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Evgenii Semenov, Fedor Sukochev,
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INVARIANT BANACH LIMITS AND APPLICATIONS TO NONCOMMUTATIVE GEOMETRY

EVGENII SEMENOV, FEDOR SUKOCHEV,
ALEXANDR USACHEV AND DMITRIY ZANIN

A linear functional B on the space of bounded sequences l_∞ is called a Banach limit if it is positive, normalised and invariant under the shift operator. There are Banach limits which possess additional invariance properties. We prove that every Banach limit invariant under the Cesaro operator is also invariant under all dilation operators. We also prove the "continuous version" of this result and apply it to the theory of singular traces.

1. Introduction

Let $B(H)$ be an algebra of all bounded linear operators on a separable Hilbert space H.

Define the Lorentz ideal $\mathcal{M}_{1,\infty}$ in $B(H)$ [Lord et al. 2013] as

$$\mathcal{M}_{1,\infty} := \left\{ A \in B(H) : \sup_{n\in\mathbb{N}} \frac{1}{\log(2+n)} \sum_{k=0}^{n} \mu(k, A) < \infty \right\},$$

where $\mu(A)$ is a sequence of singular values of an operator A.

Now we give a definition of Dixmier traces on $\mathcal{M}_{1,\infty}$. We give it in the form which is equivalent (see [Sukochev et al. 2013, Theorem 40] and [Lord et al. 2013]) to the original definitions from [Dixmier 1966].

Definition 1. For every extended limit ω on l_∞ which is dilation-invariant (that is, $\omega(x(1), x(2), \ldots) = \omega(x(1), x(1), x(2), x(2), \ldots)$), the weight

$$\mathrm{Tr}_\omega(A) := \omega\left(\frac{1}{\log(2+n)} \sum_{k=0}^{n} \mu(k, A) \right), \quad 0 \le A \in \mathcal{M}_{1,\infty}, \tag{1}$$

extends by linearity to a nonnormal trace on $\mathcal{M}_{1,\infty}$.

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Recall that l_∞ is the space of all bounded real sequences $x = (x(1), x(2), \ldots)$ equipped with the norm

$$\|x\|_{l_\infty} = \sup_{n\in\mathbb{N}} |x(n)|,$$

and the usual partial order. Here $\mathbb{N}$ stands for the set of natural numbers. Also recall that the extended limit on l_∞ is a linear functional which coincides with the ordinary limit on the subspace of all convergent sequence. By the Hahn–Banach theorem, extended limits exist.

These nonnormal (or singular) traces were discovered by J. Dixmier [1966] and are termed Dixmier traces. Dixmier traces play a key role in Alain Connes' noncommutative geometry — they are used to define an integral in the noncommutative context. A computation of Dixmier traces is a highly nontrivial task. This is due to the fact that in geometrically or physically inspired examples one rarely has any information on singular values of an operator. The remedy to this is formulae relating Dixmier traces to the operator ζ-function and heat traces [Connes and Moscovici 1995; Carey et al. 2007; Lord et al. 2013; Sukochev et al. 2017]. These formulae have become central to noncommutative geometry. In particular, it was proved in [Benameur and Fack 2006, Theorem 2] that for every extended limit ω which is invariant under the logarithmic Cesaro operator M (see definition in Section 4) the formula

$$\mathrm{Tr}_\omega(A) = \lim_{t\to\infty} \frac{1}{t}\mathrm{Tr}(A^{1+\frac{1}{t}}), \quad 0 \le A \in \mathcal{M}_{1,\infty},$$

holds, provided that the limit on the right-hand side exists.

Essentially at the same time, Carey et al. proved the same formula without the assumption on the convergence of the right-hand side. However, it was proved under additional assumptions on ω. More precisely, in [Carey et al. 2003, Theorem 3.1] it was proved that

$$\mathrm{Tr}_\omega(A) = (\omega \circ \log)\left(\frac{1}{t}\mathrm{Tr}(A^{1+\frac{1}{t}})\right), \quad 0 \le A \in \mathcal{M}_{1,\infty},$$

for every extended limit ω which is M-invariant and also dilation and exponentiation-invariant (see definitions in Section 4).

A priori the latter formula is proved for a significantly smaller set of Dixmier traces than the former. Our initial motivation was to compare these two classes of traces. Surprisingly, it turned out that these two classes coincide. To tackle the problem about the classes of Dixmier traces we consider the corresponding classes of extended limits.

Definition 2. A linear functional $B \in l_\infty^*$ is said to be a Banach limit if

(1) $B \geqslant 0$, that is, $Bx \geqslant 0$ for every $x \geqslant 0$,

(2) $B\mathbb{1} = 1$, where $\mathbb{1} = (1, 1, \ldots)$,

(3) $B(Tx) = B(x)$ for every $x \in l_\infty$, where T is a translation operator, that is, $T(x(1), x(2), \ldots) = (0, x(1), x(2), \ldots)$.

