{|x-y|}\right)(2\overline{K}_1(y)e^{2\psi_1^k(y)} - \overline{K}_2(y)e^{2\psi_2^k(y)})\,dy \\ &\qquad - t^k\beta_1 u_0(x) + c_1^k \\ \psi_2^k(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\frac{1}{|x-y|}\right)(2\overline{K}_2(y)e^{2\psi_2^k(y)} - \overline{K}_1(y)e^{2\psi_1^k(y)})\,dy \\ &\qquad - t^k\beta_2 u_0(x) + c_2^k. \end{aligned} \tag{2-6}$$

For $|x| \ge 1$ this is equivalent to

$$\begin{aligned} \psi_1^k(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\frac{|x|}{|x-y|}\right)(2\overline{K}_1(y)e^{2\psi_1^k(y)} - \overline{K}_2(y)e^{2\psi_2^k(y)})\,dy + c_1^k, \\ \psi_2^k(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\frac{|x|}{|x-y|}\right)(2\overline{K}_2(y)e^{2\psi_2^k(y)} - \overline{K}_1(y)e^{2\psi_1^k(y)})\,dy + c_2^k. \end{aligned} \tag{2-7}$$

Since $\|\mathbf{v}^k\| \to \infty$, we necessarily have

$$\max\{\sup \psi_1^k, \sup \psi_2^k\} \to \infty.$$

Without any loss of generality we assume that $\sup \psi_1^k \geq \sup \psi_2^k$. We fix $x^k \in \mathbb{R}^2$ such that

$$\sup \psi_1^k < \psi_1^k(x^k) + 1.$$

Writing

$$\widehat{\psi}_i^k := \psi_i^k + t^k \beta_i u_0, \quad \widehat{K}_i^k := \overline{K}_i e^{-2t^k \beta_i u_0},$$

we see that $(\widehat{\psi}_i^k)$ satisfies an equation of the form (5-1). Therefore, as the term $t^k \beta_i u_0$ is uniformly bounded on bounded domains, Lemma 5.2 can be applied to the sequence (ψ_i^k).

If x^k is bounded then, up to a subsequence, $x^k \to x^\infty$.

We consider the following three cases.

Case 1: $x^\infty \in \mathbb{R}^2 \setminus \{P_\ell : \ell = 1, 2, \dots, m\}$.

By Lemma 5.2 (see also [Jost et al. 2006; Lucia and Nolasco 2002]) we have

$$\max\{\sigma_1(x^\infty), \sigma_2(x^\infty)\} \geq 1,$$

where the blow-up value at a point P is defined by

$$\sigma_i(P) := \lim_{r\to 0} \lim_{k\to\infty} \frac{1}{2\pi} \int_{B_r(P)} \overline{K}_i e^{2\psi_i^k} dx, \quad i = 1, 2.$$

This contradicts (2-3) as $\sigma_i(x^\infty) \leq \bar{\beta}_i < 1$.

Case 2: $x^\infty \in \{P_\ell : \ell = 1, 2, \dots, m\}$.

Without loss of generality we assume that $x^\infty = P_1$. Notice that

$$\overline{K}_i(x) = \frac{f_i(x)}{|x - P_1|^{2\beta_{i,1}}}, \quad i = 1, 2,$$

for some positive continuous functions f_1 and f_2 in a small neighborhood of the point P_1. In particular, the functions $w_i^k(x) := \psi_i^k(x - P_1)$ satisfies the conditions of Lemma 5.2 for some $R > 0$, and we get

$$\sigma_1(x^\infty) \geq 1 - \beta_{11}, \quad \text{or } \sigma_2(x^\infty) \geq 1 - \beta_{2,1},$$

a contradiction to (2-3).

Case 3: $|x^k| \to \infty$.

We set

$$\widetilde{\psi}_i^k(x) = \psi_i^k\left(\frac{x}{|x|^2}\right), \quad \widetilde{K}_i(x) = \frac{1}{|x|^4} \overline{K}_i\left(\frac{x}{|x|^2}\right) \quad \text{on } \mathbb{R}^2 \setminus \{0\}, \quad i = 1, 2,$$

and extend them continuously at the origin. Then $\widetilde{\psi}_i^k$ satisfies

$$\begin{aligned}\widetilde{\psi}_1^k(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\frac{|y|}{|x-y|}\right)\big(2\widetilde{K}_1(y)e^{2\widetilde{\psi}_1^k(y)} - \widetilde{K}_2(y)e^{2\widetilde{\psi}_2^k(y)}\big)\,dy + c_1^k \\ \widetilde{\psi}_2^k(x) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\frac{|y|}{|x-y|}\right)\big(2\widetilde{K}_2(y)e^{2\widetilde{\psi}_2^k(y)} - \widetilde{K}_1(y)e^{2\widetilde{\psi}_1^k(y)}\big)\,dy + c_2^k,\end{aligned} \tag{2-8}$$

for $x \in B_1$. Since $\widetilde{K}_i(0) > 0$ for $i = 1, 2$, and

$$\widetilde{\psi}_1^k(\tilde{x}_k) \to \infty, \quad \tilde{x}_k := \frac{x_k}{|x_k|^2} \to 0,$$

one obtains a contradiction as in Case 1.

We conclude the proposition. □

Proof of Theorem 1.1. It follows from Proposition 2.1 and Schauder fixed point theorem that the operator T has a fixed point, say (v_1, v_2). Then setting

$$u_i := v_i + \beta_i u_0 + c_{i,v_i}, \quad i = 1, 2,$$

one sees that (u_1, u_2) is a solution to (1-6)–(1-7). □

3. Nonexistence results

We show that Theorem 1.1 is not true if the assumption (1-8) is replaced by (1-9). Let us fix $\beta_1, \dots, \beta_7 \in (0, 1)$ such that the assumptions $\mathcal{A}(1)$ to $\mathcal{A}(5)$ hold:

($\mathcal{A}1$) $\beta_4 + \sum_{\ell=1}^4 \beta_\ell = 2$.

($\mathcal{A}2$) $\beta_2 + \beta_3 < \beta_1$.

($\mathcal{A}3$) $\beta_4 < \frac{1}{3}$.

($\mathcal{A}4$) $\beta_4 + \sum_{\ell=5}^7 \beta_\ell = 2$.

($\mathcal{A}5$) $\beta_4 + \beta_5 < 1$.

It is easy to see that $\mathcal{A}(1)$ and $\mathcal{A}(2)$ imply that

($\mathcal{A}6$) $\beta_4 + \beta_1 > 1$ and $\beta_4 + \beta_\ell < 1$ for $\ell = 2, 3$.

We shall show an nonexistence result to the Toda system (1-1) satisfying (1-5) for the following choice of $\{\beta_{i,\ell}\}$:

$$\beta_{1,\ell} := \begin{cases} \beta_\ell & \text{for } \ell = 1, 2, 3, 4, \\ 0 & \text{for } \ell = 5, 6, 7, \end{cases} \qquad \beta_{2,\ell} := \begin{cases} 0 & \text{for } \ell = 1, 2, 3, 4, \\ \beta_\ell & \text{for } \ell = 5, 6, 7. \end{cases} \tag{3-1}$$

Let us point out that we can choose $\{\beta_\ell\}$ satisfying $\mathcal{A}(1)$ to $\mathcal{A}(5)$ in such a way that $\{\beta_{i,\ell}\}$ satisfy (1-9) with $m = 7$, $i = 1, 2$. For instance, one can simply take

$$\begin{aligned} \beta_1 &= 1 - \varepsilon, & \beta_2 &= \beta_3 = \tfrac{1}{2} - \varepsilon, \\ \beta_4 &= \tfrac{3}{2}\varepsilon, & \beta_5 &= 1 - \tfrac{5}{2}\varepsilon, \\ \beta_6 &= \beta_7 = \tfrac{1}{2}(1+\varepsilon), & \varepsilon &\in \big(0, \tfrac{2}{9}\big). \end{aligned}$$

For these β_ℓ one has

$$\sum_{\ell=1}^{7} \beta_{1,\ell} = (1 + \beta_{1,1}) - \tfrac{1}{2}\varepsilon,$$

and hence $\{\beta_{i,\ell}\}$ does not satisfy (1-8).

We begin with the following nonexistence result for a singular Liouville equation.

Lemma 3.1. *Let $\beta_\ell \in (0, 1)$ with $\ell = 1, 2, 3, 4$ be such that $\mathcal{A}(1)$ to $\mathcal{A}(3)$ hold. Let P_1, P_2, P_3 be fixed three distinct points in $\mathbb{R}^2$. Then, for $|P_4|$ large enough, there exists no continuous solution to*

$$\begin{aligned} -\Delta u &= \prod_{\ell=1}^{4} \frac{1}{|x - P_\ell|^{2\beta_\ell}} e^{2u} \qquad \textit{in } \mathbb{R}^2, \\ u(x) &= -2\beta_4 \log|x| + O(1) \quad \textit{as } |x| \to \infty. \end{aligned} \tag{3-2}$$

Proof. Assume by contradiction that there exists a sequence of solutions (u^k) to (3-2) with

$$P_4 = P_{4,k}, \ |P_4| \to \infty \quad \text{as } k \to \infty.$$

Notice that the assymptotic behavior

$$u^k(x) = -2\beta_4 \log|x| + O_k(1) \quad \text{as } |x| \to \infty$$

is equivalent to

$$\int_{\mathbb{R}^2} \frac{K_0(x)}{|x - P_4|^{2\beta_4}} e^{2u^k}\, dx = 4\pi\beta_4, \qquad K_0(x) := \prod_{\ell=1}^{3} \frac{1}{|x - P_\ell|^{2\beta_\ell}}.$$

Step 1: We have

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_R^c} \frac{K_0(x)}{|x - P_4|^{2\beta_4}} e^{2u^k(x)}\, dx = 0.$$

To prove this we use Kelvin transform. Up to a small translation, we can assume that none of P_1, P_2, P_3 is the origin. We set

$$\tilde{u}^k(x) := u^k\left(\frac{x}{|x|^2}\right) - 2\beta_4 \log|x| + c^k, \quad x \neq 0,$$

for some $c^k \in \mathbb{R}$. Then setting $Q_\ell := P_\ell/(|P_\ell|^2)$ for $\ell = 1, 2, 3, 4$ we see that

$$-\Delta\tilde{u}^k(x) = \frac{1}{|x|^4}\prod_{\ell=1}^{4}\frac{1}{|x/(|x|^2) - Q_\ell/(|Q_\ell|^2)|^{2\beta_\ell}} e^{2u^k(x/(|x|^2))} \quad \text{in } \mathbb{R}^2 \setminus \{0\}.$$

Using that $|x||y||x/(|x|^2) - y/(|y|^2)| = |x-y|$, $\mathcal{A}(1)$ and for suitably chosen c^k, we obtain

$$\begin{aligned} -\Delta\tilde{u}^k(x) &= |x|^{2\beta_4}\prod_{\ell=1}^{4}\frac{1}{|x-Q_\ell|^{2\beta_\ell}} e^{2\tilde{u}^k(x)} && \text{in } \mathbb{R}^2 \setminus \{0\}, \\ \tilde{u}^k(x) &= -2\beta_4 \log|x| + O_k(1) && \text{as } |x| \to \infty. \end{aligned}$$

In fact, as $\tilde{u}^k = O_k(1)$ in B_1, it satisfies the above equation at the origin as well, that is,

$$-\Delta\tilde{u}^k(x) = \frac{|x|^{2\beta_4}}{|x-Q_4|^{2\beta_4}} f(x) e^{2\tilde{u}^k(x)} \quad \text{in } \mathbb{R}^2, \qquad f(x) := \prod_{\ell=1}^{3}\frac{1}{|x-Q_\ell|^{2\beta_\ell}}.$$

As $|P_4| \to \infty$, we have that $Q_4 \to 0$. By $\mathcal{A}(3)$ one gets

$$\int_{\mathbb{R}^2} \frac{|x|^{2\beta_4}}{|x-Q_4|^{2\beta_4}} f(x) e^{2\tilde{u}^k(x)} = 4\pi\beta_4 \le 2\pi(1-\beta_4-\varepsilon) \tag{3-3}$$

for some $\varepsilon > 0$. Hence, by Lemma 5.1 we obtain

$$\tilde{u}^k \le C \quad \text{in } B_\delta \text{ for some } \delta > 0.$$

Step 1 follows immediately from the relation

$$\int_{B_R^c} \frac{K_0(x)}{|x-P_4|^{2\beta_4}} e^{2u^k(x)}\,dx = \int_{B_{\frac{1}{R}}} \frac{|x|^{2\beta_4}}{|x-Q_4|^{2\beta_4}} f^k(x) e^{2\tilde{u}^k(x)}\,dx.$$

Step 2: No blow-up occurs on bounded domains, that is, for every $R > 0$,

$$u^k - \beta_4 \log|P_4| \le C(R) \quad \text{on } B_R.$$

Writing $\bar{u}^k = u^k - \beta_4 \log|P_4|$ we see that

$$-\Delta\bar{u}^k = K_0 K_1 e^{2\bar{u}^k} \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} K_0 K_1 e^{2\bar{u}^k}\,dx = 4\pi\beta_4,$$

where

$$K_0(x) := \prod_{\ell=1}^{3}\frac{1}{|x-P_\ell|^{2\beta_\ell}}, \qquad K_1 := \frac{|P_4|^{2\beta_4}}{|x-P_4|^{2\beta_4}}.$$

It follows that $K_1 \to 1$ in $C^0_{\text{loc}}(\mathbb{R}^2)$ as $k \to \infty$, and K_0 does not depend on k.

Assume by contradiction that $\bar{u}^k$ is not locally uniformly bounded from above. Then, as blow-up points are discrete, there exists $\delta > 0$ such that

$$\max_{B_\delta(x_0)} \bar{u}^k = \bar{u}^k(x^k) \to \infty, \quad x^k \to x_0,$$

for some $x_0 \in \mathbb{R}^2$. If $x_0 \notin \{P_1, P_2, P_3\}$, then one can show that

$$4\pi\beta_4 \geq \lim_{r\to 0}\lim_{k\to\infty} \int_{B_r(x_0)} K_0 K_1 e^{2\bar{u}^k}\,dx \geq 4\pi,$$

a contradiction as $\beta_4 < 1$. Thus, $x_0 = P_{\ell_0}$ for some $\ell_0 \in \{1, 2, 3\}$, and in fact, the set of all blow-up points is a subset of $\{P_1, P_2, P_3\}$. We fix $R > 0$ such that $\overline{B}_{2R}(x_0) \cap \{P_1, P_2, P_3\} = \{x_0\}$. Then $\bar{u}^k$ is uniformly bounded from above in $B_{2R}(x_0) \setminus B_{R/2}(x_0)$. Using this, and as $\bar{u}^k$ satisfies the integral equation

$$\bar{u}^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1+|y|}{|x-y|}\right) K(y) e^{2\bar{u}^k(y)}\,dy + C^k, \quad K := K_0 K_1,$$

for some $C^k \in \mathbb{R}$, we get that

$$|\bar{u}^k(x) - \bar{u}^k(y)| \leq C \quad \text{for every } x, y \in \partial B_R(x_0).$$

Hence, by the remark after Lemma 5.2 we have (this can be shown easily by a local Pohozaev type identity to the above integral equation satisfied by $\bar{u}_k$)

$$\sigma(x_0) = \lim_{r\to 0}\lim_{k\to\infty} \frac{1}{2\pi}\int_{B_r(x_0)} K_0 K_1 e^{2\bar{u}^k}\,dx = 2(1-\beta_{\ell_0}).$$

Thus $2\beta_4 \geq \sigma(x_0) = 2(1-\beta_{\ell_0})$. This and $\mathcal{A}(6)$ imply that $\ell_0 = 1$, that is, P_1 is the only blow-up point. In particular, $\bar{u}^k \to -\infty$ locally uniformly outside P_1. Therefore, by Step 1 and (3-3) we get

$$2\beta_4 = \frac{1}{2\pi}\lim_{k\to\infty}\int_{\mathbb{R}^2} K_0 K_1 e^{2\bar{u}^k}\,dx = \sigma(x_0) = 2(1-\beta_1),$$

a contradiction to $\mathcal{A}(6)$. This finishes Step 2.

Since $\bar{u}^k$ is locally uniformly bounded from above, up to a subsequence, either $\bar{u}^k \to -\infty$ locally uniformly, or $\bar{u}^k \to \bar{u}$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$. In the first case we get a contradiction to

$$\int_{\mathbb{R}^2} K_0 K_1 e^{2\bar{u}^k}\,dx = 4\pi\beta_4, \quad \lim_{R\to\infty}\lim_{k\to\infty}\int_{B_R^c} K_0 K_1 e^{2\bar{u}^k}\,dx = 0,$$

thanks to Step 1. Therefore, only the later case can occur, and the limit function $\bar{u}$ satisfies

$$-\Delta\bar{u} = K_0 e^{2\bar{u}} \text{ in } \mathbb{R}^2, \quad K_0 = \prod_{\ell=1}^{3} \frac{1}{|x-P_\ell|^{2\beta_\ell}}.$$

Again by Step 1, we have that

$$\int_{\mathbb{R}^2} K_0 e^{2\bar{u}}\,dx = 4\pi\beta_4,$$

which is equivalent to

$$\bar{u}(x) = -2\beta_4 \log|x| + O(1) \quad \text{as } |x| \to \infty.$$

Thus,

$$w(x) := \bar{u}(x) - \sum_{\ell=1}^{3} \beta_\ell \log|x - P_\ell|$$

satisfies (1-3) with $m = 3$, where $\beta_1, \beta_2, \beta_3$ satisfy $\mathcal{A}(2)$. This contradicts the necessary condition (1-4) in Theorem A. □

Remark. Problem (3-2) is super critical under the assumptions $\mathcal{A}(1)$ and $\mathcal{A}(2)$. To be more precise, if one uses fixed point arguments (as described in Section 4) to prove the lemma, then one would not be able to rule-out a blow-up phenomena around the point P_1. This is due to the fact that the energy of a singular bubble at P_1 is $4\pi(1-\beta_1)$, which is smaller than the total energy $4\pi\beta_4$.

The supercriticality of the Problem (3-2) under $\mathcal{A}(1)$ and $\mathcal{A}(2)$ can also be seen from the point of view of singular Moser–Trudinegr inequality; see, e.g., [Adimurthi and Sandeep 2007; Battaglia and Malchiodi 2016; Malchiodi and Ruiz 2011; Chen 1990; Troyanov 1991].

Now we are in a position to prove nonexistence of solution to the Toda system (1-1)–(1-5) for the choice of $\{\beta_{i,\ell}\}$ as in (3-1). More precisely, we have:

Lemma 3.2. *Let $\beta_\ell \in (0, 1)$ with $\ell = 1, \dots, 7$ be such that $\mathcal{A}(1)$ to $\mathcal{A}(5)$ hold. Let $\{\beta_{i,\ell} : i = 1, 2,\ \ell = 1, \dots, 7\}$ be as in (3-1). Let $P_1, \dots, P_4$ be such that Problem (3-2) has no solution. Let P_5 be a fixed point (different from $P_1, \dots, P_4$). Then for $|P_6|, |P_7|$ large ($P_6 \neq P_7$) there exists no solution to (1-6) with $m = 7$ such that*

$$u_i(x) = -\beta_4 \log|x| + O(1) \quad \textit{as } |x| \to \infty, \quad i = 1, 2.$$

Proof. We assume by contradiction that there is a sequence of solutions $(\hat{u}_i^k)$ to (1-6) with

$$P_\ell = P_{\ell,k}, \quad |P_\ell| \xrightarrow{k\to\infty} \infty \quad \text{for } \ell = 6, 7.$$

Then, setting

$$u_1^k := \hat{u}_1^k, \quad u_2^k := \hat{u}_2^k - \beta_6 \log|P_6| - \beta_7 \log|P_7|,$$

we see that (u_i^k) satisfies

$$
(3\text{-}4)\qquad
\begin{aligned}
-\Delta u_1^k &= 2K_1 e^{2u_1^k} - K_2 e^{2u_2^k} && \text{in } \mathbb{R}^2,\\
-\Delta u_2^k &= 2K_2 e^{2u_2^k} - K_1 e^{2u_1^k} && \text{in } \mathbb{R}^2,\\
\int_{\mathbb{R}^2} K_i e^{2u_i^k}\,dx &= 2\pi\beta_4 && i=1,2,\\
|P_\ell| &\xrightarrow{k\to\infty} \infty && \ell=6,7,
\end{aligned}
$$

where

$$K_1(x) := \prod_{\ell=1}^{4} \frac{1}{|x-P_\ell|^{2\beta_\ell}}, \qquad K_2(x) := |P_6|^{2\beta_6}|P_7|^{2\beta_7} \prod_{\ell=5}^{7} \frac{1}{|x-P_\ell|^{2\beta_\ell}}.$$

Notice that K_1 does not depend on k, $K_1 \in L^1(\mathbb{R}^2)$, thanks to the assumption $\beta_4 < 1$, and

$$K_2 \to |x-P_5|^{-2\beta_5} \text{ locally uniformly in } \mathbb{R}^2 \setminus \{P_5\} \quad \text{as } k \to \infty.$$

We claim that $u_1^k \to u$ locally uniformly in $\mathbb{R}^2$, where u satisfies

$$(3\text{-}5)\qquad -\Delta u = 2K_1 e^{2u} \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} K_1 e^{2u}\,dx = 2\pi\beta_4.$$

Then one can show that $u(x) = -2\beta_4 \log|x| + O(1)$ as $|x| \to \infty$. In particular, $\bar{u}(x) = u(x) + \frac{1}{2}\log 2$ is a solution to the Problem (3-2), a contradiction to our assumption on $P_1, \dots, P_4$ that the Problem (3-2) has no solution.

We prove the claim in few steps.

Step 1: We have

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_R^c} K_1 e^{2u_1}\,dx = 0.$$

The proof is very similar to that of Step 1 in Lemma 3.1. Here we give a sketch of it.

We set

$$\tilde{u}_1^k(x) = u_1^k\left(\frac{x}{|x|^2}\right) - \beta_4 \log|x| + c^k,$$

so that $\tilde{u}_1^k$ satisfies ($\widetilde{K}$ does not depend on k)

$$-\Delta \tilde{u}_1^k = \widetilde{K} e^{2\tilde{u}_1^k} - g^k \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \widetilde{K} e^{2\tilde{u}^k}\,dx = 4\pi\beta_4, \quad \int_{\mathbb{R}^2} g^k\,dx = 2\pi\beta_4,$$
$$g^k, \widetilde{K} > 0 \text{ in } \mathbb{R}^2, \quad \widetilde{K}(x) \xrightarrow{|x|\to 0} 1.$$

Now we can apply Lemma 5.1 with $\beta = 0$, thanks to the assumption $\mathcal{A}(3)$, to get that $\tilde{u}_1^k \le C$ in a neighborhood of the origin. Step 1 follows.

Setting

$$S_i := \{x \in \mathbb{R}^2 : \text{ there is a sequence } x^k \to x \text{ such that } u_i^k(x^k) \to \infty\}, \quad i = 1, 2,$$

we shall show that $S_1 \cup S_2 = \varnothing$. We start with the following:

Step 2: $S_1 \subseteq \{P_1, \dots, P_4\}$ and $S_2 \subseteq \{P_5\}$.

For $x_0 \in S_1 \cup S_2$ we can write

$$K_i(x) = \frac{c_i + o(1)}{|x - x_0|^{2\alpha_i}}, \quad c_i > 0, \quad o(1) \xrightarrow{x \to x_0} 0, \quad i = 1, 2,$$

where $\alpha_1 \in \{0, \beta_1, \dots, \beta_4\}$, $\alpha_2 \in \{0, \beta_5\}$ and $\alpha_1\alpha_2 = 0$. By Lemma 5.1 and $\mathcal{A}(3)$ one gets $S_1 \subseteq \{P_1, \dots, P_4\}$ and $S_2 \subseteq \{P_5\}$.

Step 3: $S_1 \cup S_2 = \varnothing$.

It is well-known that u_i^k satisfies the integral equation

$$u_i^k(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{1+|y|}{|x-y|}\right) \left(2K_i(y)e^{2u_i^k(y)} - K_{3-i}(y)e^{2u_{3-i}^k(y)}\right) dy + C^k, \; i = 1, 2.$$

For $x_0 \in S_1 \cup S_2$ let $R > 0$ be such that $\overline{B}_R(x_0) \cap (S_1 \cup S_2) = \{x_0\}$, and x_0 is the only singularity for K_1, K_2 on $\overline{B}_R(x_0)$. Then, from the above integral representation, one can show that

$$|u_i^k(x) - u_i^k(y)| \le C \text{ for every } x, y \in \partial B_R(x_0), \quad i = 1, 2.$$

In particular, u_i^k and K_i satisfy all the assumptions in Lemma 5.2. Therefore, if $S_2 = \{P_5\}$, then as $\sigma_1(P_5) = 0$, we must have $\sigma_2(P_5) = 1 - \beta_5$. This implies that

$$\beta_4 \ge \sigma_2(P_5) = 1 - \beta_5,$$

a contradiction to $\mathcal{A}(5)$. Hence, $S_2 = \varnothing$.

Now we assume that $\beta_{\ell_0} \in S_1$ for some $\ell_0 \in \{1, \dots, 4\}$. Then, in a similar way we get that $\beta_4 \ge 1 - \beta_{\ell_0}$. In fact, by $\mathcal{A}(6)$, a strict inequality holds, that is, $\beta_4 > 1 - \beta_{\ell_0}$. Since

$$u_1^k \to -\infty \quad \text{locally uniformly in } \mathbb{R}^2 \setminus S_1,$$

we must have that the cardinality of S_1 is at least 2, thanks to Step 1. Taking $P_{\ell_1} \in S_1$ with $\ell_1 \in \{1, \dots, 4\} \setminus \{\ell_0\}$, and again using that $\sigma(P_{\ell_1}) = 1 - \beta_{\ell_1}$, we obtain

$$\beta_4 \ge \sigma(P_{\ell_0}) + \sigma(P_{\ell_1}) = 2 - \beta_{\ell_0} - \beta_{\ell_1},$$

a contradiction to $\mathcal{A}(1)$.

We conclude Step 3.

Step 4: $u_1^k \to \bar{u}_1$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$ where $\bar{u}_1$ satisfies (3-5).

Since $S_1 \cup S_2 = \varnothing$, up to a subsequence, one of the following holds:

(i) $u_i^k \to \bar{u}_i$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$ for $i = 1, 2$.

(ii) $u_1^k \to \bar{u}_1$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$ and $u_2^k \to -\infty$ locally uniformly in $\mathbb{R}^2$.

(iii) $u_2^k \to \bar{u}_2$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$ and $u_1^k \to -\infty$ locally uniformly in $\mathbb{R}^2$.

(iv) $u_i^k \to -\infty$ locally uniformly in $\mathbb{R}^2$ for $i = 1, 2$.

It follows from Step 1, and the integral condition $\int_{\mathbb{R}^2} K_1 e^{2u_1}\,dx = 2\pi\beta_4$ that either (i) or (ii) holds, and $\bar{u}_1$ satisfies the integral condition

$$\int_{\mathbb{R}^2} K_1 e^{2\bar{u}_1}\,dx = 2\pi\beta_4.$$

Now we assume by contradiction that (i) holds. Then the limit functions $(\bar{u}_1, \bar{u}_2)$ satisfy the system

$$\begin{aligned} -\Delta\bar{u}_1 &= 2K_1 e^{2\bar{u}_1} - \overline{K}_2 e^{2\bar{u}_2} \quad \text{in } \mathbb{R}^2,\\ -\Delta\bar{u}_2 &= 2\overline{K}_2 e^{2\bar{u}_2} - K_1 e^{2\bar{u}_1} \quad \text{in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2} K_1 e^{2\bar{u}_1}\,dx &= 2\pi\beta_4, \quad \int_{\mathbb{R}^2} \overline{K}_2 e^{2\bar{u}_2}\,dx =: 2\pi\gamma \le 2\pi\beta_4, \end{aligned} \tag{3-6}$$

where $\overline{K}_2(x) := |x - P_5|^{-2\beta_5}$ is the limit of K_2 as $k \to \infty$. Then one has

$$\lim_{|x|\to\infty} \frac{\bar{u}_2(x)}{\log|x|} = -(2\gamma - \beta_4),$$

and together with $\overline{K}_2 e^{2\bar{u}_2} \in L^1(\mathbb{R}^2)$ we have $\beta_5 + 2\gamma - \beta_4 > 1$. Hence, $\beta_4 + \beta_5 > 1$, a contradiction to $\mathcal{A}(5)$.

Thus, (ii) holds, and (3-6) reduces to a single equation (3-5). □

4. Higher order singular Liouville equation

The proof of Theorem 1.2 is very similar to that of Theorem 1.1 (see also [Hyder et al. 2019]). Here we give a sketch of it.

Writing

$$w(x) = u(x) - \sum_{\ell=1}^{m} \beta_\ell \log|x - P_\ell|,$$

Theorem 1.2 is equivalent to prove the existence of solution $u \in C^0(\mathbb{R}^n)$ to

$$(-\Delta)^{n/2} u = K e^{nu} \text{ in } \mathbb{R}^n, \quad K(x) := \prod_{\ell=1}^{m} \frac{1}{|x - P_\ell|^{n\beta_\ell}}, \tag{4-1}$$

satisfying the asymptotic behavior

$$u(x) = -\beta \log|x| + O(1) \text{ as } |x| \to \infty, \quad \beta := 2 - \sum_{\ell=1}^{m} \beta_\ell. \tag{4-2}$$

As before we fix $u_0 \in C^\infty(\mathbb{R}^n)$ such that $u_0(x) = -\log|x|$ for $|x| \geq 1$, and we look for a solution u to (4-1) of the form

$$u = \beta u_0 + v + c,$$

where c is a normalizing constant and $v \in X$, where

$$X := C_0(\mathbb{R}^n) = \left\{ v \in C^0(\mathbb{R}^n) : v(x) \xrightarrow{|x|\to\infty} 0 \right\}, \quad \|v\| := \max_{x\in\mathbb{R}^n} |v(x)|.$$

Then u satisfies (4-1) if and only if $v = u - \beta u_0 - c$ satisfies

$$(-\Delta)^{n/2} v = \overline{K} e^{nv+c} - \beta(-\Delta)^{n/2} u_0 \text{ in } \mathbb{R}^n, \quad \overline{K} := K e^{n\beta u_0}. \tag{4-3}$$

The function $\overline{K}$ satisfies

$$\lim_{|x|\to\infty} |x|^{2n} \overline{K}(x) = 1. \tag{4-4}$$

For $v \in X$, we fix $c_v \in \mathbb{R}$ so that

$$\int_{\mathbb{R}^n} \overline{K}(x) e^{n(v(x)+c_v)} = \beta \gamma_n. \tag{4-5}$$

We define a compact operator

$$T : X \to X, \quad v \mapsto \bar{v},$$

$$\bar{v}(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) \overline{K}(y) e^{n(v(y)+c_v)}\, dy - \beta u_0(x), \quad x \in \mathbb{R}^n. \tag{4-6}$$

It follows that $\bar{v} \in C^0(\mathbb{R}^n)$ (in fact, Hölder continuous), and by (4-5)

$$\bar{v}(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{|x|}{|x-y|}\right) \overline{K}(y) e^{n(v(y)+c_v)}\, dy \quad \text{for } |x| > 1.$$

We claim that there exists $C > 0$ such that

$$\|v\|_X \leq C \text{ for every } (v,t) \in X \times [0,1] \text{ satisfying } v = tT(v). \tag{4-7}$$

Then by Schauder fixed point theorem the operator T has a fixed point v in X, and consequently we get a continuous solution to (4-1) satisfying (4-2).

To prove (4-7) we assume by contradiction that there exists $(v^k, t^k) \in X \times [0,1]$ such that $\|v^k\|_X \to \infty$ and $v^k = t^k T(v^k)$, that is

$$v^k(x) = \frac{t^k}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) \overline{K}(y) e^{n(v^k(y)+c_{v^k})}\, dy - t^k \beta u_0(x). \tag{4-8}$$

Then we can choose $x^k \in \mathbb{R}^n$ so that

$$\sup_{x\in\mathbb{R}^n} \psi^k(x) \leq \psi^k(x^k) + 1 \xrightarrow{k\to\infty} \infty, \quad \psi^k(x) := v^k(x) + c_{v^k} + \frac{1}{n}\log t^k.$$

The crucial ingredients to obtain a contradiction are Theorem 5.3, and the relation

(4-9) $$\beta = 2 - \sum_{\ell=1}^{m} \beta_\ell = 2 - \beta_\ell - \sum_{\ell \neq j} \beta_\ell < 2(1-\beta_j) \text{ for every } j = 1, 2, \dots, m,$$

which follows from the second condition in (1-4). Up to a subsequence, we distinguish the following two cases:

Case 1: $x^k \to x^\infty \in \mathbb{R}^n$.

In a small neighborhood of x^∞ we have for some $c_0 > 0$

$$\overline{K}(x) = \frac{c_0 + o(1)}{|x - x^\infty|^{n\alpha}}, \quad o(1) \xrightarrow{x \to x^\infty} 0,$$

where $\alpha \in \{0, \beta_1, \dots, \beta_m\}$. Using (4-8)–(4-9) one gets a contradiction as in [Hyder et al. 2019]; see also [Aviles 1986; Wei and Ye 2008].

Case 2: $|x^k| \to \infty$.

Setting

$$\widetilde{\psi}^k(x) := \psi^k\left(\frac{x}{|x|^2}\right), \quad \tilde{x}^k := \frac{x^k}{|x^k|^2} \to 0,$$

we obtain $\widetilde{\psi}_k(\tilde{x}^k) \to \infty$, and $\widetilde{\psi}_k$ satisfies

$$\widetilde{\psi}^k(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right) \widetilde{K}(y) e^{n\widetilde{\psi}^k(y)}\, dy + c^k \quad \text{in } B_1,$$

where

$$\widetilde{K}(x) := \frac{1}{|x|^{2n}} \overline{K}\left(\frac{x}{|x|^2}\right), \quad c^k := c_{v^k} + \frac{1}{n} \log t^k.$$

Note that $\widetilde{K}$ is smooth around the origin and

$$\widetilde{K}(x) \xrightarrow{|x| \to 0} 1,$$

one can proceed as in Case 1. Thus, $\psi^k \le C$ on $\mathbb{R}^n$, and we have (4-7).

5. Some useful lemmas

The following lemma is a generalizations of Brezis–Merle [1991] type results; compare [Bartolucci and Tarantello 2002b, Theorem 5].

Lemma 5.1. *Let (u^k) be a sequence of solutions to*

$$-\Delta u^k = \frac{f^k(x)}{|x|^{2\alpha}} e^{2u^k} - g^k \quad \text{in } B_1, \qquad \int_{B_1} \frac{f^k(x)}{|x|^{2\alpha}} e^{2u^k} dx \le 2\pi(1 - \alpha - \varepsilon),$$

for some $\varepsilon > 0$ and $\alpha \in [0, 1)$. Assume that $g^k \ge 0$, $\|g^k\|_{L^1(B_1)} \le C$, $0 \le f^k \le C$ and $\inf_{B_1 \setminus B_\delta} f^k \ge C_\delta^{-1}$ for some $0 < \delta < \frac{1}{3}$. Then u^k is locally uniformly bounded from above in B_1.

Proof. We write $u^k = v^k + h^k$, where h^k is harmonic in B_1 and

$$v^k(x) := \frac{1}{2\pi}\int_{B_1} \log\left(\frac{2}{|x-y|}\right)\left(\frac{f^k(y)}{|y|^{2\alpha}}e^{2u^k(y)} - g^k(y)\right) dy.$$

Since $g_k \geq 0$, by Jensen's inequality one gets that

$$\int_{B_1} e^{2pv^k(x)}\, dx \leq C(p), \quad p \in \left[1, \frac{1}{1-\alpha-\varepsilon/2}\right].$$

Notice that

$$\int_{B_1\setminus B_\delta} (h^k)^+\, dx \leq \int_{B_1\setminus B_\delta} ((u^k)^+ + |v^k|)\, dx \leq C.$$

Since $\delta < \frac{1}{3}$, fixing $\delta + \frac{1}{3} < r_1 < r_2 < 1-\delta$ we see that

$$\partial B_t(x) \subset B_1 \setminus B_\delta \quad \text{for every } x \in \overline{B}_\delta,\ r_1 \leq t \leq r_2.$$

Therefore, by the mean value theorem,

$$2\pi(r_2 - r_1)h^k(x) = \int_{r_1}^{r_2}\int_{\partial B_t(x)} h^k(y)\, d\sigma(y)\, dt \leq \int_{B_1\setminus B_\delta} (h^k)^+\, dy \leq C.$$

Thus, $\int_{B_1}(h^k)^+\, dx \leq C$. If $\rho^k := \int_{B_{1/2}} |h^k|\, dx \leq C$ then we have

$$h^k \to h \quad \text{in } C^2_{\mathrm{loc}}(B_1), \quad \Delta h = 0 \quad \text{in } B_1.$$

In particular, (h^k) is bounded in $C^0_{\mathrm{loc}}(B_1)$. If $\rho^k \to \infty$, then

$$\frac{h^k}{\rho^k} \to h \quad \text{in } C^2_{\mathrm{loc}}(B_1), \quad \Delta h = 0, \quad h < 0 \quad \text{in } B_1.$$

This shows that (h^k) is locally uniformly bounded from above in B_1. This leads to

$$\int_{B_r} e^{2pu^k}\, dx \leq C_r \int_{B_r} e^{2pv^k}\, dx \leq C(p, r, \varepsilon, \alpha), \quad 0 < r < 1,\ p \in \left[1, \frac{1}{1-\alpha-\varepsilon/2}\right].$$

Using this uniform bound, and Hölder inequality with $p = 1/(1-\alpha-\varepsilon/2)$, one gets $v^k \leq C$ in B_r for $0 < r < 1$, and the lemma follows. □

A strong version (precise quantization value of σ_1, σ_2) of the following lemma is proven in [Lin et al. 2015; 2018b]. See [Lucia and Nolasco 2002] for a Pohozaev type identity for regular SU(3) Toda system.

