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University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

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COHOMOLOGICAL KERNELS OF PURELY INSEPARABLE FIELD EXTENSIONS

ROBERTO ARAVIRE, BILL JACOB AND MANUEL O'RYAN

Let F be a field of characteristic p and L a finite purely inseparable extension. The kernel $H_p^{n+1}(L/F) = \ker(H_p^{n+1}F \rightarrow H_p^{n+1}L)$ has been described by Sobiech, and independently when $p = 2$ by Aravire, Laghribi, and O'Ryan. When L has exponent 1, the kernel is the sum of the kernels of the simple subextensions, but when L has larger exponent it is significantly more complex. This paper determines $H_{p^m}^{n+1}(L/F)$ for $L = F(\sqrt[p^e]{x})$ and all $m, n, e \geq 1$. Whereas the results when $m = 1$ used the theory of differential forms, the results for $m > 1$ require the de Rham Witt complex. The $m > 1$ case clarifies why the “messy” generators in the $m = 1$ case arose, as they emerge from relations in the de Rham Witt complex. As a corollary, when L is modular over F and m exceeds the exponent of L , the kernel $H_{p^m}^2(L/F)$ is the sum of the kernels of the simple subextensions.

Introduction

We assume throughout F is a field of characteristic p . We consider finite extensions L of F that are purely inseparable and modular. This means that L is a tensor product of simple purely inseparable extensions, that is, for some $x_1, x_2, \dots, x_s \in F$ which are p -independent, $L = F(\alpha_1, \alpha_2, \dots, \alpha_s)$, where $\alpha_i^{p^{r_i}} = x_i$ for $r_i \in \mathbb{N}$. The problem of interest is to determine $\ker(i_{L/F} : H_{p^m}^{n+1}F \rightarrow H_{p^m}^{n+1}L)$ for $m, n, s \geq 1$. When $m = 1$ this problem has a comprehensive solution due to Sobiech [2018], which coincides with the independent work of Aravire, Laghribi, and O'Ryan [Aravire et al. 2019] when $p = 2$. Therefore our interest here is in the case $m > 1$. This paper treats the case where $s = 1$ and all m, n , and as a corollary obtains the result in the case for all m and s where $n + 1 = 2$. The general case of $n + 1 \geq 3$ is significantly more complicated and requires an analysis of the kernel of de Rham Witt modules $\ker : W_m \Omega_F^n \rightarrow W_m \Omega_{F(\sqrt[p^e]{x})}^n$. This will be addressed in subsequent work.

As in [Sobiech 2018; Aravire et al. 2019], the results depend upon the theory of differential forms in characteristic p . We assume familiarity with the differential

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modules Ω_F^n and the exact sequence

$$0 \rightarrow \nu_F(n) \rightarrow \Omega_F^n \xrightarrow{\wp} \Omega_F^n / d\Omega_F^{n-1} \rightarrow H_p^{n+1} F \rightarrow 0$$

which defines the two groups $\nu_F(n)$ and $H_p^{n+1} F$. Kato [1982] showed that $\nu_F(n) \cong K_n^M F / pK_n^M F$ (Milnor K-theory) and when $n = 1$ the group $H_p^2 F \cong \text{Br}_2 F$ is the p -torsion in the Brauer group of F (first observed in different notation by Witt). The map $\wp$ is $\wp = \Phi - 1$, where

$$\Phi \left(a \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \right) = a^p \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \pmod{d\Omega_F^{n-1}}$$

is the Frobenius map. A good reference for this background is [Gille and Szamuely 2006, Chapter 9].

A finite extension L/F is *modular* if there exist p -independent elements

$$x_1, \dots, x_s \in F \quad \text{with} \quad L = F(\sqrt[p^r]{x_1}, \dots, \sqrt[p^r]{x_s}).$$

If $r = \max\{r_i \mid 1 \leq i \leq s\}$ then r is called the *exponent* of L (over F .) The exponent 1 case has a particularly nice answer and motivated this work (see [Aravire and Baeza 2003; 2013] for $p = 2$, and [Sobiech 2018] for $p \geq 2$.) Here we have $L = F(\sqrt[p]{x_1}, \dots, \sqrt[p]{x_s})$ and the result is

$$\ker(H_p^{n+1} F \rightarrow H_p^{n+1} L) = \sum_{i=1}^s \overline{\Omega_F^{n-1} \wedge dx_i} = \sum_{i=1}^s \ker(H_p^{n+1} F \rightarrow H_p^{n+1} F(\sqrt[p]{x_i})),$$

where the bar denotes cohomology classes. We remark that in the separable case counterexamples to results like this abound and have an interesting history. For example it is known that the analogue of this result fails for separable triquadratic extensions where there can be “nonobvious” elements in the kernel (see [Amitsur et al. 1979; Elman et al. 1980]) and also for separable elementary abelian extensions of degree p^2 when p is odd (see [Tignol 1987]). These results played a key role in finding indecomposable division algebras (see [Tignol 1987; Jacob 1991]) and provide motivation for looking for generalizations.

In [Sobiech 2018; Aravire et al. 2019], the kernel $\ker(H_p^{n+1} F \rightarrow H_p^{n+1} L)$ is computed for arbitrary modular L , but it does not have a simple characterization. In fact, even when $s = 1$ the kernel has a messy description (see Theorem 4.1 below.) The main result of this paper is that this can be remedied if one looks instead at $H_{p^m}^{n+1} F$, where m is the exponent of the extension L . When $L = F(\sqrt[p^m]{x})$, Theorem 4.2 gives the direct analogue of the exponent 1 case,

$$\ker(H_{p^m}^{n+1} F \rightarrow H_{p^m}^{n+1} L) = \overline{W_m \Omega_F^{n-1} \wedge d[x]_m}.$$

In this description, for $x \in F$, we denote $[x]_m = (x, 0, \dots, 0) \in W_m F$ the multiplicative representative in the ring of Witt vectors $W_m F$, and $W_m \Omega_F^n$ is the de Rham Witt

module discussed below (see [Illusie 1979] for details). With this, by considering the image under the injection $V^{m-1} : H_p^{n+1} F \rightarrow H_{p^m}^{n+1} F$, one obtains a simplification of the of the cases where m is less than the exponent of L and a better understanding of the origin of the formulae that emerged in [Sobiech 2018; Aravire et al. 2019] when $m = 1$. As a consequence, when $n + 1 = 2$ one obtains a purely cohomological approach to an old result of A. A. Albert [1939] (see Corollary 4.3 below). Suppose $x_1, x_2, \dots, x_s \in F$ are p -independent and set $L = F(\alpha_1, \alpha_2, \dots, \alpha_s)$, where $\alpha_i^{p^{r_i}} = x_i$. Set $\tilde{m} := \max\{r_i \mid 1 \leq i \leq s\}$ and define $L_i := F(\alpha_i)$. If $m \geq \tilde{m}$ then

$$\ker(H_{p^m}^2 F \rightarrow H_{p^m}^2 L) = \sum_{i=1}^s p^{m-r_i} \cdot \overline{W_m F \cdot d[x_i]_m} = \sum_{i=1}^s \ker(H_{p^m}^2 F \rightarrow H_{p^m}^2 L_i).$$

When interpreted in terms of p -algebras this gives Albert's result that if an algebra is split in such a modular extension then it is similar to a corresponding tensor product of cyclic algebras. Of course, the main interest in Theorem 4.2 is that one has results when $n + 1 > 2$.

In order to treat the case of $m > 1$ it is necessary to use the de Rham Witt complex. Although the use of the differential modules Ω_F^n has become fairly standard in work in characteristic p , the de Rham Witt $W_m F$ -modules, $W_m \Omega_F^n$, are not so familiar. Details about $W_m \Omega_F^n$ can be found in [Illusie 1979] and also in Section 2 of [Aravire et al. 2018], which gives a detailed introduction and whose results are referenced here. Here is a rough guide about how to think about them.

Loosely speaking one gets intuition by thinking about these modules as $\Omega_{W_m F}^n$, but in fact they are a quotient module. The relations defining the quotient are introduced for several reasons: one is to make sure that the shift operator $V : W_m F \rightarrow W_{m+1} F$ generalizes to $V : W_m \Omega_F^n \rightarrow W_{m+1} \Omega_F^n$ (we have $W_m \Omega_F^0 = W_m F$), and a second is to assure that the derivative $dx^p = px^{p-1} dx$ interacts correctly with the other arithmetic in $W_m \Omega_F^n$. One particularly important relation is for all $x \in W_{m-s} F$ with $m > s \geq 1$ and $a \in F$, we have $V^s(x) d[a]_m = V^s(x[a]_{m-s}^{p^s-1}) dV^s[a]_{m-s} \in W_m \Omega_F^1$ (see Lemma 3.1 below.) This relation generalizes a defining relation for $W_m \Omega_F^n$ and provides the key as to why $\bar{\omega} \in \ker(H_p^{n+1} F \rightarrow H_{p^m}^{n+1} F(\sqrt[p^m]{x}))$ satisfies $V^{m-1}(\bar{\omega}) \in W_m \Omega_F^{n-1} \wedge d[x]_m$.

The Frobenius $\Phi : W_m F \rightarrow W_m F$ defined by $\Phi(a_1, \dots, a_m) = (a_1^p, \dots, a_m^p)$ extends to $W_m \Omega_F^n$ but in its most general form shifts levels down, $\Phi : W_m \Omega_F^n \rightarrow W_{m-1} \Omega_F^n$. When $m = 1$ (where the Frobenius is also known as the inverse Cartier operator), the shift is remedied by working (mod $d\Omega_F^{n-1}$), that is, we have $\Phi : \Omega_F^n \rightarrow \Omega_F^n / d\Omega_F^{n-1}$ as described above. The Frobenius on the de Rham Witt modules can also be set up to avoid the shift and this version is $\Phi : W_m \Omega_F^n \rightarrow W_m \Omega_F^n / dV^{m-1} \Omega_F^{n-1}$.

To study the cohomology groups $H_p^{n+1} F$, Izhboldin [1991] defined groups $Q^n(F, m)$ using generators and relations, mimicking the arithmetic in $W_m \Omega_F^n$ (there is a natural surjection $W_m \Omega_F^n \rightarrow H_{p^m}^{n+1} F$, see [Aravire et al. 2018, Section 2])

for discussion). One of his main results is the exact sequence [Izhboldin 1991, Theorem C],

$$0 \rightarrow K_n^M F / p^m K_n^M F \rightarrow Q^n(F, m) \xrightarrow{\wp} Q^n(F, m) \rightarrow H_{p^m}^{n+1} F \rightarrow 0.$$

However for explicit computations, such as those given in this paper, since the $Q^n(F, m)$ are not $W_m F$ -modules we will use the following exact sequence to study the group $H_{p^m}^{n+1} F$,

$$W_m \Omega_F^n \xrightarrow{\wp} \frac{W_m \Omega_F^n}{d W_m \Omega_F^{n-1}} \rightarrow H_{p^m}^{n+1} F \rightarrow 0$$

described in [Aravire et al. 2018, Theorem 2.31]. Here, as in the case $m = 1$, $\wp = \Phi - 1$. Two particularly important properties of Φ in this context are that $\Phi \circ V = V \circ \Phi = p \cdot$ (note that the level reduction for Φ is used here), and if $\omega \in W_m \Omega_F^n$ then $\bar{\omega} = \overline{\Phi(\omega)} \in H_{p^m}^{n+1} F$.

Finally, in order to build up results by induction on m it is necessary to use the analogue of the exact sequence of Witt vectors

$$0 \rightarrow W_{m-s} F \xrightarrow{V^s} W_m F \xrightarrow{R^s} W_s F \rightarrow 0,$$

where

$$V^s(a_1, \dots, a_{m-s}) = (0, \dots, 0, a_1, \dots, a_{m-s})$$

and

$$R^s(a_1, \dots, a_m) = (a_1, \dots, a_s) \in W_s F.$$

In the case of the de Rham Witt Complex, because of the relations, the map V^s fails to be injective so one does not have an exact analogue, but we do have the following exact sequence [Illusie 1979, Proposition 3.2.],

$$(V^s W_{m-s} \Omega_F^n + dV^s W_{m-s} \Omega_F^{n-1}) \rightarrow W_m \Omega_F^n \xrightarrow{R^s} W_s \Omega_F^n \rightarrow 0,$$

where the first map is subgroup inclusion (of the images.) This sequence will suffice to enable the induction required for this paper, for fortunately $dV^s W_{m-s} \Omega_F^{n-1}$, which can be problematic in other contexts, vanishes in the cohomology group $H_{p^m}^{n+1} F$.

1. Background on differential forms and the $m = 1$ case

In this and in the next section we compute in the module Ω_F^n . Although technically the Frobenius $\Phi : \Omega_F^n \rightarrow \Omega_F^n / d\Omega_F^{n-1}$ we abuse notation and use a lift so that we can write $\Phi : \Omega_F^n \rightarrow \Omega_F^n$. We use notation introduced in [Kato 1982] as follows: We fix $\{b_i \mid i \in I\}$ to be a p -basis for F with a well-ordered index set I . We let $\Sigma_F(n)$ be the set of all increasing functions $\gamma : \{1, 2, \dots, n\} \rightarrow I$ and we order $\Sigma_F(n)$

lexicographically so that $\gamma < \beta$ if and only if there is some $j \in \{1, 2, \dots, n\}$ such that $\gamma(j) < \beta(j)$ while $\gamma(i) \leq \beta(i)$ for all $i \leq j$. For $\gamma \in \Sigma_F(n)$ we denote by

$$\frac{db_\gamma}{b_\gamma} := \frac{db_{\gamma(1)}}{b_{\gamma(1)}} \wedge \cdots \wedge \frac{db_{\gamma(n)}}{b_{\gamma(n)}}.$$

It is well-known that $\{b_\gamma^{-1}db_\gamma \mid \gamma \in \Sigma_F(n)\}$ is a F -basis for Ω_F^n and therefore every element $\omega \in \Omega_F^n$ can be expressed uniquely as a (finite) sum $\omega = \sum_{\gamma \in \Sigma_F(n)} a_\gamma b_\gamma^{-1}db_\gamma$. With this fixed basis we can define

$$\Phi\left(\sum_{\gamma \in \Sigma_F(n)} a_\gamma \frac{db_\gamma}{b_\gamma}\right) := \sum_{\gamma \in \Sigma_F(n)} a_\gamma^p \frac{db_\gamma}{b_\gamma}.$$

In prior work on this problem, [Sobiech 2018; Aravire et al. 2019] and other authors use the notation $\omega^{[p]}$ to denote what is defined here as $\Phi(\omega)$. We have chosen to use the Φ notation here for these basis dependent expressions because it will be consistent with how we use Φ in the de Rham Witt complex. In all applications, because the computations only matter (mod $d\Omega_F^{n-1}$) the choice of p -basis will be immaterial.

We need more technicalities about p -bases and these are laid out next.

Notation 1.1. We suppose that $\{b_i \mid i \in I\}$ is an ordered p -basis for F . As I is well-ordered we use $1 \in I$ as its least element. In case I has a greatest element, we use $\mathcal{M} \in I$ as its greatest element. We denote by $F_{<i} := F^p(b_k \mid k < i)$. We denote by $\Omega_{F, <\gamma}^n$ the F -subspace of Ω_F^n spanned by all $b_\beta^{-1}db_\beta$ with $\beta < \gamma$. Whenever $\omega = \sum_{\gamma \in \Sigma_F(n)} a_\gamma b_\gamma^{-1}db_\gamma$ and $\beta \in \Sigma_F(n)$ is the maximum index with $a_\beta \neq 0$, we call $a_\beta b_\beta^{-1}db_\beta$ the maximal summand of ω .

With this notation we can give a fundamental tool, referred to as Kato's lemma [1982, Lemma 2].

Lemma 1.2 (Kato's lemma). *Suppose F has characteristic p and $F^* = F^{*(p-1)}$. Let $y \in F$ and $\gamma \in \Sigma_F(n)$. If $(y^p - y)b_\gamma^{-1}db_\gamma \in \Omega_{F, <\gamma}^n + d\Omega_F^{n-1}$ then $y b_\gamma^{-1}db_\gamma = v + x_1^{-1}dx_1 \wedge \cdots \wedge x_n^{-1}dx_n$, where each $x_i \in F^p(b_j \mid j \leq \gamma(i))$ and $v \in \Omega_{F, <\gamma}^n$.*

It is useful to have a weaker version of Kato's lemma that does not include the hypothesis that $F^* = F^{*(p-1)}$. This has been noted by many authors. Here is one way to do this; it appears as [Arason et al. 2007, Lemma 3.2] and also appears as [Sobiech 2018, Lemma 3.2]. We recall that $\text{dlog} : K_n^M F \rightarrow \Omega_F^n$ is the homomorphism defined on generators by $\text{dlog}(\ell(t_1) \otimes \cdots \otimes \ell(t_n)) = t_1^{-1}dt_1 \wedge \cdots \wedge t_n^{-1}dt_n$.

Lemma 1.3. *Suppose F has characteristic p . Let $y \in F$ and $\gamma \in \Sigma_F(n)$. If $(y^p - y)b_\gamma^{-1}db_\gamma \in \Omega_{F, <\gamma}^n + d\Omega_F^{n-1}$ then $y b_\gamma^{-1}db_\gamma = v + \text{dlog}(\tau)$, where $v \in \Omega_{F, <\gamma}^n$ and $\tau \in K_n^M F$.*

We need one more technical result dependent upon this notation.

Lemma 1.4 [Arason et al. 2007, Lemma 3.1]. *For F and $\{b_i \mid i \in I\}$ as above, assume*

$$\sum_{\gamma \leq \beta} c_\gamma \frac{db_\gamma}{b_\gamma} \in d\Omega_F^{n-1}$$

where $c_\beta \neq 0$. Then there exist $M_{ij} \in F_{<\beta(i)}$, for $1 \leq i \leq n$ and $1 \leq j \leq p-1$, such that

$$c_\beta = \sum_{i=1}^n \sum_{j=1}^{p-1} M_{ij} b_{\beta(i)}^j.$$

We next consider a purely inseparable simple extension $L = F(\alpha)$ where $\alpha^{p^r} = b_1$. It is well-known that if $b_1, b_2, \dots$ is an ordered p -basis for F , then $b_2, b_3, \dots, \alpha$ is a p -basis for $F(\alpha)$ for which we select the ordering with α labeled as the last element of this basis (so its index is $\mathcal{M}$ and the index set for the L -basis is $(I - \{1\}) \cup \{\mathcal{M}\}$.) In this setting we will use $\Sigma'_F(i)$ to denote those elements $\gamma \in \Sigma_F(i)$ where the $\gamma(1) > 1 \in I$. We note then that $\Sigma_L(n+1) = \Sigma'_F(n+1) \cup \Sigma'_F(n) \cdot \alpha$, where $\Sigma'_F(n) \cdot \alpha$ denotes the elements $\beta \in \Sigma_L(n+1)$ with $\beta(1) > 1 \in I$ and $\beta(n+1) = \mathcal{M}$. We note that elements $\beta \in \Sigma'_F(n) \cdot \alpha$ are in one-to-one correspondence with elements of $\delta \in \Sigma'_F(n)$ and it is convenient to view them this way. In fact we label them as $\beta = \delta \cup \{\mathcal{M}\}$ where according to the notation just given, $(\delta \cup \{\mathcal{M}\})(i) = \delta(i)$ for $i \leq n$ and $(\delta \cup \{\mathcal{M}\})(n+1) = \mathcal{M}$. We also remark that in proving results using these p -bases we may assume they are finite. This is because one can use elements and the expressions that describe the situation over a finitely generated field over the prime field, from which the result follows for all fields.

This next lemma is needed to prove Lemma 1.6, which is the main tool we develop in this section.

Lemma 1.5. *Suppose F has characteristic p , $r \geq 1$, and $L = F(\alpha)$, where $\alpha^{p^r} = b_1 \in F - F^p$ with $b_2, b_3, \dots, \alpha$ the ordered p -basis for L given above (so the index of α is the largest). Suppose $\omega \in \Omega_L^n$ and*

$$d\omega = \sum_{\beta \in \Sigma'_F(n+1)} c_\beta \frac{db_\beta}{b_\beta} + \sum_{\delta \in \Sigma'_F(n)} c_\delta \frac{db_\delta}{b_\delta} \wedge \frac{d\alpha}{\alpha} \quad \text{for } c_\beta, c_\delta \in L.$$

Suppose that the maximal summand of $d\omega$ comes from an index $\tilde{\beta} \in \Sigma'_F(n+1)$ (that is, $c_{\tilde{\beta}} \neq 0$, and $\tilde{\beta} > \delta \cup \{\mathcal{M}\}$ for all $\delta \in \Sigma'_F(n)$ with $c_\delta \neq 0$.) Then the maximal summand $c_{\tilde{\beta}} b_{\tilde{\beta}}^{-1} db_{\tilde{\beta}}$ of $d\omega$ is the leading summand of some $d\omega_0$ where $\omega_0 \in L_0 \cdot \Omega_F^n$ and $L_0 = F(\alpha^p)$.

Proof. As $\Omega_L^{n+1} = L \cdot \Omega_F^{n+1} \oplus L \cdot \Omega_F^n \wedge \alpha^{-1} d\alpha$ and $L = \bigoplus_{i=0}^{p-1} \alpha^i L_0$, we find

$$\Omega_L^{n+1} = \left(\bigoplus_{i=0}^{p-1} \alpha^i L_0 \cdot \Omega_F^{n+1} \right) \oplus \left(\bigoplus_{i=0}^{p-1} \alpha^i L_0 \cdot \Omega_F^n \wedge \frac{d\alpha}{\alpha} \right).$$

We write $\omega = \sum_{i=0}^{p-1} \alpha^i \omega_i + \sum_{i=0}^{p-1} \alpha^i \omega'_i \wedge \alpha^{-1} d\alpha$ for $\omega_i \in L_0 \cdot \Omega_F^n$ and $\omega'_i \in L_0 \cdot \Omega_F^{n-1}$. Then

$$d\omega = \sum_{i=0}^{p-1} \alpha^i d\omega_i + \sum_{i=0}^{p-1} \alpha^i ((-1)^n i \omega_i + d\omega'_i) \wedge \frac{d\alpha}{\alpha}.$$

According to Lemma 1.4, as the maximum summand of $d\omega$ is $c_{\tilde{\beta}} b_{\tilde{\beta}}^{-1} db_{\tilde{\beta}}$, we have $c_{\tilde{\beta}} = \sum_{i=1}^n \sum_{j=1}^{p-1} M_{ij} b_{\beta(i)}^j$, where $M_{ij} \in L_{<\beta(i)}$ for $1 \leq i \leq n$ and $1 \leq j \leq p-1$. As $L_{<\beta(i)} := L^p(b_k \mid k < \beta(i))$, and as the index of the basis element α is $\mathcal{M}$, the largest in the ordering of the p -basis for L , we have $\beta(i)$ less than this index and it follows that each $L_{<\beta(i)} \subseteq L_0$. It follows that $c_{\tilde{\beta}} \in L_0$. By the directness of the sum in which the summands of $d\omega$ occur, as none of the summands of $d\omega$ coming from $\alpha^i d\omega_i$ or $\alpha^i ((-1)^n i \omega_i + d\omega'_i) \wedge \alpha^{-1} d\alpha$ where $1 \leq i < p$ have coefficients that lie in L_0 , this maximum summand must be the maximum summand of $d\omega_0$ or $d\omega'_0$. But by hypothesis, this leading summand cannot involve $\alpha^{-1} d\alpha$ so it must come from $d\omega_0$. So $\omega_0 \in L_0 \cdot \Omega_F^n$ is the desired element. This proves the lemma. $\square$

This next result is used repeatedly. It is our key application of Kato's lemma.

Lemma 1.6. *Suppose F has characteristic p and $r \geq 1$. Let $L = F(\alpha)$, where $\alpha^{p^r} = b_1 \in F - F^p$ with $b_2, b_3, \dots, \alpha$ a p -basis for L as above (so the index of α is the largest). Suppose $\eta \in \Omega_L^{n-1} \wedge \alpha^{-1} d\alpha \subseteq \Omega_L^n$ and $\wp(\eta) \in L \cdot \Omega_F^n + d\Omega_L^{n-1}$. Then $\eta = \eta_0 + d\log(\tau) \in \Omega_L^n$ for some $\eta_0 \in L \cdot \Omega_F^n$ and $\tau \in K_n^M L$.*

Proof. As noted above we may assume F has a finite p -basis. We assume $\eta = (\sum_{\delta \in \Sigma'_F(n-1)} y_\delta b_\delta^{-1} db_\delta) \wedge \alpha^{-1} d\alpha$, where $0 \neq y_\delta \in L$ and proceed by induction on the size of the largest δ index (which can also be viewed as $\delta \cup \{\mathcal{M}\} \in \Sigma_L(n)$.) By our hypothesis we can write

$$(*) \quad \sum_{\delta \in \Sigma'_F(n-1)} \wp(y_\delta) \frac{db_\delta}{b_\delta} \wedge \frac{d\alpha}{\alpha} = \sum_{\beta \in \Sigma'_F(n)} z_\beta \frac{db_\beta}{b_\beta} + d\omega \in \Omega_L^n$$

for $z_\beta \in L$ and $\omega \in \Omega_L^{n-1}$. We must analyze two cases, the first where the maximum of the indices $\delta \cup \{\mathcal{M}\}$ and β in both sums in $(*)$ occurs among the β and the second where it occurs among the $\delta \cup \{\mathcal{M}\}$ (note that these two sets of indices are disjoint).

In the first case we assume the maximum occurs at $\tilde{\beta}$ among the $\beta \in \Sigma'_F(n)$ and so that $z_{\tilde{\beta}} \tilde{\beta}^{-1} d\tilde{\beta}$ is the maximum summand of $\sum_{\beta \in \Sigma'_F(n)} z_\beta b_\beta^{-1} db_\beta$. The maximality assumption then shows that $d\omega$ has maximum summand $-z_{\tilde{\beta}} \tilde{\beta}^{-1} d\tilde{\beta}$, where $\tilde{\beta} \in \Sigma'_F(n)$. According to Lemma 1.5 there is some $\omega_0 \in L_0 \cdot \Omega_F^{n-1} \subset \Omega_L^{n-1}$ such that $d\omega$ and $d\omega_0$ have the same maximum summand. Since $L_0 = F(\alpha^p)$ we know $d\omega_0 \in L_0 \cdot \Omega_F^n \subset \Omega_L^n$ (note we are viewing $d\omega_0 \in \Omega_L^n$ and therefore $d\alpha^p = 0 \in \Omega_L^1$). Expressing $d\omega_0 = -z_{\tilde{\beta}} \tilde{\beta}^{-1} d\tilde{\beta} + \sum_{\beta \in \Sigma'_F(n), \beta < \tilde{\beta}} z'_\beta \cdot b_\beta^{-1} db_\beta$,

where $z'_\beta \in L_0$, it follows that

$$\begin{aligned} \sum_{\delta \in \Sigma'_F(n-1)} \wp(y_\delta) \frac{db_\delta}{b_\delta} \wedge \frac{d\alpha}{\alpha} &= z_{\tilde{\beta}} \frac{d\tilde{\beta}}{\tilde{\beta}} + \sum_{\beta \in \Sigma'_F(n), \beta < \tilde{\beta}} z_\beta \frac{db_\beta}{b_\beta} + d\omega + d\omega_0 - d\omega_0 \\ &= \sum_{\beta \in \Sigma'_F(n), \beta < \tilde{\beta}} (z_\beta + z'_\beta) \frac{db_\beta}{b_\beta} + d(\omega - \omega_0) \in \Omega_L^n. \end{aligned}$$

Iterating this process shows we can assume the maximum of the $\delta \cup \{\mathcal{M}\}$, β in $(*)$ occurs at some $\tilde{\delta} \cup \{\mathcal{M}\}$ among the $\tilde{\delta} \in \Sigma'_F(n-1)$, that is, we are in the second case.

We now express $\eta = \eta' + y_{\tilde{\delta}} b_{\tilde{\delta}}^{-1} db_{\tilde{\delta}} \wedge \alpha^{-1} d\alpha$ where $\tilde{\delta} \cup \{\mathcal{M}\}$ is the maximal index. Then our assumption about equation $(*)$ shows that

$$\wp(y_{\tilde{\delta}}) \frac{db_{\tilde{\delta}}}{b_{\tilde{\delta}}} \wedge \frac{d\alpha}{\alpha} \in \Omega_{L, < \tilde{\delta} \cup \{\mathcal{M}\}}^n + d\Omega_L^{n-1},$$

so by Lemma 1.3 we can express $y_{\tilde{\delta}} b_{\tilde{\delta}}^{-1} db_{\tilde{\delta}} \wedge \alpha^{-1} d\alpha = v + \text{dlog}(\tau)$ for $v \in \Omega_{L, < \tilde{\delta} \cup \{\mathcal{M}\}}^{n-1}$ and $\tau \in K_n^M L$. We decompose $v = v_1 + v_2$, where $v_1 \in L \cdot \Omega_F^n$ and $v_2 \in \Omega_{L, < \tilde{\delta}}^{n-1} \wedge \alpha^{-1} d\alpha$. Since $\eta' = \eta - y_{\tilde{\delta}} b_{\tilde{\delta}}^{-1} db_{\tilde{\delta}} \wedge \alpha^{-1} d\alpha = \eta - (v_1 + v_2) - \text{dlog}(\tau)$ we can replace η by $\eta - v_1 - \text{dlog}(\tau) = \eta' + v_2 \in \Omega_{L, < \tilde{\delta}}^{n-1} \wedge \alpha^{-1} d\alpha$. We have removed the maximal summand of type $\tilde{\delta} \cup \{\mathcal{M}\}$ from η and the original hypotheses still hold. This reduction completes the proof. $\square$

In the next definition we use $\wp = \Phi - 1 : \Omega_F^n \rightarrow \Omega_F^n$. We recall our convention that $\Phi : \Omega_F^n \rightarrow \Omega_F^n$ is defined relative to a fixed p -basis of F . However, as the groups considered in this definition all contain $d\Omega_F^{n-1}$ they are independent of this choice of p -basis. Again, we remark that the purpose of the convention for Φ is to facilitate expressing our computations and in the end no ambiguities are introduced.

Definition 1.7. (i) We define $B_1 \Omega_F^n := d\Omega_F^{n-1}$ and recursively for $r \geq 1$,

$$B_{r+1} \Omega_F^n = \Phi(B_r \Omega_F^n) + B_r \Omega_F^n.$$

(ii) Assume $x \in F - F^p$ and $n \geq 0$. We define $\Gamma_F^n := \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx$.

The groups $B_{r+1} \Omega_F^n$ were first defined in [Illusie 1979] and played a significant role in [Izhboldin 1991] where Izhboldin showed that his group $Q^n(F, 1) \cong \Omega_F^n / (\bigcup_{r=1}^\infty B_r \Omega_F^n)$. In particular, if $\omega \in B_r \Omega_F^n$ for some r , then $\bar{\omega} = 0 \in H_p^{n+1} F$. In view of this we note as $B_{r+1} \Omega_F^n = \Phi(B_r \Omega_F^n) + B_r \Omega_F^n$ that $\wp(\Omega_F^n) + d\Omega_F^{n-1} = \wp(\Omega_F^n) + B_r \Omega_F^n$ for all r .

The recursive definition shows that $B_r \Omega_F^n$ is generated by elements $\Phi^{i-1}(d\omega)$, where $1 \leq i \leq r$ and $\omega \in \Omega_F^{n-1}$. One readily checks $x^{p^r} B_r \Omega_F^n \subseteq B_r \Omega_F^n$. Suppose $L = F(\sqrt[p^e]{x})$. This means that $\overline{x^j B_r \Omega_F^n} \subseteq \ker(H_p^{n+1} F \rightarrow H_p^{n+1} L)$ whenever

$1 \leq r \leq e$. The generators for $x^j B_r \Omega_F^n$ can be expressed as

$$x^j \Phi^i \left(ds \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_n}{t_n} \right) = x^j s^{p^{i-1}} \frac{ds}{s} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_n}{t_n},$$

where $1 \leq i \leq r$ and $s, t_2, \dots, t_n \in F$.

The theorem of Aravire, Laghribi, O’Ryan, and Sobiech [Sobiech 2018, Theorem 3.16; Aravire et al. 2019, Theorem 4.1] shows that these give a generating set for $\ker(H_p^{n+1} F \rightarrow H_p^{n+1} L)$. We next state their result in this notation. For simplicity we give it in the case where $L = F(\sqrt[p^{e_1}]{x_1}, \dots, \sqrt[p^{e_s}]{x_s})$ for p -independent elements $x_1, \dots, x_s \in F$ and the e_i all are a constant value e . (Their general result does not require constant values.)

Theorem 1.8 (Aravire–Laghribi–O’Ryan–Sobiech). *Suppose $L = F(\sqrt[p^e]{x_1}, \dots, \sqrt[p^e]{x_s})$ for p -independent elements $x_1, \dots, x_s \in F$. Then*

$$\ker(H_p^{n+1} F \rightarrow H_p^{n+1} L) = \sum_{1 \leq r \leq e} \sum_{0 < j_k < p^r} \overline{x_1^{j_1} \cdots x_s^{j_s} B_r \Omega_F^n}.$$

We close this section by considering the simple purely inseparable extensions of degree p . When $p = 2$ this next result is implicit in [Aravire and Baeza 2003] and the general case is implicit in the work of Sobiech. It is pulled out here because it provides a model for the results considered in the next section.

Lemma 1.9. *Suppose $\omega_i \in \Omega_F^n$ and $\sum_{i=1}^{p-1} x^i \Phi(\omega_i) \in \Gamma_F^n$. Then*

$$\sum_{i=1}^{p-1} i^{-1} x^i \Phi(d\omega_i) \in \Gamma_F^{n+1}.$$

Proof. We express $\sum_{i=1}^{p-1} x^i \Phi(\omega_i) = \wp(\omega_0) + d\hat{\theta} + \hat{\psi} \wedge dx$ for $\omega_0 \in \Omega_F^n$ and $\hat{\theta}, \hat{\psi} \in \Omega_F^{n-1}$. We consider $(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i) \wedge \alpha^{-1} d\alpha \in \Omega_L^{n+1}$, where $L = F(\alpha)$ with $\alpha^p = x \in F$. As $i_{L/F}(\hat{\psi} \wedge dx) = 0 \in \Omega_L^{n+1}$, we have

$$\begin{aligned} (**) \quad \wp \left(\left(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i \right) \wedge \frac{d\alpha}{\alpha} \right) &= \left(\wp(\omega_0) - \sum_{i=1}^{p-1} x^i \Phi(\omega_i) + \sum_{i=1}^{p-1} \alpha^i \omega_i \right) \wedge \frac{d\alpha}{\alpha} \\ &= \left(\sum_{i=1}^{p-1} \alpha^i \omega_i - d\hat{\theta} \right) \wedge \frac{d\alpha}{\alpha} \in \Omega_L^{n+1}. \end{aligned}$$

As $(\sum_{i=1}^{p-1} \alpha^i \omega_i) \wedge \alpha^{-1} d\alpha = d(\sum_{i=1}^{p-1} i^{-1} \alpha^i \omega_i) - \sum_{i=1}^{p-1} i^{-1} \alpha^i d\omega_i$, $d\hat{\theta} \wedge \alpha^{-1} d\alpha = d(\hat{\theta} \wedge \alpha^{-1} d\alpha)$ and each $\omega_i \in \Omega_F^n$ we obtain $\wp((\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i) \wedge \alpha^{-1} d\alpha) \in L \cdot \Omega_F^{n+1} + d\Omega_L^n$. Hence by Lemma 1.6 applied to $L = F(\alpha)$ we see that

$$\left(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i \right) \wedge \frac{d\alpha}{\alpha} = \eta + d\log(\tau), \quad \text{where } \eta \in L \cdot \Omega_F^{n+1} \text{ and } \tau \in K_{n+1}^M L.$$

We have $\wp(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i) \wedge \alpha^{-1} d\alpha = \wp(\eta) + d\theta$ for $\theta \in \Omega_L^n$. We can write $\eta = \eta_0 + \sum_{i=1}^{p-1} \alpha^i \eta_i$, where $\eta_i \in \Omega_F^{n+1}$ and

$$\theta = \theta_0 + \sum_{i=1}^{p-1} \alpha^i \theta_i + \left(\theta'_0 + \sum_{i=1}^{p-1} \alpha^i \theta'_i \right) \wedge \frac{d\alpha}{\alpha} \in \Omega_L^n \quad \text{for } \theta_i \in \Omega_F^n \text{ and } \theta'_i \in \Omega_F^{n-1}.$$

With this, expanding $\wp(\omega_0 - \sum_{i=1}^{p-1} \alpha^i \omega_i) \wedge \alpha^{-1} d\alpha = \wp(\eta) + d\theta$ in Ω_L^{n+1} using the first equality in (**) gives $2p$ equations:

$$0 = \wp(\eta_0) + \sum_{i=1}^{p-1} x^i \Phi(\eta_i) + d\theta_0,$$

$$0 = \alpha^i (-\eta_i + d\theta_i) \quad \text{for } 1 \leq i \leq p-1,$$

$$\left(\wp(\omega_0) - \sum_{i=1}^{p-1} x^i \Phi(\omega_i) \right) \wedge \frac{d\alpha}{\alpha} = d\theta'_0 \wedge \frac{d\alpha}{\alpha},$$

$$\alpha^i \omega_i \wedge \frac{d\alpha}{\alpha} = \alpha^i (i\theta_i + d\theta'_i) \wedge \frac{d\alpha}{\alpha} \quad \text{for } 1 \leq i \leq p-1.$$

Each of these can be replaced by an equality in Ω_F^n by adding $\Omega_F^n \wedge dx$ or $\Omega_F^{n-1} \wedge dx$ terms. We find

$$\sum_{i=1}^{p-1} x^i \Phi(\eta_i) = -\wp(\eta_0) - d\theta_0 + \psi_{00} \wedge dx,$$

$$\eta_i = d\theta_i + \psi_{0i} \wedge dx \quad \text{for } 1 \leq i < p-1,$$

$$\sum_{i=1}^{p-1} x^i \Phi(\omega_i) = \wp(\omega_0) - d\theta'_0 + \psi_{10} \wedge dx,$$

$$\omega_i = i\theta_i + d\theta'_i + \psi_{1i} \wedge dx \quad \text{for } 1 \leq i < p-1.$$

The second and fourth equations give $\eta_i = i^{-1} d\omega_i + \psi_i \wedge dx$ for some $\psi_i \in \Omega_F^n$. By the first equation $\sum_{i=1}^{p-1} i^{-1} x^i \Phi(d\omega_i) \in \wp(\Omega_F^{n+1}) + d\Omega_F^n + \Omega_F^n \wedge dx = \Gamma_F^{n+1}$. $\square$

We now have the tools to determine the kernel of $H_p^{n+1} F \rightarrow H_p^{n+1} L$ where $L = F(\alpha)$ for $\alpha^p = x \in F$. The result is due to Sobiech [2018, Proposition 3.6] and when $p = 2$ the result is due to Aravire and Baeza [2003, Lemma 2.18]. We give the proof because it illustrates in a much less technical fashion the technique of proof of the main Theorem 4.2 (as well as the base of the induction for that result) and also illustrates how Lemma 1.9 is used.