We denote the set of all Banach limits by $\mathfrak{B}$. It is easy to see that $\mathfrak{B}$ is a closed convex set on the unit sphere of l_∞^*. It follows directly from the definition of Banach limit that

$$\liminf_{n\to\infty} x(n) \leqslant Bx \leqslant \limsup_{n\to\infty} x(n)$$

for every $x \in \ell_\infty$, $B \in \mathfrak{B}$. In particular, $Bx = \lim_{n\to\infty} x(n)$ for every convergent sequence $x \in l_\infty$. Thus, Banach limits are indeed the extended limits.

The existence of Banach limits was established by S. Mazur. It first appeared in the book of S. Banach [1932]. Since then Banach limits have proved to be a useful tool in functional analysis.

Since we are motivated by the problem involving Dixmier traces generated by extended limits with additional invariance properties, it is reasonable to consider subclasses of Banach limits which posses some additional invariance properties (besides the translation invariance). To the best of our knowledge the existence of such invariant Banach limits was firstly established by W. Eberlein [1950]. Recently, invariant Banach limits have been used to study the asymptotics of the Lebesgue constants of the Walsh system [Astashkin and Semenov 2016]. Let us introduce the operators of interest.

Let the Cesaro operator $C : l_\infty \to l_\infty$ be given by the formula

$$(Cx)(n) = \frac{1}{n}\sum_{k=1}^{n} x(k), \quad n \geq 1,\ x \in l_\infty.$$

A Banach limit B is said to be Cesaro-invariant if $B = B \circ C$. The set of all Cesaro-invariant Banach limits is denoted by $\mathfrak{B}(C)$. A thorough study of such Banach limits can be found in [Semenov and Sukochev 2010].

For every $m \in \mathbb{N}$, let the dilation operator $\sigma_m : l_\infty \to l_\infty$ be given by the formula

$$(\sigma_m x)(n) = x\left(\left\lceil \frac{n}{m} \right\rceil\right), \quad n \geq 1,\ x \in l_\infty.$$

Similarly, a Banach limit B is said to be dilation-invariant if $B = B \circ \sigma_m$ for some $m \in \mathbb{N}$. The set of all σ_m-invariant Banach limits is denoted by $\mathfrak{B}(\sigma_m)$.

Using the Markov–Kakutani theorem it was shown in [Dodds et al. 2003, Theorem 4.2] that there exists a Banach limit which is invariant under the Cesaro operator and all dilation operators σ_m, $m \in \mathbb{N}$. It is known that for every $m \in \mathbb{N}$, $m \geq 2$ there exists $B \in \mathfrak{B}(\sigma_m)$ such that $B \notin \mathfrak{B}(C)$ [Alekhno et al. 2016]. However, the question of whether Cesaro-invariance of the Banach limit implies its dilation invariance was left open in [Alekhno et al. 2016]. Theorem 3 is the main result of the paper.

Theorem 3. $\mathfrak{B}(C) \subset \mathfrak{B}(\sigma_m)$ *for all* $m \in \mathbb{N}$.

This theorem tells us that every Cesaro-invariant Banach limit is in fact dilation-invariant. Having this result at hand, we prove in Proposition 13 that every M-invariant extended limit is in fact dilation- and exponentiation-invariant.

There is another important and surprising consequence of Theorem 3.

It was proved in [Sukochev et al. 2014] (and in [Gayral and Sukochev 2014] in a more general context) that if one considers P_α-invariant extended limits (instead of dilation-invariant) in the formula (1), then one obtains a proper subclass of Dixmier traces. This subclass is denoted by $\mathcal{D}_P$. The paper [Sukochev and Usachev 2015] has made the first attempt to provide a condition which guarantees that all traces from the class $\mathcal{D}_P$ take the same value (on an operator from the weak-$\mathcal{L}_1$, which is a proper subset of $\mathcal{M}_{1,\infty}$). However, that condition is somewhat cumbersome and is difficult to verify (see [Sukochev and Usachev 2015, Theorem 3.9]).

As a consequence of Theorem 3 we prove that the classes $\mathcal{D}_M$ (of all Dixmier traces generated by M-invariant extended limits) and $\mathcal{D}_P$ coincide (on the weak-$\mathcal{L}_1$). This fact enables a neat characterisation of the class $\mathcal{D}_P$ in terms of the Cesaro operator C (see Corollary 19).

The proof of Theorem 3 is based on the key estimate established in Proposition 4.

In Section 2, we prove Proposition 4. In Section 3, we finalise the proof of Theorem 3. In Section 4 we establish the "continuous version" of Theorem 3. In Section 5 we apply the results of Section 4 to the study of singular traces.

2. Proof of the key estimate

The idea behind the proof of Proposition 4 is to construct a bijection ϕ on the set of integers from 1 to ij, with a certain control of $\phi(l)$ in terms of $l \in \{1, \dots, ij\}$ (see Lemma 8).

