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JAMES DIBBLE

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Each compact Riemannian manifold with no conjugate points admits a family of functions whose integrals vanish exactly when central Busemann functions split linearly. These functions vanish when all central Busemann functions are sub- or superharmonic. When central Busemann functions are convex or concave, they must be totally geodesic. These yield generalizations of the splitting theorems of O'Sullivan and Eberlein for manifolds with no focal points and, respectively, nonpositive curvature.

1. Introduction

A key insight in Burago and Ivanov's [1994] proof that each Riemannian torus with no conjugate points is flat is that the asymptotic norm of its fundamental group $\pi_1(\mathbb{T}^k) \cong \mathbb{Z}^k$ is Riemannian, in the sense that it is generated by an inner product. In the case of an arbitrary compact Riemannian manifold N whose fundamental group has center $Z(\pi_1(N))$ of rank k, O'Sullivan [1974] showed that, when N has no focal points, it must be foliated by totally geodesic and flat k-dimensional toruses and, moreover, be covered by an isometric product with a flat $\mathbb{T}^k$. This generalized a theorem of Wolf [1964] about manifolds with nonpositive sectional curvature. It was later shown by Eberlein [1982] that each compact manifold with nonpositive sectional curvature and $Z(\pi_1(N))$ of rank k is finitely covered by a diffeomorphic product with $\mathbb{T}^k$. The aim of this paper is to explore conditions, between having no conjugate points and no focal points, that allow for variations on these results.

Associated to each $z_0, z_1 \in Z(\pi_1(N))$ is a function $F_{z_0 z_1} : N \to \mathbb{R}$ defined by (3-1); the linear splitting of central Busemann functions on $\hat{N}$, where $\pi : \hat{N} \to N$ is the universal covering map, is one of a number of conditions equivalent to the vanishing of all $\int_N F_{z_0 z_1} \, d\mathrm{vol}_N$.

Theorem 1.1. *Let N be a compact Riemannian manifold with no conjugate points and Z a subgroup of $Z(\pi_1(N))$. Then, the following are equivalent*:

(i) $\int_N F_{z_0 z_1} \, d\mathrm{vol}_N = 0$ *for all* $z_0, z_1 \in Z$.

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(ii) $B(z_0, z_1) = 1/\mathrm{vol}(N) \int_N h(\omega_{z_0}, \omega_{z_1})\, d\mathrm{vol}_N$ *for all* $z_0, z_1 \in Z$.

(iii) $h(\omega_{z_0}(x), \omega_{z_1}(x)) = B(z_0, z_1)$ *for all* $x \in N$ *and all* $z_0, z_1 \in Z$.

(iv) $\omega_{\sum_i m_i z_i} = \sum_i m_i \omega_{z_i}$ *for all* $z_i \in Z$ *and all* $m_i \in \mathbb{Z}$.

The notation that appears in Theorem 1.1 is defined in the next two sections. The proof is partly an application of Green's identity.

It follows from the main result of [Dibble 2019a] that any subgroup Z of $Z(\pi_1(N))$ is isomorphic to $\mathbb{Z}^k$ for some $0 \leq k \leq \dim(N)$. When statements (i)–(iv) in Theorem 1.1 hold, one may define two integral formulations of an inner product that generates the asymptotic norm of Z. Moreover, a number of topological properties, which are known in the case of no focal points, hold (cf. [O'Sullivan 1974]).

Theorem 1.2. *Let N be a compact Riemannian manifold with no conjugate points and Z a subgroup of $Z(\pi_1(N))$ for which statements* (i)–(iv) *in Theorem 1.1 hold. Then, so do the following*:

(a) *The asymptotic norm $\|\cdot\|_\infty$ of Z with respect to any isomorphism $Z \to \mathbb{Z}^k$ is Riemannian.*

(b) *If $w_1, \ldots, w_k$ generate Z and $H_1, \ldots, H_k$ are corresponding horospheres, then $\hat{H} = \bigcap_{i=1}^k H_i$ is a simply connected submanifold of $\hat{N}$.*

(c) *$\hat{N}$ is diffeomorphic to $\hat{H} \times \mathbb{R}^k$.*

(d) *There exists a sequence of normal covering maps*

$$\hat{H} \times \mathbb{R}^k \xrightarrow{\psi_0} N_0 \times \mathbb{T}^k \xrightarrow{\phi_0} N,$$

with respective deck transformation groups $\pi_1(N_0) \times Z$ and Γ, such that ψ_0 is a product map, N_0 is orientable, $\pi_1(N_0)$ is a normal subgroup of $\pi_1(N)$ containing the commutator subgroup $[\pi_1(N), \pi_1(N)]$, and the sequences

$$0 \to \pi_1(N_0) \times Z \to \pi_1(N) \to \Gamma \to 0$$

and

$$0 \to (\pi_1(N_0)/[\pi_1(N), \pi_1(N)]) \times Z \to H_1(N, \mathbb{Z}) \to \Gamma \to 0$$

are exact.

Because $[\pi_1(N), \pi_1(N)] \subseteq \pi_1(N_0)$, the covering map ϕ_0 is abelian.

It is clear from (3-1) that the integrals $\int_N F_{z_0 z_1}\, d\mathrm{vol}_N$ vanish when central Busemann functions are harmonic. Lemma 5.1 states that sub- or superharmonic central Busemann functions must be harmonic, which implies the following.

Corollary 1.3. *Let N be a compact Riemannian manifold with no conjugate points and Z a subgroup of $Z(\pi_1(N))$. If each Busemann function associated with Z is sub- or superharmonic, then statements* (i)–(iv) *in Theorem 1.1 and the conclusions of Theorem 1.2 hold.*

Since they must be harmonic, concave or convex central Busemann functions must have vanishing Hessian and therefore be totally geodesic, in the sense that they map geodesics to geodesics. It is well known that Busemann functions are convex when N has no focal points or, more narrowly, nonpositive sectional curvature [Eschenburg 1977]. Thus, the following generalizes the splitting theorems of O'Sullivan and Eberlein.

Theorem 1.4. *Let N be a compact Riemannian manifold with no conjugate points and Z a subgroup of $Z(\pi_1(N))$. If each Busemann function associated with Z is convex or concave, then, in addition to the conclusions of Theorem 1.2, the following hold*:

(a) *$\hat{N}$ is isometric to $\hat{H} \times \mathbb{R}^k$, where Z acts on each $\mathbb{R}^k$-fiber by translations.*

(b) *N is foliated by totally geodesic and flat k-dimensional toruses.*

(c) *There exists a sequence of Riemannian covering maps*

$$\hat{H} \times \mathbb{R}^k \xrightarrow{\psi_1} N_1 \times \mathbb{T}^k \xrightarrow{\phi_1} N,$$

where N_1 is orientable, ψ_1 is a product, ϕ_1 has finitely many sheets, and each map restricts on each $\mathbb{R}^k$- or $\mathbb{T}^k$-fiber to a totally geodesic and locally isometric immersion onto a leaf of the $\mathbb{T}^k$-foliation below.

It is not claimed in Theorem 1.4(c) that $N_1 \times \mathbb{T}^k$ has a product metric, and the statement cannot be strengthened to the isometric splitting of a finite cover, as there are examples in [Lawson and Yau 1972; Eberlein 1980; 1981] of compact $N_1 \times \mathbb{T}^k$ with nonpositive curvature that are not finitely covered by isometric products with flat k-dimensional toruses.

Eberlein additionally proved that every compact manifold having nonpositive sectional curvature and fundamental group with nontrivial center is of a canonical form. This result also generalizes, provided one modifies the definition of a canonical manifold in [Eberlein 1982] appropriately. Let $\hat{H}$ be a complete and simply connected manifold with no conjugate points, Γ_0 a properly discontinuous group of isometries of $\hat{H}$, where $\hat{H}/\Gamma_0$ is compact, and $\rho : \Gamma_0 \to \mathbb{T}^k$ a homomorphism whose kernel contains no nontrivial elements with fixed points. Define an action of Γ_0 on $\hat{H} \times \mathbb{T}^k$ by $\Phi(\xi, \hat{x}) = (\Phi(\hat{x}), \rho(\Phi) \cdot \xi)$ for all $\Phi \in \Gamma_0$, $\xi \in \mathbb{T}^k$, and $\hat{x} \in \hat{H}$. Then, the quotient $(\hat{H} \times \mathbb{T}^k)/\Gamma_0$ is called a *k-canonical* manifold.

Theorem 1.5. *Let N be a compact Riemannian manifold with no conjugate points and Z a subgroup of $Z(\pi_1(N))$. If each Busemann function associated with Z is convex or concave, then N is a k-canonical manifold for $k = \operatorname{rank} Z$. When $Z = Z(\pi_1(N))$, Γ_0 is centerless.*

The proof of Theorem 1.5 will be omitted, as it exactly follows Eberlein's argument, using Theorem 1.4(a) in the place of Lemma 1 of [Eberlein 1982].

