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Using topological degree theory, we obtain the existence of circle patterns with prescribed combinatorial type and obtuse exterior intersection angles on surfaces of finite topological type. As consequences, several generalizations of circle pattern theorem are obtained.

1. Introduction

To study the geometry and topology of 3-manifolds, Thurston [1979, Chapter 13] states the circle pattern theorem regarding the existence and uniqueness of circle patterns on higher genus surfaces with prescribed combinatorial type and nonobtuse exterior intersection angles. In recent years, the circle patterns have played significant roles in various problems in combinatorics [Schramm 1992; 1993; Liu and Zhou 2016], discrete and computational geometry [Stephenson 2005; Dai et al. 2008], deformation theory [He and Liu 2013; Huang and Liu 2017], minimal surfaces [Bobenko et al. 2006], and many others.

Assume that \mathcal{T} is a triangulation of a compact oriented surface S (possibly with boundary) with a constant curvature metric μ . A circle pattern \mathcal{P} on (S, μ) is a collection of oriented circles. We say \mathcal{P} is \mathcal{T} -type if there exists a geodesic triangulation $\mathcal{T}(\mu)$ of (S, μ) such that $\mathcal{T}(\mu)$ is isotopic to \mathcal{T} and the vertices of $\mathcal{T}(\mu)$ coincide with the centers of the circles in \mathcal{P} . Let V, E, F be the sets of vertices, edges and triangles of \mathcal{T} . In this paper, we will focus on these \mathcal{T} -type circle patterns $\mathcal{P} = \{C_v : v \in V\}$ such that C_u and C_w intersect with each other whenever there exists an edge between u and w . In this situation we have the exterior intersection angle $\Theta(e) \in [0, \pi)$ for every $e \in E$. We refer to Stephenson's monograph [2005] for more details on circle patterns.

Given a function $\Theta : E \rightarrow [0, \pi)$ defined on the edge set of \mathcal{T} , let us consider the following question: Does there exist a \mathcal{T} -type circle pattern whose exterior intersection angle function is given by Θ ? In addition, if it does exist, to what extent

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is the circle pattern unique? Under the condition that S has empty boundary and $0 \leq \Theta \leq \pi/2$, a celebrated answer to this question is the following circle pattern theorem.

Theorem 1.1 [Thurston 1979, Chapter 13]. *Let \mathcal{T} be a triangulation of an oriented closed surface S of genus $g > 0$. Suppose that $\Theta : E \rightarrow [0, \pi/2]$ is a function satisfying the following conditions:*

- (i) *If e_1, e_2, e_3 form a null-homotopic closed curve in S , and if $\sum_{l=1}^3 \Theta(e_l) \geq \pi$, then these three edges form the boundary of a triangle of \mathcal{T} .*
- (ii) *If e_1, e_2, e_3, e_4 form a null-homotopic closed curve in S , then $\sum_{l=1}^4 \Theta(e_l) < 2\pi$.*

Then there exists a constant curvature (equal to 0 for $g = 1$ and equal to -1 for $g > 1$) metric μ on S such that (S, μ) supports a \mathcal{T} -type circle pattern \mathcal{P} with the exterior intersection angles given by Θ . Moreover, the pair (μ, \mathcal{P}) is unique up to isometries if $g > 1$, and up to similarities if $g = 1$.

Recently, Ge, Hua and Zhou [Ge et al. \geq 2020] obtained a more general result, which considered the case that S had possibly nonempty boundary and Θ was possibly larger than $\pi/2$. Before stating the result, let us introduce some terminologies. Assume that S is of the topological type (g, h) , i.e., S is of genus g and has boundary consisting of h disjoint simple closed curves. A closed (not necessarily simple) curve γ in S is said to be *pseudo-Jordan* if $S \setminus \gamma$ contains at least one simply connected component \mathbb{K}_γ such that $\partial\mathbb{K}_\gamma = \gamma$, and an *enclosing set* $A_\gamma \subset V$ of γ consists of all vertices covered by \mathbb{K}_γ . Similarly, an arc λ in S is said to be *semi-pseudo-Jordan* if there exists an open arc $\tilde{\lambda} \subset \partial S$ such that $\lambda \cup \tilde{\lambda}$ is a pseudo-Jordan curve in S , and a *semi-enclosing set* W_λ of λ consists of all vertices covered by $\mathbb{K}_{\lambda \cup \tilde{\lambda}}$. What is more, we say a pseudo-Jordan (or semi-pseudo-Jordan) curve or arc is *nonvacant* if one of its enclosing sets (or semi-enclosing sets) is nonempty. Denote by V_∂ and E_∂ the sets of boundary vertices and edges of \mathcal{T} . Ge, Hua and Zhou’s result [\geq 2020] was stated as follows.

Theorem 1.2. *Let \mathcal{T} be a triangulation of a surface S of topological type (g, h) . Suppose that $\Theta : E \rightarrow [0, \pi)$ and $\varphi : V_\partial \rightarrow [0, \pi)$ are two functions satisfying the following properties:*

(C1) *If the edges e_1, e_2, e_3 form the boundary of a triangle of \mathcal{T} , then*

$$I(e_1) + I(e_2)I(e_3) \geq 0, \quad I(e_2) + I(e_1)I(e_3) \geq 0, \quad I(e_3) + I(e_1)I(e_2) \geq 0,$$

where $I(e_\eta) = \cos \Theta(e_\eta)$ for $\eta = 1, 2, 3$.

(C2) *The Gauss–Bonnet inequality (resp. equality) holds:*

$$\sum_{v \in V_\partial} \varphi(v) > 2\pi \chi(S) \quad \left(\text{resp.} \quad \sum_{v \in V_\partial} \varphi(v) = 2\pi \chi(S) \right).$$

(C3) When the edges e_1, e_2, \dots, e_s form a nonvacant pseudo-Jordan curve in S , then $\sum_{l=1}^s \Theta(e_l) < (s - 2)\pi$.

(C4) When the edges e_1, \dots, e_s form a nonvacant semi-pseudo-Jordan arc λ in S , for any nonempty semi-enclosing set W_λ of λ , then

$$\sum_{v \in W_\lambda \cap V_\partial} \varphi(v) + \sum_{l=1}^s (\pi - \Theta(e_l)) > \pi.$$

Then there exists a hyperbolic (resp. Euclidean) metric μ on S so that (S, μ) has boundary consisting of h disjoint simple piecewise-geodesic closed curves with turning angles assigned by φ and supports a \mathcal{T} -type circle pattern \mathcal{P} whose exterior intersection angle function is Θ . Moreover, the pair (\mathcal{P}, μ) is unique up to isometries (resp. similarities).