For $x \in l_\infty$, we set

$$\alpha(x) = \limsup_{i\to\infty} \max_{0\le j\le i} |x(i+j) - x(i)|.$$

Define a function $h : [0, 1] \to \left[0, \frac{1}{2}\right]$ by setting

$$h(t) = \tfrac{1}{2} \sum_{k=1}^{\infty} \frac{\min\{kt, 1\}}{2^k}, \quad t \in [0, 1]. \tag{2}$$

The following result contains the key estimate for the seminorm α on $l_\infty/\{c\mathbb{1}\}$.

Proposition 4. *If* $x \in l_\infty$ *is such that* $0 \le x \le 1$, *then*

$$\alpha(Cx) \le h(\alpha(x)).$$

Notation 5. For $i \ge 5$, let $r = 1 + \lfloor \log_2(i) \rfloor$.

(1) Let $P_{r+1} = \{1\}$. For $1 \le k \le r$, let

$$P_k = \left\{ m \in \{1, \dots, i\} : \left\lfloor \frac{i}{2^k} \right\rfloor + 2 \le m \le \left\lfloor \frac{i}{2^{k-1}} \right\rfloor + 1 \right\}.$$

(2) Define a function F on $\{1, \dots, i\}$ by setting

$$F(m) = 2^k m, \quad m \in P_k,\ 1 \le k \le r+1.$$

(3) For $1 \le j \le i$, set

$$a(l) = \left\lceil \frac{l}{j} \right\rceil, \quad b(l) = i + \left\lceil \frac{l}{i} \right\rceil, \quad 1 \le l \le ij.$$

(4) Here and throughout the paper symbols $\lfloor\ \rfloor$ and $\lceil\ \rceil$ stand for rounding a real number down and up, respectively.

We need another auxiliary result.

Lemma 6. *If $G \subset \{1, \dots, i\}$, then*

$$\max_{m \in G} F(m) \ge i + |G|.$$

Proof. Let

$$D = \{1 \le m \le i : F(m) \le l\}, \quad l = \max_{m \in G} F(m).$$

Since $F(m) \le l$ for every $m \in G$, it follows that $G \subset D$.

If $l \ge 2i$, then there is nothing to prove. Further, we assume that $l < 2i$. This implies $1 \notin D$. Indeed, if $1 \in D$, then $2^{r+1} = F(1) \le l$. However,

$$2^{r+1} = 2 \cdot 2^{1+\lfloor \log_2(i) \rfloor} \ge 2i.$$

We now write

$$|D| = \sum_{k=1}^{r} |D \cap P_k|.$$

If $m \in D \cap P_k,\ 1 \le k \le r$, then $2^k m \le l$ and, so, $m \le l/2^k$. Thus,

$$\left\lfloor \frac{i}{2^k} \right\rfloor + 2 \le m \le \left\lfloor \frac{l}{2^k} \right\rfloor.$$

Therefore,

$$|D \cap P_k| = \left| \left\{ \left\lfloor \frac{i}{2^k} \right\rfloor + 2, \dots, \left\lfloor \frac{l}{2^k} \right\rfloor \right\} \right| = \left\lfloor \frac{l}{2^k} \right\rfloor - \left\lfloor \frac{i}{2^k} \right\rfloor - 1, \quad 1 \le k \le r.$$

Hence,

$$|D| = \sum_{k=1}^{r} |D \cap P_k| = \sum_{k=1}^{r} \left(\left\lfloor \frac{l}{2^k} \right\rfloor - \left\lfloor \frac{i}{2^k} \right\rfloor - 1 \right)_+ \le \sum_{k=1}^{r} \frac{l-i}{2^k} \le l - i.$$

Thus,

$$|G| \le |D| \le l - i = \max_{m \in C} F(m) - i. \qquad \square$$

To proceed further we need the following result from graph theory [Berge 2001, Chapter 10].

Theorem 7 (marriage theorem). *Let Γ_1 and Γ_2 be finite sets such that $|\Gamma_1| = |\Gamma_2|$. For $A \subset \Gamma_1$, set*

$$M(A) = \{l' \in \Gamma_2 : l' \text{ is adjacent to some } l \in A\}.$$

If, for all $A \subset \Gamma_1$, we have $|M(A)| \geq |A|$, then there exists a bijection $\phi : \Gamma_1 \to \Gamma_2$ such that every $l \in \Gamma_1$ is linked to $\phi(l)$.

The following result uses the above theorem to construct a bijection on the set of integers from 1 to ij.

Lemma 8. *For $1 \leq j \leq i$ and $r = 1 + \lfloor \log_2(i) \rfloor$, there exist*

(1) *a partition $\{A_k\}_{k=1}^{r+1}$ of the set $\{1, \dots, ij\}$ such that*

$$|A_k| \leq \frac{ij}{2^k} + j \quad \text{for all } 1 \leq k \leq r+1.$$

(2) *a bijection $\phi : \{1, \dots, ij\} \to \{1, \dots, ij\}$ such that*

$$a(l) \leq b(\phi(l)) \leq 2^k a(l) \quad \text{for all } l \in A_k \text{ for all } 1 \leq k \leq r+1.$$

Proof. **Step 1:** For $1 \leq k \leq r+1$, set

$$A_k = \{1 \leq l \leq ij : a(l) \in P_k\}.$$

For every $1 \leq k \leq r+1$, we have $|A_k| = j|P_k|$. For $1 \leq k \leq r$, we have

$$|A_k| = j|P_k| \leq j \cdot \left(\left\lfloor \frac{i}{2^{k-1}} \right\rfloor - \left\lfloor \frac{i}{2^k} \right\rfloor \right) \leq \frac{ij}{2^k} + j.$$

For $k = r+1$, we have $|A_{r+1}| = j$. This proves the first assertion.