2. Preliminaries

Throughout this paper, (N, h) will denote a compact, connected, n-dimensional, and C^r Riemannian manifold for $r \geq 2$. The covering metric on $\hat{N}$ will be denoted by $\hat{h}$. The metric h will have no conjugate points, which by definition means that the exponential map on each tangent space is nonsingular. It follows that, for each $p \in \hat{N}$, $\exp_p : T_p\hat{N} \to \mathbb{R}^n$ is a diffeomorphism. Thus, N is aspherical and, consequently, $\pi_1(N)$ is torsion free [Hurewicz 1995]. It is well known that a complete manifold with nonpositive sectional curvature has no focal points, that a complete manifold with no focal points has no conjugate points, and that neither of these implications is reversible within the space of compact manifolds of any fixed dimension at least two [Gulliver 1975].

For any tangent vector v, denote by γ_v the geodesic $t \mapsto \exp(tv)$. Corresponding to each unit vector $v \in T\hat{N}$ is the Busemann function $b_v : \hat{N} \to \mathbb{R}$ defined by $b_v(x) = \lim_{t\to\infty}[t - d(\gamma_v(t), x)]$. It will be convenient to generalize the idea of a Busemann function to arbitrary tangent vectors by setting

$$b_v = \begin{cases} \|v\| b_{v/\|v\|} & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases}$$

for any $v \in T\hat{N}$. It was essentially shown by Busemann [1955] that associated to each $z \in Z(\pi_1(N))$ is a unique constant-length vector field ω_z on N with the property that each $\gamma_{\omega_z(x)}|_{[0,1]}$ is a closed geodesic representing z in $\pi_1(N, x)$. Denote by $\hat{\omega}_z$ the lift of ω_z to $\hat{N}$. A Busemann function b_v is *central* if $v = \hat{\omega}_z(p)$ for some $z \in Z(\pi_1(N))$ and $p \in \hat{N}$. Following the arguments in [Burago and Ivanov 1994] and [Heber 1994], one finds that, for each $z \in Z(\pi_1(N))$, the corresponding central Busemann functions have gradient field $\hat{\omega}_z$. Arbitrary Busemann functions are known only to be C^1, except for manifolds with bounded asymptote, a class which includes those with no focal points, where they are C^2 [Eschenburg 1977]. However, for $z \in Z(\pi_1(N))$, the inverse function theorem implies that ω_z is C^{r-1} and, consequently, that each central Busemann function is C^r [Dibble 2019b].

For each $v \in T\hat{N}$, a horosphere of v is, by definition, a level set of the Busemann function b_v. When $z \in Z(\pi_1(N))$, the horospheres of $\hat{\omega}_z(\hat{x})$ are the leaves of the normal distribution to $\hat{\omega}_z$ for any $\hat{x} \in \hat{N}$ and, consequently, form a C^r foliation of $\hat{N}$ by hypersurfaces. In this way, one may speak of a horosphere H of z itself. For each such H, the normal bundle NH is a trivial line bundle; since no point of $\hat{N}$ is focal to H, the exponential map on NH is a diffeomorphism onto $\hat{N}$. In this way, $\hat{N} \cong H \times \mathbb{R}$.

The action of $\pi_1(N)$ on $\hat{N}$ by deck transformations will be denoted $(\alpha, x) \mapsto \alpha(x)$. The following is a special case of an important lemma of Ivanov and Kapovitch [2014]. However, as they point out, the methods of Croke and Schroeder [1986]

suffice to prove the lemma for central elements. In particular, the narrow statement here requires only that the metric be C^2. The more general statement in [Ivanov and Kapovitch 2014] requires C^k regularity for some k depending on n.

Lemma 2.1. *Let $\alpha \in \pi_1(N)$ and $z \in Z(\pi_1(N))$. If γ_v is an axis of z, then $b_v(\alpha(\hat{x})) - b_v(\hat{x})$ is independent of $\hat{x} \in \hat{N}$.*

For any choices of $p, q \in \hat{N}$, $b_{\omega_{z_1}(p)}$ and $b_{\omega_{z_1}(q)}$ differ by a constant. Thus, the function $B(z_0, z_1) = b_{\omega_{z_1}(p)}(z_0(x)) - b_{\omega_{z_1}(p)}(\hat{x})$, defined on $Z(\pi_1(N)) \times Z(\pi_1(N))$, is also independent of the choice of p. Loosely speaking, $B(z_0, z_1)$ is the change in the Busemann functions of z_1 in the direction of z_0.

Lemma 2.1 implies, by the argument in [Ivanov and Kapovitch 2014], the virtual splitting of cyclic subgroups of $Z(\pi_1(N))$.

Theorem 2.2 (Ivanov and Kapovitch). *For each nontrivial $z \in Z(\pi_1(N))$, there exists a finite-index subgroup G of $\pi_1(N)$ isomorphic to a direct product $G' \times \mathbb{Z}$, under which identification z corresponds to a generator of the $\mathbb{Z}$-factor.*

A simple consequence is that, when $Z(\pi_1(N))$ has rank at least k, a finite-index subgroup of $\pi_1(N)$ splits as a product with $\mathbb{Z}^k$.

Corollary 2.3. *If $z_1, \dots, z_k$ are independent elements of $Z(\pi_1(N))$, then there exists a finite-index subgroup G of $\pi_1(N)$ isomorphic to a direct product $G' \times \mathbb{Z}^k$, under which identification the $\mathbb{Z}^k$-factor is generated by elements of the form $z_1^{m_1}, \dots, z_k^{m_k}$ for $m_i \geq 1$.*

Proof. Let $\eta_i = (\pi_i, \sigma_i) : G_i \to G_i' \times \mathbb{Z}$ be the isomorphisms guaranteed by Theorem 2.2, where each G_i' is taken to be a subgroup of G_i on which π_i is the identity. Let $G = \bigcap_{i=1}^k G_i$, $G' = \bigcap_{i=1}^k G_i'$, and $\pi' = \pi_k \circ \cdots \circ \pi_1$. Then, $\eta = (\pi', \sigma_1, \dots, \sigma_k) : G \to G' \times \mathbb{Z}^k$ is a homomorphism. Note that

$$\operatorname{Ker} \eta \subseteq \bigcap_{i=1}^{k} \operatorname{Ker} \sigma_i \subseteq \bigcap_{i=1}^{k} G_i' = G'.$$

Since each π_i is the identity on G', η must be one-to-one. Since G has finite index, there exist smallest $m_i \geq 1$ such that $z_i^{m_i} \in G$. It follows that $\eta(G)$ is isomorphic to $G' \times \mathbb{Z}^k$ in such a way that the $z_i^{m_i}$ generate the $\mathbb{Z}^k$-factor. □

For any $z \in Z(\pi_1(N))$, write $\|z\|_\infty = B(z, z)^{1/2}$. If Z is any subgroup of $Z(\pi_1(N))$, then $Z \cong \mathbb{Z}^k$ for some $0 \leq k \leq n$ [Dibble 2019a]. With respect to a fixed isomorphism $Z \to \mathbb{Z}^k$, $\|\cdot\|_\infty$ agrees with the asymptotic norm of the orbit metric on Z obtained from its action on $\hat{N}$. That is, if d denotes the induced $\mathbb{Z}^k$-equivariant distance function on $\mathbb{Z}^k$, then, up to the identification of Z with $\mathbb{Z}^k$,

$$\|z\|_\infty = \lim_{m \to \infty} \frac{d(0, mz)}{m},$$

and this extends uniquely to a norm on $\mathbb{R}^k$ (see Section 8.5 of [Burago et al. 2001] for details). Following [Burago and Ivanov 1994], one would like to define an inner product on $\mathbb{R}^k$ that extends B with respect to this identification. In fact, elementary arguments show that B satisfies many of the properties of an inner product: $B(z_0, mz_1) = mB(z_0, z_1) = B(z_0, mz_1)$ for all $m \in \mathbb{Z}$; by Corollary 4.2 of [Ivanov and Kapovitch 2014], $B(z_0 + z_1, z_2) = B(z_0, z_2) + B(z_1, z_2)$; and, significantly, B satisfies the Cauchy–Schwarz inequality,

$$|B(z_0, z_1)| \leq \|z_0\|_\infty \|z_1\|_\infty, \tag{2-1}$$

with equality if and only if z_0 and z_1 are rationally related, in the sense that $mz_0 = \ell z_1$ for some $m, \ell \in \mathbb{Z}$ not both zero. However, it is not clear that, in general, B is symmetric or additive in its second slot. For the purpose of showing that $\|\cdot\|_\infty$ is generated by an inner product, those two conditions are equivalent: additivity would follow from symmetry, and, if additivity held, then the symmetrization of B would extend to an inner product generating $\|\cdot\|_\infty$.