Theorem 1.10 (Aravire–Baeza–Sobiech). *If $L = F(\alpha)$ for $\alpha^p = x \in F$ then $\ker(H_p^{n+1} F \rightarrow H_p^{n+1} L) = \Omega_F^{n-1} \wedge dx$.*

Proof. Here we have

$$\Omega_{F(\alpha)}^n = \bigoplus_{i=0}^{p-1} \alpha^i \cdot i_{L/F}(\Omega_F^n) \oplus \bigoplus_{i=0}^{p-1} \alpha^i \cdot i_{L/F}(\Omega_F^{n-1}) \wedge \frac{d\alpha}{\alpha},$$

where we recall $i_{L/F}(\Omega_F^n) \cong \Omega_F^n / (\Omega_F^{n-1} \wedge dx)$.

Assume $\omega \in \Omega_F^n$ with $\bar{\omega} \in \ker(H_p^{n+1}F \rightarrow H_p^{n+1}L)$. We then have $i_{L/F}(\omega) = \wp(\eta) + d\theta$, where $\eta \in \Omega_L^n$ and $\theta \in \Omega_L^{n-1}$. We express

$$\begin{aligned} \eta &= \sum_{i=0}^{p-1} \alpha^i \eta_i + \sum_{i=0}^{p-1} \alpha^i \eta'_i \wedge \frac{d\alpha}{\alpha}, \quad \text{where } \eta_i \in \Omega_F^n \text{ and } \eta'_i \in \Omega_F^{n-1}, \\ \theta &= \sum_{i=0}^{p-1} \alpha^i \theta_i + \sum_{i=0}^{p-1} \alpha^i \theta'_i \wedge \frac{d\alpha}{\alpha}, \quad \text{where } \theta_i \in \Omega_F^{n-1} \text{ and } \theta'_i \in \Omega_F^{n-2}. \end{aligned}$$

We then have

$$\begin{aligned} i_{L/F}(\omega) &= \wp(\eta_0) + \sum_{i=1}^{p-1} (x^i \Phi(\eta_i) - \alpha^i \eta_i) + \wp(\eta'_0) \wedge \frac{d\alpha}{\alpha} + \sum_{i=1}^{p-1} (x^i \Phi(\eta'_i) - \alpha^i \eta'_i) \wedge \frac{d\alpha}{\alpha} \\ &\quad + \sum_{i=0}^{p-1} \alpha^i d\theta_i + \sum_{i=1}^{p-1} i\alpha^i \cdot \theta_i \wedge \frac{d\alpha}{\alpha} + \sum_{i=0}^{p-1} \alpha^i d\theta'_i \wedge \frac{d\alpha}{\alpha} \\ &= \left(\wp(\eta_0) + \sum_{i=1}^{p-1} x^i \Phi(\eta_i) + d\theta_0 \right) - \left(\sum_{i=1}^{p-1} \alpha^i (\eta_i - d\theta_i) \right) \\ &\quad + \left(\left(\wp(\eta'_0) + \sum_{i=1}^{p-1} x^i \Phi(\eta'_i) + d\theta'_0 \right) - \left(\sum_{i=1}^{p-1} \alpha^i (\eta'_i - i\theta_i - d\theta'_i) \right) \right) \wedge \frac{d\alpha}{\alpha}. \end{aligned}$$

By the linear independence properties of the decomposition of Ω_L^n we obtain two sets of p equations. From the first set we find for $1 \leq i \leq p-1$ that $\eta_i = d\theta_i + \psi_{i1}$ for some $\psi_{i1} \in \Omega_F^{n-1} \wedge dx$. Applying d to the second set we have $i d\theta_i = d\eta'_i + d\psi_{i2}$ for some $\psi_{i2} \in \Omega_F^{n-2} \wedge dx$. Together these give $\eta_i = i^{-1} d\eta'_i + \psi_i$, where $\psi_i \in \Omega_F^{n-1} \wedge dx$. From the second set we have $\sum_{i=1}^{p-1} x^i \Phi(\eta'_i) \in \wp(\Omega_F^{n-1}) + d\Omega_F^{n-2} + \Omega_F^{n-2} \wedge dx$ and so by Lemma 1.9 we know $\sum_{i=1}^{p-1} i^{-1} x^i \Phi(d\eta'_i) \in \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx$. But as each $\Phi(\psi_i) \in \Omega_F^{n-1} \wedge dx$, substituting $d\eta'_i = i\eta - i\psi_i$ shows

$$\sum_{i=1}^{p-1} i^{-1} i \cdot x^i \Phi(\eta_i) \in \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx.$$

Finally from the first equation we obtain $i_{L/F}(\omega) = \wp(\eta_0) + \sum_{i=1}^{p-1} x^i \Phi(\eta_i) + d\theta_0$ so $\omega \in \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx$ proving the theorem. $\square$

2. Generalizing lemmas on differential forms

The main goal of this section is to generalize Lemma 1.9 (see Theorem 2.2 below) so that it can be used to compute $\ker(H_{p^m}^{n+1}F \rightarrow H_{p^m}^{n+1}L)$ when $L = F(\alpha)$ for $\alpha^{p^e} = x \in F - F^p$. All notational conventions from the preceding section remain in force and we recall $\Gamma_F^n = \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx$. Extending this we define

$$\tilde{\Gamma}_F^n := \sum_{i=1}^{p-1} x^i B_2 \Omega_F^n + \sum_{j=1, (j,p)=1}^{p^2-1} x^j \Phi^2(\Omega_F^n) + \Gamma_F^n.$$

The next lemma will help simplify computations involving $\sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n$.

Lemma 2.1. *Suppose $e \geq 2$.*

- (i) $\sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n \subseteq \sum_{\ell=2}^e \sum_{i=1, (i,p)=1}^{p^{\ell-1}-1} x^i B_\ell \Omega_F^n + \Gamma_F^n$.
- (ii) $\sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n \subseteq \sum_{i=1}^{p-1} x^i B_2 \Omega_F^n + \sum_{j=1, (j,p)=1}^{p^2-1} x^j \Phi^2(\Omega_F^n) + \Gamma_F^n = \tilde{\Gamma}_F^n$.
- (iii) $d(\sum_{i=1}^{p^{e-1}-1} x^i B_e \Omega_F^n + \sum_{j=1, (j,p)=1}^{p^2-1} x^j \Phi^2(\Omega_F^n)) \subseteq \Omega_F^n \wedge dx$.

Proof. We note for $\chi \in \Omega_L^n$ and $s \geq 1$ that $\Phi^s(\chi) = \wp(\sum_{j=0}^{s-1} \Phi^j(\chi)) + \chi$. For the first statement, we know $B_e \Omega_F^n$ is generated by elements $\Phi^r(d\omega)$ where $0 \leq r \leq e-1$ and $\omega \in \Omega_F^{n-1}$. Consider a generator $x^{i_0 p^s} \Phi^r(d\omega)$ of $\sum_{i=1}^{p^{e-1}-1} x^i B_e \Omega_F^n$, where $(i_0, p) = 1$ and $s \geq 0$. If $s > r$ then $x^{i_0 p^s} \Phi^r(d\omega) = \Phi^r(d(x^{i_0 p^{s-r}} \omega)) \in \wp(\Omega_F^n) + d\Omega_F^{n-1}$. If $s < r$ then $x^{i_0 p^s} \Phi^r(d\omega) = \Phi^{r-s}(x^{i_0} \Phi^s(d\omega)) \in \wp(\Omega_F^n) + x^{i_0} \Phi^s(d\Omega_F^{n-1})$. If $s = r$ then $x^{i_0 p^s} \Phi^r(d\omega) = \Phi^r(x^{i_0} d\omega) \in \wp(\Omega_F^n) + x^{i_0} d\Omega_F^{n-1}$. However,

$$x^{i_0} d\Omega_F^{n-1} \subseteq d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx.$$

The first statement follows from this.

For the second statement we show for $0 \leq r < e$ that $x^i \Phi^r(d\omega) \in \tilde{\Gamma}_F^n$, where by (i) we may assume $(i, p) = 1$. If $r = 0$ and $i > 0$ then

$$x^i d\omega = d(x^i \omega) - (-1)^n i x^{i-1} \omega \wedge dx \in \Gamma_F^n.$$

If $r = 1$ and $i = i_0 + i_1 p$ then

$$x^i \Phi(d\omega) = x^{i_0} \Phi(x^{i_1} d\omega) = x^{i_0} \Phi(d(x^{i_1} \omega) - (-1)^n i_1 x^{i_1-1} \omega \wedge dx).$$

But $\Phi(\Omega_F^{n-1} \wedge dx) \subseteq \Omega_F^{n-1} \wedge dx$ so we are done in this case. For $r \geq 2$ and $i = i_0 + i_1 p + i_2 p^2$, where $0 < i_0 < p$ and $0 \leq i_1 < p$, we have $x^i \Phi^r(d\omega) = x^{i_0+i_1 p} \Phi^2(x^{i_2} \Phi^{r-2}(d\omega))$. As $i_0 \neq 0$ we have $x^i \Phi^r(d\omega) \in \sum_{j=1, (j,p)=1}^{p^2-1} x^j \Phi^2(\Omega_F^n)$. From this the second statement follows. The final statement is clear as $d\Phi(\omega) = 0$ for all $\omega \in \Omega_F^n$. $\square$

The next result extends Lemma 1.9. The proof follows the same format except that one computes in a purely inseparable extension of degree p^r instead of degree p .

Theorem 2.2. *Let $x \in F - F^p$ and $r, e \geq 1$. Suppose*

$$\sum_{i=1, (i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) \in \Gamma_F^n + \sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n,$$

where $\omega_i \in \Omega_F^n$. Then

$$\sum_{i=1, (i,p)=1}^{p^r-1} i^{-1} x^i \Phi^r(d\omega_i) \in \Gamma_F^{n+1} + \sum_{i=0}^{p^{r-1}-1} x^i B_r \Omega_F^{n+1}.$$

Proof. By hypothesis we can express

$$\sum_{i=1, (i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) = \wp(\omega_0) + d\hat{\theta} + \hat{\psi} \wedge dx + \tilde{\Theta},$$

where $\omega_0 \in \Omega_F^n$, $\hat{\theta}, \hat{\psi} \in \Omega_F^{n-1}$ and $\tilde{\Theta} \in \sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n$. We set $L = F(\alpha)$ with $\alpha^{p^r} = x \in F - F^p$ and define

$$\hat{\omega} := \left(\omega_0 - \sum_{j=0}^{r-1} \sum_{i=1, (i,p)=1}^{p^r-1} \alpha^{ip^j} \Phi^j(\omega_i) \right) \wedge \frac{d\alpha}{\alpha} \in \Omega_L^{n+1}.$$

As $i_{L/F}(\hat{\psi} \wedge dx) = 0 \in \Omega_L^{n+1}$ we have

$$\begin{aligned} \wp(\hat{\omega}) &= \left(\wp(\omega_0) - \sum_{i=1, (i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) + \sum_{i=1, (i,p)=1}^{p^r-1} \alpha^i \omega_i \right) \wedge \frac{d\alpha}{\alpha} \\ &= \left(\sum_{i=1, (i,p)=1}^{p^r-1} \alpha^i \omega_i - d\hat{\theta} - \tilde{\Theta} \right) \wedge \frac{d\alpha}{\alpha} \in \Omega_L^{n+1}. \end{aligned}$$

We note $\alpha^i \omega_i \wedge \alpha^{-1} d\alpha = d(i^{-1} \alpha^i \omega_i) - (-1)^n i^{-1} \alpha^i d\omega_i$ which shows

$$\left(\sum_{i=1, (i,p)=1}^{p^r-1} \alpha^i \omega_i \right) \wedge \frac{d\alpha}{\alpha} \in L \cdot \Omega_F^{n+1} + d\Omega_L^n.$$

Also, $d\hat{\theta} \wedge \alpha^{-1} d\alpha = d(\hat{\theta} \wedge \alpha^{-1} d\alpha) \in d\Omega_L^n$. We find that

$$\left(\sum_{i=1, (i,p)=1}^{p^r-1} \alpha^i \omega_i - d\hat{\theta} \right) \wedge \frac{d\alpha}{\alpha} \in L \cdot \Omega_F^{n+1} + d\Omega_L^n$$

and from this we have

$$\left(\wp(\hat{\omega}) - \tilde{\Theta} \wedge \frac{d\alpha}{\alpha} \right) \in L \cdot \Omega_F^{n+1} + d\Omega_L^n.$$

By Lemma 2.1(i), $\Theta \in \sum_{\ell=2}^e \sum_{i=1, (i,p)=1}^{p^{\ell-1}-1} x^i B_\ell \Omega_F^n + \Gamma_F^n$ and we can write

$$\Theta = \sum_{\ell=1}^{e-1} \sum_{i=1, (i,p)=1}^{p^{\ell-1}-1} x^i \Phi^\ell(d\hat{\theta}_{\ell i}) + \gamma$$

for $\hat{\theta}_{\ell i} \in \Omega_F^{n-1}$ and $\gamma \in \Gamma_F^n$. We recall for $\chi \in \Omega_L^n$ and $s \geq 1$ that $\Phi^s(\chi) = \wp(\sum_{j=0}^{s-1} \Phi^j(\chi)) + \chi$. If $r > \ell$ we have

$$\begin{aligned} (*_A) \quad x^i \Phi^\ell(d\hat{\theta}_{\ell i}) \wedge \frac{d\alpha}{\alpha} &= \Phi^\ell(\alpha^{ip^{r-\ell}} d\hat{\theta}_{\ell i}) \wedge \frac{d\alpha}{\alpha} = \Phi^\ell\left(d\left(\alpha^{ip^{r-\ell}} \hat{\theta}_{\ell i} \wedge \frac{d\alpha}{\alpha}\right)\right) \\ &= \wp\left(\psi_{\ell i} \wedge \frac{d\alpha}{\alpha}\right) + d\left(\alpha^{ip^{r-\ell}} \hat{\theta}_{\ell i} \wedge \frac{d\alpha}{\alpha}\right), \end{aligned}$$

where

$$\psi_{\ell i} := \sum_{j=0}^{\ell-1} \Phi^j(d(\alpha^{ip^{r-\ell}} \hat{\theta}_{\ell i})) = \sum_{j=0}^{\ell-1} \alpha^{ip^{r+j-\ell}} \Phi^j(d\hat{\theta}_{\ell i}) = \sum_{j'=r-\ell}^{r-1} \alpha^{ip^{j'}} \Phi^{j'+\ell-r}(d\hat{\theta}_{\ell i}).$$

When $1 \leq r \leq \ell$ and $(i, p) = 1$ we have

$$\begin{aligned} (*_B) \quad x^i \Phi^\ell(d\hat{\theta}_{\ell i}) \wedge \frac{d\alpha}{\alpha} &= \Phi^r(\alpha^i \Phi^{\ell-r}(d\hat{\theta}_{\ell i})) \wedge \frac{d\alpha}{\alpha} = \Phi^r\left(\alpha^i \Phi^{\ell-r}(d\hat{\theta}_{\ell i}) \wedge \frac{d\alpha}{\alpha}\right) \\ &= \wp\left(\psi_{\ell i} \wedge \frac{d\alpha}{\alpha}\right) + d(i^{-1} \alpha^i \Phi^{\ell-r}(d\hat{\theta}_{\ell i})), \end{aligned}$$

where

$$\psi_{\ell i} := \sum_{j=0}^{r-1} \Phi^j(\alpha^i \Phi^{\ell-r}(d\hat{\theta}_{\ell i})) = \sum_{j=0}^{r-1} \alpha^{ip^j} \Phi^j(\Phi^{\ell-r}(d\hat{\theta}_{\ell i})) = \sum_{j=0}^{r-1} \alpha^{ip^j} \Phi^{j+\ell-r}(d\hat{\theta}_{\ell i})$$

in this case.

We define Ψ by

$$(*_C) \quad \Psi := \sum_{\ell=1}^{e-1} \sum_{\substack{i=1 \\ (i,p)=1}}^{p^{\ell-1}-1} \psi_{\ell i} = \sum_{\ell=1}^{e-1} \sum_{\substack{i=1 \\ (i,p)=1}}^{p^{\ell-1}-1} \sum_{j=\max\{0, r-\ell\}}^{r-1} \alpha^{ip^j} \Phi^{j+\ell-r}(d\hat{\theta}_{\ell i}).$$

Using the latter sum we can express $\Psi = \sum_{j=0}^{r-1} \sum_{i=0, (i,p)=1}^{p^{r-j}-1} \alpha^{ip^j} \hat{\psi}_{ji}$, where $\hat{\psi}_{ji} \in \Omega_F^n$ for $0 \leq j \leq r-1$ and $0 \leq i \leq p^{r-j}$ are obtained by combining the like α^{ip^j} -summands. It is possible, however, as we only have a bound of $i < p^\ell$ (not the bound of $i < p^{r-j}$) that we have $ip^j = i_0 p^j + k p^r$ for some $k > 0$ and $i_0 < p^{r-j}$. In this case we have $\alpha^{ip^j} \Phi^{j+\ell-r}(d\hat{\theta}_{\ell i}) = \alpha^{i_0 p^j} x^k \Phi^{j+\ell-r}(d\hat{\theta}_{\ell i})$ appearing in the sum used to obtain $\hat{\psi}_{ji_0}$. This shows that for each $\hat{\psi}_{ji}$ we have $d\hat{\psi}_{ji} \in \Omega_F^n \wedge dx$.

The two expressions for $x^i \Phi^\ell(d\hat{\theta}_{\ell i}) \wedge \alpha^{-1} d\alpha$, $(*_A)$ and $(*_B)$, compared to the definition of Ψ in $(*_C)$, show $\tilde{\Theta} \wedge \alpha^{-1} d\alpha - \wp(\Psi \wedge \alpha^{-1} d\alpha) \in d\Omega_L^{n-1}$. As

$$\wp(\hat{\omega}) - \tilde{\Theta} \wedge \frac{d\alpha}{\alpha} \in L \cdot \Omega_F^{n+1} + d\Omega_L^n,$$

taken together these show

$$\wp\left(\hat{\omega} - \Psi \wedge \frac{d\alpha}{\alpha}\right) \in L \cdot \Omega_F^{n+1} + d\Omega_L^{n-1}.$$

Lemma 1.6 applied to $L = F(\alpha)$ gives $\hat{\omega} - \Psi \wedge \alpha^{-1}d\alpha = \text{dlog}(\tau) + \eta$, where $\tau \in K_{n+1}^M L$ and $\eta \in L \cdot \Omega_F^{n+1}$. From this we have,

$$\begin{aligned} & \wp\left(\hat{\omega} - \Psi \wedge \frac{d\alpha}{\alpha}\right) \\ &= \wp\left(\hat{\omega} - \sum_{j=0}^{r-1} \sum_{\substack{i=1 \\ (i,p)=1}}^{p^{r-j}-1} \alpha^{ip^j} \hat{\psi}_{ji} \wedge \frac{d\alpha}{\alpha}\right) \\ &= \left(\wp(\omega_0) - \sum_{\substack{i=1 \\ (i,p)=1}}^{p^r-1} x^i \Phi^r(\omega_i) + \sum_{\substack{i=1 \\ (i,p)=1}}^{p^r-1} \alpha^i \omega_i - \sum_{j=0}^{r-1} \sum_{\substack{i=1 \\ (i,p)=1}}^{p^{r-j}-1} \wp(\alpha^{ip^j} \hat{\psi}_{ji})\right) \wedge \frac{d\alpha}{\alpha} \\ &= \left(\wp(\omega_0) - \sum_{i=1, (i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) + \sum_{i=1, (i,p)=1}^{p^r-1} \alpha^i \omega_i + \sum_{i=0}^{p^r-1} \alpha^i \Psi_i\right) \wedge \frac{d\alpha}{\alpha} \\ &= \wp(\eta) + d\theta, \end{aligned}$$

where the $\Psi_i \in \Omega_F^n$ arise to fit the equation by combining all $\wp(\alpha^{ip^j} \hat{\psi}_{ji})$ summands and we have $\eta \in L \cdot \Omega_F^{n+1}$ and $\theta \in \Omega_L^n$. The one thing we need to know about the Ψ_i is that, because we know this for the $\hat{\psi}_{ji}$, we have $d\Psi_i \in \Omega_F^n \wedge dx$ for all i . Expanding $\eta = \sum_{i=0}^{p^r-1} \alpha^i \eta_i$ and $\theta = \sum_{i=0}^{p^r-1} \alpha^i \theta_i + \sum_{i=0}^{p^r-1} \alpha^i \theta'_i \wedge \alpha^{-1}d\alpha$ for $\eta_i \in \Omega_F^{n+1}$, $\theta_i \in \Omega_F^n$ and $\theta'_i \in \Omega_F^{n-1}$ gives the $2p^r$ equations, one for each summand in $\Omega_L^n = (\bigoplus_{i=0}^{p^r-1} \alpha^i F \cdot \Omega_F^n) \oplus (\bigoplus_{i=0}^{p^r-1} \alpha^i F \cdot \Omega_F^{n-1} \wedge \alpha^{-1}d\alpha)$:

$$\begin{aligned} 0 &= \wp(\eta_0) + \sum_{k=1}^{p-1} x^k \Phi(\eta_{kp^{r-1}}) + d\theta_0, \\ 0 &= \alpha^{ip^{r-1}} \left(-\eta_{ip^{r-1}} + d\theta_{ip^{r-1}} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{ip^{r-2}+kp^{r-1}}) \right), \quad 1 \leq i \leq p-1, \\ &\vdots \\ (*1) \quad 0 &= \alpha^{q'p^2} \left(-\eta_{q'p^2} + d\theta_{q'p^2} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{q'p+kp^{r-1}}) \right), \quad 1 \leq q' \leq p^{r-2}-1, \\ &\quad (q', p) = 1, \\ 0 &= \alpha^{qp} \left(-\eta_{qp} + d\theta_{qp} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{q+kp^{r-1}}) \right), \quad 1 \leq q \leq p^{r-1}-1, \\ &\quad (q, p) = 1, \\ 0 &= \alpha^j (-\eta_j + d\theta_j), \quad 1 \leq j \leq p^r-1, (j, p) = 1, \end{aligned}$$

and

$$\begin{aligned}
 & \left(\wp(\omega_0) - \sum_{i=1, (i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) + \Psi_0 \right) \wedge \frac{d\alpha}{\alpha} = d\theta'_0 \wedge \frac{d\alpha}{\alpha}, \\
 & \alpha^{ip^{r-1}} \Psi_{ip^{r-1}} \wedge \frac{d\alpha}{\alpha} = \alpha^{ip^{r-1}} d\theta'_{ip^{r-1}} \wedge \frac{d\alpha}{\alpha}, \quad 1 \leq i \leq p-1, \\
 & (*)_2 \quad \vdots \\
 & \alpha^{qp} \Psi_{qp} \wedge \frac{d\alpha}{\alpha} = \alpha^{qp} d\theta'_{qp} \wedge \frac{d\alpha}{\alpha}, \quad 1 \leq q \leq p^{r-1}-1, (q, p) = 1, \\
 & \alpha^j(\omega_j + \Psi_j) \wedge \frac{d\alpha}{\alpha} = \alpha^j(d\theta'_j + j\theta_j) \wedge \frac{d\alpha}{\alpha}, \quad 1 \leq j \leq p^r-1, (j, p) = 1.
 \end{aligned}$$

We note that in using each equation we keep in mind that $\ker(i_{L/F} : \Omega_F^n \rightarrow \Omega_L^n) = \Omega_F^{n-1} \wedge dx$.

The bottom equations from $(*)_1$ show that $\eta_j = d\theta_j + \chi'_j \wedge dx$ for some χ'_j . As $d\Psi_j \in \Omega_F^n \wedge dx$, by the bottom equations of $(*)_2$ we have

$$d\theta_j = j^{-1}(d\omega_j + d\Psi_j) + \chi''_j \wedge dx = j^{-1}d\omega_j + \chi'''_j \wedge dx \quad \text{for some } \chi''_j, \chi'''_j.$$

These give $\eta_j = j^{-1}d\omega_j + \chi_j \wedge dx$ for some χ_j whenever $1 \leq j \leq p^r-1$ and $(j, p) = 1$.

If $1 \leq q \leq p^{r-1}-1$ and $(q, p) = 1$ the second to last equations of $(*)_1$ show for some $\chi_{qp} \in \Omega_F^{n-1}$ that

$$\eta_{qp} = d\theta_{qp} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{q+kp^{r-1}}) + \chi_{qp} \wedge dx.$$

If $1 \leq q' \leq p^{r-2}-1$ and $(q', p) = 1$ the third to last equations of $(*)_1$ show for some $\chi_{q'p^2}, \chi'_{q'p^2} \in \Omega_F^{n-1}$ and $\Theta_{q'p^2} \in \sum_{k=0}^{p-1} x^k B_2 \Omega_F^{n+1}$, that

$$\begin{aligned}
 \eta_{q'p^2} &= d\theta_{q'p^2} + \sum_{k'=0}^{p-1} x^{k'} \Phi(\eta_{q'p+k'p^{r-1}}) + \chi'_{q'p^2} \wedge dx \\
 &= d\theta_{q'p^2} + \sum_{k'=0}^{p-1} x^{k'} \Phi \left(d\theta_{q'p+k'p^{r-1}} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{q'+k'p^{r-2}+kp^{r-1}}) + \chi_{q'p+k'p^{r-1}} \wedge dx \right) \\
 &\quad + \chi'_{q'p^2} \wedge dx \\
 &= \Theta_{q'p^2} + \sum_{k'=0}^{p-1} \sum_{k=0}^{p-1} x^{k'+kp} \Phi^2(\eta_{q'+k'p^{r-2}+kp^{r-1}}) + \chi_{q'p^2} \wedge dx \\
 &= \Theta_{q'p^2} + \sum_{k=0}^{p^2-1} x^k \Phi^2(\eta_{q'+kp^{r-2}}) + \chi_{q'p^2} \wedge dx.
 \end{aligned}$$

Repeating this process, substituting iteratively into the preceding equations in $(*)_1$, when the second set of equations is reached we have for $1 \leq i \leq p-1$,

$$\eta_{ip^{r-1}} = \Theta_{ip^{r-1}} + \sum_{k=0}^{p^{r-1}-1} x^k \Phi^{r-1}(\eta_{i+kp}) + \chi_{ip^{r-1}} \wedge dx,$$

where $\Theta_{ip^{r-1}} \in \sum_{i=0}^{p^{r-2}} x^i B_{r-1} \Omega_F^{n+1}$. At last, we find when the first equation is reached we substitute these expressions for the $\eta_{ip^{r-1}}$ where $1 \leq i \leq p-1$ and find that

$$\sum_{i=1, (i,p)=1}^{p^r-1} x^i \Phi^r(\eta_i) \in \wp(\Omega_F^{n+1}) + d\Omega_F^n + \Omega_F^n \wedge dx + \sum_{k=0}^{p^{r-1}-1} x^k B_r \Omega_F^{n+1}.$$

Using $\eta_i = i^{-1} d\omega_i + \chi_i \wedge dx$ gives that

$$\sum_{i=1, (i,p)=1}^{p^r-1} i^{-1} x^i \Phi^r(d\omega_i) \in \wp(\Omega_F^{n+1}) + d\Omega_F^n + \Omega_F^n \wedge dx + \sum_{k=0}^{p^{r-1}-1} x^k B_r \Omega_F^{n+1},$$

as required. $\square$

In the applications the set up is slightly different than they are in the hypotheses of Theorem 2.2. The following corollary takes care of this for us.

Corollary 2.3. *Let $x \in F - F^p$ and $r, e \geq 1$. Suppose*

$$\sum_{i=1, (i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) \in \Gamma_F^n + \sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n + \sum_{i=1, (i,p)=1}^{p^{r+1}-1} x^i \Phi^{r+1}(\Omega_F^n),$$

where $\omega_i \in \Omega_F^n$. Then

$$\sum_{i=1, (i,p)=1}^{p^{r-1}-1} i^{-1} x^i \Phi^r(d\omega_i) \in \Gamma_F^{n+1} + \sum_{i=1}^{p^{r-1}-1} x^i B_r \Omega_F^{n+1}.$$

Proof. We express $\sum_{i=1, (i,p)=1}^{p^r-1} x^i \Phi^r(\omega_i) = \gamma + \sum_{i=1}^{p^{r+1}-1} x^i \Phi^{r+1}(\theta_i)$, where $\gamma \in \Gamma_F^n + \sum_{i=1}^{p^{e-1}-1} x^i B_e \Omega_F^n$ and $\theta_i \in \Omega_F^n$. If $\tilde{\omega}_i = \omega_i - \sum_{k=0}^{p-1} \Phi(\theta_{i+kp^r})$ we then have $\sum_{i=1}^{p^{r-1}-1} x^i \Phi^r(\tilde{\omega}_i) \in \Gamma_F^n + \sum_{i=0}^{p^{e-1}-1} x^i B_e \Omega_F^n$. But we have $d\tilde{\omega}_i = d\omega_i$, so the result follows by Theorem 2.2. $\square$

3. Background on the de Rham Witt complex

As noted in the introduction, the groups $H_{p^m}^{n+1} F$ for $m \geq 1$ require for their definition, either the groups $Q^n(F, m)$ as defined by Izhboldin, or the de Rham Witt groups $W_m \Omega_F^n$. Aside from the proof of Theorem 3.13 below, we work with the latter. In

this section we collect results about $W_m\Omega_F^n$ needed to prove our main theorem. As references we cite [Illusie 1979] and Section 2 of [Aravire et al. 2018].

The module $W_m\Omega_F^n$ is a quotient of $\Omega_{W_mF}^n$ where W_mF is the ring of length m Witt vectors over F . As such it is generated by elements of the form $a \cdot db_1 \wedge \cdots \wedge db_n$, where $a, b_1, \dots, b_n \in W_mF$. For $a \in F$ we define $[a]_m := (a, 0, \dots, 0) \in W_mF$ and note that these “representatives” are multiplicative but not additive. There are important interactions between these generators and the shift $V : W_m\Omega_F^n \rightarrow W_{m+1}\Omega_F^n$, a few of which seem unnatural until one is used to them. We do have $V(a) \cdot dV(b) = V(a \cdot db)$, as the first defining relations for $W_m\Omega_F^n$ ensure this. But one needs to be careful, for V and d do not commute. The relation just given shows that $V(dy) = V([1]_m \cdot dy) = V([1]_m) \cdot dV(y) = p \cdot dV(y)$, for although $[1]_m = 1 \in W_mF$ its shift $V([1]_m) = p \in W_mF$. The generalization of this is given in Lemma 3.2(i).

The second defining relation, $V(a)d[x]_m = V(a[x]_{m-1}^{p-1})dV[x]_{m-1}$, may appear odd at first but is critical to the theory. Using the Witt vector relation, $[b]_m V(a) = V([b^p]_{m-1}a)$, this relation shows that V commutes with the logarithmic derivative of multiplicative elements as follows: $V(a)[x]_m^{-1}d[x]_m = V(a[x]_{m-1}^{-p})d[x]_m = V(a[x]_{m-1}^{-p}[x]_{m-1}^{p-1})dV([x]_{m-1}) = V(a[x]_{m-1}^{-1}d[x]_{m-1})$. The logarithmic derivative also links Milnor K-theory to the de Rham Witt groups as follows. For a generator $\tau = \ell(a_1) \otimes \cdots \otimes \ell(a_n)$ of $K_n^M F$ we define

$$\mathrm{dlog}_m(\tau) := \frac{d[a_1]_m}{[a_1]_m} \wedge \cdots \wedge \frac{d[a_n]_m}{[a_n]_m} \in W_m\Omega_F^n.$$

By [Illusie 1979, 3.23, p. 580] when $n = 1$ the map sends products to sums, and then the defining relations for $W_m\Omega_F^n$ ensure one obtains a well-defined homomorphism $\mathrm{dlog}_m : K_n^M F \rightarrow W_m\Omega_F^n$.

The next result was mentioned in the introduction. It appears as [Aravire et al. 2018, Lemma 2.2] and its proof is by induction using the second defining relation.

Lemma 3.1. *For all $x \in W_{m-s}F$ with $m > s \geq 1$ and $a \in F$ we have*

$$V^s(x) d[a]_m = V^s(x[a]_{m-s}^{p^s-1}) dV^s[a]_{m-s} \in W_m\Omega_F^1.$$

The Frobenius $\Phi : W_mF \rightarrow W_mF$ given by $\Phi(a_1, \dots, a_m) = (a_1^p, \dots, a_m^p)$ extends with a level shift to $\Phi : W_m\Omega_F^m \rightarrow W_{m-1}\Omega_F^m$ (see [Illusie 1979]). In analogy with the $m = 1$ case where $\Phi : \Omega_F^n \rightarrow \Omega_F^n/d\Omega_F^{n-1}$, there is a lift to $\Phi : W_m\Omega_F^m \rightarrow W_m\Omega_F^m/dV^{m-1}\Omega_F^n$. We will use the same notation for both maps as the one in use will be clear from context. Since $p \cdot (a_1, \dots, a_m) = (0, a_1^p, \dots, a_{m-1}^p) \in W_mF$ we have $p \cdot = V \circ \Phi = \Phi \circ V$ on W_mF . The definitions on $W_m\Omega_F^n$ are set so that this rule generalizes as given in Lemma 3.2(iii) below. In addition to the s -fold composite of V , $V^s : W_{m-s}\Omega_F^n \rightarrow W_m\Omega_F^n$, there are restriction maps $R^s : W_m\Omega_F^n \rightarrow W_s\Omega_F^n$ which generalize the same map on Witt vectors given by $R^s(a_1, \dots, a_m) = (a_1, \dots, a_s)$. This map also behaves properly with respect

to logarithmic derivatives where one has $R^s(a[x]_m^{-1}d[x]_m) = R^s(a)[x]_s^{-1}d[x]_s$. Finally we note that Φ behaves well with respect to logarithmic derivatives of multiplicative elements. According to [Illusie 1979, Proposition 2.18.5, p. 564] we have $\Phi(d[x]_m) = [x]_{m-1}^{p-1}d[x]_{m-1}$, so interpreted according to the lift gives $\Phi([x]_m^{-1}d[x]_m) = [x]_m^{p-1}d[x]_m/[x]_m^p = [x]_m^{-1}d[x]_m$. Since Φ is multiplicative we have $\Phi(\omega \wedge [x]_m^{-1}d[x]_m) = \Phi(\omega) \wedge [x]_m^{-1}d[x]_m$ for $\omega \in W_m\Omega_F^{n-1}$. There are other basic interactions between, d , V and Φ which we summarize for reference in this next lemma (see [Aravire et al. 2018, Lemma 2.9] or [Illusie 1979] for more details).