Step 2: Let

$$\Gamma_1 = \Gamma_2 = \{1, \dots, ij\}.$$

We connect $l \in \Gamma_1$ and $l' \in \Gamma_2$ if

$$b(l') \leq F(a(l)).$$

For a fixed $A \subset \Gamma_1$, let

$$G = \{m \in \{1, \dots, i\} : m = a(l) \text{ for some } l \in A\}.$$

For every $m \in G$, there exist at most j elements $l \in A$ such that $a(l) = m$. In particular, we have $|A| \leq j|G|$. We have

$$M(A) = \{l' \in \Gamma_2 : b(l') \leq F(m) \text{ for some } m \in G\}.$$

In the sequence b, every value is repeated exactly i times. It is immediate that

$$|M(A)| = i \cdot |\{1 \le m' \le j : m' + i \le F(m) \text{ for some } m \in G\}|.$$

In other words, we have

$$|M(A)| = i \cdot \min\{j, \max_{m \in C} F(m) - i\}.$$

By Lemma 6, we have

$$|M(A)| \ge i \cdot \min\{j, |G|\}.$$

Since $ij \ge j|G|$ and $i|G| \ge j|G|$, it follows that

$$|M(A)| \ge j|G| \ge |A|.$$

By the marriage theorem, there exists a bijection $\phi : \Gamma_1 \to \Gamma_2$ such that every $l \in \Gamma_1$ is adjacent to $\phi(l) \in \Gamma_2$. That is, we have

$$b(\phi(l) \le F(a(l)), \quad 1 \le l \le ij.$$

If $l \in A_k$, then $a(l) \in P_k$ and

$$b(\phi(l)) \le 2^k a(l), \quad l \in A_k.$$

This proves the right-hand side inequality in the second assertion. Noting that $a(l) \le i \le b(\phi(l))$ for all $l \in \{1, \dots, ij\}$, we complete the proof. □

We need another auxiliary result, which roughly tells us what happens to the value of the functional α if one dilates the interval over which the maximum is taken.

Lemma 9. *Let $y \in l_\infty$ be such that*

$$\sup_{i \ge 1} \max_{0 \le j \le i} |y(i + j) - y(i)| \le \alpha.$$

If $l_1 \le l_2 \le 2^k l_1$, then

$$|y(l_1) - y(l_2)| \le k\alpha.$$

Proof. If $l_1 \le l_2 \le 2^k l_1$, then there exists $0 \le k' < k$ such that $2^{k'} l_1 \le l_2 \le 2^{k'+1} l_1$. We have

$$\begin{aligned} y(l_2) - y(l_1) &= (y(l_2) - y(2^{k'} l_1)) + (y(2^{k'} l_1) - y(l_1)) \\ &= (y(l_2) - y(2^{k'} l_1)) + \sum_{p=0}^{k'-1} (y(2^{p+1} l_1) - y(2^p l_1)). \end{aligned}$$

Therefore,

$$|y(l_1) - y(l_2)| \le |y(l_2) - y(2^{k'} l_1)| + \sum_{p=0}^{k'-1} |y(2^{p+1} l_1) - y(2^p l_1)|.$$

By assumption on y, we have

$$|y(l_2) - y(2^{k'}l_1)| \le \alpha, \quad |y(2^{p+1}l_1) - y(2^p l_1)| \le \alpha, \quad 0 \le p < k'.$$

Therefore,

$$|y(l_1) - y(l_2)| \le \alpha + \sum_{p=0}^{k'-1} \alpha = (k'+1)\alpha \le k\alpha. \qquad \square$$

Proof of Proposition 4. Let $\alpha > \alpha(x)$. Fix $n \ge 0$ such that

$$\sup_{i \ge n} \max_{0 \le j \le i} |x(i+j) - x(i)| \le \alpha.$$

Set $y(i) = x(i)$ for $i \ge n$ and $y(i) = x(n)$ for $i \le n$. We have

$$\sup_{i \ge 1} \max_{0 \le j \le i} |y(i+j) - y(i)| \le \alpha.$$

Note that we still have $0 \le y \le 1$.