This section ends with a technical lemma about the horospheres of central Busemann functions. Although it is in many ways unclear how well behaved they are, they may in some cases be used to construct fundamental domains of π. A preliminary lemma is first presented.

Lemma 2.4. *If $z \in Z(\pi_1(N))$ is primitive, in the sense that $z \neq mz'$ for all $|m| > 1$ and $z' \in \pi_1(N)$, then, for each $\hat{x} \in \hat{N}$, $\pi \circ \gamma_{\hat{\omega}(\hat{x})}$ is injective on $[0, \|z\|_\infty)$.*

Proof. Assume that $\pi \circ \gamma_{\hat{\omega}(x)}(t_0) = \pi \circ \gamma_{\hat{\omega}(x)}(t_1)$ for $0 \leq t_0 < t_1 < \|z\|_\infty$. Write $T = t_1 - t_0$. Without loss of generality, replace x with $\gamma_{\hat{\omega}(x)}(t_0)$, so that

$$\pi \circ \gamma_{\hat{\omega}(x)}(0) = \pi \circ \gamma_{\hat{\omega}(x)}(T) = \pi \circ \gamma_{\hat{\omega}(x)}(\|z\|_\infty).$$

Since $\operatorname{inj}(\pi(x)) > 0$, $T = (a/b)\|z\|_\infty$ for some relatively prime $a, b \in \mathbb{Z}$ with $|b| > 1$. Because a and b are relatively prime, there exist $m, n \in \mathbb{Z}$ such that $ma = nb + 1$. Since $\pi \circ \gamma_{\hat{\omega}(x)}((ma/b)\|z\|_\infty) = \pi \circ \gamma_{\hat{\omega}(x)}(0)$, $\gamma_{\omega(\pi(x))}|_{[0,(1/b)\|z\|_\infty]}$ is a closed geodesic representing an element w of $\pi_1(N)$ with $z = bw$, which is a contradiction. □

Lemma 2.5. *Let $z \in Z(\pi_1(N))$ be primitive and H be a horosphere of z. Then, there exist disjoint open subsets $U_1, \ldots, U_K$ of H such that, for $V_i = U_i \times [0, \|z\|_\infty)$ and $V = \bigcup_{i=1}^K V_i$, π is injective on V and $\operatorname{vol}_h(N \setminus \pi(V)) = 0$.*

Proof. It follows from $\|z\|_\infty$-periodicity that there exists $\varepsilon > 0$ such that, for each $x \in H$, the exponential map is injective on $B(x, \varepsilon) \times [0, \|z_0\|_\infty) \subseteq NH$. By compactness, one may choose a finite subset $\{W_1, \ldots, W_K\}$ of $\{B(x, \varepsilon) \mid x \in H\}$ with the property that $\{\pi(W_i \times [0, \|z_0\|_\infty])\}$ is an open cover of N. Let $U_1 = W_1$. The remaining U_i are defined inductively: Suppose $U_1, \ldots, U_{i-1}$ have been defined and satisfy $U_j \subseteq W_j$. For each $1 \leq j \leq i - 1$, there exist at most finitely many $\alpha_{j,1}, \ldots, \alpha_{j,M_j} \in \pi_1(N)$

such that $\alpha_{j,m}(\hat{U}_j \times [0, \|z\|_\infty]) \cap W_i \neq \varnothing$. Let

$$U_i = W_i \setminus \bigcup_{j=1}^{i-1} \bigcup_{m=1}^{M_j} \alpha_{j,m}(\hat{U}_j \times [0, \|z\|_\infty]).$$

The sets $U_1, \ldots, U_K$ constructed in this way have the desired properties. □

3. Integral formulations of B

Let $z_0, z_1 \in Z(\pi_1(N))$. The function $F_{z_0 z_1} : N \to \mathbb{R}$ in Theorem 1.2 will be defined by an equivariant construction on $\hat{N}$. Denote by Δ the Laplace–Beltrami operator. Fix $\hat{p} \in \hat{N}$ and $v_i = \hat{\omega}_{z_i}(\hat{p})$. For $x \in N$, let $\hat{x} \in \pi^{-1}(x)$ and $v = \hat{\omega}_{z_0}(\hat{x})$, and define $F_{z_0 z_1}(x)$ to be the average value of $b_{v_1} \Delta b_{v_0}$ along $\gamma_v|_{[0,1]}$ with respect to arclength:

$$F_{z_0 z_1}(x) = \oint_{\gamma_v|_{[0,1]}} b_{v_1} \Delta b_{v_0}\, ds. \tag{3-1}$$

More specifically, $F_{z_0 z_1} = 0$ if z_0 is the identity, and

$$F_{z_0 z_1}(x) = \frac{1}{\|z_0\|_\infty} \int_{\gamma_v|_{[0,1]}} b_{v_1} \Delta b_{v_0}\, ds$$

otherwise. To see that $F_{z_0 z_1}$ is well defined, first recall the following classical result.

Lemma 3.1 (Green's identity). *Let (M, g) be a C^2 Riemannian manifold with boundary ∂M and ν an outward-pointing unit normal vector field along ∂M. If $\phi, \psi : M \to \mathbb{R}$ are C^2 functions, then*

$$\int_M \phi \Delta \psi \, d\mathrm{vol}_M + \int_M g(\nabla \phi, \nabla \psi)\, d\mathrm{vol}_M = \int_{\partial M} \phi g(\nabla \psi, \nu)\, d\mathrm{vol}_{\partial M}.$$

This generalizes in a straightforward way to manifolds with corners. A simple consequence is that, for a central Busemann function b_v, Δb_v measures the change in volume of its horospheres under its gradient flow.

Lemma 3.2. *Let $z \in Z(\pi_1(N))$ be nontrivial. Let b_v be a corresponding central Busemann function and $T \in \mathbb{R}$. For each $t \in \mathbb{R}$, denote by H_t the horosphere of z along which $b_v = t$. Let $U \subset H_T$ be any open set with C^r boundary and compact closure, and let $U_t = U \times [T, T+t]$ with respect to the splitting $\hat{N} \cong H_T \times \mathbb{R}$. Then,*

$$\frac{d}{dt} \mathrm{vol}_{H_{T+t}}(U \times \{T+t\}) = \frac{1}{\|z\|_\infty} \int_{U \times \{T+t\}} \Delta b_v \, d\mathrm{vol}_{H_{T+t}}.$$

Proof. Denote by ν the outward-pointing unit normal vector field along the C^r portion of ∂U_t, so that, except at corners, $\nu = -\hat{\omega}_z / \|z\|_\infty$ along $U \times \{T\}$, $\nu = \hat{\omega}_z / \|z\|_\infty$

along $U \times \{T+t\}$, and $\hat{h}(v, \hat{\omega}_z) = 0$ along $\partial U \times [0, \|z\|_\infty]$. By Green's identity,

$$\begin{aligned}\int_{U_t} \Delta b_v \, d\mathrm{vol}_{\hat{N}} &= \int_{\partial U_t} \hat{h}(\hat{\omega}_z, \nu) \, d\mathrm{vol}_{\partial U_t} \\ &= \|z\|_\infty [\mathrm{vol}_{H_{T+t}}(U \times \{T+t\}) - \mathrm{vol}_{H_T}(U \times \{T\})].\end{aligned}$$

It follows from the coarea formula that

$$\int_{U_t} \Delta b_v \, d\mathrm{vol}_{\hat{N}} = \int_T^{T+t} \int_{U \times \{T+s\}} \Delta b_v \, d\mathrm{vol}_{H_{T+s}} \, ds.$$

The result follows immediately. □

If $\hat{x}' \in \pi^{-1}(x)$ and $v' = \hat{\omega}_{z_0}(\hat{x}')$, then $\hat{x}' = \alpha(x)$ for some $\alpha \in \pi_1(N)$. Note that, for all t, $\Delta b_{v_0}(\gamma_v(t)) = \Delta b_{v_0}(\gamma_{v'}(t))$ and, by Lemma 2.1,

$$b_{v_1}(\gamma_v(t)) - b_{v_1}(\gamma_{v'}(t)) = B(\alpha, z_1) \in \mathbb{R}.$$