Lemma 5.2 [Lin et al. 2015; Lin et al. 2018b]. *Let* (u_1^k, u_2^k) *be a sequence of solutions to*

$$
(5\text{-}1)\qquad
\begin{aligned}
-\Delta u_1^k &= 2\frac{K_1^k}{|x|^{2\alpha_1}} g e^{2u_1^k} - \frac{K_2^k}{|x|^{2\alpha_2}} e^{2u_2^k} \quad \text{in } B_1,\\
-\Delta u_2^k &= 2\frac{K_2^k}{|x|^{2\alpha_2}} e^{2u_2^k} - \frac{K_1^k}{|x|^{2\alpha_1}} e^{2u_1^k} \quad \text{in } B_1,\\
\int_{B_1} \frac{K_i^k}{|x|^{2\alpha_i}} e^{2u_i^k}\, dx &\le C, \qquad i = 1, 2,\\
|u_i^k(x) - u_i^k(y)| &\le C, \quad \text{for every } x, y \in \partial B_1, \quad i = 1, 2,\\
\|K_i^k\|_{C^3(B_1)} &\le C, \quad 0 < \frac{1}{C} \le K_i^k \quad \text{in } B_1, \quad i = 1, 2,
\end{aligned}
$$

for some $\alpha_1, \alpha_2 < 1$, *and* B_1 *is the unit ball in* $\mathbb{R}^2$. *Assume that* 0 *is the only blow-up point*, *that is*,

$$\sup_{B_1\setminus B_\varepsilon} u_i^k \le C(\varepsilon) \quad \text{for every } 0 < \varepsilon < 1, \quad i = 1, 2.$$

Then setting

$$\sigma_i := \lim_{r\to 0} \lim_{k\to\infty} \frac{1}{2\pi} \int_{B_r} \frac{K_i^k(x)}{|x|^{2\alpha_i}} e^{2u_i^k(x)}\, dx, \quad i = 1, 2,$$

we have

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_1(1-\alpha_1) + \sigma_2(1-\alpha_2).$$

In particular, *if* $(\sigma_1, \sigma_2) \neq (0, 0)$ *then*

$$\sigma_1 \ge 1-\alpha_1 \quad \text{or } \sigma_2 \ge 1-\alpha_2.$$

Remark. If $\alpha_1 = \alpha_2 = \alpha$, $K_1^k = K_2^k$ and $u_1^k = u_2^k$ in the above lemma, then $\sigma_1 = \sigma_2 = 2(1-\alpha)$.

Theorem 5.3 [Hyder et al. 2019; Prajapat and Tarantello 2001]. *Let u be a normal solution to*

$$(5\text{-}2)\qquad (-\Delta)^{n/2} u = |x|^{n\alpha} e^{nu} \text{ in } \mathbb{R}^n, \quad \Lambda := \int_{\mathbb{R}^n} |x|^{n\alpha} e^{nu}\, dx < \infty,$$

for some $\alpha > -1$ *and* $n \ge 2$, *that is*, u *satisfies the integral equation*

$$u(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right) |y|^{n\alpha} e^{nu(y)}\, dy + C,$$

for some $C \in \mathbb{R}$. *Then* $\Lambda = \Lambda_1(1+\alpha)$, $\Lambda_1 := 2\gamma_n$.

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CONVERGENCE OF MEAN CURVATURE FLOW IN HYPER-KÄHLER MANIFOLDS

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Inspired by work of Leung and Wan (*J. Geom. Anal.* 17:2 (2007) 343–364), we study the mean curvature flow in hyper-Kähler manifolds starting from hyper-Lagrangian submanifolds, a class of middle-dimensional submanifolds, which contains the class of complex Lagrangian submanifolds. For each hyper-Lagrangian submanifold, we define a new energy concept called the *twistor energy* by means of the associated twistor family (i.e., 2-sphere of complex structures). We will show that the mean curvature flow starting at any hyper-Lagrangian submanifold with sufficiently small twistor energy will exist for all time and converge to a complex Lagrangian submanifold for one of the hyper-Kähler complex structure. In particular, our result implies some kind of energy gap theorem for hyper-Kähler manifolds which have no complex Lagrangian submanifolds.

1. Introduction

Let $(M, \bar{g})$ be a hyper-Kähler $4n$-manifold, i.e., the holonomy group is contained in $\mathrm{Sp}(n)$. Or equivalently, there exist distinct, $\bar{g}$-compatible complex structures $\{J_d\}_{d=1,2,3}$ which satisfy the *quaternion relations*:

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -\mathrm{Id}.$$

Then each hyper-Kähler manifold M admits a 2-sphere of complex structures called the *twistor family*

$$\sum_d c_d J_d \quad \text{for } (c_1, c_2, c_3) \in \mathbb{S}^2 \subset \mathbb{R}^3.$$

Throughout this paper, we assume that $(M, \bar{g})$ has bounded geometry (i.e., the injectivity radius, curvatures and derivatives of the curvatures are uniformly bounded). Typical examples of hyper-Kähler manifolds are a K3 surface and a compact torus $\mathbb{T}^4$ (in fact, any Calabi–Yau 4-manifold is hyper-Kähler since $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ and these are only compact 4-dimensional examples). Beauville [1983] constructed two distinct deformation classes of hyper-Kähler's in $4n$-dimension for every $n > 1$.

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Moreover, Grady [1999; 2003] constructed two additional deformation classes in dimensions 12 and 20. Each deformation class has representatives which are moduli spaces of semistable sheaves on projective K3 surfaces or abelian surfaces or modifications of such moduli spaces.

In this paper, we show the existence and convergence result for the mean curvature flow (MCF) in hyper-Kähler manifolds when the initial data is very small. There is no doubt that for studying the MCF, Lagrangian is one of the good class of submanifolds in a Kähler–Einstein manifold. Indeed, from Smoczyk's result [1996], the Lagrangian property is preserved under the MCF, and it gives a lot of benefits for computations of evolution equations, by identifying the extrinsic normal bundle with the intrinsic tangent bundle via the complex structure. Nevertheless, we would like to consider another class of submanifolds, called "hyper-Lagrangian submanifolds" as displayed below. This class includes Lagrangian submanifolds in hyper-Kähler 4-manifolds.

1A. ***Main result.*** A natural counterpart of the Lagrangian condition in hyper-Kähler manifolds is the "complex Lagrangian": for $J \in \mathbb{S}^2$, let Ω_J be a holomorphic symplectic form (i.e., nondegenerate J-holomorphic 2-form) with respect to J. For a $2n$-dimensional real submanifold $L \subset M$, we say that L is *complex Lagrangian* if $\Omega_J|_L = 0$ for some $J \in \mathbb{S}^2$. From a basic fact of hyper-Kähler geometry, we find that there exists a J-orthogonal element $K \in \mathbb{S}^2$ such that Ω_J can be expressed as

$$\Omega_J = \bar{\omega}_{JK} - \sqrt{-1}\bar{\omega}_K,$$

where $\bar{\omega}_{JK} = \bar{g}(JK\cdot\,,\cdot)$, $\bar{\omega}_K = \bar{g}(K\cdot\,,\cdot)$ are real symplectic forms for JK and K respectively. So the condition $\Omega_J|_L = 0$ means that two symplectic forms $\bar{\omega}_{JK}$ and $\bar{\omega}_K$ vanish at the same time for any J-orthogonal $K \in \mathbb{S}^2$.

However, this "bi-Lagrangian" condition is so strong that any complex Lagrangian submanifold L in M automatically becomes a (minimal) complex submanifold (see [Hitchin 1999]). So, following the idea of Leung and Wan [2007], we relax the assumption by using rich geometry on M. We say that L is *hyper-Lagrangian* if $\Omega_{\Psi(x)}|_L = 0$ at every point $x \in L$ for some varying complex structure $\Psi : L \to \mathbb{S}^2$. Then this map Ψ is called the *complex phase*. In particular, complex Lagrangian is a special case when we can take Ψ as a constant map. Leung and Wan [2007] showed that if the initial submanifold L_0 is hyper-Lagrangian, then $L_t := F_t(L)$ is still hyper-Lagrangian under the MCF $F_t : L \to M$, and then the complex phase Ψ_t evolves according to the coupled flow

$$\begin{cases} \frac{d}{dt}F_t = H_t, \\ \frac{d}{dt}\Psi_t = \Delta_t\Psi_t, \end{cases} \tag{1-1}$$

where $\Delta_t\Psi_t$ denotes the tension field of Ψ_t with respect to the evolving metric $g_t := F_t^*\bar{g}$. We will call (1-1) the *hyper-Lagrangian mean curvature flow* (HLMCF).

Like other success stories of coupled flows (cf. [Müller 2012; Smoczyk 2000]), the two geometric flows (1-1) can interact with each other to reveal better properties than either had by itself. For any hyper-Lagrangian submanifold $F : L \to M$, we introduce the *twistor energy* of L as the Dirichlet energy of the complex phase Ψ with respect to the induced metric $g := F^*\bar{g}$:

$$\mathcal{T}(L) := \int_L |\nabla\Psi|^2 \, d\mu,$$

where $d\mu$ denotes the Riemannian volume of g. Intuitively, the twister energy measures the deviation from L being complex Lagrangian. We can show that any hyper-Lagrangian submanifold which is "almost" complex Lagrangian can be deformed to a genuine one in the following sense:

Theorem 1.1 (convergence of the HLMCF). *Let $(M, \bar{g})$ be a hyper-Kähler $4n$-manifold with bounded geometry. Suppose L is a hyper-Lagrangian submanifold with the complex phase Ψ_0 which is smoothly immersed into M. Then for any V_0, Λ_0 and $\delta_0 > 0$, there exists $\varepsilon_0 = \varepsilon_0(n, V_0, \Lambda_0, \delta_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M)) > 0$ such that if L satisfies*

$$\mathrm{Vol}(L_0) \leqslant V_0, \quad |A|(0) \leqslant \Lambda_0, \quad \lambda_1(\Delta_L)(0) \geqslant \delta_0, \quad \mathcal{T}(L_0) \leqslant \varepsilon_0,$$

then the hyper-Lagrangian mean curvature flow (1-1) starting from L converges smoothly, exponentially fast to a complex Lagrangian submanifold in M for one of the hyper-Kähler complex structures on M.

In the above theorem, we need not assume that M has a complex Lagrangian submanifold, so it also gives an existence result for such a submanifold as well as the stability along the MCF. Although generic K3 surfaces do not have holomorphic curves at all, it is also interesting to understand this situation from a geometric analytic point of view. Applying our theorem, one can immediately see that the twistor energy causes some gap: for any V_0, Λ_0 and $\delta_0 > 0$ we define

$$\mathcal{L}(V_0, \Lambda_0, \delta_0) := \left\{ L \subset M \;\middle|\; \begin{array}{l} L \text{ is a hyper-Lagrangian submanifold,} \\ \mathrm{Vol}(L) \leqslant V_0,\ |A| \leqslant \Lambda_0,\ \lambda_1(\Delta_L) \geqslant \delta_0 \end{array} \right\}.$$

Then we have the following:

Corollary 1.2 (energy gap theorem). *Assume a $4n$-dimensional hyper-Kähler manifold M with bounded geometry has no complex Lagrangian submanifolds. Then for any V_0, Λ_0 and $\delta_0 > 0$, there exists a constant $c = c(n, V_0, \Lambda_0, \delta_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M)) > 0$ such that*

$$\inf_{L \in \mathcal{L}(V_0, \Lambda_0, \delta_0)} \mathcal{T}(L) \geqslant c.$$

The proof of Theorem 1.1 is based on [Li 2012] for the Lagrangian mean curvature flow (LMCF). In [Li 2012], the crucial step is to establish the exponential estimate for the L^2-norm of mean curvature vector H by using the fact that each submanifold L_t is Lagrangian, which is not valid for our case. Instead, we take an alternative approach from the view point of the theory of harmonic map flow. A key observation is that the L^2-norm of H is bounded by the twistor energy (see Proposition 2.4):

$$\int_{L_t} |H_t|^2 \, d\mu_t \leqslant 2\mathcal{T}(L_t).$$

So the problem comes down to establishing the exponential estimate for the twistor energy, which is indeed possible along the same line as the usual harmonic map flow (see Lemma 3.4). Note that for the harmonic map flow into positively curved targets, the flow possibly forms singularities in finite time even if it has small initial Dirichlet energy [Chen and Ding 1990]. We overcome this by showing Proposition 2.5.

1B. ***Examples and relation to other results.*** This paper is entirely written for hyper-Lagrangian submanifolds of arbitrary dimension. But after we posted the preprint, we noticed Qiu and Sun's result [2019] which states that every hyper-Lagrangian except surface must be a complex Lagrangian, so the concept of the hyper-Lagrangian is meaningful only when $n = 1$. However, we emphasize that our results are new even when $n = 1$. Contrary to the higher-dimensional case, the concept of hyper-Lagrangian surface is universal and enables us to make a systematic study of several conditions for submanifolds preserved under the MCF. We can see that every surface L in a hyper-Kähler 4-manifold M admits a canonical complex phase map $\Psi : L \to \mathbb{S}^2$ defined by

$$J_\Psi e_1 = e_2, \qquad J_\Psi e_3 = -e_4,$$

where $\{e_1, e_2, e_3, e_4\}$ is any oriented orthonormal frame on TM such that $\{e_1, e_2\}$ is an oriented frame on TL and $\{e_3, e_4\}$ is an orthonormal frame for the normal bundle. Indeed, the map Ψ is independent of the choice for such a frame. In the following, we will explain each class of submanifolds separately while considering what shape each complex phase is (see also [Leung and Wan 2007]).

1B.1. *Symplectic mean curvature flow.* First, we consider symplectic surfaces. Yau asked (for instance, see [Wang 2001]) "*how can a symplectic submanifold be deformed to a holomorphic one?*" Since a symplectic surface remains symplectic along the MCF in a Kähler–Einstein surface (see [Chen and Li 2001; Wang 2001]), one expects that the symplectic mean curvature flow (SMCF) is applicable to Yau's question. It seems that the convergence of the SMCF with small initial data has not been accomplished yet in the general case, whereas we know several partial results. For instance, our theorem generalizes Han and Sun's result [2012, Corollary 4.6]: we express Ψ as a map $\boldsymbol{a} : L \to \mathbb{R}^3$, i.e., $\boldsymbol{a}$ is a coefficient of Ψ with respect to $\{J_d\}$,

$$J_\Psi = \sum_d a_d J_d, \qquad \boldsymbol{a} := (a_1, a_2, a_3).$$

By using the quaternion relations, we see that

$$\cos\alpha := \bar{\omega}_{J_3}(e_1, e_2) = \bar{g}(J_3 e_1, e_2) = a_3. \tag{1-2}$$

Hence the condition that L is symplectic with respect to $\bar{\omega}_{J_3}$ is equivalent to saying that the image $\Psi(L)$ is contained in the hemisphere

$$\mathbb{S}^2_+ := \{(c_1, c_2, c_3) \in \mathbb{S}^2 \subset \mathbb{R}^3 | c_3 > 0\}.$$

Then the (local) angle α defined by (1-2) is called the *Kähler angle*. Applying the maximum principle to the evolution equation of $\boldsymbol{a}$, we find that the hemisphere condition is preserved under the HLMCF (see Corollary 3.2), which is essentially a restatement of the fact as explained above that *if the initial surface is symplectic, then the surface is still symplectic along the mean curvature flow*. In [Han and Sun 2012], they showed the convergence of the SMCF under the stronger assumption that the ambient Kähler surface M has zero sectional curvature and the initial L^2-norm of A is very small. Also there is a convergence result for the SMCF in Kähler–Einstein surfaces with positive Ricci curvature by Han and Li [2005], where the positivity of the extrinsic curvature was essentially used. Theorem 1.1 indicates that the MCF method is still valid for Yau's question, and makes the first step in this direction.

1B.2. *Lagrangian mean curvature flow.* Next, we explain the Lagrangian case. If L is Lagrangian with respect to $\bar{\omega}_{J_o}$ for a fixed $J_o \in \mathbb{S}^2$, then without loss of generality, we may assume $J_3 = J_o$. By the Lagrangian condition, we find that L has the J_3-orthogonal complex phase J_Ψ which can be expressed as

$$J_\Psi(x) = \cos\theta(x) J_1 + \sin\theta(x) J_2 \tag{1-3}$$

for some multivalued function $\theta : L \to \mathbb{R}$. Moreover, the functions θ and $\bar{\omega}_{J_o}$ are related by the formula

$$i_H \bar{\omega}_{J_o} = d\theta. \tag{1-4}$$

So θ is nothing but the *Lagrangian angle*. In particular, we often consider the following special cases:

(1) The form $i_H \bar{\omega}_{J_o}$ is exact, or equivalently, θ is a single-valued function.

(2) The submanifold L is *almost calibrated*, i.e., L satisfies (1) and $\cos\theta > 0$.

As it is Lagrangian, these two conditions are preserved under the MCF [Smoczyk 1999; Chen and Li 2001; Wang 2001]. The convergence result for the LMCF with small initial data was obtained by Li [2012, Theorem 1.2]. He showed this similar convergence result to Theorem 1.1 under the assumption (1) (but, we need not assume (2)) and that the initial L^2-norm of H is very small. So Theorem 1.1 is still meaningful even if L_0 is Lagrangian since we need not assume (1) in our theorem.

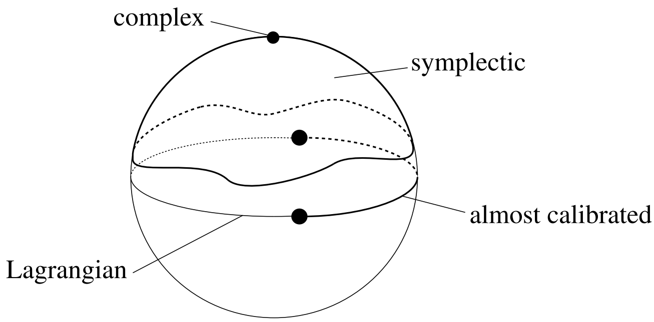

Figure 1. Image of the complex phase Ψ in $\mathbb{S}^2$.

Finally, we again emphasize the benefit of the hyper-Lagrangian submanifolds. In fact, the hyper-Lagrangian structure gives one a comprehensive view point to understand the concepts of symplectic surfaces or (almost calibrated) Lagrangian submanifolds in hyper-Kähler 4-manifolds. Figure 1 shows the correspondence between each of these concepts and the image of the complex phase map $\Psi : L \to \mathbb{S}^2$.

1B.3. *Holomorphic curves in K3.* On any polarized K3 surface (M, H) (with $H \not\simeq \mathcal{O}_M$), it is known that there exists at least one holomorphic curve which belongs to the linear system $|mH|$ for all $m \geqslant 1$ (Bogomolov, Mumford, Mori and Mukai [Mori and Mukai 1983]). Due to the Lefschetz theorem, the existence of such an H is equivalent to saying that the *Néron–Severi lattice*

$$\mathrm{NS}(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z})$$

is nonempty. Moreover, Chen [1999] proved the existence of infinitely many holomorphic curves on general K3 surfaces. Then we can take any small perturbation of the holomorphic curves as an initial data in Theorem 1.1.

1C. ***Organization of the paper.*** Our article will be organized as follows. We will first recall some results discovered by Leung and Wan [2007] and prove formulas relating the mean curvature vector (or second fundamental form) with the complex phase which are needed in the rest of the article. In Section 3, we study the behavior of the twistor energy and first eigenvalue along the HLMCF, and then establish some parabolic estimates. Finally, we give the proof of Theorem 1.1 in the last part of Section 3.

2. Hyper-Lagrangian submanifolds

In this section, we recall some results about hyper-Lagrangian submanifolds studied in [Leung and Wan 2007]. Let M be a hyper-Kähler $4n$-manifold and $L \subset M$ a real submanifold of dimension $2n$. In this section, we show the indices $(i, j, \alpha, \beta$, etc.)

run in the following manner:

$$i, j = 1, \dots, 2n, \quad \alpha, \beta = 2n+1, \dots, 4n, \quad A, B = 1, \dots, 4n,$$
$$\nu, \lambda = 1, \dots n, \quad \mu, \rho = n+1, \dots, 2n.$$

Definition 2.1. A submanifold L is called hyper-Lagrangian if $\Omega_{\Psi(x)}|_L = 0$ at every point $x \in L$ for some $\Psi : L \to \mathbb{S}^2$. Then Ψ is called the complex phase. In particular, a hyper-Lagrangian submanifold is called complex Lagrangian if we can take Ψ as a constant map.

Let $\Phi : L \to \mathbb{S}^2$ be a smooth map such that $\Phi(x)$ is orthogonal to $\Psi(x)$ for each $x \in L$. We can take a special orthonormal frame $\{e_i\}$ for TL satisfying

$$J_\Psi e_{2\nu-1} = e_{2\nu}.$$

Then $\{e_{i+2n} := J_\Phi e_i\}$ is an orthonormal frame for the normal bundle satisfying

$$J_\Psi e_{2\mu-1} = -e_{2\mu}.$$

Then $\{e_A\}$ defines a frame of TM. For a hyper-Lagrangian submanifold L with the complex phase Ψ, we denote the associated almost-complex structure by J_Ψ. Then the complex phase J_Ψ acts on TL, and determines an almost-complex structure on L. However, hyper-Lagrangian is a strong condition which imposes a lot of restrictions on the structural equations. For instance, let $\{\varphi_{AB}\}$ be the connection forms with respect to $\{e_A\}$, i.e., $\overline{\nabla} e_A = \varphi_{AB} e_B$. Then the structure theorem of hyper-Lagrangian submanifolds (see [Leung and Wan 2007, Theorem 4.1]) implies

$$\begin{aligned} \varphi_{2\nu-1,2\lambda-1} &= \varphi_{2\nu,2\lambda}, & \varphi_{2\nu,2\lambda-1} &= -\varphi_{2\nu-1,2\lambda}, \\ \varphi_{2\mu-1,2\rho-1} &= -\varphi_{2\mu,2\rho}, & \varphi_{2\mu,2\rho-1} &= \varphi_{2\mu-1,2\rho}. \end{aligned} \tag{2-1}$$

As a consequence, we obtain the following:

Theorem 2.2 [Leung and Wan 2007, Corollary 4.2]. *The complex phase Ψ induces an integrable Kähler structure $(J_\Psi, \bar{g}|_L)$ on L with holomorphic normal bundle.*

We set

$$e'_\nu = \tfrac{1}{2}(e_{2\nu-1} - \sqrt{-1}e_{2\nu}), \quad e''_\nu = \tfrac{1}{2}(e_{2\nu-1} + \sqrt{-1}e_{2\nu}),$$

$$e'_\mu = \tfrac{1}{2}(e_{2\mu-1} + \sqrt{-1}e_{2\mu}), \quad e''_\mu = \tfrac{1}{2}(e_{2\mu-1} - \sqrt{-1}e_{2\mu}).$$

Then $\{e'_\nu, e'_\mu\}$ defines a complex basis referred to as the *canonical frame adapted to* (Ψ, Φ). Correspondingly, we take the basis $\{\zeta_A\}$ dual to $\{e_A\}$ and set

$$\zeta'_\nu = \zeta_{2\nu-1} + \sqrt{-1}\zeta_{2\nu}, \quad \zeta''_\nu = \zeta_{2\nu-1} - \sqrt{-1}\zeta_{2\nu},$$

$$\zeta'_\mu = \zeta_{2\mu-1} - \sqrt{-1}\zeta_{2\mu}, \quad \zeta''_\mu = \zeta_{2\mu-1} + \sqrt{-1}\zeta_{2\mu}.$$

With this basis, Ω_Ψ can be written as

$$\Omega_\Psi = -\sqrt{-1}\sum_{\nu,\mu} \zeta'_\nu \wedge \zeta'_\mu.$$

Leung and Wan [2007, Theorem 4.5] found the formula relating the mean curvature vector H and the complex phase Ψ as follows:

Proposition 2.3. *We have*

$$i_H \Omega_\Psi + 2\sqrt{-1}\partial\Psi = 0. \tag{2-2}$$

In particular, the above proposition shows that a hyper-Lagrangian submanifold L is minimal if and only if the complex phase Ψ is antiholomorphic. Meanwhile, by using the formula (2-2), one can obtain a bound for $|H|$ by means of the energy density of the complex phase Ψ:

Proposition 2.4. *We have*

$$|H|^2 \leqslant 2|\nabla\Psi|^2.$$

Proof. For a fixed $x \in L$, we set

$$J'_1 = J_\Psi(x), \quad J'_2 = J_\Phi(x), \quad J'_3 = J'_1 J'_2.$$

We would like to call it the *canonical basis adapted to* (Ψ, Φ) *at* x. Then we set the coefficient $\boldsymbol{a}' = (a'_1, a'_2, a'_3)$ as $J_\Psi = \sum_d a'_d J'_d$. We take a local representation of Ψ:

$$\Theta(p) = \frac{a'_1(p) + \sqrt{-1}a'_2(p)}{1 - a'_3(p)}$$

via stereographic projection. Then the formula (2-2) yields that

$$i_H \Omega_\Psi + 2\sqrt{-1}\partial\Theta = 0 \quad \text{at } x.$$

From the construction, we know that

$$a'_1(x) = 1, \quad a'_2(x) = a'_3(x) = 0, \quad \Theta(x) = 1.$$

Also since L is hyper-Lagrangian with the complex phase Ψ, the derivative $\overline{\nabla} J_\Psi$ is spanned by J'_2 and J'_3 at x, so

$$da'_1|_x = 0.$$

Thus we have

$$\begin{aligned}
&\partial\Theta|_x = \sqrt{-1}\partial a'_2|_x + \partial a'_3|_x,\\
&|\partial\Theta|^2 \leqslant 2(|\partial a'_2|^2 + |\partial a'_3|^2) = |da'_2|^2 + |da'_3|^2 = |\nabla \boldsymbol{a}'|^2
\end{aligned}$$

at x. On the other hand, if we set $H = -\sum_\alpha H^\alpha e_\alpha$, one can easily observe that

$$i_H\Omega_\Psi = -\sqrt{-1}\sum_\mu (H^{2\mu-1} - \sqrt{-1}H^{2\mu})\zeta'_\mu,$$
$$|i_H\Omega_\Psi|^2 = 2|H|^2.$$

So we have

$$|H|^2 = 2|\partial\Theta|^2 \leqslant 2|\nabla \boldsymbol{a}'|^2.$$

We note that $\boldsymbol{a}'$ and Θ heavily depend on the choice of the basis (J'_1, J'_2, J'_3) whereas $\boldsymbol{a}$ only depends on the background basis (J_1, J_2, J_3). However, the point is that the norm $|\nabla \boldsymbol{a}'|^2$ is independent of the choice of an orthogonal basis (J'_1, J'_2, J'_3) since the Euclidean metric on $\mathbb{R}^3$ is invariant under the standard $O(3)$-action. So we have $|\nabla \boldsymbol{a}'| = |\nabla \boldsymbol{a}| = |\nabla\Psi|$ and $|H|^2 \leqslant 2|\nabla\Psi|^2$. □

We also note that the quantity $|\nabla\Psi|$ has the following three equivalent definitions:

- We regard the complex phase Ψ as a map $\boldsymbol{a} : L \to \mathbb{S}^2 \subset \mathbb{R}^3$, and define $|\nabla\Psi|$ as the energy density of $\boldsymbol{a}$:
$$|\nabla \boldsymbol{a}|^2 = \sum_d |\nabla a_d|_g^2.$$
- We define $|\nabla\Psi|$ as the energy density of $\Psi : L \to \mathbb{S}^2$, i.e., a map into $\mathbb{S}^2$ (also see (3-2)).
- We define $|\nabla\Psi|$ as the norm of the covariant derivative of J_Ψ along L:
$$|\overline{\nabla} J_\Psi|^2 = \sum_{i,A,B} \bar{g}((\overline{\nabla}_i J)(e_A), e_B)^2,$$
where $\overline{\nabla}$ denotes the Levi-Civita connection on the ambient space $(M, \bar{g})$. Then, taking into account the fact that $\{J_d\}$ is parallel and $\langle J_d, J_e\rangle_{\bar{g}} = 4n\delta_{de}$, we have $\overline{\nabla} J_\Psi = \sum_d da_d \otimes J_d$ and $|\overline{\nabla} J_\Psi| = 2\sqrt{n}|\nabla \boldsymbol{a}|$.

As for the relation to the second fundamental form A, we have the following:

Proposition 2.5. *In the canonical frame adapted to* (Ψ, Φ), *the quantity* $|\overline{\nabla} J_\Psi|^2$ *is expressed as*

$$|\overline{\nabla} J_\Psi|^2 = 4\sum_{i,\nu,\mu}[(h^{2\mu-1}_{2\nu,i} - h^{2\mu}_{2\nu-1,i})^2 + (h^{2\mu-1}_{2\nu-1,i} + h^{2\mu}_{2\nu,i})^2],$$

where $h^\alpha_{ij} := \bar{g}(e_i, \overline{\nabla}_j e_\alpha)$. *In particular, we have*

$$|\nabla\Psi| \leqslant c(n)|A|.$$

Proof. Set $J_{i,A,B} := \bar{g}(\overline{\nabla}_i J_\Psi(e_A), e_B)$ for simplicity. We compute

$$\begin{aligned}(\overline{\nabla} J_\Psi)(e_{2\nu-1}) &= \overline{\nabla}(e_{2\nu}) - J_\Psi(\overline{\nabla} e_{2\nu-1}) \\ &= \sum_j \varphi_{2\nu,j} e_j + \sum_\alpha \varphi_{2\nu,\alpha} e_\alpha - J_\Psi\Bigl(\sum_j \varphi_{2\nu-1,j} e_j + \sum_\alpha \varphi_{2\nu-1,\alpha} e_\alpha\Bigr).\end{aligned}$$

By using (2-1), we know that the first and third terms cancel each other out. So

$$(\overline{\nabla} J_\Psi)(e_{2\nu-1}) = \sum_\mu [(\varphi_{2\nu,2\mu-1} - \varphi_{2\nu-1,2\mu}) e_{2\mu-1} + (\varphi_{2\nu,2\mu} + \varphi_{2\nu-1,2\mu-1}) e_{2\mu}],$$

and hence

$$J_{i,2\nu-1,j} = 0, \quad J_{i,2\nu-1,2\mu-1} = -h^{2\mu-1}_{2\nu,i} + h^{2\mu}_{2\nu-1,i}, \quad J_{i,2\nu-1,2\mu} = -h^{2\mu}_{2\nu,i} - h^{2\mu-1}_{2\nu-1,i}.$$

In the same way, we can compute other terms by using (2-1) as follows:

$$\begin{aligned} J_{i,2\nu,j} &= 0, & J_{i,2\nu,2\mu-1} &= h^{2\mu}_{2\nu,i} + h^{2\mu-1}_{2\nu-1,i}, & J_{i,2\nu,2\mu} &= -h^{2\mu-1}_{2\nu,i} + h^{2\mu}_{2\nu-1,i}, \\ J_{i,2\mu-1,\alpha} &= 0, & J_{i,2\mu-1,2\nu-1} &= h^{2\mu-1}_{2\nu,i} - h^{2\mu}_{2\nu-1,i}, & J_{i,2\mu-1,2\nu} &= -h^{2\mu}_{2\nu,i} - h^{2\mu-1}_{2\nu-1,i}, \\ J_{i,2\mu,\alpha} &= 0, & J_{i,2\mu,2\nu-1} &= h^{2\mu-1}_{2\nu-1,i} + h^{2\mu}_{2\nu,i}, & J_{i,2\mu,2\nu} &= h^{2\mu-1}_{2\nu,i} - h^{2\mu}_{2\nu-1,i}. \end{aligned}$$

So we obtain the desired formula. □

3. Hyper-Lagrangian mean curvature flow

3A. ***Evolution of the coefficient vector.*** We regard the complex phase Ψ as a map into $\mathbb{S}^2 \subset \mathbb{R}^3$ and write $\boldsymbol{a} = (a_1, a_2, a_3)$. We compute the evolution equation of $\boldsymbol{a}$ when Ψ evolves along the generalized harmonic map flow $\frac{d}{dt}\Psi = \Delta_t \Psi$.

Lemma 3.1. *Along the HLMCF,* $\boldsymbol{a}$ *satisfies*

$$\left(\frac{d}{dt} - \Delta_t\right)\boldsymbol{a} = |\nabla \boldsymbol{a}|^2 \boldsymbol{a}. \tag{3-1}$$

Proof. We take a polar coordinate (θ, φ) of $\mathbb{S}^2$ and express $\boldsymbol{a}$ as

$$\boldsymbol{a} = \begin{pmatrix} \cos\Psi^\theta \sin\Psi^\varphi \\ \sin\Psi^\theta \sin\Psi^\varphi \\ \cos\Psi^\varphi \end{pmatrix},$$

where we write $\Psi^\theta = \theta \circ \Psi$, $\Psi^\varphi = \varphi \circ \Psi$ for simplicity. Then

$$\frac{d}{dt}\Psi = \frac{d}{dt}\Psi^\theta \cdot \frac{\partial}{\partial\theta} \circ \Psi + \frac{d}{dt}\Psi^\varphi \cdot \frac{\partial}{\partial\varphi} \circ \Psi.$$

Let $(x^1, \dots, x^{2n})$ be a local coordinate in L. Recall the definition of the tension field of Ψ:

$$\Delta\Psi = \sum_{i,j=1}^{n} g^{ij}\hat{\nabla}_i\hat{\nabla}_j\Psi^\theta \cdot \frac{\partial}{\partial\theta} \circ \Psi + \sum_{i,j=1}^{n} g^{ij}\hat{\nabla}_i\hat{\nabla}_j\Psi^\varphi \cdot \frac{\partial}{\partial\varphi} \circ \Psi \in C^\infty(\Psi^{-1}T\mathbb{S}^2),$$

where $\hat{\nabla}$ denotes the canonical connection on $\Psi^{-1}T\mathbb{S}^2$ associated to g and the standard metric $\tilde{g}$ on $\mathbb{S}^2$. Then

$$\hat{\nabla}_i\hat{\nabla}_j\Psi^\alpha = \nabla_i\nabla_j\Psi^\alpha + \sum_{\beta,\gamma=\theta,\varphi} \tilde{\Gamma}^\alpha_{\beta\gamma}(\Psi)\frac{\partial\Psi^\beta}{\partial x^i} \cdot \frac{\partial\Psi^\gamma}{\partial x^j}, \qquad \alpha = \theta, \varphi,$$

where $\tilde{\Gamma}^\alpha_{\beta\gamma}$ denotes the Christoffel symbol with respect to $\tilde{g}$. We can easily compute

$$\begin{aligned} &\tilde{g}_{\theta\theta} = \sin^2\varphi, && \tilde{g}_{\theta\varphi} = 0, && \tilde{g}_{\varphi\varphi} = 1, \\ &\tilde{\Gamma}^\theta_{\theta\theta} = \tilde{\Gamma}^\varphi_{\varphi\varphi} = 0, && \tilde{\Gamma}^\theta_{\theta\varphi} = \frac{\cos\varphi}{\sin\varphi}, && \tilde{\Gamma}^\varphi_{\theta\theta} = -\sin\varphi\cos\varphi. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{d}{dt}\Psi^\theta &= \sum_{i,j=1}^{n} g^{ij}\hat{\nabla}_i\hat{\nabla}_j\Psi^\theta = \Delta\Psi^\theta + \frac{\cos\Psi^\varphi}{\sin\Psi^\varphi} \cdot \langle\nabla\Psi^\theta, \nabla\Psi^\varphi\rangle_g, \\ \frac{d}{dt}\Psi^\varphi &= \sum_{i,j=1}^{n} g^{ij}\hat{\nabla}_i\hat{\nabla}_j\Psi^\varphi = \Delta\Psi^\varphi - \sin\Psi^\varphi\cos\Psi^\varphi \cdot |\nabla\Psi^\theta|^2_g. \end{aligned}$$

Since

$$\begin{aligned} \Delta a_3 &= -\sin\Psi^\varphi \cdot \Delta\Psi^\varphi - \cos\Psi^\varphi \cdot |\nabla\Psi^\varphi|^2_g, \\ |\nabla \boldsymbol{a}|^2 &= \sin^2\Psi^\varphi \cdot |\nabla\Psi^\theta|^2_g + |\nabla\Psi^\varphi|^2_g, \end{aligned} \tag{3-2}$$

we have

$$\frac{d}{dt}a_3 = -\sin\Psi^\varphi \cdot \frac{d}{dt}\Psi^\varphi = \Delta a_3 + |\nabla\boldsymbol{a}|^2 a_3.$$

We can compute the evolution equation of a_1 and a_2 in the similar way. □

Applying the maximum principle to (3-1), we obtain

Corollary 3.2 (also see [Leung and Wan 2007, Theorem 5.1]). *If L_0 satisfies $a_3 > c$ for some constant $c \in (0, 1)$ then $a_3 > c$ holds along the HLMCF L_t for all $t \in [0, T]$. In particular, the hemisphere condition $\Psi(L) \subset \mathbb{S}^2_+$ is preserved under the HLMCF.*

3B. ***L^2-estimates.*** Let $L \subset M$ be a hyper-Lagrangian submanifold with the complex phase Ψ.

Definition 3.3. We define the twistor energy of L as the Dirichlet energy of the complex phase:

$$\mathcal{T}(L) := \int_L |\nabla\Psi|^2 \, d\mu.$$

By using (3-1), we can obtain the exponential estimate for the twistor energy:

Lemma 3.4 (exponential estimate for the twistor energy). *For the HLMCF L_t,*

$$\frac{d}{dt}\mathcal{T}(L_t) \leqslant (-2\lambda_1(t) + C(n)\max_{L_t}|H||A| + 2\max_{L_t}|\nabla\Psi|^2)\cdot\mathcal{T}(L_t),$$

where $\lambda_1(t) > 0$ denotes the first eigenvalue of the Laplacian Δ_t.