If $0 \le j \le i$, then

$$\begin{aligned}(Cy)(i+j) - (Cy)(i) &= \frac{1}{i+j}\sum_{k=1}^{i+j} y(k) - \frac{1}{i}\sum_{k=1}^{i} y(k) \\ &= \frac{1}{i(i+j)}\left(i\sum_{k=1}^{i+j} y(k) - (i+j)\sum_{k=1}^{i} y(k)\right) \\ &= \frac{1}{i(i+j)}\left(i\sum_{k=i+1}^{i+j} y(k) - j\sum_{k=1}^{i} y(k)\right).\end{aligned}$$

It follows from the definition of a and b that

$$\begin{aligned} j\sum_{k=1}^{i} y(k) &= \sum_{l=1}^{ij} y(a(l)) = \sum_{k=1}^{r+1}\sum_{l \in A_k} y(a(l)), \\ i\sum_{k=i+1}^{i+j} y(k) &= \sum_{l=1}^{ij} y(b(l)) = \sum_{l=1}^{ij} y(b(\phi(l))) = \sum_{k=1}^{r+1}\sum_{l \in A_k} y(b(\phi(l))),\end{aligned}$$

where the last equality is given by Lemma 8(1), and the second-last equality is given by Lemma 8(2). Therefore,

$$\left| i\sum_{k=i+1}^{i+j} y(k) - j\sum_{k=1}^{i} y(k) \right| \le \sum_{k=1}^{r+1}\sum_{l \in A_k} |y(a(l)) - y(b(\phi(l)))|. \tag{3}$$

By Lemma 8(2) and Lemma 9, we have

$$|y(a(l)) - y(b(\phi(l)))| \le k\alpha, \quad l \in A_k,\ 1 \le k \le r+1.$$

On the other hand, we have $0 \le y \le 1$. Hence,

$$|y(a(l)) - y(b(\phi(l)))| \le 1, \quad l \in A_k,\ 1 \le k \le r+1.$$

Combining these two estimates, we infer that

$$\sum_{l \in A_k} |y(a(l)) - y(b(\phi(l)))| \le |A_k| \cdot \min\{k\alpha, 1\}, \quad 1 \le k \le r+1.$$

Substituting this formula into (3), we obtain

$$\begin{aligned}
\left| i \sum_{k=i+1}^{i+j} y(k) - j \sum_{k=1}^{i} y(k) \right| &\le \sum_{k=1}^{r+1} |A_k| \cdot \min\{k\alpha, 1\} \\
&\le i \sum_{k=1}^{r+1} \left(\frac{ij}{2^k} + j \right) \min\{k\alpha, 1\} \\
&\le ij \sum_{k=1}^{r+1} \frac{\min\{k\alpha, 1\}}{2^k} + j \sum_{k=1}^{r+1} 1 \\
&\le i(j+i) \cdot \tfrac{1}{2} \sum_{k=1}^{\infty} \frac{\min\{k\alpha, 1\}}{2^k} + 2i \log_2(i),
\end{aligned}$$

where the second inequality comes from Lemma 8(1).

In other words, we have

$$|(Cy)(i) - (Cy)(i+j)| \le \frac{2i \log_2(i)}{i^2} + \tfrac{1}{2} \sum_{k=1}^{\infty} \frac{\min\{k\alpha, 1\}}{2^k}.$$

Therefore,

$$\alpha(Cx) = \alpha(Cy) \le \tfrac{1}{2} \sum_{k=1}^{\infty} \frac{\min\{k\alpha, 1\}}{2^k} = h(\alpha).$$

Since $\alpha > \alpha(x)$ is arbitrary, the assertion follows. □

3. Proof of the main result

The following lemma is the key technical component of the proof of Theorem 11.

Lemma 10. *Let $h : [0, 1] \to [0, 1]$ be as in* (2).

(1) *For all $t \in (0, 1]$, we have $h(t) < t$.*

(2) *For all $t \in [0, 1]$, we have $h_n(t) \to 0$ as $n \to \infty$, where $h_0 = h$ and $h_n = h \circ h_{n-1}$ for $n = 1, 2, \ldots$.*

Proof. For $k > \frac{1}{\alpha}$, we have

$$\min\left\{k, \frac{1}{\alpha}\right\} < k.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{\min\{k, \frac{1}{\alpha}\}}{2^k} < \sum_{k=1}^{\infty} \frac{k}{2^k} = 2.$$

Multiplying by $\frac{1}{2}\alpha$, we obtain the first part of the assertion.

By the first part and the definition of h_n, the sequence $\{h_n(t)\}_{n\geq 0}$ is monotone and, hence, convergent. Denote the limit by s. By continuity of h, we have

$$\begin{aligned} h(s) &= h(\lim_{n\to\infty} h_n(t)) = \lim_{n\to\infty} h(h_n(t)) \\ &= \lim_{n\to\infty} h_{n+1}(t) = s. \end{aligned}$$

By the first part, this inequality is impossible unless $s = 0$. This completes the second part of the assertion. □

It is possible to construct $x \in \{0, 1\}^{\mathbb{N}}$ such that $\liminf_{m\to\infty}(C^n x)(m) = 0$ and $\limsup_{m\to\infty}(C^n x)(m) = 1$ for every $n \in \mathbb{N}$. The following result describes a certain regularity property of the Cesaro operator.

Theorem 11. *If $x \in l_\infty$, then*

$$\alpha(C^n x) \to 0, \quad n \to \infty.$$

Proof. It follows directly from the definition of the functional α that $\alpha(x+c\mathbb{1}) = \alpha(x)$ and $\alpha(cx) = |c|\alpha(x)$ for every constant $c \in \mathbb{R}$.