In this paper, we shall prove the following result which generalizes the existence part of the above theorem.

Theorem 1.3. Let \mathcal{T} be a triangulation of a surface S of topological type (g, h) . Suppose that $\Theta : E \rightarrow [0, \pi)$ and $\varphi : V_\partial \rightarrow [0, \pi)$ are two functions such that (C2), (C3), (C4) and the following condition are satisfied.

(R1) If the edges e_1, e_2 , and e_3 form the boundary of a triangle of \mathcal{T} , and if $\sum_{i=1}^3 \Theta(e_i) > \pi$, then $\Theta(e_1) + \Theta(e_2) < \pi + \Theta(e_3)$, $\Theta(e_2) + \Theta(e_3) < \pi + \Theta(e_1)$, $\Theta(e_3) + \Theta(e_1) < \pi + \Theta(e_2)$.

Then there exists a hyperbolic (resp. Euclidean) metric on S so that (S, μ) has boundary consisting of h disjoint simple piecewise-geodesic closed curves with turning angles assigned by φ and supports a \mathcal{T} -type circle pattern \mathcal{P} whose exterior intersection angle function is Θ .

Remark 1.4. The condition (C1) implies (R1). On the other hand, we can easily find examples showing that the converse does not hold. That means our result is strictly stronger than the existence part of [Theorem 1.2](#).

Let M and N be two manifolds with the same dimension and let f be a map between M and N . Let $\Lambda \subset M$ be a relatively compact open subset. Suppose that $y \in N \setminus f(\partial\Lambda)$ is a regular value of f and $\Lambda \cap f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$. The topological degree of y and f in Λ is

$$\text{deg}(f, \Lambda, y) := \sum_{l=1}^m \text{sgn}(f, x_l),$$

where $\text{sgn}(f, x_i) = 1$ in the case that the tangent map d_{x_i} preserves orientation and $\text{sgn}(f, x_i) = -1$ in the other case. In general, if $\text{deg}(f, \Lambda, y) \neq 0$, then $\Lambda \cap f^{-1}(y) \neq \emptyset$. More details can be found in [\[Guillemin and Pollack 1974; Hirsch 1976; Milnor 1965\]](#) and the [Appendix](#).

For each $r = (\rho_1, \dots, \rho_{|V|}) \in \mathbb{R}_+^{|V|}$, Thurston’s famous construction produces a corresponding curvature $\text{Th}(\Theta, r)$. What is more, there is a curvature $K[\varphi]$ produced by the function φ above (see Section 3 for the details). The strategy to prove our result above is to show that there is a relatively compact open subset Ω of $\mathbb{R}_+^{|V|}$ such that

$$\deg(\text{Th}(\Theta, \cdot), \Omega, K[\varphi]) \neq 0.$$

The paper is organized as follows: In Section 2, we introduce basic properties of three-circle configurations. In Section 3, using topological degree theory, we prove Theorem 1.3. In Section 4, we establish some results on planar circle patterns. The last section is an appendix regarding some results from manifold theory.

Throughout this paper, we denote by $|\cdot|$ the cardinality of a set and denote by $\chi(\cdot)$ the Euler characteristic of a topological space. We denote by $d_E(\cdot, \cdot)$ and $d_H(\cdot, \cdot)$ the distances in Euclidean and hyperbolic geometries, respectively.

2. Preliminaries

In this section we collect several lemmas on three-circle configurations. It should be pointed out that the nonobtuse versions of these results have been established by Thurston [1979].

Lemma 2.1. *Suppose that $\Theta_1, \Theta_2, \Theta_3 \in [0, \pi)$ are three angles satisfying*

$$\Theta_1 + \Theta_2 + \Theta_3 \leq \pi,$$

or

$$\Theta_1 + \Theta_2 + \Theta_3 > \pi, \quad \Theta_1 + \Theta_2 < \pi + \Theta_3, \quad \Theta_2 + \Theta_3 < \pi + \Theta_1, \quad \Theta_3 + \Theta_1 < \pi + \Theta_2.$$

Then for any three positive numbers ρ_1, ρ_2, ρ_3 , there exists a configuration of three intersecting circles in both Euclidean and hyperbolic geometries, unique up to isometry, having radii ρ_1, ρ_2, ρ_3 and meeting in exterior intersection angles $\Theta_1, \Theta_2, \Theta_3$.

Proof. In Euclidean geometry, set

$$\begin{aligned} l_1 &= \sqrt{\rho_2^2 + \rho_3^2 + 2\rho_2\rho_3 \cos \Theta_1}, \\ l_2 &= \sqrt{\rho_1^2 + \rho_3^2 + 2\rho_1\rho_3 \cos \Theta_2}, \\ l_3 &= \sqrt{\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos \Theta_3}. \end{aligned}$$

To prove the lemma, it suffices to verify that l_1, l_2, l_3 satisfy the triangle inequalities. We divide the proof into the following cases:

(E1) $\Theta_1 + \Theta_2 + \Theta_3 \leq \pi$. To show the triangle equalities is equivalent to checking

$$(2-1) \quad \sin^2 \Theta_1 \rho_2^2 \rho_3^2 + \sin^2 \Theta_2 \rho_1^2 \rho_3^2 + \sin^2 \Theta_3 \rho_1^2 \rho_2^2 + 2\xi_1 \rho_1^2 \rho_2 \rho_3 + 2\xi_2 \rho_1 \rho_2^2 \rho_3 + 2\xi_3 \rho_1 \rho_2 \rho_3^2 > 0,$$

where $\xi_i = \cos \Theta_i + \cos \Theta_j \cos \Theta_k$ for distinct subscripts $i, j, k \in \{1, 2, 3\}$. Note that

$$\xi_1 = \cos \Theta_1 + \cos(\Theta_2 + \Theta_3) + \sin \Theta_2 \sin \Theta_3 \geq 2 \cos \frac{\Theta_1 + \Theta_2 + \Theta_3}{2} \cos \frac{\Theta_1 - \Theta_2 - \Theta_3}{2} \geq 0.$$

Similarly, $\xi_2 \geq 0$ and $\xi_3 \geq 0$. Thus (2-1) holds.