Proof. First, we recall the evolution of the Riemannian metric on L (for instance, see [Chen and Li 2001]):

$$\frac{d}{dt}g_{ij} = -2H^\alpha h^\alpha_{ij}.$$

By using this and the expression of the energy density as the norm of the coefficient vector $|\nabla\Psi|^2 = |\nabla\boldsymbol{a}|^2$, we compute

$$\begin{aligned}\frac{d}{dt}\int_L |\nabla\boldsymbol{a}|^2 \, d\mu_t \\ = 2\int_L \langle\nabla\frac{d}{dt}\boldsymbol{a}, \nabla\boldsymbol{a}\rangle \, d\mu_t + \int_L \sum_d \frac{d}{dt}g^{ij}\nabla_i a_d \nabla_j a_d \, d\mu_t - \int_L |\nabla\boldsymbol{a}|^2|H|^2 \, d\mu_t.\end{aligned}$$

We estimate each term separately. The first term is

$$\begin{aligned}2\int_L \langle\nabla\frac{d}{dt}\boldsymbol{a}, \nabla\boldsymbol{a}\rangle \, d\mu_t &= 2\int_L \langle\nabla((\Delta + |\nabla\boldsymbol{a}|^2)\boldsymbol{a}), \nabla\boldsymbol{a}\rangle \, d\mu_t \\ &= -2\int_L |\Delta\boldsymbol{a}|^2 \, d\mu_t - 2\int_L |\nabla\boldsymbol{a}|^2\langle\boldsymbol{a}, \Delta\boldsymbol{a}\rangle \, d\mu_t \\ &\leqslant -2\lambda_1\int_L |\nabla\boldsymbol{a}|^2 \, d\mu_t + 2\int_L |\nabla\boldsymbol{a}|^4 \, d\mu_t \\ &\leqslant -2\lambda_1\int_L |\nabla\boldsymbol{a}|^2 \, d\mu_t + 2\max_{L_t}|\nabla\boldsymbol{a}|^2\int_L |\nabla\boldsymbol{a}|^2 \, d\mu_t,\end{aligned}$$

where we used the formula

$$0 = \left\langle\frac{d}{dt}\boldsymbol{a}, \boldsymbol{a}\right\rangle = \langle(\Delta + |\nabla\boldsymbol{a}|^2)\boldsymbol{a}, \boldsymbol{a}\rangle = \langle\Delta\boldsymbol{a}, \boldsymbol{a}\rangle + |\nabla\boldsymbol{a}|^2,$$

which can be proved easily by differentiating $|\boldsymbol{a}|^2 = 1$ in t. For the second term, we have

$$\left|\int_L \sum_d \frac{d}{dt} g^{ij} \nabla_i a_d \nabla_j a_d \, d\mu_t\right| = \left|2\int_L \sum_d H^\alpha h^\alpha_{ij} \nabla_i a_d \nabla_j a_d \, d\mu_t\right|$$
$$\leqslant C(n) \max_{L_t} |H||A| \cdot \int_L |\nabla \boldsymbol{a}|^2 \, d\mu_t. \qquad \square$$

The above lemma says that we need to control λ_1 in order to obtain a bound for the twistor energy. So we establish the exponential estimate for λ_1 as follows:

Lemma 3.5 (exponential estimate for the first eigenvalue). *Along the HLMCF*, *the first eigenvalue $\lambda_1(t)$ satisfies*

$$\frac{d}{dt}\lambda_1 \geqslant -\big(\max_{L_t} |H|^2 + C(n) \max_{L_t} |H||A|\big) \cdot \lambda_1.$$

Proof. Let f be an eigenfunction with respect to λ_1, i.e., f satisfies

$$-\Delta_t f = \lambda_1 f, \qquad \int_L f^2 \, d\mu_t = 1.$$

Then the first eigenvalue λ_1 is

$$\lambda_1 = \int_L |\nabla f|^2 \, d\mu_t.$$

Differentiating $\int_L f^2 \, d\mu_t = 1$ in t, we have

$$\int_L \left(2\frac{d}{dt} f \cdot f - f^2 |H|^2\right) d\mu_t = 0.$$

Thus we can compute

$$\begin{aligned}\frac{d}{dt}\lambda_1 &= 2\int_L \left\langle \nabla \frac{d}{dt} f, \nabla f\right\rangle d\mu_t + \int_L \frac{d}{dt} g^{ij} \nabla_i f \nabla_j f \, d\mu_t - \int_L |\nabla f|^2 |H|^2 \, d\mu_t \\ &= -2\int_L \frac{d}{dt} f \cdot \Delta f \, d\mu_t + 2\int_L H^\alpha h^\alpha_{ij} \nabla_i f \nabla_j f \, d\mu_t \\ &\qquad + \int_L f \Delta f \cdot |H|^2 \, d\mu_t + \int_L f \langle \nabla f, \nabla |H|^2 \rangle \, d\mu_t.\end{aligned}$$

Using the relation $-\Delta f = \lambda_1 f$, we find that the first term and the third term cancel each other out. The second term can be estimated as

$$\left|2\int_L H^\alpha h^\alpha_{ij} \nabla_i f \nabla_j f \, d\mu_t\right| \leqslant C(n) \max_{L_t} |H||A| \cdot \lambda_1.$$

The fourth term is

$$\begin{aligned}\int_L f\langle\nabla f, \nabla|H|^2\rangle\,d\mu_t &= -\int_L (f\Delta f + |\nabla f|^2)|H|^2\,d\mu_t\\ &= \lambda_1\int_L f^2|H|^2\,d\mu_t - \int_L |\nabla f|^2|H|^2\,d\mu_t\\ &\geqslant -\max_{L_t}|H|^2\cdot\lambda_1.\end{aligned}$$

Thus we obtain the desired result. □

3C. *C^0-estimates.* In order to get the C^0-estimates from L^2, the notion of a noncollapsing geodesic ball is convenient. Roughly speaking, the volume of each geodesic ball in L is bounded from below by that of the Euclidean geodesic ball of the same radius. Let N be a compact Riemannian m-manifold.

Definition 3.6. We say that

(1) A geodesic ball $B(x, \rho)$ in N is called κ-noncollapsed if

$$\frac{\mathrm{Vol}(B(y,s))}{s^m} \geqslant \kappa$$

whenever $B(y, s) \subset B(x, \rho)$.

(2) A compact Riemannian manifold N is called κ-noncollapsed on the scale r if every geodesic ball $B(x, s)$ is κ-noncollapsed for $s \leqslant r$.

Lemma 3.7. *Let (E, h, D) be a vector bundle with a fiber metric h and a compatible connection D over a compact Riemannian manifold N. Assume that N is κ-noncollapsed on the scale r. For any smooth section $\sigma \in C^\infty(E)$, if*

$$|D\sigma| \leqslant \Lambda, \qquad \int_N |\sigma|^2\,d\mu \leqslant \varepsilon \leqslant r^{m+2},$$

then

$$\max_N|\sigma| \leqslant (\Lambda + \kappa^{-1/2})\varepsilon^{1/(m+2)}.$$

Proof. Assume that $|\sigma|$ attains its maximum at a point $x_0 \in N$ and the statement does not hold, i.e.,

$$|\sigma(x_0)| > (\Lambda + \kappa^{-1/2})\varepsilon^{1/(m+2)}.$$

Then by setting $\delta := \varepsilon^{1/(m+2)}$, we get

$$\Lambda\delta = \Lambda\varepsilon^{1/(m+2)} < |\sigma(x_0)|.$$

Thus for any $x \in B(x_0, \delta)$, we have

$$|\sigma(x)| \geqslant |\sigma(x_0)| - \Lambda\delta > 0.$$

Integrating on $B(x_0, \delta)$ yields that

$$\varepsilon \geqslant \int_{B(x_0,\delta)} |\sigma|^2\, d\mu \geqslant (|\sigma(x_0)| - \Lambda\delta)^2 \operatorname{Vol}(B(x_0, \delta)) \geqslant (|\sigma(x_0)| - \Lambda\delta)^2 \kappa\delta^m,$$

where we used $\delta = \varepsilon^{1/(m+2)} \leqslant r$ and the assumption that N is κ-noncollapsed on the scale r in the last inequality. So putting $\delta = \varepsilon^{1/(m+2)}$ into the above yields that $|\sigma(x_0)| \leqslant (\Lambda + \kappa^{-1/2})\varepsilon^{1/(m+2)}$, contradicting the assumption. □

Now we go back to our situation, so let L_t be the HLMCF in a hyper-Kähler $4n$-manifold M. Lemma 3.7 indicates that it is important to study the evolution of the volume ratio along the flow.

Lemma 3.8 (volume ratio estimate). *If L_0 is κ_0-noncollapsed on the scale r_0, then for any small geodesic ball $B_t(x, \rho)$ in L_t with radius $\rho \in (0, r_0)$, we have*

$$\operatorname{Vol}(B_t(x, \rho)) \geqslant \kappa_0 e^{-(2n+1)E(t)} \rho^{2n},$$

where $E(t)$ is given by

$$E(t) := \int_0^t (\max_{L_s} |H|^2 + \max_{L_s} |A|\,|H|)\, ds.$$

Proof. Let γ_t be a length-minimizing unit-speed geodesic with respect to $g(t)$ joining p to $q \in B_t(p, \rho)$. Then for every t_0 we have

$$d_t(p, q) = \operatorname{Length}_{g(t)}(\gamma_t) \leqslant \operatorname{Length}_{g(t)}(\gamma_{t_0}),$$

and equality holds when $t = t_0$, which implies that

$$\frac{d}{dt} d_t(p, q)|_{t=t_0} = \frac{d}{dt}\operatorname{Length}_{g(t)}(\gamma_t)|_{t=t_0} = \frac{d}{dt}\operatorname{Length}_{g(t)}(\gamma_{t_0})|_{t=t_0}.$$

Thus we can compute

$$\left|\frac{d}{dt} d_t(p, q)\right| = \left|\frac{1}{2}\int_0^{d_t(p,q)} \frac{dg_t}{dt}\left(\frac{d}{ds}\gamma_t, \frac{d}{ds}\gamma_t\right) ds\right| \leqslant \max_{L_t} |A|\,|H| \cdot d_t(p, q).$$

This implies that

$$e^{-E(t)} d_0(p, q) \leqslant d_t(p, q) \leqslant d_0(p, q) e^{E(t)}, \qquad d\mu_t \geqslant e^{-E(t)} d\mu_0.$$

Since L_0 is κ_0-noncollapsed on the scale r_0, for $\rho \leqslant r_0$, we have

$$\operatorname{Vol}(B_t(p, \rho)) = \int_{B_t(p,\rho)} d\mu_t \geqslant \int_{B_0(p, e^{-E(t)}\rho)} e^{-E(t)}\, d\mu_0 \geqslant \kappa_0 e^{-(2n+1)E(t)} \rho^{2n}. \quad \square$$

3D. ***Some parabolic estimates for the HLMCF.*** In this subsection, we prove some parabolic estimates for the HLMCF. The first lemma says that the HLMCF does not change a lot in short time intervals.

Lemma 3.9. *If L_0 satisfies*

$$|A|(0) \leqslant \Lambda, \quad |\nabla\Psi|(0) \leqslant P, \quad \lambda_1(0) \geqslant \delta,$$

then there exists $T = T(n, \Lambda, \overline{\mathrm{Rm}})$ such that the HLMCF L_t satisfies

$$|A|(0) \leqslant 2\Lambda, \quad |\nabla\Psi|(t) \leqslant 2P, \quad \lambda_1(t) \geqslant \tfrac{2}{3}\delta, \quad t \in [0, T].$$

Proof. The estimate of $|A|$ follows from [Han and Sun 2012, Lemma 2.2]. Then the estimate of λ_1 follows from the exponential estimate for λ_1. Finally, we establish the estimate for $|\nabla\Psi|$. By the Bochner identity, the Gauss equation and Proposition 2.5, we can compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta_t\right)|\nabla\Psi|^2 &= -2|\nabla^2\Psi|^2 + \mathrm{Rm}^{\mathbb{S}^2} *(\nabla\Psi)^4 + \overline{\mathrm{Rm}} *(\nabla\Psi)^2 + A^2 * (\nabla\Psi)^2 \\ &\leqslant C(n, \Lambda, \overline{\mathrm{Rm}})|\nabla\Psi|^2.\end{aligned}$$

Applying the maximum principle, we obtain

$$|\nabla\Psi|(t) \leqslant e^{\frac{1}{2}C(n,\Lambda,\overline{\mathrm{Rm}})t}|\nabla\Psi|(0) \leqslant e^{\frac{1}{2}C(n,\Lambda,\overline{\mathrm{Rm}})t} P,$$

so we may take $T \leqslant 2\log 2/(C(n, \Lambda, \overline{\mathrm{Rm}}))$. □

We can obtain not only the usual smoothing estimates for A, but also for Ψ with the help of Proposition 2.5.

Lemma 3.10 (smoothing estimates). *Suppose along the HLMCF, we have*

$$\sup_{L_t}|A| \leqslant \Lambda, \quad t \in [0, T],$$

for some $T > 0$. Then for each $l \geqslant 1$, there exist constants $\Lambda_l = \Lambda_l(n, \Lambda, \overline{\mathrm{Rm}}, T)$ such that

$$\sup_{L_t}|\nabla^l A| \leqslant \frac{\Lambda_l}{t^{l/2}}, \quad t \in (0, T].$$

Further, for any $t_0 \in (0, T]$, there exist constants $P_l = P_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)$ such that

$$\sup_{L_t}|\nabla^l \Psi_*| \leqslant P_l, \quad t \in [t_0, T],$$

where $\Psi_* = \nabla\Psi$ is the differential map of the complex phase $\Psi : L \to \mathbb{S}^2$.

Proof. The estimate of A follows from [Han and Sun 2012, Theorem 3.1]. Then for any $t_0 \in (0, T]$ we have

$$\sup_{L_t} |\nabla^l A| \leqslant \frac{\Lambda_l}{(t_0/2)^{l/2}}, \quad t \in [t_0/2, T].$$

We use this estimate to show the estimate of Ψ_*. Note also that $|\Psi_*|$ has a uniform bound $|\Psi_*| \leqslant c(n)|A| \leqslant c(n)\Lambda$ by Proposition 2.5.

In order to derive the estimate of Ψ_*, we first compute the time derivative of $|\nabla^l \Psi_*|^2$ along the generalized harmonic map flow. A straight calculation shows that for each $l \geqslant 0$ we get the formula

$$\begin{aligned}\frac{d}{dt}\nabla^l \Psi_* = \Delta(\nabla^l \Psi_*) &+ \sum_{r+i+j+k=l} \tilde{\nabla}^r \operatorname{Rm}^{\mathbb{S}^2} *(\Psi_*)^r * \nabla^i \Psi_* * \nabla^j \Psi_* * \nabla^k \Psi_* \\ &+ \sum_{r+i_i+\cdots+i_l+j=l} \overline{\nabla}^r\, \overline{\operatorname{Rm}} * \nabla^{i_1-1} A * \cdots * \nabla^{i_l-1} A * \nabla^j \Psi_* \\ &+ \sum_{i+j+k=l} \nabla^i A * \nabla^j A * \nabla^k \Psi_*,\end{aligned}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection on $T\mathbb{S}^2$. It follows that for $t \in [t_0/2, T]$ we have

$$\begin{aligned}(3\text{-}3)\quad &\frac{d}{dt}|\nabla^l \Psi_*|^2 \\ &= A^2 *(\nabla^l \Psi_*)^2 + 2\left\langle \frac{d}{dt}\nabla^l \Psi_*, \nabla^l \Psi_* \right\rangle \\ &\leqslant \Delta|\nabla^l \Psi_*|^2 - 2|\nabla^{l+1}\Psi_*|^2 + C \sum_{0 \leqslant i+j+k \leqslant l} |\nabla^i \Psi_*||\nabla^j \Psi_*||\nabla^k \Psi_*||\nabla^l \Psi_*|,\end{aligned}$$

where $C = C(n, \Lambda, \overline{\operatorname{Rm}}, t_0, T)$ is a constant. From (3-3) we have

$$\frac{d}{dt}|\Psi_*|^2 \leqslant \Delta|\Psi_*|^2 - 2|\nabla\Psi_*|^2 + c_1$$

and

$$\frac{d}{dt}|\nabla\Psi_*|^2 \leqslant \Delta|\nabla\Psi_*|^2 - 2|\nabla^2\Psi_*|^2 + c_2|\nabla\Psi_*|^2 + c_3,$$

where $c_k = c_k(n, \Lambda, \overline{\operatorname{Rm}}, t_0, T)$, $k = 1, 2, 3$, are constants. Set

$$F := (t - t_0/2)|\nabla\Psi_*|^2 + \alpha|\Psi_*|^2,$$

where α is a constant which will be determined later. It is not difficult to see

$$\left(\frac{d}{dt} - \Delta\right) F \leqslant (-2\alpha + 1 + Tc_2)|\nabla\Psi_*|^2 + \alpha c_1 + Tc_3.$$

Then we choose $\alpha = (1 + TC_2)/2$ to get

$$\left(\frac{d}{dt} - \Delta\right)F \leqslant \left(\frac{1 + Tc_2}{2}\right)c_1 + Tc_3.$$

Applying the maximum principle, we have

$$F(t) \leqslant F(0) \leqslant \left(\frac{1 + Tc_2}{2}\right)\Lambda^2 = C_1(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T), \quad t \in [t_0/2, T].$$

Hence we get

$$|\nabla \Psi_*|^2 \leqslant \frac{C_1}{t - t_0/2}, \quad t \in (t_0/2, T].$$

It follows that

$$\sup_{L_t} |\nabla \Psi_*| \leqslant \frac{6C_1}{t_0} = P_1(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T), \quad t \in [2t_0/3, T].$$

This proves the case $l = 1$.

For $l \geqslant 2$, our proof is by induction. Assume that the following estimate holds for each $0 \leqslant m \leqslant l - 1$:

$$\sup_{L_t} |\nabla^m \Psi_*| \leqslant \frac{(m+1)(m+2)C_m(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)}{t_0}, \quad t \in [((m+1)/(m+2))t_0, T].$$

Then by (3-3) we have

$$\frac{d}{dt}|\nabla^{l-1}\Psi_*|^2 \leqslant \Delta|\nabla^{l-1}\Psi_*|^2 - 2|\nabla^l \Psi_*|^2 + c_4$$

and

$$\frac{d}{dt}|\nabla^l \Psi_*|^2 \leqslant \Delta|\nabla^l \Psi_*|^2 - 2|\nabla^{l+1}\Psi_*|^2 + c_5|\nabla^l \Psi_*|^2 + c_6,$$

for $t \in [(l/(l+1))t_0, T]$, where $c_k = c_k(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)$, $k = 4, 5, 6$, are constants which are controlled by the lower order estimates. As for $l = 1$, using the maximum principle we see

$$|\nabla^l \Psi_*|^2 \leqslant \frac{C_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)}{t - (l/(l+1))t_0}, \qquad t \in ((l/(l+1))t_0, T].$$

Therefore we obtain the desired bound

$$|\nabla^l \Psi_*|^2 \leqslant \frac{(l+1)(l+2)C_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)}{t_0} =: P_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0, T)$$

for $t \in [((l+1)/(l+2))t_0, T]$. ☐

Remark 3.11. From the smoothing estimates, for any $t_0 \in (0, T)$ we have

$$\sup_{L_t}|\nabla^l A| \leqslant \Lambda_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0), \quad \sup_{L_t}|\nabla^l \Psi| \leqslant P_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0), \quad t \in [t_0/2, t_0].$$

In particular, we have bounds for the derivatives $|\nabla^l A|$ and $|\nabla^l \Psi|$ for $l \geqslant 1$ at $t = t_0$. On the other hand, as in the proof of the above lemma, it is not difficult to see that we have bounds which only depend on n, $A(t_0)$ and $\Psi_*(t_0)$ (including their higher order derivatives):

$$\sup_{L_t}|\nabla^l A| \leqslant \Lambda_l(n, A(t_0), \overline{\mathrm{Rm}}), \quad \sup_{L_t}|\nabla^l \Psi| \leqslant P_l(n, A(t_0), \Psi_*(t_0), \overline{\mathrm{Rm}}), \quad t \in [t_0, T].$$

Combining both estimates on $[t_0, T]$, we obtain T-independent estimates

$$\sup_{L_t}|\nabla^l A| \leqslant \Lambda_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0), \quad \sup_{L_t}|\nabla^l \Psi| \leqslant P_l(n, \Lambda, \overline{\mathrm{Rm}}, t_0), \quad t \in [t_0, T].$$

We often use this property without mentioning it in later arguments.

3E. ***Convergence of the flow.*** Now we are ready to prove the main theorem.

Theorem 3.12 (Theorem 1.1). *Let $(M, \bar{g})$ be a hyper-Kähler $4n$-manifold with bounded geometry. Suppose L is a hyper-Lagrangian submanifold with the complex phase Ψ_0 which is smoothly immersed into M. Then for any V_0, Λ_0 and $\delta_0 > 0$, there exists $\varepsilon_0 = \varepsilon_0(n, V_0, \Lambda_0, \delta_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M)) > 0$ such that if L satisfies*

$$\mathrm{Vol}(L_0) \leqslant V_0, \quad |A|(0) \leqslant \Lambda_0, \quad \lambda_1(\Delta_L)(0) \geqslant \delta_0, \quad \mathcal{T}(L_0) \leqslant \varepsilon_0,$$

then the hyper-Lagrangian mean curvature flow starting from L converges smoothly, exponentially fast to a complex Lagrangian submanifold in M for one of the hyper-Kähler complex structure on M.

Proof. Step 1 (reduction from L^2 to C^0): In the first step, we see that after a short period of time, the parabolicity of the flow improves the initial L^2-condition for $\nabla\Psi$ to the C^0-condition. From Proposition 2.5 and Lemma 3.9, we know that L_t satisfies

$$|A|(t) \leqslant 2\Lambda_0, \quad |\nabla\Psi|(t) \leqslant c(n)\Lambda_0, \quad \lambda_1(t) \geqslant \tfrac{2}{3}\delta_0, \quad t \in [0, T_0]$$

for $T_0 = T_0(n, \Lambda_0, \overline{\mathrm{Rm}})$. So Lemma 3.4 implies the following exponential estimate for the twistor energy:

$$\mathcal{T}(L_t) \leqslant e^{ct}\mathcal{T}(L_0) \leqslant \varepsilon_0 e^{ct}, \quad t \in [0, T_0]$$

for some $c = c(n, \Lambda_0) > 0$. Therefore we can choose $t_0 = t_0(n, \Lambda_0) \in (0, T_0]$ so that

$$\mathcal{T}(L_t) \leqslant 2\varepsilon_0, \quad t \in [0, t_0].$$

On the other hand, by the smoothing estimates, we know that for any $l \geqslant 1$,

$$|\nabla^l A|(t) \leqslant C_l(n, \Lambda_0, \overline{\mathrm{Rm}}), \qquad t \in [t_0/2, t_0], \tag{3-4}$$

and also

$$|\nabla^2 \Psi|(t) \leqslant c(n, \Lambda_0, \overline{\mathrm{Rm}}), \qquad t \in [t_0/2, t_0].$$

In order to get the estimate for the energy density $|\nabla\Psi|$, we need to establish the noncollapsing estimate for L_t first. By [Chen and He 2010, Proposition 2.2] and (3-4), we know that the injectivity radius of L is bounded from below along the HLMCF

$$\mathrm{inj}(L_t) \geqslant \iota(n, \Lambda_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M)) > 0, \quad t \in [t_0/2, t_0].$$

Meanwhile, the Gauss equation implies that

$$|\mathrm{Rm}| \leqslant C(\Lambda_0, \overline{\mathrm{Rm}}), \qquad t \in [t_0/2, t_0].$$

So in the same way as the proof of [Li 2012, Theorem 1.1], the volume comparison theorem shows there exist $\kappa = \kappa(n, \Lambda_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M))$ and $r = r(n, \Lambda_0, \overline{\mathrm{Rm}}, \mathrm{inj}(M))$ such that L_t is κ-noncollapsed on the scale r for all $t \in [t_0/2, t_0]$. So Lemma 3.7 implies that

$$|\nabla\Psi|(t) \leqslant (c + \kappa^{-1/2})(2\varepsilon_0)^{\frac{1}{2n+2}} =: \eta, \quad t \in [t_0/2, t_0],$$

where we take ε_0 sufficiently small so that $2\varepsilon_0 \leqslant r^{2n+2}$.

Step 2 (ε_0-regularity): We set

$$\mathcal{A}(\kappa, r, \Lambda, P, \delta) := \left\{ L \subset M \;\middle|\; \begin{array}{l} L \text{ is a hyper-Lagrangian submanifold,} \\ L \text{ is } \kappa\text{-noncollapsed on the scale } r, \\ |A| \leqslant \Lambda, \quad |\nabla\Psi| \leqslant P, \quad \lambda_1(\Delta_L) \geqslant \delta \end{array} \right\}.$$

Without loss of generality, we regard $L_{t_0/2}$ as the initial data of the HLMCF, so

$$L_t \in \mathcal{A}(\kappa, r, \Lambda, \eta, \delta), \quad t \in [0, t_0/2],$$

where $\Lambda := 2\Lambda_0$, $\eta := (c+\kappa^{-1/2})(2\varepsilon_0)^{1/(2n+2)}$, $\delta := \frac{2}{3}\delta_0$. So Lemma 3.9 combining with the volume ratio estimate (see Lemma 3.8) implies that we can choose a small $T^* > 0$ such that

$$L_t \in \mathcal{A}\big(\tfrac{1}{3}\kappa, r, 6\Lambda, 2\eta^{\frac{1}{2n+2}}, \tfrac{1}{3}\delta\big), \quad t \in [0, T^*].$$

Let T^* be the maximal time such that the above estimate holds. Then in order to prove the long-time existence of the flow, it suffices to prove the following ε_0-regularity:

Claim 3.13. There exists a small $\eta > 0$ (and hence small $\varepsilon_0 > 0$) such that

$$L_t \in \mathcal{A}\big(\tfrac{2}{3}\kappa, r, 3\Lambda, \eta^{\frac{1}{2n+2}}, \tfrac{1}{2}\delta\big), \quad t \in [0, T^*].$$

Indeed, if $T^* < \infty$ then from the claim we have $L_t \in \mathcal{A}\big(\frac{2}{3}\kappa, r, 3\Lambda, \eta^{1/(2n+2)}, \frac{1}{2}\delta\big)$ for $t \in [0, T^*]$. By using Lemma 3.9 and the volume ratio estimate again, we find that there exists $\tilde{T} > T^*$ such that $L_t \in \mathcal{A}\big(\frac{1}{3}\kappa, r, 6\Lambda, 2\eta^{1/(2n+2)}, \frac{1}{3}\delta\big)$ for $t \in [0, \tilde{T}]$, contradicting the maximality of T^*.

First, we establish an estimate for $|\nabla\Psi|$. We know that

$$\lambda_1(t) \geqslant \tfrac{1}{3}\delta, \quad t \in [0, T^*].$$

So if we choose $\eta > 0$ small so that

$$\lambda_1(t) \geqslant \tfrac{1}{4}\delta + C(n) \cdot 3\Lambda \cdot 2\eta^{\frac{1}{2n+2}} + (2\eta^{\frac{1}{2n+2}})^2, \quad t \in [0, T^*],$$

then the exponential estimate for the twistor energy (see Lemma 3.4) implies

$$\mathcal{T}(L_t) \leqslant e^{-\frac{\delta}{2}t}\mathcal{T}(L_0) \leqslant \eta^2 V_0 e^{-\frac{\delta}{2}t}, \quad t \in [0, T^*].$$

By Lemma 3.9, there exists some $t^* = t^*(n, \Lambda, \overline{\mathrm{Rm}}) \in (0, T^*)$ such that

$$|\nabla\Psi| \leqslant 2\eta \leqslant \eta^{\frac{1}{2n+2}}, \quad t \in [0, t^*],$$

for $\eta \leqslant \frac{1}{2}$. On the other hand, since $|A|(t) \leqslant 6\Lambda$ for $t \in [0, T^*]$, the smoothing estimates imply that

$$|\nabla^2\Psi| \leqslant C(n, \Lambda, \overline{\mathrm{Rm}}), \quad t \in [t^*, T^*].$$

Thus we obtain

$$|\nabla\Psi|(t) \leqslant C(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}}) \cdot \eta^{\frac{1}{n+1}} e^{-\frac{\delta t}{4n+4}}, \qquad t \in [t^*, T^*]. \tag{3-5}$$

So we can choose $\eta > 0$ small so that

$$C(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}}) \cdot \eta^{\frac{1}{2n+2}} \leqslant 1$$

and obtain

$$|\nabla\Psi|(t) \leqslant \eta^{\frac{1}{2n+2}}, \quad t \in [0, T^*].$$

Next, we compute $|A|$. By the smoothing estimates, for any $l \geqslant 1$, we have

$$|\nabla^l A| \leqslant C_l(n, \Lambda, \overline{\mathrm{Rm}}), \quad t \in [t^*, T^*].$$

Thus we also have

$$|\nabla^l H| \leqslant C_l(n, \Lambda, \overline{\mathrm{Rm}}), \quad t \in [t^*, T^*].$$

From Proposition 2.4 and (3-5), we know that $|H|$ also decreases exponentially fast. So integrating by parts, we have

$$\int_{L_t} |\nabla^2 H|^2 \, d\mu_t \leqslant \int_{L_t} |H||\nabla^4 H| \, d\mu_t \leqslant C(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}})\eta^{\frac{1}{n+1}} e^{-\frac{\delta t}{4n+4}}$$

for $t \in [t^*, T^*]$. So we have

$$|\nabla^2 H| \leqslant c(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}})\eta^{\frac{1}{2(n+1)^2}} e^{-\frac{\delta t}{8(n+1)^2}}, \qquad t \in [t^*, T^*].$$

We recall the evolution equation of A along the MCF (see [Chen and Li 2001])

$$\frac{d}{dt} h^{\alpha}_{ij} = \nabla_i \nabla_j H^{\alpha} - H^{\beta} h^{\beta}_{jk} h^{\alpha}_{ik} + H^{\beta} \bar{R}_{\alpha j \beta i} + h^{\beta}_{ij} b^{\beta}_{\alpha},$$

where $b^{\beta}_{\alpha} = \bar{g}\big(\frac{d}{dt} e_{\alpha}, e_{\beta}\big) = \bar{g}(\bar{\nabla}_H e_{\alpha}, e_{\beta})$. Note that b^{β}_{α} is antisymmetric since

$$0 = \frac{d}{dt}(\bar{g}(e_{\alpha}, e_{\beta})) = b^{\beta}_{\alpha} + b^{\alpha}_{\beta}.$$

Then it follows that

$$h^{\alpha}_{ij} h^{\beta}_{ij} b^{\alpha}_{\beta} = 0.$$

So we compute

$$2|A| \frac{d}{dt}|A| = \frac{d}{dt}|A|^2 \leqslant c(n)(|\nabla^2 H||A| + |H||A||\overline{\mathrm{Rm}}| + |H||A|^3).$$

Dividing both sides by $|A|$, we have

$$\frac{d}{dt}|A| \leqslant c(n)(|\nabla^2 H| + |H||\overline{\mathrm{Rm}}| + |H||A|^2). \tag{3-6}$$

Meanwhile, Lemma 3.9 shows that

$$|A|(t) \leqslant 2\Lambda, \quad t \in [0, t^*].$$

So integrating (3-6) in t and using the exponential decay of $|H|$, we have

$$\begin{aligned} |A|(t) &\leqslant |A|(t^*) + c(n) \int_{t^*}^{t} (|\nabla^2 H| + |H||\overline{\mathrm{Rm}}| + |H||A|^2)\, ds \\ &\leqslant 2\Lambda + c(n)\left[c\eta^{\frac{1}{2(n+1)^2}} \frac{16(n+1)^2}{\delta} + (C(\overline{\mathrm{Rm}}) + 64\Lambda^2) \cdot c\eta^{\frac{1}{n+1}} \frac{8(n+1)}{\delta} \right]. \end{aligned}$$

Thus we can take $\eta > 0$ sufficiently small so that

$$|A|(t) \leqslant 3\Lambda, \quad t \in [0, T^*].$$

Then we establish the estimate for $\lambda_1(t)$. Since $\lambda_1(0) \geqslant \delta$, Lemma 3.9 shows that

$$\lambda_1(t) \geqslant \tfrac{2}{3}\delta, \quad t \in [0, t^*].$$

Thus the exponential estimate for λ_1 combined with the exponential decay of $|H|$ implies that

$$\begin{aligned}\lambda_1(t) &\geqslant \exp\left[-\int_{t^*}^{t}(\max_{L_s}|H|^2 + C(n)\max_{L_s}|H||A|)\,ds\right]\lambda_1(t^*)\\ &\geqslant \exp\left[-c^2\eta^{\frac{2}{n+1}}\frac{4(n+1)}{\delta} - C(n)\cdot 3\Lambda\cdot c\eta^{\frac{1}{n+1}}\frac{8(n+1)}{\delta}\right]\lambda_1(t^*).\end{aligned}$$

If we take $\eta > 0$ sufficiently small, then

$$\lambda_1(t) \geqslant \tfrac{1}{2}\delta, \quad t \in [0, T^*].$$

We can prove a noncollapsing estimate of L_t in the same way as λ_1, by using the volume ratio estimate.

Step 3 (exponential convergence of the flow): From Step 2, we have a uniform bound for A. So the standard bootstrapping arguments combined with Simon's theorem [1983] imply the smooth convergence of the MCF $L_t \to L_\infty$. Moreover, we have already seen that for a fixed sufficiently small $\eta > 0$, we have

$$|\nabla\Psi(t)| \leqslant C(n, \Lambda, \kappa, r, V_0, \overline{\mathrm{Rm}})\cdot\eta^{\frac{1}{n+1}}e^{-\frac{\delta t}{4n+4}} \searrow 0.$$

In particular, Proposition 2.4 implies that H_t converges exponentially fast to $H_\infty = 0$, and hence L_∞ is minimal.

As for the generalized harmonic map flow, we have also the uniform bounds $|\nabla^l\Psi| \leqslant C_l$ for all $l \geqslant 1$. Thus there exists a subsequence $\{\Psi_{t_i}\}$ which converges to a smooth map $\Psi_\infty : L \to \mathbb{S}^2$ and L_∞ inherits a hyper-Lagrangian structure with the complex phase Ψ_∞. Since $|\nabla\Psi_\infty| = 0$, the map Ψ_∞ should be a constant. Finally, we show that the complex phase Ψ_∞ which arises from the generalized harmonic map flow does not depend on the choice of the subsequence $\{\Psi_{t_i}\}$ by contradiction. So we assume that there exist two distinct constant phase maps Ψ_∞ and Ψ'_∞ which arise in this way. We take a small geodesic ball in $B \subset \mathbb{S}^2$ centered at Ψ_∞ so that $\Psi'_\infty \notin B$. Since $\{\Psi_{t_i}\}$ converges to Ψ_∞ we know that $\Psi_{t_i}(L) \subset B$ for i large enough. We fix such an i and consider the generalized harmonic map flow Ψ'_t starting from the data (L_{t_i}, Ψ_{t_i}). Then a simple maximum principle argument (see Corollary 3.2) shows that $\Psi'_t(L) \subset B$ for all $t \in [0, \infty)$ whereas $\{\Psi'_t\}$ should have a convergent subsequence to $\Psi'_\infty \notin B$, which is a contradiction. This completes the proof. □

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THE TWO-DIMENSIONAL ANALOGUE OF THE LORENTZIAN CATENARY AND THE DIRICHLET PROBLEM

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We generalize in Lorentz–Minkowski space $\mathbb{L}^3$ the two-dimensional analogue of the catenary of Euclidean space. We solve the Dirichlet problem when the bounded domain is mean convex and the boundary data has a spacelike extension to the domain. We also classify all singular maximal surfaces of $\mathbb{L}^3$ invariant by a uniparametric group of translations and rotations.

1. Introduction and motivation

The purpose of this paper is to investigate the physical problem of characterizing the surfaces in Lorentz–Minkowski space with lowest gravity center and solve the corresponding Dirichlet problem. The existence of a variety of causal vectors in the Lorentzian setting leads to several issues that need to be fixed. Firstly, we recall this problem in the Euclidean space in order to motivate our definitions. Let $\mathbb{R}^2$ be the Euclidean plane with canonical coordinates (x, y) where the y-axis indicates the gravity direction. Consider the physical problem of finding the curve in the halfplane $y > 0$ with the lowest gravity center. If the curve is $y = u(x)$, then u satisfies the equation

$$\frac{u''}{1+u'^2} = \frac{1}{u}. \tag{1}$$

The solution of this equation is known as the catenary

$$u(x) = \frac{1}{a}\cosh(ax+b), \qquad a, b \in \mathbb{R}, a \neq 0.$$

The equation (1) can be expressed in terms of the curvature κ of the curve as

$$\kappa = \frac{\langle \boldsymbol{n}, \vec{a}\rangle}{y}, \tag{2}$$

where $\boldsymbol{n}$ is the unit normal vector and $\vec{a} = (0, 1)$. In particular, (2) prescribes the angle that the vector $\boldsymbol{n}$ makes with the vertical direction.

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The generalization in Euclidean 3-space $\mathbb{R}^3$ of the property of the catenary is to find surfaces in the halfspace $z > 0$ with the lowest gravity center. If (x, y, z) denote the canonical coordinates of $\mathbb{R}^3$ and z indicates the direction of the gravity, these surfaces characterize by means of the equation

$$H = \frac{\langle N, \vec{a}\rangle}{z},$$

where H is the mean curvature of the surface and $\vec{a} = (0, 0, 1)$. The surface is called the *two-dimensional analogue of the catenary* ([Böhme et al. 1980; Dierkes and Huisken 1990]). Historically, this problem goes back to early works of Lagrange and Poisson on the equation that models a heavy surface in a vertical gravitational field. If we embed $\mathbb{R}^2$ as the xz-plane by identifying the y-axis of $\mathbb{R}^2$ with the z-axis of $\mathbb{R}^3$, and we rotate the catenary with respect to the x-axis, we obtain the catenoid $a^2(y^2 + z^2) = \cosh^2(x)$, which is the only nonplanar rotational minimal surface of $\mathbb{R}^3$.

More generally, given a constant $\alpha \in \mathbb{R}$, a surface in the halfspace $z > 0$ is called a singular minimal surface if it satisfies

$$H = \alpha \frac{\langle N, \vec{a}\rangle}{z}. \tag{3}$$

The theory of singular minimal surfaces has been intensively studied in the works of Bemelmans, Dierkes and Huisken, among others (see, for example, [Bemelmans and Dierkes 1987; Böhme et al. 1980; Dierkes 1988a; 2003; Dierkes and Huisken 1990; López 2018a; 2018b; 2019b; Nitsche 1986]). Now that we have presented the problem in the Euclidean space, we proceed to generalize it in the Lorentz–Minkowski space. As in the Euclidean case, we begin with the one-dimensional case. Let $\mathbb{L}^2$ be the Lorentz–Minkowski plane defined as the affine (x, y)-plane $\mathbb{R}^2$ endowed with the metric $dx^2 - dy^2$. Here we use the usual terminology of the Lorentz–Minkowski space: see [O'Neill 1983] as a general reference and [López 2014] for curves and surfaces in Lorentz–Minkowski space. In what follows, we will assume that for a given set, the causal character is the same in all its points, that is, we do not admit the existence of points with different causal character.

A first issue is that the notion of gravity in $\mathbb{L}^2$ does not make sense because the y-coordinate represents the time in the Lorentzian context. Thus we need to view the initial problem as a problem of finding curves in $\mathbb{L}^2$ with prescribed angle between the normal vector and a fixed direction, such as it was shown in (2). There appear two new issues. Firstly there are three types of curves in $\mathbb{L}^2$ owing to its causal character, namely, spacelike, timelike and lightlike and the behavior of each of these curves is completely different. Because we are only interested in the Riemannian sense, we will only consider spacelike curves.