Lemma 3.2. *Suppose $\text{char}(F) = p$. Then:*

- (i) $Vd = pdV$ as functions $W_{m-1}\Omega_F^{n-1} \rightarrow W_m\Omega_F^n$.
- (ii) $\Phi dV = d$ as functions $W_m\Omega_F^{n-1} \rightarrow W_m\Omega_F^n$.
- (iii) $p \cdot = \Phi \circ V = V \circ \Phi$ as functions $W_m\Omega_F^n \rightarrow W_m\Omega_F^n$.
- (iv) $d\Phi = p\Phi d$ as functions $W_m\Omega_F^n \rightarrow W_{m-1}\Omega_F^{n+1}$.
- (v) If $\omega_1 \in W_m\Omega_F^r$ and $\omega_2 \in W_{m-1}\Omega_F^s$ then $\omega_1 V(\omega_2) = V(R^{m-1} \circ \Phi(\omega_1)\omega_2) \in W_m\Omega_F^{r+s}$.

For computations involving $\Phi : W_m\Omega_F^n \rightarrow W_m\Omega_F^n$, we will use logarithmic derivatives like what we used in the previous sections for the case of $m = 1$. Since we are working with the lift of Φ these representative values are well-defined (mod $dV^{m-1}(\Omega_F^{n-1})$), which is absorbed into the calculations when used. By the above we can use

$$\Phi\left(a \frac{d[b_1]_m}{[b_1]_m} \wedge \cdots \wedge \frac{d[b_n]_m}{[b_n]_m}\right) = \Phi(a) \frac{d[b_1]_m}{[b_1]_m} \wedge \cdots \wedge \frac{d[b_n]_m}{[b_n]_m} \quad \text{for } a, b_1, \dots, b_n \in F.$$

Since Φ and V commute we can then use

$$\Phi\left(V^s\left(a \frac{d[b_1]_{m-s}}{[b_1]_{m-s}} \wedge \cdots \wedge \frac{d[b_n]_{m-s}}{[b_n]_{m-s}}\right)\right) = V^s\left(\Phi(a) \frac{d[b_1]_{m-s}}{[b_1]_{m-s}} \wedge \cdots \wedge \frac{d[b_n]_{m-s}}{[b_n]_{m-s}}\right).$$

These two formulae suffice for our computations in this and the next section.

The shift map on the ring of Witt vectors gives an exact sequence

$$0 \rightarrow W_{m-s}F \xrightarrow{V^s} W_mF \xrightarrow{R^s} W_sF \rightarrow 0.$$

The analogue for the de Rham Witt complex is the following result from [Illusie 1979, Proposition 3.2, p. 568], upon which the induction used to prove the main theorem will be based.

Theorem 3.3. *Let $R^s : W_m\Omega_F^n \rightarrow W_s\Omega_F^n$ be the restriction. Then the following sequence is exact,*

$$0 \rightarrow (V^s W_{m-s}\Omega_F^n + dV^s W_{m-s}\Omega_F^{n-1}) \hookrightarrow W_m\Omega_F^n \xrightarrow{R^s} W_s\Omega_F^n \rightarrow 0,$$

where $(V^s W_{m-s} \Omega_F^n + dV^s W_{m-s} \Omega_F^{n-1})$ is viewed as its image inside $W_m \Omega_F^n$.

We remark that it is *not* a consequence of Theorem 3.3 that V^s is injective — indeed it is not and its kernel plays a special role in the sequel. In the theorem only its image in $W_m \Omega_F^n$ is considered which also takes into account relations between $V^s W_{m-s} \Omega_F^n$ and $dV^s W_{m-s} \Omega_F^{n-1}$ in $W_m \Omega_F^n$.

We next generalize the definition of the groups $B_r \Omega_F^n$ to the de Rham Witt complex. These are discussed in detail in [Aravire et al. 2018] and are important in relating the $Q^n(F, m)$ groups of Izhboldin to the de Rham Witt complex. (We also remark that Izhboldin must have understood such details, but decided to spell out his work in a different way.)

Definition 3.4. For $r, m, n \geq 1$, $B_r W_m \Omega_F^n \subseteq W_m \Omega_F^n$ is defined to be

$$B_r W_m \Omega_F^n := \ker(V^r : W_m \Omega_F^n \rightarrow W_{m+r} \Omega_F^n).$$

For convenience, $B_0 W_m \Omega_F^n = \{0\}$.

According to [Illusie 1979, 3.21.1.4, p. 579], for all $r, m, n \geq 1$ we have

$$\ker(V^r : W_m \Omega_F^n \rightarrow W_{m+r} \Omega_F^n) = \Phi^r(dV^m W_r \Omega_F^{n-1}).$$

This formula uses the shifted Φ and Φ^r is the r -fold composite of these shifted Φ . We note that one consequence of the relation $\Phi dV = d$ is that $\Phi(dV^s(W_{m-s} \Omega_F^n)) = dV^{s-1}(W_{m-s} \Omega_F^n)$. So for $r \leq m$ the definitions give $B_r W_m \Omega_F^n = dV^{m-r}(W_r \Omega_F^n)$, so in particular $B_1 W_m \Omega_F^n = dV^{m-1} \Omega_F^{n-1}$ and $B_m W_m \Omega_F^n = dW_m \Omega_F^{n-1}$. When $r \leq m$ this also gives that $B_r W_m \Omega_F^n = \Phi(B_{r-1} W_m \Omega_F^n)$.

This next lemma collects for reference the basic properties of $B_r W_m \Omega_F^n$. The Φ here are the unshifted Φ which is not a problem because $dV^{m-1} \Omega_F^{n-1} \subseteq B_r W_m \Omega_F^n$ for all $r \geq 1$.

Lemma 3.5. (i) $B_r W_m \Omega_F^n = \Phi^r(dV^m W_r \Omega_F^{n-1})$.

(ii) $B_1 W_m \Omega_F^n = dV^{m-1} \Omega_F^{n-1}$.

(iii) For $r < m$, $B_r W_m \Omega_F^n = dV^{m-r} W_r \Omega_F^{n-1}$.

(iv) $B_m W_m \Omega_F^n = dW_m \Omega_F^{n-1}$.

(v) For $r > m$, $B_r W_m \Omega_F^n = \Phi(B_{r-1} W_m \Omega_F^n) = \Phi^{r-m}(dW_m \Omega_F^{n-1})$.

Proof. (i) is the result from [Illusie 1979] just cited. (ii) is (i) with $r = 1$ together with Lemma 3.2(ii). (iii) follows from (i) and (ii) as Lemma 3.2(ii) gives $\Phi dV = d$. (iv) is (iii) with $r = m$, and (v) follows from (iv) using (i). $\square$

It is important to understand how the $B_r W_m \Omega_F^n$ interact with V . This is given next.

Lemma 3.6. For all r, m, n we have $V(B_r W_m \Omega_F^n) \subseteq B_{r-1} W_{m+1} \Omega_F^n$.

Proof. We have the commutative diagram

$$\begin{array}{ccccc}
 B_r W_m \Omega_F^n & \longrightarrow & W_m \Omega_F^n & \xrightarrow{V^r} & W_{m+r} \Omega_F^n \\
 \downarrow V & & \downarrow V & & \downarrow = \\
 \ker(V^{r-1}) & \longrightarrow & W_{m+1} \Omega_F^n & \xrightarrow{V^{r-1}} & W_{m+r} \Omega_F^n
 \end{array}$$

and since $B_{r-1} W_{m+1} \Omega_F^n = \ker(V^{r-1} : W_{m+1} \Omega_F^n \rightarrow W_{m+r} \Omega_F^n)$ the result follows by definition. $\square$

We remark that the shift of indices in Lemma 3.6 will play an important role in the sequel. The next result generalizes well-known properties of $B_r \Omega_F^n$ to the more general case.

Lemma 3.7. *For all r, m, n :*

- (i) *If $a \in F$ then $[a]_m^{p^r} B_r W_m \Omega_F^n \subseteq B_r W_m \Omega_F^n$.*
- (ii) *If $x \in F$ and $r \geq m$ then $[x]_m^{p^{r-m}} B_r W_m \Omega_F^n \subseteq B_r W_m \Omega_F^n + W_m \Omega_F^{n-1} \wedge d[x]_m$.*
- (iii) *For all i , $[x]_m^i B_{m-1} W_m \Omega_F^n \subseteq B_{m-1} W_m \Omega_F^n + W_m \Omega_F^{n-1} \wedge d[x]_m$.*

Proof. For (i) we proceed by induction on r . We have $B_1 W_m \Omega_F^n = dV^{m-1} \Omega_F^{n-1}$ by definition. Let $\omega \in \Omega_F^{n-1}$ and then

$$d([a]_m^p V^{m-1}(\omega)) = p[a]_m^{p-1} V^{m-1}(\omega) d[a]_m + [a]^p dV^{m-1}(\omega)$$

so as $pV^{m-1}(\omega) = 0$ we find $[a]^p dV^{m-1}(\omega) \in dV^{m-1} W_m \Omega_F^{n-1}$ giving the result when $r = 1$.

Next, assuming $[a]_m^{p^r} B_r W_m \Omega_F^n \subseteq B_r W_m \Omega_F^n$, we use Lemma 3.5(v) and the inductive hypothesis to see that

$$\begin{aligned}
 [a]_m^{p^{r+1}} B_{r+1} W_m \Omega_F^n &= [a]_m^{p^{r+1}} \Phi(B_r W_m \Omega_F^n) = \Phi([a]_m^{p^r} B_r W_m \Omega_F^n) \\
 &\subseteq \Phi(B_r W_m \Omega_F^n) = B_{r+1} W_m \Omega_F^n.
 \end{aligned}$$

This gives (i).

For (ii) we note if $r = m$ and $\omega \in W_m \Omega_F^{n-1}$, by Lemma 3.5(iv) $B_m W_m \Omega_F^n = dW_m \Omega_F^{n-1}$ so we have $[x]_m d\omega = d([x]_m \omega) - (-1)^n \omega \wedge d[x]_m$ and the result follows in that case. The same argument applies when $r = m - 1$ as $B_{r-1} W_m \Omega_F^n = V W_{m-1} \Omega_F^n$ and $[x]_m^i V(\omega) = V([x]_{m-1}^i \omega)$, giving (iii). To finish (ii), by induction, assuming the result for $r \geq m$ we note that for a generator $\Phi^{r+1-m}(d\omega)$ of $B_{r+1} W_m \Omega_F^n \pmod{B_r W_m \Omega_F^n}$ we have

$$\begin{aligned}
 [x]_m^{p^{r+1-m}} \Phi^{r+1-m}(d\omega) &= \Phi^{r+1-m}([x]_m d\omega) = \Phi^{r+1-m}(d([x]_m \omega) - \omega \wedge d[x]_m) \\
 &\in B_{r+1} W_m \Omega_F^n + W_m \Omega_F^{n-1} \wedge d[x]_m.
 \end{aligned}$$

This gives (ii). $\square$

According to Lemma 3.5(i) we have

$$\ker(V^r : W_m \Omega_F^n \rightarrow W_{m+r} \Omega_F^n) = \Phi^r(dV^m W_r \Omega_F^{n-1}).$$

This characterization gives a set of generators for the $B_r W_m \Omega_F^n$ which is spelled out next.

Theorem 3.8. *For $r \geq m$, $B_r W_m \Omega_F^n$ is generated by*

$$B_m W_m \Omega_F^n \quad \text{and} \quad [s]_m^{p^t} \frac{d[s]_m}{[s]_m} \wedge \mathrm{dlog}_m(\tau),$$

$$\text{where } t \leq r - m, s \in F \text{ and } \tau \in K_{n-1}^M F \quad (\text{Milnor K-theory}).$$

In particular, $B_r W_m \Omega_F^n = B_r W_m \Omega_F^1 \wedge \mathrm{dlog}_m(K_{n-1}^M F) + B_m W_m \Omega_F^n$.

Proof. We proceed by induction on $r \geq m$, the case of $r = m$ being trivial. Applying d to Theorem 3.3 when $s = 1$ we find that $B_m W_m \Omega_F^n = dW_m \Omega_F^{n-1}$ is generated by elements $d[s]_m \wedge \mathrm{dlog}_m(\tau)$ for $s \in F$ and $\tau \in K_{n-1}^M F$ over $B_{m-1} \Omega_F^n = dV W_{m-1} \Omega_F^{n-1}$. As $\Phi(d[s]_m \wedge \mathrm{dlog}_m(\tau)) = [s]_m^p [s]_m^{-1} d[s]_m \wedge \mathrm{dlog}_m(\tau)$ we are done for $r = m + 1$ as $B_{m+1} W_m \Omega_F^n = \Phi(B_m W_m \Omega_F^n)$ and these will give generators over $\Phi(B_{m-1} W_m \Omega_F^n) = B_m W_m \Omega_F^n$. For $r > m + 1$, as $B_r W_m \Omega_F^n = \Phi(B_{r-1} W_m \Omega_F^n)$ it suffices to note by the properties of Φ that

$$\Phi\left([s]_m^{p^t} \frac{d[s]_m}{[s]_m} \wedge \mathrm{dlog}_m(\tau)\right) = [s]_m^{p^{t+1}} \frac{d[s]_m}{[s]_m} \wedge \mathrm{dlog}_m(\tau).$$

This gives the first statement. The second statement follows from the first, proving the result. $\square$

We now recall the machinery that is used to define the groups $H_{p_m}^{n+1} F$. These results are discussed in detail in [Aravire et al. 2018, Section 2]. To begin we recall the definition of Izhboldin's groups.

Definition 3.9 [Izhboldin 1991, p. 139]. The group $Q^n(F, m)$ is the quotient $A_m^n / \mathcal{J}_m^n$, where we define $A_m^n := W_m F^+ \otimes F^* \otimes \cdots \otimes F^*$ and $\mathcal{J}_m^n$ is the subgroup generated by the tensors

- (i) $a \otimes t_1 \otimes \cdots \otimes t_n$, where $a \in W_m F$ and $t_i = t_j$ for some $i \neq j$, and
- (ii) $V^s[a^\ell]_{m-s} \otimes a \otimes t_2 \otimes \cdots \otimes t_n$, where $a, t_2, \dots, t_n \in F^*$, $0 \leq s < m$, and $\ell \geq 1 \in \mathbb{N}$.

The next result shows how the Izhboldin groups are linked to the de Rham Witt complex. For this, one checks (see [Aravire et al. 2018, Section 2]), that $q_m^n : W_m \Omega_F^n \rightarrow Q^n(F, m)$ defined on generators by

$$q_m^n\left(a \frac{d[t_1]_m}{[t_1]_m} \wedge \cdots \wedge \frac{d[t_n]_m}{[t_n]_m}\right) = [a \otimes t_1 \otimes \cdots \otimes t_n] \in Q^n(F, m)$$

gives a well-defined homomorphism. Further, Izhboldin showed for all m and n that there is a well-defined homomorphism $\delta_m^n : K_n^M F \rightarrow Q^n(F, m)$ defined on generators

by $\delta_m^n(\ell(t_1) \otimes \cdots \otimes \ell(t_n)) = [1 \otimes t_1 \otimes \cdots \otimes t_n]$. It is clear by the definitions that $\delta_m^n(\tau) = q_m^n \circ \text{dlog}_m(\tau)$ for all $\tau \in K_n^M F$. We define $B_\infty W_m \Omega_F^n := \bigcup_{r \geq 1} B_r W_m \Omega_F^n$. By [Aravire et al. 2018, Theorem 2.27] we have the following.

Theorem 3.10. *The homomorphism $q_m^n : W_m \Omega_F^n \rightarrow Q^n(F, m)$ induces an isomorphism $Q^n(F, m) \cong W_m \Omega_F^n / B_\infty W_m \Omega_F^n$.*

Following [Aravire et al. 2018], using $\Phi(B_{r-1} W_m \Omega_F^n) \subseteq B_r W_m \Omega_F^n$, for $r \geq m$ we have a well defined map

$$\wp := (\Phi - 1) : \frac{W_m \Omega_F^n}{B_{r-1} W_m \Omega_F^n} \rightarrow \frac{W_m \Omega_F^n}{B_r W_m \Omega_F^n}.$$

whose kernel will temporarily be labeled $\ker_{m,r}^n$ and whose cokernel will temporarily be labeled $\text{coker}_{m,r}^n$. With this notation we have the following result [Aravire et al. 2018, Lemma 2.30].

Lemma 3.11. *For all n, m and $r \geq m$ the following diagram is commutative with down-arrows induced by the inclusions $B_r W_m \Omega_F^n \subseteq B_{r+1} W_m \Omega_F^n \subseteq B_{r+2} W_m \Omega_F^n$ and induced isomorphisms as indicated.*

$$\begin{array}{ccccccc} \ker_{m,r}^n & \longrightarrow & \frac{W_m \Omega_F^n}{B_{r-1} W_m \Omega_F^n} & \xrightarrow{\wp} & \frac{W_m \Omega_F^n}{B_r W_m \Omega_F^n} & \longrightarrow & \text{coker}_{m,r}^n \\ \cong \downarrow & & \downarrow & & \downarrow & & \cong \downarrow \\ \ker_{m,r+1}^n & \longrightarrow & \frac{W_m \Omega_F^n}{B_r W_m \Omega_F^n} & \xrightarrow{\wp} & \frac{W_m \Omega_F^n}{B_{r+1} W_m \Omega_F^n} & \longrightarrow & \text{coker}_{m,r+1}^n \end{array}$$

Combining Lemma 3.11 with Theorem 3.10 and passing to direct limits we obtain the following result [Aravire et al. 2018, Theorem 2.31]. The first exact sequence provides the basis for our computations and the second is due to Izhboldin.

Theorem 3.12. *For all r with $m \leq r \leq \infty$ we have $\ker_{m,r}^n \cong K_n^M F / p^m K_m^M F$ and each $\text{coker}_{m,r}^n \cong H_{p^m}^{n+1} F$. In particular we have exact sequences*

$$0 \longrightarrow K_n F / p^m K_n F \longrightarrow \frac{W_m \Omega_F^n}{B_{m-1} W_m \Omega_F^n} \xrightarrow{\wp} \frac{W_m \Omega_F^n}{B_m W_m \Omega_F^n} \longrightarrow H_{p^m}^{n+1} F \longrightarrow 0,$$

and

$$0 \longrightarrow K_n F / p^m K_n F \longrightarrow Q^n(F, m) \xrightarrow{\wp} Q^n(F, m) \longrightarrow H_{p^m}^{n+1} F \longrightarrow 0.$$

As $B_m W_m \Omega_F^n = d W_m \Omega_F^{n-1}$, the characterization of $H_{p^m}^{n+1} F$ given in the introduction

$$W_m \Omega_F^n \xrightarrow{\wp} \frac{W_m \Omega_F^n}{d W_m \Omega_F^{n-1}} \rightarrow H_{p^m}^{n+1} F \rightarrow 0$$

is a consequence of the first exact sequence of Theorem 3.12. Whenever $\omega \in W_m \Omega_F^n$ we use $\bar{\omega}$ to denote the class it determines in $W_m \Omega_F^n / (\wp(W_m \Omega_F^n) + dW_m \Omega_F^{n-1}) \cong H_{p^m}^{n+1} F$.

Although the shift-restriction short exact sequence for Witt Vectors doesn't generalize completely to the de Rham Witt complex, it does generalize nicely in cohomology. Although this was known by Izhboldin, we include a proof to illustrate the tools. The injectivity of V is critical in the sequel.

Theorem 3.13. *For all $m \geq 2$ we have exact sequences*

$$0 \rightarrow H_{p^{m-1}}^{n+1} F \xrightarrow{V} H_{p^m}^{n+1} F \xrightarrow{R} H_p^{n+1} F \rightarrow 0.$$

In particular, for $1 \leq \ell < m$ the maps $V^\ell : H_{p^{m-\ell}}^{n+1} F \rightarrow H_{p^m}^{n+1} F$ are injective.

Proof. By Izhboldin's work [1991, Proposition 6.3], the following diagram has exact rows and columns,

$$\begin{array}{ccccccc} & & & & K_n^M F & \xrightarrow{1} & K_n^M F \longrightarrow 0 \\ & & & & \delta_m^n \downarrow & & \delta_1^n \downarrow \\ 0 & \longrightarrow & Q^n(F, m-1) & \xrightarrow{V} & Q^n(F, m) & \xrightarrow{R} & Q^n(F, 1) \longrightarrow 0 \\ & & \wp \downarrow & & \wp \downarrow & & \wp \downarrow \\ 0 & \longrightarrow & Q^n(F, m-1) & \xrightarrow{V} & Q^n(F, m) & \xrightarrow{R} & Q^n(F, 1) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \wp \downarrow \\ & & H_{p^{m-1}}^{n+1}(F) & \longrightarrow & H_{p^m}^{n+1}(F) & \longrightarrow & H_p^{n+1}(F) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The standard chase gives the required exact sequence. □

To characterize cohomological kernels we need the next definition.

Definition 3.14. For $x \in F - F^p$ we set

$$\mathcal{L}_{m,e}^{n+1} := \sum_{r=1}^e \sum_{i=1, (i,p)=1}^{p^r-1} \overline{[x]_m^i B_r W_m \Omega_F^n} \subseteq H_{p^m}^{n+1} F,$$

$$\text{and } K_{m,e}^{n+1} := \ker(i_{L/F} : H_{p^m}^{n+1} F \rightarrow H_{p^m}^{n+1} F(\sqrt[p^e]{x})).$$

We next describe the $V^\ell(x B_r W_{m-\ell} \Omega_F^n)$ images in $H_{p^m}^{n+1} F$.

Lemma 3.15. *Suppose $m > \ell \geq t$. Then for $x, s \in F$ we have*

$$V^\ell([x]_{m-\ell}[s]_{m-\ell}^{p^t-1}d[s]_{m-\ell}) \equiv p^{\ell-t} \cdot [xs^{p^t}]_m \frac{d[x]_m}{[x]_m} \pmod{\wp(W_m\Omega_F^n) + B_m W_m\Omega_F^n}.$$

In particular, whenever $r \leq m$,

$$\overline{V^\ell([x]_{m-\ell}B_r W_{m-\ell}\Omega_F^n)} \subseteq p^{m-r} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1}F$$

and for all $m \geq e$, $\overline{\mathcal{L}_{m,e}^{n+1}} \subseteq p^{m-e} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1}F$.

Proof. First we note that as $\ell \geq t$,

$$\begin{aligned} V^\ell([x]_{m-\ell}\Phi^t(d[s]_{m-\ell})) &= V^\ell([x]_{m-\ell}[s]_{m-\ell}^{p^t-1}d[s]_{m-\ell}) \\ &= V^\ell([xs^{p^t-1}]_{m-\ell})V^\ell(d[s]_{m-\ell}) \\ &= V^\ell([xs^{p^t-1}]_{m-\ell}[s]_{m-\ell}^{1-p^\ell})d[s]_m \\ &= V^\ell([xs^{p^t-1}s^{1-p^\ell}]_{m-\ell}[s]_{m-\ell}^{p^\ell})\frac{d[s]_m}{[s]_m} \\ &= V^\ell([xs^{p^t}]_{m-\ell})\frac{d[s]_m}{[s]_m}, \end{aligned}$$

where the second line follows using Lemma 3.1. Next we have

$$\begin{aligned} V^\ell([xs^{p^t}]_{m-\ell})\frac{d[s]_m}{[s]_m} &\equiv V^\ell([xs^{p^t}]_{m-\ell}^{p^\ell})\frac{d[s]_m}{[s]_m} \pmod{\wp(W_m\Omega_F^n) + B_m W_m\Omega_F^n} \\ &= p^\ell \cdot [xs^{p^t}]_m \frac{d[s]_m}{[s]_m} = p^{\ell-t} \cdot [xs^{p^t}]_m \frac{d[s^{p^t}]_m}{[s^{p^t}]_m} \\ &\equiv p^{\ell-t} \cdot [xs^{p^t}]_m \frac{d[x]_m}{[x]_m} \pmod{\wp(W_m\Omega_F^n) + B_m W_m\Omega_F^n}, \end{aligned}$$

where we used $\wp = \Phi - 1$ in the first line and recall that $B_m W_m\Omega_F^n = dW_m\Omega_F^{n-1}$ in the third. These show that if $m > \ell \geq t$ then

$$\overline{V^\ell([x]_{m-\ell}\Phi^t(d[s]_{m-\ell}))} \in p^{\ell-t} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^2F$$

and from this, $\overline{V^\ell([x]_{m-\ell}B_{t+(m-\ell)}W_{m-\ell}\Omega_F^n)} \subseteq p^{\ell-t} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1}F$ using the second statement of Theorem 3.8. As $m + (t - \ell) \leq m$, setting $r = t + m - \ell$ so $\ell - t = m - r$ this can be reformulated as

$$\overline{V^\ell([x]_{m-\ell}B_r W_{m-\ell}\Omega_F^n)} \subseteq p^{m-r} \cdot \overline{W_m\Omega_F^{n-1} \wedge d[x]_m} \subseteq H_{p^m}^{n+1}F$$

whenever $r \leq m$. Using the fact that $r \leq e$ in the definition of $\mathcal{L}_{m,e}^{n+1}$, the result just given shows when $m \geq e$ that $\overline{\mathcal{L}_{m,e}^{n+1}} \subseteq p^{m-e} \cdot \overline{W_e\Omega_F^{n-1} \wedge d[x]_e} \subseteq H_{p^e}^{n+1}F$. This proves the lemma. $\square$

This final lemma considers what happens when we pass to $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$. We remark that part (iii) in this lemma describes the case where special computations will be required to carry out the induction in the proof of the main theorem in the next section.

Lemma 3.16. *Suppose $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$ and $e \geq m > \ell$.*

(1) *Then $V^\ell(\mathcal{L}_{m-\ell,e}^{n+1}) \subseteq K_{m,e}^{n+1}$.*

(2) *Suppose $t \leq e + \ell - m$. Then*

$$i_{L/F} \left(\overline{V^\ell \left([x^i s^{p^t}]_{m-\ell} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right)} \right) = 0 \in H_{p^m}^2 L.$$

(3) *When $t = e + \ell - m + 1$ we have*

$$i_{L/F} \left(\overline{V^\ell \left([x^i s^{p^t}]_{m-\ell} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right)} \right) = -\overline{V^{m-1} \left(i x^i s^{p^t} \frac{d\alpha}{\alpha} \right)} \in H_{p^m}^2 L.$$

Proof. For (i) we have to show that $i_{L/F}(\overline{V^\ell([x]_{m-\ell}^i B_r W_{m-\ell} \Omega_F^n)}) = \{0\} \subseteq H_{p^m}^{n+1} L$ for $1 \leq r \leq e$. For $r \leq m$, $B_r W_m \Omega_F^n = dV^{m-r} W_r \Omega_F^{n-1}$ and therefore $[x]_m^i B_r W_m \Omega_F^n \subseteq dW_m \Omega_F^{n-1} + W_m \Omega_F^{n-1} \wedge d[x]_m$ which as $x \in L^{p^e}$ gives the result in this case. For $r \geq m$, by Theorem 3.8 the generators of $V^\ell([x]_{m-\ell}^i B_r W_{m-\ell} \Omega_F^n)$ are of the form

$$V^\ell \left([x^i]_{m-\ell} [s]_{m-\ell}^{p^t} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \wedge \text{dlog}_{m-\ell}(\tau) \right),$$

where $t \leq r - (m - \ell)$, $s \in F$ and $\tau \in K_{n-1}^M F$. From this, $\ell + e - t \geq \ell + e - (r - (m - \ell)) = e - r + m$, so as $e - r \geq 0$ we find $\ell + e - t \geq m$. We have the following in $H_{p^m}^2 L$ whenever $t < r \leq e$:

$$\begin{aligned} i_{L/F} \left(\overline{V^\ell \left([x^i s^{p^t}]_{m-\ell} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right)} \right) &= \overline{V^\ell \left([\alpha^{ip^e} s^{p^t}]_{m-\ell} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right)} \\ &= \overline{V^\ell \left([\alpha^{ip^{e-t}} s]_{m-\ell}^{p^t} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right)} \\ &= \overline{V^\ell \left([\alpha^{ip^{e-t}} s]_{m-\ell} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right)} \\ &= -ip^{e-t} \overline{V^\ell \left([\alpha^{ip^{e-t}} s]_{m-\ell} \frac{d[\alpha]_{m-\ell}}{[\alpha]_{m-\ell}} \right)}. \end{aligned}$$

This latter element is 0 as long as $\ell + e - t \geq m$. This gives the desired result when $r \geq m$. Part (ii) is a consequence of the latter computation.

For part (iii), when $t = e + \ell - m + 1$, as $m - \ell - 1 \geq 0$ we have by (ii)

$$\begin{aligned}
 i_{L/F} \left(\overline{V^\ell \left([x^i s^{p^t}]_{m-\ell} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right)} \right) &= -i p^{e-t} \overline{V^\ell \left([\alpha^i p^{e-t} s]_{m-\ell} \frac{d[\alpha]_{m-\ell}}{[\alpha]_{m-\ell}} \right)} \\
 &= -i p^{m-\ell-1} \overline{V^\ell \left([\alpha^i p^{m-\ell-1} s]_{m-\ell} \frac{d[\alpha]_{m-\ell}}{[\alpha]_{m-\ell}} \right)} \\
 &= -i \overline{V^{m-1} \left(\alpha^i p^{2(m-\ell-1)} s p^{m-\ell-1} \frac{d\alpha}{\alpha} \right)} \\
 &= -\overline{V^{m-1} \left(i \alpha^i p^e s p^{e+\ell-m+1} \frac{d\alpha}{\alpha} \right)} \\
 &= -\overline{V^{m-1} \left(i x^i s^{p^t} \frac{d\alpha}{\alpha} \right)}.
 \end{aligned}$$

This gives (iii). $\square$

4. The main result

We begin by restating Theorem 1.8 in the case of a simple extension. Although our notation is slightly different, this is the result from [Sobiech 2018, Theorem 3.16] and when $p = 2$ it is [Aravire et al. 2019, Theorem 3.1]. This formulation is convenient for the proof of Theorem 4.2.

Theorem 4.1 (Aravire–Laghribi–O’Ryan–Sobiech). *Suppose $e \geq 1$ and $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$. Suppose $\omega \in \Omega_F^n$ with $\bar{\omega} \in \ker : H_p^{n+1} F \rightarrow H_p^{n+1} L$. Then*

$$\omega \in \sum_{r=1}^e \sum_{\substack{i=1 \\ (i,p)=1}}^{p^{r-1}-1} x^i B_r \Omega_F^n + \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx.$$

We now give the main theorem. Set $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$ and $e \geq 1$. We recall

$$\begin{aligned}
 K_{m,e}^{n+1} &:= \ker(i_{L/F} : H_{p^m}^{n+1} F \rightarrow H_{p^m}^{n+1} L) \\
 \text{and } \mathcal{L}_{m,e}^{n+1} &:= \sum_{r=1}^e \sum_{i=1, (i,p)=1}^{p^r-1} [x]_m^i B_r W_m \Omega_F^n \subseteq W_m \Omega^n F.
 \end{aligned}$$

The result is that $K_{m,e}^{n+1} = \sum_{\ell=0}^{m-1} \overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})}$. By Lemma 3.16, $\overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})} \subseteq K_{m,e}^{n+1}$, so our task is to show that $K_{m,e}^{n+1} \subseteq \sum_{\ell=0}^{m-1} \overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})}$. The proof is inductive and the theorem is stated that way for the reader’s convenience. The idea behind the induction is to combine Theorem 4.1 with the exact sequence in Theorem 3.3. The subtlety in the argument is that one could proceed with the induction in a simplistic fashion, but then the value of the e in $V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})$ would grow by 1 at

each increment in m . This would yield a result, but not what is desired; in particular part (iv), the key to the result described in the introduction, would not be obtained.

Theorem 4.2. *Suppose $\omega \in W_m \Omega_F^n$ where $m, n \geq 1$. Set $L = F(\alpha)$ where $\alpha^{p^e} = x \in F - F^p$ and $e \geq 1$. Set $K_{m,e}^{n+1} := \ker(i_{L/F} : H_{p^m}^{n+1} F \rightarrow H_{p^m}^{n+1} L)$.*

- (i) *If $\bar{\omega} \in K_{m,e}^{n+1}$ then $\bar{\omega} \in \overline{\mathcal{L}_{m,e}^{n+1}} + V(K_{m-1,e}^{n+1})$.*
- (ii) *If $\bar{\omega} \in \overline{\mathcal{L}_{m,e}^{n+1}} + V(K_{m-1,e}^{n+1})$ then $\bar{\omega} \in \sum_{\ell=0}^{m-1} \overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})}$.*
- (iii) *For all $m \geq 1$, $K_{m,e}^{n+1} = \sum_{\ell=0}^{m-1} \overline{V^\ell(\mathcal{L}_{m-\ell,e}^{n+1})}$.*
- (iv) *When $m \geq e$ we have $K_{m,e}^{n+1} = p^{m-e} \cdot W_m \Omega_F^{n-1} \wedge d[x]_m \subseteq H_{p^m}^{n+1} F$.*
- (v) *For all $m, n \geq 1$,*

$$\ker(H_{p^m}^{n+1} F \rightarrow H_{p^m}^{n+1} L) = \ker(H_{p^m}^2 F \rightarrow H_{p^m}^2 L) \wedge \overline{\text{dlog}_m(K_{n-1}^M F)}.$$

Proof. The proof proceeds by induction on m . When $m = 1$, since there are no V terms parts (i), (ii) and (iii) follow by Theorem 4.1. When $m = 1$ part (iv) is Theorem 1.10. For $m > 1$ we start with part (i) and assume the theorem for $m - 1$. Suppose $\bar{\omega} \in K_{m,e}^{n+1}$. As $\overline{R^{m-1}(\omega)} = 0 \in H_{p^{m-1}}^{n+1} F$, by the $m - 1$ case part (iii) we know $R^{m-1}(\omega) \in \sum_{\ell=0}^{m-2} \overline{V^\ell(\mathcal{L}_{m-\ell-1,e}^{n+1})} + \wp(W_{m-1} \Omega_F^n) + dW_{m-1} \Omega_F^{n-1}$. By Theorem 3.8, for $r \geq m - \ell - 1$, if $t \leq r - (m - \ell - 1)$ and $s \in F$ we have $[s^{p^t}]_{m-\ell-1} [s]_{m-\ell-1}^{-1} d[s]_{m-\ell-1} \in B_r W_{m-\ell-1} \Omega_F^1$ and are a set of generators over $dW_m F$. We can lift such a generator to $[s^{p^t}]_{m-\ell} [s]_{m-\ell}^{-1} d[s]_{m-\ell}$ and note that as $t \leq r + 1 - (m - \ell)$ it is a generator for $B_{r+1} W_{m-\ell} \Omega_F^1$. So these lift to $W_{m-\ell} \Omega_F^n$ lie in $\sum_{r=1}^e \sum_{i=1}^{p^r-1} \sum_{(i,p)=1} [x]_{m-\ell}^i B_{r+1} W_{m-\ell} \Omega_F^n \subseteq \mathcal{L}_{m-\ell,e+1}^{n+1}$. We note the shift from e to $e + 1$ occurs because of the increase of r to $r + 1$ in the subscript of B_{r+1} (even though in the sum we have $r \leq e$.) The technical details of this proof involve dealing with this shift from e to $e + 1$ in the subscript of $\mathcal{L}_{m-\ell,e+1}^{n+1}$ that are a result of this lifting of the inductive hypothesis.