Moreover, for $T = b_{v_0}(\hat{x})/\|z_0\|_\infty^2$, $U^\varepsilon = B(\gamma_v(-T), \varepsilon) \subseteq H_0$, and $U^\varepsilon_{\|z_0\|_\infty} = U^\varepsilon \times [T, T + \|z_0\|_\infty]$, an application of Lemma 3.2 shows that

$$\begin{aligned}\int_{\gamma_v|_{[0,1]}} \Delta b_{v_0} dt &= \lim_{\varepsilon \to 0} \int_{U^\varepsilon_{\|z_0\|_\infty}} \Delta b_{v_0} \, d\mathrm{vol}_{\hat{N}} \\ &= \lim_{\varepsilon \to 0} \|z_0\|_\infty [\mathrm{vol}_{H_{T+\|z_0\|_\infty}}(U^\varepsilon \times \{T+\|z_0\|_\infty\}) - \mathrm{vol}_{H_T}(U^\varepsilon \times \{T\})] = 0.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\gamma_{v'}|_{[0,1]}} b_{v_1} \Delta b_{v_0} dt &= \int_{\gamma_v|_{[0,1]}} [b_{v_1} + B(\alpha, z_1)] \Delta b_{v_0} ds \\ &= \int_{\gamma_v|_{[0,1]}} b_{v_1} \Delta b_{v_0} ds + B(\alpha, z_1) \int_{\gamma_v|_{[0,1]}} \Delta b_{v_0} ds = \int_{\gamma_v|_{[0,1]}} b_{v_1} \Delta b_{v_0} ds.\end{aligned}$$

So, $F_{z_0 z_1}$ is well defined. Similarly, $F_{z_0 z_1}$ is independent of the choice of $\hat{p}$, as replacing $\hat{p}$ with $\hat{q}$ induces a constant change in b_{v_1} of $b_{v_1}(\hat{q}) - b_{v_1}(\hat{p})$ while leaving Δb_{v_0} unchanged.

For any $\hat{p}$ as above, let S be a fundamental domain of π constructed using a horosphere H_T of v_0 as in Lemma 2.5, and write $S_t = S \times [t, t + \|z_0\|_\infty]$ for each $t \in [T, T + \|z_0\|_\infty]$. Then,

$$\begin{aligned}\int_N F_{z_0 z_1} \, d\mathrm{vol}_N &= \int_{S_T} \int_{\gamma_{\hat{\omega}_{z_0}(\hat{x})}|_{[0,1]}} b_{v_1} \Delta b_{v_0} ds \; d\mathrm{vol}_{\hat{N}} \\ &= \int_T^{T+\|z_0\|_\infty} \int_{S \times \{t\}} \int_{\gamma_{\hat{\omega}_{z_0}(\hat{x})}|_{[0,1]}} b_{v_1} \Delta b_{v_0} ds \; d\mathrm{vol}_{H_t} \; dt \\ &= \int_T^{T+\|z_0\|_\infty} \int_{S_t} b_{v_1} \Delta b_{v_0} \, d\mathrm{vol}_{\hat{N}} \; dt.\end{aligned} \tag{3-2}$$

Theorem 3.3. *Let $z_0, z_1 \in Z(\pi_1(N))$. Then,*

$$B(z_0, z_1) = \frac{1}{\operatorname{vol}(N)} \int_N [h(\omega_{z_0}, \omega_{z_1}) + F_{z_0 z_1}]\, d\operatorname{vol}_N .$$

Proof. By Theorem 2.2, there exist a finite covering map $\psi : \tilde{N} \to N$, with covering metric $\tilde{h}$, and a primitive $\tilde{z}_0 \in Z(\pi_1(\tilde{N}))$ such that $\psi_*(\tilde{z}_0) = z_0$. For each $i = 1, 2$, denote by $\tilde{\omega}_i$ the lift of ω_i to $\tilde{N}$. Then,

$$\frac{1}{\operatorname{vol}(N)} \int_N h(\omega_{z_0}, \omega_{z_1})\, d\operatorname{vol}_N = \frac{1}{\operatorname{vol}_{\tilde{h}}(\tilde{N})} \int_{\tilde{N}} \tilde{h}(\tilde{\omega}_0, \tilde{\omega}_1)\, d\operatorname{vol}_{\tilde{h}} .$$

Fix $\hat{x} \in \hat{N}$ and $v_i = \hat{\omega}_{z_i}(\hat{x})$. Let $T \in \mathbb{R}$. For the horospheres H_t of z_0, let $U_1, \ldots, U_K$ be the open subsets of H_T obtained by applying Lemma 2.5 to the covering $\hat{N} \to \tilde{N}$, and write $S = \bigcup_{i=1}^K U_i$ and $S_t = S \times [t, t + \|z\|_\infty]$. If each U_i has C^r boundary, then Green's identity shows that

$$\int_{S_t} \hat{h}(\hat{\omega}_{z_0}, \hat{\omega}_{z_1})\, d\operatorname{vol}_{\hat{N}} = \int_{\partial S_t} b_{v_1} \hat{h}(\hat{\omega}_{z_0}, \nu)\, d\operatorname{vol}_{\partial S_t} - \int_{S_t} b_{v_1} \Delta b_{v_0}\, d\operatorname{vol}_{\hat{N}} . \tag{3-3}$$

Applying Lemma 2.1, one has that

$$\begin{aligned} \int_{\partial S_t} b_{v_1} \hat{h}(\hat{\omega}_{z_0}, \nu)\, d\operatorname{vol}_{\partial S_t} &= \|z_0\|_\infty \left[\int_{S \times \{t + \|z_0\|_\infty\}} b_{v_1}\, d\operatorname{vol}_{H_{\|z_0\|_\infty}} - \int_{S \times \{t\}} b_{v_1}\, d\operatorname{vol}_{H_0} \right] \\ &= \|z_0\|_\infty B(z_0, z_1) \operatorname{vol}_{H_t}(S \times \{t\}). \end{aligned}$$

Suppose that z_0 is not the identity, as the result holds otherwise. Substituting the above into (3-3), integrating both sides from T to $T + \|z_0\|_\infty$, and using the fact that $\int_{S_t} \hat{h}(\hat{\omega}_{z_0}, \hat{\omega}_{z_1})\, d\operatorname{vol}_{\hat{N}} = \int_{\tilde{N}} \tilde{h}(\tilde{\omega}_0, \tilde{\omega}_1)\, d\operatorname{vol}_{\tilde{h}}$ yields

$$\begin{aligned} B(z_0, z_1) &= \frac{1}{\operatorname{vol}(N)} \int_N h(\omega_{z_0}, \omega_{z_1})\, d\operatorname{vol}_N \\ &\qquad + \frac{1}{\|z_0\|_\infty \operatorname{vol}(N)} \int_T^{T + \|z_0\|_\infty} \int_{S_t} b_{v_1} \Delta b_{v_0}\, d\operatorname{vol}_{\hat{N}}\, dt \\ &= \frac{1}{\operatorname{vol}(N)} \int_N h(\omega_{z_0}, \omega_{z_1})\, d\operatorname{vol}_N + \frac{1}{\operatorname{vol}(N)} \int_N F_{z_0 z_1}\, d\operatorname{vol}_N , \end{aligned} \tag{3-4}$$

the latter equality following from (3-2). In the general case, one may apply a similar argument to a union of open sets $U_{i,m} \subseteq U_i$ that have C^r boundary and whose measures converge to that of U_i as $m \to \infty$. □

Corollary 3.4. *Let $z_0, z_1 \in Z(\pi_1(N))$. Then, the following hold*:

(a) $B(z_0, z_1) = B(z_1, z_0)$ *if and only if* $\int_N F_{z_0 z_1}\, d\operatorname{vol}_N = \int_N F_{z_1 z_0}\, d\operatorname{vol}_N$.

(b) $B(z_0, z_1) = 1/\operatorname{vol}(N) \int_N h(\omega_{z_0}, \omega_{z_1})\, d\operatorname{vol}_N$ *if and only if* $\int_N F_{z_0 z_1}\, d\operatorname{vol}_N = 0$.

Let Z be any subgroup of $Z(\pi_1(N))$ of rank k, and fix an isomorphism $D : Z \to \mathbb{Z}^k$. Denote by $e_1, \ldots, e_k$ the standard basis for $\mathbb{R}^k$, and let $w_i = D^{-1}(e_i)$. Let $f : \mathbb{T}^k \to N$ be any fixed C^1 map satisfying $f_*(\alpha_i) = w_i$, where $\alpha_1, \ldots, \alpha_k$ are generators for $\pi_1(\mathbb{T}^k)$. Since N is aspherical, and therefore an Eilenberg–Mac Lane space, such maps are guaranteed to exist, although in the case of no conjugate points they may be constructed more concretely by iteratively exponentiating around loops (see Section 3.3 of [Dibble 2019b]).