A second issue is the choice of the axis with respect to which we measure the angle of the normal vector $\boldsymbol{n}$. Notice that in the Euclidean plane both axes are indistinct but in $\mathbb{L}^2$ the y-axis and the x-axis are not interchangeable by a rigid motion. Thus arises the problem of which axis is to be fixed. Since for a spacelike curve, the vector $\boldsymbol{n}$ is timelike, we will measure the angle between $\boldsymbol{n}$ and the y-axis, which is also timelike. This is also justified because it makes sense to define the angle between two timelike vectors ([O'Neill 1983, p. 144]). After all these considerations, let us proceed.

Let $\gamma = \gamma(s)$ be a spacelike curve parametrized by the arc-length $s \in I$ and contained in the halfplane $y > 0$ of $\mathbb{L}^2$. The curvature κ of γ is defined by $\gamma''(s) = \kappa(s)\boldsymbol{n}(s)$ where $\boldsymbol{n}$ is a unit normal vector of γ. Here we are assuming $\kappa \neq 0$. Motivated by (2), we ask for those spacelike curves of $\mathbb{L}^2$ that satisfy the same equation (2) where $\boldsymbol{a} = (0, 1)$. If γ is a graph $y = u(x)$, then $\gamma(x) = (x, u(x))$, which is not parametrized by the arc-length. Then $\boldsymbol{n} = (u', 1)/\sqrt{1-u'^2}$, $\langle \boldsymbol{n}, \vec{a}\rangle = -1/\sqrt{1-u'^2}$ and

$$\kappa(x) = -\frac{1}{1-u'^2}\langle \gamma''(x), \boldsymbol{n}(x)\rangle = \frac{u''(x)}{(1-u'(x)^2)^{3/2}}.$$

Let us observe that $u'^2 < 1$ because γ is a spacelike curve. The equation (2) is now

$$\frac{u''}{1-u'^2} = -\frac{1}{u}, \tag{4}$$

which will be the Lorentzian model of (1) that we are looking for. The spacelike condition $u'^2 - 1 < 0$ is an extra hypothesis compared to the Euclidean case. For example, $u(x) = \sinh(x)$, with $u > 0$, solves (4), but $u'^2 > 1$. So, the corresponding curve $y = u(x)$ is a timelike curve. In contrast, because we are assuming that the curve is spacelike, the right solution of (4) is

$$u(x) = \frac{1}{a}\sin(ax+b), \quad x \in \left(-\frac{b}{a}, \pi - \frac{b}{a}\right), \tag{5}$$

where $a \neq 0$ $a, b \in \mathbb{R}$. This curve will be the analogue catenary in $\mathbb{L}^2$. As in the Euclidean case, we introduce a constant $\alpha \in \mathbb{R}$ and we consider the analogous equation of (2), namely,

$$\kappa = \alpha\frac{\langle \boldsymbol{n}, \vec{a}\rangle}{\langle p, \vec{a}\rangle} = -\alpha\frac{\langle \boldsymbol{n}, \vec{a}\rangle}{y}, \tag{6}$$

where $p = (x, y) \in \mathbb{L}^2$. For instance, the curve (5) is the solution for $\alpha = -1$.

Following the same steps as in the Euclidean setting, we embed $\mathbb{L}^2$ in the Lorentz–Minkowski 3-space $\mathbb{L}^3$. Here $\mathbb{L}^3$ is the affine Euclidean 3-space endowed with the metric $dx^2 + dy^2 - dz^2$. Then $\mathbb{L}^2$ is identified with the xz-plane, the y-axis of $\mathbb{L}^2$ with the z-axis of $\mathbb{L}^3$ and the vector $(0, 1) \in \mathbb{L}^2$ with $\vec{a} = (0, 0, 1)$. Definitively, the objects of our study in this paper are described in the following definition.

Definition 1.1. Let α be a nonzero real number. A spacelike surface S in the halfspace $z > 0$ of $\mathbb{L}^3$ is called an α-singular maximal surface if it satisfies

$$H(p) = \alpha \frac{\langle N(p), \vec{a}\rangle}{\langle p, \vec{a}\rangle} = -\alpha \frac{\langle N(p), \vec{a}\rangle}{z}, \qquad (p \in S), \tag{7}$$

where N is a unit normal vector field on S and H is the mean curvature.

Here H is the trace of the second fundamental form of S, that is, the sum of the principal curvatures. We will omit the constant α if it is understood from the context. Recently, these surfaces have been studied in [Martínez and Martínez-Triviño 2020] relating the Riemannian and the Lorentzian settings by means of a Calabi type correspondence.

In view of (2), and as a motivation of this paper, the case $\alpha = -1$ in (7) is the corresponding *two-dimensional analogue of the Lorentzian catenary*. Other known examples appear when $\alpha = 2$ because in such a case, the surface is a minimal surface in the steady state space ([López 2017]). Another special example is the hyperbolic plane $\mathbb{H}^2(r) = \{p \in \mathbb{L}^3 : \langle p, p\rangle = -r^2, z > 0\}$, $r > 0$. This surface has mean curvature $H = 2/r$ for $N(p) = p/r$. It is clear that $\mathbb{H}^2(r)$ satisfies (7) for $\alpha = 2$. Even more, $\mathbb{H}^2(r)$ satisfies (7) *for any* vector $\vec{a}$.

On the other hand, we extend a similar property that has the catenary in Euclidean space. Indeed, we take the catenary (5) and we rotate with respect to the x-axis. The rotations that leave the x-axis pointwise fixed are described by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

For a curve $z = u(x)$, namely, $\gamma(x) = (x, 0, u(x))$, $x \in I \subset \mathbb{R}$, contained in the xz-plane, the corresponding rotational surface S is parametrized by

$$X(x, \theta) = (x, u(x)\sinh\theta, u(x)\cosh\theta), \quad \theta \in \mathbb{R}. \tag{8}$$

If $u(x) = \sin(ax + b)/a$, it is not difficult to see that the corresponding rotational surface (8) has zero mean curvature, that is, S is a maximal surface of $\mathbb{L}^3$. This surface is called the catenoid of second kind or the hyperbolic catenoid in the literature.

Remark 1.2. If we rotate the curve $u(x) = \sinh(ax + b)/a$, the timelike solution of (4), with respect to the x-axis, the rotational surface is a timelike surface with zero mean curvature ([López 2000]). Similarly, any vertical straight line is a timelike curve that satisfies (2) and if we rotate with respect to the x-axis, we obtain a (timelike) plane parallel to the yz-plane, which has zero mean curvature everywhere.

As a conclusion, the generalization in $\mathbb{L}^3$ of the two-dimensional analogue of the catenary, or more generally, singular maximal surfaces in Lorentz–Minkowski space $\mathbb{L}^3$, is carried out for spacelike surfaces and the angle between N and $\vec{a}$ is measured with respect to the (timelike) z-axis. We have also discussed that there are other possibilities to generalize the initial problem in $\mathbb{L}^3$, although all of them less justified, such as for example, changing the axis $\vec{a} = (0, 0, 1)$ to $(1, 0, 0)$ (spacelike) or $(1, 0, 1)$ (lightlike). Also, we may consider timelike surfaces and measuring the angle between N with respect to an axis of $\mathbb{L}^3$.

In this paper we will also be interested in solving the Dirichlet problem of the singular maximal surface equation. Since a spacelike surface is locally the graph of a function $z = u(x, y)$, the nonparametric form of (7) is

$$\operatorname{div} \frac{Du}{\sqrt{1-|Du|^2}} = \alpha \frac{1}{u\sqrt{1-|Du|^2}}, \tag{9}$$

with the spacelike condition $|Du| < 1$. The left-hand side of this equation is the mean curvature of the graph $z = u(x, y)$ computed with respect to the upwards orientation

$$N = \frac{1}{\sqrt{1-|Du|^2}}(Du, 1).$$

Comparing (9) with the Riemannian case [Dierkes 1988a; 1988b; 2003; López 2019a]), this equation is not uniformly elliptic and, as a consequence, we must ensure that $|Du|$ is bounded away from 1.

This paper is organized as follows. In Section 2 we classify all singular maximal surfaces that are invariant by a uniparametric group of translations and of rotations. In Section 3 we describe the solutions of (7) that are invariant by rotations about the z-axis and finally, in Section 4 we solve the Dirichlet problem associated to (9) for mean convex domains and arbitrary boundary data.

2. Invariant singular maximal surfaces

In this section we classify and describe all singular maximal surfaces that are invariant by a uniparametric group of translations or of rotations of $\mathbb{L}^3$. Firstly, we notice that some transformations of the affine Euclidean space $\mathbb{R}^3$ preserve the singular maximal surface equation. To fix the terminology, a vector $\vec{v} \in \mathbb{R}^3$ is called horizontal direction if it is parallel to the xy-plane and it is called vertical if it is parallel to the z-axis.

It is clear that a solution of (7) is invariant by a translation along a horizontal direction, that is, if S is an α-singular maximal surface, then $S + \vec{v}$ is also an α-singular maximal surface, where $\vec{v}$ is a horizontal vector of $\mathbb{R}^3$. Similarly, the same property holds if we rotate S with respect to a vertical direction because the term $\langle N, \vec{a}\rangle$ and the denominator z in (7) are invariant by this type of rotation.

Finally, if $\lambda > 0$ is a positive real number, and $T_\lambda(p) = p_0 + \lambda(p - p_0)$ is the dilation with center $p_0 \in \mathbb{R}^2 \times \{0\}$, then $T_\lambda(S)$ is an α-singular maximal surface.

Remark 2.1. We point out that a rigid motion of $\mathbb{L}^3$ does not preserve (7) in general because the denominator z may change in general by the motion.

As we have announced, a natural source of examples of singular maximal surfaces of $\mathbb{L}^3$ is found in the class of invariant surfaces by a uniparametric group of rigid motions. The key point is that (7), which locally is the partial differential (9), changes into an ordinary differential equation. In particular, by standard theory, there always is a solution for any initial conditions.

Surfaces invariant by translations. We begin the study of the surfaces invariant by a uniparametric group of translations. Since the rulings generated by this group are straight lines contained in the surface, and the surface is spacelike, then any ruling is a spacelike line. Thus the vector generating the group of translation must be spacelike. Let $\vec{v}$ be a unit spacelike vector and consider a surface S invariant by the group of translations generated by $\vec{v}$. Then S parametrizes as $X(s,t) = \gamma(s) + t\vec{v}$, where γ is a planar spacelike curve of $\mathbb{L}^3$ contained in a (timelike) orthogonal plane to $\vec{v}$. The equation (7) gives

$$\kappa \det(\gamma', \vec{v}, \boldsymbol{n}) = \alpha \frac{\det(\gamma', \vec{v}, \vec{a})}{\gamma_3 + t v_3},$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and $\vec{v} = (v_1, v_2, v_3)$. We consider the orientation in γ so $\gamma' \times \vec{v} = \boldsymbol{n}$. Since $\boldsymbol{n}$ is a unit timelike vector, the above equation is now

$$\kappa(\gamma_3 + t v_3) + \alpha\langle \boldsymbol{n}, \vec{a}\rangle = 0. \tag{10}$$

This is a polynomial equation on t, hence

$$\kappa v_3 = 0, \quad \kappa\gamma_3 + \alpha\langle \boldsymbol{n}, \vec{a}\rangle = 0.$$

Since $\kappa \neq 0$, we deduce that $v_3 = 0$ and $\kappa\gamma_3 + \alpha\langle \boldsymbol{n}, \vec{a}\rangle = 0$. Then $\vec{v}$ is a horizontal vector and γ is a planar curve contained in a vertical plane. After a horizontal translation and a rotation about the z-axis, we assume that this plane is the xz-plane which can be identified with $\mathbb{L}^2$. Furthermore, the equation $\gamma_3 + \alpha\langle \boldsymbol{n}, \vec{a}\rangle = 0$ means that γ satisfies, as a planar curve of $\mathbb{L}^2$, the one-dimensional singular maximal surface equation (6). The converse of this result is immediate.

Proposition 2.2. *Let S be an α-singular maximal surface of $\mathbb{L}^3$ invariant by a uniparametric group of translations generated by $\vec{v}$ and denote by γ its generatrix. Then $\vec{v}$ is a horizontal vector, γ is contained in a plane orthogonal to $\vec{v}$ and γ, as a planar curve, satisfies (6). Conversely, if γ is a curve in $\mathbb{L}^2$ that satisfies (6) and, if we embed this curve in the xz-plane as usual, then the surface $X(s,t) = \gamma(s) + t(0,1,0)$ is an α-singular maximal surface.*

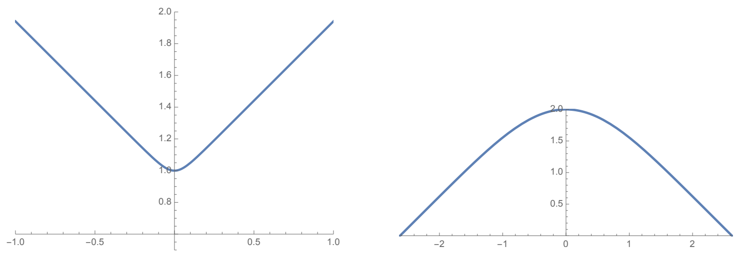

Figure 1. Solutions of (11). Left: $\alpha = 1$. Right: $\alpha = -2$.

In view of this proposition, consider the one-dimensional case of (7). Let $\gamma(s) = (x(s), y(s))$ be a spacelike curve in $\mathbb{L}^2$ that satisfies (6). Since γ is spacelike, then $x'^2 - y'^2 > 0$, in particular, $x'(s) \neq 0$ for every s and thus γ is globally the graph of a function $u = u(x),\ x \in I \subset \mathbb{R}$. The equation (6) now becomes

$$\frac{u''}{1-u'^2} = \alpha \frac{1}{u}, \quad u > 0,\ u'^2 < 1. \tag{11}$$

It is possible to find some explicit solutions of (11) by simple quadratures. In the introduction we have seen that if $\alpha = -1$, the solution is $u(x) = \sin(ax+b)/a$, where $a \neq 0,\ a, b \in \mathbb{R}$ and where x is defined in some interval to ensure that $u > 0$. If $\alpha = 1$, it is easy to find that the solution of (11) is

$$u(x) = \frac{1}{a}\sqrt{1 + a^2x^2 + 2abx + b^2}, \quad a, b \in \mathbb{R},\ a > 0.$$

After a change of variable, this function u is written as $u(x) = \sqrt{1+a^2x^2}/a,\ a > 0$. It is immediate that u is the upper branch of the hyperbola $a^2(x^2 - y^2) = -1$. This curve, viewed as a planar curve in $\mathbb{L}^2$, has nonzero constant curvature $\kappa = a$. The generated surface by Proposition 2.2 is the right-cylinder of $a^2(x^2 - z^2) = -1$.

Remark 2.3. As for the catenary $u(x) = \sin(ax+b)/a$, if we rotate the curve $u(x) = \sqrt{1+a^2x^2}/a$ with respect to the x-axis, we obtain the hyperbolic plane $\mathbb{H}^2(1/a)$.

Remark 2.4. Similarly as in the case $\alpha = -1$, there is a timelike solution of (11) by replacing the spacelike condition $u'^2 < 1$ by $u'^2 > 1$. The solution is now $u(x) = \sqrt{a^2x^2 - 1}\,/a$, where $a > 0$ and $x > 1/a$. The function u is the positive part of the hyperbola $x^2 - y^2 = 1/a^2$, which is a timelike curve. If we rotate about the x-axis, the generated surface is $x^2 + y^2 - z^2 = 1/a^2$. This surface is the (upper part of) the de Sitter space $\mathbb{S}^2_1(1/a) = \{p \in \mathbb{L}^3 : \langle p, p\rangle = 1/a^2\}$. This surface satisfies (7) when $\alpha = 2$ and plays the same role as the hyperbolic plane in the family of timelike surfaces of $\mathbb{L}^3$.

We now describe the geometric properties of the solutions of (11). See Figure 1.

Theorem 2.5. *Let $u = u(x)$ be a solution of (11), $x \in I$, where $I \subset \mathbb{R}$ is the maximal domain of u. Then u is symmetric about a vertical line and $I = \mathbb{R}$ if $\alpha > 0$ or I is a bounded interval if $\alpha < 0$. Furthermore*:

(1) *Case $\alpha > 0$. The function u is convex with a unique global minimum,*

$$\lim_{r\to\infty} u(r) = \infty \quad \textit{and} \quad \lim_{r\to\infty} u'(r) = 1.$$

(2) *Case $\alpha < 0$. The function u is concave with a unique global maximum. If $I = (-b, b)$, then*

$$\lim_{r\to b} u(r) = 0 \quad \textit{and} \quad \lim_{r\to b} u'(r) = -1.$$

Proof. If u has a critical point at $r = r_o$, then $u''(r_o) = \alpha/u(r_o)$ has the same sign as α. Hence, there is one critical point at most that will be a global minimum (resp. maximum) if $\alpha > 0$ (resp. $\alpha < 0$).

Claim: There exists a critical point of u.

Suppose now that the claim is proved and we finish the proof of the theorem. After a change in the variable x, we suppose that $x = 0$ is the critical point, $u'(0) = 0$. Then u is the solution of (11) with initial conditions $u(0) = u_0 > 0$ and $u'(0) = 0$. It is clear that $u(-s)$ is also a solution of the same initial value problem, so $u(s) = u(-s)$ by uniqueness. This proves that u is symmetric about the y-axis.

Multiplying (11) by u', we obtain a first integral

$$\frac{1}{1-u'^2} = \mu u^{2\alpha}, \tag{12}$$

for some positive constant $\mu > 0$.

(1) Case $\alpha > 0$. Since $u(x) \geq u_0$, we deduce from (11) that u' and u'' are bounded functions and this implies the maximal domain is $\mathbb{R}$. Since u is a convex function, then $u(r) \to \infty$ as $r \to \infty$ and from (12), we conclude that $u'(r) \to 1$ as $r \to \infty$.

(2) Case $\alpha < 0$. By symmetry, $I = (-b, b)$ for some $b \leq \infty$. Since u is a positive concave function, then $b < \infty$. Using the concavity of u again, and because $u'^2 < 1$, then the graph of u must meet the x-axis, that is, $\lim_{r\to b} u(r) = 0$. From (12), we deduce $\lim_{r\to b} u'(r)^2 = 1$, and by concavity, $\lim_{r\to b} u'(r) = -1$.

We now prove the claim. The proof is by contradiction. Assume that the sign of u' is constant and denote $I = (a, b)$ with $-\infty \leq a < b \leq \infty$.

(1) Case $\alpha > 0$. We suppose that $u' > 0$ in I (similar argument if u' is negative). Since u is increasing and u' and u'' are bounded near $r = b$, we deduce that $b = \infty$ by standard theory. If $-\infty < a$, then $\lim_{r\to a} u(r) = 0$ because otherwise we could extend u beyond $r = a$ because u' and u'' would be bounded functions. Therefore $\lim_{r\to a} u'(r)^2 = 1$ by (12). Since $u' > 0$, this limit is just 1. This is a contradiction

because u' is an increasing function and we would have $u' > 1$ in I, which is not possible by the spacelike condition.

Thus $a = -\infty$. Since u is increasing and $u > 0$ in $\mathbb{R}$, we find $\lim_{r\to-\infty} u(r) = c \geq 0$. Because $u' > 0$ and $u'' > 0$, then $\lim_{r\to-\infty} u'(r) = \lim_{r\to-\infty} u''(r) = 0$. However, by (11), and letting $r \to -\infty$, we have that $u''(r)$ goes to $\alpha/c \neq 0$ if $c > 0$ or to ∞ if $c = 0$, obtaining a contradiction.

(2) Case $\alpha < 0$. We suppose that $u' > 0$ in I (similar argument if u' is negative). Since u' and u'' are bounded for r close to b, then $b = \infty$ and by concavity, we deduce that $-\infty < a$. If u is bounded from above with $\lim_{r\to\infty} u(r) = c > 0$, then $\lim_{r\to\infty} u'(r) = 0$ and since $u'' < 0$, then $\lim_{r\to\infty} u''(r) = 0$. By (11), we find $\lim_{r\to\infty} u''(r) = \alpha/c < 0$, a contradiction. Thus $\lim_{r\to\infty} u(r) = \infty$. By using (12), we conclude $\lim_{r\to\infty} u'(r)^2 = 1$, so this limit is 1; this is a contradiction because u' is a decreasing function and we would have $u' > 1$ in the interval I, which is not possible. □

Surfaces of revolution with respect to a spacelike axis and a lightlike axis. The second source of examples of singular maximal surfaces are the surfaces invariant by a uniparametric group of rotations. A difference between the Euclidean and the Lorentzian settings is that in $\mathbb{L}^3$ there are three types of surfaces of revolution depending on whether the rotational axis is spacelike, timelike or lightlike. Section 3 is devoted to the surfaces of revolution whose rotation axis is timelike because this type of surface will play a special role in the solvability of the Dirichlet problem in Section 4. In this section we investigate the cases where the rotation axis is spacelike and lightlike.

We point out that there is not an a priori relation between the rotation axis L and the vector $\vec{a} = (0, 0, 1)$ of (7). This implies that if we apply a rigid motion to prescribe the rotation axis, then the vector $\vec{a}$ does change; see also Remark 2.1.

First we consider the case that the axis is spacelike.

Proposition 2.6. *Let S be a spacelike surface of $\mathbb{L}^3$ invariant by the uniparametric group of rotations about a spacelike axis L. Suppose that S satisfies lrefeqL where $\vec{a}$ is a timelike vector. Then either $\vec{a}$ is orthogonal to L, or S is the hyperbolic plane $\mathbb{H}^2(r)$ with $\vec{a}$ an arbitrary timelike vector.*

Proof. After a rigid motion of $\mathbb{L}^3$ we assume that L is the x-axis. This rigid motion changes the vector $\vec{a}$ in (7) and $\vec{a}$ must be considered an arbitrary (timelike) vector. Let $\vec{a} = (a, b, c)$ denote the new vector $\vec{a}$ in (7) after the rigid motion. Since $\vec{a}$ is timelike, then $c \neq 0$.

Using the expression of a parametrization (8) of S and after some computations, (7) is a polynomial equation on $\{1, \sinh\theta, \cos\theta\}$. Since these functions are linearly independent, all three coefficients (which are functions on the variable s) must vanish, obtaining

$$-c(1-u'^2+uu'')+\alpha c(1-u'^2)=0$$
$$-b(1-u'^2+uu'')+\alpha b(1-u'^2)=0$$
$$as(1-u'^2+uu'')-\alpha a u u'(1-u'^2)=0.$$

Since $c\neq 0$, we find $a(uu'-s)=b(uu'-s)=0$. If $uu'-s\neq 0$, then $a=b=0$, proving that $\vec{a}=(0,0,c)$, hence L is orthogonal to the x-axis and the result is proved. The other possibility is $uu'-s=0$. Solving this equation, we find $u(s)=\sqrt{s^2+r^2}$, $r>0$. Then $X(s,\theta)=(s,\sqrt{s^2+r^2}\sinh\theta,\sqrt{s^2+r^2}\cos\theta)$ and it is immediate that this surface is the hyperbolic plane $\mathbb{H}^2(r)$. $\square$

As a consequence of Proposition 2.6, and besides the hyperbolic plane as a special case, we can assume that $\vec{a}=(0,0,1)$ in the singular maximal surface (7), and that the rotation axis is the x-axis. In such a case, the proof of Proposition 2.6 gives immediately that (7) is

$$\frac{u''}{1-u'^2}=(\alpha-1)\frac{1}{u}.$$

This equation is just (11). Identifying the Lorentzian plane $\mathbb{L}^2$ with the plane of equation $y=0$, we have obtained the following result.

Proposition 2.7. *Any rotational α-singular maximal surface in $\mathbb{L}^3$ about the x-axis is generated by a planar curve in $\mathbb{L}^2$ that satisfies the one-dimensional $(\alpha-1)$-singular maximal surface equation. Conversely, any planar curve in $\mathbb{L}^2$ that satisfies (11) is the generating curve of an $(\alpha+1)$-singular maximal surface invariant by all rotations about the x-axis.*

Example 2.8. We know that the solution of (11) for $\alpha=1$ is the hyperbola $u(x)=\sqrt{1+a^2x^2}/a$, $a>0$. As a consequence of Proposition 2.7, the only 2-singular maximal surface that is invariant by the rotations about the x-axis is the surface $x^2+y^2-z^2=-1/a^2$, $z>0$. This surface is the hyperbolic plane $\mathbb{H}^2(1/a)$. Another solution of (11) appeared in the introduction for $\alpha=-1$. Then the surface generated is the hyperbolic catenoid of $\mathbb{L}^3$.

We finish this section considering singular maximal surfaces of revolution about a lightlike axis. Again, we have in mind that if we fix the rotation axis, then the vector $\vec{a}$ in (7) is arbitrary. If the rotation axis is determined by the vector $(1,0,1)$, the parametrization of the surface is

$$X(s,t)=\begin{pmatrix}1-\frac{t^2}{2} & t & \frac{t^2}{2}\\ -t & 1 & t\\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2}\end{pmatrix}\begin{pmatrix}u(s)+s\\ 0\\ u(s)-s\end{pmatrix},\quad t\in\mathbb{R}, \tag{13}$$

for some function $u = u(s)$, $s \in I \subset \mathbb{R}$. The spacelike condition on the surface is equivalent to $u' > 0$.

Proposition 2.9. *Let S be a spacelike surface of $\mathbb{L}^3$ invariant by the uniparametric group of rotations about a lightlike axis L. Suppose S satisfies (7) where $\vec{a}$ is a timelike vector. Then either $\vec{a}$ is orthogonal to L, or S is the hyperbolic plane $\mathbb{H}^2(r)$ with $\vec{a}$ an arbitrary vector. More precisely, if L is generated by the vector* $(1, 0, 1)$, *S is parametrized by (13) and if $\alpha \neq 2$, then $\vec{a} = (1, b, 1)$, $b \neq 0$, and we have the following possibilities*:

(1) *If $\alpha = \frac{3}{2}$, then $u(s) = m\log(s)$, $m > 0$.*

(2) *If $\alpha \neq \frac{3}{2}$, then $u(s) = ms^{3-2\alpha}/(3-2\alpha)$, $m > 0$.*

In particular, hyperbolic planes $\mathbb{H}^2(r)$ are the only α-singular maximal surfaces in $\mathbb{L}^3$ satisfying (7) with $\vec{a} = (0, 0, 1)$ and invariant by the group of rotations about the lightlike axis generated by the vector $(1, 0, 1)$.

Proof. A straightforward computation of (7) for the surface (13) concludes that this equation is a polynomial equation on t of degree 2. Thus the coefficients corresponding to the variable t must vanish, obtaining

$$\begin{aligned} 2u'((\alpha+1)s(a+c)+(a-c)(u+\alpha su'))-su''((a-c)u+s(a+c)) &= 0\\ b(su''-2(1-\alpha)u') &= 0\\ (a-c)(su''-2(1-\alpha)u') &= 0. \end{aligned}$$

From the second and third equation, if $su''-2(1-\alpha)u' \neq 0$, we have $b=0$ and $a=c$, obtaining that $\vec{a}$ is a lightlike vector, which is not possible. Thus $su''-2(1-\alpha)u'=0$. The solution of this equation depends on the value of α.

(1) Case $\alpha = \frac{3}{2}$. Here, $u(s) = m\log(s)$ with $m > 0$. The first equation yields $(a-c)m^2(1+\log(s) = 0$, that is, $a = c$ and $\vec{a} = (a, b, a)$, $b \neq 0$.

(2) Case $\alpha \neq \frac{3}{2}$. Here, $u(s) = ms^{3-2\alpha}/(3-2\alpha)$ with $m > 0$. Now the first equation simplifies into $(a-c)(2-\alpha)s^{5-4\alpha} = 0$. If $\alpha = 2$, then $u(s) = -m/s$ and it is not difficult to see that this surface is the hyperbolic plane $\mathbb{H}^2(2\sqrt{m})$. If $\alpha \neq 2$, then $a = c$, so $\vec{a} = (a, b, a)$, $b \neq 0$. □

3. Surfaces of revolution about the z-axis

In this section we study the surfaces of revolution with timelike axis L. Again, the same observations as in the previous section hold in the sense that there is not an a priori relation between the vector $\vec{a}$ and the axis L. The first result that we will prove is that, indeed, L must be parallel to the vector $\vec{a}$.

Proposition 3.1. *Let S be an α-singular maximal surface in $\mathbb{L}^3$ that is invariant by the uniparametric group of rotations about a timelike axis L. Suppose that S*

satisfies (7) where $\vec{a}$ is now an arbitrary timelike vector. Then either L and $\vec{a}$ are parallel, or S is the hyperbolic plane $\mathbb{H}^2(m)$ and $\vec{a}$ is an arbitrary timelike vector.

Proof. After a rigid motion, we suppose that the rotation axis is the z-axis. Let $\vec{a}=(a,b,c)$ after this motion. The surface S parametrizes as $X(r,\theta)=(r\cos\theta, r\sin\theta, u(r))$, $r\in I\subset\mathbb{R}^+$, $\theta\in\mathbb{R}$, $u>0$ and $u'^2<1$. The computation of (7) gives a polynomial equation on the trigonometric functions $\{1,\sin\theta,\cos\theta\}$. Thus all three coefficients must vanish, obtaining

$$a(ru''+(\alpha+1)u'(1-u'^2))=0$$
$$b(ru''+(\alpha+1)u'(1-u'^2))=0$$
$$c(\alpha r(1-u'^2)+u(ru''+u'(1-u'^2)))=0.$$

If $ru''-(\alpha+1)u'(1-u'^2)\neq 0$, then $a=b=0$, proving that $\vec{a}=(0,0,c)$, hence L and $\vec{a}$ are parallel.

Suppose now that $ru''+(\alpha+1)u'(1-u'^2)=0$. Recall that $c\neq 0$ because $\vec{a}$ is a timelike vector. Combining this with the third equation, we find $uu'-r=0$. Solving this equation we obtain $u(r)=\sqrt{r^2+m^2}$, $m>0$, and the corresponding surface is the hyperbolic plane $\mathbb{H}^2(m)$. □

By Proposition 3.1, and after a horizontal translation, we will assume that the rotation axis is the z-axis and $\vec{a}=(0,0,1)$ in (7). We know that $X(r,\theta)=(r\cos\theta, r\sin\theta, u(r))$, where $r\in I\subset\mathbb{R}^+$, $\theta\in\mathbb{R}$ and $u>0$. By the proof of Proposition 3.1, (7) is written as

$$\frac{u''}{(1-u'^2)^{3/2}}+\frac{u'}{r\sqrt{1-u'^2}}=\frac{\alpha}{u\sqrt{1-u'^2}}, \tag{14}$$

or equivalently,

$$\frac{u''}{1-u'^2}+\frac{u'}{r}=\frac{\alpha}{u}. \tag{15}$$

We are interested in those solutions that meet the z-axis, that is, when $r=0$ is contained in the domain of the solution. Let us observe that (14) is singular at $r=0$ and thus the existence of solutions is not a direct consequence of standard ODE theory.

Multiplying (14) by r, and integrating by parts, we wish to establish the existence of a classical solution of

$$\begin{cases} \left(r\dfrac{u'}{\sqrt{1-u'^2}}\right)'=r\dfrac{\alpha}{u\sqrt{1-u'^2}}, & r\in(0,\delta),\\ u(0)=u_0>0, \quad u'(0)=0. \end{cases} \tag{16}$$

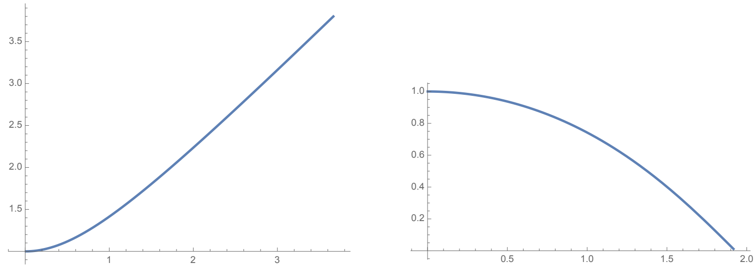

Figure 2. Solutions of (16). Left: case $\alpha > 0$, shown with $\alpha = 2$. Right: case $\alpha < 0$, shown with $\alpha = -1$.

Define the functions $\phi : (-1, 1) \to \mathbb{R}$ and $f : \mathbb{R}^+ \times (-1, 1) \to \mathbb{R}$ by

$$\phi(y) = \frac{y}{\sqrt{1-y^2}}, \qquad f(x, y) = \frac{\alpha}{x\sqrt{1-y^2}}.$$

Let $\delta > 0$. It is clear that a function $u \in C^2([0, \delta])$ is a solution of (16) if and only if $(r\phi(u'))' = rf(u, u')$ and $u(0) = u_0,\ u'(0) = 0$. Let $\mathcal{B} = (C^1([0, \delta]), \|\cdot\|)$ be the Banach space of the continuously differentiable functions on $[0, \delta]$ endowed with the usual norm

$$\|u\| = \|u\|_\infty + \|u'\|_\infty.$$

Define the operator $\mathsf{T} : \mathcal{B} \to \mathcal{B}$ by

$$(\mathsf{T}u)(r) = u_0 + \int_0^r \phi^{-1}\left(\int_0^s \frac{t}{s} f(u, u')\, dt\right) ds.$$

Notice that a fixed point of the operator T is a solution of the initial value problem (16). Indeed, $(\mathsf{T}u)' = \phi^{-1}\big(\frac{1}{r}\int_0^r tf(u, u')\, dt\big)$ and

$$r\phi(\mathsf{T}u') \int_0^r tf(u, u')\, dt,$$

obtaining the result. Moreover, $\mathsf{T}u(0) = u_0$ and

$$\phi(\mathsf{T}u)'(0) = \lim_{r\to 0} \frac{1}{r}\int_0^r tf(u, u')\, dt = \lim_{r\to 0} rf(u, u') = 0,$$

where in the second identity we have used the L'Hôpital rule. Thus, $(\mathsf{T}u)'(0) = 0$.

The existence of solutions of (16) follows now standard techniques of radial solutions for some equations of mean curvature type ([Bereanu et al. 2009; Corsato et al. 2015]). In Figure 2 we show the solutions of (16) when α is positive and negative.

Theorem 3.2. *The initial value problem* (16) *has a solution* $u \in C^2([0, \delta])$ *for some* $\delta > 0$ *that depends continuously on the initial data.*

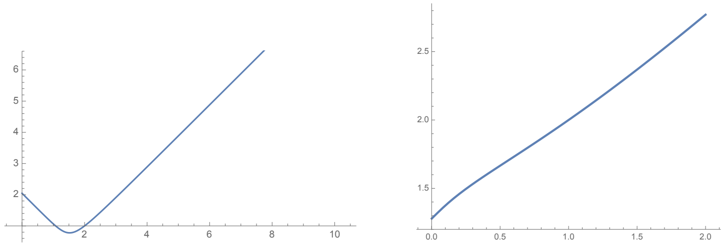

Figure 3. Solutions of (15), case $\alpha > 0$ and $u_0' \neq 0$. Left: case $u_0' = -1$. Right: case $u_0' = 1$.

Proof. In order to find a fixed point of T, we prove that T is a contraction in $\mathcal{B}$ for some $\delta > 0$ to be chosen. The functions f and ϕ^{-1} are locally Lipschitz continuous of constant $L > 1$ in $[u_0 - \epsilon, u_0 + \epsilon] \times [-\epsilon, \epsilon]$ and $[-\epsilon, \epsilon]$ respectively, provided $\epsilon < \{u_0, 1\}$. Since $\phi^{-1}(y) = y/\sqrt{1+y^2}$, then $L < 1$. Then for all $u, v \in \overline{B(0,\epsilon)}$ and for all $r \in [0, \delta]$,

$$\begin{aligned}
|(\mathsf{T}u)(r) - (\mathsf{T}v)(r)| &\leq L \int_0^r \left| \int_0^s \frac{t}{s}(f(u,u') - f(v,v'))\,dt \right| \\
&\leq L^2 \int_0^r \int_0^s \frac{t}{s} \|u - v\|\,dt = \frac{L^2}{4} r^2 \|u - v\|. \\
|(\mathsf{T}u)'(r) - (\mathsf{T}v)'(r)| &\leq \frac{L}{r} \left| \int_0^r t(f(u,u') - f(v,v'))\,dt \right| \\
&\leq \frac{L^2}{r} \int_0^r t\|u - v\|\,dt = \frac{L^2}{2} r \|u - v\|.
\end{aligned}$$

By choosing $\delta > 0$ small enough, we deduce that T is a contraction in the closed ball $\overline{B(0,\delta)} \subset \mathcal{B}$. Thus the Schauder point fixed theorem proves the existence of one fixed point of T, so the existence of a local solution of the initial value problem (16). This solution belongs to $C^1([0,\delta]) \cap C^2((0,\delta])$. The C^2-regularity up to 0 is verified directly by using the L'Hôpital rule because (14) leads to

$$\lim_{r\to 0} u''(r) + \lim_{r\to 0} \frac{u'(r)}{r} = \frac{\alpha}{u_0},$$

that is,

$$\lim_{r\to 0} u''(r) = \frac{\alpha}{2u_0}. \tag{17}$$

The continuous dependence of local solutions on the initial data is a consequence of the continuous dependence of the fixed points of T. $\square$

In the following result we describe the geometric properties of the rotational solutions of (15). See Figures 2, 3 and 4.

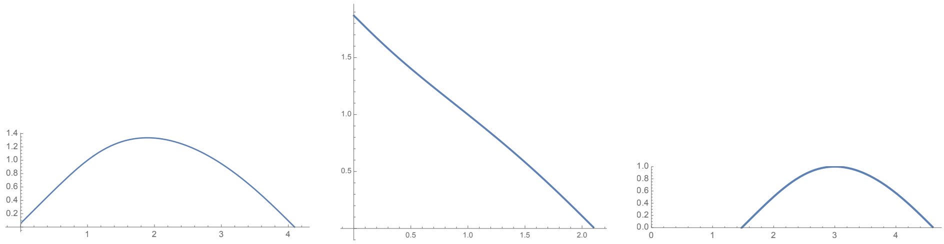

Figure 4. Solutions of (15), case $\alpha < 0$ and $u_0' \neq 0$. Left: case $u_0' = 1$. Middle: case $u_0' = -1$. Right: a solution that does not meet the rotation axis

Theorem 3.3. *Let u be a solution of (14) with $u > 0$ and $u'^2 < 1$.*

(1) *Case $\alpha > 0$. The maximal domain of u is $(0, \infty)$. Let $u_0' = \lim_{r\to 0} u'(r)$. Then we have the following cases: $u_0' = 0$ and the function u is increasing; $u_0' = -1$ and u has a unique critical point which is a global minimum; $u_0 = 1$ and the function is increasing. In all cases,*

$$\lim_{r\to\infty} u(r) = \infty. \tag{18}$$

Also, the function $u(r) = \sqrt{\alpha} r$ is a solution of (14).