(E2) $\Theta_1 + \Theta_2 + \Theta_3 > \pi$, $\Theta_1 + \Theta_2 < \pi + \Theta_3$, $\Theta_1 + \Theta_3 < \pi + \Theta_2$, $\Theta_3 + \Theta_2 < \pi + \Theta_1$.

In the complex plan \mathbb{C} we find the three points

$$z_1 = \rho_1 \exp(i\pi - i\Theta_2), \quad z_2 = \rho_2 \exp(i\Theta_1 - i\pi), \quad z_3 = \rho_3.$$

It is easy to see

$$d_E(z_2, z_3) = \sqrt{\rho_2^2 + \rho_3^2 + 2\rho_2 \rho_3 \cos \Theta_1} = l_1,$$

$$d_E(z_1, z_3) = \sqrt{\rho_1^2 + \rho_3^2 + 2\rho_1 \rho_3 \cos \Theta_2} = l_2,$$

$$d_E(z_1, z_2) = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 \cos(\Theta_1 + \Theta_2)} > \sqrt{\rho_1^2 + \rho_2^2 + 2\rho_1 \rho_2 \cos \Theta_3} = l_3.$$

As a result, we obtain

$$l_1 + l_2 = d_E(z_2, z_3) + d_E(z_1, z_3) \geq d_E(z_1, z_2) > l_3.$$

Similarly, $l_1 + l_3 > l_2$ and $l_2 + l_3 > l_1$. We thus prove the triangle equalities.

In hyperbolic geometry, the lemma has been established in [Zhou 2017] by computation. For completeness, here we give an independent proof. Set

$$l_1 = \cosh^{-1}(\cosh \rho_2 \cosh \rho_3 + \sinh \rho_2 \sinh \rho_3 \cos \Theta_1),$$

$$l_2 = \cosh^{-1}(\cosh \rho_1 \cosh \rho_3 + \sinh \rho_1 \sinh \rho_3 \cos \Theta_2),$$

$$l_3 = \cosh^{-1}(\cosh \rho_1 \cosh \rho_2 + \sinh \rho_1 \sinh \rho_2 \cos \Theta_3).$$

It suffices to verify that l_1, l_2, l_3 satisfy the triangle inequalities. As before, we divide the proof into the following cases.

(H1) $\Theta_1 + \Theta_2 + \Theta_3 \leq \pi$. To obtain the proof, we need to show

$$(2-2) \quad \sin^2 \Theta_1 x_2^2 x_3^2 + \sin^2 \Theta_2 x_1^2 x_3^2 + \sin^2 \Theta_3 x_1^2 x_2^2 + (2 + 2 \cos \Theta_1 \cos \Theta_2 \cos \Theta_3) x_1^2 x_2^2 x_3^2 + 2\xi_1 a_2 a_3 x_1^2 x_2 x_3 + 2\xi_2 a_1 a_3 x_1 x_2^2 x_3 + 2\xi_3 a_1 a_2 x_1 x_2 x_3^2 > 0,$$

where $a_\eta = \cosh \rho_\eta$ and $x_\eta = \sinh \rho_\eta$ for $\eta = 1, 2, 3$. Under the assumption that $\Theta_1 + \Theta_2 + \Theta_3 \leq \pi$, we have shown that $\xi_1 \geq 0$, $\xi_2 \geq 0$ and $\xi_3 \geq 0$. Hence (2-2) holds.

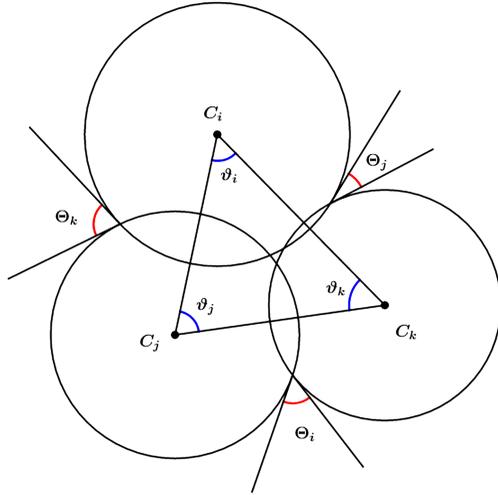


Figure 1. A configuration of three circles.

(H2) $\Theta_1 + \Theta_2 + \Theta_3 > \pi$, $\Theta_1 + \Theta_2 < \pi + \Theta_3$, $\Theta_1 + \Theta_3 < \pi + \Theta_2$, $\Theta_2 + \Theta_3 < \pi + \Theta_1$.
 In the hyperbolic disk \mathbb{D} we find the three points

$$w_1 = \tanh \frac{\rho_1}{2} \exp(i\pi - i\Theta_2), \quad w_2 = \tanh \frac{\rho_2}{2} \exp(i\Theta_1 - i\pi), \quad w_3 = \tanh \frac{\rho_3}{2}.$$

Similarly, it is easy to see

$$\begin{aligned} d_H(w_2, w_3) &= \cosh^{-1}(\cosh \rho_2 \cosh \rho_3 + \sinh \rho_2 \sinh \rho_3 \cos \Theta_1) = l_1, \\ d_H(w_1, w_3) &= \cosh^{-1}(\cosh \rho_1 \cosh \rho_3 + \sinh \rho_1 \sinh \rho_3 \cos \Theta_2) = l_2, \\ d_H(w_1, w_2) &= \cosh^{-1}(\cosh \rho_1 \cosh \rho_2 - \sinh \rho_1 \sinh \rho_2 \cos(\Theta_1 + \Theta_2)) \\ &> \cosh^{-1}(\cosh \rho_1 \cosh \rho_2 + \sinh \rho_1 \sinh \rho_2 \cos \Theta_3) = l_3. \end{aligned}$$

As a result, we obtain

$$l_1 + l_2 = d_H(w_2, w_3) + d_H(w_1, w_3) \geq d_H(w_1, w_2) > l_3.$$

Similarly, $l_1 + l_3 > l_2$ and $l_2 + l_3 > l_1$. We thus finish the proof. □

As in [Figure 1](#), let $\vartheta_1, \vartheta_2, \vartheta_3$ denote the corresponding inner angles of the triangle formed by the centers of the three circles. The following two lemmas were obtained in [\[Zhou 2017\]](#):