For the proof we will carefully select $\omega_0 \in W_m \Omega_F^n$ with $R^{m-1}(\omega_0) = R^{m-1}(\omega)$. For such ω_0 , by Theorem 3.3, there exist $\hat{\omega} \in \Omega_F^n$ and $\hat{\omega}' \in \Omega_F^{n-1}$ with

$$\omega = \omega_0 + V^{m-1}(\hat{\omega}) + dV^{m-1}(\hat{\omega}') \in W_m \Omega_F^n.$$

As $\overline{dV^{m-1}(\hat{\omega}')} = 0 \in H_{p^m}^{n+1} F$ we have

$$\bar{\omega} = \overline{\omega_0 + V^{m-1}(\hat{\omega}) + dV^{m-1}(\hat{\omega}')} = \overline{\omega_0 + V^{m-1}(\hat{\omega})} \in K_{m,e}^{n+1}.$$

By the previous paragraph

$$R^{m-1}(\omega_0) = R^{m-1}(\omega) \in \sum_{\ell=0}^{m-2} \overline{V^\ell(\mathcal{L}_{m-\ell-1,e}^{n+1})} + \wp(W_{m-1} \Omega_F^n) + dW_{m-1} \Omega_F^{n-1}$$

and we can choose the lifts of $\mathcal{L}_{m-\ell-1,e}^{n+1}$ elements to lie in

$$\sum_{r=1}^e \sum_{i=1, (i,p)=1}^{p^r-1} [x]_{m-\ell}^i B_{r+1} W_{m-\ell} \Omega_F^n \subseteq \mathcal{L}_{m-\ell,e+1}^{n+1}.$$

Pulling out the $r = e$ summand we decompose

$$\begin{aligned} & \sum_{r=1}^e \sum_{i=1, (i,p)=1}^{p^r-1} [x]_{m-\ell}^i B_{r+1} W_{m-\ell} \Omega_F^n \\ &= \sum_{i=1, (i,p)=1}^{p^e-1} [x]_{m-\ell}^i B_{e+1} W_{m-\ell} \Omega_F^n + \sum_{r=1}^{e-1} \sum_{i=1, (i,p)=1}^{p^r-1} [x]_{m-\ell}^i B_{r+1} W_{m-\ell} \Omega_F^n, \end{aligned}$$

where the second sum lies in $\mathcal{L}_{m-\ell,e}^{n+1}$. So we can choose ω_0 with

$$\omega_0 \in \sum_{\ell=0}^{m-2} V^\ell(\mathcal{L}_{m-\ell,e+1}^{n+1}) + \wp(W_m \Omega_F^n) + dW_m \Omega_F^{n-1},$$

where we can further decompose $\omega_0 = \omega'_0 + \omega''_0$ with $\omega'_0 = \sum_{\ell=0}^{m-2} V^\ell(\omega'_{0\ell})$ with carefully chosen $\omega'_{0\ell} \in \sum_{i=1, (i,p)=1}^{p^{e-1}} [x]_{m-\ell}^i B_{e+1} W_{m-\ell} \Omega_F^n$ (specified below) and where $\omega''_0 \in \sum_{\ell=0}^{m-2} V^\ell(\mathcal{L}_{m-\ell,e}^{n+1}) + \wp(W_m \Omega_F^n) + dW_m \Omega_F^{n-1}$.

We claim that if further ω'_0 is chosen in such a way that $\overline{\omega'_0} \in \overline{\mathcal{L}_{m,e}^{n+1}} + \overline{V(W_{m-1} \Omega_F^n)}$ part (i) will be proved. For as $\omega'_0 = \sum_{\ell=0}^{m-2} V^\ell(\omega'_{0\ell})$ we would then have $\overline{\omega'_0} \in \overline{\mathcal{L}_{m,e}^{n+1}} + \overline{V(W_{m-1} \Omega_F^n)}$. By the set up,

$$\omega = \omega_0 + V^{m-1}(\hat{\omega}) + dV^{m-1}(\hat{\omega}') = \omega'_0 + \omega''_0 + V^{m-1}(\hat{\omega}) + dV^{m-1}(\hat{\omega}'),$$

where $\omega''_0 \in \mathcal{L}_{m,e}^{n+1} + V(W_{m-1} \Omega_F^n)$. So given the claim we would have

$$\bar{\omega} \in \overline{\mathcal{L}_{m,e}^{n+1}} + \overline{V(W_{m-1} \Omega_F^n)}.$$

We express $\bar{\omega} = \bar{\omega}' + V(\bar{\omega}'')$, where $\omega' \in \mathcal{L}_{m,e}^{n+1}$ and $\omega'' \in W_{m-1} \Omega_F^n$. By Lemma 3.16 we know $\bar{\omega}' \in K_{m,e}^{n+1}$, so as $\bar{\omega} \in K_{m,e}^{n+1}$ we also have $V(\bar{\omega}'') \in K_{m,e}^{n+1}$. By Theorem 3.13, the injectivity of $V : H_p^{n+1} L \rightarrow H_p^{n+1} L$ shows that $\bar{\omega}'' \in K_{m-1,e}^{n+1}$. By induction we find $\bar{\omega}'' \in \overline{\mathcal{L}_{m-1,e}^{n+1}} + V(K_{m-2,e}^{n+1}) \subseteq K_{m-1,e}^{n+1}$. From this, $\bar{\omega} = \bar{\omega}' + V(\bar{\omega}'') \in \overline{\mathcal{L}_{m,e}^{n+1}} + V(K_{m-1,e}^{n+1})$ and (i) follows.

We next carefully choose ω_0 and ω'_0 to meet the requirements. According to Theorem 3.8, since we are working with $B_{e+1} W_{m-\ell} \Omega_F^n$ taken (mod $B_e W_{m-\ell} \Omega_F^n$), we know $\omega'_0 = \sum_{\ell=0}^{m-2} V^\ell(\omega'_{0\ell})$ where $\omega'_{0\ell} \in \sum_{i=1, (i,p)=1}^{p^{e-1}} [x]_{m-\ell}^i B_{e+1} W_{m-\ell} \Omega_F^n$ can be expanded as a sum

$$\omega'_{0\ell} := \sum_{i=1, (i,p)=1}^{p^{e-1}} [x]_{m-\ell}^i \left(\sum_j [s_{\ell ij}]_{m-\ell}^{p^{e+1-(m-\ell)}} \frac{d[s_{\ell ij}]_{m-\ell}}{[s_{\ell ij}]_{m-\ell}} \wedge \text{dlog}_{m-\ell}(\tau_{\ell ij}) \right)$$

for $s_{\ell ij} \in F$ and $\tau_{\ell ij} \in K_{n-1}^M F$, where the j depend upon the ℓ, i and their number may vary. For reference below, we note using the $\ell = 0$ terms, these definitions give

$$R^1(\omega'_0) = R^1(\omega'_{00}) = \sum_{i=1, (i,p)=1}^{p^e-1} x^i \left(\sum_j s_{0ij}^{p^{e+1-m}} \frac{ds_{0ij}}{s_{0ij}} \wedge \text{dlog}_1(\tau_{0ij}) \right).$$

Our goal is to show that $R^1(\omega'_0) \in \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx + \sum_{i=1}^{p^e-1} x^i B_e \Omega_F^n$, for then it will follow we can modify ω'_{00} so that $\overline{\omega'_{00}} \in \overline{\mathcal{L}_{m,e}^{n+1}} + \overline{V(W_{m-1}\Omega_F^n)}$ and we would have what we need. To do this we have to analyze

$$i_{L/F}(\overline{\omega'_0}) = \sum_{\ell=0}^{m-2} i_{L/F}(\overline{V^\ell(\omega'_{0\ell})}) \in H_{p^m}^{n+1} L$$

more closely, in particular each summand $i_{L/F}(\overline{V^\ell(\omega'_{0\ell})})$ including those with $\ell > 0$. Applying Lemma 3.16(iii) when $t = e + \ell - m + 1$ we have

$$\overline{i_{L/F} \left(V^\ell \left([x^i s^{p^t}]_{m-\ell} \frac{d[s]_{m-\ell}}{[s]_{m-\ell}} \right) \right)} = \overline{-V^{m-1} \left(i x^i s^{p^t} \frac{d\alpha}{\alpha} \right)} \in H_{p^m}^{n+1} F.$$

From this we find

$$\begin{aligned} \overline{i_{L/F}(\omega'_0)} &= \overline{\sum_{\ell=0}^{m-2} i_{L/F} \left(V^\ell \left(\sum_{\substack{i=1 \\ (i,p)=1}}^{p^e-1} [x]_{m-\ell}^i \left(\sum_j [s_{\ell ij}]_{m-\ell}^{p^{e+1-(m-\ell)}} \frac{d[s_{\ell ij}]_{m-\ell}}{[s_{\ell ij}]_{m-\ell}} \wedge \text{dlog}_{m-\ell}(\tau_{\ell ij}) \right) \right) \right)} \\ &= \overline{-V^{m-1} \left(\sum_{\ell=0}^{m-2} \sum_{\substack{i=1 \\ (i,p)=1}}^{p^e-1} i x^i \sum_j \left(s_{\ell ij}^{p^{e+1-(m-\ell)}} \frac{d\alpha}{\alpha} \wedge \text{dlog}_1(\tau_{\ell ij}) \right) \right)} \\ &= \overline{V^{m-1} \left(\left(\sum_{\ell=0}^{m-2} \hat{\omega}_\ell \right) \wedge \frac{d\alpha}{\alpha} \right)}, \end{aligned}$$

where using $\alpha^{-1} d\alpha \wedge \text{dlog}_1(\tau_{\ell ij}) = (-1)^{n-1} \text{dlog}_1(\tau_{\ell ij}) \wedge \alpha^{-1} d\alpha$ we define

$$\begin{aligned} \hat{\omega}_\ell &:= (-1)^n \sum_{i=1, (i,p)=1}^{p^e-1} i x^i \left(\sum_j (s_{\ell ij}^{p^{e+1-(m-\ell)}} \wedge \text{dlog}_1(\tau_{\ell ij})) \right) \\ &= (-1)^n \sum_{i=1, (i,p)=1}^{p^e-1} i x^i \Phi^{p^{e+1-(m-\ell)}}(\omega_{\ell i}). \end{aligned}$$

and we define $\omega_{\ell i} := \sum_j (s_{\ell ij} \wedge \text{dlog}_1(\tau_{\ell ij}))$. Altogether this means we have

$$\sum_{\ell=0}^{m-2} \hat{\omega}_\ell = (-1)^n \sum_{\ell=0}^{m-2} \sum_{\substack{i=1 \\ (i,p)=1}}^{p^e-1} i x^i \Phi^{p^{e+1-(m-\ell)}}(\omega_{\ell i}).$$

We have

$$0 = \overline{i_{L/F}(\omega)} = \overline{i_{L/F}(\omega_0)} + \overline{i_{L/F}(V^{m-1}(\hat{\omega}))} = \overline{i_{L/F}(\omega'_0)} + \overline{i_{L/F}(\omega''_0)} + \overline{i_{L/F}(V^{m-1}(\hat{\omega}))}.$$

By Lemma 3.16(ii) we have $\overline{i_{L/F}(\omega''_0)} = 0$ so we find $\overline{i_{L/F}(\omega'_0)} + \overline{i_{L/F}(V^{m-1}(\hat{\omega}))} = 0$. Combining this with the previous shows

$$V^{m-1} \left(\overline{i_{L/F} \left(\sum_{\ell=0}^{m-2} \hat{\omega}_\ell \right) \wedge \frac{d\alpha}{\alpha}} \right) + V^{m-1}(\overline{i_{L/F}(\hat{\omega})}) = 0.$$

The injectivity of $V^{m-1} : H_p^{n+1} L \rightarrow H_{p^m}^{n+1} L$ gives

$$\overline{i_{L/F}(\hat{\omega}) + i_{L/F} \left(\sum_{\ell=0}^{m-2} \hat{\omega}_\ell \right) \wedge \frac{d\alpha}{\alpha}} = 0 \in H_p^{n+1} L.$$

From this we have

$$i_{L/F}(\hat{\omega}) + i_{L/F} \left(\sum_{\ell=0}^{m-2} \hat{\omega}_\ell \right) \wedge \frac{d\alpha}{\alpha} \in \wp(\Omega_L^n) + d\Omega_L^{n-1}.$$

We know

$$L = F(\alpha) = \bigoplus_{i=0}^{p^e-1} \alpha^i F \quad \text{and} \quad \Omega_L^n = L \cdot \Omega_F^n \oplus L \cdot \Omega_F^{n-1} \wedge \frac{d\alpha}{\alpha}.$$

Expanding $i_{L/F}(\hat{\omega}) + i_{L/F} \left(\sum_{\ell=0}^{m-2} \hat{\omega}_\ell \right) \wedge \alpha^{-1} d\alpha = \wp(\eta) + d\theta \in \Omega_L^n$, where $\eta = \sum_{i=0}^{p^e-1} \alpha^i \eta_i + \sum_{i=0}^{p^e-1} \alpha^i \eta'_i \wedge \alpha^{-1} d\alpha$ and $\theta = \sum_{i=0}^{p^e-1} \alpha^i \theta_i + \sum_{i=0}^{p^e-1} \alpha^i \theta'_i \wedge \alpha^{-1} d\alpha$ for $\eta_i \in \Omega_F^n$, $\theta_i, \eta'_i \in \Omega_F^{n-1}$ and $\theta'_i \in \Omega_F^{n-2}$ gives the $2p^e$ equations, one for each summand in $\Omega_L^n = (\bigoplus_{i=0}^{p^e-1} \alpha^i F \cdot \Omega_F^n) \oplus (\bigoplus_{i=0}^{p^e-1} \alpha^i F \cdot \Omega_F^{n-1} \wedge \alpha^{-1} d\alpha)$:

$$\begin{aligned} i_{L/F}(\hat{\omega}) &= \wp(\eta_0) + \sum_{k=1}^{p-1} x^k \Phi(\eta_{kp^{e-1}}) + d\theta_0, \\ 0 &= \alpha^{ip^{e-1}} \left(-\eta_{ip^{e-1}} + d\theta_{ip^{e-1}} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{ip^{e-2}+kp^{e-1}}) \right), \quad 1 \leq i \leq p-1, \\ (*_i) \quad &\vdots \\ 0 &= \alpha^{qp} \left(-\eta_{qp} + d\theta_{qp} + \sum_{k=0}^{p-1} x^k \Phi(\eta_{q+kp^{e-1}}) \right), \quad 1 \leq q \leq p^{e-1}-1, \quad (q, p) = 1, \\ 0 &= \alpha^j (-\eta_j + d\theta_j), \quad 1 \leq j \leq p^e-1, \quad (j, p) = 1, \end{aligned}$$

and

$$\begin{aligned}
i_{L/F} \left(\sum_{\ell=0}^{m-2} \hat{\omega}_\ell \right) \wedge \frac{d\alpha}{\alpha} &= \left(\wp(\eta'_0) + \sum_{k=1}^{p-1} x^k \Phi(\eta'_{kp^{e-1}}) + d\theta'_0 \right) \wedge \frac{d\alpha}{\alpha}, \\
0 &= \alpha^{ip^{e-1}} \left(-\eta'_{ip^{e-1}} + d\theta'_{ip^{e-1}} + \sum_{k=0}^{p-1} x^k \Phi(\eta'_{ip^{e-2}+kp^{e-1}}) \right) \wedge \frac{d\alpha}{\alpha}, \quad 1 \leq i \leq p-1, \\
(*_{ii}) \quad &\vdots \\
0 &= \alpha^{qp} \left(-\eta'_{qp} + d\theta'_{qp} + \sum_{k=0}^{p-1} x^k \Phi(\eta'_{q+kp}) \right) \wedge \frac{d\alpha}{\alpha}, \quad 1 \leq q \leq p^{e-1}-1, (q, p) = 1, \\
0 &= \alpha^j \left(-\eta'_j + d\theta'_j + j\theta_j \right) \wedge \frac{d\alpha}{\alpha}, \quad 1 \leq j \leq p^e-1, (j, p) = 1.
\end{aligned}$$

We note that in using each equation we keep in mind that $\ker(i_{L/F} : \Omega_F^n \rightarrow \Omega_L^n) = \Omega_F^{n-1} \wedge dx$.

Using the equations in $(*_{ii})$ collectively, starting from the bottom and iteratively solving for each η'_j in terms of the $d\theta'_i$ and the θ_j for $(j, p) = 1$, and substituting to find $\sum_{\ell=0}^{m-2} \hat{\omega}_\ell$, gives

$$\sum_{\ell=0}^{m-2} \hat{\omega}_\ell \in \wp(\Omega_F^{n-1}) + d\Omega_F^{n-2} + \Omega_F^{n-2} \wedge dx + \sum_{i=1}^{p^e-1} x^i B_e \Omega_F^{n-1} + \sum_{\substack{j=1 \\ (j,p)=1}}^{p^{e-1}-1} x^j \Phi^e(\Omega_F^{n-1}).$$

By the above,

$$\sum_{\ell=0}^{m-2} \hat{\omega}_\ell = (-1)^n \sum_{\ell=0}^{m-2} \sum_{\substack{i=1 \\ (i,p)=1}}^{p^e-1} i x^i \Phi^{p^{e+1}-(m-\ell)}(\omega_{\ell i}).$$

By Corollary 2.3 we have for the minimum $e+1-(m-\ell)$ -value of $e+1-m < e$ when $\ell = 0$,

$$\sum_{i=1, (i,p)=1}^{p^e-1} i^{-1} i x^i \Phi^{p^{e+1}-m}(d\omega_{0i}) \in \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx + \sum_{i=1}^{p^e-1} x^i B_e \Omega_F^n.$$

We note that $\Phi^{p^{e+1}-m}(d\omega_{0i}) = \sum_j s_{rij}^{p^{e+1}-m} s_{0ij}^{-1} ds_{0ij} \wedge d\log_1(\tau_{0ij})$. This gives

$$\begin{aligned}
R^1(\omega'_{00}) &= \sum_{i=1, (i,p)=1}^{p^e-1} x^i \left(\sum_j s_{0ij}^{p^{e+1}-m} \frac{ds_{0ij}}{s_{0ij}} \wedge d\log_1(\tau_{0ij}) \right) \\
&= \sum_{\substack{i=1 \\ (i,p)=1}}^{p^e-1} x^i \Phi^{p^{e+1}-m}(d\omega_{0i}) \in \wp(\Omega_F^n) + d\Omega_F^{n-1} + \Omega_F^{n-1} \wedge dx + \sum_{i=1}^{p^e-1} x^i B_e \Omega_F^n.
\end{aligned}$$

As $\sum_{i=1}^{p^e-1} x^i B_e \Omega_F^n \subseteq R^1(\mathcal{L}_{m,e}^{n+1})$ we can express $\overline{\omega'_0} = \bar{\chi} + \overline{V(\hat{\omega}'_{00})} \in H_{p^{m-\ell}}^{n+1} F$, where $\bar{\chi} \in \overline{\mathcal{L}_{m,e}^{n+1}}$. From this we can select

$$\overline{\omega'_0} = \sum_{\ell=0}^{m-2} \overline{V^\ell(\omega'_{0\ell})} = \bar{\chi} + \overline{V(\hat{\omega}'_{00})} + \sum_{\ell=1}^{m-2} \overline{V^\ell(\hat{\omega}'_{0\ell})} \in \overline{\mathcal{L}_{m,e}^{n+1}} + \overline{V(W_{m-1}\Omega_F^n)},$$

giving the above claim required to prove (i).

For (ii) we have $\omega \in W_m \Omega_F^n$ with $\bar{\omega} \in K_{m,e}^{n+1}$ and $\bar{\omega} \in \overline{\mathcal{L}_{m,e}^{n+1}} + \overline{V(K_{m-1,e}^{n+1})}$. By induction using part (iii) for $m-1$ we have $K_{m-1,e}^{n+1} = \sum_{\ell=0}^{m-2} \overline{V^\ell(\mathcal{L}_{m-1-\ell,e}^{n+1})}$. Part (ii) follows combining these. Part (iii) is immediate by parts (i) and (ii). For (iv), when $m \geq e$, by Lemma 3.15 we have $\mathcal{L}_{m,e}^{n+1} \subseteq p^{m-e} \cdot W_m \Omega_F^{n-1} \wedge d[x]_m \subseteq H_{p^m}^{n+1} F$. Part (iv) of the theorem now follows from part (iii).

For (v), when $r \leq m$ we have

$$[x]_m^i B_r W_m \Omega_F^n = [x]_m^i dV^{m-r}(W_r \Omega_F^{n-1}) \subseteq W_m \Omega_F^{n-1} F \wedge d[x]_m + dW_m \Omega_F^{n-1},$$

and when $r > m$ by Theorem 3.8 we have

$$B_r W_m \Omega_F^n = B_r W_m \Omega_F^1 \wedge \text{dlog}_m(K_{n-1}^M F) + B_m W_m \Omega_F^n.$$

As $\mathcal{L}_{m,e}^{n+1} := \sum_{r=1}^e \sum_{i=1, (i,p)=1}^{p^r-1} [x]_m^i B_r W_m \Omega_F^n \subseteq W_m \Omega_F^n F$, the result follows by part (iii). $\square$

As noted in the introduction, using Theorem 4.2 one obtains a characterization of the kernel of purely inseparable modular extensions when $n+1=2$. A direct generalization of this result is not possible for $n+1 \geq 3$ (see Remark 4.4 below).

Corollary 4.3. *Suppose $x_1, x_2, \dots, x_s \in F$ are p -independent and set*

$$L = F(\alpha_1, \alpha_2, \dots, \alpha_s), \quad \text{where } \alpha_i^{p^{r_i}} = x_i.$$

Set $\tilde{m} := \max\{r_i \mid 1 \leq i \leq s\}$ and define $L_i := F(\alpha_i)$. If $m \geq \tilde{m}$ then

$$\ker(H_{p^m}^2 F \rightarrow H_{p^m}^2 L) = \sum_{i=1}^s p^{m-r_i} \cdot \overline{W_m F \cdot d[x_i]_m} = \sum_{i=1}^s \ker(H_{p^m}^2 F \rightarrow H_{p^m}^2 L_i).$$

Proof. The proof proceeds by induction on s , where the case $s=1$ follows from Theorem 4.2. By the $s-1$ case,

$$\ker(H_{p^m}^2 L_1 \rightarrow H_{p^m}^2 L) = \sum_{i=2}^s p^{m-r_i} \cdot \overline{W_m L_1 \wedge d[x_i]_m}.$$

So if $\omega \in W_m \Omega_F^1$ and $i_{L/F}(\bar{\omega}) = 0$ then we can express

$$i_{L_1/F}(\bar{\omega}) = \sum_{i=2}^s p^{m-r_i} \cdot \overline{\psi_i \wedge \frac{d[x_i]_m}{[x_i]_m}}, \quad \text{where } \psi_i \in W_m L_1.$$

Since $L_1 = F(\alpha_1)$ and $\alpha_1^{p^{r_1}} \in F$ we know that $\Phi^{r_1}(\psi_i) \in W_m F$. As

$$\overline{\psi_i \wedge \frac{d[x_i]_m}{[x_i]_m}} = \overline{\Phi^{r_1}(\psi_i) \wedge \frac{d[x_i]_m}{[x_i]_m}} \in H_{p^m}^2 L_1,$$

we find

$$\overline{\omega_1} := \overline{\omega} - \overline{\sum_{i=2}^s p^{m-r_i} \cdot \Phi^{r_1}(\psi_i) \wedge \frac{d[x_i]_m}{[x_i]_m}} \in \ker(H_{p^m}^2 F \rightarrow H_{p^m}^2 L_1).$$

By the $s = 1$ case we have $\overline{\omega_1} \in p^{m-r_1} \cdot \overline{W_m \Omega_F^{n-1} \wedge d[x_1]_m}$ and the first equation follows. The second equation is an immediate consequence of the first in view of the $s = 1$ case. $\square$

Remark 4.4. (i) The result in Corollary 4.3(i) coincides with the cohomological interpretation of Albert's work on p -algebras, see for example [Albert 1939, Theorem 28, p. 108] or [Jacobson 1996, 4.2.16, p. 162].

(ii) Although Theorem 4.2 is valid for all m and n , it is not possible to generalize Corollary 4.3 directly when $m \geq 2$ and $n \geq 3$. For example,

$$\overline{W_2 F \cdot d[x]_2 \wedge d[y]_2} \subseteq \ker(H_{p^2}^3 F \rightarrow H_{p^2}^3 F(\sqrt[p]{x}, \sqrt[p]{y}))$$

and it is possible for an element in $\overline{W_2 F \cdot d[x]_2 \wedge d[y]_2}$ to have order p^2 . However, all elements in

$$\ker(H_{p^2}^3 F \rightarrow H_{p^2}^3 F(\sqrt[p]{x})) + \ker(H_{p^2}^3 F \rightarrow H_{p^2}^3 F(\sqrt[p]{y}))$$

have order p . Hence the result in Corollary 4.3(i) cannot be generalized without taking into account additional elements of this type. There are other complexities as well. The characterization of these kernels is the subject of ongoing work.

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ROBERTO ARAVIRE
 INSTITUTO DE CIENCIAS EXACTAS Y NATURALES, FACULTAD DE CIENCIAS
 UNIVERSIDAD ARTURO PRAT
 IQUIQUE
 CHILE
 raravire@unap.cl

BILL JACOB
 MATHEMATICS DEPARTMENT
 UNIVERSITY OF CALIFORNIA, SANTA BARBARA
 SANTA BARBARA, CA
 UNITED STATES
 jacob@math.ucsb.edu

MANUEL O’RYAN
 INSTITUTO DE MATEMATICA Y FISICA
 UNIVERSIDAD DE TALCA
 AVENIDA LIRCAY S/N
 TALCA
 CHILE
 moryan@inst-mat.otalca.cl

KUPERBERG AND TURAEV–VIRO INVARIANTS IN UNIMODULAR CATEGORIES

FRANCESCO COSTANTINO, NATHAN GEER,
BERTRAND PATUREAU-MIRAND AND VLADIMIR TURAEV

We give a categorical setting in which Penrose graphical calculus naturally extends to graphs drawn on the boundary of a handlebody. We use it to introduce invariants of 3-manifolds presented by Heegaard splittings. We recover Kuperberg invariants when the category comes from an involutory Hopf algebra and Turaev–Viro invariants when the category is semisimple and spherical.

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1. Introduction

A remarkable achievement of low-dimensional topology in the last 30 years was a discovery of deep relations between topology and the theory of monoidal (tensor) categories. This development was initiated by V. Jones' introduction of his famous knot polynomial; by now it encompasses many aspects of low-dimensional topology including 3-manifold invariants, representations of mapping class groups of surfaces, topological quantum field theories in dimensions 2 and 3, etc. In particular, it was shown that monoidal categories satisfying certain conditions and carrying appropriate additional structures give rise to topological invariants of 3-dimensional manifolds, see [Turaev 1994; Turaev and Virelizier 2017]. This has instigated extensive research in the theory of monoidal categories aiming at construction (and eventually classification) of monoidal categories with required properties. At the

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same time, this development has provoked an appearance of parallel approaches not involving monoidal categories but using related algebraic objects. One such approach is due to G. Kuperberg [1991] who derived invariants of 3-manifolds from involutory Hopf algebras. The initial aim of this paper was to recover Kuperberg's invariants in terms of monoidal categories. To this end we introduce here a new construction of 3-manifold invariants from monoidal categories. We show that our method produces both the Kuperberg invariants and the standard Turaev–Viro invariants. Other generalizations of Kuperberg invariants of 3-manifolds were considered by Kashaev and Virelizier [2019].

The first main result of our paper is that in a general categorical setting there exists an invariant of graphs on the boundary of a handlebody.

With some additional categorical structure we show that this invariant extends to an invariant of bichrome handlebody graphs which are graphs on the boundary of a handlebody which are the union of two subgraphs: blue and red. Here the red curves have the nice property: any edge of the graph can be slid over a red curve. This is used when we define a 3-manifold invariant. In particular, we use Heegaard decompositions of 3-manifolds which are unions of two handlebodies glued along their common boundary. Such a decomposition is encoded by a set of disjoint simple closed curves on the boundary of a handlebody. These curves form a complete set of meridians for the second handlebody and are drawn in red.

On the algebraic side, we use two main tools — the so-called modified traces (m-traces) and the chromatic morphisms. The m-traces generalize the usual trace of endomorphisms of objects of a monoidal category to situations where the standard trace is not defined. The m-traces first appeared in [Geer et al. 2011a; 2013a; 2013b] and have been successfully used to produce 3-manifold invariants, see for example [Beliakova et al. 2018b; Costantino et al. 2014; Geer et al. 2011b]. The chromatic morphisms are introduced here.

2. Statements of main results and open problems

2A. The invariant F' . We first introduce the notation used in the main statements, for more details see Section 3. Let $\mathcal{C}$ be a pivotal $\mathbb{k}$ -category, where $\mathbb{k}$ is a field. Let F be the Penrose functor (defined using the Penrose graphical calculus) from the category of planar $\mathcal{C}$ -colored ribbon graphs to $\mathcal{C}$, see for example [Geer et al. 2013b]. Finally, let t be a modified trace (or m-trace for short) on an ideal $\mathcal{I}$ in $\mathcal{C}$. Then t induces an invariant of $\mathcal{I}$ -colored spherical graphs denoted by F' , see [Geer et al. 2013b]. This invariant can be computed by composing t and F on a cutting presentation of the graph (see Equation (8)).

We assume the m -trace is *nondegenerate*, i.e., for any object $P \in \mathcal{I}$, the pairing:

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P) \times \mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1}) \rightarrow \mathbb{k} \quad \text{given by} \quad (x, y) \mapsto t_P(xy)$$

is nondegenerate. Given a basis $\{x_i\}$ of $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P)$ let $\{y_i\}$ be the dual basis of $\mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1})$ for this pairing. Let

$$(1) \quad \Omega_P \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P) \otimes_{\mathbb{k}} \mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1}) \quad \text{be given by} \quad \Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i,$$

where $\otimes_{\mathbb{k}}$ is the standard tensor product of vector spaces over $\mathbb{k}$. The following proposition is the first item of Proposition 4.2.

Proposition 2.1. *The element Ω_P is independent of the choice of the basis of $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P)$.*

By a *multihandlebody* we will mean a disjoint union of a finite number of oriented handlebodies. A $\mathcal{C}$ -*colored ribbon graph* on a multihandlebody is a graph on the boundary of the multihandlebody such that each edge is colored with an object of $\mathcal{C}$ and each vertex is thickened to a coupon colored with a morphism of $\mathcal{C}$. All coupons have a top and a bottom sides which in our pictures will be the horizontal sides of the coupons. Since our graphs are drawn on a surface they have a natural framing and therefore can be considered as ribbon graphs in the usual sense. When all the colors of such a $\mathcal{C}$ -colored ribbon graph are in the ideal $\mathcal{I}$ we say that the graph is $\mathcal{I}$ -*colored*. Let

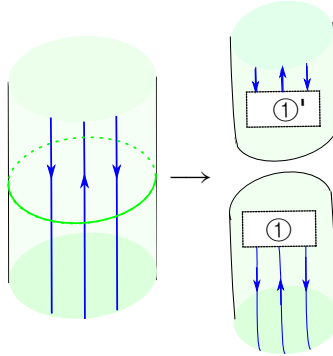
$$\mathcal{H}_{\mathcal{I}} = \{(H, \Gamma) : \Gamma \text{ is a nonempty } \mathcal{I}\text{-colored graph on a multihandlebody } H\}.$$

In what follows we extend the colorings of coupons multilinearly. In particular, we can color an ordered matching pair of coupons with $\Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i$ and we represent such a coupon with the adjacent edges by the figure

$$(2) \quad \begin{array}{c} \text{Diagram 1: A coupon box labeled } \mathbb{1}' \text{ with four blue lines (two entering from the top, two exiting from the bottom) passing through it.} \end{array} = \sum_i \begin{array}{c} \text{Diagram 2: Two adjacent coupon boxes, the left one labeled } x_i \text{ and the right one labeled } y_i, \text{ with four blue lines (two entering from the top, two exiting from the bottom) passing through them.} \end{array}$$

To state our first theorem we define a cutting operation on colored graphs. Let $(H, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ and D be a disk in H bounded by a simple oriented curve $\partial D \subset \partial H$ which intersects the edges of Γ nontrivially and transversely without meeting the coupons of Γ . Cutting (H, Γ) along D we obtain a new multihandlebody graph $(\mathrm{cut}_D(H), \mathrm{cut}_D(\Gamma)) \in \mathcal{H}_{\mathcal{I}}$, where $\mathrm{cut}_D(\Gamma)$ is obtained by cutting the edges of Γ intersecting ∂D and then joining the cut points into two new coupons in $\partial(\mathrm{cut}_D(H))$ (one on each side of the cut). The coupons are colored as in Equation (2), see the

following figure:



Note that H and $\text{cut}_D(H)$ can have different numbers of connected components and the orientation of $\text{cut}_D(H)$ is induced by the one of H .

Remark, if $(B^3, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ then Γ is a $\mathcal{I}$ -colored spherical graph in the domain of the invariant F' defined in Equation (8).

The following theorem (proved in Section 4) extends F' to the full set $\mathcal{H}_{\mathcal{I}}$.

Theorem 2.2. *Let $\mathcal{C}$ be a pivotal $\mathbb{k}$ -category equipped with an ideal $\mathcal{I}$ in $\mathcal{C}$ and a nondegenerate m -trace on $\mathcal{I}$. Then there exists a unique mapping*

$$F' : \mathcal{H}_{\mathcal{I}} \rightarrow \mathbb{k}$$

satisfying the following four properties.

(1) Invariance. *The element $F'(H, \Gamma)$ of $\mathbb{k}$ depends only on the orientation preserving diffeomorphism class of the pair $(H, \Gamma) \in \mathcal{H}_{\mathcal{I}}$.*

(2) Extension of F' . *For any $\mathcal{I}$ -colored ribbon graph (B^3, Γ) in the 3-ball B^3 , we have*

$$F'(B^3, \Gamma) = F'(\Gamma).$$

(3) Disjoint union of multihandlebodies. *For any $(H_1, \Gamma_1), (H_2, \Gamma_2) \in \mathcal{H}_{\mathcal{I}}$ we have*

$$F'(H_1 \sqcup H_2, \Gamma_1 \sqcup \Gamma_2) = F'(H_1, \Gamma_1) F'(H_2, \Gamma_2).$$

(4) Cutting along a disk. *Cutting any $(H, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ along a disk D as described before the statement of the theorem, we always have $F'(\text{cut}_D(H), \text{cut}_D(\Gamma)) = F'(H, \Gamma)$.*

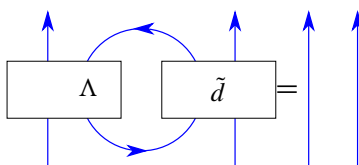
The hypotheses of the above theorem are quite mild. Indeed, by Theorem 5.5 of [Geer et al. 2018] they are satisfied in all the following examples (see [Geer et al. 2013a]): representations of factorizable ribbon Hopf algebras, finite groups and their quantum doubles, Lie (super)algebras, the $(1, p)$ minimal model in conformal field theory, and quantum groups at a root of unity.

Following the ideas of [De Renzi et al. 2018], we extend the invariant of multihandlebody graphs F' to so-called bichrome multihandlebody graphs as follows. A *bichrome handlebody graph* is a graph on the boundary of a multihandlebody which is split as a disjoint union of two subgraphs: blue and red. The blue subgraph is a $\mathcal{C}$ -colored graph on the multihandlebody. The red subgraph is a collection of disjoint simple closed unoriented curves (which are not required to be $\mathcal{C}$ -colored). We refer to these curves as *red circles*. We say a bichrome handlebody graph is *admissible* if its blue subgraph is $\mathcal{I}$ -colored and meets each connected component of the multihandlebody.

Definition 2.3. Let G be an object of $\mathcal{I}$. Set $\Lambda = \sum_i x_i y_i \in \text{End}_{\mathcal{C}}(G \otimes G^*)$, where $\Omega_{G \otimes G^*} = \sum x_i \otimes_{\mathbb{K}} y_i$. A *chromatic morphism* for G is a morphism $\tilde{d} : G \otimes G \rightarrow G \otimes G$ such that

$$(3) \quad (\text{Id}_G \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_G) \circ (\Lambda \otimes \tilde{d}) \circ (\text{Id}_G \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_G) = \text{Id}_{G \otimes G}.$$

This equation is represented pictorially as



where all blue strands are colored by G .

We use the word chromatic here because the morphism $\tilde{d}$ is used to change a red circle into a blue graph, as explained in Theorem 2.4. In the case of Hopf algebras this corresponds to the evaluation of the integral on the red circles (see Section 6).

A *generator* of an ideal $\mathcal{I}$ is an object $G \in \mathcal{I}$ such that for any $P \in \mathcal{I}$ there exists morphisms $f_j : G \rightarrow P$ and $g_j : P \rightarrow G$ indexed by a finite set J such that

$$(4) \quad \text{Id}_P = \sum_{j \in J} f_j g_j.$$

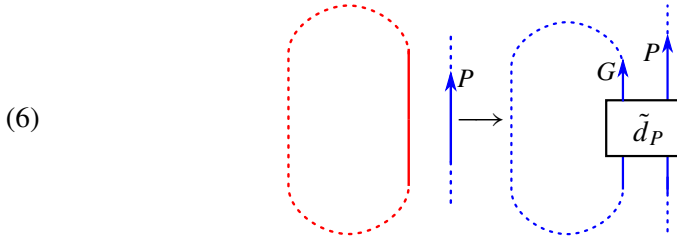
Remark there are many different notions of a generator of an ideal. Here we use the word generator because in the case of Hopf algebras G will be the projective generator.