(2) *Case $\alpha < 0$. The maximal domain of u is (a, b) with $0 \leq a < b < \infty$ and*

$$\lim_{r\to b} u(r) = 0, \quad \lim_{r\to b} u'(r) = -1.$$

If $a > 0$, then u has a global maximum and

$$\lim_{r\to a} u(r) = 0, \quad \lim_{r\to a} u'(r) = 1.$$

If $a = 0$, let $u_0' = \lim_{r\to 0} u'(r)$. Then we have the following cases: $u_0' = 0$ and u is a decreasing function; $u_0' = -1$ and u is a decreasing function; $u_0' = 1$ and u has a global maximum.

Proof. We observe that if u has a critical point at $r_o \geq 0$, then (15) implies $u''(r_o) = \alpha/u(r_o) \neq 0$, hence all critical points are all maxima or are all minima. Thus there is one critical point at most. In such a case, this point is a global minimum (resp. maximum) if $\alpha > 0$ (resp. $\alpha < 0$).

Claim A. If the graph of u meets the x-axis at $r_* > 0$, then $\alpha < 0$ and $\lim_{r\to r_*} u'(r)^2 = 1$.

The proof follows by multiplying (15) by $2u'$ and integrating. Then

$$\log(1 - u'(r)^2) + 2\alpha \log u(r) = 2\alpha \int^r \frac{u'(t)^2}{t}\, dt + \mu, \quad \mu \in \mathbb{R}.$$

In a neighborhood of r_*, the right-hand side of the above equation is finite. Since $\log(u(r)) \to -\infty$ as $r \to r_*$, the same occurs with $\lim_{r\to r_*} \log(1-u'(r)^2)$, proving that $u'(r)^2 \to 1$ as $r \to r_*$. Moreover, the case $\alpha > 0$ is not possible because the left-hand side would be $-\infty$.

Claim B. If the graph of u meets the y-axis, then $\lim_{r\to 0} u'(r)=0$ or $\lim_{r\to 0} u'(r)^2=1$.

Denote $u'_0=\lim_{r\to 0} u'(r)$. From Theorem 3.2, we know the existence of solutions when $u'_0=0$. Suppose now $u'_0 \neq 0$. By contradiction, we assume that ${u_0'}^2 \neq 1$. For $\delta > 0$ close to 0 and by (16),

$$\frac{ru'(r)}{\sqrt{1-u'(r)^2}} - \frac{\delta u'(\delta)}{\sqrt{1-u'(\delta)^2}} = \int_\delta^r \frac{\alpha t}{u\sqrt{1-u'^2}}\, dt. \tag{19}$$

Since ${u_0'}^2 \neq 1$, letting $r \to 0$ we have

$$\frac{u'(\delta)}{\sqrt{1-u'(\delta)^2}} = \frac{1}{\delta}\int_0^\delta \frac{\alpha t}{u\sqrt{1-u'^2}}\, dt.$$

Letting $\delta \to 0$ and by the L'Hôpital rule, we deduce

$$\lim_{\delta\to 0} \frac{u'(\delta)}{\sqrt{1-u'(\delta)^2}} = \lim_{\delta\to 0} \frac{\alpha\delta}{u\sqrt{1-u'^2}} = 0,$$

hence $u'_0=0$, a contradiction.

In particular, Claim B implies that it is not possible to find solutions of the initial value problem (16) when ${u_0'}^2 \in (0, 1)$.

From now, we will denote by $u(a)$ and $u'(a)$ (and similarly for $r=b$), the limit of $u(r)$ and $u'(r)$ at $r=a$.

Claim C. If $a > 0$ (resp. $b < \infty$), then $u(a) =$ (resp. $u(b) = 0$).

Suppose that $a > 0$ (and similarly for $b < \infty$). If $u(a) \neq 0$, then u'' is bounded around $r=a$ by (15). Since u' and u'' are bounded functions, we could extend the solution u beyond $r=a$, a contradiction.

We are in position to prove the theorem.

(1) Case $\alpha > 0$. Suppose that $u' > 0$ in all its domain. Since u' and u'' are bounded functions by (15), then the value of b in I is $b=\infty$. If $a > 0$, this implies that $u(a) = 0$ by Claim C and this is a contradiction by Claim A. This proves that $I = (0, \infty)$.

Suppose that the sign of u' is negative in all its domain. Then (15) implies that u is a concave function and thus $b < \infty$ because u is decreasing. Then $u(b)=0$, which is not possible by Claim A.

With the above arguments, we have proved that if u' has a constant sign, then $u' > 0$, $a=0$ and either $u'_0=0$ or $u'_0=1$. In the case where u' changes sign, then there is a unique critical point at some point $r=r_o > 0$, which is a global minimum.

In this case, $u' > 0$ for $r > r_o$. Since u' and u'' are bounded, then $b = \infty$. The case $a > 0$ is forbidden by Claim C. Thus $a = 0$. Since $u' < 0$ for $r < r_o$, Claim B asserts $u'_0 = -1$.

We prove (18). Since u is increasing close to ∞, let $c = \lim_{r\to\infty} u(r)$. If $c < \infty$, then $u'(r) \to 0$ as $r \to \infty$ and using (15), $\lim_{r\to\infty} u''(r) = \alpha/c > 0$, a contradiction. Thus $c = \infty$.

Finally, by a direct computation, we observe that $u(r) = \sqrt{\alpha} r$ is a solution of (14).

(2) Case $\alpha < 0$. Suppose that $u' < 0$ in all its domain. Let $c = \lim_{r\to b} u(r) \geq 0$. If $b = \infty$, then $u'(r) \to 0$ and (15) would imply that $\lim_{r\to\infty} u''(r)$ is either α/c if $c > 0$ or ∞ if $c = 0$, a contradiction. Thus $b < \infty$, hence $u(b) = 0$. By Claim A, $u'(b) = -1$. If $a > 0$, then $u(a) > 0$ because u is decreasing; by Claim C, this is a contradiction. Thus $a = 0$. By Claim B and because u is decreasing, we have two possibilities, namely, $u'_0 = 0$ and $u'_0 = -1$.

Suppose now that $u' > 0$ in all its domain. Then u is a concave function by (15). By Claim C, we have $b = \infty$. In the other end of the interval I, namely $r = a$, we have $a = 0$, $u'_0 = 1$ or $a > 0$, $u(a) = 0$ and $u'(a) = 1$. In both cases, as $u'' < 0$, we find $\lim_{r\to\infty} u'(r) = \lim_{r\to\infty} u''(r) = 0$ and $\lim_{r\to\infty} u(r) = \infty$. By (19)

$$\frac{u'(r)}{\sqrt{1-u'(r)^2}} = \frac{1}{r}\int_\delta^r \frac{\alpha t}{u\sqrt{1-u'^2}}\,dt + \mu.$$

Letting $r \to \infty$, the left-hand side is 0. However, and applying twice the L'Hôpital rule, the limit of the right-hand side is

$$\lim_{r\to\infty} \frac{\alpha r}{u(r)} + \mu = \lim_{r\to\infty} \frac{\alpha}{u'(r)} + \mu = \infty,$$

giving a contradiction.

So, if $u' > 0$ at some point, there is a critical point r_o of u, which will be the global maximum of u. Then $a \geq 0$ with $u'(a) = 1$ because u is increasing in (a, r_o). $\square$

Remark 3.4. If $\alpha < 0$, there exist solutions that do not meet the rotation axis; see Figure 4 (right). This case appears if $0 < a < b < \infty$, where the function u has a global maximum and $u'(a) = 1 = -u'(b)$. This extends the same property of the solution of (4), where the part of the function $u = u(x)$ given in (5) that lies over the x-axis is formed by successive bounded intervals.

4. The Dirichlet problem

The Dirichlet problem of the singular maximal surface equation asks if given a positive function $\varphi : \partial\Omega \to \mathbb{R}$ defined in a bounded domain $\Omega \subset \mathbb{R}^2$, there exists a smooth positive function $u : \overline{\Omega} \to \mathbb{R}$ such that (9) holds in Ω, $u = \varphi$ on $\partial\Omega$

and $|Du| < 1$ on $\overline{\Omega}$. Since any curve in a spacelike surface must be spacelike, the graph Γ of φ is spacelike. The problem is to determine the type of function φ and the boundary $\partial\Omega$ for the solvability of the Dirichlet problem. It is expected that the sign α in (9) plays an important role because we have seen in Sections 2 and 3 the contrast of the behavior of the invariant solutions of (7) depending on whether α is positive or negative.

Following similar ideas in [Jenkins and Serrin 1968; Serrin 1969], we will solve the Dirichlet problem if the domain Ω is mean convex. In fact, we will establish the Dirichlet problem in the n-dimensional case, or equivalently, we will find singular maximal hypersurfaces in the $(n+1)$-dimensional Lorentz–Minkowski space $\mathbb{L}^{n+1}$ with prescribed boundary data.

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is said to be mean convex if $\partial\Omega$ has nonnegative mean curvature $H_{\partial\Omega}$ with respect to the inward orientation. In the case $n = 2$, the mean convexity property is equivalent to the convexity of Ω, but in arbitrary dimensions, the mean convexity is less restrictive than convexity.

The Dirichlet problem is now formulated as follows. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $\alpha \neq 0$ a given constant. Let $\varphi : \partial\Omega \to \mathbb{R}$ be a positive spacelike smooth function. The problem is finding a classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $u > 0$ in $\overline{\Omega}$, of

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1-|Du|^2}}\right) = \frac{\alpha}{u\sqrt{1-|Du|^2}} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ |Du| < 1 & \text{in } \overline{\Omega}. \end{cases} \tag{20}$$

We solve the Dirichlet problem when the boundary data φ has a spacelike extension in $\overline{\Omega}$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded mean convex domain with smooth boundary $\partial\Omega$. Assume that $\alpha < 0$. If $\varphi \in C^2(\overline{\Omega})$ is a positive function with* $\max_{\overline{\Omega}}|D\varphi| < 1$, *then there is a unique positive solution u of* (20).

The proof of Theorem 4.1 is accomplished by using the Schauder theory of a priori global estimates, the method of continuity and the Leray–Schauder fixed point theorem. Applying these techniques, we find all elements for proving Theorem 4.1. As usual, we will utilize the distance function d to $\partial\Omega$ to construct a barrier function ([Gilbarg and Trudinger 1983; Jenkins and Serrin 1968; Ladyzhenskaya and Uraltseva 1961]).

The C^0 estimates will be obtained by comparing the solution of (20) with the rotational examples studied in Section 3: here the hypothesis $\alpha < 0$ will be essential because if $\alpha > 0$ it is not possible to prevent that $|u| \to 0$ for a solution u. For the C^1 estimates, we need to prove that $|Du|$ is bounded away from 1 which will be deduced by using barrier functions. Finally, the hypothesis $\alpha < 0$ will be also used when we apply the implicit function theorem for the existence of the linearized problem associated to (20).

The maximum principle for elliptic equations of divergence type implies the following result.

Proposition 4.2 (touching principle). *Let Σ_1 and Σ_2 be two α-singular maximal surfaces. If Σ_1 and Σ_2 have a common tangent interior point and Σ_1 lies above Σ_2 around p, then Σ_1 and Σ_2 coincide at an open set around p.*

We also need to formulate the comparison principle in the context of α-singular maximal surfaces. We write the equation of (20) in classical notation. Define the operator

$$Q[u] = (1-|Du|^2)\Delta u + u_i u_j u_{ij} - \frac{\alpha(1-|Du|^2)}{u} \tag{21}$$
$$= a_{ij}(Du)u_{ij} + \boldsymbol{b}(u, Du),$$

where

$$a_{ij} = (1-|Du|^2)\delta_{ij} + u_i u_j, \quad \boldsymbol{b} = -\frac{\alpha(1-|Du|^2)}{u}.$$

Here $u_i = \partial u/\partial x_i$, $1 \le i \le n$, and we assume the summation convention of repeated indices. It is immediate that u is a solution of (20) if and only if $Q[u] = 0$. The ellipticity of the operator Q is clear because if $A = (a_{ij})$ and $\xi \in \mathbb{R}^n$, then

$$(1-|p|^2)|\xi|^2 \le \xi^t A\xi = (1-|p|^2)|\xi|^2 + \langle p, \xi\rangle^2 \le |\xi|^2. \tag{22}$$

Moreover, this shows that Q is not uniformly elliptic. We recall the comparison principle ([Gilbarg and Trudinger 1983, Theorem 10.1]).

Proposition 4.3 (comparison principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $Q[u] \ge Q[v]$ and $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.*

Notice that if $\alpha < 0$, the classical theory implies the uniqueness of solutions of the Dirichlet problem.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha < 0$. The solution of (20), if it exists, is unique.*

In arbitrary dimension, any horizontal translation and any dilation from a point of $\mathbb{R}^n \times \{0\}$ preserve (20). Similarly, Theorem 3.3 holds where now (16) is

$$\left(r\frac{u'}{\sqrt{1-u'^2}}\right)' = r^{n-1}\frac{\alpha}{u\sqrt{1-u'^2}}.$$

We establish the solvability of (20) in the particular case that Ω is a ball of $\mathbb{R}^n$ and φ is a positive constant.

Proposition 4.5. *Let $\alpha < 0$ and $B_R \subset \mathbb{R}^n$ be a round ball of radius $R > 0$. If $c > 0$, then there is a unique radial solution u of (20) in B_R with $u = c$ on ∂B_R.*

Proof. After a horizontal translation, we suppose that the origin $O \in \mathbb{R}^n$ is the center of B_R. By Theorem 3.2, let $v = v(r)$ be the solution of (16) with $v(0) = 1$. Recall that Theorem 3.3 asserts that the maximal domain of v is a ball B_b for some $b > 0$ with $v(b) = 0$. In the (r, v)-plane, consider the line $x_{n+1} = cr/R$. Since v is a decreasing function, the graph of v meets this line at one point $r = r_o$, $u(r_o) = cr_o/R$. If $\lambda = R/r_o$, then $u_\lambda(r) = \lambda u(r/\lambda)$ is a solution of (20) with $u_\lambda(R) = c$. □

Following a standard scheme, we start by finding C^0 estimates by using the rotational solutions of (7). In the following result, we do not require the mean convexity of Ω.

Proposition 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha < 0$. If u is a positive solution of (20), there exists a constant $C_1 = C_1(\alpha, \Omega, \varphi) > 0$ such that*

$$\min_{\partial\Omega} \varphi \leq u \leq C_1 \quad \text{in } \Omega. \tag{23}$$

Proof. Since the right-hand side of (20) is negative, then $\inf_\Omega u = \min_{\partial\Omega} \varphi$ by the maximum principle. This proves the left inequality of (23).

For the upper estimate of (23), we consider the radial solution v of (16) with $v(0) = 1$ and let $\{v_\lambda : \lambda > 0\}$ where $v_\lambda(r) = \lambda v(r/\lambda)$. Denote B_R the maximal domain of v, with $v(R) = 0$ and let Σ_λ denote the graph of v_λ. Take $\lambda > 0$ sufficiently big so the graph S of u is included in the domain of the halfspace $x_{n+1} > 0$ bounded by $\Sigma_\lambda \cup B_{\lambda R}$. Let λ decrease to 0 until the first time λ_0 that Σ_λ meets Σ_u. By the maximum principle, the first contact must occur at some boundary point of S. Then this point is a point of $\partial\Omega$ or a point of ∂S. Since ∂S is the graph of φ, this value λ_0 depends on Ω and φ. Consequently, $u \leq v_{\lambda_0} \leq \sup_\Omega v_{\lambda_0}$. The proof finishes by letting $C_1 = \sup_\Omega v_{\lambda_0}$, which depends only on α, Ω and φ. □

The next step to prove Theorem 4.1 is the derivation of estimates for $|Du|$. This is done by first proving the supremum of $|Du|$ is attained at some boundary point. In the next result, the assumption that α is negative is essential.

Proposition 4.7 (interior gradient estimates). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\alpha < 0$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a positive solution of (20), then*

$$\max_{\overline{\Omega}} |Du| = \max_{\partial\Omega} |Du|.$$

Proof. Let $v^k = u_k$, $1 \leq k \leq n$. By differentiating $Q[u] = 0$ with respect to x_k, we find for each k,

$$((1-|Du|^2)\delta_{ij} + u_i u_j) v^k_{ij} + 2\left(u_i \Delta u + u_j u_{ij} - \frac{\alpha u_i}{u}\right) v^k_i + \frac{\alpha(1-|Du|^2)}{u^2} v^k = 0. \tag{24}$$

The equation (24) is a linear elliptic equation in v^k. Because $\alpha < 0$, the coefficient for v^k is negative and the maximum principle ([Gilbarg and Trudinger 1983, Theorem 3.7]) implies that $|v^k|$, and hence $|Du|$, does not have an interior maximum. In particular, the maximum of $|Du|$ on $\overline{\Omega}$ is attained at some boundary point, proving the result. □

As a consequence of Proposition 4.7, the problem of finding a priori estimates of $|Du|$ reduces to get these estimates along $\partial\Omega$. With this purpose, we prove that u admits barriers from above and from below along $\partial\Omega$. We now use the assumption of the mean convexity of Ω.

Proposition 4.8 (boundary gradient estimates). *Let $\Omega \subset \mathbb{R}^n$ be a bounded mean convex domain and $\alpha < 0$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a positive solution of (20), then there is a constant*

$$C_2 = C_2(\alpha, \Omega, C_1, \|\varphi\|_{1;\overline{\Omega}}, \|\varphi\|_{2;\overline{\Omega}}) < 1$$

such that

$$\max_{\partial\Omega} |Du| \le C_2.$$

Proof. We consider the operator $Q[u]$ defined in (21). For a lower barrier for u, we take the solution v^0 of the Dirichlet problem of the maximal surface equation in Ω with the same boundary φ. The function v^0 is the solution of (20) for $\alpha = 0$ whose existence is assured ([Bartnik and Simon 1982/83, Theorem 4.1]). Then

$$Q[v^0] = -\frac{\alpha(1 - |Dv^0|^2)}{v^0} > 0 = Q[u].$$

Since $v^0 = u$ on $\partial\Omega$, we conclude $v^0 < u$ in Ω by the comparison principle.

We now construct an upper barrier for u by means of the distance function in a small tubular neighborhood of $\partial\Omega$ in Ω.

Consider the distance function $d(x) = \operatorname{dist}(x, \partial\Omega)$ and let $\epsilon > 0$ sufficiently small so $\mathcal{N}_\epsilon = \{x \in \overline{\Omega} : d(x) < \epsilon\}$ is a tubular neighborhood of $\partial\Omega$. We parametrize $\mathcal{N}_\epsilon$ using normal coordinates $x \equiv (t, \pi(x)) \in \mathcal{N}_\epsilon$, where $x \equiv \pi(x) + t\nu(\pi(x))$ for some $t \in [0, \epsilon)$, $\pi : \mathcal{N}_\epsilon \to \partial\Omega$ is the orthogonal projection and ν is the unit normal vector to $\partial\Omega$ pointing to Ω. A straightforward computation gives that d is C^2, $|Dd|(x) = 1$, and $\Delta d(x) \le -(n-1)H_{\partial\Omega}(\pi(x))$ for all $x \in \mathcal{N}_\epsilon$. Because Ω is mean convex, then $\Delta d(x) \le 0$.

Define in $\mathcal{N}_\epsilon$ a function $w = h \circ d + \varphi$, where we use the same symbol φ for a spacelike extension of φ into Ω. The function h is defined as

$$h(t) = a \log(1 + kb^2 t), \qquad b, k > 0, \qquad a = \frac{C_1}{\log(1+b)}, \tag{25}$$

where C_1 is the constant that appears in (23) and b and k will be chosen later. Let us observe that $w > 0$ and that we require that $|Dw| < 1$. The computation of $Q[w]$

leads to

$$Q[w] = a_{ij}(h'' d_i d_j + h' d_{ij} + \varphi_{ij}) - \frac{\alpha}{w}(1 - |Dw|^2).$$

From $|Dd| = 1$, it follows that $\langle D(Dd)_x \xi, Dd(x)\rangle = 0$ for all $\xi \in \mathbb{R}^n$. If $\{e_i\}_i$ is the canonical basis of $\mathbb{R}^n$ and $\xi = e_i$, we find $d_{ij} d_j = 0$. Thus

$$\begin{aligned} w_i w_j d_{ij} &= (h' d_i + \varphi_i)(h' d_j + \varphi_j) d_{ij} = (h'^2 d_i + 2h' \varphi_i) d_j d_{ij} + \varphi_i \varphi_j d_{ij} \\ &= \varphi_i \varphi_j d_{ij} \le \varphi_i^2 \lambda_i^d \le 0, \end{aligned}$$

where λ_i^d are the eigenvalues of $D^2 d$, which all are not positive because $D^2 d$ is negative semidefinite. Using this inequality, the definition of a_{ij} in (21) and (22), it follows that

$$a_{ij} d_{ij} = (1 - |Dw|^2)\Delta d + w_i w_j d_{ij} \le (1 - |Dw|^2)\Delta d \le 0.$$

Again (22) implies $a_{ij} d_i d_j \ge 1 - |Dw|^2$ and $a_{ij} \varphi_{ij} \le |D^2 \varphi|$, where $|D^2 \varphi| = \sum_{ij} \sup_{\overline{\Omega}} |\varphi_{ij}|$. Since $h' > 0$ and $\Delta d \le 0$, we find

$$\begin{aligned} Q[w] &\le h''(1 - |Dw|^2) + h' \Delta d (1 - |Dw|^2) - \frac{\alpha}{w}(1 - |Dw|^2) + a_{ij} \varphi_{ij} \\ &\le \left(h'' - \frac{\alpha}{w}\right)(1 - |Dw|^2) + |D^2 \varphi|. \end{aligned} \tag{26}$$

We now study the spacelike condition $|Dw| < 1$. The computation of $|Dw|$ and the Cauchy–Schwarz inequality gives

$$|Dw|^2 = h'^2 + |D\varphi|^2 + 2h' \langle Dd, D\varphi \rangle \le (h' + |D\varphi|)^2.$$

Because $h' > 0$ and h' is decreasing on t, we deduce

$$|Dw| \le h' + |D\varphi| \le h'(0) + |D\varphi| \le akb^2 + \mu \quad \text{in } \overline{\Omega}, \tag{27}$$

where $\mu = \|D\varphi\|_{0;\overline{\Omega}} < 1$. Fix a constant δ with the property $\mu < \delta < 1$. Then it is possible to choose k sufficiently small in (27) so $|Dw| \le akb^2 + \mu < \delta$. Let $\beta = 1 - \delta^2$. If $h'' - \alpha/w < 0$, then (26) implies

$$Q[w] \le \beta \left(h'' - \frac{\alpha}{w}\right) + \|D^2 \varphi\|_{0;\overline{\Omega}}. \tag{28}$$

The right-hand side in (28) is a function defined in $\partial\Omega \times [0, \epsilon]$. Let $\varphi_0 = \min_{\overline{\Omega}} \varphi > 0$ and we evaluate this function at $t = 0$, obtaining

$$\beta\left(-ak^2 b^4 - \frac{\alpha}{\varphi}\right) + \|D^2 \varphi\|_{0;\overline{\Omega}} \le \beta\left(-\frac{(\delta - \mu)^2}{a} - \frac{\alpha}{\varphi_0}\right) + \|D^2 \varphi\|_{0;\overline{\Omega}}.$$

If b is sufficiently big, then $a \to 0$, hence the right-hand side in this inequality is negative. By compactness of $\partial\Omega \times [0, \epsilon]$ and by continuity, let us take b sufficiently large in (28) so $Q[w] < 0$. Even more, we require b so large that $1/(kb) < \epsilon$. We now change the tubular neighborhood $\mathcal{N}_\epsilon$ by replacing ϵ by $\epsilon = 1/(kb)$ and we denote this by $\mathcal{N}_\epsilon$ again.

In order to assure that w is a local upper barrier in $\mathcal{N}_\epsilon$ for the Dirichlet problem (20), we need to have

$$u \le w \quad \text{in } \partial\mathcal{N}_\epsilon. \tag{29}$$

In $\partial\mathcal{N}_\epsilon \cap \partial\Omega$, the distance function is $d = 0$, so $w = \varphi = u$. On the other hand, in $\partial\mathcal{N}_\epsilon \setminus \partial\Omega$, and because $\epsilon = 1/(kb)$, we find

$$w = h(\epsilon) + \varphi = \frac{C_1}{\log(1+b)} \log(1 + kb^2\epsilon) + \varphi = C_1 + \varphi.$$

By Proposition 4.6, we have $u \le C_1$ and we deduce $u < w$ in $\mathcal{N}_\epsilon \setminus \partial\Omega$. Definitively, we find $Q[w] < 0 = Q[u]$ and $u \le w$ in $\partial\mathcal{N}_\epsilon$, concluding that $u \le w$ in $\mathcal{N}_\epsilon$ by the comparison principle.

Consequently, we have proved the existence of lower and upper barriers for u in $\mathcal{N}_\epsilon$, namely, $v^0 \le u \le w$ in $\mathcal{N}_\epsilon$. Hence we deduce

$$\max_{\partial\Omega} |Du| \le C_2 := \max\{\|Dw\|_{0;\bar{\Omega}}, \|Dv^0\|_{0;\bar{\Omega}}\}$$

and both values $\|Dw\|_{0;\bar{\Omega}}$, $\|Dv^0\|_{0;\bar{\Omega}}$ depend only on the initial data of the Dirichlet problem. This completes the proof of proposition. □

With all of the above ingredients, we are in position to prove Theorem 4.1.

Proof of Theorem 4.1. We establish the solvability of the Dirichlet problem (20) by the method of continuity (see [Gilbarg and Trudinger 1983, Section 17.2]). Define the family of Dirichlet problems parametrized by $t \in [0, 1]$

$$\mathcal{P}_t : \begin{cases} Q_t[u] = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where

$$Q_t[u] = (1 - |Du|^2)\Delta u + u_i u_j u_{ij} - \frac{\alpha t(1 - |Du|^2)}{u}.$$

The graph Σ_{u_t} of a solution of u_t is a $(t\alpha)$-singular maximal surface. Notice that if $t = 0$, $Q_0[u] = 0$ is the maximal surface equation and the solution of $\mathcal{P}_0$ is the function v^0 that appeared in Proposition 4.8. As usual, let

$$\mathcal{A} = \{t \in [0, 1] : \text{there exists } u_t \in C^{2,\gamma}(\bar{\Omega}),\, u_t > 0,\, Q_t[u_t] = 0,\, u_{t|\partial\Omega} = \varphi\}.$$

The proof consists of showing that $1 \in \mathcal{A}$. For this, we prove that $\mathcal{A}$ is a nonempty open and closed subset of $[0, 1]$.

(1) The set $\mathcal{A}$ is not empty. This is because $0 \in \mathcal{A}$ since v^0 is the solution of $\mathcal{P}_0$.

(2) The set $\mathcal{A}$ is open in [0, 1]. Given $t_0 \in \mathcal{A}$, we need to prove that there is an $\epsilon > 0$ such that $(t_0-\epsilon, t_0+\epsilon)\cap[0, 1] \subset \mathcal{A}$. Define the map $T(t, u) = Q_t[u]$ for $t \in \mathbb{R}$ and $u \in C^{2,\gamma}(\overline{\Omega})$. Then $t_0 \in \mathcal{A}$ if and only if $T(t_0, u_{t_0}) = 0$. If we show that the derivative of Q_t with respect to u, say $(DQ_t)_u$, at the point u_{t_0} is an isomorphism, then from the implicit function theorem we have the existence of an open set $\mathcal{V} \subset C^{2,\gamma}(\overline{\Omega})$, with $u_{t_0} \in \mathcal{V}$, and a C^1 function $\xi : (t_0-\epsilon, t_0+\epsilon) \to \mathcal{V}$ for some $\epsilon > 0$, such that $\xi(t_0) = u_{t_0} > 0$ and $T(t, \xi(t)) = 0$ for all $t \in (t_0-\epsilon, t_0+\epsilon)$; this guarantees that $\mathcal{A}$ is an open set of [0, 1].

The proof that $(DQ_t)_u$ is one-to-one is equivalent to proving that for any $f \in C^{\gamma}(\overline{\Omega})$, there is a unique solution $v \in C^{2,\gamma}(\overline{\Omega})$ of the linear equation $Lv := (DQ_t)_u(v) = f$ in Ω and $v = \varphi$ on $\partial\Omega$. The computation of L was done in Proposition 4.7, obtaining

$$Lv = (DQ_t)_u v = a_{ij}(Du)v_{ij} + \boldsymbol{b}_i(u, Du, D^2u)v_i + \boldsymbol{c}(u, Du)v,$$

where a_{ij} are defined in (21) and

$$\boldsymbol{b}_i = 2\left(\Delta u - \frac{\alpha t}{u}\right)u_i + 2u_j u_{ij}, \qquad \boldsymbol{c} = \frac{\alpha t(1-|Du|^2)}{u^2}.$$

Since $\alpha < 0$, $\boldsymbol{c} \le 0$ and the existence and uniqueness is assured by standard theory ([Gilbarg and Trudinger 1983, Theorem 6.14]).

The set $\mathcal{A}$ is closed in [0, 1]. Let $\{t_k\} \subset \mathcal{A}$ with $t_k \to t \in [0, 1]$. For each $k \in \mathbb{N}$, there exists $u_{t_k} \in C^{2,\gamma}(\overline{\Omega})$, $u_{t_k} > 0$, such that $Q_{t_k}[u_{t_k}] = 0$ in Ω and $u_{t_k} = \varphi$ in $\partial\Omega$. Define the set

$$\mathcal{S} = \{u \in C^{2,\gamma}(\overline{\Omega}) : \text{there exists } t \in [0, 1] \text{ such that } Q_t[u] = 0 \text{ in } \Omega, u_{|\partial\Omega} = \varphi\}.$$

Then $\{u_{t_k}\} \subset \mathcal{S}$. If we prove that the set $\mathcal{S}$ is bounded in $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in [0, \gamma]$, and since $a_{ij} = a_{ij}(Du)$ in (21), then Schauder theory proves that $\mathcal{S}$ is bounded in $C^{2,\beta}(\overline{\Omega})$, in particular, $\mathcal{S}$ is precompact in $C^2(\overline{\Omega})$ (see Theorem 6.6 and Lemma 6.36 in [Gilbarg and Trudinger 1983]). Thus there exists a subsequence $\{u_{k_l}\} \subset \{u_{t_k}\}$ converging in $C^2(\overline{\Omega})$ to some $u \in C^2(\overline{\Omega})$. Since $T : [0, 1] \times C^2(\overline{\Omega}) \to C^0(\overline{\Omega})$ is continuous, it follows that $Q_t[u] = T(t, u) = \lim_{l\to\infty} T(t_{k_l}, u_{k_l}) = 0$ in Ω. Moreover, $u_{|\partial\Omega} = \lim_{l\to\infty} u_{k_l|\partial\Omega} = \varphi$ on $\partial\Omega$, so $u \in C^{2,\gamma}(\overline{\Omega})$ and consequently, $t \in \mathcal{A}$.

The above reasoning asserts that $\mathcal{A}$ is closed in [0, 1] provided we find a constant M independent of $t \in \mathcal{A}$, such that

$$\|u_t\|_{C^1(\overline{\Omega})} = \sup_{\Omega}|u_t| + \sup_{\Omega}|Du_t| \le M.$$

However the C^0 and C^1 estimates for the function u_1, that is, when the parameter t is $t = 1$, are enough as we now see.

The C^0 estimates for u_t follow with the comparison principle. Indeed, let $t_1 < t_2$, $t_i \in [0, 1]$, $i = 1, 2$. Then $Q_{t_1}[u_{t_1}] = 0$ and

$$Q_{t_1}[u_{t_2}] = -\frac{(t_1 - t_2)\alpha(1 - |Du_{t_2}|^2)}{u_{t_2}} < 0 = Q_{t_1}[u_{t_1}]$$

because $\alpha < 0$. Since $u_{t_1} = \varphi = u_{t_2}$ on $\partial\Omega$, the comparison principle yields $u_{t_1} < u_{t_2}$ in Ω. This proves that the solutions u_{t_i} are ordered in an increasing sense according the parameter t. By (23), we find

$$\sup_\Omega u_t \le \sup_\Omega u_1 \le C_1. \tag{30}$$

In order to derive the gradient estimates for the solution u_t, the same computations obtained in Proposition 4.8 conclude that $\sup_{\partial\Omega}|Du_t|$ is bounded by a constant depending on α, Ω, φ and $\|u_t\|_{0;\bar\Omega}$. Now (30) implies that the value $\|u_t\|_{0;\bar\Omega}$ is bounded by C_1, which depends only on α, φ and Ω, but not on t.

The above three steps prove the existence part in Theorem 4.1. The uniqueness is consequence of Proposition 4.4 and this completes the proof of theorem. □

A consequence of Theorem 4.1 is the solvability of the Plateau problem if $\alpha < 0$ in the following situation.

Corollary 4.9. *Let Γ be a spacelike $(n-1)$-submanifold of $\mathbb{L}^{n+1}$ with a one-to-one orthogonal projection C on the hyperplane of equation $x_{n+1} = 0$ such that C is the boundary of a mean convex simply-connected domain Ω. Let $\alpha < 0$. If Γ has a spacelike extension to a graph on Ω, then there exists a unique α-singular maximal hypersurface S spanning Γ. Moreover, S is a graph on Ω.*

Proof. Theorem 4.1 asserts the existence of an α-singular maximal hypersurface S whose boundary is Γ and S is a graph on Ω. Assume that M is another such hypersurface. The property that M is spacelike implies that the orthogonal projection $p : \mathbb{R}^{n+1} \to \mathbb{R}^n = \mathbb{R}^n \times \{0\}$, $p(x) = (x_1, \ldots, x_n)$ is a local diffeomorphism between M and Ω. In particular, $p : M \to \Omega$ is a covering map and since Ω is simply connected, the map p is a diffeomorphism, in particular, M is a graph on Ω. Finally, the uniqueness of (7) when α is negative concludes that $M = S$. □

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SCHWARZ D-SURFACES IN Nil_3

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We construct a two parameter family of complete embedded triply periodic minimal surfaces in Nil_3, which are the analogues of the three parameter family of the Schwarz D-surfaces in the Euclidean three space.

1. Introduction

Take a polygon in $\mathbb{R}^3$ consisting of three pairs of oppositely parallel edges of a cube (Figure 1) and solve the Plateau problem with respect to this polygon to get an embedded minimal disk, and then reflect this surface successively with respect to the edges. Continuing this process indefinitely one gets an embedded completely triply periodic minimal surface, which is called the Schwarz D-surface.

In this paper we construct a two parameter family of embedded complete triply periodic minimal surfaces in Nil_3, which are the analogues of this Schwarz D-surfaces in the Euclidean three space. In Nil_3, a reflection with respect to a geodesic is not necessarily an isometry, however, reflections with respect to the horizontal or vertical geodesic are isometries. As in the case in [Shin et al. 2018], we suitably take a geodesic polygon consisting of the segments of horizontal and vertical geodesics only to get an embedded minimal disk. The polygon for the Schwarz D-surface in Figure 1 consists of the traces of the integral curves of the vector fields ∂x, ∂y and ∂z, starting from (0, 0, 0) in the order of ∂x, ∂y, ∂z, ∂x, ∂y and ∂z completing a Hamiltonian circuit. Our geodesic polygon consists of the traces of the integral curves of the vector fields $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, starting from (0, 0, 0) in the order of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ completing a Hamiltonian circuit (see Figure 2). On the other hand, the geodesic polygon for the Schwarz CLP-surface in Nil_3 in our earlier paper [Shin et al. 2019] consists of the traces of the integral curves of the vector fields $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ too, however, in the order of $\boldsymbol{e}_1$, $\boldsymbol{e}_3$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{e}_2$. Then our construction consists of the successive reflections with respect to boundary geodesics. The

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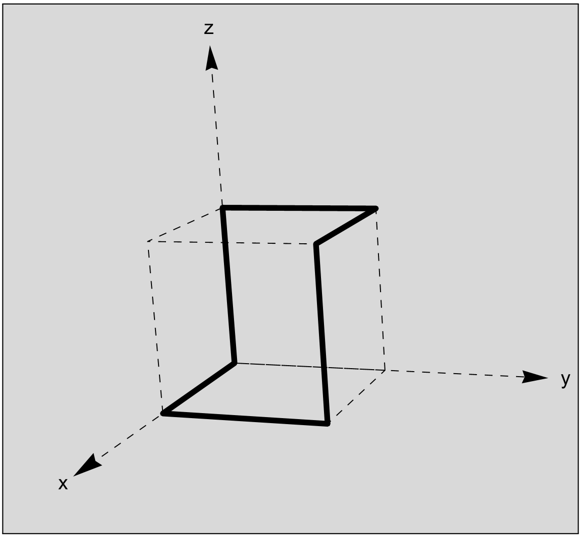

Figure 1. Geodesic polygon for Schwarz D-surface.

different surfaces obtained after reflections are still smooth because the surfaces considered are conformal embeddings of a disk that extends continuously up to the boundary and each piece of boundary is sent to a piece of the geodesic line along which the successive reflections are to be applied. Since the glued surface with the reflected surface is smooth along the geodesic line of the reflection, a crucial issue in our construction lies in how to make the pieces fit together along the boundary geodesic to form a smooth embedded surface.

2. Nil_3

The three-dimensional Heisenberg group is the Lie group $(\mathbb{R}^3, *)$, with $*$ defined as

$$(x, y, z) * (x', y', z') = \big(x + x', y + y', z + z' + \tfrac{1}{2}(xy' - x'y)\big).$$

The identity element is (0, 0, 0) and the inverse element of (x, y, z) is $(-x, -y, -z)$. We denote by Nil_3 the three-dimensional Heisenberg group endowed with the left-invariant metric

$$g = dx^2 + dy^2 + \big(dz + \tfrac{1}{2}(ydx - xdy)\big)^2.$$

It is a Riemannian fibration over the Euclidean plane $\mathbb{R}^2$ with the projection

$$\pi : (x, y, z) \mapsto (x, y).$$

For the left-invariant orthonormal frame field

$$\boldsymbol{e}_1 := \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad \boldsymbol{e}_2 := \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad \boldsymbol{e}_3 := \frac{\partial}{\partial z},$$

the Koszul formula gives

$$\begin{array}{lll} \nabla_{\boldsymbol{e}_1}\boldsymbol{e}_1 = 0, & \nabla_{\boldsymbol{e}_2}\boldsymbol{e}_1 = -\frac{1}{2}\boldsymbol{e}_3, & \nabla_{\boldsymbol{e}_3}\boldsymbol{e}_1 = -\frac{1}{2}\boldsymbol{e}_2, \\ \nabla_{\boldsymbol{e}_1}\boldsymbol{e}_2 = \frac{1}{2}\boldsymbol{e}_3, & \nabla_{\boldsymbol{e}_2}\boldsymbol{e}_2 = 0, & \nabla_{\boldsymbol{e}_3}\boldsymbol{e}_2 = \frac{1}{2}\boldsymbol{e}_1, \\ \nabla_{\boldsymbol{e}_1}\boldsymbol{e}_3 = -\frac{1}{2}\boldsymbol{e}_2, & \nabla_{\boldsymbol{e}_2}\boldsymbol{e}_3 = \frac{1}{2}\boldsymbol{e}_1, & \nabla_{\boldsymbol{e}_3}\boldsymbol{e}_3 = 0, \end{array}$$

from which one can see that any integral curve of the vector field $\boldsymbol{e}_i$, $i = 1, 2, 3$, is a geodesic.