Lemma 2.2. *Assume that $a, b, c \in (0, +\infty]$. In both Euclidean and hyperbolic geometries, we have*

$$\begin{aligned} \lim_{(\rho_1, \rho_2, \rho_3) \rightarrow (0, a, b)} \vartheta_1 &= \pi - \Theta_1, \\ \lim_{(\rho_1, \rho_2, \rho_3) \rightarrow (0, 0, c)} \vartheta_1 + \vartheta_2 &= \pi, \\ \lim_{(\rho_1, \rho_2, \rho_3) \rightarrow (0, 0, 0)} \vartheta_1 + \vartheta_2 + \vartheta_3 &= \pi. \end{aligned}$$

Lemma 2.3. Fixing $\Theta_1, \Theta_2, \Theta_3 \in [0, \pi)$, in hyperbolic geometry, for any $\epsilon > 0$, there exists a positive number L such that for any positive ρ_1, ρ_2, ρ_3 satisfying $\rho_1 > L$,

$$\vartheta_1 < \epsilon.$$

As a consequence, we have

$$\lim_{\rho_1 \rightarrow +\infty} \vartheta_1 = 0.$$

Remark 2.4. Let $\Omega \subset [0, \pi)^3$ be the set of all vectors $(\Theta_1, \Theta_2, \Theta_3)$ satisfying $\Theta_1 + \Theta_2 + \Theta_3 \leq \pi$ or $\Theta_1 + \Theta_2 + \Theta_3 > \pi$, $\Theta_1 + \Theta_2 < \pi + \Theta_3$, $\Theta_2 + \Theta_3 < \pi + \Theta_1$, $\Theta_1 + \Theta_3 < \pi + \Theta_2$. Lemma 2.3 still holds if $(\Theta_1, \Theta_2, \Theta_3)$ varies in a compact subset of Ω .

3. Proof of Theorem 1.3

Thurston’s construction. Recall that \mathcal{T} is a triangulation of S with the vertex set V , the edge set E and the face set F and Θ is a weight associated to the edge set. Assume that $V = \{v_1, \dots, v_{|V|}\}$. For each radius vector $r = (\rho_1, \dots, \rho_{|V|}) \in \mathbb{R}_+^{|V|}$, it assigns each vertex v_i a positive number ρ_i . Using the two systems of data Θ and r , we obtain a hyperbolic (or Euclidean) cone metric on the surface S as follows.

Each triangle $\Delta(v_i v_j v_k)$ of \mathcal{T} is associated with a hyperbolic (or Euclidean) triangle formed by centers of three hyperbolic circles of radii ρ_i, ρ_j, ρ_k with exterior intersection angles $\Theta([v_i, v_j])$, $\Theta([v_j, v_k])$, $\Theta([v_k, v_i])$. Precisely, let l_{ij}, l_{jk}, l_{ki} be the three lengths of this triangle. Then

$$l_{ij} = \cosh^{-1}(\cosh \rho_i \cosh \rho_j + \sinh \rho_i \sinh \rho_j \cos \Theta([v_i, v_j]))$$

in hyperbolic background geometry, or

$$l_{ij} = \sqrt{\rho_i^2 + \rho_j^2 + 2\rho_i \rho_j \cos \Theta([v_i, v_j])}$$

in Euclidean background geometry. Similarly, we can get the formulas for l_{jk} and l_{ki} . Under the condition (R1), Lemma 2.1 implies that l_{ij}, l_{jk}, l_{ki} satisfy the triangle inequalities. Thus the above procedure works well.

Gluing all these hyperbolic triangles along the common edges produces a hyperbolic cone metric on S with possible cone singularities at vertices of \mathcal{T} . For $i = 1, 2, \dots, |V|$, set K_i as the discrete curvature at v_i . More precisely,

$$K_i := \begin{cases} 2\pi - \sigma(v_i) & \text{if } v_i \in V \setminus V_\partial, \\ \pi - \sigma(v_i) & \text{if } v_i \in V_\partial, \end{cases}$$

where $\sigma(v_i)$ is the sum of inner angles at v_i for all hyperbolic (or Euclidean) triangles incident to v_i . Clearly, $K_1, \dots, K_{|V|}$ are smooth functions of Θ and r .

This gives rise to the map $\text{Th}(\Theta, \cdot)$:

$$\begin{aligned} \mathbb{R}_+^{|V|} &\rightarrow \mathbb{R}^{|V|}, \\ (\rho_1, \rho_2, \dots, \rho_{|V|}) &\mapsto (K_1, K_2, \dots, K_{|V|}). \end{aligned}$$

Using the prescribed function $\varphi : V_\partial \rightarrow [0, \pi]$ in [Theorem 1.3](#), we define a vector

$$K[\varphi] = (k_1, \dots, k_{|V|}),$$

where

$$k_i = \begin{cases} 0 & \text{if } v_i \in V \setminus V_\partial, \\ \varphi(v_i) & \text{if } v_i \in V_\partial. \end{cases}$$

The main goal now is to show that $K[\varphi]$ belongs to the image of the map $\text{Th}(\Theta, \cdot)$. Actually, if there exists a radius vector

$$r^* = (\rho_1^*, \dots, \rho_{|V|}^*)$$

such that $\text{Th}(\Theta, r^*) = K[\varphi]$, then it produces a hyperbolic (or Euclidean) metric on S . Drawing the circle centered at v_i with the radius ρ_i^* for each v_i , we obtain the desired circle pattern realizing (\mathcal{T}, Θ) .

Topological degree. To this end, we will use the topological degree theory. The readers can refer to the [Appendix](#) for some basic knowledge on this subject.

First let us deal with the hyperbolic geometry case. We need to find a relatively compact open set $\Omega \subset \mathbb{R}_+^{|V|}$ and determine the topological degree

$$\deg(\text{Th}(\Theta, \cdot), \Omega, K[\varphi]).$$

Once we show

$$\deg(\text{Th}(\Theta, \cdot), \Omega, K[\varphi]) \neq 0,$$

[Theorem A.9](#) then implies that $K[\varphi]$ is in the image of $\text{Th}(\Theta, \cdot)$. Now let us compute $\deg(\text{Th}(\Theta, \cdot), \Omega, K[\varphi])$ via homotopy method. For any $t \in [0, 1]$, let $\Theta(t) = t\Theta$. Because $\Theta(t)$ satisfies the condition (R1), [Lemma 2.1](#) implies that $\text{Th}(\Theta(t), \cdot)$ is well defined. Setting $\text{Th}_t = \text{Th}(\Theta(t), \cdot)$, gives a continuous homotopy from $\text{Th}(\Theta, \cdot)$ to Th_0 , where $\text{Th}_0 = \text{Th}(0, \cdot)$.