Theorem 3.5. *Let $z_0, z_1 \in Z$ and $\beta_1 = f^*(\omega_{z_1}^\flat)$ (i.e., $\beta_1(v) = h(\omega_{z_1}, f_*(v))$ for all $v \in T\mathbb{T}^k$). Then, $B(z_0, z_1) = \int_{\mathbb{T}^k} \beta_1 \circ D(z_0)\, d\mathrm{vol}_g$ for the standard flat metric g on $\mathbb{T}^k$.*

Proof. Suppose $k \geq 2$, as otherwise the result is clear. Denote by $\phi : \mathbb{R}^k \to \mathbb{T}^k$ the covering map that quotients by $\mathbb{Z}^k$. Let $V = D(z_0) = \sum_{i=1}^m v_i e_i$ be a constant vector field on $\mathbb{T}^k$. Suppose $V \neq 0$, so that some $v_i \neq 0$. Let $u_i = V$ and, for each $j \neq i$, let u_j be the vector in $\mathbb{R}^k$ whose only nonzero entries are $-v_j$ in the i-th component and v_i in the j-th component. Let P be the parallelepiped in $\mathbb{R}^k$ determined by $u_1, \ldots, u_k$. Then, P is the union of $v_i^{k-2} \sum_{j=1}^k v_j^2 = v_i^{k-2} \|V\|_{\mathbb{R}^k}^2$ fundamental domains of ϕ. Denote by Q the face of P that contains the origin but not u_i, and, for each $p \in Q$, denote by α_p the geodesic $t \mapsto p + tV/\|V\|_{\mathbb{R}^k}$.

The map f lifts to a map $F : \mathbb{R}^k \to \hat{N}$ such that $f \circ \phi = \pi \circ F$. Let $\hat{V} = \phi^{-1}(V)$, so that

$$h(\omega_{z_1}, f_*(V)) = \hat{h}(\hat{\omega}_{z_1}, F_*(\hat{V})) = \hat{h}(\hat{\omega}_{z_1}, F_*(\alpha_x'(t))).$$

The coarea formula implies that

$$\begin{aligned}
\int_{\mathbb{T}^k} h(\omega_{z_1}, f_*(V))\, d\,\mathrm{vol}_g &= \frac{1}{v_i^{k-2}\|V\|_{\mathbb{R}^k}} \int_Q \left[\int_0^{\|V\|_{\mathbb{R}^k}} \hat{h}(\hat{\omega}_{z_1}, F_*(\alpha_x'(t)))dt \right] d\,\mathrm{vol}_Q \\
&= \frac{1}{v_i^{k-2}\|V\|_{\mathbb{R}^k}} \int_Q [b_{z_1} \circ F(x+V) - b_{z_1} \circ F(x)]\, d\,\mathrm{vol}_Q \\
&= \frac{1}{v_i^{k-2}\|V\|_{\mathbb{R}^k}} \int_Q [b_{z_1}(z_0(F(x))) - b_{z_1}(F(x))]\, d\,\mathrm{vol}_Q \\
&= \frac{1}{v_i^{k-2}\|V\|_{\mathbb{R}^k}} \int_Q B(z_0, z_1)\, d\,\mathrm{vol}_Q \\
&= B(z_0, z_1).
\end{aligned}$$

If $V = 0$, then z_0 is the identity, and one obtains the same equality. □

Remark 3.6. For $N = \mathbb{T}^k$ and $z \in \mathbb{Z}^k$, $z = D(\omega_z) = \int_{\mathbb{T}^k} \omega_z\, d\,\mathrm{vol}_g$, where g is the standard flat metric on $\mathbb{T}^k$. This continuously extends the direction at infinity in [Burago and Ivanov 1994] to the leaves of the Heber foliation [1994] of the unit sphere bundle of $\mathbb{T}^k$.

4. Proof of Theorems 1.1 and 1.2

As before, let Z be a subgroup of $Z(\pi_1(N))$, fix an isomorphism $D : Z \to \mathbb{Z}^k$, and let $w_i = D^{-1}(e_i)$, so that $w_1, \dots, w_k$ generate Z. Write $\mathscr{B}_0 = \{\omega_z \mid z \in Z\}$, and denote by $\mathscr{B}$ the vector space of finite formal linear combinations of elements of $\mathscr{B}_0$ with real coefficients. (For each $\hat{x} \in \hat{N}$, $\mathscr{B}$ may be identified with the space of finite formal combinations of Busemann functions in Z that vanish at $\hat{x}$.) Roughly speaking, one may extend D to a linear direction at infinity $\mathscr{D} : \mathscr{B} \to \mathbb{R}^k$ by setting $\mathscr{D}\big(\sum_i a_i \omega_{z_i}\big) = \sum_i a_i D(z_i)$. In the other direction, there is the inclusion $\iota : \mathbb{Z}^k \to \mathscr{B}$ defined by $\iota\big(\sum_i m_i e_i\big) = \omega_{\sum_i m_i w_i}$.

For each $\zeta_1, \zeta_2 \in \mathscr{B}$ (i.e., $\zeta_i = \sum_j a_{ij}\omega_{z_{ij}}$ for $z_{ij} \in Z$ and $a_{ij} \in \mathbb{R}$), let $\beta_i = f^*(\zeta_i^\flat)$. Define

$$G(\zeta_1, \zeta_2) = \frac{1}{2}\int_{\mathbb{T}^k} [\beta_1 \circ \mathscr{D}(\zeta_2) + \beta_2 \circ \mathscr{D}(\zeta_1)]\, d\mu_g.$$

It is clear that G is symmetric and bilinear. By Corollary 3.4(a) and Theorem 3.5, when (i) holds,

$$G(\omega_{z_0}, \omega_{z_1}) = B(z_0, z_1) = B(z_1, z_0).$$

At the same time, one may define a semi-inner product H on $\mathscr{B}$ by

$$H(\zeta_1, \zeta_2) = \frac{1}{\mathrm{vol}(N)}\int_N h(\zeta_1, \zeta_2)\, d\,\mathrm{vol}_N\,.$$

Again assuming (i), Corollary 3.4(b) implies that

$$H(\omega_{z_0}, \omega_{z_1}) = B(z_0, z_1) = B(z_1, z_0).$$

In this case, since they agree on $\mathscr{B}_0 \times \mathscr{B}_0$, G and H define the same semi-inner product on $\mathscr{B}$ and, consequently, the same seminorm. Overloading notation, write $\|\cdot\|_\infty = \|\cdot\|_G = \|\cdot\|_H$.

Lemma 4.1. *Let Z be a subgroup of $Z(\pi_1(N))$ such that $\omega_{\sum_i m_i z_i} = \sum_i m_i \omega_{z_i}$ for all $z_i \in Z$ and all $m_i \in \mathbb{Z}$. Then, for all $z_0, z_1 \in Z$, the following hold*:

(a) $[\omega_{z_0}, \omega_{z_1}] = 2\nabla_{\omega_{z_0}}\omega_{z_1}$ (*i.e.*, $\nabla_{\omega_{z_0}}\omega_{z_1} = -\nabla_{\omega_{z_1}}\omega_{z_0}$).

(b) $h([\omega_{z_0}, \omega_{z_1}], \omega_{z_0}) = h([\omega_{z_0}, \omega_{z_1}], \omega_{z_1}) = 0$.

(c) $h(\omega_{z_0}(x), \omega_{z_1}(x)) = B(z_0, z_1)$ *for all* $x \in N$.

Proof. One has that

$$\begin{aligned} 0 &= \nabla_{\omega_{z_0+z_1}}\omega_{z_0+z_1} = \nabla_{\omega_{z_0}}\omega_{z_0} + \nabla_{\omega_{z_0}}\omega_{z_1} + \nabla_{\omega_{z_1}}\omega_{z_0} + \nabla_{\omega_{z_1}}\omega_{z_1} \\ &= \nabla_{\omega_{z_0}}\omega_{z_1} + \nabla_{\omega_{z_1}}\omega_{z_0}, \end{aligned}$$

so $\nabla_{\omega_{z_0}}\omega_{z_1} = -\nabla_{\omega_{z_1}}\omega_{z_0}$. This proves (a). Moreover,

$$0 = \omega_{z_1}[h(\omega_{z_0}, \omega_{z_0})] = 2h(\nabla_{\omega_{z_1}}\omega_{z_0}, \omega_{z_0}) = -2h(\nabla_{\omega_{z_0}}\omega_{z_1}, \omega_{z_0}) = -2\omega_{z_0}[h(\omega_{z_1}, \omega_{z_0})].$$

One may deduce (b) from the first three equalities above. Let $\hat{x} \in \hat{N}$, and write $\gamma = \gamma_{\hat{\omega}_{z_0}(\hat{x})}$. Since $h(\hat{\omega}_{z_1}, \hat{\omega}_{z_0})$ is constant along γ, one finds that

$$\begin{aligned} B(z_0, z_1) &= b_{z_1}(z_0(x)) - b_{z_1}(x) \\ &= \int_0^1 h((\hat{\omega}_{z_0} \circ \gamma)(t), (\hat{\omega}_{z_1} \circ \gamma)(t))\, dt = h(\hat{\omega}_{z_0}(x), \hat{\omega}_{z_1}(x)), \end{aligned}$$

which proves (c). □

It is now possible to prove Theorem 1.1. Statements (i) and (ii) are equivalent by Corollary 3.4(b). By Lemma 4.1(c), (iv) implies (iii), and it is clear that (iii) implies (ii). Therefore, the proof can be completed by showing that (i) implies (iv).