Let $\gamma^i_{(a,b,c)}(t)$ be the integral curves of $\boldsymbol{e}_i$ with $\gamma^i_{(a,b,c)}(0) = (a, b, c)$, $i = 1, 2, 3$. Then, since each $\boldsymbol{e}_i$ is left invariant, one can see that the traces $\{\gamma^1_{(0,0,0)}\}$ of the integral curve $\gamma^1_{(0,0,0)}(t)$ is the x-axis $\{(t, 0, 0)\}$, the trace $\{\gamma^2_{(0,0,0)}\}$ of the integral curve $\gamma^2_{(0,0,0)}(t)$ is the y-axis $\{(0, t, 0)\}$ and the trace $\{\gamma^3_{(0,0,0)}\}$ of the integral curve $\gamma^3_{(0,0,0)}(t)$ is the z-axis $\{(0, 0, t)\}$. In fact, the trace $\{\gamma^i_{(a,b,c)}\}$ is the fiber of the fibration π through (a, b, c).

Let $\mathcal{R}^i_{(a,b,c)}$ be the reflection with respect to the geodesic $\{\gamma^i_{(a,b,c)}\}$, $i = 1, 2, 3$. The following are proved in [Shin et al. 2018]:

$$\begin{aligned} \mathcal{R}^1_{(0,0,0)}(x, y, z) &= (x, -y, -z), \\ \mathcal{R}^2_{(0,0,0)}(x, y, z) &= (-x, y, -z), \\ \mathcal{R}^3_{(0,0,0)}(x, y, z) &= (-x, -y, z). \end{aligned}$$

Proposition 1.

$$\begin{aligned} \mathcal{R}^1_{(a,b,c)}(x, y, z) &= (x, -y + 2b, -z - bx + ab + 2c), \\ \mathcal{R}^2_{(a,b,c)}(x, y, z) &= (-x + 2a, y, -z + ay - ab + 2c), \\ \mathcal{R}^3_{(a,b,c)}(x, y, z) &= (-x + 2a, -y + 2b, z + bx - ay). \end{aligned}$$

Proof. We give a proof of the first formula. The other formulae follow similarly. Note first that, since $\boldsymbol{e}_1$ is a left invariant vector field, one has

$$(-a, -b, -c) * \{\gamma^1_{(a,b,c)}\} = \{(t, 0, 0)\},$$

the x-axis. Then one has

$$\begin{aligned} \mathcal{R}^1_{(a,b,c)}(x, y, z) &= (a, b, c) * \mathcal{R}^1_{(0,0,0)}((-a, -b, -c) * (x, y, z)) \\ &= (x, 2b - y, 2c - z - bx + ab). \end{aligned}$$

□

Then one can see that every reflection $\mathcal{R}^i_{(a,b,c)}$ is an isometry. Moreover,

$$\begin{aligned}
\mathcal{R}^1_{(a,b,c)} \circ \mathcal{R}^3_{(a,b,c)} &= \mathcal{R}^3_{(a,b,c)} \circ \mathcal{R}^1_{(a,b,c)} = \mathcal{R}^2_{(a,b,c)},\\
\mathcal{R}^2_{(a,b,c)} \circ \mathcal{R}^3_{(a,b,c)} &= \mathcal{R}^3_{(a,b,c)} \circ \mathcal{R}^2_{(a,b,c)} = \mathcal{R}^1_{(a,b,c)},\\
\mathcal{R}^1_{(a,b,c)} \circ \mathcal{R}^2_{(a,b,c)} &= \mathcal{R}^2_{(a,b,c)} \circ \mathcal{R}^1_{(a,b,c)} = \mathcal{R}^3_{(a,b,c)}
\end{aligned}$$

and

$$\begin{array}{llll}
(1) & d\mathcal{R}^1_{(a,b,c)}(\boldsymbol{e}_1) = \boldsymbol{e}_1, & d\mathcal{R}^1_{(a,b,c)}(\boldsymbol{e}_2) = -\boldsymbol{e}_2, & d\mathcal{R}^1_{(a,b,c)}(\boldsymbol{e}_3) = -\boldsymbol{e}_3,\\
(2) & d\mathcal{R}^2_{(a,b,c)}(\boldsymbol{e}_1) = -\boldsymbol{e}_1, & d\mathcal{R}^2_{(a,b,c)}(\boldsymbol{e}_2) = \boldsymbol{e}_2, & d\mathcal{R}^2_{(a,b,c)}(\boldsymbol{e}_3) = -\boldsymbol{e}_3,\\
(3) & d\mathcal{R}^3_{(a,b,c)}(\boldsymbol{e}_1) = -\boldsymbol{e}_1, & d\mathcal{R}^3_{(a,b,c)}(\boldsymbol{e}_2) = -\boldsymbol{e}_2, & d\mathcal{R}^3_{(a,b,c)}(\boldsymbol{e}_3) = \boldsymbol{e}_3.
\end{array}$$

Furthermore, since an isometry is uniquely defined by its differential at one point (for the proof of this fact, see for example [Petersen 1998, p. 137]) one has:

Proposition 2. *Let I be an isometry in Nil_3 satisfying* (i), $i = 1, 2, 3$ *of the above conditions with $I(a, b, c) = (a, b, c)$. Then $I = \mathcal{R}^i_{(a,b,c)}$.* □

One can also see that the translation along the fiber

$$T_a(x, y, z) := (x, y, z + a)$$

is also an isometry. Then one has the following relations:

$$\begin{aligned}
\mathcal{R}^i_{(0,0,c)} \circ T_a &= T_{-a+2c} \circ \mathcal{R}^i_{(0,0,0)}, \qquad i = 1, 2,\\
\mathcal{R}^3_{(0,0,0)} \circ T_a &= T_a \circ \mathcal{R}^3_{(0,0,0)}.
\end{aligned}$$

Finally, note that for any constant a, the Euclidean planes

$$\{(a, y, z)\}, \quad \{(x, a, z)\}, \quad \{(x, y, a)\}$$

are minimal surfaces in Nil_3.

3. A construction

Our construction follows the procedure: Partition the whole space into cylindrical regions

$$C_{n,m} := \{(x, y, z) \mid (2n-1)p \le x \le (2n+1)p,\ (2m-1)q \le y \le (2m+1)q\}.$$

Then construct a smooth minimal surface $D_{0,0}$ embedded in $C_{0,0}$ with $\partial D_{0,0} \subset \partial C_{0,0}$. Since $\mathcal{R}^3_{(np,mq,0)}(C_{0,0}) = C_{n,m}$, there exists a smooth minimal surface $D_{n,m} := \mathcal{R}^3_{(np,mq,0)}(D_{0,0})$ embedded in $C_{n,m}$. Then we show that the surfaces $D_{n,m}$ are joined smoothly along the boundary segments to get the complete smooth minimal surface $D := \bigcup_{n,m\in\mathbb{Z}} D_{n,m}$.

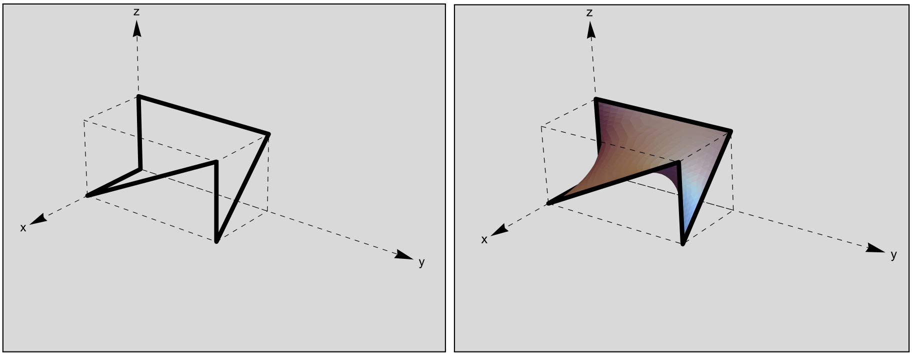

Figure 2. Geodesic polygon $\Gamma_{p,q}$ and minimal disk D_0.

Let us now begin the construction. We first construct a minimal surface embedded in the cylindrical region

$$C_{0,0} := \{(x, y, z) : -p \le x \le p, -q \le y \le q\}.$$

For constants $p, q > 0$, let $\Gamma_{p,q}$ be the geodesic polygon consisting of the six geodesic segments

$$\begin{aligned}
\gamma^1_{(0,0,0)}(s) &= (s, 0, 0), && 0 \le s \le p,\\
\gamma^2_{(p,0,0)}(s) &= \bigl(p, s, \tfrac{1}{2}ps\bigr), && 0 \le s \le q,\\
\gamma^3_{(p,q,0)}(s) &= (p, q, s), && 0 \le s \le \tfrac{1}{2}pq,\\
\gamma^1_{(0,q,\frac{1}{2}pq)}(s) &= \bigl(s, q, \tfrac{1}{2}(p-s)q\bigr), && 0 \le s \le p,\\
\gamma^2_{(0,0,\frac{1}{2}pq)}(s) &= \bigl(0, s, \tfrac{1}{2}pq\bigr), && 0 \le s \le q,\\
\gamma^3_{(0,0,0)}(s) &= (0, 0, s), && 0 \le s \le \tfrac{1}{2}pq.
\end{aligned}$$

which is contained in the boundary of the parallelepiped

$$C_0 := \bigl\{(x, y, z) : 0 \le x \le p,\ 0 \le y \le q,\ 0 \le z \le \tfrac{1}{2}pq\bigr\},$$

which is mean-convex since it is bounded by six minimal surfaces

$$\{(0, y, z)\},\ \ \{(p, y, z)\},\ \ \{(x, 0, z)\},\ \ \{(x, q, z)\},\ \ \{(x, y, 0)\},\ \ \bigl\{\bigl(x, y, \tfrac{1}{2}pq\bigr)\bigr\}.$$

Then $\Gamma_{p,q}$ spans an embedded minimal disk D_0 lying inside of C_0 (see Figure 2). Let

$$D_i = \mathcal{R}^i_{(0,0,0)}(D_0), \quad i = 1, 2, 3.$$

Then one has

$$\begin{aligned}
&D_0 \cap D_1 = \{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid 0 \le s \le p\},\\
&D_0 \cap D_2 = \{(0, 0, 0)\},\\
&D_0 \cap D_3 = \left\{\gamma^3_{(0,0,0)}(s) = (0, 0, s) \mid 0 \le s \le \tfrac{1}{2}pq\right\},\\
&D_1 \cap D_2 = \left\{\gamma^3_{(0,0,0)}(s) = (0, 0, s) \mid -\tfrac{1}{2}pq \le s \le 0\right\},\\
&D_1 \cap D_3 = \{(0, 0, 0)\},\\
&D_2 \cap D_3 = \{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid -p \le s \le 0\}.
\end{aligned}$$

Let

$$D := D_0 \cup D_1 \cup D_2 \cup D_3$$

which is smooth along $\{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid 0 \le s \le p\}$ since

$$D_1 = \mathcal{R}^1_{(0,0,0)}(D_0),$$

smooth along $\left\{\gamma^3_{(0,0,0)}(s) = (0, 0, s) \mid 0 \le s \le \frac{1}{2}pq\right\}$ since

$$D_3 = \mathcal{R}^3_{(0,0,0)}(D_0),$$

smooth along $\left\{\gamma^3_{(0,0,0)}(s) = (0, 0, s) \mid -\frac{1}{2}pq \le s \le 0\right\}$ since

$$D_2 = \mathcal{R}^2_{(0,0,0)}(D_0) = \mathcal{R}^3_{(0,0,0)}(\mathcal{R}^1_{(0,0,0)}(D_0)) = \mathcal{R}^3_{(0,0,0)}(D_1)$$

and smooth along $\{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid -p \le s \le 0\}$ since

$$D_2 = \mathcal{R}^2_{(0,0,0)}(D_0) = \mathcal{R}^1_{(0,0,0)}(\mathcal{R}^3_{(0,0,0)}(D_0)) = \mathcal{R}^1_{(0,0,0)}(D_3).$$

Since

$$D_1 \cup D_2 = \mathcal{R}^1_{(0,0,0)}(D_0 \cup D_3),$$

we can see that D is also smooth along the geodesic

$$\{\gamma^1_{(0,0,0)}(s) = (s, 0, 0) \mid -p \le s \le p\}$$

including the corner point $(0, 0, 0)$ of D_0. Hence the surface D is a smooth minimal surface. Moreover, one can see that D is embedded in the region

$$\left\{(x, y, z) : -p \le x \le p,\ -q \le y \le q,\ -\tfrac{1}{2}pq \le z \le \tfrac{1}{2}pq\right\}$$

and that

$$\mathcal{R}^1_{(0,0,0)}(D) = \mathcal{R}^2_{(0,0,0)}(D) = \mathcal{R}^3_{(0,0,0)}(D) = D.$$

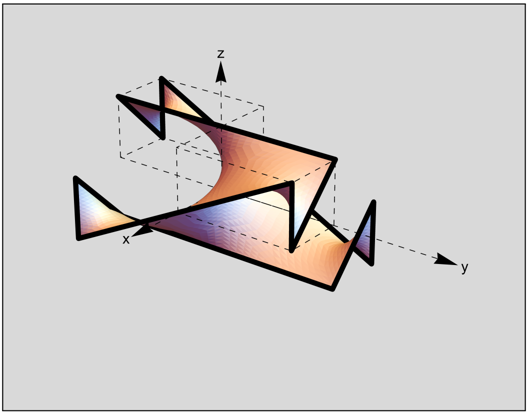

Figure 3. $D_0 \cup D_1 \cup D_2 \cup D_3$

The boundary of D is the Jordan curve consisting of the geodesic segments

$$\begin{array}{llll}
\gamma^2_{(0,0,-\frac{1}{2}pq)}(s), & -q \le s \le q, & \gamma^1_{(0,q,-\frac{1}{2}pq)}(s), & -p \le s \le 0, \\
\gamma^3_{(-p,q,0)}(s), & -\frac{1}{2}pq \le s \le 0, & \gamma^2_{(-p,0,0)}(s), & -q \le s \le q, \\
\gamma^3_{(-p,-q,0)}(s), & 0 \le s \le \frac{1}{2}pq, & \gamma^1_{(0,-q,\frac{1}{2}pq)}(s), & -p \le s \le 0, \\
\gamma^2_{(0,0,\frac{1}{2}pq)}(s), & -q \le s \le q, & \gamma^1_{(0,q,\frac{1}{2}pq)}(s), & 0 \le s \le p, \\
\gamma^3_{(p,q,0)}(s), & 0 \le s \le \frac{1}{2}pq, & \gamma^2_{(p,0,0)}(s), & -q \le s \le q, \\
\gamma^3_{(p,-q,0)}(s), & -\frac{1}{2}pq \le s \le 0, & \gamma^1_{(0,-q,-\frac{1}{2}pq)}(s), & 0 \le s \le p.
\end{array}$$

Now we let

$$D_{0,0} := \bigcup_{k \in \mathbb{Z}} T_{kpq}(D).$$

For each $k \in \mathbb{Z}$, the surface $T_{kpq}(D)$ intersects $T_{(k+1)pq}(D)$ on the common boundary

$$\left\{\gamma^2_{(0,0,(k+\frac{1}{2})pq)}(s) = \left(0,\, s,\, \left(k+\tfrac{1}{2}\right)pq\right) \mid -q \le s \le q\right\}.$$

Since $\mathcal{R}^2_{(0,0,(k+\frac{1}{2})pq)} \circ T_{kpq} = T_{(k+1)pq} \circ \mathcal{R}^2_{(0,0,0)}$, we have

$$\mathcal{R}^2_{(0,0,(k+\frac{1}{2})pq)}(T_{kpq}(D)) = T_{(k+1)pq}(\mathcal{R}^2_{(0,0,0)}(D)) = T_{(k+1)pq}(D),$$

that is, $T_{(k+1)pq}(D)$ is the reflection of $T_{kpq}(D)$ with respect to the common boundary. Therefore each $T_{(k+1)pq}(D)$ is smoothly joined to $T_{kpq}(D)$ and $D_{0,0}$ is a smooth embedded minimal surface in the cylinder $C_{0,0}$.

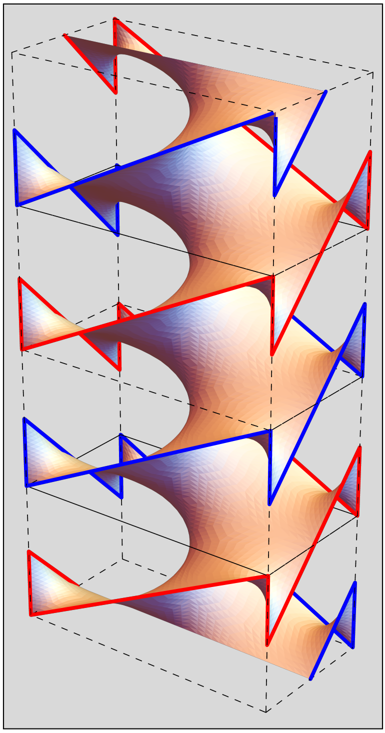

Figure 4. $D_{0,0}$

The boundary $\partial D_{0,0} = D_{0,0} \cap \partial C_{0,0}$ of $D_{0,0}$ consists of the following geodesic segments:

$$\begin{aligned}
&\gamma^1_{(0,\pm q,kpq+\frac{1}{2}pq)}(t), && -p \le t \le p,\\
&\gamma^2_{(\pm p,0,kpq)}(t) && -q \le t \le q,\\
&\gamma^3_{(\pm p,\pm q,kpq)}(t), && 0 \le t \le \tfrac{1}{2}pq,\\
&\gamma^3_{(\pm p,\mp q,kpq)}(t), && -\tfrac{1}{2}pq \le t \le 0
\end{aligned}$$

for $k \in \mathbb{Z}$. It is clear that $T_{lpq}(D_{0,0}) = D_{0,0},\ l \in \mathbb{Z}$. We also note that

$$\mathcal{R}^i_{(0,0,0)}(D_{0,0}) = D_{0,0}, \quad i = 1, 2, 3,$$

since $\mathcal{R}^i_{(0,0,0)} \circ T_{kpq} = T_{-kpq} \circ \mathcal{R}^i_{(0,0,0)}$ for $i = 1, 2$, and $\mathcal{R}^3_{(0,0,0)} \circ T_{kpq} = T_{kpq} \circ \mathcal{R}^3_{(0,0,0)}$.

We now reflect the surface $D_{0,0}$ with respect to the vertical geodesic $\gamma^3_{(np,mq,0)}$ and let

$$D_{n,m} := \mathcal{R}^3_{(np,mq,0)}(D_{0,0}), \quad m, n \in \mathbb{Z}.$$

Then $D_{n,m}$ is an embedded minimal surface in the cylinder

$$\begin{aligned} C_{n,m} &:= \mathcal{R}^3_{(np,mq,0)}(C_{0,0}) \\ &= \{(x,y,z) \mid (2n-1)p \le x \le (2n+1)p,\ (2m-1)q \le y \le (2m+1)q\} \end{aligned}$$

with the boundary $\partial D_{n,m} = D_{n,m} \cap \partial C_{n,m}$. Since

$$\begin{aligned} \mathcal{R}^3_{(np,mq,0)}\big(0, \pm q, kpq + \tfrac{1}{2}pq\big) &= \big(2np, (2m \mp 1)q, (k \mp n)pq + \tfrac{1}{2}pq\big), \\ \mathcal{R}^3_{(np,mq,0)}(\pm p, 0, kpq) &= ((2n \mp 1)p, 2mq, (k \pm m)pq), \\ \mathcal{R}^3_{(np,mq,0)}(\pm p, \pm q, kpq) &= ((2n \mp 1)p, (2m \mp 1)q, (k \mp n \pm m)pg), \\ \mathcal{R}^3_{(np,mq,0)}(\pm p, \mp q, kpq) &= ((2n \mp 1)p, (2m \pm 1)q, (k \pm n \pm m)pg) \end{aligned}$$

and

$$d\mathcal{R}^3_{(np,mq,0)}(\boldsymbol{e}_1) = -\boldsymbol{e}_1, \quad d\mathcal{R}^3_{(np,mq,0)}(\boldsymbol{e}_2) = -\boldsymbol{e}_2, \quad d\mathcal{R}^3_{(np,mq,0)}(\boldsymbol{e}_3) = \boldsymbol{e}_3$$

we can see that the boundary $\partial D_{n,m}$ of $D_{n,m}$ is the union of the geodesic segments

$$\begin{array}{ll} \gamma^1_{(2np,(2m\pm1)q,kpq+\frac{1}{2}pq)}(t), & -p \le t \le p, \\ \gamma^2_{((2n\pm1)p,2mq,kpq)}(t), & -q \le t \le q, \\ \gamma^3_{((2n\pm1)p,(2m\pm1)q,kpq)}(t), & \begin{cases} 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is even}, \\ -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is odd}, \end{cases} \\ \gamma^3_{((2n\pm1)p,(2m\mp1)q,kpq)}(t), & \begin{cases} -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is even}, \\ 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is odd}. \end{cases} \end{array}$$

Now, we show that any boundary geodesic segment in $\partial D_{n,m}$ is the common boundary segments of exactly two such surfaces and these surfaces are joined smoothly across the common boundary segments. We will consider each type of boundary segments listed above separately.

(i) The boundary segment

$$\gamma^1_{(2np,(2m+1)q,kpq+\frac{1}{2}pq)}(t), \quad -p \le t \le p$$

of $D_{n,m}$ is also a boundary segment of $D_{n,m+1}$ only. Moreover, since ${\mathcal{R}^3_{(np,mq,0)}}^{-1} = \mathcal{R}^3_{(np,mq,0)}$ and $D_{0,0}$ is invariant under the translations T_{lpq} and the reflections $\mathcal{R}^i_{(0,0,0)}$,

$$\begin{aligned} D_{n,m+1} &= \mathcal{R}^3_{(np,(m+1)q,0)}(D_{0,0}) \\ &= \mathcal{R}^3_{(np,(m+1)q,0)}\big(T_{(2k+1)pq}(\mathcal{R}^1_{(0,0,0)}(D_{0,0}))\big) \\ &= \big(\mathcal{R}^3_{(np,(m+1)q,0)} \circ T_{(2k+1)pq} \circ \mathcal{R}^1_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}\big)(D_{n,m}). \end{aligned}$$

For the isometry $\psi = \mathcal{R}^3_{(np,(m+1)q,0)} \circ T_{(2k+1)pq} \circ \mathcal{R}^1_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}$, a direct computation shows that

$$\psi\big(2np, (2m+1)q, kpq + \tfrac{1}{2}pq\big) = \big(2np, (2m+1)q, kpq + \tfrac{1}{2}pq\big)$$

and

$$d\psi(\boldsymbol{e}_1) = \boldsymbol{e}_1, \quad d\psi(\boldsymbol{e}_2) = -\boldsymbol{e}_2, \quad d\psi(\boldsymbol{e}_3) = -\boldsymbol{e}_3.$$

Therefore

$$\psi = \mathcal{R}^1_{(2np,(2m+1)q,kpq+\frac{1}{2}pq)} \quad \text{and} \quad D_{n,m+1} = \mathcal{R}^1_{(2np,(2m+1)q,kpq+\frac{1}{2}pq)}(D_{0,0}).$$

Hence $D_{n,m+1}$ and $D_{n,m}$ are joined smoothly across the common boundary segments

$$\gamma^1_{(2np,(2m+1)q,kpq+\frac{1}{2}pq)}(t), \quad -p \le t \le p.$$

The boundary segment

$$\gamma^1_{(2np,(2m-1)q,kpq+\frac{1}{2}pq)}(t), \quad -p \le t \le p$$

of $D_{n,m}$ is the common boundary segment of $D_{n,m-1}$ and $D_{n,m}$ and they are joined smoothly across these segments by the same argument taking $m-1$ instead of m.

(ii) The boundary segment

$$\gamma^2_{((2n+1)p,2mq,kpq)}(t), \quad -q \le t \le q$$

of $D_{n,m}$ is the common boundary segments of $D_{n+1,m}$ and $D_{n,m}$. Moreover,

$$\begin{aligned} D_{n+1,m} &= \mathcal{R}^3_{((n+1)p,mq,0)}(D_{0,0}) \\ &= \mathcal{R}^3_{((n+1)p,mq,0)}(T_{2kpq}(\mathcal{R}^2_{(0,0,0)}(D_{0,0}))) \\ &= (\mathcal{R}^3_{((n+1)p,mq,0)} \circ T_{2kpq} \circ \mathcal{R}^2_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)})(D_{n,m}). \end{aligned}$$

For the isometry $\psi = \mathcal{R}^3_{((n+1)p,mq,0)} \circ T_{2kpq} \circ \mathcal{R}^2_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}$,

$$\psi((2n+1)p, 2mq, kpq) = ((2n+1)p, 2mq, kpq)$$

and

$$d\psi(\boldsymbol{e}_1) = -\boldsymbol{e}_1, \quad d\psi(\boldsymbol{e}_2) = \boldsymbol{e}_2, \quad d\psi(\boldsymbol{e}_3) = -\boldsymbol{e}_3.$$

Therefore $\psi = \mathcal{R}^2_{((2n+1)p,2mq,kpq)}$ and $D_{n+1,m} = \mathcal{R}^2_{((2n+1)p,2mq,kpq)}(D_{n,m})$. Thus $D_{n+1,m}$ and $D_{n,m}$ are joined smoothly across the common boundary segments

$$\gamma^2_{((2n+1)p,2mq,kpq)}(t), \quad -q \le t \le q.$$

The boundary segment

$$\gamma^2_{((2n-1)p,2mq,kpq)}(t), \quad -q \le t \le q$$

of $D_{n,m}$ is the common boundary segment of $D_{n-1,m}$ and $D_{n,m}$ and they are joined smoothly across these segments by the same argument taking $n-1$ instead of n.

(iii) The boundary segment

$$\gamma^3_{((2n+1)p,(2m+1)q,kpq)}(t), \qquad \begin{cases} 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is even,} \\ -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is odd,} \end{cases}$$

of $D_{n,m}$ is the common boundary segment of $D_{n+1,m+1}$ and $D_{n,m}$. Moreover,

$$\begin{aligned} D_{n+1,m+1} &= \mathcal{R}^3_{((n+1)p,(m+1)q,0)}(D_{0,0}) \\ &= \mathcal{R}^3_{((n+1)p,(m+1)q,0)}(T_{2(n-m)pq}(\mathcal{R}^3_{(0,0,0)}(D_{0,0}))) \\ &= (\mathcal{R}^3_{((n+1)p,(m+1)q,0)} \circ T_{2(n-m)pq} \circ \mathcal{R}^3_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)})(D_{n,m}). \end{aligned}$$

For the isometry $\psi = \mathcal{R}^3_{((n+1)p,(m+1)q,0)} \circ T_{2(n-m)pq} \circ \mathcal{R}^3_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}$,

$$\psi((2n+1)p,\,(2m+1)q,\,kpq) = ((2n+1)p,\,(2m+1)q,\,kpq)$$

and

$$d\psi(\boldsymbol{e}_1) = -\boldsymbol{e}_1, \quad d\psi(\boldsymbol{e}_2) = -\boldsymbol{e}_2, \quad d\psi(\boldsymbol{e}_3) = \boldsymbol{e}_3.$$

Therefore $\psi = \mathcal{R}^3_{((2n+1)p,(2m+1)q,kpq)}$ and $D_{n+1,m+1} = \mathcal{R}^3_{((2n+1)p,(2m+1)q,kpq)}(D_{n,m})$. This implies that $D_{n+1,m+1}$ and $D_{n,m}$ are joined smoothly across the common boundary segments

$$\gamma^3_{((2n+1)p,(2m+1)q,kpq)}(t), \qquad \begin{cases} 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is even,} \\ -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is odd.} \end{cases}$$

The boundary segment

$$\gamma^3_{((2n-1)p,(2m-1)q,kpq)}(t), \qquad \begin{cases} 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is even,} \\ -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is odd} \end{cases}$$

of $D_{n,m}$ is the common boundary segment of $D_{n-1,m-1}$ and $D_{n,m}$ and they are joined smoothly across these segments by the same argument taking $n-1$, $m-1$ instead of n, m.

(iv) The boundary segment

$$\gamma^3_{((2n+1)p,(2m-1)q,kpq)}(t), \qquad \begin{cases} -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is even,} \\ 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is odd} \end{cases}$$

of $D_{n,m}$ is the common boundary segments of $D_{n+1,m-1}$ and $D_{n,m}$. Moreover,

$$\begin{aligned} D_{n+1,m-1} &= \mathcal{R}^3_{((n+1)p,(m-1)q,0)}(D_{0,0}) \\ &= \mathcal{R}^3_{((n+1)p,(m-1)q,0)}(T_{-2(n+m)pq}(\mathcal{R}^3_{(0,0,0)}(D_{0,0}))) \\ &= (\mathcal{R}^3_{((n+1)p,(m-1)q,0)} \circ T_{-2(n+m)pq} \circ \mathcal{R}^3_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)})(D_{n,m}). \end{aligned}$$

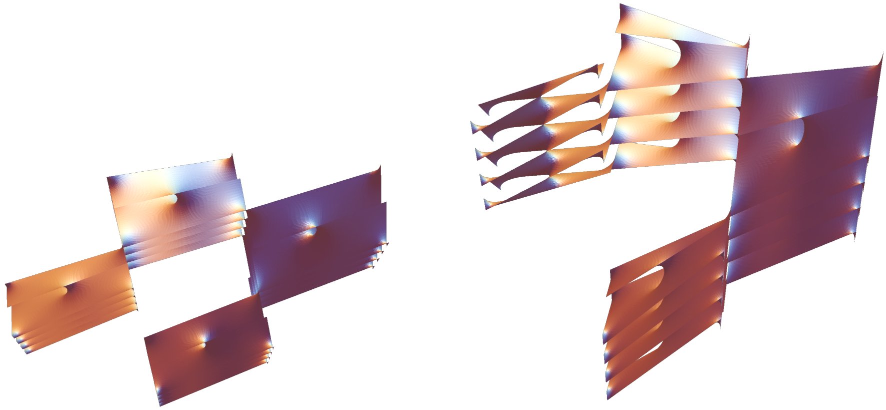

Figure 5. Two views of a part of M.

For the isometry $\psi = \mathcal{R}^3_{((n+1)p,(m-1)q,0)} \circ T_{-2(n+m)pq} \circ \mathcal{R}^3_{(0,0,0)} \circ \mathcal{R}^3_{(np,mq,0)}$,

$$\psi((2n+1)p, (2m-1)q, kpq) = ((2n+1)p, (2m-1)q, kpq)$$

and

$$d\psi(\boldsymbol{e}_1) = -\boldsymbol{e}_1, \quad d\psi(\boldsymbol{e}_2) = -\boldsymbol{e}_2, \quad d\psi(\boldsymbol{e}_3) = \boldsymbol{e}_3.$$

Therefore $\psi = \mathcal{R}^3_{((2n+1)p,(2m-1)q,kpq)}$ and $D_{n+1,m-1} = \mathcal{R}^3_{((2n+1)p,(2m-1)q,kpq)}(D_{n,m})$. This implies that $D_{n+1,m-1}$ and $D_{n,m}$ are joined smoothly across the common boundary segments

$$\gamma^3_{((2n+1)p,(2m-1)q,kpq)}(t), \qquad \begin{cases} -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is even,} \\ 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is odd.} \end{cases}$$

The boundary segment

$$\gamma^3_{((2n-1)p,(2m+1)q,kpq)}(t), \qquad \begin{cases} -\frac{1}{2}pq \le t \le 0 & \text{if } n+m \text{ is even,} \\ 0 \le t \le \frac{1}{2}pq & \text{if } n+m \text{ is odd} \end{cases}$$

of $D_{n,m}$ is the common boundary segment of $D_{n-1,m+1}$ and $D_{n,m}$ and they are joined smoothly across this segments by the same argument taking $n-1$, $m+1$ instead of n, m.

We have shown that each boundary segments of the minimal surface $D_{n,m}$ are joined smoothly to the neighboring surfaces. For any n, m, above argument shows that $D_{n,m} \cup D_{(n+1),m}$ becomes a smooth minimal surface which contains the vertical geodesic $\gamma^3_{((2n+1)p,(2m\pm1)q,0)}$ in its boundary. Moreover, since

$$D_{n,(m+1)} \cup D_{(n+1),(m+1)} = \mathcal{R}^3_{((2n+1)p,(2m+1)q,0)}(D_{(n+1),m} \cup D_{n,m}),$$

$D_{n,m} \cup D_{(n+1),m} \cup D_{n,(m+1)} \cup D_{(n+1),(m+1)}$ is smooth along the vertical geodesic $\gamma^3_{((2n+1)p,(2m+1)q,0)}$ which includes the corner points of the boundary of $D_{n,m}$.

Since the surfaces $D_{n,m}$ for $n, m \in \mathbb{Z}$ do not intersect with others at interior points, we can conclude that

$$M := \bigcup_{n,m\in\mathbb{Z}} D_{n,m}$$

is a smooth embedded complete minimal surface in Nil_3.

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COMPACTNESS OF CONSTANT MEAN CURVATURE SURFACES IN A THREE-MANIFOLD WITH POSITIVE RICCI CURVATURE

AO SUN

We prove a compactness theorem for constant mean curvature surfaces with area and genus bound in a three-manifold with positive Ricci curvature. As an application, we give a lower bound of the first eigenvalue of constant mean curvature surfaces in a three-manifold with positive Ricci curvature.

1. Introduction

Let M be a three-dimensional manifold and $\Sigma \subset M$ be a surface. Let H be the mean curvature of Σ. We say Σ is a constant mean curvature (CMC) surface if H is a constant. In particular, if H is constant 0, Σ is a minimal surface. There are many examples of CMC surfaces in $\mathbb{R}^3$; see [Meeks et al. 2016]. Recently, Zhou and Zhu [2019] proved the existence of embedded CMC hypersurfaces in closed $(n+1)$-dimensional manifolds with $2 \le n \le 6$.

We will prove the following compactness theorem for embedded CMC surfaces in a three-dimensional manifold with positive Ricci curvature:

Theorem 1.1. *Let M be a three-dimensional compact manifold with positive Ricci curvature and no boundary. Suppose $\Sigma_i \subset M$ is a sequence of closed embedded CMC surfaces with constant mean curvature H_i, satisfying the following conditions*:

(1) $|H_i| \le H_0$ *for some constant* H_0.

(2) *The genus of Σ_i is uniformly bounded.*

(3) *The area of Σ_i is uniformly bounded.*

Then either

(1) *there is a self-touching smoothly immersed CMC surface Σ with finitely many neck pinching points, such that a subsequence of Σ_i converges to Σ in C^k topology for any $k \ge 2$ apart from those neck pinching points*, *or*

(2) *there is an embedded minimal surface Σ such that Σ_i converges to Σ with multiplicity* 2.

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Keywords: constant mean curvature surfaces, compactness.

Here we say Σ is *self-touching* if at any nonembedded point $p \in \Sigma$, there is a small r such that $B_r(p) \cap \Sigma$ is a union of two disks D_1, D_2, and D_1 can be written as a graph of function ϕ over D_2 where $\phi \geq 0$ on D_2. Intuitively this means that Σ is immersed but cannot cross itself. *Neck pinching* points are special touching points. We will give the precise definition in Section 4. Intuitively one can imagine that we are pinching a piece of a plasticine into two pieces, and just at the moment they are detached, the point connecting them is a neck pinching point.

Compactness theorem. The compactness theorem for minimal surfaces was first developed by Choi and Schoen [1985]. They proved the compactness theorem of embedded minimal surfaces in a three-dimensional manifold with positive Ricci curvature. Later their result was generalized to many other situations. For example, White [1987] generalized the compactness theorem to surfaces which are stationary for parametric elliptic functionals, and Colding and Minicozzi [2012] generalized the compactness theorem to self-shrinkers in $\mathbb{R}^3$. We will follow the key ideas of these papers.

There are two main ingredients in the proof by Choi and Schoen. The first ingredient is a curvature estimate for minimal surfaces. Then we can get uniform curvature bound on Σ_i apart from finitely many points, so we can find a subsequence of Σ_i converging smoothly to a limit surface Σ apart from finitely many points. Here we need to generalize the curvature estimate to CMC surfaces, and get an uniform curvature estimate only depending on the uniform mean curvature bound H_0.

The second ingredient is showing the multiplicity of the convergence is no more than two. In particular, if the multiplicity is one, then by a result of Allard [1972] the convergence is smooth. There are two methods to show the multiplicity is no more than two. Choi and Schoen argued by constructing a family of functions which contradict the eigenvalue estimate in [Choi and Wang 1983]; another method by White and by Colding and Minicozzi argued that if the multiplicity is more than two, then the linearized operator has a positive Jacobi field, which is impossible if M has positive Ricci curvature. We will follow the second method, because we do not have an eigenvalue estimate for CMC surfaces. In our case, a key observation is that although CMC surfaces and minimal surfaces satisfy different equations, their linearized operators are the same. Hence we may conduct the same argument as the minimal surfaces case.

If the limit surface is minimal, then the CMC surfaces may approach it on both sides with different orientation, and the differential operator is not the same as the differential operator of minimal surfaces. Thus, the convergence may not be multiplicity 1. However, if the multiplicity of the convergence is more than 2, then we can still find two sheets with the same orientation, and so the differential operator is again the same as the differential operator of minimal surfaces. Again we can obtain a positive Jacobi field to argue for contradiction.

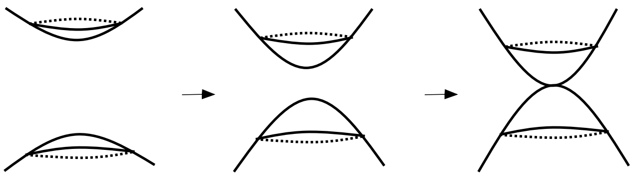

Figure 1. Kissing process.