Let $\tilde{d}$ be a chromatic morphism for a generator G of an ideal $\mathcal{I}$. If $P \in \mathcal{I}$ and $\text{Id}_P = \sum_j f_j g_j$ as in Equation (4) we define

$$(5) \quad \tilde{d}_P = \sum_{j \in J} (\text{Id}_G \otimes f_j) \tilde{d} (\text{Id}_G \otimes g_j) : G \otimes P \rightarrow G \otimes P.$$

We prove the following theorem in Section 5C.

Theorem 2.4. *Let $\mathcal{C}$ be a pivotal $\mathbb{k}$ -category equipped with a nondegenerate m -trace on an ideal $\mathcal{I}$ and a chromatic morphism $\tilde{d}$ on a generator G . Then there exist a unique extension of F' to admissible bichrome handlebody graphs which is preserved under the following transformation making a red circle blue in the presence of a nearby blue edge colored with an object P as shown in the figure*



Moreover, if (H, Γ) is a bichrome handlebody graph then $F'(H, \Gamma)$ only depends on the orientation preserving diffeomorphism class of (H, Γ) .

Later we will see that F' is also invariant under sliding an edge of Γ over a red circle, see Proposition 5.4.

2B. The invariant $\mathcal{K}$. Let $\mathcal{C}$ be a pivotal $\mathbb{k}$ -category equipped with a nondegenerate m -trace t on an ideal $\mathcal{I}$ and a chromatic morphism on a generator G . Since t is nondegenerate, there exists a morphism $h : G \rightarrow G$ such that $t_G(h) \neq 0$. By renormalizing the m -trace we can assume $t_G(h) = 1$. Let $\mathbf{O}_G$ be the ribbon graph in $\mathbb{R}^2$ formed by the braid closure of the coupon filled with h . Consider the bichrome handlebody graph $(B^3, \mathbf{O}_G)$ where $\mathbf{O}_G$ is blue and viewed as a graph on the boundary of B^3 . By definition, one has $F'(B^3, \mathbf{O}_G) = F'(\mathbf{O}_G) = t_G(h) = 1$.

Let M be a closed connected orientable 3-manifold. Next we use some standard topological definitions, see Section 5 for more details. A *Heegaard diagram* of M is a prescription for a Heegaard splitting $M = H_\alpha \cup_\Sigma H_\beta$ determined by upper and lower reducing sets of bounding circles $\{\alpha_i\}$ and $\{\beta_i\}$ in $\Sigma = \partial H_\alpha = \partial H_\beta$. A Heegaard diagram determines an admissible bichrome handlebody graph on H_α where the red subgraph is the set of circles $\{\beta_i\}$ (on the boundary of H_α) and the blue subgraph is $\mathbf{O}_G$ embedded in a small disk in ∂H_α . We call such a handlebody graph a *bichrome diagram* for M . See Figure 1 for an example.

Next we state the main theorem of this paper which will be proved in Section 5C.

Theorem 2.5. *If (H, Γ) is a bichrome diagram for a closed connected orientable 3-manifold M , then $F'(H, \Gamma)$ only depends of the diffeomorphism class of M .*

We denote the invariant of Theorem 2.5 by $\mathcal{K}_{\mathcal{C}}(M)$. Let us now discuss some examples.

2C. Hopf algebras and Kuperberg's invariant. We use standard terminology of Hopf algebras, for details see Section 6. Let A be a finite dimensional unibalanced

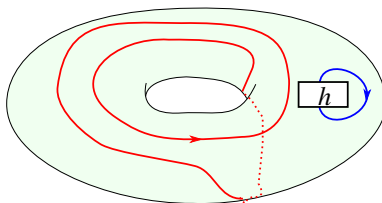


Figure 1. Bichrome diagram for the lens space $L(2, 1)$.

unimodular pivotal Hopf algebra. Let $A\text{-mod}$ be the category of finite dimensional modules over A . Let Proj be the ideal of projective objects in $A\text{-mod}$. The Hopf algebra A itself with its left regular representation is a generator of Proj .

In Section 6 we prove the following theorem:

Theorem 2.6. *There exists a nondegenerate m -trace on Proj in $A\text{-mod}$ and a chromatic morphism $\tilde{d}$ on the generator A .*

Let us discuss the hypothesis on A briefly. The existence of the chromatic morphism comes from the theory of (co)integrals in Hopf algebras. The requirement that A is pivotal implies that the category $A\text{-mod}$ is pivotal. The unimodularity of A ensures that the ideal Proj has a nondegenerate right m -trace. That A is unibalanced implies that this m -trace is also a left m -trace, see [Beliakova et al. 2018a].

Theorems 2.5 and 2.6 yield an invariant $\mathcal{K}_{A\text{-mod}}$. From Lemmas 6.2 and 6.3, the chromatic morphism is essentially determined by the integral $\lambda : A \rightarrow \mathbb{k}$ and cutting along a bounding circle is determined by the cointegral $\Lambda \in A$, where $\lambda(\Lambda) = 1$.

In Section 6C we will prove the following:

Theorem 2.7. *If A is involutive (the square of the antipode is the identity map) then*

$$\mathcal{K}_{A\text{-mod}}(M) = \text{Ku}_A(M),$$

where $\text{Ku}_A(M)$ is the Kuperberg invariant associated to A (see [Kuperberg 1991]).

2D. Turaev–Viro invariant. Let $\mathcal{C}$ be a finite semisimple spherical $\mathbb{k}$ -category (see [Barrett and Westbury 1999]). Here we claim that such a category satisfies the hypothesis of Theorem 2.5. If the dimension of $\mathcal{C}$ (see Equation (7)) is not zero in $\mathbb{k}$, the resulting 3-manifold invariant $\mathcal{K}_{\mathcal{C}}$ is the Turaev–Viro invariant $\text{TV}_{\mathcal{C}}$ [Turaev and Viro 1992; Barrett and Westbury 1996] associated to $\mathcal{C}$.

Let $\{S_i\}_{i \in I}$ be a set of representatives of the isomorphism classes of simple objects of $\mathcal{C}$. Then $G = \bigoplus_{i \in I} S_i$ is a generator. By definition of a semisimple spherical $\mathbb{k}$ -category, the quantum trace $\text{t} = \text{qTr}_{\mathcal{C}}$ is a nondegenerate m -trace on $\mathcal{I} = \mathcal{C}$. Then it follows that

$$\left\{ x_i = \frac{1}{\text{qdim}(S_i)} \overleftarrow{\text{coev}}_{S_i} \right\}_{i \in I} \quad \text{and} \quad \{ y_i = \overrightarrow{\text{ev}}_{S_i} \}_{i \in I}$$

are dual bases of $\text{Hom}_{\mathcal{C}}(\mathbb{1}, G \otimes G^*)$ and $\text{Hom}_{\mathcal{C}}(G \otimes G^*, \mathbb{1})$, respectively. Using the expansion $\Omega_{G \otimes G^*} = \sum_{i \in I} x_i \otimes_{\mathbb{k}} y_i$, it is straightforward to check that

$$\tilde{d} = \sum_{i \in I} \text{qdim}(S_i) \text{Id}_{S_i} \otimes \text{Id}_G$$

is a chromatic morphism for G .

In [Barrett and Westbury 1996], the Turaev–Viro invariant $\text{TV}_{\mathcal{C}}$ is generalized to the context of a finite semisimple spherical $\mathbb{k}$ -category $\mathcal{C}$ when the following scalar D , called the dimension of $\mathcal{C}$, does not vanish in $\mathbb{k}$:

$$(7) \quad D = \sum_{i \in I} \text{qdim}(S_i)^2 \in \mathbb{k}.$$

In Section 7 we prove the following theorem.

Theorem 2.8. *If $\mathcal{C}$ is a finite semisimple nondegenerate spherical $\mathbb{k}$ -category with the chromatic morphism $\tilde{d}$ and generator G then the invariant $\mathcal{K}_{\mathcal{C}}$ is proportional to the Turaev–Viro invariant of closed 3-manifolds associated to $\mathcal{C}$,*

$$\text{TV}_{\mathcal{C}} = D^{-1} \mathcal{K}_{\mathcal{C}}.$$

2E. Open Problems. A strong point of our approach is its great generality. Besides the categories studied here, it certainly applies in other settings. Here we list (from least to most general) three further categories where our constructions should work:

- (1) the categories of finite dimensional modules over nice (quantum) Lie superalgebras, see [Scheunert and Zhang 2001; 2005],
- (2) the categories of finite dimensional modules over nice quasi-Hopf algebras, see [Bulacu and Caenepeel 2003; 2012; Hausser and Nill 1999; Panaite and Van Oystaeyen 2000],
- (3) general unimodular finite tensor categories, see [Shimizu 2019].

Here the adjective “nice” means that the category satisfies the hypothesis of Theorem 2.4, in particular, admits an m-trace and a chromatic morphism. The theory of [Geer et al. 2018] should imply the existence of an m-trace in the above contexts. Here, we may need to choose an appropriate pivotal structure to make the m-trace two sided, and it may be useful to work with the ideal of projective modules. It seems likely that the references listed above can help to construct chromatic morphisms in the categories in question. It also looks plausible that the results of [Geer et al. 2013a] may help to generalize our approach to nonunimodular categories.

In a different direction, recall that Kuperberg [1996] used framings of 3-manifolds to generalize the invariant in [Kuperberg 1991] to arbitrary finite dimensional Hopf algebras. As explained above, a finite dimensional unibalanced unimodular pivotal

Hopf algebra A gives rise to a framing-independent 3-manifold invariant $\mathcal{K}_{A\text{-mod}}$ which is computed in a way similar to the invariants in [Kuperberg 1996] using an integral and a cointegral (see Section 6). With Theorem 2.7 in mind, we ask if $\mathcal{K}_{A\text{-mod}}(M) = \text{Ku}_A(M, f)$ for some framing f of a 3-manifold M ? Is the Kuperberg invariant associated to a *unibalanced unimodular pivotal Hopf algebra* framing-independent? (This is known not to be true for all finite dimensional Hopf algebras.)

3. Algebraic setup

3A. Pivotal and ribbon categories. In this paper, we consider strict tensor categories with tensor product $\otimes$ and unit object $\mathbb{1}$. Let $\mathcal{C}$ be such a category. The notation $V \in \mathcal{C}$ means that V is an object of $\mathcal{C}$.

The category $\mathcal{C}$ is a *pivotal category* if it has duality morphisms

$$\begin{aligned} \overleftarrow{\text{coev}}_V: \mathbb{1} &\rightarrow V \otimes V^*, & \overleftarrow{\text{ev}}_V: V^* \otimes V &\rightarrow \mathbb{1}, \\ \overrightarrow{\text{coev}}_V: \mathbb{1} &\rightarrow V^* \otimes V, & \overrightarrow{\text{ev}}_V: V \otimes V^* &\rightarrow \mathbb{1}, \end{aligned}$$

which satisfy compatibility conditions (see for example [Barrett and Westbury 1999; Geer et al. 2013a]).

3B. $\mathbb{k}$ -categories. Let $\mathbb{k}$ be a field. A $\mathbb{k}$ -category is a category $\mathcal{C}$ such that its hom-sets are left $\mathbb{k}$ -modules, the composition of morphisms is $\mathbb{k}$ -bilinear, and the canonical $\mathbb{k}$ -algebra map $\mathbb{k} \rightarrow \text{End}_{\mathcal{C}}(\mathbb{1})$, $k \mapsto k \text{Id}_{\mathbb{1}}$ is an isomorphism. A *tensor $\mathbb{k}$ -category* is a tensor category $\mathcal{C}$ such that $\mathcal{C}$ is a $\mathbb{k}$ -category and the tensor product of morphisms is $\mathbb{k}$ -bilinear.

3C. M -traces on ideals in pivotal categories. Let $\mathcal{C}$ be a pivotal $\mathbb{k}$ -category. Here we recall the definition of an m -trace on an ideal in $\mathcal{C}$, for more details see [Geer et al. 2013a; Geer et al. 2013b]. By a *ideal* of $\mathcal{C}$ we mean a full subcategory, $\mathcal{I}$, of $\mathcal{C}$ such that:

Closed under tensor products. If V is an object of $\mathcal{I}$ and W is any object of $\mathcal{C}$, then $V \otimes W$ and $W \otimes V$ are objects of $\mathcal{I}$.

Closed under retracts. If V is an object of $\mathcal{I}$, W is any object of $\mathcal{C}$, and there exists morphisms $f: W \rightarrow V$, $g: V \rightarrow W$ such that $gf = \text{Id}_W$, then W is an object of $\mathcal{I}$.

An m -trace on an ideal $\mathcal{I}$ is a family of linear functions

$$\{t_V: \text{End}_{\mathcal{C}}(V) \rightarrow \mathbb{k}\}_{V \in \mathcal{I}}$$

such that following two conditions hold:

Cyclicity. If $U, V \in \mathcal{I}$ then for any morphisms $f: V \rightarrow U$ and $g: U \rightarrow V$ in $\mathcal{C}$ we have

$$t_V(gf) = t_U(fg).$$

Partial trace properties. If $U \in \mathcal{I}$ and $W \in \mathcal{C}$ then for any $f \in \text{End}_{\mathcal{C}}(U \otimes W)$ and $g \in \text{End}_{\mathcal{C}}(W \otimes U)$ we have

$$\begin{aligned} \text{t}_{U \otimes W}(f) &= \text{t}_U \left((\text{Id}_U \otimes \overrightarrow{\text{ev}}_W)(f \otimes \text{Id}_{W^*})(\text{Id}_U \otimes \overleftarrow{\text{coev}}_W) \right), \\ \text{t}_{W \otimes U}(g) &= \text{t}_U \left((\overleftarrow{\text{ev}}_W \otimes \text{Id}_U)(\text{Id}_{W^*} \otimes g)(\overrightarrow{\text{coev}}_W \otimes \text{Id}_U) \right). \end{aligned}$$

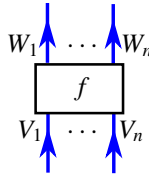
As above, an m -trace is *nondegenerate* if for any object $P \in \mathcal{I}$, the following pairing is nondegenerate:

$$\text{Hom}_{\mathcal{C}}(\mathbb{1}, P) \times \text{Hom}_{\mathcal{C}}(P, \mathbb{1}) \rightarrow \mathbb{k} \text{ given by } (x, y) \mapsto \text{t}_P(xy).$$

Remark that using the pivotal structure and the partial trace property one can show the nondegeneracy condition is equivalent to the following: for all $P \in \mathcal{I}$ and $V \in \text{Obj}(\mathcal{C})$, the pairing $\text{Hom}_{\mathcal{C}}(V, P) \times \text{Hom}_{\mathcal{C}}(P, V) \rightarrow \mathbb{k}$ given by $(x, y) \mapsto \text{t}_P(xy)$ is nondegenerate.

3D. Projective objects. Many of our examples use an m -trace on the ideal of projective objects. An object P of $\mathcal{C}$ is *projective* if for any epimorphism $p: X \rightarrow Y$ and any morphism $f: P \rightarrow Y$ in $\mathcal{C}$, there exists a morphism $g: P \rightarrow X$ in $\mathcal{C}$ such that $f = pg$. An object Q of $\mathcal{C}$ is *injective* if for any monomorphism $i: X \rightarrow Y$ and any morphism $f: X \rightarrow Q$ in $\mathcal{C}$, there exists a morphism $g: Y \rightarrow Q$ in $\mathcal{C}$ such that $f = gi$. Denote by Proj the full subcategory of projective objects. In a pivotal category projective and injective objects coincide (see [Geer et al. 2013b]). Also, Proj is an ideal.

3E. Invariants of colored ribbon graphs. Let $\mathcal{C}$ be a pivotal $\mathbb{k}$ -category. A morphism $f: V_1 \otimes \cdots \otimes V_n \rightarrow W_1 \otimes \cdots \otimes W_m$ in $\mathcal{C}$ can be represented by a box and arrows:



which are called *coupons*. All coupons have a top and a bottom sides which in our pictures will be the horizontal sides of the coupons. By a ribbon graph in an oriented manifold Σ , we mean an oriented compact surface embedded in Σ which is decomposed into elementary pieces: bands, annuli, and coupons (see [Turaev 1994]) and is the thickening of an oriented graph. In particular, the vertices of the graph lying in $\text{Int } \Sigma = \Sigma \setminus \partial \Sigma$ are thickened to coupons. A $\mathcal{C}$ -colored ribbon graph is a ribbon graph whose (thickened) edges are colored by objects of $\mathcal{C}$ and whose coupons are colored by morphisms of $\mathcal{C}$. The intersection of a $\mathcal{C}$ -colored ribbon graph in Σ with $\partial \Sigma$ is required to be empty or to consist only of vertices of

valency 1. When Σ is a surface the ribbon graph is just a tubular neighborhood of the graph.

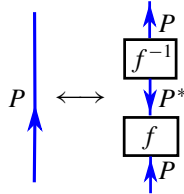
A $\mathcal{C}$ -colored ribbon graph in $\mathbb{R}^2$ (resp. $S^2 = \mathbb{R}^2 \cup \{\infty\}$) is called *planar* (resp. *spherical*). Let Rib be the category of planar $\mathcal{C}$ -colored ribbon graphs and let $F : \text{Rib} \rightarrow \mathcal{C}$ be the pivotal functor¹ (defined using the Penrose graphical calculus) corresponding to $\mathcal{C}$, see for example [Geer et al. 2013b]. Let $\mathcal{L}_{\text{adm}}$ be all spherical $\mathcal{C}$ -colored ribbon graphs L such that L is the braid closure of a $(1,1)$ -ribbon graph T_V whose open edge is colored with an object $V \in \mathcal{I}$.

Given an m-trace t on $\mathcal{I}$ we can renormalize F to an invariant

$$(8) \quad F' : \mathcal{L}_{\text{adm}} \rightarrow \mathbb{k} \quad \text{given by} \quad F'(L) = t_V(F(T_V)),$$

where T_V is any $(1,1)$ -ribbon graph as above. The properties of the m-trace imply F' is a isotopy invariant of L , see [Geer et al. 2013b].

Remark 3.1. If $\mathcal{I}$ is an ideal and $P \in \mathcal{I}$ then Lemma 2 of [Geer et al. 2013b] implies that $P^* \in \mathcal{I}$. Moreover, the pivotal structure gives an isomorphism $f : P \rightarrow P^{**}$ for all P . This morphism can be used to change the orientation of an edge of a graph as shown in the following diagram:



4. An invariant of $\mathcal{I}$ -colored graphs on a multihandlebody

In this section we prove Proposition 2.1 and Theorem 2.2.

4A. Algebraic preliminaries. The following lemma contains standard facts from linear algebra; we leave the proof to the reader.

Lemma 4.1. *Let X_j and Y_j be finite dimensional $\mathbb{k}$ -modules, for $j = 1, 2$. Let $\langle \cdot, \cdot \rangle_{X_j, Y_j} : X_j \otimes_{\mathbb{k}} Y_j \rightarrow \mathbb{k}$ be a pairing whose right and left kernels are zero. Given a basis $\{x_i^j\}$ of X_j let $\{y_i^j\}$ be the dual basis of Y_j determined by $\langle x_i^j, y_j^j \rangle_{X_j, Y_j} = \delta_{i,j}$. Then*

- (1) *the element $\Omega_j = \sum_i x_i^j \otimes_{\mathbb{k}} y_i^j \in X_j \otimes Y_j$ is independent of the choice of the basis $\{x_i^j\}$,*

¹We call this functor the pivotal functor because such a functor is associated to each pivotal category. However, the functor does not preserve the duality.

(2) if $h : X_1 \rightarrow X_2$ and $k : Y_2 \rightarrow Y_1$ are $\mathbb{k}$ -linear maps such that $\langle h(x), y \rangle_{X_2, Y_2} = \langle x, k(y) \rangle_{X_1, Y_1}$ for all $x \in X_1$ and $y \in Y_2$ then

$$(9) \quad (h \otimes \text{Id}_{Y_1})(\Omega_1) = (\text{Id}_{X_2} \otimes k)(\Omega_2).$$

Proposition 4.2. Recall the element $\Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i$ defined in Equation (1). Let $\Lambda_P = \sum_i x_i y_i \in \text{End}_{\mathcal{C}}(P)$. We have:

(1) The element Ω_P is independent of the choice of the basis of $\text{Hom}_{\mathcal{C}}(\mathbb{1}, P)$.

(2) If $P' \in \mathcal{I}$ and $\phi : P \rightarrow P'$ is a morphism then

$$(\phi \otimes_{\mathbb{k}} \text{Id}_{\mathbb{1}})\Omega_P = \Omega_{P'}(\text{Id}_{\mathbb{1}} \otimes_{\mathbb{k}} \phi) \quad \text{and} \quad \phi \circ \Lambda_P = \Lambda_{P'} \circ \phi.$$

(3) If $f : \mathbb{1} \rightarrow P$ and $g : P \rightarrow \mathbb{1}$ are morphisms then

$$\text{t}_P(fg) = \sum_i \text{t}_P(fy_i)\text{t}_P(x_i g).$$

Proof. We use the above lemma to prove this proposition. Let

$$\begin{aligned} X_1 &= \text{Hom}_{\mathcal{C}}(\mathbb{1}, P), & Y_1 &= \text{Hom}_{\mathcal{C}}(P, \mathbb{1}), \\ X_2 &= \text{Hom}_{\mathcal{C}}(\mathbb{1}, P'), & Y_2 &= \text{Hom}_{\mathcal{C}}(P', \mathbb{1}). \end{aligned}$$

For $j = 1, 2$, define the bilinear pairing $\langle \cdot, \cdot \rangle_{X_j, Y_j} : X_j \otimes Y_j \rightarrow \mathbb{k}$ by

$$\langle x, y \rangle_{X_1, Y_1} = \text{t}_P(x \circ y) \quad \text{and} \quad \langle x, y \rangle_{X_2, Y_2} = \text{t}_{P'}(x \circ y), \quad \text{respectively,}$$

for $x \in X_j$ and $y \in Y_j$. Now the first statement of the proposition is a direct consequence of first statement of Lemma 4.1. Similarly, the second statement of the proposition follows from the second statement of the lemma: let

$$\begin{aligned} h : \text{Hom}(\mathbb{1}, P) &\rightarrow \text{Hom}(\mathbb{1}, P') & \text{be given by} & \quad f \mapsto \phi \circ f \\ k : \text{Hom}(P', \mathbb{1}) &\rightarrow \text{Hom}(P, \mathbb{1}) & \text{be given by} & \quad g \mapsto g \circ \phi \end{aligned}$$

then Equation (9) becomes the first equality in the second statement of the proposition. If we write this equality explicitly we get

$$\sum_i (\phi \circ x_i) \otimes y_i = \sum_i x'_i \otimes (y'_i \circ \phi),$$

where $\Omega_{P'} = \sum_i x'_i \otimes_{\mathbb{k}} y'_i$ and $\Lambda_{P'} = \sum_i x'_i y'_i \in \text{End}_{\mathcal{C}}(P')$ and the second equality follows.

Concerning the last statement of the proposition, let $\{x_i\}$ be a basis of $X_1 = \text{Hom}_{\mathcal{C}}(\mathbb{1}, P)$ and let $\{y_i\}$ be the dual basis of Y_1 determined by $\langle \cdot, \cdot \rangle_{X_1, Y_1}$. Using these bases we can find coefficients a_i and b_i such that $f = \sum a_i x_i$ and $g = \sum b_i y_i$. Then

$$\text{t}_P(fg) = \sum_{i,j} a_i b_j \text{t}_P(x_i y_j) = \sum_i a_i b_i.$$

But we also have that $\mathfrak{t}_P(fy_i) = \sum_j a_j \mathfrak{t}_P(x_j y_i) = a_i$ and similarly $\mathfrak{t}_P(x_i g) = b_i$ so the third statement of the proposition follows. $\square$

4B. Proof of Theorem 2.2. By the genus of a multihandlebody we mean the sum of the genera of all its components. We will show that the last three properties of Theorem 2.2 determine a well defined invariant by inducting on the genus of the multihandlebody. We first do this for a fixed multihandlebody then at the end of the proof we show it is invariant under orientation preserving diffeomorphisms.

The base induction case is graphs on the disjoint union of 3-balls. The extension of F' Property (2) defines F' for graphs up to isotopy on the boundary of the 3-ball. The disjoint union of handlebodies Property (3) extends the invariant F' uniquely to the disjoint union of 3-balls. To conclude the base case we need to show that F' satisfies the cutting-along-a-disk Property (4) of Theorem 2.2. Let $(B^3, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ be a $\mathcal{I}$ -colored graph on the boundary of the 3-ball. Let D be a disk properly embedded in the 3-ball such that the boundary of D is a simple curve γ on S^2 (which intersects Γ transversely).

Orient γ arbitrarily and up to isotopy assume it is the equator of S^2 . The curve γ intersects Γ in several points which correspond to the tensor product of objects of $\mathcal{I}$. We denote by P this tensor product. Let Γ_l and Γ_u be the graphs in the lower and upper hemispheres, respectively (here we use the orientation of γ to distinguish them). Let $(\text{cut}_{\gamma}(B^3), \text{cut}_{\gamma}(\Gamma)) = (B^3, \Gamma'_l) \sqcup (B^3, \Gamma'_u)$ be the $\mathcal{I}$ -colored graph on the disjoint union of two 3-balls obtained by cutting along the disk where the new graphs are obtained by closing Γ_l and Γ_u with coupons determined by $\Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i$ as described in the statement above the theorem.

Then $F(\Gamma_l) : \mathbb{1} \rightarrow P$ and $F(\Gamma_u) : P \rightarrow \mathbb{1}$ are morphisms and by definition

$$F'(B^3, \Gamma) = \mathfrak{t}_P(F(\Gamma_l)F(\Gamma_u)).$$

But by Part (3) of Proposition 4.2 we have

$$\mathfrak{t}_P(F(\Gamma_l)F(\Gamma_u)) = \sum_i \mathfrak{t}_P(F(\Gamma_l)y_i)\mathfrak{t}_P(x_i F(\Gamma_u))$$

where by definition $F'(\Gamma'_l) = \mathfrak{t}_P(F(\Gamma_l)y_i)$ and $F'(\Gamma'_u) = \mathfrak{t}_P(x_i F(\Gamma_u))$. This shows that we can cut along D^2 and also that the choice of the orientation on $\gamma = \partial D^2$ does not change the result of F' as the left hand side does not depend on γ . This concludes the base induction step.

Let us assume F' is well defined and satisfies Properties (2), (3) and (4) of Theorem 2.2 for all multihandlebodies with genus strictly less than $g \geq 1$. Let $(H, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ where H is a multihandlebody of genus g . Choose a disk D which is properly embedded in H such that the boundary of D is an essential simple circle γ on ∂H intersecting Γ transversely. Cutting along D produces a genus $g-1$ multihandlebody $(\text{cut}_{\gamma}(H), \text{cut}_{\gamma}(\Gamma))$ which for simplicity we also denote by $\text{cut}_{\gamma}(\Gamma)$. Then

Property (4) and induction says we can assign the value $F'(H, \Gamma) = F'(\text{cut}_\gamma(\Gamma))$. We need to show that this value is well defined, i.e., independent of the disk which is cut along and of the orientation of its boundary chosen to draw the picture used in the description of the property. To do this, let D and D' be two discs which bound essential simple oriented circles γ and γ' , respectively. We want to show that by cutting along D or D' we obtain the same value for F' . We prove this in two steps.

First, let us assume γ and γ' are disjoint. Then one can choose disjoint discs D and D' which both cut the multihandlebody to produce genus $g - 1$ multihandlebodies. We have

$$F'(\text{cut}_{\gamma'}(\Gamma)) = F'(\text{cut}_{\gamma'}(\text{cut}_\gamma(\Gamma))) = F'(\text{cut}_{\gamma'}(\text{cut}_\gamma(\Gamma))) = F'(\text{cut}_\gamma(\Gamma)),$$

where the first and the last equalities hold because Property (3) holds in genus less than g . This in particular proves that the orientation of γ is irrelevant: choose γ' to be parallel to γ but oriented the opposite way.

Second, it follows from [Masur and Minsky 2004] that for any two essential bounding curves γ and γ' , there exists a sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \gamma'$ of circles that bound discs D_i with $D_i \cap D_{i+1} = \emptyset$. Hence the first step implies the value of F' is constant on this sequence:

$$F'(\text{cut}_{\gamma_0}(\Gamma)) = F'(\text{cut}_{\gamma_1}(\Gamma)) = \dots = F'(\text{cut}_{\gamma_n}(\Gamma)).$$

This completes the induction step.

We are now left to prove that F' is invariant under the action of the group of orientation preserving diffeomorphisms of H up to isotopy. Again, we argue by induction on the genus. Since this group is trivial for B^3 (i.e., genus 0), there is nothing to prove in the base case. Assume F' is invariant for all multihandlebodies of genus less than g . Let (H, Γ) be a $\mathcal{I}$ -colored graph on a multihandlebody of genus g . If γ is a simple essential curve bounding in H then applying the cut-along-a-disc property of F' we get

$$F'(H, \Gamma) = F'(\text{cut}_\gamma(H), \text{cut}_\gamma(\Gamma)).$$

If $f : H \rightarrow H$ is a positive self-diffeomorphism, then we have:

$$F'(f(H), f(\Gamma)) = F'(H, f(\Gamma)) = F'(\text{cut}_{f(\gamma)}(H), \text{cut}_{f(\gamma)}(f(\Gamma))).$$

But f induces a diffeomorphism between the genus $g - 1$ multihandlebody graphs $(\text{cut}_\gamma(H), \text{cut}_\gamma(\Gamma))$ and $(\text{cut}_{f(\gamma)}(H), \text{cut}_{f(\gamma)}(f(\Gamma)))$ so by induction we have

$$\begin{aligned} F'(H, \Gamma) &= F'(\text{cut}_\gamma(H), \text{cut}_\gamma(\Gamma)) = F'(\text{cut}_{f(\gamma)}(H), \text{cut}_{f(\gamma)}(f(\Gamma))) \\ &= F'(f(H), f(\Gamma)). \end{aligned}$$

Thus, we have proved Theorem 2.2.

5. Invariant of bichrome graphs and of 3-manifolds

5A. Bichrome graphs on handlebodies. Recall by a *multihandlebody* we will mean a disjoint union of a finite number of oriented connected handlebodies. By the *genus* of a multihandlebody we mean the sum of the genera of all its components. Recall a *bichrome handlebody graph* is a pair (H, Γ) where H is a multihandlebody and Γ is a graph on ∂H (seen up to isotopy) composed of two disjoint subgraphs $\Gamma = \Gamma_{\text{blue}} \sqcup \Gamma_{\text{red}}$ where Γ_{blue} is an $\mathcal{I}$ -colored graph and $\Gamma_{\text{red}} \subset \partial H$ a disjoint union of simple closed unoriented circles. When convenient, we denote a bichrome handlebody graph by Γ with the understanding that it is on a multihandlebody. Recall, we say a bichrome handlebody graph is *admissible* if its blue subgraph is $\mathcal{I}$ -colored and meets each connected component of the multihandlebody. Let $\mathcal{H}_b$ be the set of orientation preserving diffeomorphism classes of admissible bichrome handlebody graphs.

Definition 5.1. Let H be a multihandlebody.

- (1) We say that a set of simple closed curves (referred to as circles) on ∂H *bounds* in H if the circles bound a disjoint union of disks embedded in H .
- (2) A bounding set of circles is a *reducing set* if the complement of the circles is a disjoint union of spheres with holes.
- (3) The *complexity* of a circle γ on $\partial H \subset H$ is the minimal number of intersections of γ with any reducing set of circles.

Definition 5.2 (red capping and digging moves). Let (H, Γ) and (H', Γ') be bichrome handlebody graphs. We say that (H, Γ) is obtained from (H', Γ') by a *red capping move* along a red circle $c \subset \Gamma'_{\text{red}}$ if there is a properly embedded disc D in H' such that

- (1) $D \cap c = \{\text{pt}\}$,
- (2) H is obtained from H' by gluing a 2-handle along c , and
- (3) $\Gamma = \Gamma' \setminus c$, where we identify $\partial H'$ set minus a neighborhood of c with a subset of ∂H .

Conversely, we say (H', Γ') is obtained from (H, Γ) by a *red digging move* (resulting in the red circle c). See Figure 2 for a visual representation of these moves.

Proposition 5.3. *A red circle in a bichrome handlebody graph is the result of a red digging move if and only if it has complexity one.*

Proof. Let c be a red circle in a bichrome handlebody graph (H', Γ') . If c has complexity one then there exists a reducing set S of circles such that $c \cap S$ is exactly one point. Now gluing a 2-handle along c one gets a multihandlebody H , and the

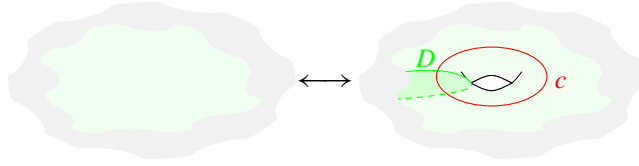


Figure 2. Red digging / capping moves.

graph $\Gamma := \Gamma' \setminus c$ is obtained by a red capping move which is the inverse of the sought digging move.

Reciprocally, if c is obtained by a red digging move then it intersects exactly once the disc D in Definition 5.2. $\square$

Proposition 5.4. *Let $(H, \Gamma), (H, \Gamma') \in \mathcal{H}_b$ be bichrome graphs such that Γ' is obtained from Γ by sliding a blue or red edge of Γ over one of its simple red circles. Then Γ and Γ' are related by a sequence of red digging and capping moves.*

Proof. Let $\Gamma = \Gamma_{\text{blue}} \cup \Gamma_{\text{red}}$ and let $c \subset \Gamma_{\text{red}}$ be the simple circle on which we want to slide an edge e (where e is contained in either Γ_{blue} or Γ_{red}). We claim that up to applying one red digging move which transforms (H, Γ) into a new bichrome graph (H_1, Γ_1) and creates a new red circle $c' \subset \Gamma_1$, we can reduce to the case where c is a red circle created by a red digging move. Proving this claim would imply the proposition because after applying a red capping move on c , one gets a bichrome graph (H_2, Γ_2) in which e can be slid, by an isotopy, over the disc added by the red capping. Then redigging along the same disc and recapping along c' produces exactly the bichrome graph (H, Γ') obtained by sliding e over c .

We now prove the claim by describing a suitable red digging move. Let $I \subset \partial H$ be a parametrized segment, i.e., the image of an embedding $i : [-1, 1] \hookrightarrow \partial H$ such that $\partial I \cap \Gamma = i(\{0\})$ is formed by a single point belonging to c . Let I' be a properly embedded arc in H obtained by slightly pushing I inside H (keeping fixed ∂I). Also, let

$$\begin{aligned} H_1 &= H \setminus \text{Tub}(I'), & c' &= \partial \text{Tub}(i(1)), \\ (\Gamma_1)_{\text{blue}} &= \Gamma_{\text{blue}}, & (\Gamma_1)_{\text{red}} &= \Gamma_{\text{red}} \sqcup c', \end{aligned}$$

where Tub denotes a tubular neighborhood and we identify $\partial H \setminus \text{Tub}(\{i(\pm 1)\})$ with a subset of ∂H_1 .

Then (H_1, Γ_1) is obtained from (H, Γ) by a red digging move along I which creates the red circle c' . But now, with the notation of Definition 5.2, c intersects the disc D bound by $I \cup I'$ exactly once so it has complexity 1 and thus, by Proposition 5.3 it is the result of a red digging. $\square$

5B. Heegaard splittings. A Heegaard splitting for a closed oriented connected 3-manifold M is an ordered triple $(H_\alpha, H_\beta, \Sigma)$ such that the following conditions hold:

- (1) Σ is a closed surface embedded in M , (2) H_α and H_β are oriented handlebodies positively embedded in M , (3) $\partial H_\beta = \Sigma = \partial H_\alpha = H_\alpha \cap H_\beta$, (4) $M = H_\alpha \cup H_\beta$ and (5) Σ is oriented as the boundary of H_α (with the outgoing vector first convention).

A *Heegaard diagram* for M compatible with this Heegaard splitting is a triple $(\Sigma, \{\alpha_i\}, \{\beta_i\})$, where $\{\alpha_i\}$ and $\{\beta_i\}$ are minimal reducing sets of circles bounding in H_α and H_β , respectively. Then one can recover H_α , H_β and $M \simeq H_\alpha \cup_\Sigma H_\beta$ up to diffeomorphism from the Heegaard diagram.