Suppose (i) holds. Write $\alpha = \omega_{\sum_i m_i z_i}$, $\beta = \sum_i m_i \omega_{z_i}$, and $\zeta = \alpha - \beta$. Then,

$$\begin{aligned} \|\zeta\|_\infty^2 &= G(\zeta, \zeta) \\ &= \int_{\mathbb{T}^k} h(\zeta, D(\zeta))\, d\mathrm{vol}_g = \int_{\mathbb{T}^k} h(\zeta, 0)\, d\mathrm{vol}_g = 0, \end{aligned}$$

which means that

$$0 = \mathrm{vol}(N) H(\zeta, \zeta) = \int_N h(\zeta, \zeta)\, d\mathrm{vol}_h = \int_N [\|\alpha\|_h^2 + \|\beta\|_h^2 - 2h(\alpha, \beta)]\, d\mathrm{vol}_h\,.$$

Thus,

$$\int_N h(\alpha, \beta)\, d\mathrm{vol}_h = \int_N \tfrac{1}{2}(\|\alpha\|_h^2 + \|\beta\|_h^2)\, d\mathrm{vol}_h\,. \tag{4-1}$$

By the Cauchy–Schwarz inequality,

$$h(\alpha, \beta) \le \|\alpha\|_h \|\beta\|_h,$$

so

$$\int_N (\|\alpha\|_h^2 + \|\beta\|_h^2)\, d\mathrm{vol}_h \le \int_N 2\|\alpha\|_h^2 \|\beta\|_h^2\, d\mathrm{vol}_h\,.$$

At the same time, $2\|\alpha\|_h \|\beta\|_h \le \|\alpha\|_h^2 + \|\beta\|_h^2$. Thus,

$$\int_N 2\|\alpha\|_h^2 \|\beta\|_h^2\, d\mathrm{vol}_h = \int_N (\|\alpha\|_h^2 + \|\beta\|_h^2)\, d\mathrm{vol}_h\,,$$

which implies that $\|\alpha\|_h = \|\beta\|_h$ on N. Substituting into (4-1), one finds that

$$\int_N h(\alpha, \beta)\, d\mathrm{vol}_h = \int_N \|\alpha\|_h^2 \|\beta\|_h^2\, d\mathrm{vol}_h$$

and, consequently, that $h(\alpha, \beta) = \|\alpha\|_h \|\beta\|_h$ on N. It follows that $\alpha = \beta$, which is (iv). This proves Theorem 1.1.

Theorem 1.2(a) is a consequence of the following lemma and the fact that, when (i)–(iv) hold, G agrees with H.

Lemma 4.2. *Let Z be a subgroup of $Z(\pi_1(N))$ of rank k. If G is positive semidefinite, then*

$$G(\iota(z_0), \iota(z_1)) = G(\omega_{z_0}, \omega_{z_1}) = \tfrac{1}{2}[B(z_0, z_1) + B(z_1, z_0)]$$

extends to an inner product on $\mathbb{R}^k$ that induces the asymptotic norm $\|\cdot\|_\infty$ of Z with respect to the isomorphism $D : Z \to \mathbb{Z}^k$.

Proof. Since G is positive semidefinite, it is a semi-inner product and induces a seminorm $\|\cdot\|_G$ on $\mathscr{B}$. If $\zeta \in \mathrm{Ker}(\mathscr{D})$, then it follows from the definition of G that $\|\zeta\|_G = 0$. Conversely, suppose $\|\zeta\|_G = 0$. Since $\|\iota(\cdot)\|_G = \|\cdot\|_\infty$, there exists $c > 0$ such that $\|\iota(x)\|_G \geq c\|x\|_{\mathbb{R}^k}$ for all $x \in \mathbb{Z}^k$. Write $W = \mathrm{span}\{\omega_{w_1}, \ldots, \omega_{w_k}\}$. Since $\mathscr{D}|_W : W \to \mathbb{R}^k$ is an invertible linear map, there exists $C > 0$ such that $\|w\|_G \leq C\|\mathscr{D}(w)\|_{\mathbb{R}^k}$ for all $w \in W$. There exist $K_i \to \infty$ and $v_i \in W$ with $\|\mathscr{D}(v_i)\|_{\mathbb{R}^k} \leq 1$ such that $\mathscr{D}(K_i\zeta + v_i) \in \mathbb{Z}^k$. Since $\mathscr{D} \circ \iota$ is the identity map on $\mathbb{Z}^k$, one has that

$$\mathscr{D}(K_i\zeta + v_i - \iota \circ \mathscr{D}(K_i\zeta + v_i)) = 0,$$

and, consequently,

$$|\,\|K_i\zeta + v_i\|_G - \|\iota \circ \mathscr{D}(K_i\zeta + v_i)\|_G\,| \leq \|K_i\zeta + v_i - \iota \circ \mathscr{D}(K_i\zeta + v_i)\|_G = 0.$$

It follows that

$$c\|\mathscr{D}(K_i\zeta + v_i)\|_{\mathbb{R}^k} \leq \|\iota \circ \mathscr{D}(K_i\zeta + v_i)\|_G = \|K_i\zeta + v_i\|_G \leq \|K_i\zeta\|_G + \|v_i\|_G \leq C.$$

Therefore, $\|\mathscr{D}(\zeta)\|_{\mathbb{R}^k} \leq (1 + C/c)/K_i \to 0$ as $i \to \infty$ and, consequently, $\mathscr{D}(\zeta) = 0$. Thus, $\mathrm{Ker}(\mathscr{D}) = \{\zeta \in \mathscr{B} \mid \|\zeta\|_G = 0\}$.

Let $\tilde{\mathscr{B}}$ be the vector space obtained as the quotient of $\mathscr{B}$ by $\mathrm{Ker}(\mathscr{D})$, $\tilde{G}$ the corresponding inner product on $\mathscr{B}$, and $\|\cdot\|_{\tilde{G}}$ the induced norm. The map $\mathscr{D}$ descends to a map $\tilde{\mathscr{D}}$ on $\tilde{\mathscr{B}}$ with trivial kernel. By construction, $\tilde{\mathscr{D}}([\omega_{w_i}]) = e_i$ for each i, so $\tilde{\mathscr{D}}$ is surjective. From this, we have that $\tilde{\mathscr{D}}$ is a linear isomorphism, and $\tilde{\mathscr{D}}^{-1}(\mathbb{Z}^k) = \left\{\sum_i m_i[\omega_{w_i}] \mid m_i \in \mathbb{Z}\right\} = \{[\omega_z] \mid z \in Z\}$ projects to a dense subset of the unit sphere in $\tilde{\mathscr{B}}$. Since

$$\|[\omega_z]\|^2_{\tilde{G}} = \tilde{G}([\omega_z], [\omega_z]) = G(\omega_z, \omega_z) = B(z, z) = \|z\|^2_\infty = \|\tilde{\mathscr{D}}([\omega_z])\|^2_\infty$$

for all $z \in Z$, it follows by continuity that $\tilde{\mathscr{D}} : (\tilde{\mathscr{B}}, \|\cdot\|_{\tilde{G}}) \to (\mathbb{R}^k, \|\cdot\|_\infty)$ is an isomorphism of normed spaces. Consequently, $\tilde{\mathscr{D}}_*(\tilde{G})$ is an inner product on $\mathbb{R}^k$ that induces $\|\cdot\|_\infty$. On $\mathbb{Z}^k$, $\tilde{\mathscr{D}}_*(\tilde{G}) = G \circ \iota$. □

By the following lemma, when (i)–(iv) hold, the involutive C^{r-1} distribution $\bigcap_{i=1}^k \hat{\omega}_{w_i}^\perp$ has constant dimension $n-k$, and, by Frobenius's theorem, it foliates $\hat{N}$ by C^r $(n-k)$-dimensional submanifolds, each of which is contained in an intersection of the form $\bigcap_{i=1}^k H_i$ for horospheres $H_1, \ldots, H_k$ of $w_1, \ldots, w_k$, respectively.