Thus, by considering

$$y = \frac{x + \|x\|_\infty}{2\|x\|_\infty}$$

instead of x, we may assume without loss of generality that $0 \leq x \leq 1$.

By Proposition 4, we have

$$\alpha(Cx) \leq h(\alpha(x)).$$

Repeating this inequality n times, we obtain

$$\alpha(C^n x) \leq h_n(\alpha(x)), \quad n \geq 1.$$

By Lemma 10, we have

$$h_n(t) \to 0, \quad n \to \infty \text{ for every } t \in [0, 1].$$

This completes the proof. □

Proof of Theorem 3. Let $B = B \circ C$. We have

$$|B(x) - B(\sigma_m x)| = |B(C^n x) - B(C^n \sigma_m x)|.$$

Much as in [Semenov and Sukochev 2010, Lemma 16], it can be shown that for all $m, n \geq 1$ we have

$$C^n\sigma_m - \sigma_m C^n : l_\infty \to c_0.$$

Thus,

$$|B(x) - B(\sigma_m x)| = |(B \circ (1 - \sigma_m))(C^n x)|.$$

Therefore, we have

$$\begin{aligned} |B(x) - B(\sigma_m x)| &\leq \limsup_{k\to\infty} \left|(C^n x)(k) - (C^n x)\left(\left\lceil \frac{k}{m} \right\rceil\right)\right| \\ &\leq \limsup_{i\to\infty} \sup_{i\leq j\leq mi} |(C^n x)(i) - (C^n x)(j)| \\ &\leq \limsup_{i\to\infty} \sup_{i\leq j\leq 2^m i} |(C^n x)(i) - (C^n x)(j)|. \end{aligned}$$

Arguing as in Lemma 9, we have that

$$\limsup_{i\to\infty} \sup_{i\leq j\leq 2^m i} |(C^n x)(i) - (C^n x)(j)| \leq m \limsup_{i\to\infty} \sup_{i\leq j\leq 2i} |(C^n x)(i) - (C^n x)(j)|.$$

In this inequality, the right-hand side is simply $m\alpha(C^n x)$. It follows that

$$|B(x) - B(\sigma_m x)| \leq m\alpha(C^n x).$$

Passing $n \to \infty$ and using Theorem 11, we obtain that

$$B(x) = B(\sigma_m x).$$

Since $x \in l_\infty$ is arbitrary, it follows that $B = B \circ \sigma_m$. Since $m \geq 2$ is arbitrary,

$$B \in \bigcap_{m=2}^{\infty} \mathfrak{B}(\sigma_m)$$

and, thus,

$$\mathfrak{B}(C) \subset \bigcap_{m=2}^{\infty} \mathfrak{B}(\sigma_m). \qquad \square$$

It can be proved that the above inclusion is strict.

4. A continuous case

Let $L_\infty = L_\infty(0, \infty)$ be the space of all (classes of) Lebesgue measurable bounded functions on $(0, \infty)$ with the uniform norm. Let $C_0 \subset L_\infty$ be a subclass of functions vanishing at ∞.

For $l, \alpha > 0$ define operators $T_l,\ \sigma_\alpha,\ H$ on L_∞ as

$$(T_l f)(t) = f(t + l), \quad (\sigma_\alpha f)(t) = f\left(\frac{t}{\alpha}\right), \quad (Hf)(t) = \frac{1}{t}\int_0^t f(s)ds, \quad f \in L_\infty.$$

The notions of extended limits and their translation, dilation and H-invariance are defined in the same way as for discrete settings (see, e.g., [Carey et al. 2003; Sukochev et al. 2013]).

Let us also define the embedding operator $\pi : l_\infty \to L_\infty$ and the natural expectation operator $E : L_\infty \to l_\infty$ as follows:

$$(\pi x)(t) = x(k) \quad \text{for } t \in (k, k+1]i,\ k \geq 0;$$
$$(Ef)(n) = \int_n^{n+1} f(s)\, ds.$$

First, we prove the analogue of Theorem 3 for the continuous case.

Proposition 12. *Every H-invariant extended limit on L_∞ is σ_α-invariant for every $\alpha > 0$ and T_l-invariant for every $l > 0$.*

Proof. Let B be an H-invariant Banach limit on L_∞. It follows from [Semenov et al. 2015, Lemma 5.10] that the functional $B \circ \pi$ on l_∞ is a Cesaro-invariant Banach limit. Hence, by Theorem 3, $B \circ \pi$ is σ_m-invariant for every $m \in \mathbb{N}$.