Lemma 3.1. *In hyperbolic geometry, there exists a relatively compact open set $\Omega \subset \mathbb{R}_+^{|V|}$ such that*

$$\text{Th}_t(\mathbb{R}_+^{|V|} \setminus \Omega) \subset \mathbb{R}^{|V|} \setminus \{K[\varphi]\} \quad \text{for all } t \in [0, 1].$$

Proof. Let us exhaust $\mathbb{R}_+^{|V|}$ by an increasing sequence of relatively compact open sets $\{\Omega_n\}$. Assuming that the lemma is not true, then for each n , we obtain $t_n \in [0, 1]$ and

$$r_n = (\rho_{1,n}, \dots, \rho_{|V|,n}) \in \mathbb{R}_+^{|V|} \setminus \Omega_n$$

such that

$$\text{Th}(\Theta(t_n), r_n) = K[\varphi].$$

Namely,

$$(3-1) \quad K_i(\Theta(t_n), r_n) = \begin{cases} 0 & \text{if } v_i \in V \setminus V_\partial, \\ \varphi(v_i) & \text{if } v_i \in V_\partial. \end{cases}$$

Because $\{\Omega_n\}$ exhausts $\mathbb{R}_+^{|V|}$, there exists $i_0 \in \{1, 2, \dots, |V|\}$ such that

$$\rho_{i_0, n} \rightarrow +\infty \quad \text{or} \quad \rho_{i_0, n} \rightarrow 0.$$

In the first case, it follows from [Lemma 2.3](#) and [Remark 2.4](#), that

$$K_{i_0}(\Theta(t_n), r_n) \rightarrow \begin{cases} 2\pi & \text{if } v_{i_0} \in V \setminus V_\partial, \\ \pi & \text{if } v_{i_0} \in V_\partial. \end{cases}$$

which contradicts [\(3-1\)](#).

It remains to consider the second case. Without loss of generality, assume that $\{t_n\}$ converges to a number $t_* \in [0, 1]$. Otherwise, we pick up a convergent subsequence of $\{t_n\}$. Let $A \subset V$ be the set of vertices v_i for which $\rho_{i, n} \rightarrow 0$. Then A is a nonempty subset of V . We denote by $G(A)$ the union of η -cells ($\eta = 0, 1, 2$) of \mathcal{T} that have at least one vertex in A , and denote by $\text{Lk}(A)$ the set of pairs (e, v) of an edge e and a vertex v such that

$$v \in A; \quad \partial e \cap A = \emptyset; \quad e, v \text{ form a triangle of } \mathcal{T}.$$

Due to [\(3-1\)](#) and [Proposition 3.2](#), we obtain

$$\sum_{v_i \in A \cap V_\partial} \varphi(v_i) = - \sum_{(e, v) \in \text{Lk}(A)} (\pi - \Theta(t_*)(e)) + 2\pi \chi(G(A) \setminus \partial S) + \pi(\chi(G(A) \cap \partial S)).$$

From Thurston's construction, the radius vector r_n produces a \mathcal{T} -type circle pattern pair (μ_n, \mathcal{P}_n) realizing $\Theta(t_n)$. Without loss of generality, suppose that $G(A) \setminus \partial S$ is connected and is of topological type (g_0, h_0) . Note that

$$\chi(G(A) \setminus \partial S) = 2 - 2g_0 - h_0.$$

Meanwhile, $\chi(G(A) \cap \partial S)$ is equal to $-m$, where m is the number of open arc components of $G(A) \cap \partial S$. This yields

$$(3-2) \quad \sum_{v_i \in A \cap V_\partial} \varphi(v_i) = - \sum_{(e, v) \in \text{Lk}(A)} (\pi - \Theta(t_*)(e)) + 2\pi(2 - 2g_0 - h_0) - m\pi.$$

If $g_0 \geq 1$ or $h_0 \geq 2$, the right side of [\(3-2\)](#) is negative, which contradicts the condition that $\varphi(v_i) \geq 0$ for any $v_i \in A$.

Let $g_0 = 0$, $h_0 = 1$. Then $G(A) \setminus \partial S$ is simply connected. Suppose that $\text{Lk}(A) = \{(e_l, v_l)\}_{l=1}^s$. Using (3-2), we have

$$(3-3) \quad \sum_{v_i \in A \cap V_\partial} \varphi(v) = - \sum_{l=1}^s (\pi - \Theta(t_*)(e_l)) + 2\pi - m\pi.$$

We divide it into the following cases:

- (i) $m \geq 2$. Similar arguments to the above part lead to a contradiction.
- (ii) $m = 0$. Here, either $G(A) \cap \partial S = \emptyset$, or $G(A) \setminus \partial S$ is bounded by $G(A) \cap \partial S$. Because $G(A) \setminus \partial S$ is simply connected, the latter case occurs if and only if $(g, h) = (0, 1)$ and $A = V$. Applying the Gauss–Bonnet formula, it is easy to see

$$\sum_{i=1}^{|V|} K_i(\Theta(t_n), r_n) = 2\pi \chi(S) + \text{Area}(S, \mu_n).$$

As $n \rightarrow \infty$, the radius of every circle tends to zero. Therefore

$$\text{Area}(S, \mu_n) \rightarrow 0.$$

Combining this with (3-1), we obtain

$$\sum_{v_i \in V_\partial} \varphi(v) = 2\pi \chi(S),$$

which contradicts the condition (C2). Thus $G(A) \cap \partial S = \emptyset$, and we have $A \cap V_\partial = \emptyset$. By (3-3), we have

$$(3-4) \quad 0 = - \sum_{l=1}^s (\pi - \Theta(t_*)(e_l)) + 2\pi.$$

However, the edges e_1, e_2, \dots, e_s form a *nonvacant pseudo-Jordan curve*. According to the condition (C3), we obtain

$$- \sum_{l=1}^s (\pi - \Theta(t_*)(e_l)) + 2\pi \leq \sum_{l=1}^s \Theta(e_l) - (s - 2)\pi < 0,$$

which contradicts (3-4).