Lemma 4.3. *Let Z be a subgroup of $Z(\pi_1(N))$ of rank k for which statements* (i)–(iv) *of Theorem 1.1 hold. Then, for each $\hat{x} \in \hat{N}$, the set $\{\hat{\omega}_{w_1}(\hat{x}), \ldots, \hat{\omega}_{w_k}(\hat{x})\}$ is linearly independent.*

Proof. Suppose that, for some $c_1, \ldots, c_k \in \mathbb{R}$, $\sum_{i=1}^{k} c_i \omega_{w_i}(\hat{x}) = 0$. Then,

$$0 = \left\| \sum_{i=1}^{k} c_i \omega_{w_i}(\hat{x}) \right\|_h^2 = \sum_{i,j=1}^{k} c_i c_j h(\omega_{w_i}(\hat{x}), \omega_{w_j}(\hat{x}))$$
$$= \sum_{i,j=1}^{k} c_i c_j h(\omega_{w_i}(\hat{y}), \omega_{w_j}(\hat{y})) = \left\| \sum_{i=1}^{k} c_i \omega_{w_i}(\hat{y}) \right\|_h^2$$

for all $\hat{y} \in N$. Thus, $\sum_{i=1}^{k} c_i \omega_{w_i} = 0$ on N. Therefore,

$$0 = \left\| \sum_{i=1}^{k} c_i \omega_{w_i} \right\|_H = \left\| \sum_{i=1}^{k} c_i \omega_{w_i} \right\|_G$$

and, consequently,

$$0 = \mathscr{D}\left(\sum_{i=1}^{k} c_i \omega_{w_i} \right) = \sum_{i=1}^{k} c_i e_i .$$

This forces $c_i = 0$ for all i. □

The next lemma implies that each intersection $\bigcap_{i=1}^{k} H_i$ is contained in one leaf of the foliation and, as a consequence, its leaves are exactly those intersections.

Lemma 4.4. *Let Z be a subgroup of $Z(\pi_1(N))$ for which statements* (i)–(iv) *in Theorem 1.1 hold. Fix $\hat{x} \in \hat{N}$, and, for each i, denote by H_i the horosphere of w_i through $\hat{x}$. Then, $\hat{H} = \bigcap_{i=1}^{k} H_i$ is connected.*

Proof. The proof is by induction. Write $v_j = \hat{\omega}_{w_j(\hat{x})}$. Since b_{v_1} has nonzero gradient, the projection $\hat{N} \to \hat{H}_{z_1}$ along the integral curves of b_{v_1} is a continuous surjection, which implies that H_1 is connected. If the result holds for $\bigcap_{i=1}^{j} H_i$, then, by Lemma 4.1(c), the restriction of $b_{v_{j+1}}$ to $\bigcap_{i=1}^{j} H_i$ has nonzero gradient, so $\bigcap_{i=1}^{j+1} H_i$ is similarly connected. □

Define a map $\Psi : \hat{H} \times \mathbb{R}^k \to \hat{N}$ in the following way: For each $(\hat{y}, s_1, \ldots, s_k)$ in $\hat{H} \times \mathbb{R}^k$, let $\hat{x}_0 = \hat{y}$, and inductively define $\hat{x}_{i+1} = \gamma_{\hat{\omega}_{i+1}(\hat{x}_i)}(s_{i+1}) = \exp_{\hat{x}_i}(s_{i+1}\hat{\omega}_{i+1}(\hat{x}_i))$. Let $\Psi(\hat{y}, s_1, \ldots, s_k) = \hat{x}_k$. The splittings $\hat{N} \cong H_i \times \mathbb{R}$ ensure that $D\Psi$ is nonsingular, and, consequently, the inverse function theorem implies that Ψ is a local diffeomorphism.

Lemma 4.5. *Ψ is proper.*

Proof. It follows from Corollary 3.4 that $[B(w_i, w_j)]$ is a positive-definite symmetric matrix, so $\sum_{i=1}^k |b_i \circ \Psi(\hat{y}, m_1, \ldots, m_k)| \to \infty$ uniformly as $\sum_{i=1}^k |m_i| \to \infty$ for $m_i \in \mathbb{Z}$. By periodicity, there exists a Lipschitz constant, uniform in $\hat{y} \in \hat{N}$, for all maps of the form $\Psi(\hat{y}, \cdot)$. Thus, $\sum_{i=1}^k |b_i \circ \Psi(\hat{y}, s_1, \ldots, s_k)| \to \infty$ uniformly as $\sum_{i=1}^k |s_i| \to \infty$. Let $X \subset \hat{N}$ be compact. Then $\sum_{i=1}^k |b_i|$ is bounded on X, which implies that the projection of $\Psi^{-1}(X)$ onto the $\mathbb{R}^k$-factor is compact. At the same time, the projection of $\Psi^{-1}(X)$ onto the $\hat{H}$-factor is contained within a closed ball around X, so it too is compact. Thus, $\Psi^{-1}(X)$ is compact. □

Hadamard's global inverse function theorem implies that Ψ is a diffeomorphism, which proves parts (c) and, in turn, (b) of Theorem 1.2.

For a fixed $\hat{x} \in \hat{N}$, let $\hat{H}$ be the intersection of horospheres $\bigcap_{i=1}^k H_i$ containing $\hat{x}$. Following the argument in [O'Sullivan 1974], set $G_0 = \{g \in \pi_1(N) \mid g(\hat{x}) \in \hat{H}\}$. By Lemma 2.1, G_0 is normal, contains the commutator subgroup $[\pi_1(N), \pi_1(N)]$, and acts freely and properly discontinuously on $\hat{H}$ by isometries. Note that the subgroup G_0' of G_0 consisting of orientation-preserving elements has all of those same properties, and that the quotient space $N_0 = \hat{H}/G_0'$ is orientable. The subgroup G generated by G_0' and Z is isomorphic to $G_0' \times Z$, and the quotient space $\hat{H}/G$ is diffeomorphic to $N_0 \times \mathbb{T}^k$. One obtains normal covering maps

$$\hat{H} \times \mathbb{R}^k \xrightarrow{\psi_0} N_0 \times \mathbb{T}^k \xrightarrow{\phi_0} N.$$

Note that ψ_0 may be assumed to be a product and that $G_0' = \pi_1(N_0)$. Let Γ denote the quotient group $\pi_1(N)/G$. By construction,

$$0 \to \pi_1(N_0) \times Z \to \pi_1(N) \to \Gamma \to 0$$

is a short exact sequence, which implies that

$$0 \to (\pi_1(N_0)/[\pi_1(N), \pi_1(N)]) \times Z \to H_1(N, \mathbb{Z}) \to \Gamma \to 0$$

is as well. This proves Theorem 1.2(d).

5. Proof of Theorem 1.4

A straightforward volume argument shows that, whenever a central Busemann function is everywhere subharmonic or everywhere superharmonic, it must be harmonic.

Lemma 5.1. *Let N be a compact Riemannian manifold with no conjugate points and b_v a central Busemann function on $\hat{N}$. If b_v is either sub- or superharmonic, then it is harmonic.*

Proof. If b_v vanishes identically, the result is clear. Suppose b_v corresponds to a nontrivial element of $Z(\pi_1(N))$. Let T, U, and, for each $t \in \mathbb{R}$, H_t be as in

Lemma 3.2. Then,

$$\frac{d}{dt}\operatorname{vol}_{H_{T+t}}(U\times\{T+t\}) = \frac{1}{\|z\|_\infty}\int_{U\times\{T+t\}} \Delta b_v \, d\operatorname{vol}_{H_{T+t}}.$$

Since the right-hand side is either nonnegative or nonpositive, $\operatorname{vol}_{H_{T+t}}(U\times\{T+t\})$ is $\|z\|_\infty$-periodic, and U and T are arbitrary, Δb_v must vanish identically. □

If b_v is a convex or concave central Busemann function, then it is subharmonic or, respectively, superharmonic. By Lemma 5.1, it is therefore harmonic. Since every convex or concave harmonic function has vanishing Hessian, one obtains the following.

Lemma 5.2. *Let b_v be a central Busemann function on $\hat{N}$. If b_v is convex or concave, then it is totally geodesic.*

For the remainder of this section, the hypotheses of Theorem 1.4 will be assumed. In this case, the conclusions of Theorem 1.1 hold. As before, let $w_1, \ldots, w_k$ generate Z.