Recall in Section 2B given a Heegaard diagram $(\Sigma, \{\alpha_i\}, \{\beta_i\})$ of a closed connected orientable 3-manifold M we defined a bichrome diagram for M : a bichrome handlebody graph $(H, \Gamma_{\text{blue}} \cup \Gamma_{\text{red}})$, where $H = H_\alpha$, $\Gamma_{\text{red}} = \{\beta_i\}$ and Γ_{blue} is the planar ribbon graph $\mathbf{O}_G$ which is the braid closure of the coupon filled with h such that $t_G(h) = 1$. (Observe that since $H \setminus \Gamma_{\text{red}}$ is connected, the position of $\mathbf{O}_G$ is unique up to isotopy.).

Theorem 5.5. *If (H, Γ) and (H', Γ') are two bichrome diagrams for M , then they are related by a finite sequence of red digging and capping moves.*

Proof. It is well known that (H, Γ_{red}) and $(H', \Gamma'_{\text{red}})$ can be connected by a finite sequence of stabilization moves (and their inverses) as well as handle slide moves (see [Singer 1933]). Clearly a stabilization move is a special case of a red digging move (and its inverse is a red capping move). Furthermore by Proposition 5.4 we can achieve handle slides via isotopy and red digging and capping moves. Finally, since $\Gamma_{\text{blue}} = \mathbf{O}_G$ is contained in a disc in ∂H we can always operate the above moves. $\square$

5C. Proof of Theorems 2.4 and 2.5. We need to prove a few lemmas.

Lemma 5.6. *Let $P, P' \in \mathcal{I}$ and set $\Lambda_{P' \otimes G^*} = \sum_i y_i x_i \in \text{End}_{\mathcal{C}}(P' \otimes G^*)$, where $\Omega_{P' \otimes G^*} = \sum x_i \otimes_{\mathbb{K}} y_i$. Then*

$$(10) \quad (\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ (\Lambda_{P' \otimes G^*} \otimes \tilde{d}_P) \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P) = \text{Id}_{P' \otimes P}.$$

Proof. Since G is a generator of $\mathcal{I}$ there exist $g_i : P \rightarrow G$, $f_i : G \rightarrow P$ and $g'_i : P' \rightarrow G$, $f'_i : G \rightarrow P'$ such that $\sum_i f_i \circ g_i = \text{Id}_P$ and $\sum_i f'_i \circ g'_i = \text{Id}_{P'}$. Then we have

$$\begin{aligned} \text{Id}_{P' \otimes P} &= \sum_{i,j} (f'_i \otimes f_j) \circ \text{Id}_{G \otimes G} \circ (g'_i \otimes g_j) \\ &= \sum_{i,j} (f'_i \otimes \overleftarrow{\text{ev}}_G \otimes f_j) \circ (\Lambda \otimes \tilde{d}) \circ (g'_i \otimes \overrightarrow{\text{coev}}_G \otimes g_j) \\ &= \sum_i (f'_i \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ (\Lambda \otimes \tilde{d}_P) \circ (g'_i \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P) \end{aligned}$$

$$\begin{aligned}
&= \sum_i (\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ ((f'_i \otimes \text{Id}_{G^*}) \Lambda (g'_i \otimes \text{Id}_{G^*})) \otimes \tilde{d}_P \\
&\quad \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P) \\
&= \sum_i (\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ ((\Lambda_{P' \otimes G^*} \circ (f'_i g'_i \otimes \text{Id}_{G^*})) \otimes \tilde{d}_P) \\
&\quad \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P) \\
&= \sum_i (\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ (\Lambda_{P' \otimes G^*} \otimes \tilde{d}_P) \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P),
\end{aligned}$$

where the second equality uses the definition of the chromatic morphism (Equation (3)), the third equality uses the definition of $\tilde{d}_P$ (Equation (5)) and the second to last equality comes from Part (2) of Proposition 4.2 where $\phi = f'_i \otimes \text{Id}_{G^*}$. $\square$

Given a bichrome handlebody graph (H, Γ) we can produce a new bichrome handlebody graph (H', Γ') by doing a red digging move on (H, Γ) then changing the newly created red circle into a blue graph using the chromatic morphism, as in Equation (6). We say (H', Γ') is obtained from (H, Γ) by a *blue digging move*. Conversely, we say (H, Γ) is obtained from (H', Γ') by a *blue capping move*. See Figure 3 for a pictorial representation of these moves.

Lemma 5.7. *The invariant F' of $\mathcal{I}$ -colored ribbon graphs on multihandlebodies defined in Theorem 2.2 is invariant under blue digging and capping moves.*

Proof. Let (H, Γ) and (H', Γ') be bichrome handlebody graphs such that (H', Γ') is obtained from (H, Γ) by a blue digging move. Up to cutting along a reducing set of curves for H , we can assume that the component of H to which we are applying the blue digging has genus 0. To prove the statement we will compute the morphisms associated to the subsurfaces of ∂H and $\partial H'$ drawn in Figure 3.

The right hand side is just $\text{Id}_{P'} \otimes \text{Id}_P$. In order to compute the left hand side, let γ be the curve bounding the disc in the far left part of Figure 3. Cutting along γ , as discussed before Theorem 2.2, we get that the morphism associated to the left hand side is:

$$(\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ (\Lambda_{P' \otimes G^*} \otimes \tilde{d}_P) \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P).$$

But by Lemma 5.6 this morphism is equal to $\text{Id}_{P'} \otimes \text{Id}_P$. $\square$

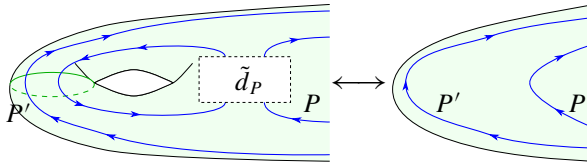


Figure 3. *Blue digging and capping moves.* Here one can assume the orientation of the left most strand is as shown in the figure because of Remark 3.1.

Proof of Theorem 2.4. To prove the theorem we will prove a slightly stronger statement:

Statement. There exists a unique extension of F' to an invariant of admissible bichrome handlebody graphs satisfying Equation (6) and invariant under the blue capping and digging moves.

We prove the statement by induction on the number of red circles. When there are no red circles the statement is just Lemma 5.7.

Now suppose the statement is true for all admissible bichrome handlebody graphs with $n - 1$ red circles. Let (H, Γ) be an admissible bichrome handlebody graph with n red circles. There exists a not self-interesting path γ_0 in $\partial H \setminus \Gamma$ going from a point on a blue edge to a point on one red circle β_0 . Using this path we can pull a small segment of the blue edge to β_0 then use the chromatic morphism to change β_0 into a blue graph. We obtain an admissible bichrome handlebody graph (H, Γ_{γ_0}) with $n - 1$ red circles. Then by induction $F'(H, \Gamma_{\gamma_0})$ is well defined, so if F' exists it is unique. To prove F' exists we need to show it is independent of the choice of the path γ_0 . Let γ_1 be another such path going from a point on a blue edge to a point on one of the red circles β_1 and let (H, Γ_{γ_1}) be the admissible bichrome handlebody graph obtained by using γ_1 to make β_1 blue. We consider two cases.

Case 1. $\beta_0 = \beta_1$. Here we have two subcases. First, suppose the red circle $\beta_0 = \beta_1$ has complexity one. By Proposition 5.3 we have β_0 is the result of a digging move. Therefore, when we use either γ_0 or γ_1 with the chromatic morphism to change β_0 to a blue graph and we arrive at a diagram of the form given in the left side of Figure 3. In both cases we can do a blue capping move to arrive at the same $\mathcal{I}$ -colored ribbon graph on a multihandlebody with $n - 1$ red circles. Thus, induction implies

$$F'(H, \Gamma_{\gamma_0}) = F'(H, \Gamma_{\gamma_1})$$

so in this subcase the extension of F' does not depend on the choice of γ_0 .

Second, assume the red curve $\beta_0 = \beta_1$ has any complexity. Apply a blue digging move to Γ along a small interval I intersecting β_0 in a single point. The result is a new bichrome handlebody graph (H', Γ') with n red circles in which the image β'_0 of β_0 is a red curve with complexity 1. Since this digging move only modified (H, Γ) in a neighborhood of I , we can identify γ_0 and γ_1 as paths in $\partial H' \setminus \Gamma'$ and we can apply the chromatic morphism to β'_0 either through γ_0 and γ_1 getting, respectively, bichrome handlebody graphs Γ'_{γ_0} and Γ'_{γ_1} with $n - 1$ red components. By the preceding case $F'(H', \Gamma'_{\gamma_0}) = F'(H', \Gamma'_{\gamma_1})$. But now observing that Γ'_{γ_0} (resp. Γ'_{γ_1}) is obtained from Γ_{γ_0} (resp. Γ_{γ_1}) by a blue digging move, we have

$$F'(H, \Gamma_{\gamma_0}) = F'(H', \Gamma'_{\gamma_0}) = F'(H', \Gamma'_{\gamma_1}) = F'(H, \Gamma_{\gamma_1}).$$

Case 2. $\beta_0 \neq \beta_1$. In this situation we have two subcases. First, suppose γ_0 and γ_1 are nonintersecting. In (H, Γ_{γ_0}) (resp. (H, Γ_{γ_1})) we can use γ_1 (resp. γ_0) to change β_1 (resp. β_0) into a blue graph and obtain an admissible bichrome handlebody graph (H, Γ_1) with $n - 2$ red circles. Then by induction we have:

$$F'(H, \Gamma_{\gamma_0}) = F'(H, \Gamma_1) = F'(H, \Gamma_{\gamma_1}).$$

Second, suppose $\gamma_0 \cap \gamma_1 \neq \emptyset$; we claim that there exists another path γ'_1 connecting Γ_{blue} to β_1 such that $\gamma_0 \cap \gamma'_1 = \emptyset$ so that by the preceding subcase we have $F'(H, \Gamma_{\gamma_0}) = F'(H, \Gamma_{\gamma'_1})$ then Case 1 implies $F'(H, \Gamma_{\gamma'_1}) = F'(H, \Gamma_{\gamma_1})$ and the proof follows. To prove our claim observe that since $\gamma_0 \cap \gamma_1 \neq \emptyset$ then γ_0 and γ_1 are contained in the same connected component R of $\partial H \setminus \Gamma$. Moreover, R is an open orientable surface which is the interior of a compact surface with at least 3 distinct boundary components: $\partial_{\text{blue}} \subset \Gamma_{\text{blue}}$, β_0 and β_1 . Then γ_0 and γ_1 are embedded arcs in R connecting ∂_{blue} to β_0 and β_1 , respectively. But $R \setminus \gamma_0$ is connected as γ_0 intersects the closed curve β_0 once. Thus, there exists another path in $R \setminus \gamma_0$ connecting ∂_{blue} and β_1 . $\square$

Proof of Theorem 2.5. By Theorem 5.5 it is sufficient to prove that F' is invariant under red digging and capping moves. Let (H, Γ) and (H', Γ') be bichrome handlebody graphs such that (H', Γ') is obtained from (H, Γ) by a red digging move. Suppose c is the red circle created in this move. Let (H', Γ'') be the bichrome handlebody graph obtained from using an edge of the blue graph and the chromatic morphism to change the red circle c into a blue graph. Then by definition the composition of these two moves is a blue digging move. Thus, by Lemma 5.7 we have

$$F'(H, \Gamma) = F'(H', \Gamma'').$$

Moreover, since (H', Γ') and (H', Γ'') differ by an isotopy and a move represented in Equation (6) then Theorem 2.4 implies

$$F'(H', \Gamma') = F'(H', \Gamma'').$$

Combining the last two equalities we have $F'(H, \Gamma) = F'(H', \Gamma')$ which concludes the lemma. $\square$

6. Details on Hopf algebras and Kuperberg invariants

In this section we prove Theorems 2.6 and 2.7. First, we briefly recall some well known facts about Hopf algebras, see for example [Radford 2012].

6A. Hopf algebra preliminaries. Let A be a finite-dimensional Hopf algebra over a field $\mathbb{k}$ with a multiplication $m : A \otimes A \rightarrow A$, a unit $\eta : \mathbb{k} \rightarrow A$, a coproduct $\Delta : A \rightarrow A \otimes A$, a counit $\epsilon : A \rightarrow \mathbb{k}$, an antipode $S : A \rightarrow A$. A *right integral* of

A is a linear form $\lambda \in A^*$ satisfying $\lambda f = f(1_A) \cdot \lambda$ for every $f \in A^*$. This means that $(\lambda \otimes \text{Id}_A)(\Delta(x)) = \lambda(x) \cdot 1_A$ for every $x \in A$. A *left* (resp. *right*) *cointegral* of A is a vector $\Lambda \in A$ satisfying $x\Lambda = \epsilon(x)\Lambda$ (resp. $\Lambda x = \epsilon(x)\Lambda$) for every $x \in A$. Since A is finite-dimensional, right integrals form a 1-dimensional ideal in A^* and left cointegrals form a 1-dimensional ideal in A . Moreover, every nonzero right integral $\lambda \in A^*$ and every nonzero left cointegral $\Lambda \in A$ satisfy $\lambda(\Lambda) \neq 0$. We fix a choice of a right integral $\lambda \in A^*$ and of a left cointegral $\Lambda \in A$ satisfying $\lambda(\Lambda) = 1$. In this section we use sumless Sweedler notation to describe the application of the coproduct, for example $\Delta^3(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$.

The Hopf algebra A being *unimodular* means that $S(\Lambda) = \Lambda$, or equivalently, Λ is both a right and left cointegral. We say A is *pivotal* if there exists $g \in A$ such that $S^2(x) = gxg^{-1}$ for all $x \in A$. Let Proj be the ideal of projective modules over A (for a definition of projective object see Section 3D). By Theorem 1 of [Beliakova et al. 2018a] every finite dimensional unimodular pivotal Hopf algebra has a left m -trace on Proj . Such a Hopf algebra is *unibalanced* if this left m -trace is also a right m -trace (see [Beliakova et al. 2018a]).

6B. Proof of Theorem 2.6. Let A be a finite dimensional unibalanced unimodular pivotal Hopf algebra over a field $\mathbb{k}$ and $A\text{-mod}$ be the category of its finite dimensional left modules. Denote A as the *regular representation* of A , which is the left A -module structure on A itself determined by the action $L : A \rightarrow \text{End}_{\mathbb{k}}(A)$ given by $L_h(x) = hx$ for all $h, x \in A$. Let t be the nondegenerate m -trace on Proj with normalization: $t_A(f_\Lambda \circ \epsilon) = 1$, where $f_\Lambda : \mathbb{k} \rightarrow A$ denotes the unique morphism determined by $f_\Lambda(1) = \Lambda$.

Lemma 6.1. *The regular representation A is a generator of Proj in $A\text{-mod}$.*

Proof. Let P be an indecomposable projective A -module. Then P being projective implies it is a direct summand of a free module $\bigoplus_n A$. But P is indecomposable so P is a direct summand of A . Since Krull-Schmidt Theorem holds in $A\text{-mod}$ we have that every element of Proj is a direct sum of indecomposable projective modules and the lemma follows. $\square$

For $P \in \text{Proj}$, observe that the map $\Lambda'_P : P \rightarrow P$ given by the left action of Λ on P is a morphism because Λ is both a left and right cointegral. On the other hand, consider the morphism $\Lambda_P : P \rightarrow P$ given by $\Lambda_P = x_i y_i$, where $\Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i$ is defined in Proposition 2.1.

Lemma 6.2. $\Lambda_P = \Lambda'_P.$

Proof. First, consider the case when P is the generator A . Since the space of cointegrals is one dimensional then $\text{Hom}_{A\text{-mod}}(\mathbb{1}, A)$ is one dimensional and generated by the morphism $f_\Lambda : \mathbb{k} \rightarrow A$ determined by $f_\Lambda(1) = \Lambda$. Combining this with the fact

that t is nondegenerate (with the normalization given above) we have $\Omega_A = f_\Lambda \otimes_{\mathbb{k}} \epsilon$. Thus, for all $x \in A$ we have

$$\Lambda_A(x) = f_\Lambda \epsilon(x) = \epsilon(x) \Lambda = x \Lambda = \Lambda'_A(x).$$

So the lemma holds for $P = A$.

Let P be an object in Proj . Since A is a generator there exist $f_j : A \rightarrow P$ and $g_j : P \rightarrow A$ such that $\text{Id}_P = \sum_{j \in J} f_j g_j$. From Proposition 4.2 Part (2) we have

$$\Omega_P = \sum_{j \in J} ((f_j \otimes \text{Id}_\mathbb{1}) \Omega_A (\text{Id}_\mathbb{1} \otimes g_j)) = \sum_{j \in J} ((f_j \circ f_\Lambda) \otimes (\epsilon \circ g_j)).$$

So by definition of Λ_P , for $y \in P$, we have

$$\begin{aligned} \Lambda_P(y) &= \sum_{j \in J} (f_j \circ f_\Lambda \circ \epsilon \circ g_j)(y) = \sum_{j \in J} f_j(\epsilon(g_j(y)) \Lambda) \\ &= \sum_{j \in J} f_j(\Lambda g_j(y)) \\ &= \sum_{j \in J} \Lambda f_j(g_j(y)) = \Lambda.y, \end{aligned}$$

and the lemma follows. $\square$

In [De Renzi et al. 2018] it is shown that $f_{\lambda, 1_A} : A \rightarrow A^* \otimes A$ defined by $1_A \mapsto \lambda \otimes 1_A$ is a morphism.

Lemma 6.3. *The morphism $\tilde{d} : A \otimes A \rightarrow A \otimes A$ defined by*

$$\tilde{d} = (\vec{\text{ev}}_A \otimes \text{Id}_{A \otimes A})(\text{Id}_A \otimes f_{\lambda, 1_A} \otimes \text{Id}_A)(\text{Id}_A \otimes \Delta)$$

is a chromatic morphism for the generator A .

Proof. First, we set $h = \lambda(S(\Lambda_{(2)}))\Lambda_{(1)}$ and prove

$$(11) \quad h = \lambda(S(\Lambda_{(2)}))\Lambda_{(1)} = 1_A.$$

To prove this we consider

$$\begin{aligned} S(h) &= (\text{Id}_A \otimes \lambda)(S \otimes S)(\Delta(\Lambda)) = (\text{Id}_A \otimes \lambda)(\Delta^{\text{op}}(S(\Lambda))) \\ &= (\lambda \otimes \text{Id}_A)(\Delta(S(\Lambda))) \\ &= \lambda(S(\Lambda))1_A = \lambda(\Lambda)1_A = 1_A. \end{aligned}$$

Then A being finite dimensional implies S is invertible and so Equation (11) follows.

For $x \in X$ let L_x be the left action of x on A or A^* . We also need the following equation

$$(12) \quad \overleftarrow{\text{ev}}_A \circ (\text{Id}_{A^*} \otimes L_x) \circ f_{\lambda \otimes 1_A} = \lambda(x) \cdot \epsilon$$

which is proved, for all $x \in A$, in the proof of Lemma 3.8 of [De Renzi et al. 2018].

Let g be the morphism on the left side of Equation (3). Let $x, y \in A$ then

$$\begin{aligned}
 g(x \otimes y) &= (\text{Id}_A \otimes \overleftarrow{\text{ev}}_A)(\Lambda \otimes \text{Id}_A)(\text{Id}_A \otimes f_{\lambda \otimes 1_A} \otimes \text{Id}_A)(x \otimes \Delta(y)) \\
 &= \Lambda_{(1)}x \otimes (\overleftarrow{\text{ev}}_A(L_{\Lambda_{(2)}} \otimes \text{Id}_A)f_{\lambda \otimes 1_A}(y_{(1)}))y_{(2)} \\
 &= \Lambda_{(1)}x \otimes (\overleftarrow{\text{ev}}_A(\text{Id}_{A^*} \otimes L_{S(\Lambda_{(2)})})f_{\lambda \otimes 1_A}(y_{(1)}))y_{(2)} \\
 &= \Lambda_{(1)}x \otimes \lambda(S(\Lambda_{(2)}))\epsilon(y_{(1)})y_{(2)} \\
 &= \lambda(S(\Lambda_{(2)}))\Lambda_{(1)}x \otimes y \\
 &= x \otimes y,
 \end{aligned}$$

where the fourth equality follows from Equation (12), the fifth from definition of the Hopf algebra, i.e., $(\epsilon \otimes \text{Id}_A)\Delta = \text{Id}_A$ and the last equality follows from Equation (11). $\square$

Thus, the results of this subsection complete the proof of Theorem 2.6.

6C. Proof of Theorem 2.7. In the last subsection we showed $A\text{-mod}$ satisfies the hypotheses of Theorems 2.4 and 2.5. Thus we have:

Theorem 6.4. *The category of finite dimensional modules over a finite dimensional unibalanced unimodular pivotal Hopf algebra A gives rise to an invariant of bichrome handlebody graphs F' and an invariant of 3-manifolds $\mathcal{K}_{A\text{-mod}}$.*

We will now show how to compute the invariant $\mathcal{K}_{A\text{-mod}}$. Let M be a closed connected orientable 3-manifold M . Here we use the generator $G = A$ and choose $\mathbf{O}_G$ to be the ribbon graph formed by the braid closure of the coupon filled with $h = f_\Lambda \circ \epsilon$, see Section 2B. Let (H_α, Γ) be a bichrome diagram determined by a Heegaard splitting $M = H_\alpha \cup_\Sigma H_\beta$ with lower and upper minimal reducing sets of circles $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_1, \dots, \beta_g\}$, respectively. Using the blue graph $\mathbf{O}_G$ and the chromatic morphism we change all the red circles $\{\beta_i\}$ into a blue graph. Now $\{\alpha_i\}$ is a minimal reducing set of circles on H_α and so using Property (4) of Theorem 2.2 we can cut along the discs in H_α bounded by these circles to obtain a graph Γ' on the boundary of the 3-ball. By definition

$$(13) \quad \mathcal{K}_{A\text{-mod}}(M) = F'(B^3, \Gamma').$$

See Figure 4 for an example.

For the rest of the subsection, we assume A is involutive (the square of the antipode is the identity map). In this case, the invariant $\mathcal{K}_{A\text{-mod}}(M)$ can be computed as follows. To do this we need the following two lemmas. The proof of the first is straightforward and we leave it to the reader.

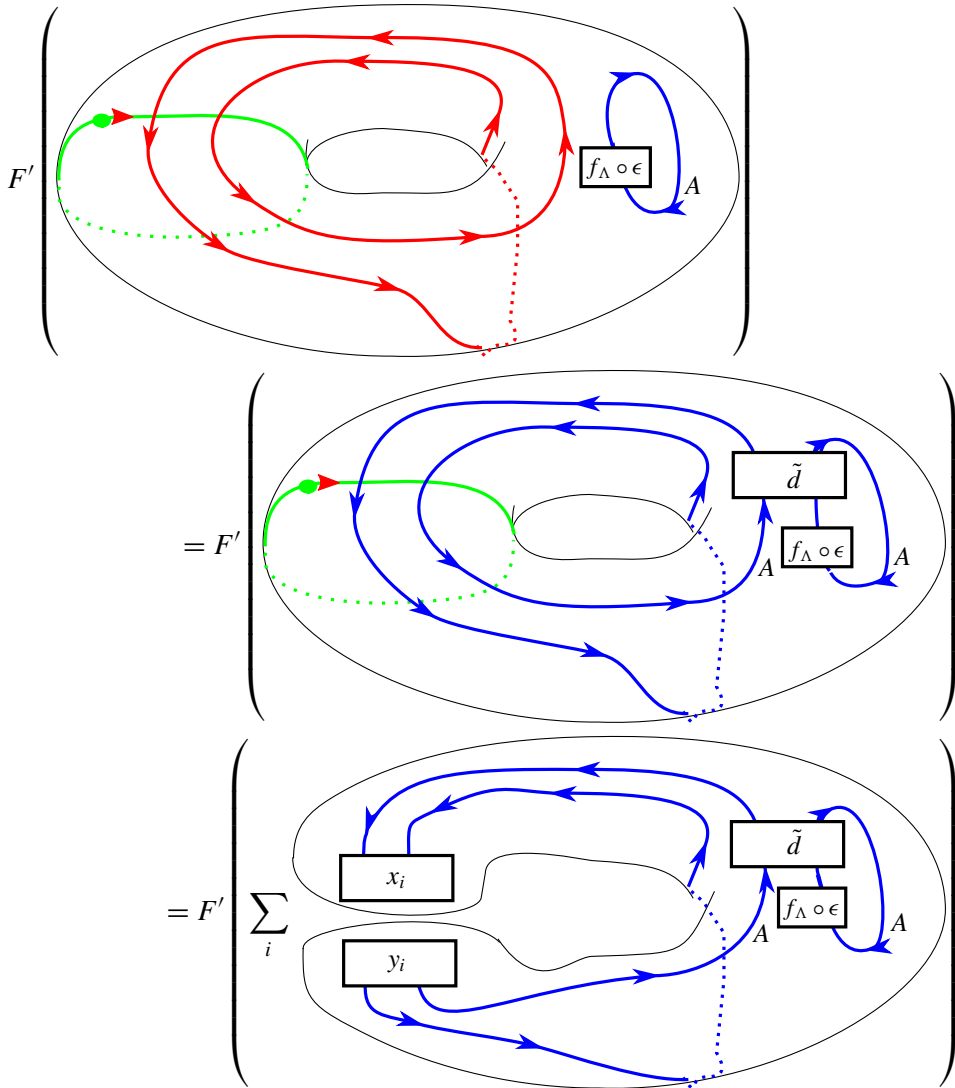


Figure 4. Here we compute $\mathcal{K}_{A\text{-mod}}$ for $L(2, 1)$. The first equality comes from using the chromatic morphism to transform the red circle into a blue graph. Then cutting along the meridian (depicted on the left) one obtains a graph $\Gamma' \subset \partial B^3$. By definition $\mathcal{K}_{A\text{-mod}}(L(2, 1)) = F'(B^3; \Gamma')$.

Lemma 6.5. *Since A is involutive then we can choose the pivotal structure of $A\text{-mod}$ to be trivial and there exist a forgetful pivotal functor from $A\text{-mod}$ to the category of vector spaces $\text{Vect}_{\mathbb{k}}$. Moreover, we have the following commutative*

diagram:

$$\begin{array}{ccc}
 \text{Rib}_{A\text{-mod}} & \xrightarrow{\text{Forgetful}} & \text{Rib}_{\text{Vect}_{\mathbb{k}}} \\
 \downarrow F & & \downarrow F_{\text{Vect}_{\mathbb{k}}} \\
 A\text{-mod} & \xrightarrow{\text{Forgetful}} & \text{Vect}_{\mathbb{k}}
 \end{array}$$

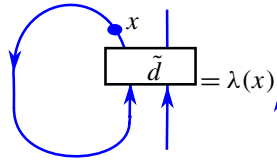
where the horizontal arrows are given by the forgetful functors and the vertical arrows are the pivotal functor from planar colored ribbon graphs Rib to $A\text{-mod}$ or $\text{Vect}_{\mathbb{k}}$.

The next lemma says that evaluating red circles with the chromatic morphism is essentially the integral.

Lemma 6.6. *For all $x \in A$ the following equality holds in $\text{Vect}_{\mathbb{k}}$*

$$(\overleftarrow{\text{ev}}_A \otimes \text{Id}_A)(\text{Id}_{A^*} \otimes L_x \otimes \text{Id}_A)(\text{Id}_{A^*} \otimes \tilde{d})(\overrightarrow{\text{coev}}_A \otimes \text{Id}_A) = \lambda(x) \text{Id}_A$$

this equation is represented pictorially as



where all the strands are colored by A and the dot (or bead) is labeled with x .

Proof. The lemma follows from the definition of the chromatic morphism and Equation (12). $\square$

Now we are ready to compute $\mathcal{K}_{A\text{-mod}}(M)$ when A is involutive and prove Theorem 2.7. To do this we continue the computations given above. In particular, by Equation (13) it suffices to determine the value of $F'(B^3, \Gamma')$ as follows. Let T_A be a (1,1)-ribbon graph whose closure is Γ' . Then Equation (8) implies

$$F'(B^3, \Gamma') = t_A(F(T_A)).$$

Recall the $A\text{-mod}$ morphism $f_\Lambda \circ \epsilon$ defined at the beginning of Section 6B. Since $t_A(f_\Lambda \circ \epsilon) = 1$ the proof of Theorem 2.7 is concluded after proving the following claim:

Claim 6.7. $F(T_A) = \text{Ku}_A(M) \cdot (f_\Lambda \circ \epsilon).$

Proof of Claim 6.7. Lemma 6.5 implies that

$$\text{Forgetful}(F(T_A)) = F_{\text{Vect}_{\mathbb{k}}}(\text{Forgetful}(T_A)).$$

Thus, it suffices to prove

$$(14) \quad F_{\text{Vect}_{\mathbb{k}}}(\text{Forgetful}(T_A)) = \text{Ku}_A(M) \cdot \text{Forgetful}(f_\Lambda \circ \epsilon)$$

since Forgetful is a faithful functor. For the rest of the proof, we will work in the category of $\mathcal{Vect}_{\mathbb{k}}$, allowing us to consider left multiplication by an element $x \in A$ as a morphism in $\mathcal{Vect}_{\mathbb{k}}$. Moreover, to simplify notation we identify each morphism in $A\text{-mod}$ with its underlying linear map.

Recall, by definition, when cutting one of the α -circles we replace the blue circles crossing the disk with a ordered matching pair of coupons filled with a sum

$$\sum_i x_i \otimes_{\mathbb{k}} y_i$$

as in Equation (2). By Lemma 6.2 this morphism is equal to the left action by Λ . We now explain how to represent this action with elements called beads.

For each circle in $\{\alpha_i\}$ choose an orientation and a base point (the following procedure is independent of these choices). As above, use the chromatic morphism to change all the red circles $\{\beta_i\}$ into blue graphs (note this induces an orientation on each circle β_i). Then instead of cutting along the bounding circles $\{\alpha_i\}$ we decorate the blue graph with certain sums of elements of A called beads as follows. Each circle α_j intersects the set of upper circles $\{\beta_i\}$ transversely. Let $c_1, \dots, c_m$ be these crossings in the order that they are encountered if we travel from the base point along α_j in the positively oriented direction. For the k -th crossing, set $p_k = 0$ if the tangent vectors of the lower circle α_j and the upper circle, in that order, form a positively oriented basis for the tangent space at c_k , otherwise set $p_k = 1$. We assign to the k -th crossing the bead $S^{p_k}(\Lambda_{(k)})$ where

$$\Delta^m(\Lambda) = \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(m)}.$$

Notice, since $\mathcal{Vect}_{\mathbb{k}}$ has a trivial ribbon structure, then the left hand side of Equation (14) only depends on the abstract graph of T_A or equivalently Γ' . Thus, we can compute $F_{\mathcal{Vect}_{\mathbb{k}}}(\text{Forgetful}(T_A))$ as follows. Each upper circle β_i has an orientation and a base point (determined by where the chromatic morphism is applied). Starting at this point and following the orientation we collect the beads to obtain a word a_i of A written from right to left. Doing this for all upper circles we obtain g beads: $a_1, \dots, a_g$. For each β_i , apply Lemma 6.6 where x is the bead a_i to obtain

$$F_{\mathcal{Vect}_{\mathbb{k}}}(\text{Forgetful}(T_A)) = \lambda(a_1)\lambda(a_2) \cdots \lambda(a_g) \cdot (f_{\Lambda} \circ \epsilon).$$

For an example of this computation see Figure 5. But when A is involutive, by definition the Kuperberg invariant is

$$\text{Ku}_A(M) = \lambda(a_1)\lambda(a_2) \cdots \lambda(a_g).$$

Thus, we have completed the proof of the claim. $\square$

$$\begin{aligned}
& \text{Forgetful} \left(F \sum_i \left(\text{Diagram 1} \right) \right) \\
&= F_{\mathcal{V}\text{ect}_{\mathbb{K}}} \left(\sum_i \left(\text{Diagram 2} \right) \right) \\
&= F_{\mathcal{V}\text{ect}_{\mathbb{K}}} \left(\text{Diagram 3} \right) \\
&= F_{\mathcal{V}\text{ect}_{\mathbb{K}}} \left(\text{Diagram 4} \right) = \lambda(S(\Lambda_{(2)})S(\Lambda_{(1)})) \cdot (f_{\Lambda} \circ \epsilon).
\end{aligned}$$

Figure 5. In Figure 4 we showed $\mathcal{K}_{A\text{-mod}}(L(2, 1)) = F'(B^3, \Gamma')$; here we continue this computations when A is involutive. The first drawing depicts the 1-1 tangle obtained by cutting Γ' along the black disk. In the first equality, we pass to the category $\mathcal{V}\text{ect}_{\mathbb{K}}$ using Lemma 6.5. Since $\mathcal{V}\text{ect}_{\mathbb{K}}$ has a symmetric braiding, the crossings make sense. The third equality re-expresses the cutting through beads (see Lemma 6.2). Finally, we collect the beads and apply Lemma 6.6.

7. The semisimple case and the TV invariant

Here we prove Theorem 2.8. Recall the notation of Section 2D.

Proof of Theorem 2.8. Let $\mathcal{T}$ be a triangulation of a closed connected orientable 3-manifold M . Let t be a maximal tree of edges of $\mathcal{T}$, then t contains all the v vertices of $\mathcal{T}$. We define a Heegaard diagram as follows. Let H_β be a regular neighborhood of the 1-skeleton of $\mathcal{T}$ and H_α its complement, then $M = H_\alpha \cup H_\beta$ is a Heegaard splitting. The β -circles are meridians of the edges of $\mathcal{T}$ not in the tree t and the α -circles bound discs formed by the 2-dimensional faces of $\mathcal{T}$. As in Section 5B this Heegaard diagram gives a bichrome handlebody graph (H_α, Γ) where the red graph is the β -circles.

By definition of the chromatic morphism we see that the value of a red unknot is the nonzero dimension D of $\mathcal{C}$ (see Equation (7)). Thus, the value of $F'(H_\alpha, \Gamma)$ does not change if we multiply it by $1/D$ and at the same time add a red unknot to the bichrome handlebody graph (H_α, Γ) . With this in mind, we construct a new bichrome handlebody graph as follows. Starting with (H_α, Γ) place a red unknot on the boundary of the neighborhood of each edge of the tree t . Let e be a leaf of t , i.e., an edge of t such that one of its vertices has degree 1. Then we can slide the new red unknot associated to e over all the red meridians of the edges adjacent to this vertex. After these slidings the new unknot becomes a red meridian around e . Continuing this process on the leaves of $t \setminus \{e\}$, we obtain a bichrome handlebody graph (H_α, Γ') where each neighborhood of an edge of $\mathcal{T}$ has a red meridian. Combining the fact that F' satisfies the sliding property for red circles and the discussion above we have

$$(15) \quad \mathcal{K}_{\mathcal{C}}(M) = F'(H_\alpha, \Gamma) = \frac{1}{D^{v-1}} F'(H_\alpha, \Gamma'),$$

where v is the number of vertices of $\mathcal{T}$.

Now we compute $F'(H_\alpha, \Gamma')$. Use the chromatic morphism to make all the red circles blue. By definition of the chromatic morphism each red circle is changed to a blue circle colored with G or equivalently the weighted sum $\sum_i \text{qdim}(S_i) S_i$ (note this happens for a meridian of each edge of $\mathcal{T}$). Then cutting along the discs formed by the 2-dimensional faces of $\mathcal{T}$ we obtain a set of spherical tetrahedra (indexed by the set of tetrahedra of $\mathcal{T}$) whose edges are all colored by G and whose four 3-legged coupons are filled with morphisms coming from the cutting. Using the fact that G splits as an orthogonal direct sum of simple objects one can see that each of these spherical tetrahedra is a sum indexed by colorings of the edges of T by elements of $\{S_i\}$. Moreover, each component of this sum is proportional to the corresponding usual $6j$ -symbol. Therefore, we have

$$\text{TV}_{\mathcal{C}}(\mathcal{T}) = \frac{1}{D^v} F'(H_\alpha, \Gamma'),$$

where $\mathrm{TV}_{\mathcal{C}}(M) = \mathrm{TV}_{\mathcal{C}}(\mathcal{T})$ is the TV invariant associated to the triangulation $\mathcal{T}$ of M , see [Turaev 1994]. Combining this with Equation (15), we have $\frac{1}{D}\mathcal{K}_{\mathcal{C}}(M) = \mathrm{TV}_{\mathcal{C}}(M)$. $\square$

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FRANCESCO COSTANTINO
 INSTITUT DE MATHÉMATIQUES DE TOULOUSE III - PAUL SABOTIER
 TOULOUSE
 FRANCE
francesco.costantino@math.univ-toulouse.fr

NATHAN GEER
 DEPARTMENT OF MATHEMATICS AND STATISTICS
 UTAH STATE UNIVERSITY
 LOGAN, UT
 UNITED STATES
nathan.geer@gmail.com

BERTRAND PATUREAU-MIRAND
 UMR 6205, LMBA
 UNIVERSITÉ DE BRETAGNE-SUD
 CAMPUS TOHANNIC
 VANNES
 FRANCE
bertrand.patureau@univ-ubs.fr

VLADIMIR TURAEV
 DEPARTMENT OF MATHEMATICS
 INDIANA UNIVERSITY
 BLOOMINGTON, IN
 UNITED STATES
vturaev@yahoo.com

A NEW EQUIVALENCE BETWEEN SUPER HARISH-CHANDRA PAIRS AND LIE SUPERGROUPS

FABIO GAVARINI

It is known that there exists a natural functor Φ from Lie supergroups to super Harish-Chandra pairs. A functor going backwards, that associates a Lie supergroup with each super Harish-Chandra pair, yielding an equivalence of categories, was found by Koszul (1983), and later generalized by several authors.