Direct verification shows that the operator $H \circ \pi \circ E - H$ maps L_∞ into C_0 and $H \circ \sigma_\alpha = \sigma_\alpha \circ H$ for every $\alpha > 0$. Using these facts together with H-invariance of B we obtain

$$\begin{aligned} B \circ \sigma_m &= B \circ H \circ \sigma_m = B \circ \sigma_m \circ H \\ &= B \circ \sigma_m \circ H \circ \pi \circ E \\ &= B \circ H \circ \sigma_m \circ \pi \circ E. \end{aligned}$$

Using that $\sigma_m \circ \pi = \pi \circ \sigma_m$ for every $m \in \mathbb{N}$, H-invariance of B and σ_m-invariance of $B \circ \pi$ we obtain

$$\begin{aligned} B \circ \sigma_m &= B \circ \pi \circ \sigma_m \circ E \\ &= B \circ \pi \circ E \\ &= B \circ H \circ \pi \circ E \\ &= B \circ H = B. \end{aligned}$$

Hence, B is σ_m-invariant for every $m \in \mathbb{N}$.

Since $\sigma_m \circ \sigma_{1/m} = \mathrm{Id}$, it follows that

$$B \circ \sigma_{1/m} = B \circ \sigma_m \circ \sigma_{1/m} = B$$

and, so, B is $\sigma_{1/m}$-invariant for every $m \in \mathbb{N}$. Next, since $\sigma_m \circ \sigma_{1/n} = \sigma_{m/n}$, it follows that B is σ_q-invariant for every positive $q \in \mathbb{Q}$.

Further, for every $\alpha > 0$, positive $q \in \mathbb{Q}$ and every $f \in L_\infty$, using σ_q- and H-invariance of B, we obtain

$$\begin{aligned}
|(B \circ \sigma_\alpha)(f) - B(f)| &= |(B \circ \sigma_\alpha)(f) - (B \circ \sigma_q)(f)| \\
&= |(B \circ (H \circ \sigma_\alpha - H \circ \sigma_q))(f)| \\
&\le \|(H \circ \sigma_\alpha - H \circ \sigma_q)(f)\| \\
&= \sup_{t>0} \left| \frac{\alpha}{t} \int_0^{\frac{t}{\alpha}} f(s)ds - \frac{q}{t} \int_0^{\frac{t}{q}} f(s)ds \right| \\
&= \sup_{t>0} \left| \frac{\alpha - q}{t} \int_0^{\frac{t}{\alpha}} f(s)ds + \frac{q}{t} \int_{\frac{t}{\alpha}}^{\frac{t}{q}} f(s)ds \right| \\
&\le 2\frac{|\alpha - q|}{\alpha} \|f\|.
\end{aligned}$$

Letting $q \to \alpha,\ q \in \mathbb{Q}$, we obtain $|(B \circ \sigma_\alpha)(f) - B(f)| = 0$ for every $f \in L_\infty$. That is, B is σ_α-invariant for every $\alpha > 0$. This proves the first claim. The second claim follows directly from [Carey et al. 2003, Proposition 1.3(5)]. □

For $\alpha > 0$ define operators P_α and M on L_∞ as

$$(P_\alpha f)(t) = f(t^\alpha), \quad (Mf)(t) = \frac{1}{\log(1+t)} \int_0^t f(s)\frac{ds}{s}.$$

It was proved in [Carey et al. 2003, Theorem 1.5] that there exists an extended limit on L_∞ which is M-, σ_α- and P_α-invariant for every $\alpha > 0$. Proposition 13 and Corollary 15 below are refinements of this theorem.

Proposition 13. *Every M-invariant extended limit on L_∞ is P_α-invariant for every $\alpha > 0$ and σ_α-invariant for every $\alpha > 0$.*

Proof. It follows from [Carey et al. 2003, Proposition 1.4] that for every M-invariant extended limit γ, the extended limit $\gamma \circ \log$ defined on L_∞ by the formula

$$(\gamma \circ \log)(t \mapsto f(t)) := \gamma(t \mapsto f(\log t))$$

is H-invariant. Thus, by Proposition 12, $\gamma \circ \log$ is σ_α-invariant for every $\alpha > 0$. Again, by [Carey et al. 2003, Proposition 1.4] the functional γ is P_α-invariant for every $\alpha > 0$. The second claim follows from directly from [Carey et al. 2003, Proposition 1.3(6)]. □

The following results show that there are infinitely many M- and H-invariant extended limits.

Proposition 14. *The cardinality of the set of extreme points of the set of all M-invariant extended limits on L_∞ and that of all H-invariant extended limits on L_∞ is $2^{2^{\mathbb{N}}}$.*

Proof. It was proved in [Alekhno et al. 2018, Theorem 4.7] that the cardinality of the set of all extreme points of $\mathfrak{B}(C)$ equals $2^{2^{\mathbb{N}}}$. It is easy to see that for every $B \in \mathfrak{B}(C)$ the functional $B \circ P$ is an H-invariant extended limit on L_∞. Since the operator P is linear, it follows that if B is an extreme point of the set $\mathfrak{B}(C)$, then $B \circ P$ is an extreme point of the set of all H-invariant extended limits on L_∞. Thus, the cardinality of the set of all extreme points of the set of all H-invariant extended limit on L_∞ equals $2^{2^{\mathbb{N}}}$.