- (iii) $m = 1$. Similarly, it follows from (3-3) that

$$(3-5) \quad \sum_{v_i \in A \cap V_\partial} \varphi(v_i) = - \sum_{l=1}^s (\pi - \Theta(t_*)(e_l)) + \pi.$$

But the edges e_1, e_2, \dots, e_s form a *semi-pseudo-Jordan arc* so that A is a nonempty

semi-enclosing set. Under the condition (C4), we have

$$\sum_{v_i \in A \cap V_\partial} \varphi(v_i) + \sum_{l=1}^s (\pi - \Theta(t_*)(e_l)) \geq \sum_{v_i \in A \cap V_\partial} \varphi(v_i) + \sum_{l=1}^s (\pi - \Theta(e_l)) > \pi,$$

which contradicts (3-5). □

The following result was obtained by Ge, Hua and Zhou [[≥ 2020](#)]. It plays a significant role in the above part.

Proposition 3.2. *Let A , $\text{Lk}(A)$ and $G(A)$ be as above. Then*

$$\sum_{v_i \in A} K_i(\Theta(t_n), r_n) \rightarrow - \sum_{(e, v) \in \text{Lk}(A)} (\pi - \Theta(t_*)(e)) + 2\pi \chi(G(A) \setminus \partial S) + \pi \chi(G(A) \cap \partial S).$$

Remark 3.3. The Euler characteristic $\chi(G(A) \cap \partial S)$ is equal to the opposite of the number of open arc components of $G(A) \cap \partial S$.

It remains to consider the Euclidean geometry case. We easily derive the following combinatorial Gauss–Bonnet formula:

$$(3-6) \quad \sum_{i=1}^{|V|} K_i = 2\pi \chi(S).$$

We use $Y \subset \mathbb{R}^{|V|}$ to denote the hyperplane determined by (3-6) and use $X \subset \mathbb{R}_+^{|V|}$ to denote the set of radius vectors $r = (\rho_1, \dots, \rho_{|V|})$ satisfying

$$\sum_{i=1}^{|V|} \rho_i = 1.$$

It is easy to see that

$$\dim(X) = \dim(Y) = |V| - 1.$$

We have the restriction curvature map $\text{Rh}(\Theta, \cdot)$:

$$\begin{aligned} X &\rightarrow Y, \\ (\rho_1, \rho_2, \dots, \rho_{|V|}) &\mapsto (K_1, K_2, \dots, K_{|V|}). \end{aligned}$$

Analogously, we can construct the homotopy $\text{Rh}_t = \text{Rh}(\Theta(t), \cdot)$ from $\text{Rh}(\Theta, \cdot)$ to $\text{Rh}(0, \cdot)$. Furthermore, similar arguments to the proof of [Lemma 3.1](#) give the following lemma.

Lemma 3.4. *In Euclidean geometry, there exists a relatively compact open set $\Lambda \subset X$ such that*

$$\text{Rh}_t(X \setminus \Lambda) \subset Y \setminus \{K[\varphi]\} \quad \text{for all } t \in [0, 1].$$

We use the strategy of Zhou [[2017](#)] for the proof of the following result.

Theorem 3.5. *Suppose that Θ and φ satisfy the conditions of [Theorem 1.3](#). In hyperbolic geometry, we have*

$$\deg(\text{Th}(\Theta, \cdot), \Omega, K[\varphi]) = 1 \quad \text{or} \quad \deg(\text{Th}(\Theta, \cdot), \Omega, K[\varphi]) = -1.$$

In Euclidean geometry, we have

$$\deg(\text{Rh}(\Theta, \cdot), \Lambda, K[\varphi]) = 1 \quad \text{or} \quad \deg(\text{Rh}(\Theta, \cdot), \Lambda, K[\varphi]) = -1.$$

Proof. In hyperbolic background geometry, due to [Theorem A.8](#),

$$\deg(\text{Th}(\Theta, \cdot), \Omega, K[\varphi]) = \deg(\text{Th}_0, \Omega, K[\varphi]).$$

According to [Theorem 1.2](#), $\Omega \cap \text{Th}_0^{-1}(K[\varphi])$ consists of a unique point. Meanwhile, from [\[Ge et al. \$\geq\$ 2020\]](#), the Jacobian matrix of the map Th_0 is diagonally dominant. Hence $K[\varphi]$ is a regular value of Th_0 . As a result,

$$\deg(\text{Th}_0, \Omega, K[\varphi]) = 1 \quad \text{or} \quad \deg(\text{Th}_0, \Omega, K[\varphi]) = -1.$$

In Euclidean geometry, the conclusion follows verbatim from the hyperbolic case. □

Proof of [Theorem 1.3](#). It follows from [Theorem 3.5](#) and [Theorem A.9](#) that $K[\varphi]$ lies in the image of the map $\text{Th}(\Theta, \cdot)$. By Thurston’s construction, there exists a constant curvature metric μ on S such that (S, μ) supports a \mathcal{T} -type circle pattern \mathcal{P} with the exterior intersection angles given by Θ . □

4. Further discussion

Cone metrics with prescribed curvatures. For a nonempty subset A of V , recall that $G(A)$ is the union of η -cells ($\eta = 0, 1, 2$) of \mathcal{T} that have at least one vertex in A , and $\text{Lk}(A)$ is the set of pairs (e, v) of an edge e and a vertex v such that

$$v \in A, \quad \partial e \cap A = \emptyset, \quad \text{the set of } e, v \text{ form a triangle of } \mathcal{T}.$$

The following result is a generalization of [Theorem 1.3](#).

Theorem 4.1. *Let \mathcal{T} be a triangulation of a surface S of topological type (g, h) . Assume that $\Theta : E \rightarrow [0, \pi)$ is a function satisfying (R1) and $K = (k_1, k_2, \dots, k_{|V|})$ is a vector satisfying*

$$(4-1) \quad k_i < \begin{cases} 2\pi & \text{if } v_i \in V \setminus V_\partial, \\ \pi & \text{if } v_i \in V_\partial, \end{cases}$$

and

$$(4-2) \quad \sum_{v_i \in A} k_i > - \sum_{(e,v) \in \text{Lk}(A)} (\pi - \Theta(e)) + 2\pi \chi(G(A) \setminus \partial S) + \pi \chi(G(A) \cap \partial S)$$

for any nonempty subset A of V . Then there exists a hyperbolic cone metric μ in S

such that (S, μ) has discrete curvatures assigned by K and supports a \mathcal{T} -type circle pattern \mathcal{P} with exterior intersection angles given by Θ .