As an application of the compactness theorem (Theorem 1.1), we obtain a lower bound for the first eigenvalue of CMC surfaces.

Theorem 1.2. *Let M be a three-dimensional manifold with positive Ricci curvature. Suppose there is no embedded minimal surface in M which is the multiplicity 2 limit of a sequence of CMC surfaces. Then for any embedded CMC surface with area bound V, genus bound G and mean curvature bound $|H| \leq H_0$, we have the first eigenvalue lower bound*:

$$\lambda \geq \frac{\min \operatorname{Ric} - HC}{2}, \tag{1-1}$$

where C is a constant depending on M, V, G, H_0.

Touching phenomenon. The touching phenomenon does not occur in minimal surfaces due to the maximum principle, but it may occur in CMC surfaces, especially in a convergence process. The appearance of touching points makes the convergence of CMC surfaces much more complicated.

The touching phenomenon is natural in our physical world. For example, one can observe many soap bubbles touching each other. More complicated examples appear in general three-manifold rather than $\mathbb{R}^3$, and we give some examples in Section 5.

In general touching points in the limit do not influence the smooth convergence in our main theorem (Theorem 1.1) if they are generated when two parts of the surface are kissing each other. One can imagine the convergence is smooth on each piece; see Figure 1.

However, neck pinching points are generated with some topological changes, so smooth convergence cannot cross these points; see Figure 2.

The neck pinching phenomenon is very common in geometric analysis. For example, the neck pinching phenomenon appears in geometric flows, such as mean curvature flow (see [Gang and Sigal 2009]) and Ricci flow (see [Angenent and Knopf 2004]). In order to deal with the nonsmoothness of the flow across these neck pinching points, one needs to do surgery for the geometric flows. For example

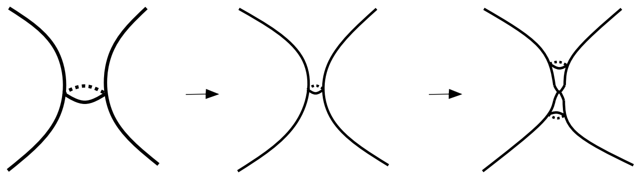

Figure 2. Neck pinching process.

Perelman [2002] studied surgery of Ricci flows, and Brendle and Huisken [2016] studied surgery of mean curvature flows in $\mathbb{R}^3$.

Organization of the paper. In Section 2 we will prove the Choi–Schoen type curvature estimate for CMC surfaces. We will just follow Choi and Schoen's proof. A similar estimate for CMC surfaces in $\mathbb{R}^3$ appears in [Zhang 2005].

In Section 3 we discuss the linearized equation and the linearized operator.

In Section 4 we prove the main compactness theorem. We follow the idea from [White 1987] and [Colding and Minicozzi 2012].

In Section 5 we present some touching examples of CMC surfaces.

In Section 6, as an application of the compactness theorem, we prove a lower bound for the first eigenvalue of CMC surfaces in a three-manifold.

2. Curvature estimate

In this section we generalize the Choi–Schoen curvature estimate for minimal surfaces to CMC surfaces. We first need some tools.

Tools for curvature estimate. The first lemma is a Simon's type inequality for CMC surfaces. We need to keep track of the mean curvature term.

Lemma 2.1. *Let Σ be a CMC surface with mean curvature H, $|H| \le H_0$. Then*

$$\Delta_\Sigma |A|^2 \ge -C(\delta^2 + |A|^2)^2, \tag{2-1}$$

where C is a universal constant, and δ is quadratic under the rescaling of the M, i.e., suppose we rescale the metric g to $\tilde{g} = \sigma g$, then δ becomes $\tilde{\delta} = \sigma^{-1}\delta$.

Proof. See [Ilias et al. 2012, Theorem 3.1]. □

The next lemma generalizes the monotonicity formula for minimal surfaces to CMC surfaces.

Lemma 2.2. *Let M be a closed three-manifold with sectional curvature bounded by k and injective radius bounded from below by i_0. Let $\Sigma \subset M$ be a CMC surface with mean curvature H, $|H| \le H_0$. Let f be a function on Σ satisfying $\Delta_\Sigma f \ge -\lambda t^{-2} f$, where λ is a fixed constant and $t < \min\{i_0, 1/\sqrt{k}\}$. Then we have*

$$f(x_0) \le \frac{e^{\lambda + C(H_0,k)t/2}}{\pi} \int_{B_t(x_0)\cap\Sigma} f. \tag{2-2}$$

Before we prove this lemma, let us state a lemma of the famous Laplacian comparison theorem in three-manifold. See [Colding and Minicozzi 2011, Chapter 7, Lemma 7.1] for a proof.

Lemma 2.3. *Suppose that M is a closed three-manifold with sectional curvature bounded by k and injective radius bounded from below by i_0. Let $x \in M$ be a fixed point, and r be the distance function from x. Then for $r < \min\{i_0, 1/\sqrt{k}\}$ and any vector X with $|X| = 1$,*

$$\left|\mathrm{Hess}_r(X, X) - \frac{1}{r}\langle X - \langle X, Dr\rangle Dr, X - \langle X, Dr\rangle Dr\rangle\right| \le \sqrt{k}. \tag{2-3}$$

Here D is the gradient on M.

Proof of Lemma 2.2. Let $y \in \Sigma$ be a point with $r(y) < \min\{i_0, 1/\sqrt{k}\}$. We choose a local orthonormal frame E_1, E_2. Then by Laplacian comparison (Lemma 2.3), we have

$$\left|\mathrm{Hess}_r(E_1, E_1) - \frac{1}{r}\langle E_1 - \langle E_1, Dr\rangle Dr, E_1 - \langle E_1, Dr\rangle Dr\rangle\right| \le \sqrt{k}, \tag{2-4}$$

$$\left|\mathrm{Hess}_r(E_2, E_2) - \frac{1}{r}\langle E_2 - \langle E_2, Dr\rangle Dr, E_2 - \langle E_2, Dr\rangle Dr\rangle\right| \le \sqrt{k}. \tag{2-5}$$

Adding these two inequalities and noting Σ is a CMC surface, we get (compare to the minimal surfaces case in [Colding and Minicozzi 2011, Chapter 7, (7.2)])

$$|\Delta_\Sigma r^2 - 4 - \langle \nabla^\perp r^2, H\boldsymbol{n}\rangle| \le 4\sqrt{k}r. \tag{2-6}$$

Noting $|Dr| \le 1$, we get

$$|\Delta_\Sigma r^2 - 4| \le (4\sqrt{k} + 2H_0)r = \alpha r. \tag{2-7}$$

where C only depends on k, H_0. Let us define

$$F(s) = \frac{1}{s^2}\int_{B_s(x_0)\cap\Sigma} f.$$

We can differentiate it for almost every $s < t$

$$F'(s) = -\frac{2}{s^3}\int_{B_s(x_0)\cap\Sigma} f + \frac{1}{s^2}\int_{\partial B_s(x_0)\cap\Sigma} \frac{f}{|\nabla_\Sigma r|}. \tag{2-8}$$

Here we use the co-area formula; see [Colding and Minicozzi 2011, p. 24, (1.59)]. Let us estimate the first term on the right-hand side. Using inequality (2-7) and integrating by parts gives

$$
\begin{aligned}
(2\text{-}9)\quad -\frac{2}{s^3}\int_{B_s(x_0)\cap\Sigma} f &= -\frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma} 4f \\
&\geq -\frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma} \Delta_\Sigma(r^2-s^2)f - \frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}\alpha r f \\
&= -\frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}(r^2-s^2)\Delta_\Sigma f \\
&\qquad -\frac{1}{2s^3}\int_{\partial B_s(x_0)\cap\Sigma}\nabla_\Sigma(r^2)f - \frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}\alpha r f.
\end{aligned}
$$

So we get the following inequality

$$
\begin{aligned}
(2\text{-}10)\quad F'(s) &\geq \frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}(s^2-r^2)\Delta_\Sigma f + \frac{1}{s^2}\int_{\partial B_s(x_0)\cap\Sigma}\frac{1-|\nabla_\Sigma r|^2}{|\nabla_\Sigma r|}f - \frac{\alpha}{2}F(s) \\
&\geq \frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}(s^2-r^2)\Delta_\Sigma f - \frac{\alpha}{2}F(s) \\
&\geq -\frac{1}{2s^3}\int_{B_s(x_0)\cap\Sigma}(s^2-r^2)t^{-2}\lambda f - \frac{\alpha}{2}F(s) \\
&\geq -\frac{\lambda}{t}F(s) - \frac{\alpha}{2}F(s).
\end{aligned}
$$

Hence $e^{(\lambda/t+\alpha/2)s}F(s)$ is monotone nondecreasing. Then we can conclude that

$$
(2\text{-}11)\qquad f(x_0) \leq \frac{e^{C+\alpha t/2}}{\pi}\int_{B_t(x_0)\cap\Sigma} f. \qquad \square
$$

Choi–Schoen type estimate.

Theorem 2.4. *Let M be a three-dimensional manifold. Let $p \in M$ and $r > 0$ such that $B_r(p)$ has a compact closure in M. Let Σ be a compact immersed CMC surface with mean curvature H in M such that $B_r(p)\cap\partial\Sigma = \varnothing$. Here $|H| \leq H_0$. Then there exists $\varepsilon_0 > 0$ depending on the geometry of $B_r(p)$ and H_0 such that if*

$$\int_{\Sigma\cap B_r(p)} |A|^2 \leq \varepsilon_0.$$

and $r \leq \varepsilon_0$, then

$$
(2\text{-}12)\qquad \max\nolimits_{0\leq\sigma\leq r}\sigma^2\sup\nolimits_{B_{r-\sigma}(p)}|A|^2 \leq C = C(H_0, B_r(p)).
$$

Proof. We follow the idea of Choi and Schoen. Since $\sigma^2\sup_{B_{r-\sigma}(p)}|A|^2$ vanishes on ∂B_r, the supremum of $\sigma^2\sup_{B_{r-\sigma}(p)}|A|^2$ must be achieved inside B_r. Let σ_0 be the number such that

$$\sigma_0^2 \sup_{B_{r-\sigma_0}(p)}|A|^2 = \max_{0\le\sigma\le r} \sigma^2 \sup_{B_{r-\sigma}(p)}|A|^2.$$

and let $q \in B_{r-\sigma_0}(p)$ be chosen to satisfy

$$|A|^2(q) = \sup_{B_{r-\sigma_0}(q)}|A|^2.$$

Then

$$\sup_{B_{\frac{1}{2}\sigma_0}(q)}|A|^2 \le 4|A|^2(q). \tag{2-13}$$

If $\sigma_0^2|A|^2(q) \le 4$, then the inequality holds. So we only need to consider the case $\sigma_0^2|A|^2(q) > 4$. Now we rescale the metric ds^2 on M by setting $\tilde{d}s^2 = |A|^2(q)ds^2$, and we denote the balls and the quantities under the rescaled metric with tilde. $\sigma_0^2|A|^2(q) > 4$ implies that $\partial\Sigma \cap \widetilde{B}_1(q) = \varnothing$. Inequality (2-13) implies that

$$\sup_{\widetilde{B}_1(q)}|\widetilde{A}|^2 \le 4.$$

By Simons' inequality (Lemma 2.1),

$$\tilde{\Delta}_\Sigma|\widetilde{A}|^2 \ge -C(\delta^2 + |A|^2)^2.$$

Note here $\delta^2 \le C\sigma_0^2 \le C\varepsilon_0^2$. Together with the inequality $\sup_{\widetilde{B}_1(q)}|\widetilde{A}|^2 \le 4$ we get

$$\tilde{\Delta}_\Sigma u \ge -Cu \text{ on } \widetilde{B}_1(q),$$

where $u = \delta^2 + |\widetilde{A}|^2$ and C is a universal constant. So the monotonicity formula (Lemma 2.2) gives

$$u(x_0) \le \frac{e^{C+\widetilde{\alpha}/2}}{\pi} \int_{\widetilde{B}_1(q)\cap\Sigma} u. \tag{2-14}$$

Noting $\widetilde{\alpha} \le \alpha\sigma_0 \le \alpha\varepsilon_0$, we have

$$|A|^2(q) \le u(q) \le \frac{e^{C+\alpha\varepsilon_0/2}}{\pi} \int_{\widetilde{B}_1(q)\cap\Sigma} (\delta^2 + |A|^2) \le C\varepsilon_0. \tag{2-15}$$

where we use the conformal invariance of the integral of $|A|^2$ and the area bound of CMC surface. If ε_0 is small enough, we will get a contradiction since $|\widetilde{A}|^2(q) = 1$. Thus we finish the proof. □

3. Linearized equation

Let Σ be a CMC surface in M. Let us define a differential operator L such that

$$Lu = \Delta_\Sigma u + \mathrm{Ric}(\boldsymbol{n}, \boldsymbol{n})u + |A|^2 u. \tag{3-1}$$

Here u is a function on Σ. We call L the *linearized operator*. In this section we study some properties of this operator.

Difference of two CMC surfaces. Let M be a three-dimensional manifold. Suppose $\Sigma_1, \Sigma_2 \subset M$ are two CMC surfaces, with mean curvature H_1, H_2 respectively.

Theorem 3.1. *Suppose Σ_2 is a graph over Σ_1, i.e.,*

$$\Sigma_2 = \{x + \varphi \boldsymbol{n} : x \in \Sigma_1\}.$$

Then φ satisfies the second order elliptic equation

$$L\varphi - (H_2 - H_1) = \operatorname{div}(a\nabla\varphi) + b \cdot \nabla\varphi + c\varphi. \tag{3-2}$$

Here a, b, c turns to 0 as $\|\varphi\|_{C^2}$ goes to 0.

This can be viewed as a noninfinitesimal version of the second variational formula. The computations are in the Appendix. Intuitively, one can imagine the second order variational formula gives the second order derivative of minimal surfaces, and the difference formula here gives the Taylor expansion of the minimal surfaces up to the second order. In particular letting $\varphi \to 0$, we will again get the second variational formula.

Stability of linearized operator. Suppose Σ is a CMC surface. We say the linearized operator L of Σ is *stable* if for any function u on Σ,

$$\int_\Sigma uLu \le 0. \tag{3-3}$$

Otherwise we say L is unstable. Note this definition of stability is not the same as the stability of the CMC surface itself, since when we talk about the stability of a CMC surface Σ, we only consider the variational fields which preserve the (local) volume enclosed by Σ.

For a surface Σ in a three-dimensional manifold M with positive Ricci curvature, letting $u \equiv 1$ we see that

$$\int_\Sigma uLu = \int_\Sigma |A|^2 + \operatorname{Ric}(\boldsymbol{n}, \boldsymbol{n}) > 0.$$

Hence L is always unstable.

Recall a Jacobi field on Σ is a variational field $f\boldsymbol{n}$ such that $Lf = 0$. The following lemma shows that a positive Jacobi field implies the stability of L.

Lemma 3.2. *Suppose there is a positive function u on Σ such that $Lu = 0$. Then L is stable.*

Proof. Let $w = \log u$. Then $\Delta_\Sigma w = -|A|^2 - \operatorname{Ric}(\boldsymbol{n}, \boldsymbol{n}) - |\nabla_\Sigma w|^2$.

Let v be any smooth function on Σ. Multiplying both sides of the above identity by v^2 gives

$$\begin{aligned}(3\text{-}4)\quad \int_\Sigma v^2(|A|^2+\mathrm{Ric}(\boldsymbol{n},\boldsymbol{n}))+\int_\Sigma |\nabla_\Sigma w|^2v^2 \\ &= -\int_\Sigma v^2\Delta_\Sigma w = 2\int_\Sigma v\langle\nabla_\Sigma v, \nabla_\Sigma w\rangle \\ &\le 2\int_\Sigma |v||\nabla_\Sigma w||\nabla_\Sigma v| \le \int_\Sigma v^2|\nabla_\Sigma w|^2+\int_\Sigma |\nabla_\Sigma v|^2.\end{aligned}$$

Then integration by parts gives

$$\int_\Sigma vLv \le 0. \tag{3-5}$$

□

4. Compactness theorem

In this section, we will prove the main compactness theorem.

Smooth limit. We first show that there is a reasonable smooth limit under the conditions in Theorem 1.1.

Theorem 4.1. *Let M be a three-dimensional compact manifold with positive Ricci curvature and no boundary. Suppose $\Sigma_i \subset M$ is a sequence of closed embedded CMC surfaces with constant mean curvature H_i, satisfying the following conditions*:

(1) $|H_i| \le H_0$ *for some constant* H_0.

(2) *The genus of Σ_i is uniformly bounded.*

(3) *The area of Σ_i is uniformly bounded.*

Then there is a self-touching smoothly immersed CMC surface Σ such that a subsequence of Σ_i converges to Σ in C^k topology for any $k \ge 2$ apart from a finite singular set $\mathcal{S}$.

Proof. We follow [Choi and Schoen 1985] and [Colding and Minicozzi 2012]. First of all, since the mean curvature, the area and the genus of Σ_i is uniformly bounded, by the Gauss–Bonnet theorem the total curvature of Σ_i is also uniformly bounded by a constant C. For each positive integer m, take a finite covering $\{B_{r_m}(y_j)\}$ of M such that each point of M is covered at most h times by balls in this covering, and $\{B_{r_m/2}(y_j)\}$ is still a covering of M. Here we set $r_m = 2^{-m}\varepsilon_0$ and h only depends on M. Then we have

$$\sum_j \int_{\Sigma_i\cap B_{r_m}(y_j)} |A|^2 \le hC$$

Therefore for each i there are at most hC/ε_0 number of balls such that

$$\int_{\Sigma_i \cap B_{r_m}(y_j)} |A|^2 \geq \varepsilon_0$$

By passing to a subsequence of Σ_i we can always assume that all the Σ_i have the same balls with total curvature $\geq \varepsilon_0$. Call the center of these balls $\{x_{1,m}, \dots, x_{l,m}\}$, where l is an integer at most hC/ε_0. Then on the balls other than $B_{x_{k,m}(r_m)}$, by Theorem 2.4 we have a uniformly point-wise curvature bound for Σ_i. Passing to a subsequence we may assume that the Σ_i converge smoothly on a half size of those balls to Σ. Since the Σ_i are embedded, the limit Σ is self-touching in the balls other than $B_{x_{k,m}(r_m)}$.

We can continue this process as m increases. Finally by a diagonal argument we can get a subsequence $\{\Sigma_i\}$, converging smoothly everywhere to Σ apart from those points $x_1, \dots, x_l$ which are the limit of those $\{x_{1,m}\}, \dots, \{x_{l,m}\}$. Moreover, since there is no maximum principle for CMC surfaces, the limit is only immersed. However if we consider the compactness for each connected components in any fixed ball, we can see the limit is self-touching away from $x_1, \dots, x_l$. □

Next we will show that Σ is actually smooth everywhere. We will follow White to prove that the singularities are removable. The main ingredient is a more delicate curvature estimate near the singularities.

Lemma 4.2. *Suppose Σ is a properly self-touching CMC surface in $B_R(x_0) \setminus \{x_0\}$ with mean curvature $|H| \leq H_0$. Then there exists $\varepsilon = \varepsilon(H_0, R, x_0) > 0$ such that if $\int_\Sigma |A|^2 \leq \varepsilon$, there exists C such that*

$$|A(x)|(\operatorname{dist}(x, x_0)) \leq C. \tag{4-1}$$

Proof. We show this by contradiction. If the criterion is not true, we can find a sequence of points $x_n \in ((B_R(x_0) \setminus B_{1/n}(x_0)) \cap \Sigma)$ such that

$$|A(x_n)|^2\Big(\operatorname{dist}(x, x_0) - \frac{1}{n}\Big) \to +\infty.$$

Otherwise we will have uniform bound for $|A(x)|^2\big(\operatorname{dist}(x, x_0) - \frac{1}{n}\big)$ for a sequence of $n \to \infty$, then passing to limit we will have a uniform bound for $|A(x)|^2 \operatorname{dist}(x, x_0)$.

We can choose $z_n \in ((B_R(x_0) \setminus B_{1/n}(x_0)) \cap \Sigma)$ such that $|A(z_n)|^2\big(\operatorname{dist}(z_n, x_0) - \frac{1}{n}\big)$ achieves its maximum. Note that $|A(x)|^2\big(\operatorname{dist}(x, x_0) - \frac{1}{n}\big)$ equals 0 on $\partial B_{1/n}(x_0) \cap \Sigma$, so $d_n := \operatorname{dist}(z_n, x_0) - \frac{1}{n} > 0$.

We rescale $B_{d_n/2}(z_n)$ by $|A(z_n)|$, and denote the set $\{x \in \Sigma : \operatorname{dist}(x, z_n) \leq d_n/2\}$ after rescaling by $\widetilde{\Sigma}_n$. We will use tilde to denote the quantities on this new surface. Moreover, since $|A(z_n)| \to \infty$, the limit of the rescaling of $B_{d_n/2}(z_n)$ will converge to $\mathbb{R}^3$, so we can assume n is sufficiently large such that $\widetilde{\Sigma}_n$ actually lives in $\mathbb{R}^3$ with a metric which is perturbed from the standard Euclidean metric.

$\widetilde{\Sigma}_n$ satisfies the following properties:

(i) $|\widetilde{A}(0)| = 1$.

(ii) Since

$$|A(z_n)|^2 d_n \to +\infty,$$

we know that, for any fixed $R > 0$, $\widetilde{\Sigma}_n \cap \partial B_R(0) \neq \varnothing$ in $\mathbb{R}^3$ if n is large enough, and $\partial \widetilde{\Sigma}_n \cap B_R(0) = \varnothing$ if n is large enough.

(iii) For any $x' = |A(z)|x \in \widetilde{\Sigma}_n$, we have

$$|A(x)|\left(\operatorname{dist}(x - x_0)\operatorname{dist} - \frac{1}{n}\right) \leq |A(z)| d_n.$$

Since $\operatorname{dist}(x, z) \leq d_n/2$, we have $\operatorname{dist}(x, x_0) - \frac{1}{n} \geq d_n/2$, thus $|A(x)| \leq 2|A(z)|$, $|\widetilde{A}(x')| \leq 2$.

By the uniform curvature bound of $\widetilde{\Sigma}_n$, for each $R > 0$, there exists a subsequence (still denoted by $\widetilde{\Sigma}_n$) converging smoothly on $B_R(0)$ to a complete surface $\widetilde{\Sigma}$. By checking the equation after rescaling, we see that the limit $\widetilde{\Sigma}$ must be a minimal surface, i.e., $\widetilde{H} = 0$.

Since the rescaling does not change the integral of the squared curvature, we have

$$\int_{B_R(0)\cap\widetilde{\Sigma}} |A|^2 \leq \varepsilon.$$

Thus $\widetilde{\Sigma}$ has to be a plane if ε is small enough; see [White 1987, p. 249]. This is a contradiction to the condition that $|\widetilde{A}(0)| = 1$. □

Theorem 4.3. *The limit surface in Theorem 4.1 is smoothly immersed. Moreover, for $y \in \mathcal{S}$ a nonembedded point, in a small neighborhood of y, Σ is a union of two disks which are touching at y.*

Proof. We only need to prove that Σ is smooth around the singular set $\mathcal{S}$. Suppose $y \in \mathcal{S}$ is a singularity. We may assume r small enough such that $\int_{B_r(y)\cap\Sigma} |A|^2 \leq \varepsilon$ (Note $\Sigma \backslash \mathcal{S}$ has finite total curvature since the Σ_i's have uniformly bounded total curvature).

By Lemma 4.2, there is a constant C such that, for any $x \in B_r(y) \cap \Sigma$,

$$|A(x)| \operatorname{dist}(x, y) \leq C.$$

Now we choose a sequence $r_i \to 0$ and rescale $B_r(y)$ and Σ_i by $1/r_i$ and denote it by $\widetilde{\Sigma}_i$. Note the curvature bound

$$|A(x)| \operatorname{dist}(x, y) \leq C$$

is invariant under rescaling, so this uniform curvature bound indicates that $\widetilde{\Sigma}_i$ smoothly converges to a complete surface $\widetilde{\Sigma}$ in $\mathbb{R}^3 \setminus \{0\}$; see [White 1987].

Now for K a compact subset of $\mathbb{R}^3 \setminus \{0\}$,

$$\int_{\widetilde{\Sigma}_i \cap K} |A|^2 = \int_{\Sigma_i \cap r_i K} |A|^2 \to 0 \quad \text{as } r_i \to 0.$$

This implies that $\widetilde{\Sigma}$ is a union of planes. Thus $\Sigma \cap B_r(0)$ is actually a union of disks and punctured disks.

Now let Σ denote one of its connected components which is a punctured disk. Since $\widetilde{\Sigma}_i$ converges to the plane in $\mathbb{R}^3 \setminus \{0\}$, we can assume for some i that $\widetilde{\Sigma}_i$ can be written as a graph φ_i of that plane. Without loss of generality, let the plane be the xy plane in $\mathbb{R}^3$. By the computations in the Appendix, in B_1, φ_i satisfies an elliptic equation over the tangent plane:

$$L\varphi_i - (H_2 - H_1) = \operatorname{div}(a\nabla\varphi_i) + b \cdot \nabla\varphi_i + c\varphi_i. \tag{4-2}$$

Here all terms are defined on $\mathbb{R}^2 \cap B_1(0)$. Again, when i is large, each term on the right-hand side goes to 0. Then by the implicit function theorem, if we fixed the normal direction to point upwards, we can solve $\varphi_{i,t}$ for boundary data

$$\varphi_{i,t} = \varphi_i + t$$

on $\partial B_1(0)$. Then the graphs of $\varphi_{i,t}$ foliate a region of $(B_1(0) \cap \mathbb{R}^2) \times \mathbb{R}$. Since we fixed the direction of normal vectors, we can apply the maximal principle, which indicates that the leaf such that $\varphi_{i,t}(0) = 0$ lies on one side of $\widetilde{\Sigma}_i$. As a result, any sequence of dilations of Σ must converge to the same limit plane, which is just the tangent plane of that leaf at 0.

Thus $\Sigma \cup \{0\}$ is a C^1 graph of a function v in a neighborhood of 0. Since v is a $C^{2,\alpha}$ solution to an elliptic equation except 0, v is actually $C^{2,\alpha}$ everywhere. Hence $\Sigma \cup \{0\}$ is a smooth disk.

We have already shown that $\Sigma \cup \{0\}$ is a union of smooth disks. So Σ is an immersed surface, with locally finitely many curvature concentration points. By the maximal principle, at each touching point Σ consists of two disks which are touching at that point. So Σ is a smoothly self-touching immersed surface. □

Smooth convergence. In this subsection we first assume that the limit surface is not minimal, and discuss the situation when the limit surface is minimal at the end.

We will show the convergence is smooth apart from neck pinching points. Note we have already shown the convergence is smooth apart from $\mathcal{S}$, so we only need to show smooth convergence across points in $\mathcal{S}$ with density 1 (i.e., locally Σ is one disk) and points in $\mathcal{S}$ which are not neck pinching.

We first define neck pinching points. From Theorem 4.3 we know that for any points $y \in \mathcal{S}$ with density more than one, locally Σ is the union of two disks D_1, D_2, and Σ_i can be written as graphs G_i^1, G_i^2 of functions φ_i^1, φ_i^2 over $D_1 \setminus \{y\}$ and $D_2 \setminus \{y\}$ respectively.

Definition 4.4. We say y is a *neck pinching* point if there exists $r_0 > 0$ such that for $0 < r < r_0$, G_i^1 and G_i^2 do not lie in the same connected components of $\Sigma_i \cap B_r$ for at most finitely many Σ_i's.

Now we prove the convergence is smooth apart from these neck pinching points. The main ingredient is to show the convergence has multiplicity one. Then by the regularity theorem of Allard [1972] (see also [Choi and Schoen 1985] and [Colding and Minicozzi 2012]), we can show the convergence is smooth across those singularities with density 1. Finally we show that even for a singularity with density greater than 1, if it is not a neck pinching point we can still argue that the convergence is smooth across it.

Theorem 4.5. *The multiplicity of the convergence in Theorem 4.1 is one when the limit surface is not minimal.*

We follow the idea in [Colding and Minicozzi 2012]. The key ingredient is to show that if the convergence has multiplicity greater than 1, there exists a positive Jacobi field on Σ, which is a contradiction.

Proof. We argue as in [Choi and Schoen 1985] that we only need to consider the case that M is simply connected, and self-touching Σ is two sided (note although in [Choi and Schoen 1985] this argument is for closed embedded surfaces, it can be adapted to self-touching surfaces). If the convergence has multiplicity more than 1, then Σ_i's can be decomposed into several sheets of graphs on $\Sigma \setminus \mathcal{S}$. Since Σ is two-sided, we can label the graphs by height, and let the highest sheet of Σ_i be written as the graph of w_i^+, let the lowest sheet of Σ_i be written as the graph of w_i^-, and let $w_i = w_i^+ - w_i^-$. Fix a point p not in $\mathcal{S}$, and let $u(x) = w(x)/w(p)$. Then $u(p) = 1$ and $u > 0$ on $\Sigma \setminus \mathcal{S}$. Moreover, although w_i^- and w_i^+ do not satisfy a linear elliptic equation, their difference does. Hence u_i satisfies a linear elliptic equation. Then Harnack inequality implies C^α bound for u_i's and then standard elliptic theory gives $C^{2,\alpha}$ bound. Then by the Arezela–Ascoli theorem, a subsequence (still denoted by u_i) converges uniformly in C^2 on a compact subset of $\Sigma \setminus \mathcal{S}$ to a nonnegative function u on $\Sigma \setminus \mathcal{S}$ such that

$$Lu = 0, \quad u(p) = 1. \tag{4-3}$$

Next we show u can be extended smoothly across $\mathcal{S}$ to a solution of $Lu = 0$. Again we follow the idea in [White 1987] and [Colding and Minicozzi 2012]. We only need to show u is bounded around each singularity y, then by the standard elliptic theory u extends smoothly. Suppose u_i satisfies the linearized equation

$$L(u_i) = \operatorname{div}(a_i \cdot \nabla u_i) + b_i \cdot \nabla u_i + c_i u_i.$$

Then choose exponential normal coordinates over $B_\varepsilon(y) \subset \Sigma$ and a cylinder N over $B_\varepsilon(y) \cap \Sigma$; when ε is small, the implicit function theorem gives a foliation of graphs v_t over $B_\varepsilon(y) \cap \Sigma$ in N so that

$$v_0(x) = 0 \text{ for all } x \in B_\varepsilon(y) \quad \text{and} \quad v_t(x) = t \text{ for all } x \in \partial B_\varepsilon(y).$$

By the Harnack inequality, $t/C_i \le v_t \le C_i t$ for some $C_i > 0$. Since the right-hand side of the linearized equation turns to 0 as $i \to \infty$, C_i actually has uniform bound. Then by the maximum principle, u_i is bounded on $B_\varepsilon(y)$ by a multiple of its supremum on $B_\varepsilon(y) \setminus B_{\varepsilon/2}(y)$. Hence u has a removable of singularity at p.

So there exists a nonnegative solution u of the linearized operator $Lu = 0$. By $u(p) = 1$, Harnack inequality implies that u is positive everywhere. Then by Lemma 3.2, Σ is stable. However, plugging in a test function constant 1 implies that no immersed CMC surface in positive Ricci three-manifold can be stable, which is a contradiction. Then we conclude that the convergence has multiplicity one. □

By [Allard 1972], this theorem implies smooth convergence across those density 1 points. It remains to show smooth convergence across those touching singularities which are not neck pinching singularities.

Theorem 4.6. *The convergence is smooth apart from those neck pinching singularities.*

Proof. Let $y \in \mathcal{S}$ be a singularity with density greater than 1; then by Theorem 4.3 locally Σ is the union of two disks D_1, D_2. Then by the definition of pinching points, we know that if y is not a pinching point, locally $\Sigma_i = G_i^1 \cup G_i^2$ is the union of two graphs over $D_1 \setminus \{y\}$, $D_2 \setminus \{y\}$ respectively. Thus we only need to apply previous analysis to each graph G_i^j to get smooth convergence across y. □

Finally we discuss the situation that the limit Σ is an embedded minimal surface. Now multiplicity 2 convergence may happen because the CMC surfaces can converge to Σ from both sides with different orientation. However, if the convergence is of multiplicity larger than 2, there are at least two graphs that have the same orientation. Repeating the argument for these graphs, we again get a positive Jacobi field, which is a contradiction. Therefore, the convergence has at most multiplicity 2.

Combining all the ingredients in this section we conclude the main theorem (Theorem 1.1).

5. Touching examples

In this section we give some examples of touching points of CMC surfaces in three-dimensional manifolds.

Example 5.1 (kissing itself). Let us consider a sphere $\mathbb{S}_R$ with radius R in $\mathbb{R}^3$. By quotient a $\mathbb{Z}^3$ action of $\mathbb{R}^3$, we get a torus $\mathbb{T}^3$, and the image of $\mathbb{S}_R$ in $\mathbb{T}^3$ is an embedded CMC surface when R sufficiently small. Now we increase the radius of $\mathbb{S}_R$. Then for some specific R_0, in $\mathbb{T}^3$, $\mathbb{S}_{R_0}$ will kiss itself thus form a touching point. This is not a neck pinching point.

The touching set may be very large. For example, we can consider a cylinder $\mathcal{C}_R$ with radius R in $\mathbb{R}^3$. Using the same construction, we can see for some radius R_0, $\mathcal{C}_{R_0}$ kisses itself at a straight line, which is a one-dimensional curve.

Example 5.2 (unduloid neck pinching). An unduloid is a one periodic CMC surface in $\mathbb{R}^3$. See [Hadzhilazova et al. 2007] for a detailed discussion of unduloids.

The unduloid has two parameters a, c to determine its shape; see [Hadzhilazova et al. 2007, Theorem 3.1]. When $a \to 0, c \to 1/H$, we can see the family of unduloids will smoothly converge to the union of spheres apart from the touching points of spheres. This is an example of neck pinching singularity. One can see that the smooth convergence cannot cross these neck pinching points because the topology changes in the limit.

Of course, we can quotient $\mathbb{R}^3$ by some $\mathbb{Z}^3$ actions to make this example be an example in a closed three-manifold.

The reader may notice that these examples do not lie in a Ricci positive three-manifold. It is not known whether the touching behavior of CMC surfaces in positive Ricci three-manifolds is simpler or not. So we suggest the following conjectures:

Conjecture 5.3. A self-touching CMC surface in a three-dimensional manifold with positive Ricci curvature cannot carry infinitely many touching points.

Conjecture 5.4. For a CMC surface in a three-dimensional manifold with a one-dimensional touching set, the touching set must be a geodesic of the ambient space.

Another interesting observation is that a touching point of a self-touching CMC surface can be generated by both kissing and neck pinching process. For example, in $\mathbb{T}^3$, a sphere kissing itself can be generated by both the first example and the second example above. So a very natural question is whether any touching can be generated by both process? Some observations suggest the answer is probably “no”:

Example 5.5. Consider two spheres in $\mathbb{R}^3$ kissing at a single point p. Aleksandrov [1958] proved that any embedded CMC surface in $\mathbb{R}^3$ must be a standard sphere. They cannot be the limit of a sequence of embedded CMC surfaces; hence p cannot be a neck pinching point of a sequence of embedded CMC surfaces.

We suggest the following conjecture.

Conjecture 5.6. Suppose M is a compact three-manifold with positive Ricci curvature. Assume that $S_1 \cup S_2$ is the union of two embedded CMC spheres in M kissing at p. Then p cannot be a neck pinching point.

6. Eigenvalue estimate of CMC surfaces with small $|H|$

In this section we discuss an application of our main theorem. We will give a lower bound of the first eigenvalue of CMC surfaces in a positive Ricci three-manifold with small $|H|$.

The main idea is a method by Choi and Wang [1983]. They used an identity by Reilly to estimate the first eigenvalue of minimal surface in three-manifold. The main issue for generalizing their method to CMC surfaces is that we may not be able to control the term involving mean curvature (in the minimal surface case, this term vanishes). So we need a more delicate estimate for each term in Reilly's identity.

We first recall the proof by Choi and Wang [1983]. They used a formula by Reilly. For u a smooth function defined on a bounded domain Ω we have

$$\int_\Omega \big(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - (\Delta u)^2\big) = \int_{\partial\Omega} \big(A((\nabla u)^\top, (\nabla u)^\top) - 2u_n \Delta_{\partial\Omega} u + H u_n^2\big), \tag{6-1}$$

where u_n is the normal derivative and H is the mean curvature on $\partial\Omega$. Then they applied this formula when $\partial\Omega$ is minimal, where u is the harmonic function solving

$$\Delta_\Omega u = 0 \quad \text{and} \quad u|_{\partial\Omega} = f,$$

where f an eigenfunction of the first eigenvalue on $\partial\Omega$ such that $\int_{\partial\Omega} f^2 = 1$. Then they could get a first eigenvalue estimate for $\partial\Omega$, i.e., the minimal surface, in a simply connected three-dimensional manifold with positive Ricci curvature. Later, Choi and Schoen [1985] used a covering argument to extend the estimate to all closed three-manifolds with positive Ricci curvature.

Let us naively follow their method to deal with CMC surfaces. Suppose $\partial\Omega$ is a CMC surface with constant mean curvature H and the first eigenvalue of $\partial\Omega$ is λ. We will get the following inequality (see [Colding and Minicozzi 2011, p. 244]):

$$2\lambda \int_\Omega |\nabla u|^2 \geq (\min \mathrm{Ric}) \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla^2 u|^2 - \int_{\partial\Omega} A((\nabla u)^\top, (\nabla u)^\top) - H \int_{\partial\Omega} u_n^2. \tag{6-2}$$

Since A changes sign if we replace Ω by its complement, we can always assume

$$\int_{\partial\Omega} A((\nabla u)^\top, (\nabla u)^\top)$$

is nonnegative and get

$$2\lambda \int_\Omega |\nabla u|^2 \geq (\min \mathrm{Ric}) \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla^2 u|^2 - H \int_{\partial\Omega} u_n^2. \tag{6-3}$$

So our goal is to control $\int_\Omega |\nabla^2 u|^2 - H\int_{\partial\Omega} u_n^2$.

Trace theorem. In this subsection, we will transform the problem of controlling $\int_\Omega |\nabla^2 u|^2 - H\int_{\partial\Omega} u_n^2$ to the problem of getting an uniform tubular neighborhood of CMC surfaces. We need a trace theorem in three-manifolds. The idea of the proof is based on the proof in [Evans 2010].