We provide two new backwards equivalences, i.e., two different functors Ψ° and Ψ^e that construct a Lie supergroup (thought of as a special group-valued functor) out of a given super Harish-Chandra pair, so that both Ψ° and Ψ^e are quasi-inverse to the functor Φ .

1. Introduction

To every Lie supergroup G one can associate the pair $(G_0, \mathfrak{g})$ formed by the classical Lie group G_0 underlying G and the tangent Lie superalgebra $\mathfrak{g} = \text{Lie}(G)$ of G ; these two objects are “compatible” in a natural sense, so that their pair is what is called a “super Harish-Chandra pair”, or just “sHCp” for short. Overall, mapping $G \mapsto (G_0, \mathfrak{g})$ yields a functor, call it Φ , from the category of Lie supergroups — either smooth, analytic or holomorphic — to the category of super Harish-Chandra pairs — of smooth, analytic or holomorphic type respectively. Is there any functor Ψ from sHCps to Lie supergroups which is a quasi-inverse for Φ , so that the two categories are equivalent? And how explicit is such a functor (if any exist)?

An answer to this question was given by Kostant [1977] and Koszul [1983] in the real smooth case, providing an explicit quasi-inverse for Φ . Later on, Vishnyakova [2011] fixed the complex holomorphic case, and her proof works for the real analytic case as well. This result was increasingly extended to the setup of algebraic supergeometry; see [Carmeli and Fioresi 2013; Masuoka 2012; Masuoka and Shibata 2017]. It is worth remarking, though, that all these subsequent results were,

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in the end, further improvements of the original idea by Koszul (while Kostant’s method was a slight variation of that), who defined a Lie supergroup out of a sHCp $(K_+, \mathfrak{k})$ as a super-ringed space, defining the “proper” sheaf of superalgebras onto K_+ by means of $\mathfrak{k}$.

In this paper we present a new solution, namely we provide a new functor Ψ — in two different versions — from sHCps to Lie supergroups that is quasi-inverse to Φ . For this we follow the approach where, instead of thinking of supermanifolds as being super-ringed manifolds, one treats them as suitable functors, defined on the category of “Weil superalgebras”. This point of view allows to unify several different approaches to supergeometry (see [Balduzzi et al. 2013]) and also to treat the infinite-dimensional setup (see [Alldridge and Laubinger 2012]); for a broader discussion about this, we refer to the classical sources [Berezin 1987; Deligne and Morgan 1999; Leites 1980; Varadarajan 2004] and the more recent sources [Balduzzi et al. 2013; Bertram and Souvay 2014; Carmeli et al. 2011; Molotkov 2010].

Now, if we want a functor Ψ from sHCps to Lie supergroups, we need a Lie supergroup $G_{\mathcal{P}}$ for each sHCp $\mathcal{P}$; to have such a $G_{\mathcal{P}}$ (as a functor) we need a Lie group $G_{\mathcal{P}}(A)$ for each Weil superalgebra A , whose definition must be natural in A : moreover, one still has to show that the resulting functor have those additional properties that make it into a Lie supergroup. Finally, all this should aim to find a Ψ that is quasi-inverse to Φ — and this fixes ultimate bounds to the construction we are aiming towards.

Bearing all this in mind, the construction that we present goes as follows. Given a super Harish-Chandra pair $\mathcal{P} = (G_+, \mathfrak{g})$, for each Weil superalgebra A , we define a group $G_{\mathcal{P}}(A)$ abstractly, by generators and relations: this definition is natural in A , hence it yields a functor from Weil algebras to (abstract) groups, call it $G_{\mathcal{P}}$ — cf. Sections 3.1 and 3.3. As a key step in the work, we prove that $G_{\mathcal{P}}$ admits a “global splitting”, i.e., it is the direct product of G_+ times a totally odd affine superspace (isomorphic to $\mathfrak{g}_1$, the odd part of $\mathfrak{g}$): as both these are supermanifolds, it turns out that $G_{\mathcal{P}}$ itself is a supermanifold as well, hence it is a Lie supergroup because (as a functor) it is group-valued too — cf. Sections 3.2 and 3.4. One more step proves that the construction of $G_{\mathcal{P}}$ is natural in $\mathcal{P}$, so it yields a functor Ψ from sHCps to Lie supergroups: this is our candidate to be a quasi-inverse to Φ — cf. Theorems 3.2.6 and 3.4.6.

It is easy to check that $\Phi \circ \Psi$ is isomorphic to the identity functor onto sHCps. Proving that $\Psi \circ \Phi$ is isomorphic to the identity on Lie supergroups is much more demanding. For this we need to know that every Lie supergroup G has a “global splitting” on its own: this fact is more or less known among specialists, but we need it stated in a genuine geometrical form, while it is usually given in sheaf-theoretic terms — so we work it out explicitly; see Section 2.4. In fact, we find *two* different formulations of such a result: this is why, building upon them, we can provide *two* versions, Ψ^o and Ψ^e , of a functor Ψ as required.

Finally, the reader can find a more detailed treatment in the expanded version of this paper [Gavarini 2016b]. Moreover, specific examples of application can be realized by suitably adapting the constructions of *algebraic* supergroups presented in [Fioresi and Gavarini 2011; 2012; Gavarini 2014a; 2014b].

2. Preliminaries

Hereafter we fix notation and terminology for super mathematics, referring to the literature (e.g., [Alldridge et al. 2014; Alldridge and Laubinger 2012; Carmeli et al. 2011; Fioresi and Gavarini 2012; Varadarajan 2004]) for further details. We denote by $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers.

2.1. Recollection and notation.

2.1.1. Supermodules, superalgebras, Lie superalgebras. We denote by $(\mathbf{smod})_{\mathbb{K}}$ and $(\mathbf{mod})_{\mathbb{K}}$ the categories of $\mathbb{K}$ -supermodules and $\mathbb{K}$ -modules, respectively, and by $(\mathbf{salg})_{\mathbb{K}}$ and $(\mathbf{alg})_{\mathbb{K}}$ — possibly dropping $\mathbb{K}$ — the categories of all commutative $\mathbb{K}$ -superalgebras and $\mathbb{K}$ -algebras, respectively. We write $\mathbb{Z}_2 = \{0, 1\}$ for the group with two elements and $|x|$ or $p(x)$ ($\in \mathbb{Z}_2$) for the *parity* of any homogeneous x in some superspace. For $A \in (\mathbf{salg})$, $n \in \mathbb{N}$, we call $A_1^{[n]}$ the A_0 -submodule of A spanned by all products $\vartheta_1 \cdots \vartheta_n$ with $\vartheta_i \in A_1$ for all i . If $J_A := (A_1)$ is the ideal of A generated by A_1 , then $J_A = A_1^{[2]} \oplus A_1$, and $\bar{A} := A/J_A$ is a commutative superalgebra which is *totally even*, i.e., $\bar{A} \in (\mathbf{alg})$, with an obvious isomorphism $\bar{A} := A/(A_1) \cong A_0/A_1^{[2]}$. We call *Weil superalgebra* any finite-dimensional commutative $\mathbb{K}$ -superalgebra A such that $A = \mathbb{K} \oplus \mathring{A}$ where $\mathbb{K}$ is even and $\mathring{A} = \mathring{A}_0 \oplus \mathring{A}_1$ is the *nilradical* of A ; see [Balduzzi et al. 2013]. Every Weil superalgebra A is endowed with the canonical morphisms $p_A : A \rightarrow \mathbb{K}$ and $u_A : \mathbb{K} \rightarrow A$ associated with the direct sum splitting $A = \mathbb{K} \oplus \mathring{A}$; thus $p_A \circ u_A = \text{id}_{\mathbb{K}}$, so p_A is surjective and u_A is injective. Weil superalgebras over $\mathbb{K}$ form a full subcategory of $(\mathbf{salg})_{\mathbb{K}}$, denoted $(\mathbf{Wsalg})_{\mathbb{K}}$ or just $(\mathbf{Wsalg})$.

We denote by $(\mathbf{sLie})_{\mathbb{K}}$, or just $(\mathbf{sLie})$, the category of all Lie $\mathbb{K}$ -superalgebras. For any $\mathfrak{g} \in (\mathbf{sLie})_{\mathbb{K}}$, we use notation $Y^{(2)} := 2^{-1}[Y, Y]$ ($\in \mathfrak{g}_0$) for $Y \in \mathfrak{g}_1$. In particular, for $V \in (\mathbf{smod})_{\mathbb{K}}$ we consider $\mathfrak{gl}(V) := \text{End}(V)$ as a Lie $\mathbb{K}$ -superalgebra, with Lie superbracket $[A, B] := AB - (-1)^{|A||B|}BA$ for homogeneous $A, B \in \text{End}(V)$; then $Y^{(2)} = Y^2$ for odd Y . If $V := V_0 \oplus V_1$ with $V_0 := \mathbb{K}^p$ and $V_1 := \mathbb{K}^q$ we write $\text{End}(\mathbb{K}^{p|q}) := \text{End}(V)$ or $\mathfrak{gl}_{p|q} := \text{End}(V)$. Every $\mathfrak{g} \in (\mathbf{sLie})_{\mathbb{K}}$ defines a functor

$$\mathcal{L}_{\mathfrak{g}} : (\mathbf{Wsalg})_{\mathbb{K}} \rightarrow (\mathbf{Lie})_{\mathbb{K}}, \quad A \mapsto \mathcal{L}_{\mathfrak{g}}(A) := (A \otimes \mathfrak{g})_0 = (A_0 \otimes \mathfrak{g}_0) \oplus (A_1 \otimes \mathfrak{g}_1),$$

where $(\mathbf{Lie})_{\mathbb{K}}$ is the category of Lie $\mathbb{K}$ -algebras and the Lie bracket in $A \otimes \mathfrak{g}$ is $[a \otimes X, a' \otimes X'] := (-1)^{|X||a'|}aa' \otimes [X, X']$; see [Carmeli et al. 2011] for details. In particular, all this applies to $\mathfrak{g} := \text{End}(V)$.

This functorial presentation applies in particular to representations: so, if $\phi : \mathfrak{g} \rightarrow \text{End}(V)$ is a representation of $\mathfrak{g}$ onto the $\mathfrak{g}$ -module V , then it induces a natural transformation $\mathcal{L}_{\mathfrak{g}} \rightarrow \mathcal{L}_{\text{End}(V)}$.

2.1.2. Supermanifolds and supergroups. We consider “supermanifolds” (ones that are real smooth, real analytic or complex holomorphic) and the corresponding group objects (i.e., “Lie supergroups”) as described in [Balduzzi et al. 2013]. If M is any supermanifold, we denote by $|M|$ its underlying topological space and by $\mathcal{O}_M$ its *structure sheaf*. Moreover, by M_0 we denote the so-called *reduced (classical) smooth manifold* associated with M , that can be thought of as an embedded sub(super)manifold of M itself. We denote the category of (real) smooth, (real) analytic, or (complex) holomorphic supermanifolds by $(\mathbf{ssmfd})$, $(\mathbf{asmfd})$, or $(\mathbf{hsmfd})$, respectively. In most cases the distinction between these cases is immaterial, hence we shall often refer only to “supermanifolds”.

Finally, by *Lie supergroup* (real smooth, real analytic or complex holomorphic) we mean a group object in the category of supermanifolds (of the same type). These objects, together with the obvious morphisms, form a subcategory among supermanifolds, denoted $(\mathbf{Lsgrp})_{\mathbb{R}}^{\infty}$, $(\mathbf{Lsgrp})_{\mathbb{R}}^{\omega}$, and $(\mathbf{Lsgrp})_{\mathbb{C}}^{\omega}$, respectively.

2.1.3. The functorial point of view. We now recall the language of “functor(s) of points”, referring to [Balduzzi et al. 2013] for details. For two categories $\mathbf{A}$ and $\mathbf{B}$, by $[\mathbf{A}, \mathbf{B}]$ we mean the category of all functors between $\mathbf{A}$ and $\mathbf{B}$, whose morphisms are the natural transformations, while $\mathbf{A}^{\text{op}}$ will denote the *opposite category* to $\mathbf{A}$.

Given $M \in (\mathbf{ssmfd})$, its associated *functor of points* $\mathcal{F}_M : (\mathbf{ssmfd})^{\text{op}} \rightarrow (\mathbf{set})$ is defined on objects by $\mathcal{F}_M(S) := \text{Hom}(S, M)$ and on morphisms by $\mathcal{F}_M(\phi) : \mathcal{F}_M(S) \rightarrow \mathcal{F}_M(T)$, $f \mapsto (\mathcal{F}_M(\phi))(f) := f \circ \phi$, for $S, T \in (\mathbf{ssmfd})$ and $\phi \in \text{Hom}(S, T)$; the elements in $\mathcal{F}_M(S)$ are the “ S -points of M ”. Given $M, N \in (\mathbf{ssmfd})$, Yoneda’s lemma yields a full and faithful immersion $\mathcal{Y} : (\mathbf{ssmfd}) \rightarrow [(\mathbf{ssmfd})^{\text{op}}, (\mathbf{set})]$ given on objects by $M \mapsto \mathcal{F}_M$. Similar constructions hold with $(\mathbf{asmfd})$ and $(\mathbf{hsmfd})$ replacing $(\mathbf{ssmfd})$.

Following [Balduzzi et al. 2013], we consider a variant of this construction, that of “ A -points” of a supermanifold. To any $M \in (\mathbf{ssmfd})$ we associate its *Weil–Berezin (local) “functor of A -points”* $\mathcal{W}_M : (\mathbf{Wsalg}) \rightarrow (\mathcal{A}_0\text{-smfd})$, where $(\mathcal{A}_0\text{-smfd})$ is a suitable category of “ $\mathcal{A}_0$ -manifolds”. This construction is natural in M , hence it yields a functor $\mathcal{W} : (\mathbf{ssmfd}) \rightarrow [(\mathbf{Wsalg}), (\mathcal{A}_0\text{-smfd})]$, called *Shvarts embedding*, that is full and faithful. Similar constructions and results apply to analytic and to holomorphic supermanifolds as well.

Therefore one can correctly study supermanifolds via their Weil–Berezin functors. In particular, the Shvarts embedding $\mathcal{W}$ preserves products; hence also group objects: this implies (see [Balduzzi et al. 2013, §4]) that a supermanifold M is a Lie supergroup if and only if $\mathcal{W}_M$ takes values in the subcategory (among $\mathcal{A}_0$ -manifolds) of group objects (that we call “Lie $\mathcal{A}_0$ -groups”).

2.2. Lie supergroups and super Harish-Chandra pairs. Hereafter we recall the notion of a super Harish-Chandra pair (sHCp in short), referring to [Carmeli et al. 2011] for details.

2.2.1. Super Harish-Chandra pairs. We call (*smooth, analytic or holomorphic*) *super Harish-Chandra pair* over $\mathbb{K}$ any pair $(G_+, \mathfrak{g})$ such that

- (a) G_+ is a (smooth, etc.) Lie group over $\mathbb{K}$,
- (b) $\mathfrak{g} \in (\mathbf{sLie})_{\mathbb{K}}$,
- (c) $\mathrm{Lie}(G_+) = \mathfrak{g}_0$ and there is an action $\mathrm{Ad} : G_+ \rightarrow \mathrm{Aut}(\mathfrak{g})$ which extends the adjoint action of G_+ on $\mathrm{Lie}(G_+) = \mathfrak{g}_0$ and whose differential is the restriction to $\mathrm{Lie}(G_+) = \mathfrak{g}_0$ of the adjoint action of $\mathfrak{g}$.

All super Harish-Chandra pairs over $\mathbb{K}$, with their obvious morphisms, form a category, denoted $(\mathbf{sHCp})_{\mathbb{K}}$ — or $(\mathbf{sHCp})_{\mathbb{R}}^{\infty}$, $(\mathbf{sHCp})_{\mathbb{R}}^{\omega}$ or $(\mathbf{sHCp})_{\mathbb{C}}^{\omega}$ to specify that we are taking smooth, analytic or holomorphic objects.

2.2.2. The super Harish-Chandra pair of a Lie supergroup. Let G be a Lie supergroup (of either type: smooth, etc.). It is known that its underlying reduced manifold G_0 (see Section 2.1.2) is indeed a *Lie group* (either smooth, etc., like G is). Even more, the construction $G \mapsto G_0$ yields a functor from Lie supergroups (of either type) to Lie groups (of the same type). Moreover, G has a tangent Lie superalgebra $\mathrm{Lie}(G)$, whose construction is natural in G so to define a functor from Lie supergroups to Lie superalgebras (see [Carmeli et al. 2011]). In the sequel by $\mathrm{Lie}(G)$ we will mean both this Lie superalgebra and its associated functor as in Section 2.1.1. Note that for $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ one has $\mathrm{Lie}(G)(A) = \mathrm{Lie}(G(A))$, the latter being the tangent Lie algebra to the Lie group $G(A)$. Now $(G_0, \mathrm{Lie}(G))$ is a super Harish-Chandra pair — natural in G — so we have functors

$$\Phi : (\mathbf{Lsgrp})_{\mathbb{R}}^{\infty} \rightarrow (\mathbf{sHCp})_{\mathbb{R}}^{\infty}, \quad \Phi : (\mathbf{Lsgrp})_{\mathbb{R}}^{\omega} \rightarrow (\mathbf{sHCp})_{\mathbb{R}}^{\omega}, \quad \Phi : (\mathbf{Lsgrp})_{\mathbb{C}}^{\omega} \rightarrow (\mathbf{sHCp})_{\mathbb{C}}^{\omega}$$

given on objects by $G \mapsto (G_0, \mathrm{Lie}(G))$ and on morphisms by $\phi \mapsto (\phi_0, \mathrm{Lie}(\phi))$.

2.3. Special splittings for Lie supergroups. We recall now some special splittings (either “pointwise”, say, or “global” ones) for Lie supergroups: we then take them as a model to *build up* a Lie supergroup out of a given super Harish-Chandra pair.

2.3.1. Pointwise splittings for Lie supergroups. Given a Lie supergroup G , there exists a bunch of results concerning particular splittings (typically, as semidirect product) of its groups of A -points $G(A)$. They are more or less known, see, e.g., [Boseck 1989; Molotkov 2010].

To begin with, let $p : A' \rightarrow A''$ and $u : A'' \rightarrow A'$ be morphisms in $(\mathbf{Wsalg})_{\mathbb{K}}$ such that $p \circ u = \mathrm{id}_{A''}$, and let $G : (\mathbf{Wsalg})_{\mathbb{K}} \rightarrow (\mathbf{group})$ be any functor. Then $G(A)$ canonically splits into a semidirect product, namely $G(A) = \mathrm{Im}(G(u)) \rtimes \mathrm{Ker}(G(p)) \cong G(A'') \rtimes \mathrm{Ker}(G(p))$. In particular, when the functor G is in fact a *Lie supergroup*, choose $A' := A \in (\mathbf{Wsalg})_{\mathbb{K}}$, $A'' := \mathbb{K}$ with $p := p_A$ and $u := u_A$ as in Section 2.1.1. Then the splitting $G(A) \cong G(A'') \rtimes \mathrm{Ker}(G(p))$ given above now reads as a *canonical*

splitting of Lie groups $G(A) \cong G_0(\mathbb{K}) \ltimes N_G(A)$, where $G_0(\mathbb{K})$ is the ordinary Lie group underlying G and $N_G(A) := \text{Ker}(G(p_A))$.

2.3.2. Pointwise splitting for Lie superalgebras. Consider a Lie $\mathbb{K}$ -superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and its associated functor $\mathcal{L}_{\mathfrak{g}} : (\mathbf{salg})_{\mathbb{K}} \rightarrow (\mathbf{Lie})_{\mathbb{K}}$ as in Section 2.1.1. Acting like in Section 2.3.1, for any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we have $\mathcal{L}_{\mathfrak{g}}(A) = \text{Im}(\mathcal{L}_{\mathfrak{g}}(u_A)) \oplus \text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A))$, a Lie algebra splitting where the symbol “ $\oplus$ ” denotes the (internal) semidirect sum of $\text{Im}(\mathcal{L}_{\mathfrak{g}}(u_A))$ with $\text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A))$. Now, definitions give that $\text{Im}(\mathcal{L}_{\mathfrak{g}}(u_A)) \cong \mathcal{L}_{\mathfrak{g}}(\mathbb{K}) := (\mathbb{K} \otimes_{\mathbb{K}} \mathfrak{g})_0 = \mathfrak{g}_0$, and we fix the simpler notation $\mathfrak{n}_{\mathfrak{g}}(A) := \text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A))$. Then

$$(2-1) \quad \mathcal{L}_{\mathfrak{g}}(A) = \mathfrak{g}_0 \oplus \mathfrak{n}_{\mathfrak{g}}(A) \quad \text{for all } A \in (\mathbf{Wsalg})_{\mathbb{K}}.$$

In the following, we shall refer to (2-1) as to “pointwise splitting for $\mathcal{L}_{\mathfrak{g}}$ ” — or simply “for $\mathfrak{g}$ ” itself.

Now note that $A = A_0 \oplus A_1$ with $A_0 = \mathbb{K} \oplus \mathring{A}_0$ and $A_1 = \mathring{A}_1$; therefore

$$\mathcal{L}_{\mathfrak{g}}(A) := (A \otimes_{\mathbb{K}} \mathfrak{g})_0 = (A_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) = \mathfrak{g}_0 \oplus (\mathring{A}_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1),$$

from which it clearly follows that $\mathfrak{n}_{\mathfrak{g}}(A) := \text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A)) = (\mathring{A}_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (\mathring{A}_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$ for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$. This in turn entails that *the Lie algebra $\mathfrak{n}_{\mathfrak{g}}(A)$ is nilpotent*.

2.3.3. The Lie subgroup $N_G(A)$. Let again G be a Lie supergroup over $\mathbb{K}$, and $\mathfrak{g} := \text{Lie}(G)$ be its tangent Lie superalgebra. For any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, the Lie group $G(A)$ and the Lie algebra $\mathfrak{g}(A) := \mathcal{L}_{\mathfrak{g}}(A)$ — also equal to $\text{Lie}(G)(A) = \text{Lie}(G(A))$, see Section 2.2.2 — are linked by the exponential map $\exp : \mathfrak{g}(A) \rightarrow G(A)$ which is a local isomorphism (either in the smooth, analytic or holomorphic sense, as usual). Similarly for the “even counterparts” we have also the local isomorphism $\exp_0 : \mathfrak{g}_0 \rightarrow G_0(\mathbb{K})$ with $\exp_0 = \exp|_{\mathfrak{g}_0}$ if we think at $\mathfrak{g}_0$ as embedded into $\mathfrak{g}(A) := \mathcal{L}_{\mathfrak{g}}(A) = \mathfrak{g}_0 \oplus \mathfrak{n}_{\mathfrak{g}}(A)$ — see (2-1).

Now, since the Lie algebra $\mathfrak{n}_{\mathfrak{g}}(A)$ is nilpotent — see Section 2.3.2 — its image $\exp(\mathfrak{n}_{\mathfrak{g}}(A))$ for the exponential map is a (closed, connected) nilpotent Lie subgroup of $G(A)$. Furthermore, let us use notation $\mathfrak{g}(p_A) := \mathcal{L}_{\mathfrak{g}}(p_A)$ and $\mathfrak{g}(u_A) := \mathcal{L}_{\mathfrak{g}}(u_A)$, and consider the diagram

$$\begin{array}{ccc} \mathfrak{g}(A) & \xrightarrow{\exp} & G(A) \\ \mathfrak{g}(u_A) \updownarrow \mathfrak{g}(p_A) & & G(p_A) \updownarrow G(u_A) \\ \mathfrak{g}_0 & \xrightarrow{\exp_0} & G_0(\mathbb{K}) \end{array}$$

This diagram is commutative, hence in particular $G(p_A) \circ \exp = \exp_0 \circ \mathfrak{g}(p_A)$, which in turn implies $G(p_A)(\exp(\mathfrak{n}_{\mathfrak{g}}(A))) = \exp_0(\mathfrak{g}(p_A)(\mathfrak{n}_{\mathfrak{g}}(A))) = \exp_0(\{0_{\mathfrak{g}_0}\}) = \{1_{G_0(\mathbb{K})}\}$

because $\mathfrak{n}_{\mathfrak{g}}(A) := \text{Ker}(\mathfrak{g}(p_A))$; so in the end $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) \subseteq \text{Ker}(G(p_A)) =: N_G(A)$.

The fact that $\exp : \mathfrak{g}(A) \rightarrow G(A)$ is a local isomorphism, together with pointwise splittings — namely, $\mathfrak{g}(A) = \mathfrak{g}_0 \oplus \mathfrak{n}_{\mathfrak{g}}(A)$ and $G(A) = G_0(\mathbb{K}) \ltimes N_G(A)$ — and $\dim(\mathfrak{g}_0) = \dim(G_0(\mathbb{K}))$, jointly imply $\dim(\mathfrak{n}_{\mathfrak{g}}(A)) = \dim(N_G(A))$. On the other hand, as $\mathfrak{n}_{\mathfrak{g}}(A)$ is nilpotent, its exponential map — i.e., just the restriction to $\mathfrak{n}_{\mathfrak{g}}(A)$ of $\exp : \mathfrak{g}(A) \rightarrow G(A)$ — is actually a *global isomorphism* of $\mathbb{K}$ -manifolds from $\mathfrak{n}_{\mathfrak{g}}(A)$ to $\exp(\mathfrak{n}_{\mathfrak{g}}(A))$. It then follows that $\dim(\exp(\mathfrak{n}_{\mathfrak{g}}(A))) = \dim(N_G(A))$. Letting $N_G(A)^\circ$ be the connected component of $N_G(A)$, all the above eventually gives $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) = N_G(A)^\circ$.

We shall now analyze $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) = N_G(A)^\circ$, eventually proving that it coincides with $N_G(A)$.

We denote again by G the Weil–Berezin functor of $\mathcal{A}$ -points (see Section 2.1.3) of our Lie supergroup G . In detail, for $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ the group $G(A)$ of A -points of G is defined as $G(A) := G_A = \bigsqcup_{g \in |G|} G_{A,g}$, where $|G|$ is the underlying topological space of G and $G_{A,g} := \text{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{G,g}, A)$, with $\mathcal{O}_{G,g}$ being the stalk of the structure sheaf of G (now thought of as a locally ringed superspace) at the point $g \in |G|$. We adopt the canonical identification $|G| = G(\mathbb{K})$ via $g \mapsto \text{ev}_g$ with $\text{ev}_g : \mathcal{O}_{G,g} \rightarrow \mathbb{K}$ given by $f \mapsto \text{ev}_g(f) := f(g)$.

For every $g_A \in G_{A,g}$ we have $\tilde{g}_A := p_A \circ g_A$ (see Section 2.1.1) which coincides with ev_g ; moreover, the very definition gives also $\tilde{g}_A := p_A \circ g_A = G(p_A)(g_A)$. Finally, due to the splitting $A = \mathbb{K} \oplus \mathring{A}$, for every $g_A \in G(A)$, say $g_A \in G_{A,g}$, there exists also a unique map $\hat{g}_A : \mathcal{O}_{G,g} \rightarrow \mathring{A}$ such that $g_A = \tilde{g}_A + \hat{g}_A$.

Now assume $g_A \in N_G(A) := \text{Ker}(G(p_A))$. Then $G(p_A)(g_A) = 1_{G_A} \in G(A)$; therefore — by the previous analysis — we have $\tilde{g}_A = 1$, whence $g_A = 1 + \hat{g}_A$ — which can be read as the sum, in the natural sense, of maps from $\mathcal{O}_{G,1}$ to A . Now, we can rewrite our g_A as $g_A = 1 + \hat{g}_A = \exp(X_{g_A})$ with $X_{g_A} := \log(g_A) = \sum_{n=1}^{+\infty} (-1)^{n+1} (\hat{g}_A^n / n)$, where $\exp(X_{g_A}) := \sum_{n=0}^{+\infty} X_{g_A}^n / n!$ and all powers in these formulas are given by $X_{g_A}^n(f) := (X_{g_A}(f))^n$, $\hat{g}_A^n(f) := (\hat{g}_A(f))^n$, etc. All this makes sense because $\text{Im}(\hat{g}_A) \in \mathring{A}$, thus $\hat{g}_A$ is nilpotent, hence X_{g_A} is given by a finite sum and it is nilpotent, so $\exp(X_{g_A})$ is a finite sum too.

By formal properties of exponentials and logarithms, since $g_A : \mathcal{O}_{G,1} \rightarrow A$ is a (superalgebra) morphism it follows from the above expansion of g_A that $X_{g_A} : \mathcal{O}_{G,1} \rightarrow A$ is in turn a derivation; therefore — see Section 2.2.2 — $X_{g_A} \in \text{Lie}(G(A)) = (\text{Lie}(G))(A) = \mathcal{L}_{\mathfrak{g}}(A) =: \mathfrak{g}(A)$. Finally, by construction we have also $\text{Im}(X_{g_A}) \in \mathring{A}$. Along with pointwise splitting $\mathfrak{g}(A) = \mathfrak{g}_0 \oplus \mathfrak{n}_{\mathfrak{g}}(A)$ — see (2-1) for $\mathfrak{g}(A) := \mathcal{L}_{\mathfrak{g}}(A)$ — and with $\mathfrak{n}_{\mathfrak{g}}(A) = (\mathring{A}_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (\mathring{A}_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$ all this together eventually gives $X_{g_A} \in \mathfrak{n}_{\mathfrak{g}}(A)$. All in all, we get:

Proposition 2.3.4. *For any Lie supergroup G and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we have $N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$. In particular, $N_G(A)$ is connected nilpotent, and (globally) isomorphic, as a manifold, to $\mathfrak{n}_{\mathfrak{g}}(A)$.*

Proof. Our analysis above shows that each $g_A \in N_G(A)$ can be realized as $g_A = \exp(X_{g_A})$ with $X_{g_A} \in \mathfrak{n}_g(A)$; hence $N_G(A) \subseteq \exp(\mathfrak{n}_g(A))$; conversely, Section 2.3.3 yields $\exp(\mathfrak{n}_g(A)) = N_G(A)^\circ \subseteq N_G(A)$. Thus $N_G(A) = \exp(\mathfrak{n}_g(A))$ as claimed. The last part of the claim then is clear. See also [Carmeli et al. 2019] for another proof. $\square$

2.4. Global splittings for Lie supergroups. We present now two remarkable splittings for the groups $G(A)$ of “ A -points” of a Lie supergroup G ; these are natural in A , and hence give “global splittings” of G as a functor. Such a result is often stated as a splitting of the structure sheaf (e.g., in [Berezin 1987; Molotkov 2010; Vishnyakova 2011]), we provide instead a more geometrical proof.

2.4.1. Structure theorem and global splittings for Lie supergroups. Let G be a Lie supergroup over $\mathbb{K}$, whose tangent Lie superalgebra is $\mathfrak{g} := \text{Lie}(G)$, and let $A \in (\mathbf{Wsalg})_{\mathbb{K}}$. The powers $\mathring{A}^d$ of the nilradical $\mathring{A}$ of A form a descending sequence such that $\mathring{A}^N = 0$ for $N \gg 0$ (see Section 2.1.1). Then we consider

$$\mathfrak{n}_g^{(d)}(A) := (\mathring{A}^d \otimes_{\mathbb{K}} \mathfrak{g})_0 = ((\mathring{A}^d)_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus ((\mathring{A}^d)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \quad \text{for all } d \in \mathbb{N}_+,$$

which in turn yields a decreasing filtration of Lie subalgebras of $\mathfrak{n}_g(A)$, with $\mathfrak{n}_g^{(N)}(A) = 0$ for $N \gg 0$.

Consider the case of an element $\eta Y := \eta \otimes Y \in \mathfrak{n}_g(A)$ with $\eta \in A_1$, $Y \in \mathfrak{g}_1$. As $\eta^2 = 0$, if we express $\exp(\eta Y)$ as a formal series we actually have $\exp(\eta Y) = 1 + \eta Y$. Similarly, for every $cX = c \otimes X \in \mathfrak{n}_g(A)$ with $c \in A_0$, $X \in \mathfrak{g}_0$ if $c^2 = 0$ then also the formal series expression of $\exp(cX)$ reads $\exp(cX) = 1 + cX$.

For later use, we fix a $\mathbb{K}$ -basis B of $\mathfrak{g}$ of the form $B := B_0 \sqcup B_1$ with $B_0 = \{X_j\}_{j \in J}$ and $B_1 = \{Y_i\}_{i \in I}$ being $\mathbb{K}$ -bases of $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively. Moreover, we fix any total order $\leq$ on B — or, equivalently, on $I \sqcup J$ — such that all elements from B_0 are less than those of B_1 — in a nutshell, $B_0 \leq B_1$.

Now consider the $\mathbb{K}$ -algebra $\mathbb{K}\langle\langle Z_1, Z_2 \rangle\rangle$ of formal power series in the non-commutative variables Z_1 and Z_2 . The Campbell–Baker–Hausdorff formula in $\mathbb{K}\langle\langle Z_1, Z_2 \rangle\rangle$ is

$$\exp(Z_1) \cdot \exp(Z_2) = \exp(Z_1 * Z_2)$$

with $Z_1 * Z_2 := \log(\exp(Z_1) \cdot \exp(Z_2)) \in \mathbb{K}\langle\langle Z_1, Z_2 \rangle\rangle$. More precisely, the formal series expansion of $Z_1 * Z_2$ can be rearranged as a formal series $Z_1 * Z_2 = \sum_{n=1}^{+\infty} L_n(Z_1, Z_2)$, where each $L_n(Z_1, Z_2)$ is a homogeneous Lie monomial of degree n in the free Lie $\mathbb{K}$ -algebra $\langle Z_1, Z_2 \rangle_{\text{Lie}}^{\mathbb{K}}$ generated by Z_1 and Z_2 . In particular, if one replaces Z_1 and Z_2 with elements z_1 and z_2 sitting in some nilpotent Lie algebra, then all but finitely many of the $L_n(z_1, z_2)$ ’s do vanish; hence $z_1 * z_2$ can be written as a finite sum.

Our next goal is another description of $N_G(A) = \exp(\mathfrak{n}_g(A))$. We need an auxiliary result:

Lemma 2.4.2. *Let $S_1, \dots, S_\ell \in \mathfrak{n}_{\mathfrak{g}}(A)$ with $S_i \in \mathfrak{n}_{\mathfrak{g}}^{(d_i)}(A)$, $d_i \in \mathbb{N}_+$ ($i = 1, \dots, \ell$). Then there exist $T_1, \dots, T_k \in \mathfrak{n}_{\mathfrak{g}}(A)$ such that $T_j \in \mathfrak{n}_{\mathfrak{g}}^{(\partial_j)}(A)$ with $\partial_j \geq d_{a_j} + d_{b_j}$ and $a_j, b_j \in \{1, \dots, \ell\}$ ($j = 1, \dots, k$), and*

$$\exp(S_1 + \dots + S_\ell) = \exp(S_1) \cdots \exp(S_\ell) \exp(T_1) \cdots \exp(T_k)$$

Proof. Writing all exponentials as formal series (actually *finite sums*!) the claim follows at once from definitions by induction on ℓ via a straightforward application of Baker–Campbell–Hausdorff formula. $\square$

We can now provide our new description of the subgroup $N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$:

Proposition 2.4.3. *The subgroup $N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$ of $G(A)$ is generated by the set*

$$\Gamma_B := \{\exp(t_j X_j), \exp(\eta_i Y_i) \mid t_j \in \mathring{A}_0, \eta_i \in \mathring{A}_1 = A_1, \text{ for all } j \in J, i \in I\},$$

where $\{X_j\}_{j \in J} \sqcup \{Y_i\}_{i \in I} = B_0 \sqcup B_1 = B$ is the $\mathbb{K}$ -basis of $\mathfrak{g}$ chosen in Section 2.4.1.