Proof. We still use the homotopy $\text{Th}_t(\cdot) = \text{Th}(\Theta(t), \cdot)$. Similar arguments to the proof of Lemma 3.1 imply that there exists a relatively compact open set $\Omega_K \subset \mathbb{R}_+^{|V|}$ such that

$$\text{Th}_t(\mathbb{R}_+^{|V|} \setminus \Omega_K) \subset \mathbb{R}_+^{|V|} \setminus \{K\} \quad \text{for all } t \in [0, 1].$$

Set $\text{Th}_0(\cdot) = \text{Th}(0, \cdot)$. Because K satisfies the inequalities (4-1) and (4-2), it follows from [Ge et al. \geq 2020, Theorem 0.4] that K is a regular value of the map Th_0 and $\Omega_K \cap \text{Th}_0^{-1}(K)$ consists of a unique point. Therefore,

$$\deg(\text{Th}_0, \Omega_K, K) = 1 \text{ or } \deg(\text{Th}_0, \Omega_K, K) = -1.$$

Due to Theorem A.8, we obtain

$$\deg(\text{Th}(\Theta, \cdot), \Omega_K, K) = 1 \quad \text{or} \quad \deg(\text{Th}(\Theta, \cdot), \Omega_K, K) = -1.$$

Theorem A.9 then implies that K is in the image of $\text{Th}(\Theta, \cdot)$. Applying Thurston’s construction, the statement follows. \square

Similarly, we obtain the following result.

Theorem 4.2. *Let \mathcal{T} be a triangulation of a surface S of topological type (g, h) . Assume that $\Theta : E \rightarrow [0, \pi)$ is a function satisfying (R1) and $K = (k_1, k_2, \dots, k_{|V|})$ is a vector satisfying (4-1) and*

$$(4-3) \quad \sum_{v_i \in A} k_i \geq - \sum_{(e,v) \in \text{Lk}(A)} (\pi - \Theta(e)) + 2\pi \chi(G(A) \setminus \partial S) + \pi \chi(G(A) \cap \partial S)$$

for any nonempty subset A of V , where the equality holds if and only if $A = V$. Then there exists a Euclidean cone metric μ in S such that (S, μ) has discrete curvatures assigned by K and supports a \mathcal{T} -type circle pattern \mathcal{P} with exterior intersection angles given by Θ .

Planar circle patterns. Setting $(g, h) = (0, 1)$, Theorem 1.3 gives the following result.

Theorem 4.3. *Let \mathcal{T} be a triangulation of a closed topological disk. Suppose that $\Theta : E \rightarrow [0, \pi)$ and $\varphi : V_\partial \rightarrow [0, \pi)$ are two functions such that (R1), (C3) and the following conditions are satisfied:*

(T1) *The Gauss–Bonnet inequality (resp. equality) holds:*

$$\sum_{v \in V_\partial} \varphi(v) > 2\pi \quad \left(\text{resp. } \sum_{v \in V_\partial} \varphi(v) = 2\pi \right).$$

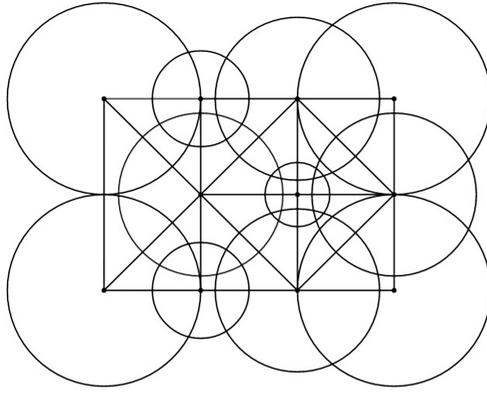


Figure 2. A circle pattern on a rectangle.

(T2) *When the edges e_1, \dots, e_s form a simple arc, joining two distinct boundary vertices, with a nonempty semi-enclosing set W , then*

$$\sum_{v \in W \cap V_\partial} \varphi(v) + \sum_{i=1}^s (\pi - \Theta(e_i)) > \pi.$$

Then there exists a convex hyperbolic (resp. Euclidean) polygon Q , so that Q has external angles given by φ and supports a \mathcal{T} -type circle pattern \mathcal{P} whose exterior intersection angle function is Θ .

Turning to the limiting case that $\varphi \equiv \pi$, the following result is straightforward.

Corollary 4.4. *Let \mathcal{T} be a triangulation of a closed topological disk with a function $\Theta : E \rightarrow [0, \pi)$ satisfying (R1), (C3). Then there exists a \mathcal{T} -type circle pattern \mathcal{P} in the hyperbolic disk, whose exterior intersection angle function is Θ and whose boundary circles are horocycles.*

Another corollary deals with circle patterns on rectangles (see Figure 2).

Corollary 4.5. *Let \mathcal{T} be a triangulation of a quadrangle. Suppose $\Theta : E \rightarrow [0, \pi)$ is a function satisfying (R1), (C3) and the following conditions:*

- (Q1) *When the edges e_1, \dots, e_s form a simple arc joining a pair of opposite corner vertices, then $\sum_{i=1}^s \Theta(e_i) < (s - 1/2)\pi$.*
- (Q2) *When the edges e_1, \dots, e_s form a simple arc which does not join a pair of opposite corner vertices, then $\sum_{i=1}^s \Theta(e_i) < (s - 1)\pi$.*

Then there exists a rectangle R which supports a circle pattern \mathcal{P} with exterior intersection angles given by Θ .

Proof. Let $V_* \subset V_\partial$ be the set of corner vertices. We define a function $\varphi : V_\partial \rightarrow [0, \pi)$ by setting

$$\varphi(v) = \begin{cases} \pi/2 & \text{if } v \in V_*, \\ 0 & \text{if } v \in V_\partial \setminus V_*. \end{cases}$$

Under the conditions (Q1) and (Q2), it is easy to see that (T1) and (T2) are satisfied. By [Theorem 4.3](#), the conclusion holds. \square

Finally, the following result generalizes Marden and Rodin’s theorem [\[1990\]](#).

Theorem 4.6. *Let \mathcal{T} be a triangulation of the sphere. Suppose that $\Theta : E \rightarrow [0, \pi)$ is a function satisfying (R1) and the following conditions:*

- (a) *There is a triangle of \mathcal{T} with boundary edges $e_a, e_b,$ and e_c such that*

$$\sum_{l=a,b,c} \Theta(e_l) < \pi.$$

- (b) *When the edges e_1, e_2, \dots, e_s form a simple closed curve which separates the vertices of \mathcal{T} , then $\sum_{l=1}^s \Theta(e_l) < (s - 2)\pi$.*

Then there exists a \mathcal{T} -type circle pattern \mathcal{P} on the Riemann sphere $\hat{\mathbb{C}}$ with the exterior intersection angles given by Θ .