Theorem 6.1. *Let $\Sigma = \partial\Omega$ be an embedded surface in a three manifold M. Suppose there is a constant δ such that $\exp_x(t\boldsymbol{n}) : \Sigma \times [-\delta, \delta] \to M$ is a diffeomorphism from $\Sigma \times [-\delta, \delta]$ to its image, and there is a constant A_0 such that $|A| \le A_0$ on Σ. Then there is a constant C only depending on M, A_0 and δ such that*

$$\int_{\partial\Omega} (u_n)^2 \le C \int_\Omega (|\nabla u|^2 + |\nabla^2 u|^2). \tag{6-4}$$

Proof. Note $(u_n)^2 \le |\nabla u|^2$, so we only need to prove a standard trace theorem

$$\int_{\partial\Omega} f^2 \le C \int_\Omega (f^2 + |\nabla f|^2).$$

By the conditions, we can pull back the metric of M to $\Sigma \times [-\delta, \delta]$, and by the uniform curvature bound, the pull back metric is uniformly closed to the standard production metric. In particular, we only need to prove the trace theorem on $\Sigma \times [-\delta, \delta]$ with product metric. Let us choose a cut-off function ζ such that $\zeta = 1$ on $\Sigma \times [-\delta/2, \delta/2]$ and is supported on $\Sigma \times [-\delta, \delta]$. Moreover we may assume that its gradient is bounded by C/δ for some constant C. Then

$$\begin{aligned} \int_{\partial\Omega} f^2\, dx' = \int_\Sigma f^2 \zeta\, dx' &= -\int_{\Sigma\times[-\delta,0]} (f^2\zeta)_{x_n}\, dx \\ &= -\int_{\Sigma\times[-\delta,0]} |f|^2 \zeta_{x_n} + 2 f f_{x_n} \zeta\, dx \\ &\le C \int_{\Sigma\times[-\delta,0]} |f|^2 + |\nabla f|^2\, dx. \end{aligned} \tag{6-5}$$

Here x_n is the normal direction (i.e., the direction on $[-\delta, \delta]$), dx' is the measure on Σ, and dx is the measure of the production metric. In the last inequality we use Young's inequality. Translating this back to M gives the desired trace theorem. □

If H is sufficiently close to 0, we can apply this trace theorem in the inequality (6-3) to get the eigenvalue lower bound

$$\lambda \ge \frac{\min \mathrm{Ric} - HC}{2}, \tag{6-6}$$

where C is a constant depending on M, A_0, δ.

Uniform bound for CMC surfaces with H close to 0. It remains to prove the pointwise curvature bound and the existence of δ in Theorem 6.1. We will use the compactness theorem to get these bounds for CMC surfaces with H small.

Theorem 6.2. *Suppose there is no embedded minimal surface in M which is the multiplicity 2 limit of a sequence of CMC surfaces. There exists $H_0 > 0$ such that an embedded CMC surface Σ with mean curvature $|H| \le H_0$, area less than V and genus less than G has curvature $|A| \le C(H_0, V, G)$*

Proof. We argue by contradiction. Suppose such H_0 does not exist. Then we can find a family of CMC surfaces Σ_i, with mean curvature $H_i \to 0$ such that a point p_i on Σ_i has curvature $|A(p_i)| \to \infty$ as $i \to \infty$. By compactness of M we may assume $p_i \to p$ for a point $p \in M$. Now, by the main theorem (Theorem 1.1), Σ_i converges to a minimal surface Σ. Since Σ is minimal, by the maximum principle there is no touching point. So the convergence is everywhere smooth. However $|A(p)|$ is finite since Σ is a smoothly embedded surface, which is a contradiction. Thus H_0 exists. □

Theorem 6.3. *Suppose there is no embedded minimal surface in M which is the multiplicity 2 limit of a sequence of CMC surfaces. There exists $H_0 > 0$ and $\delta_0 > 0$ such that for an embedded CMC surface Σ with mean curvature $|H| \le H_0$, area less than V and genus less than G, $\exp_x(t\boldsymbol{n}) : \Sigma \times [-\delta, \delta] \to M$ is a diffeomorphism.*

Proof. We argue by contradiction. Suppose such H_0, δ_0 does not exist. Then we can find a family of CMC surfaces Σ_i, with mean curvature $H_i \to 0$ and $\delta_i \to 0$ such that there is a point $p \in M$ such that $p = \exp_{x_i^j}(t_i^j \boldsymbol{n})$, $j = 1, 2$ for $x_i^j \in \Sigma_i$ and $t_i^j \in [-\delta_i, \delta_i]$. Since we have already obtained an uniform curvature bound for Σ_i, we know $\mathrm{dist}_\Sigma(x_i^1, x_i^2) \ge d$ for some constant d when i is large enough.

Again, a subsequence of Σ_i smoothly converges to a smooth embedded minimal surface Σ. By passing to a subsequence we can find two points $x^1, x^2 \in \Sigma$, with intrinsic distance $\mathrm{dist}_\Sigma(x^1, x^2) \ge d$ but extrinsic distance $\mathrm{dist}_M(x^1, x^2) = 0$. This is a contradiction by the maximum principle of minimal surfaces. □

Combining all the ingredients in this section we get the following lower bound for the first eigenvalue of CMC surfaces:

Theorem 6.4 (Theorem 1.2). *Let M be a three-manifold with positive Ricci curvature. Suppose there is no embedded minimal surface in M which is the multiplicity 2 limit of a sequence of CMC surfaces. Then for any embedded CMC surface with area bound V, genus bound G and mean curvature bound $|H| \le H_0$, we have the first eigenvalue lower bound*

$$\lambda \ge \frac{\min \mathrm{Ric} - HC}{2}, \tag{6-7}$$

where C is a constant depending on M, V, G, H_0.

Remark 6.5. An interesting question is: can we directly get the first eigenvalue lower bound for CMC surfaces? If we can, then we can prove the compactness theorem for CMC surfaces without area bound.

Appendix: Difference of two surfaces in three manifold

Here we will present some computations of the difference of two surfaces in a three-manifold. These kinds of computation have already appeared in [Kapouleas 1990] and [Colding and Minicozzi 2011] in three-dimensional Euclidean space.

Let Σ_1, Σ_2 be two surfaces in three-manifold M, and let H_1, H_2 be their mean curvatures respectively. Moreover, we assume Σ_2 can be viewed as a graph over Σ_1, i.e.,

$$\Sigma_2 = \{\exp_x(\varphi \boldsymbol{n}) : x \in \Sigma_1\},$$

where φ is a C^2 function on Σ_1.

Theorem A.1. *Suppose $\|\varphi\|_{C^2}$ is small enough, then φ satisfies a second order elliptic equation.*

Proof. Since this assertion is a local assertion, we only need to check this in a small neighborhood U of $p \in \Sigma_1$. Let us choose the Fermi coordinate x_1, x_2, x_3 in U (so we can view U as an open subset of $\mathbb{R}^3$ with non-Euclidean metric), such that

$$\Sigma_1 = \{(x_1, x_2, x_3) : x_3 = 0\}.$$

Moreover, the metric g under this coordinate satisfies $g_{i3} = 0$, $i = 1, 2$, and $(0, 0, 1)$ is the unit normal vector at each point in Σ_1. We will use $\partial_1, \partial_2, \partial_3$ to denote the vector fields defined on M with respect to the differential under this coordinate.

Σ_2 is a graph,

$$\Sigma_2 = \{(x_1, x_2, x_3) : x_3 = \varphi(x_1, x_2)\}.$$

From now on we will use tilde over quantity to denote the quantity of Σ_2. We use x_1, x_2 to parametrize Σ_2, then we have

$$\tilde{\partial}_i = \tilde{\partial}_{x_i} = \partial_i + \varphi_i \partial_3, \; i = 1, 2. \tag{A-1}$$

Then the metric on Σ_2 satisfies

$$\tilde{g}_{ij} = g_{ij} + g_{i3}\varphi_j + g_{j3}\varphi_i + g_{33}\varphi_i\varphi_j. \tag{A-2}$$

Now we compute the unit normal vector fields on Σ_2. We observe that Σ_2 can be viewed as the 0-level set of the function $\varphi(x_1, x_2) - x_3$. So we can find a normal vector field $\boldsymbol{m}$ on Σ_2:

$$\boldsymbol{m} = -\nabla^M(\varphi(x_1, x_2) - x_3) = g^{ij}(\varphi_k \delta_i^k - \delta_{3i})\partial_j. \tag{A-3}$$

Note

$$\begin{aligned}\langle \boldsymbol{m}, \boldsymbol{m}\rangle &= g^{ij}(\varphi_k\delta_i^k - \delta_{3i})g^{pq}(\varphi_k\delta_p^k - \delta_{3p})g_{qj}\\ &= g^{ij}\varphi_i\varphi_j - 2g^{3k}\varphi_k + g^{33}.\end{aligned}$$

So

$$\boldsymbol{n} = -(g^{pq}\varphi_p\varphi_q - 2g^{3l}\varphi_l + g^{33})^{-1/2}g^{ij}(\varphi_k\delta_i^k - \delta_{3i})\partial_j. \tag{A-4}$$

Now let us calculate the mean curvatures. From now on we will slightly abuse the notation when we use Einstein notation. When we use i, j in the summation we will assume they are in $\{1, 2\}$. On Σ_1, $\boldsymbol{n} = \partial_3$, so we have

$$H_1 = g^{ij}\langle \nabla_{\partial_i}\partial_j, \partial_3\rangle = g^{ij}\Gamma_{ij}^k g_{k3}. \tag{A-5}$$

On Σ_2, first we note the covariant derivative is

$$\begin{aligned}\nabla_{\tilde{\partial}_i}\tilde{\partial}_j &= \nabla_{\partial_i+\varphi_i\partial_3}(\partial_j + \varphi_j\partial_3)\\ &= \nabla_{\partial_i}\partial_j + \varphi_i\nabla_{\partial_3}\partial_j + \varphi_{ij}\partial_3 + \varphi_i\varphi_j\nabla_{\partial_3}\partial_3\\ &= \nabla_{\partial_i}\partial_j + \varphi_i\nabla_{\partial_3}\partial_j + \varphi_{ij}\partial_3.\end{aligned} \tag{A-6}$$

Here we note that ∂_3 is the direction of the geodesic starting from Σ_1; hence $\nabla_{\partial_3}\partial_3 = 0$. Then the mean curvature of Σ_2 is

$$\begin{aligned}H_2 &= \widetilde{g^{ij}}\langle \nabla_{\tilde{\partial}_i}\tilde{\partial}_j, \boldsymbol{n}\rangle\\ &= \widetilde{g^{ij}}\big\langle \nabla_{\partial_i}\partial_j + \varphi_i\nabla_{\partial_3}\partial_j + \varphi_{ij}\partial_3,\\ &\qquad - (g^{pq}\varphi_p\varphi_q - 2g^{3l}\varphi_l + g^{33})^{-1/2}g^{rs}(\varphi_k\delta_r^k - \delta_{3r})\partial_s\big\rangle\\ &= -\widetilde{g^{ij}}(g^{pq}\varphi_p\varphi_q - 2g^{3l}\varphi_l + g^{33})^{-1/2}\\ &\qquad \times g^{rs}(\varphi_k\delta_r^k - \delta_{3r})(\Gamma_{ij}^m g_{ms} + \varphi_i\Gamma_{3j}^m g_{ms} + \varphi_{ij}g_{3s})\end{aligned} \tag{A-7}$$

In conclusion, H_2 is a function of φ, $\nabla\varphi$, $\nabla^2\varphi$ and the coordinate in ambient manifold. We define a function $H(x_1, x_2, x_3, v_i, w_{ij})$ where $H(x, y, \varphi, \nabla\varphi, \nabla^2\varphi) = H_2$ as above, where (x_1, x_2, x_3) is the local coordinate. Also note $H(x, y, 0, 0, 0, 0) = H_1$. Then we have

$$\begin{aligned}H_2 - H_1 &= \varphi\int_0^1 \frac{\partial H(x_1, x_2, t\varphi, t\nabla\varphi, t\nabla^2\varphi^2)}{\partial x_3}\,dt\\ &\qquad + \varphi_i\int_0^1 \frac{\partial H(x_1, x_2, t\varphi, t\nabla\varphi, t\nabla^2\varphi^2)}{\partial v_i}\,dt\\ &\qquad + \varphi_{jk}\int_0^1 \frac{\partial H(x_1, x_2, t\varphi, t\nabla\varphi, t\nabla^2\varphi^2)}{\partial w_{jk}}\,dt.\end{aligned} \tag{A-8}$$

Let the coefficients of φ, $\nabla\varphi$, $\nabla^2\varphi$ on the right-hand side of the above identity be a function depending on φ, $\nabla\varphi$, $\nabla^2\varphi$. Then letting $\|\varphi\|_{C^2}$ go to 0 we can see the right-hand side terms will just be Lu by the second variational formula. Thus we have

$$Lu-(H_2-H_1)=\operatorname{div}(a\nabla\varphi)+b\cdot\nabla\varphi+c\varphi, \tag{A-9}$$

where a, b, c turns to 0 as $\|\varphi\|_{C^2}$ goes to 0. □

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THE RATIONAL COHOMOLOGY HOPF ALGEBRA OF A GENERIC KAC–MOODY GROUP

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In this paper we determine the rational homotopy type of the classifying space of a generic Kac–Moody group by computing its rational cohomology ring. As an application we determine the rational homology Hopf algebra of the generic Kac–Moody group.

1. Introduction

Let $A = (a_{ij})$ be an $n \times n$ Cartan matrix. By Kac [1968] and Moody [1968], it is well known that there is a Kac–Moody Lie algebra $g(A)$ associated to A. The corresponding Kac–Moody group $G(A)$ was constructed in [Kac and Peterson 1983; 1985; Kac 1985]. In this paper for convenience we consider the derived Lie algebra $g'(A)$ and the associated simply connected group $G'(A)$. But we still use the symbols $g(A)$ and $G(A)$.

Cartan matrices are divided into three types, i.e., finite type, affine type and indefinite type. A Cartan matrix A is indecomposable if A can't be written as a direct sum of two Cartan matrices A_1 and A_2. A is symmetrizable if there exists an invertible diagonal matrix D and a symmetric matrix B such that $A = DB$, see [Kac 1990] for details. A Cartan matrix A is generic if $a_{ij}a_{ji} \geq 4$ for all i, j. A is generic if and only if all its principal submatrices of rank 2 are not of finite type. All these properties for Cartan matrices can be used for the associated Kac–Moody Lie algebras and Kac–Moody groups. For example a generic Kac–Moody group is indecomposable. The Weyl group $W(A)$ of a generic Cartan matrix A is the group generated by the Weyl reflections σ_i, $1 \leq i \leq n$. It has a Coxeter presentation

$$W(A) = \langle \sigma_1, \cdots, \sigma_n \mid \sigma_i^2 = e,\ 1 \leq i \leq n \rangle.$$

A main result in this paper is that the graded Lie algebra $\pi_{\text{even}}(G(A))$ formed by even dimensional homotopy groups of a generic Kac–Moody group $G(A)$ is a free Lie algebra with infinite generators.

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Zhao and Jin [2015] determined the rational homotopy type of the indefinite Kac–Moody group $G(A)$. $G(A)$ is a Hopf space. It is important to determine the rational Hopf homotopy type of $G(A)$. This is equivalent to determine the rational cohomology Hopf algebra $H^*(G(A))$ or the dual rational homology Hopf algebra $H_*(G(A))$. It is further equivalent to determine the rational cohomology algebra $H^*(BG(A))$ of the classifying space $BG(A)$.

On the rational homotopy group $\pi_*(G(A))$, the Samelson product

$$[\cdot,\cdot]:\pi_p(G(A))\times\pi_q(G(A))\to\pi_{p+q}(G(A))$$

is defined as

$$[\alpha,\beta](s\wedge t)=\alpha(s)\beta(t)\alpha(s)^{-1}\beta(t)^{-1},\quad s\in S^p,\ t\in S^q,$$

and $(\pi_*(G(A)),[\cdot,\cdot])$ is a rational graded Lie algebra.

Let $\chi:\pi_*(G(A))\to H_*(G(A))$ be the Hurewicz morphism of graded Lie algebras. By [Milnor and Moore 1965], the induced morphism $\widetilde{\chi}:U\big(\pi_*(G(A))\big)\to H_*(G(A))$ is an isomorphism of Hopf algebras, where $U\big(\pi_*(G(A))\big)$ is the universal enveloping algebra of $\pi_*(G(A))$, and $H_*(G(A))$ is primitively generated by $\pi_*(G(A))$. So to determine the Hopf algebra structure on $H_*(G(A))$, it is enough to compute the graded Lie algebra $\pi_*(G(A))$. By combining rational homotopy theory (see [Sullivan 1977]) with the cohomology ring $H^*(G(A))$ (see [Zhao et al. 2017]), one knows that for a generic Cartan matrix A, $\pi_{\mathrm{odd}}(G(A))\cong\mathbb{Q}$ or $\{0\}$ depending on whether A is symmetrizable or not. And the decomposition $\pi_*(G(A))=\pi_{\mathrm{even}}(G(A))\oplus\pi_{\mathrm{odd}}(G(A))$ is a decomposition of Lie algebras. $H_{\mathrm{even}}(G(A))$ (i.e., the rational Chow ring of $G(A)$) is isomorphic to the universal enveloping algebra $U(\pi_{\mathrm{even}}(G(A)))$. By [Kac 1985] the Poincaré series of $H_{\mathrm{even}}(G(A))$ is

$$C_A(q)=P_{F(A)}(q)(1-q^2)^n(1-q^4)^{-\epsilon(A)}.$$

Here $P_{F(A)}(q)$ is the Poincaré series of the flag manifolds $F(A)$ associate to Kac–Moody group $G(A)$ and $\epsilon(A)=0$ for A nonsymmetrizable and $\epsilon(A)=1$ for A symmetrizable. By [Zhao et al. 2017],

$$P_{F(A)}(q)=\frac{1+q^2}{1-(n-1)q^2}.$$

Hence we can compute the Poincaré series of $H_{\mathrm{even}}(G(A))$.

For a nonsymmetrizable Cartan matrix A, we write the Poincaré series

$$C_A(q)=\frac{1}{\frac{1-(n-1)q^2}{(1-q^2)^{n-1}(1-q^4)}}$$

as

$$\frac{1}{1 - a_4 q^4 - a_6 q^6 - \cdots - a_{2i} q^{2i} - \cdots},$$

and for a symmetrizable Cartan matrix A, we write

$$C_A(q) = \frac{1}{\frac{1-(n-1)q^2}{(1-q^2)^{n-1}}}$$

as

$$\frac{1}{1 - b_4 q^4 - b_6 q^6 - \cdots - b_{2i} q^{2i} - \cdots},$$

where for $i \geq 2$,

$$a_{2i} = \sum_{k=0}^{[i/2]} (i-1-2k) \binom{n+i-2k-3}{n-3}, \quad b_{2i} = (i-1) \binom{n+i-3}{n-3}$$

are natural numbers depending on n.

These two Poincaré series are the same as the Poincaré series of the tensor Hopf algebra with a_{2i} and b_{2i} generators of degree $2i$ for each $i \geq 2$. In fact, we have

Theorem 1. *For a nonsymmetrizable* (or symmetrizable) *generic Kac–Moody group* $G(A)$, *the graded Lie algebra* $\pi_{\mathrm{even}}(G(A))$ *is a free Lie algebra with* a_{2i} (or b_{2i}) *generators of degree* $2i$ *for each* $i \geq 2$.

This result was previously conjectured by Zhao Xu-an and Jin Chunhua [2015].

The graded Lie algebra $\pi_*(G(A))$ with Samelson product is determined by $\pi_*(BG(A))$ with Whitehead product, and $\pi_*(BG(A))$ can in turn be determined by $H^*(BG(A))$. We determine $H^*(BG(A))$ by computing its Poincaré series. And we have

Theorem 2. *If* $A = (a_{ij})_{n \times n}$ *is a nonsymmetrizable generic Cartan matrix*, $n \geq 3$, *then the Poincaré series of* $H^*(BG(A))$ *is*

$$P_n(q) = q \left[\frac{(n-1)q^2 - 1}{(1-q^2)^{n-1}(1-q^4)} + 1 \right] + 1.$$

Theorem 3. *If* $A = (a_{ij})_{n \times n}$ *is a symmetrizable generic Cartan matrix*, $n \geq 2$, *then the Poincaré series of* $H^*(BG(A))$ *is*

$$Q_n(q) = \frac{1}{1-q^4} \left(q \left[\frac{(n-1)q^2 - 1}{(1-q^2)^{n-1}} + 1 \right] + 1 \right).$$

The contents of this paper are as follows: in Section 2 we give some preparatory lemmas, in Section 3 we prove Theorems 2 and 3, in Section 4 we give some results derived from these theorems, including Theorem 1.

2. Some preparatory lemmas

In the following all the Cartan matrices are assumed to be generic. All the homology and cohomology are of rational coefficients.

Let S be the set of integers $1, 2, \dots, n$, and $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the simple root system of $G(A)$. For each $I \subset S$, the matrix $A_I = (a_{ij})_{i,j\in I}$ is also a Cartan matrix. Corresponding to $I \subset S$, there is a parabolic subgroup $G_I(A)$ of $G(A)$ whose simple root system is $\Pi_I = \{\alpha_i \mid i \in I\}$. All the proper subsets of S form a category C with object $I \subset S$ and morphism $I \subset J$. By constructing classifying spaces we have a functor $F : \mathrm{C} \to \mathrm{Top}$ which sends I to $BG_I(A)$ and $I \subset J$ to the map $BG_I(A) \to BG_J(A)$.

Since we only consider the homotopy type of the Kac–Moody group we replace the group $G(A)$ (or $G_I(A)$) by its unitary form and use the same symbol.

We need the following lemmas to prove the main theorems.

Lemma 2.1. *For a Kac–Moody group $G(A)$ and $I \subset S$, the subgroup $G_I(A)$ is isomorphic to $G(A_I) \,\tilde{\times}\, T^{n-|I|}$, the semidirect product of $G(A_I)$ and $T^{n-|I|}$. As a result there is an isomorphism $H^*(BG_I(A)) \cong H^*(BG(A_I)) \times H^*(BT^{n-|I|})$.*

By this lemma, the Poincaré series of $BG_I(A)$ is obtained from the Poincaré series of $BG(A_I)$ by multiplying a factor $1/(1-q^2)^{n-|I|}$.

By [Kitchloo 1998; Broto and Kitchloo 2002], for a Cartan matrix of infinite type, the homotopy colimit of the functor F gives the homotopy type of $BG(A)$. For any $I \in \mathrm{C}$, let C_I be the full subcategory of C whose objects are proper subsets of I. If $|I| \geq 2$, then $G_I(A)$ is of infinite type. By using the result of Kitchloo to $BG_I(A)$, we get $H^*(BG_I(A)) \simeq \mathrm{colimit} F|_{\mathrm{C}_I}$. As a consequence we have:

Lemma 2.2. *Let C' be the full subcategory of C which contains only objects $\varnothing, \{1\}, \{2\}, \dots, \{n\}$. Then for a generic Kac–Moody group $G(A)$,*

$$\begin{aligned} BG(A) &\simeq \mathrm{colimit} F|_{\mathrm{C}'} \\ &\simeq BG_1(A) \cup_{BT} BG_2(A) \cup_{BT} \cdots \cup_{BT} BG_n(A) \\ &\simeq BG_{\{1,2,\dots,n-1\}}(A) \cup_{BT} BG_n(A). \end{aligned}$$

The action of Weyl group $W(A)$ (or $W_I(A)$) of $G(A)$ (or $G_I(A)$) on the maximal torus T induces the action of $W(A)$ (or $W_I(A)$) on $H^*(BT)$.

Lemma 2.3. *For a generic Kac–Moody group $G(A)$, the image of the homomorphism $Bi_I^* : H^*(BG_I(A)) \to H^*(BT)$ induced by the inclusion $i_I : T \subset G_I(A)$ is $H^*(BT)^{W_I(A)}$, i.e., the $W_I(A)$ invariants. In particular the image of the homomorphism $H^*(BG(A)) \to H^*(BT)$ is $H^*(BT)^{W(A)}$.*

This lemma is the generalization of a result of Borel [1953] for compact Lie groups. It can be proved in the inductive procedure of the proofs for the main theorems.

Lemma 2.4. *If A is a nonsymmetrizable generic Cartan matrix, then there exist $i, j, k \in S$, $i < j < k$ such that $A_{\{i,j,k\}}$ is nonsymmetrizable.*

Proof. Suppose this lemma is not true for Cartan matrix A. Then for any $i < j < k$, $A_{\{i,j,k\}}$ is symmetrizable. Hence $a_{ij}a_{jk}a_{ki} = a_{ik}a_{kj}a_{ji}$. We set $d_i = a_{1i}/a_{i1}$. For $1 < i < j$, we have $a_{1i}a_{ij}a_{j1} = a_{1j}a_{ji}a_{i1}$, hence $a_{ij}/a_{ji} = d_j/d_i$. But this means that A is symmetrizable, a contradiction. $\square$

Lemma 2.5. *Let A be a generic Cartan matrix. If A is symmetrizable, then $H^*(BT)^{W(A)} \cong \mathbb{Q}[\psi]$, where ψ is the Killing form. If A is nonsymmetrizable, then $H^*(BT)^{W(A)} \cong \mathbb{Q}$.*

This result was proved in [Zhao and Jin 2014]. In fact it is valid for an arbitrary indefinite and indecomposable Cartan matrix.

Lemma 2.6. *Let $X = X_1 \cup_{X_0} X_2$ be the push-out of the diagram $X_1 \xleftarrow{j_1} X_0 \xrightarrow{j_2} X_2$. The homomorphism $j : H^*(X_1) \oplus H^*(X_2) \to H^*(X_0)$ is given by*

$$j(u, v) = j_1^*(u) - j_2^*(v).$$

If X_1, X_2 are deformation retracts of some open subspaces of X, then there exists a short exact sequence

$$0 \to \Sigma \operatorname{coker} j \to H^*(X) \to \ker j \to 0.$$

Proof. We have the Mayer–Vietoris exact sequence

$$\cdots \to H^{*-1}(X_1) \oplus H^{*-1}(X_2) \xrightarrow{j} H^{*-1}(X_0) \xrightarrow{\delta} H^*(X) \\ \xrightarrow{i} H^*(X_1) \oplus H^*(X_2) \xrightarrow{j} H^*(X_0) \to \cdots .$$

From this sequence we get the short exact sequence

$$0 \to \operatorname{im} \delta \to H^*(X) \to \operatorname{im} i \to 0.$$

By the exactness of this sequence, we have

$$\operatorname{im} i \cong \ker j \quad \text{and} \quad \operatorname{im} \delta \cong H^{*-1}(X_0)/\ker \delta \cong H^{*-1}(X_0)/\operatorname{im} j \cong \operatorname{coker} j. \quad \square$$

Lemma 2.7. *Let A be a generic 2×2 Cartan matrix, then $H^*(BG(A)) \cong \mathbb{Q}[\psi]$, where ψ corresponds to the Killing form which has degree 4. The Poincaré series of $BG(A)$ is $1/(1 - q^4)$.*

Proof. By Lemma 2.6, we have the short exact sequence

$$0 \to \Sigma \operatorname{coker} j \to H^*(BG(A)) \to \ker j \to 0$$

with $j : H^*(BG_{\{1\}}(A)) \oplus H^*(BG_{\{2\}}(A)) \to H^*(BT)$.

The Poincaré series of coker j is

$$\frac{1}{(1-q^2)^2} - \frac{2}{(1-q^2)(1-q^4)} + \frac{1}{1-q^4} = 0.$$

Since ker j is isomorphic to $H^*(BT)^{W(A)} = \mathbb{Q}[\psi]$, its Poincaré series is $1/(1-q^4)$. Hence $H^*(BG(A)) \cong \ker j \cong \mathbb{Q}[\psi]$. □

This lemma is the special case of $n = 2$ for Theorem 3.

3. The proofs of the main theorems

The proof of Theorem 1 depends on the Poincaré series of $BG(A)$. So we prove Theorems 2 and 3 first.

In this section we denote $BG_I(A)$ by X_I for simplicity. For $I = \{i_1, i_2, \ldots, i_k\}$, we always denote X_I by $X_{i_1 i_2 \cdots i_k}$. So we have

$$X_\varnothing = BT \quad \text{and} \quad X_{12\cdots k} \simeq X_{12\cdots k-1} \cup_{BT} X_k.$$

Proof of Theorem 2. For $n = 3$, $BG(A)$ is homotopic equivalent to $X_{12} \cup_{BT} X_3$. By Lemma 2.6 we have the short exact sequence

$$0 \to \Sigma \operatorname{coker} j \to H^*(BG(A)) \to \ker j \to 0.$$

The homomorphism $j : H^*(X_{12}) \oplus H^*(X_3) \to H^*(BT)$ is given by $j(u, v) = Bi_{12}^*(u) - Bi_3^*(v)$, where Bi_{12}^* and Bi_3^* are induced by the homomorphisms $i_{12} : T \to G_{12}(A)$ and $i_3 : T \to G_3(A)$. By Lemma 2.3, we observe that ker j is the subring of Weyl group invariants. Since A is nonsymmetrizable, by Lemma 2.5, ker $j \cong \mathbb{Q}$. We have

$$\operatorname{im} j = \operatorname{im}(Bi_{12}^*) + \operatorname{im}(Bi_3^*) \quad \text{and} \quad \operatorname{im}(Bi_{12}^*) \cap \operatorname{im}(Bi_3^*) = \ker j \cong \mathbb{Q}.$$

By Lemma 2.1, the Poincaré series of $\operatorname{im}(Bi_{12}^*)$ and $\operatorname{im}(Bi_3)^*$ are

$$\frac{1}{(1-q^2)(1-q^4)} \quad \text{and} \quad \frac{1}{(1-q^2)^2(1-q^4)},$$

respectively. Combining these results, the Poincaré series of coker j is

$$\frac{1}{(1-q^2)^3} - \frac{1}{(1-q^2)(1-q^4)} - \frac{1}{(1-q^2)^2(1-q^4)} + 1 = \frac{2q^2-1}{(1-q^2)^2(1-q^4)} + 1.$$

Hence for $n = 3$ the Poincaré series of $H^*(BG(A))$ is

$$q\left[\frac{2q^2-1}{(1-q^2)^2(1-q^4)} + 1\right] + 1.$$

For $n \geq 4$, we prove this theorem by induction on n. We assume that the theorem is true for $n - 1$. Since A is nonsymmetrizable, by Lemma 2.4, without loss of generality we can assume A_{123} is nonsymmetrizable. Then $A' = A_{12\cdots n-1}$ is also nonsymmetrizable. By Lemma 2.1, $H^*(X_{1,2\cdots n-1}) \cong H^*(BG(A')) \otimes H^*(BS^1)$. By the induction assumption, the Poincaré series of $BG(A')$ is

$$P_{n-1}(q) = q\left[\frac{(n-2)q^2-1}{(1-q^2)^{n-2}(1-q^4)} + 1\right] + 1.$$

This means that the reduced cohomology $\widetilde{H}^*(BG(A'))$ concentrates in odd dimensions. Since $X \simeq X_{12\cdots n-1} \cup_{BT} X_n$, we use Lemma 2.6 to compute $H^*(BG(A))$. By the decomposition $H^*(X_{12\cdots n-1}) \cong \widetilde{H}^*(BG(A')) \otimes H^*(BS^1) \oplus \mathbb{Q} \otimes H^*(BS^1)$, we have $\ker j \cong \widetilde{H}^*(BG(A')) \otimes H^*(BS^1) \oplus \mathbb{Q}$, and $\operatorname{im} j = \operatorname{im} Bi^*_{1,2,\ldots,n-1} + \operatorname{im} Bi^*_n$. The intersection of $\operatorname{im} Bi^*_{1,2,\ldots,n-1}$ and $\operatorname{im} Bi^*_n$ is the subring of Weyl group invariants. It is isomorphic to $\mathbb{Q}$. The Poincaré series of coker j is

$$\frac{1}{(1-q^2)^n} - \frac{1}{(1-q^2)^{n-1}(1-q^4)} - \frac{1}{1-q^4} + 1.$$

Therefore the Poincaré series of $BG(A)$ is

$$\frac{q}{(1-q^2)^n} - \frac{q}{(1-q^2)^{n-1}(1-q^4)} - \frac{q}{1-q^4} + q + \frac{q}{1-q^2}(P_{n-1} - 1) + 1$$

which is equal to

$$q\left[\frac{(n-1)q^2-1}{(1-q^2)^{n-1}(1-q^4)} + 1\right] + 1. \qquad \square$$

Proof of Theorem 3. The proof of this theorem is similar to that of Theorem 2. The difference is that for the symmetrizable case, the invariants of Weyl group is generated by the Killing form which is in degree 4.

We use induction on n. If $n = 2$, by Lemma 2.7, the theorem is true. We assume that the theorem is true for $n-1$. For an $n \times n$ symmetrizable generic Cartan matrix A, $A' = A_{12\cdots n-1}$ is a symmetrizable generic Cartan matrix. By the induction assumption, the Poincaré series of $BG(A')$ is

$$Q_{n-1} = \frac{1}{1-q^4}\left(q\left[\frac{(n-2)q^2-1}{(1-q^2)^{n-2}} + 1\right] + 1\right).$$

Since $BG(A)$ is homotopy equivalent to $X_{12\cdots n-1} \cup_{BT} X_n$. By a similar Mayer–Vietoris sequence computation, we get that the Poincaré series of $BG(A)$ is

$$\begin{aligned} Q_n = {} & \frac{q}{(1-q^2)^n} - \frac{q}{(1-q^2)^{n-1}(1-q^4)} - \frac{q}{(1-q^2)(1-q^4)} + \frac{q}{(1-q^4)} \\ & + \frac{1}{(1-q^2)(1-q^4)}(Q_{n-1} - 1) + \frac{1}{1-q^4} \\ = {} & \frac{1}{1-q^4}\left[q\left(\frac{(n-1)q^2-1}{(1-q^2)^{n-1}} + 1\right) + 1\right]. \qquad \square \end{aligned}$$

Remark. In the proof of Theorem 3, we need the $H^*(BG(A))$-module structure on the Mayer–Vietoris sequences. In fact all the cohomology groups that appeared in the sequence are free $\mathbb{Q}[\psi]$-modules.

4. Some results derived from the main theorems

In this section we need some general results in algebraic topology. For details see [Whitehead 1978].

From the expressions of the Poincaré series of $BG(A)$ in Theorem 2, we can see that for the nonsymmetrizable case, the even dimensional cohomology group is $H^0(BG(A))$. Hence the cup product on $H^*(BG(A))$ is trivial and we have:

Corollary 4.1. *For a nonsymmetrizable generic $n\times n$ Cartan matrix A, the rational homotopy type of $BG(A)$ is $\bigvee_{i=2}^{\infty}\bigvee_{j=1}^{\alpha_i} S^{2i+1}$ with*

$$P_n(q)=1+\alpha_2q^5+\alpha_3q^7+\cdots+\alpha_iq^{2i+1}+\cdots.$$

Lemma 4.2. $$\alpha_i=a_{2i}\quad \textit{for all } i\geq 2.$$

Proof. By definition we have

$$\begin{aligned}\alpha_2q^4+\alpha_3q^6+\cdots&=\frac{P_n(q)-1}{q}=\frac{(n-1)q^2-1}{(1-q^2)^{n-1}(1-q^4)}+1\\&=1-\frac{1}{C_A(q)}=a_4q^4+a_6q^6+\cdots\end{aligned}$$ □

By the Milnor–Hilton theorem (see [Whitehead 1978]) and the homotopy equivalence $G(A)\simeq\Omega BG(A)$, we get:

Corollary 4.3. *The homotopy Lie algebra $\pi_*(G(A))$ with Samelson product is the free graded Lie algebra generated by $\Sigma^{-1}\widetilde{H}_*(BG(A))$.*

The Hopf algebra $H_*(G(A))$ and the tensor algebra $T\big(\Sigma^{-1}\widetilde{H}_*(BG(A))\big)$ are isomorphic.

From the expressions of the Poincaré series in Theorem 3, we can see that for the symmetrizable case, as $\mathbb{Q}[\psi]$-module the only even dimensional generator of $H^*(BG(A))$ is $1\in H^0(BG(A))$. The cup product on $H^*(BG(A))$ can be determined by the following lemma.

Lemma 4.4. *Let $R=R_0\oplus R_1$ be a ring with Z_2-gradation satisfying $ab=(-1)^{|a||b|}ba$ for homogeneous a,b and let R_1 be a free R_0-module. If the characteristics of R is not 2, then $R_1R_1=0$.*

Proof. For each $b\in R_1$, we have $bb=-bb$. Hence $b^2=0$. Let $b_1,b_2,\ldots$ be a basis of R_1 as R_0-module. Then $b_1^2=b_2^2=\cdots=0$. We have $(b_ib_j)b_i=0$ for all i,j, since b_i is a base element, so $b_ib_j=0$. As a result $R_1R_1=0$. □

Set $R_0 = H^{\text{even}}(BG(A))$ and $R_1 = H^{\text{odd}}(BG(A))$. By this lemma, we get:

Corollary 4.5. *For a generic $n \times n$ symmetrizable Cartan matrix A, the rational homotopy type of $BG(A)$ is $BS^3 \times \bigvee_{i=2}^{\infty} \bigvee_{j=1}^{\beta_i} S^{2i+1}$ with*

$$(1-q^4)Q_n = 1 + \beta_2 q^5 + \beta_3 q^7 + \cdots + \beta_i q^{2i+1} + \cdots .$$

Similarly we have

Lemma 4.6. $\beta_i = b_{2i}$ *for all* $i \geq 2$.

Corollary 4.7. *The homotopy Lie algebra $\pi_*(G(A))$ with Samelson product is the direct sum of $\pi_*(S^3)$ and the free graded Lie algebra generated by $\Sigma^{-1}\overline{H}_*(BG(A))$, where $\overline{H}_*(BG(A)) \cong \widetilde{H}_*\big(\bigvee_{i=2}^{\infty} \bigvee_{j=1}^{\beta_i} S^{2i+1}\big)$.*

The Hopf algebra $H_*(G(A))$ and the $\mathbb{Q}$-algebra $H_*(S^3) \times T(\Sigma^{-1}\overline{H}_*(BG(A)))$ are isomorphic.

Theorem 1 is a direct consequence of Corollary 4.3 and 4.7.

We also have

Proposition 4.8. *If A_1, A_2 are two generic Cartan matrices of size n_1 and n_2, then $G(A_1)$ and $G(A_2)$ are rational homotopy equivalent Hopf spaces if and only if $n_1 = n_2$ and $\epsilon(A_1) = \epsilon(A_2)$.*

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