Fix $\hat{x} \in N$, and, for each i, let $v_i = \hat{\omega}_{w_i}(\hat{x})$. Since b_{v_i} is totally geodesic, each of its horospheres is a totally geodesic submanifold of $\hat{N}$. This and the fact that $\nabla_{\hat{\omega}_{w_i}}\hat{\omega}_{w_i} = 0$ imply that $\hat{\omega}_{w_i}$ is parallel. Thus, the k-dimensional distribution $\mathscr{S} = \operatorname{span}\{\hat{\omega}_{w_1}, \ldots, \hat{\omega}_{w_1}\}$ is involutive and, by Frobenius's theorem, foliates $\hat{N}$ by k-dimensional submanifolds. Because $\mathscr{S}$ restricts on each leaf of this foliation to a globally parallel orthonormal frame, its leaves are flat and totally geodesic Euclidean spaces. It follows from de Rham's splitting theorem [1952] that $\hat{N}$ is isometric to $\hat{H}\times\mathbb{R}^k$ for any intersection of horospheres $\hat{H}$ as in Theorem 1.1. Note that Z acts on each $\mathbb{R}^k$-fiber by translations, which completes the proof of Theorem 1.4(a). The proof of part (b) follows exactly as in [O'Sullivan 1974]: the distribution $\mathscr{D}$ projects to a parallel distribution on N, the leaves of which are compact, flat, totally geodesic, and without holonomy; consequently, they must be toruses.

Unlike in the case of nonpositive curvature, it is not known that a nontrivial subgroup of the fundamental group of a compact manifold with no conjugate points has center consisting of its Clifford translations (cf. Proposition 2.3 of [Chen and Eberlein 1980] and Lemma 3 of [Eberlein 1982]). However, Theorem 1.4(c) may still be proved using Theorem 2.2 and the argument in Remark 1 of [Eberlein 1982]. Corollary 2.3 implies that N is finitely covered by a manifold $\tilde{N}$ with $\pi_1(\tilde{N}) \cong G'\times\mathbb{Z}^k$ for a subgroup G' of $\pi_1(N)$, where $w_1^{m_1}, \ldots, w_k^{m_k}$ generate the $\mathbb{Z}^k$-factor. Without loss of generality, one may replace G' with its orientation-preserving elements. By Theorem 1.4(a) and Lemma 2.1, elements of $\pi_1(N)$ preserve the horizontal and vertical foliations of $\hat{H}\times\mathbb{R}^k$. Thus, G' acts on $\hat{H}$. If this action were not free and properly discontinuous, one could construct, by an argument similar to the proof of

Lemma 2.4, elements of $\pi_1(\tilde{N})$ of arbitrarily small displacement, contradicting the compactness of N. Thus, the quotient space $N_1 = \hat{H}/G'$ is an orientable manifold.

An elementary argument shows that $\tilde{N}$ has the structure of a $\mathbb{T}^k$-bundle, with totally geodesic and flat $\mathbb{T}^k$-fibers, over N_1. The proof of Lemma 4 of [Eberlein 1982], except with C^r regularity, passes through exactly as written. Thus, one may construct a C^r section of the $\mathbb{T}^k$-bundle as in Remark 1 of [Eberlein 1982], which then implies that $\tilde{N}$ is diffeomorphic to $N_1 \times \mathbb{T}^k$ and that the covering map $N_1 \times \mathbb{T}^k \to N$ restricts on each $\mathbb{T}^k$-fiber to a totally geodesic and locally isometric immersion onto a leaf of the $\mathbb{T}^k$-foliation of N. This proves part (c).

6. Additional results and questions

This work was largely motivated by the question of whether the center theorem of Wolf [1964] and O'Sullivan [1974] generalizes to the case of no conjugate points.

Question 6.1. Let N be a compact Riemannian manifold with no conjugate points with $Z(\pi_1(N))$ of rank k.

(a) Is N foliated by totally geodesic and flat k-toruses?

(b) Does the universal covering space $\hat{N}$ split isometrically as $\hat{H} \times \mathbb{R}^k$?

As a starting point, one might try to show that the asymptotic norm on $Z(\pi_1(N))$ is Riemannian.

It is natural to look for geometric conditions, weaker than having no focal points, that ensure the linear splitting of central Busemann functions in Theorem 1.1. The following argument shows that it suffices to control the asymptotic geometry of distance spheres on $\hat{N}$: Fix a nontrivial $z \in Z(\pi_1(N))$. The C^{r-1} regularity of ω_z guarantees that the stable Jacobi tensor along each $\gamma_{\hat{\omega}_z(\hat{x})}$ is bounded. The argument in Proposition 5 of [Eschenburg 1977] shows that, at each point $\hat{x} \in \hat{N}$, the second fundamental forms of the distance spheres $\partial B(\gamma_v z(-t), t)$, for $v = \hat{\omega}_z(\hat{x})/\|\hat{\omega}_z(\hat{x})\|$, converge to that of the horosphere of z through $\hat{x}$.

For any unit vector $v \in T_{\hat{x}}\hat{N}$, define the distance function $\rho_{\hat{x}}(\cdot) = d_{\hat{N}}(\cdot, \hat{x})$ to be *asymptotically subharmonic in the direction of* v (respectively, *superharmonic*) if $\liminf_{t\to\infty} \Delta\rho_{\hat{x}}(\gamma_v(t)) \geq 0$ (respectively, $\limsup_{t\to\infty} \Delta\rho_{\hat{x}}(\gamma_v(t)) \leq 0$). On $\hat{N} \setminus \{\hat{x}\}$, let $\kappa^-_{\hat{x}}$ and $\kappa^+_{\hat{x}}$ be the functions equal to the smallest and, respectively, largest eigenvalues of Hess $\rho_{\hat{x}}$, and similarly define $\rho_{\hat{x}}$ to be *asymptotically convex in the direction of* v (respectively, *concave*) if $\liminf_{t\to\infty} \kappa^-_{\hat{x}}(\gamma_v(t)) \geq 0$ (respectively, $\limsup_{t\to\infty} \kappa^+_{\hat{x}}(\gamma_v(t)) \leq 0$). Lemmas 5.1 and 5.2 imply the following.

Proposition 6.2. *Let* $z \in Z(\pi_1(N))$. *Then, the following hold*:

(a) *If, for all* $\hat{x} \in \hat{N}$, $\rho_{\hat{x}}$ *is asymptotically subharmonic in the direction of* $v = \hat{\omega}_z(\hat{x})$, *then, for each such* v, b_v *is harmonic.*

(b) *If, for all* $\hat{x} \in \hat{N}$, $\rho_{\hat{x}}$ *is asymptotically convex in the direction of* $v = \hat{\omega}_z(\hat{x})$, *then, for each such* v, b_v *is totally geodesic.*

Similar results hold for asymptotically superharmonic or asymptotically concave central Busemann functions.

Question 6.3. Are there natural geometric conditions, other than having no focal points, that ensure central Busemann functions are asymptotically subharmonic or asymptotically convex?

Natural conditions that have been considered are Ricci curvature bounds, the effects and limitations of which are discussed thoroughly in [Eschenburg and O'Sullivan 1980].

It's also unclear whether the Heber foliation of the unit sphere bundle of a torus with no conjugate points [Heber 1994] generalizes to the case where Busemann functions in Z split linearly.

Question 6.4. Let N be a compact Riemannian manifold with no conjugate points and Z a subgroup of $Z(\pi_1(N))$ such that statements (i)–(iv) in Theorem 1.1 hold. Write

$$\mathscr{W} = \left\{ \sum_{i=1}^{k} a_i \hat{\omega}_{w_i} \mid a_i \in \mathbb{R} \right\},$$

and, for any $\hat{x} \in \hat{N}$, write

$$\mathscr{V}_{\hat{x}} = \left\{ \nabla b_v \mid v = \sum_{i=1}^{k} a_i \hat{\omega}_{w_i}(\hat{x}) \text{ for } a_i \in \mathbb{R} \right\}.$$

Is $\mathscr{V}_{\hat{x}} = \mathscr{W}$?

An elementary argument shows that, for each nonzero $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, there is a unique C^r horofunction h_a such that any sequence of rational directions that converge to a induces a sequence of Busemann functions that vanish at $\hat{x}$ and converge uniformly on compact sets to h_a. Moreover, the gradient flow of h_a is through geodesics, and its level sets are the integral submanifolds of the codimension-one involutive distribution

$$\left(\sum_{i=1}^{k} a_i \hat{\omega}_{w_i} \right)^{\perp}.$$

However, without the linear divergence of geodesics in [Heber 1994], it's not clear that, when a is irrational, $h_a = b_v$ for $v = \sum_{i=1}^{k} a_i \hat{\omega}_{w_i}(\hat{x})$.

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JAMES DIBBLE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF IOWA
IOWA CITY, IA
UNITED STATES
james-dibble@uiowa.edu
Current address:
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SOUTHERN MAINE
PORTLAND, ME
UNITED STATES
james.dibble@maine.edu

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