A circle pattern \mathcal{P} on the sphere can be stereographically projected to the plane. Since stereographic projection is conformal, the exterior intersection angles are the same. More precisely, we can construct the circle pattern on the sphere by constructing the corresponding planar pattern and then projecting it back to the sphere. Thus, to prove [Theorem 4.6](#), we only need to construct a proper planar circle pattern.

Proof of Theorem 4.6. Let Δ_∞ be the triangle of \mathcal{T} with edges e_a, e_b, e_c and let v_a, v_b, v_c be the vertices opposite to e_a, e_b, e_c . By stereographic projection, then $\mathcal{T}' = \mathcal{T} \setminus \Delta_\infty$ gives a triangulation of Δ_∞ . Moreover, we can regard $\Theta : E \rightarrow [0, \pi)$ as a function defined on the edge of \mathcal{T}' . Because of the stereographic projection construction, it suffices to show that there exists a \mathcal{T}' -type circle pattern \mathcal{P} on the unit disk with the exterior intersection angles given by Θ , which is a result of [Corollary 4.4](#). \square

Appendix

In this part we shall give a simple introduction to some results on manifolds, especially the topological degree theory. The reader can refer to [\[Guillemin and Pollack 1974; Hirsch 1976; Milnor 1965\]](#) for more background.

Let M, N be two oriented smooth manifolds. A point $x \in M$ is called critical for a C^1 map $f : M \mapsto N$ if the tangent map $df : T_x \mapsto N_{f(x)}$ is not surjective. Let C_f denote the set of critical points of f , and define $N \setminus f(C_f)$ to be the set of regular values of f .

Theorem A.1 (regular value theorem). *Let $f : M \mapsto N$ be a C^r ($r \geq 1$) map, and let $y \in N$ be a regular value of f . Then $f^{-1}(y)$ is a closed C^r submanifold of M . If $y \in \text{im}(f)$, then the codimension of $f^{-1}(y)$ is equal to the dimension of N .*

Theorem A.2 (Sard's theorem). *Let M, N be manifolds of dimensions m, n and let $f : M \mapsto N$ be a C^r map. If*

$$r \geq \max\{1, m - n + 1\},$$

then $f(C_f)$ has zero measure in N .

Assume that M and N have the same dimension. Let $\Lambda \subset M$ be a relatively compact open subset. Precisely, Λ has compact closure $\bar{\Lambda}$ in M . For a continuous map $f : M \mapsto N$ and a point $y \in N \setminus f(\partial\Lambda)$, we shall define a topological invariant $\text{deg}(f, \Lambda, y)$, called the topological degree of y and f in Λ .

First, suppose that $f \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$ and y is a regular value. When the set $\Lambda \cap f^{-1}(y)$ is empty, we set $\text{deg}(f, \Lambda, y) = 0$. If $\Lambda \cap f^{-1}(y)$ is nonempty, the regular value theorem implies that it consists of finite points. For a point $x \in \Lambda \cap f^{-1}(y)$, the sign $\text{sgn}(f, x)$ is equal to $+1$ if the tangent map $d_x f : M_x \mapsto N_y$ preserves orientation. Otherwise, $\text{sgn}(f, x) = -1$.

Definition A.3. Suppose that $\Lambda \cap f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$. We define

$$\text{deg}(f, \Lambda, y) := \sum_{l=1}^m \text{sgn}(f, x_l).$$

For simplicity, in what follows we use the notation $f \pitchfork_\Lambda y$ to denote that y is a regular value of the restriction map $f : \Lambda \mapsto N$.

Proposition A.4. *Suppose that $f_i \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$, $f_i \pitchfork_\Lambda y$ and $f_i(\partial\Lambda) \subset N \setminus \{y\}$, $i = 0, 1$. If there exists a homotopy*

$$H \in C^0(I \times \bar{\Lambda}, N)$$

such that $H(0, \cdot) = f_0(\cdot)$, $H(1, \cdot) = f_1(\cdot)$, and $H(I \times \partial\Lambda) \subset N \setminus \{y\}$, then

$$\text{deg}(f_0, \Lambda, y) = \text{deg}(f_1, \Lambda, y).$$

The following lemma is a consequence of Sard's theorem.

Lemma A.5. *For any $f \in C^0(\bar{\Lambda}, N)$ and $y \in N$, if $f(\partial\Lambda) \subset N \setminus \{y\}$, then there exist $g \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$ and $H \in C^0(I \times \bar{\Lambda}, N)$ such that*

- (i) $g \pitchfork_\Lambda y$;
- (ii) $H(0, \cdot) = f(\cdot)$, $H(1, \cdot) = g(\cdot)$;
- (iii) $H(I \times \partial\Lambda) \subset N \setminus \{y\}$.

Now we are ready to define the topological degrees for general continuous maps. Recall that $f \in C^0(\bar{\Lambda}, N)$ and $f(\partial\Lambda) \subset N \setminus \{y\}$.

Definition A.6. The topological degree of y and f in Λ is defined as

$$\deg(f, \Lambda, y) = \deg(g, \Lambda, y),$$

where g is given in [Lemma A.5](#).

Due to [Proposition A.4](#), $\deg(f, \Lambda, y)$ is well defined. Below we list several properties of the topological degree.

Theorem A.7. Let $\gamma \subset N$ be a continuous curve such that $f(\partial\Lambda) \subset N \setminus \{\gamma\}$. Then

$$\deg(f, \Lambda, \gamma(t)) = \deg(f, \Lambda, \gamma(0)) \quad \text{for all } t \in [0, 1].$$

For $i = 0, 1$, suppose that $f_i \in C^0(\bar{\Lambda}, N)$ satisfies $f_i(\partial\Lambda) \subset N \setminus \{y\}$.

Theorem A.8. If there exists $H \in C^0(I \times \bar{\Lambda}, N)$ such that

- (i) $H(0, \cdot) = f_0(\cdot)$, $H(1, \cdot) = f_1(\cdot)$,
- (ii) $H(I \times \partial\Lambda) \subset N \setminus \{y\}$,

then

$$\deg(f_0, \Lambda, y) = \deg(f_1, \Lambda, y).$$

Finally, it follows from the definition that:

Theorem A.9. If $\deg(f, \Lambda, y) \neq 0$, then $\Lambda \cap f^{-1}(y) \neq \emptyset$.

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