It is known (see, e.g., [Sukochev et al. 2013, Remark 31]) that if γ is an H-invariant extended limit on L_∞, then the extended limit $\gamma \circ \exp$ defined on L_∞ by the formula

$$(\gamma \circ \exp)(t \mapsto f(t)) := \gamma(t \mapsto f(e^t))$$

is M-invariant.

Since the mapping $\gamma \mapsto \gamma \circ \exp$ preserves the extreme points, we conclude that the cardinality of the set of extreme points of the set of all M-invariant extended limits on L_∞ is $2^{2^{\mathbb{N}}}$. □

The existence of an M-, σ_α- and P_α-invariant extended limit was proved in [Carey et al. 2003, Theorem 1.5]. The following corollary shows that this extended limit is not unique and, moreover, the set of such extended limits has the largest possible cardinality.

Corollary 15. *The cardinality of the set of all extended limits on L_∞ which are M-, σ_α- and P_α-invariant is $2^{2^{\mathbb{N}}}$.*

Proof. By [Sukochev et al. 2013, Lemma 7] the operator $M \circ \sigma_\alpha - M$ maps L_∞ into C_0. Thus, any M-invariant extended limit is automatically σ_α-invariant. Combining this with Proposition 13, we conclude that the set in the statement of the corollary contains the set of all M-invariant extended limits. Now, the result follows from Proposition 14. □

5. Applications

The result of Proposition 13 allows us to reduce M- and P_α-invariance to a single assumption of M-invariance.

The following theorem extends Theorem 2 in [Benameur and Fack 2006], relaxing the condition of the convergence of the right-hand side.

Theorem 16. *If the extended limit ω on L_∞ is such that $\omega = \omega \circ M$, then*

$$\mathrm{Tr}_\omega(A) = (\omega \circ \log)\left(\frac{1}{t}\mathrm{Tr}(A^{1+\frac{1}{t}})\right), \quad 0 \le A \in \mathcal{M}_{1,\infty}.$$

Proof. By Proposition 13, we have that $\omega = \omega \circ P_\alpha$ for all $\alpha > 0$. Hence, the assertion follows from [Carey et al. 2003, Theorem 3.1]. □

Now we turn to the comparison of the subclasses $\mathcal{D}_M$ and $\mathcal{D}_P$ of Dixmier traces (defined at the end of the Introduction).

A straightforward consequence of Proposition 13 is the following result.

Corollary 17. $\mathcal{D}_M \subset \mathcal{D}_P$ *on* $\mathcal{M}_{1,\infty}$.

Besides the Lorentz ideal, the weak-trace class ideal

$$\mathcal{L}_{1,\infty} := \left\{ A \in B(H) : \sup_{n\in\mathbb{N}} (n+1)\mu(n, A) < \infty \right\}$$

is also widely used in noncommutative geometry. This ideal also admits Dixmier traces and they are also defined by the formula (1). Thus, every Dixmier trace on $\mathcal{L}_{1,\infty}$ is a restriction of that from $\mathcal{M}_{1,\infty}$.

The following result shows the coincidence of $\mathcal{D}_M$ and $\mathcal{D}_P$ in the setting of the weak-$\mathcal{L}_1$ ideal.

Theorem 18. $\mathcal{D}_M = \mathcal{D}_P$ *on* $\mathcal{L}_{1,\infty}$.

Proof. One inclusion follows from Corollary 17 and the other one was established in [Sukochev and Usachev 2015, Theorem 3.10]. □

As a corollary we obtain the strengthening of [Sukochev and Usachev 2015, Theorem 3.9]. It provides the characterisation of the class $\mathcal{D}_P$.

Corollary 19. *All traces from the class* $\mathcal{D}_P$ *take the same value on the operator* $A \in \mathcal{L}_{1,\infty}$ *if and only if*

$$\lim_{m\to\infty} \liminf_{n\to\infty} (C^m x)(n) = \lim_{m\to\infty} \limsup_{n\to\infty} (C^m x)(n), \tag{4}$$

where $x := \left\{ \sum_{k=2^n-1}^{2^{n+1}-2} \mu(k, A) \right\}_{n\geq 0}$.

Proof. It was proved in [Semenov et al. 2015, Proposition 7.8] that all traces from $\mathcal{D}_M$ coincide on A if and only if the condition (4) holds. The assertion follows from Theorem 18. □

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EVGENII SEMENOV
VORONEZH STATE UNIVERSITY
VORONEZH
RUSSIA
nadezhka_ssm@geophys.vsu.ru

FEDOR SUKOCHEV
SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES
SYDNEY
AUSTRALIA
f.sukochev@unsw.edu.au

ALEXANDR USACHEV
DEPARTMENT OF MATHEMATICAL SCIENCES
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
GOTHENBURG
SWEDEN
Current address:
SCHOOL OF MATHEMATICS AND STATISTICS
CENTRAL SOUTH UNIVERSITY
HUNAN
CHINA
a.usachev@csu.edu.cn

DMITRIY ZANIN
SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES
SYDNEY
AUSTRALIA
d.zanin@unsw.edu.au

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