Proof. Let $n \in N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$, say $n = \exp(Z)$ with $Z \in \mathfrak{n}_{\mathfrak{g}}(A)$; clearly we can assume $Z \neq 0$. Using our fixed, ordered, $\mathbb{K}$ -basis B of $\mathfrak{g}$ our Z expands into

$$Z = \sum_{j \in J} t'_j X_j + \sum_{i \in I} \eta'_i Y_i$$

for some $t'_j \in \mathring{A}_0$ and $\eta'_i \in \mathring{A}_1$, by the very definition of $\mathfrak{n}_{\mathfrak{g}}(A)$. By Lemma 2.4.2, this implies that

$$\begin{aligned} \exp(Z) &= \exp\left(\sum_{j \in J} t'_j X_j + \sum_{i \in I} \eta'_i Y_i\right) \\ &= \overrightarrow{\prod}_{j \in J} \exp(t'_j X_j) \overrightarrow{\prod}_{i \in I} \exp(\eta'_i Y_i) \cdot \exp(Z_1^{(1)}) \cdots \exp(Z_{k_1}^{(1)}) \end{aligned}$$

for some $Z_1^{(1)}, \dots, Z_{k_1}^{(1)} \in \mathfrak{n}_{\mathfrak{g}}(A)$, where $\overrightarrow{\prod}_{j \in J}$ and $\overrightarrow{\prod}_{i \in I}$ denote ordered products. Furthermore, the lemma ensures that these $Z_h^{(1)}$'s “lie deeper”, in the decreasing filtration of $\mathfrak{n}_{\mathfrak{g}}(A)$ given by the $\mathfrak{n}_{\mathfrak{g}}^{(d)}(A)$'s, than the initial Z we started with: so we can iterate this argument and finally stop after finitely many steps. $\square$

2.4.4. Special exponentials in $G(A)$. Before going on, let us consider elements in $G(A)$ of the form $\exp(tX)$ or $\exp(\eta Y)$ — with $t \in A_0$ such that $t^2 = 0$, $\eta \in \mathring{A}_1$, $X \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_1$. Since both t and η have square zero, the formal power series expansion of both $\exp(tX)$ and $\exp(\eta Y)$ stops at first order, i.e., it reads $\exp(tX) = (1 + tX)$ and $\exp(\eta Y) = (1 + \eta Y)$ respectively. More generally, we consider elements of the form $\exp(\mathcal{X})$, $\exp(\mathcal{Y}) \in \exp(\mathfrak{n}_{\mathfrak{g}}(A)) = N_G(A)$ with $\mathcal{X} \in \mathring{A}_0 \otimes \mathfrak{g}_0$ and $\mathcal{Y} \in \mathring{A}_1 \otimes \mathfrak{g}_1 = A_1 \otimes \mathfrak{g}_1$. As $\mathring{A}_0$ and A_1 are nilpotent, the power series expansion of $\exp(\mathcal{X})$ and $\exp(\mathcal{Y})$ can be seen again as polynomials.

In the next lemma we collect some identities in $G(A)$ involving these “special exponentials”.

Lemma 2.4.5. *Let $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, $\eta, \eta', \eta'' \in A_1$, $\eta_i \in A_1$ (for all $i \in I$), $Y, Y' \in \mathfrak{g}_1$, $X \in \mathfrak{g}_0$ and $g_0 \in G_0(A)$. Then inside $G(A)$ we have*

- (a) $(1 + \eta\eta'[Y, Y']) = \exp(\eta\eta'[Y, Y']) \in G_0(A),$
- (b) $(1 + \eta Y)g_0 = g_0(1 + \eta \operatorname{Ad}(g_0^{-1})(Y)),$
 $\exp(\sum_{i \in I} \eta_i Y_i)g_0 = g_0 \exp(\sum_{i \in I} \eta_i \operatorname{Ad}(g_0^{-1})(Y_i)),$
- (c) $(1 + \eta'Y')(1 + \eta''Y'') = (1 + \eta''\eta'[Y', Y''])(1 + \eta''Y'')(1 + \eta'Y'),$
- (d) $(1 + \eta Y')(1 + \eta Y'') = (1 + \eta(Y' + Y'')) = (1 + \eta Y'')(1 + \eta Y'),$
- (e) $(1 + \eta'Y)(1 + \eta''Y) = (1 + \eta''\eta'Y^{(2)})(1 + (\eta' + \eta'')Y),$
- (f) $(1 + \eta Y)(1 + \eta'\eta''X) = (1 + \eta'\eta''X)(1 + \eta\eta'\eta''[Y, X])(1 + \eta Y)$
 $= (1 + \eta'\eta''X)(1 + \eta Y)(1 + \eta\eta'\eta''[Y, X]),$

(g) *Let $(h, k) := hkh^{-1}k^{-1}$ be the commutator of elements h and k in a group. Then*

$$((1 + \eta Y), (1 + \eta'Y')) = (1 + \eta'\eta[Y, Y']), ((1 + \eta Y), (1 + \eta Y')) = (1 + \eta(Y + Y')),$$

$$((1 + \eta'Y), (1 + \eta''Y)) = (1 + \eta''\eta'Y^{(2)})^2 = (1 + \eta''\eta'2Y^{(2)}) = (1 + \eta''\eta'[Y, Y])$$

(N.B.: taking the rightmost term in the last identity, the latter is a special case of the first).

(h) *For any $n \in \mathbb{N}_+$, there exist unique $T_0^{(n)}, T_1^{(n)} \in \langle Z_1, Z_2 \rangle_{\operatorname{Lie}}^{\mathbb{K}}$, independent of A , such that:*

- $T_0^{(n)}$ is a $\mathbb{K}$ -linear combination of Lie monomials of **even** degree greater than n ,
- $T_1^{(n)}$ is a $\mathbb{K}$ -linear combination of Lie monomials of **odd** degree greater than n ,
- setting $d_1 := \dim(\mathfrak{g}_1)$, for any $\mathcal{Y}', \mathcal{Y}'' \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$ we have

$$\exp(\mathcal{Y}') \exp(\mathcal{Y}'') = \exp(P_0^{(d_1)}(\mathcal{Y}', \mathcal{Y}'')) \exp(\mathcal{Y}' + \mathcal{Y}'' + P_1^{(d_1)}(\mathcal{Y}', \mathcal{Y}'')),$$

$$\text{with } P_0^{(d_1)} := T_0^{(1)} * T_0^{(2)} * \dots * T_0^{(d_1-1)} \text{ and } P_1^{(d_1)} := T_1^{(d_1-1)} + \dots + T_1^{(2)} + T_1^{(1)}.$$

Proof. Writing all exponentials as formal power series (actually *finite sum*, as noticed above), claims (a) through (g) follow at once from definitions, via straightforward applications of the Baker–Campbell–Hausdorff formula. Claim (a) is even simpler, since $(1 + \eta\eta'[Y, Y'])$ is just the formal power series expansion of $\exp(\eta\eta'[Y, Y'])$, and the latter belong to $\exp(\mathring{A}_0\mathfrak{g}_0) \subseteq G_0(A)$.

Claim (h) requires some more work. An equivalent formulation is that the identity

$$(2-2) \quad \mathcal{Y}' * \mathcal{Y}'' = P_0^{(d_1)}(\mathcal{Y}', \mathcal{Y}'') * (\mathcal{Y}' + \mathcal{Y}'' + P_1^{(d_1)}(\mathcal{Y}', \mathcal{Y}''))$$

holds true for some uniquely determined Lie polynomials

$$P_0^{(d_1)} := T_0^{(1)} * T_0^{(2)} * \dots * T_0^{(d_1-1)} \quad \text{and} \quad P_1^{(d_1)} := T_1^{(d_1-1)} + \dots + T_1^{(2)} + T_1^{(1)}$$

with the $T_{0/1}^{(i)}$'s having the properties mentioned above.

We start working with the product “ $*$ ” in $\langle Z_1, Z_2 \rangle_{\text{Lie}}^{\mathbb{K}}$. As a matter of terminology, we call the *order* of any nonzero Lie polynomial P in two variables the least degree of a homogeneous monomial occurring with nonzero coefficient in the standard $\mathbb{K}$ -linear expansion of P (while the zero polynomial has order $-\infty$).

First we need some technical results. For formal F, G there exist unique $R, S \in \langle F, G \rangle_{\text{Lie}}^{\mathbb{K}}$ such that

$$(2-3) \quad F * G = R * (F + G), \quad F + G = F * S * G$$

and R, S are Lie polynomials (in F and G) of order greater than 1: in fact, R is the unique solution of the equation $\exp(F) \exp(G) = \exp(R) \exp(F + G)$, while S is the unique solution of the equation $\exp(F + G) = \exp(F) \exp(S) \exp(G)$. Then the explicit expression of the product “ $*$ ” implies that both R and S have order greater than 1, as claimed; moreover, both are independent of A and $\mathfrak{g}$ whatsoever. Finally, for any Lie polynomial T in two variables there exist unique T_0 and T_1 such that $T = T_0 + T_1$ where T_0 and T_1 are $\mathbb{K}$ -linear combinations of Lie monomials of *even* and *odd* degree, respectively.

With repeated applications of (2-3) and of the identity $T = T_0 + T_1$ we eventually find the expression

$$Z_1 * Z_2 = T_0^{(1)} * T_0^{(2)} * \dots * T_0^{(n-1)} * T^{(n)} * (T_1^{(n-1)} + \dots + T_1^{(2)} + T_1^{(1)} + Z_1 + Z_2)$$

for arbitrarily big $n \in \mathbb{N}$, where each Lie polynomial T_s has order greater than s . Finally, we can rewrite this last formula, for all $n \in \mathbb{N}$, as

$$(2-4) \quad Z_1 * Z_2 = P_0^{(n)}(Z_1, Z_2) * T^{(n)} * (Z_1 + Z_2 + P_1^{(n)}(Z_1, Z_2))$$

with $P_0^{(n)} := T_0^{(1)} * T_0^{(2)} * \dots * T_0^{(n-1)}$ and $P_1^{(n)} := T_1^{(n-1)} + \dots + T_1^{(2)} + T_1^{(1)}$.

But every Lie monomial of degree $m > d_1 := \dim(\mathfrak{g}_1)$ vanishes on $A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$, hence $T^{(m-1)}$ vanishes too. It follows that for $n = d_1$ replacing $\mathcal{Y}'$ for Z_1 and $\mathcal{Y}''$ for Z_2 in (2-4) we eventually get (2-2). $\square$

We still need to introduce some auxiliary objects associated with G :

Definition 2.4.6. Let G be a Lie supergroup, as above. For any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we define:

- (a) $G^-(A) := \left\{ \prod_{s=1}^n (1 + \eta_s Y_s) \mid n \in \mathbb{N}, (\eta_s, Y_s) \in A_1 \times \mathfrak{g}_1 \forall s \in \{1, \dots, n\} \right\} (\subseteq G(A)),$
- (b) $\exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) := \{\exp(\mathcal{Y}) \mid \mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1\} (\subseteq G^-(A)),$
- (c) $N_G^{[2]}(A) := \exp(A_1^{[2]} \otimes_{\mathbb{K}} [\mathfrak{g}_1, \mathfrak{g}_1]) (\subseteq N_{G_0}(A_0) = N_G(A) \cap G_0(A)),$
- (d) for any fixed $\mathbb{K}$ -basis $\{Y_i\}_{i \in I}$ of $\mathfrak{g}_1$ (for some index set I) and any fixed total order in I ,

$$G^{\leq}(A) := \left\{ \overrightarrow{\prod}_{i \in I} (1 + \eta_i Y_i) \mid \eta_i \in A_1 \text{ for all } i \in I \right\} (\subseteq G^-(A)),$$

where $\overrightarrow{\prod}_{i \in I}$ denotes an *ordered product* with respect to the fixed total order in I . $\diamond$

Remark 2.4.7. By definition, $\exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$ contains the set of generators of $G^-(A)$; therefore, the former generates a subgroup $\langle \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \rangle$ of $G(A)$ that contains $G^-(A)$. On the other hand, for any $\sum_{s=1}^n \eta_s Y_s \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$, the formal series expansion of $\exp(\sum_{s=1}^n \eta_s Y_s)$ yields

$$\exp\left(\sum_{s=1}^n \eta_s Y_s\right) = \prod_{\sigma \in \mathcal{S}_n} (1 + \eta_{\sigma(1)} Y_{\sigma(1)} / n!) \cdot (1 + \eta_{\sigma(2)} Y_{\sigma(2)} / n!) \cdots (1 + \eta_{\sigma(n)} Y_{\sigma(n)} / n!),$$

implying $\langle \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \rangle \subseteq G^-(A)$. The outcome is that $G^-(A) = \langle \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \rangle$.

From now on, we fix a $\mathbb{K}$ -basis $\{Y_i\}_{i \in I}$ of $\mathfrak{g}_1$ (for some index set I) and we fix in I a total order, as in Definition 2.4.6(d). Our first result provides new, interesting factorizations for $G(A)$:

Proposition 2.4.8. *Let G be a Lie supergroup as above, let $\{Y_i\}_{i \in I}$ be a totally ordered $\mathbb{K}$ -basis of $\mathfrak{g}_1$ (for some total order in the set I) and let $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ be any Weil superalgebra. Then:*

(a) $G^-(A)$ coincides with the subgroup $\langle G_-^{\leq}(A) \rangle$ of $G(A)$ generated by $G_-^{\leq}(A)$ and with the subgroup $\langle \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \rangle$ generated by $\exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$;

(b) there exist set-theoretic factorizations (with respect to the group product “ $\cdot$ ”)

$$(2-5) \quad G^-(A) = N_G^{[2]}(A) \cdot G_-^{\leq}(A), \quad G^-(A) = G_-^{\leq}(A) \cdot N_G^{[2]}(A),$$

$$(2-6) \quad N_G(A) = N_{G_0}(A_0) \cdot G_-^{\leq}(A), \quad N_G(A) = G_-^{\leq}(A) \cdot N_{G_0}(A_0),$$

$$(2-7) \quad G(A) = G_0(A) \cdot G_-^{\leq}(A), \quad G(A) = G_-^{\leq}(A) \cdot G_0(A);$$

(c) there exist set-theoretic factorizations (with respect to the group product “ $\cdot$ ”)

$$(2-8) \quad G^-(A) = N_G^{[2]}(A) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1), \quad G^-(A) = \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \cdot N_G^{[2]}(A),$$

$$(2-9) \quad N_G(A) = N_{G_0}(A_0) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1), \quad N_G(A) = \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \cdot N_{G_0}(A_0),$$

$$(2-10) \quad G(A) = G_0(A) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1), \quad G(A) = \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \cdot G_0(A).$$

Proof. (a) Remark 2.4.7 proves that $\langle \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \rangle = \langle G_-^{\leq}(A) \rangle$, so we are left to prove $\langle G_-^{\leq}(A) \rangle = G^-(A)$. By definition, $G^-(A)$ is the subgroup in $G(A)$ generated by $\{(1 + \eta Y) \mid \eta \in A_1, Y \in \mathfrak{g}_1\}$: thus it is enough to prove that each $(1 + \eta Y)$ lies in $\langle G_-^{\leq}(A) \rangle$. For $Y \in \mathfrak{g}_1$ let $Y = \sum_{i \in I} c_i Y_i$ be its $\mathbb{K}$ -linear expansion with respect to the basis $\{Y_i\}_{i \in I}$ of $\mathfrak{g}_1$; then repeated applications of Lemma 2.4.5(d) yield,

$$(1 + \eta Y) = (1 + \eta \sum_{i \in I} c_i Y_i) = (1 + \sum_{i \in I} (c_i \eta) Y_i) = \prod_{i \in I} (1 + (c_i \eta) Y_i) \in G_-^{\leq}(A). \quad \square$$

(b) This part can be proved much like Proposition 4.2.7 in [Gavarini 2016a], but now applying Lemma 2.4.5.

(c) Starting with (2-8), it is enough to prove the left-hand side, i.e., $G^-(A) = N_G^{[2]}(A) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$.

First of all, the inclusion $N_G^{[2]}(A) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \subseteq G^-(A)$ follows at once from (2-5) together with claim (a). Moreover, again by claim (a) we have $G^-(A) = \langle \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \rangle$ so it is enough to prove that the product of any two generators $\exp(\mathcal{Y}')$ and $\exp(\mathcal{Y}'')$ of $G^-(A) = \langle \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \rangle$ lies in $N_G^{[2]}(A) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$. In fact, this follows at once from the identity in Lemma 2.4.5(h).

The same argument used above also applies to prove (2-9).

Finally, the following chain of identities

$$\begin{aligned} G(A) &= G_0(\mathbb{K}) \cdot N_G(A) = G_0(\mathbb{K}) \cdot N_{G_0}(A_0) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \\ &= G_0(A_0) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) = G_0(A) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1), \end{aligned}$$

using the pointwise splitting $G(A) \cong G_0(\mathbb{K}) \ltimes N_G(A)$ — see Section 2.3.1 — and (2-9), proves (2-10). $\square$

We aim to improve the previous result. We still need a technical result:

Lemma 2.4.9. *Given a Lie supergroup G and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, let $\zeta_i \in A_1$ ($i \in I$). Then*

- (a) *if $g := \prod_{i \in I} (1 + \zeta_i Y_i) \in G_0(A) \cap G_-^-(A)$, then $\zeta_i = 0$ for all $i \in I$;*
- (b) *if $g := \exp(\sum_{i \in I} \zeta_i Y_i) \in G_0(A) \cap \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$, then $\zeta_i = 0$ for all $i \in I$.*

Proof. (a) Recall (see Section 2.3.3) that, $G(A) := \coprod_{x \in |G|} \mathrm{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{|G|,x}, A)$, by definition; therefore

$$g := \prod_{i \in I} (1 + \zeta_i Y_i) = 1 + \sum_{i \in I} \zeta_i Y_i + \mathcal{O}(2)$$

makes sense, as an expansion of g where $\mathcal{O}(2)$ is a short-hand notation for “*additional summands of higher order in the ζ_i ’s*”. Let $\mathfrak{a} := (\{\zeta_i\}_{i \in I})$ be the ideal of A generated by the ζ_i ’s; then the previous expansion of g yields a similar expansion $[g]_2 := 1 + \sum_{i \in I} [\zeta_i]_2 Y_i$ inside $G(A/\mathfrak{a}^2) := \coprod_{x \in |G|} \mathrm{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{|G|,x}, A/\mathfrak{a}^2)$. Moreover, from $g \in G_0(A) \cap G_-^-(A)$ we get $[g]_2 \in G_0(A/\mathfrak{a}^2) \cap G_-^-(A/\mathfrak{a}^2)$ too, hence — thinking of $[g]_2$ as an $A/\mathfrak{a}^2$ -valued map — also $\mathrm{Im}([g]_2) \subseteq (A/\mathfrak{a}^2)_0$.

As $\{Y_i\}_{i \in I}$ is a $\mathbb{K}$ -basis of $\mathfrak{g}_1$, there exists a local system of coordinates around the unit point $1_G \in |G|$, say $\{y_i\}_{i \in I}$, such that $Y_i(y_j) = \delta_{i,j}$ for all $i, j \in I$. Then $[g]_2(y_j) := 1 + \sum_{i \in I} [\zeta_i]_2 Y_i(y_j) = [\zeta_j]_2$, in particular $[g]_2(y_j) = [\zeta_j]_2 \in (A/\mathfrak{a}^2)_1$; this together with $\mathrm{Im}([g]_2) \subseteq (A/\mathfrak{a}^2)_0$ implies $[\zeta_j]_2 = [0]_2 \in A/\mathfrak{a}^2$, i.e., $\zeta_j \in \mathfrak{a}^2 = (\{\zeta_i\}_{i \in I})^2$ for all $j \in I$: thus $\zeta_j \in \mathfrak{a}^n$ for $n \in \mathbb{N}$, $j \in I$. But $\mathfrak{a}^n = 0$ for $n \gg 0$, so $\zeta_j = 0$ for all $j \in I$.

(b) The same argument as in (a) applies again. $\square$

Finally, we are ready to state the main result of this subsection. Later on, we refer to the isomorphisms in part (a) and/or (c) of its statement as “*global splittings*”.

Theorem 2.4.10 (existence of global splittings for Lie supergroups). *Let G be a Lie supergroup, and $\mathfrak{g}$ its tangent Lie superalgebra.*

(a) *The restriction of group multiplication in G provides isomorphisms of (set-valued) functors*

$$\begin{aligned} N_G^{[2]} \times G_-^\leq &\cong G^-, & N_{G_0} \times G_-^\leq &\cong N_G, & G_0 \times G_-^\leq &\cong G, \\ G_-^\leq \times N_G^{[2]} &\cong G^-, & G_-^\leq \times N_{G_0} &\cong N_G, & G_-^\leq \times G_0 &\cong G, \\ N_G^{[2]} \times \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) &\cong G^-, & \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \times N_G^{[2]} &\cong G^-, \\ N_{G_0} \times \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) &\cong N_G, & \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \times N_{G_0} &\cong N_G, \\ G_0 \times \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) &\cong G, & \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \times G_0 &\cong G, \end{aligned}$$

with $\exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$ the set-valued functor $(\mathbf{Wsalg})_{\mathbb{K}} \rightarrow (\mathbf{sets})$ given by $A \mapsto \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$.

(b) *Setting notation $d_1 := \dim_{\mathbb{K}}(\mathfrak{g}_1) = |I|$, there exist isomorphisms of (set-valued) functors $\mathbb{A}_{\mathbb{K}}^{0|d_1} \cong G_-^\leq$ and $\mathbb{A}_{\mathbb{K}}^{0|d_1} \cong \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$, given on A -points — for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — by*

$$\begin{aligned} \mathbb{A}_{\mathbb{K}}^{0|d_1}(A) &= A_1^{d_1} \rightarrow G_-^\leq(A), & (\eta_i)_{i \in I} &\mapsto \prod_{i \in I} (1 + \eta_i Y_i), \\ \mathbb{A}_{\mathbb{K}}^{0|d_1}(A) &= A_1^{d_1} \rightarrow \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1), & (\eta_i)_{i \in I} &\mapsto \exp(\sum_{i \in I} \eta_i Y_i). \end{aligned}$$

(c) *The isomorphisms in (a) and (b) induce further isomorphisms of (set-valued) functors*

$$\begin{aligned} N_G^{[2]} \times \mathbb{A}_{\mathbb{K}}^{0|d_1} &\cong G^-, & N_{G_0} \times \mathbb{A}_{\mathbb{K}}^{0|d_1} &\cong N_G, & G_0 \times \mathbb{A}_{\mathbb{K}}^{0|d_1} &\cong G, \\ \mathbb{A}_{\mathbb{K}}^{0|d_1} \times N_G^{[2]} &\cong G^-, & \mathbb{A}_{\mathbb{K}}^{0|d_1} \times N_{G_0} &\cong N_G, & \mathbb{A}_{\mathbb{K}}^{0|d_1} \times G_0 &\cong G. \end{aligned}$$

Proof. The proof can be done along the same lines followed for Proposition 4.2.11 in [Gavarini 2016a]. The technical, accessory results that one needs to exploit are the various lemmas and propositions developed so far in the present subsection: of course they are formally different, but the overall argument is the same. $\square$

3. From super Harish-Chandra pairs to Lie supergroups

In this section we provide two different functors Ψ that are quasi-inverse to the functor Φ of Section 2.2.2. In both cases, for any super Harish-Chandra pair $\mathcal{P}$, we define as associated $\Psi(\mathcal{P}) := G_{\mathcal{P}}$ a suitable functor from Weil superalgebras to groups, and then prove that it has the “right properties”. Concretely, we follow the pattern provided by the *global splitting theorem* for Lie supergroups, which tells us two possible ways how our would-be Lie supergroup $G_{\mathcal{P}}$ should look like: this yields us, eventually, two different recipes.

3.1. Supergroup functors out of super Harish-Chandra pairs: first recipe. For later use, we fix the following notation: given $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$, $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ and $c \in A_0$ such that $c^2 = 0$, for every $X \in \mathfrak{g}_0$ we set $(1_{G_+} + cX) := \exp(cX) \in G_+(A_0)$. When no confusion is possible we will drop the subscript G_+ and simply write $(1 + cX)$ instead. Similarly, we shall presently introduce new formal elements of type “ $(1 + \eta Y) = \exp(\eta Y)$ ” with $\eta \in A_1$, $Y \in \mathfrak{g}_1$.

Definition 3.1.1. Let $\mathcal{P} := (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be a super Harish-Chandra pair over $\mathbb{K}$.

(a) We introduce a functor $G_{\mathcal{P}}^{\circ} : (\mathbf{Wsalg})_{\mathbb{K}} \rightarrow (\mathbf{group})$ as follows. For any Weil superalgebra $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we define $G_{\mathcal{P}}^{\circ}(A)$ as being the group with generators the elements of the set

$$\Gamma_A := \{g_+, (1 + \eta Y) \mid g_+ \in G_+(A), (\eta, Y) \in A_1 \times \mathfrak{g}_1\} = G_+(A) \cup \{(1 + \eta Y)\}_{(\eta, Y) \in A_1 \times \mathfrak{g}_1}$$

and relations (for $g'_+, g''_+ \in G_+(A)$, $\eta, \eta', \eta'' \in A_1$, $Y, Y', Y'' \in \mathfrak{g}_1$, $c \in \mathbb{K}$)

$$\begin{aligned} g'_+ \cdot g''_+ &= g'_+ \cdot_{G_+} g''_+, \quad (1 + \eta Y) \cdot g_+ = g_+ \cdot (1 + \eta \operatorname{Ad}(g_+^{-1})(Y)), \\ (1 + \eta'' Y) \cdot (1 + \eta' Y) &= (1_{G_+} + \eta' \eta'' Y^{(2)})_{G_+} \cdot (1 + (\eta' + \eta'') Y), \quad (1 + \eta 0_{\mathfrak{g}_1}) = 1, \\ (1 + \eta'' Y'') \cdot (1 + \eta' Y') &= (1_{G_+} + \eta' \eta'' [Y', Y''])_{G_+} \cdot (1 + \eta' Y') \cdot (1 + \eta'' Y''), \quad (1 + 0_A Y) = 1, \\ (1 + \eta Y') \cdot (1 + \eta Y'') &= (1 + \eta(Y' + Y'')), \\ (1 + (c\eta)Y) &= (1 + \eta(cY)), \end{aligned}$$

where the first line means that for generators chosen in $G_+(A)$ their product, denoted “ $\cdot$ ”, inside $G_{\mathcal{P}}^{\circ}(A)$ is the same as in $G_+(A)$, where it is denoted “ $\cdot_{G_+}$ ”; moreover, $(1_{G_+} + \eta' \eta'' Y^{(2)})_{G_+}$ and $(1_{G_+} + \eta' \eta'' [Y', Y''])_{G_+}$ are elements of the form $(1_{G_+} + cX)$ with $X \in \mathfrak{g}_0$, $c \in A_0$ and $c^2 = 0$ as mentioned above.

Furthermore, for any morphism $f : A' \rightarrow A''$ in $(\mathbf{Wsalg})_{\mathbb{K}}$ we let $G_{\mathcal{P}}^{\circ}(f) : G_{\mathcal{P}}^{\circ}(A') \rightarrow G_{\mathcal{P}}^{\circ}(A'')$ be the group morphism uniquely defined on generators—for all $g'_+ \in G_+(A')$, $\eta \in A'_1$, $Y \in \mathfrak{g}_1$ —by

$$G_{\mathcal{P}}^{\circ}(f)(g'_+) := G_+(f)(g'_+), \quad G_{\mathcal{P}}^{\circ}(f)(1 + \eta' Y) := (1 + f(\eta') Y)$$

As the defining relations of each $G_{\mathcal{P}}^{\circ}(A)$ are independent of A , such a $G_{\mathcal{P}}^{\circ}(f)$ is well defined indeed.

(b) We define a functor $G_{\mathcal{P}}^{\circ, -} : (\mathbf{Wsalg})_{\mathbb{K}} \rightarrow (\mathbf{set})$ on any object $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ by

$$G_{\mathcal{P}}^{\circ, -}(A) := \left\{ \prod_{s=1}^n (1 + \eta_s Y_s) \mid n \in \mathbb{N}, (\eta_s, Y_s) \in A_1 \times \mathfrak{g}_1 \forall s \in \{1, \dots, n\} \right\} \quad (\subseteq G_{\mathcal{P}}^{\circ}(A))$$

and on morphism in the obvious way—just like for $G_{\mathcal{P}}$.

(c) Let us fix in $\mathfrak{g}_1$ a $\mathbb{K}$ -basis $\{Y_i\}_{i \in I}$ — for some index set I — and a total order in I . We define a functor $G_-^\leq : (\mathbf{Wsalg})_{\mathbb{K}} \rightarrow (\mathbf{set})$ as follows. For $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we set

$$G_-^\leq(A) := \left\{ \prod_{i \in I} (1 + \eta_i Y_i) \mid \eta_i \in A_1 \ \forall i \in I \right\} \quad (\subseteq G_{\mathcal{P}}^{\circ, -}(A) \subseteq G_{\mathcal{P}}^{\circ}(A)),$$

where $\prod_{i \in I}$ denotes an *ordered product* — with respect to the fixed total order in I . This defines the functor $G_-^\leq$ on objects, and its definition on morphism is the obvious one (like for $G_{\mathcal{P}}^{\circ}$). $\diamond$

Remark 3.1.2. By definition, $G_{\mathcal{P}}^{\circ, -}$ and $G_-^\leq$ are subfunctors of $G_{\mathcal{P}}^{\circ}$. Moreover, every $G_{\mathcal{P}}^{\circ, -}(A)$ is the subgroup of $G_{\mathcal{P}}^{\circ}(A)$ generated by $G_-^\leq(A)$, or by $\{(1 + \eta Y)\}_{(\eta, Y) \in A_1 \times \mathfrak{g}_1}$. Thus, although $G_-^\leq$ depends on the choice of $\{Y_i\}_{i \in I}$, the supergroup subfunctor that it generates (inside $G_{\mathcal{P}}^{\circ}$) instead is *independent* of that.

The next result shows that $G_{\mathcal{P}}^{\circ}$ can also be described using a much smaller set of generators:

Proposition 3.1.3. *Let $\mathcal{P} := (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$, and fix in $\mathfrak{g}_1$ a totally ordered $\mathbb{K}$ -basis $\{Y_i\}_{i \in I}$. Then for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ the group $G_{\mathcal{P}}^{\circ}(A)$ is generated by $\Gamma_A^{\bullet} := G_+(A) \cup \{(1 + \eta_i Y_i) \mid \eta_i \in A_1, \text{ for all } i \in I\}$.*

Proof. Given $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, let $G_{\mathcal{P}}^{\bullet}(A)$ be the subgroup of $G_{\mathcal{P}}^{\circ}(A)$ generated by $\Gamma_A^{\bullet}$. We will prove that every generator of the group $G_{\mathcal{P}}^{\circ}(A)$ of the form $(1 + \eta Y)$ with $(\eta, Y) \in A_1 \times \mathfrak{g}_1$ also belongs to the subgroup $G_{\mathcal{P}}^{\bullet}(A)$: this then will prove the claim. So let $(\eta, Y) \in A_1 \times \mathfrak{g}_1$; then, in terms of the $\mathbb{K}$ -basis $\{Y_i\}_{i \in I}$ of $\mathfrak{g}_1$, our Y expands into $Y = \sum_{s=1}^k c_{j_s} Y_{j_s}$. By repeated applications of relations of the form $(1 + \eta Y') \cdot (1 + \eta Y'') = (1 + \eta(Y' + Y''))$, we find that the generator $(1 + \eta Y)$ in $G_{\mathcal{P}}^{\circ}(A)$ factors as $(1 + \eta Y) = (1 + \eta \sum_{s=1}^k c_{j_s} Y_{j_s}) = \prod_{s=1}^k (1 + c_{j_s} \eta Y_{j_s})$, where the product can be done in any order, as its factors mutually commute. Now the product on the right-hand side belongs to $G_{\mathcal{P}}^{\bullet}(A)$. $\square$

3.1.4. Another realization of $G_{\mathcal{P}}^{\circ}$. In the following, if K is any group presented by generators and relations, we write $K = \langle \Gamma \rangle / (\mathcal{R})$ if Γ is a set of free generators, $\mathcal{R}$ is a set of relations among generators and $(\mathcal{R})$ is the normal subgroup in K generated by $\mathcal{R}$. Note that, given a presentation of K of the form $K = \langle \Gamma \rangle / (\mathcal{R}) = \langle \Gamma \rangle / (\mathcal{R}_1 \cup \mathcal{R}_2)$ with $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, the double quotient theorem gives

$$(3-1) \quad K = \langle \Gamma \rangle / (\mathcal{R}) = \langle \Gamma \rangle / (\mathcal{R}_1 \cup \mathcal{R}_2) = \langle \Gamma \rangle / (\mathcal{R}_1) / (\mathcal{R}_1 \cup \mathcal{R}_2) / (\mathcal{R}_1) = \langle \bar{\Gamma} \rangle / (\bar{\mathcal{R}}_2),$$

where $\bar{\Gamma}$ and $\bar{\mathcal{R}}_2$ respectively denote the images of Γ and of $\mathcal{R}_2$ in the quotient group $\langle \Gamma \rangle / (\mathcal{R}_1)$.

Let $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$. For any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we denote by $G_+^{[2]}(A)$ the subgroup of $G_+(A)$ generated by the set $\{(1 + cX) \mid c \in A_1^{[2]}, X \in [\mathfrak{g}_1, \mathfrak{g}_1]\}$ — see Section 2.1.1 for the notation $A_1^{[2]}$. Note then that $G_+^{[2]}(A)$ is *normal* in $G_+(A)$, as one easily sees by construction (taking into account that, as $\mathcal{P} := (G_+, \mathfrak{g})$ is a

sHCp, the “adjoint” action of G_+ onto $\mathfrak{g}$ maps $[\mathfrak{g}_1, \mathfrak{g}_1]$ into itself). We consider also the three sets

$$\Gamma_A^+ := G_+(A), \quad \Gamma_A^{[2]} := G_+^{[2]}(A), \quad \Gamma_A^- := \Gamma_A^{[2]} \cup \{(1 + \eta Y)\}_{(\eta, Y) \in A_1 \times \mathfrak{g}_1}$$

and the sets of relations — for all $g_+, g'_+, g''_+ \in \Gamma_A^+$, $g_{[2]}, g'_{[2]}, g''_{[2]} \in \Gamma_A^{[2]}$, $\eta, \eta', \eta'' \in A_1$, $X \in [\mathfrak{g}_1, \mathfrak{g}_1]$, $Y, Y', Y'' \in \mathfrak{g}_1$, with $\cdot_{G_+}$ and $\cdot_{G_+^{[2]}}$ being the product in $G_+(A)$ and in $G_+^{[2]}(A)$ — given by

$$\begin{aligned} \mathcal{R}_A^+ : g'_+ \cdot g''_+ &= g'_+ \cdot_{G_+} g''_+, & \mathcal{R}_A^{[2]} : (g_{[2]})_{\Gamma_A^{[2]}} &= (g_{[2]})_{\Gamma_A^+}, \\ \mathcal{R}_A^\times : g_{[2]} \cdot g_+ &= g_+ \cdot (g_+^{-1} \cdot_{G_+} g_{[2]} \cdot_{G_+} g_+), & (1 + \eta Y) \cdot g_+ &= g_+ \cdot (1 + \eta \text{Ad}(g_+^{-1})(Y)), \\ \mathcal{R}_A^- : \left\{ \begin{array}{l} g'_{[2]} \cdot g''_{[2]} &= g'_{[2]} \cdot_{G_+^{[2]}} g''_{[2]}, \\ (1 + \eta Y) \cdot g_{[2]} &= g_{[2]} \cdot (1 + \eta \text{Ad}(g_{[2]}^{-1})(Y)), \\ (1 + \eta'' Y) \cdot (1 + \eta' Y) &= (1 + \eta' \eta'' Y^{(2)}) \cdot (1 + (\eta' + \eta'') Y), \\ (1 + \eta'' Y'') \cdot (1 + \eta' Y') &= (1 + \eta' \eta'' [Y', Y'']) \cdot (1 + \eta' Y') \cdot (1 + \eta'' Y''), \\ (1 + \eta Y') \cdot (1 + \eta Y'') &= (1 + \eta(Y' + Y'')), \\ (1 + \eta 0_{\mathfrak{g}_1}) &= 1, (1 + 0_A Y) = 1, \end{array} \right. \\ \mathcal{R}_A &:= \mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times \cup \mathcal{R}_A^{[2]} \end{aligned}$$

(in particular, note that the relations of type $\mathcal{R}_A^{[2]}$ just identify each element in $\Gamma_A^{[2]}$ with its copy inside Γ_A^+). Then we *define* a new group, by generators and relations, namely $G_p^\circ(A) := \langle \Gamma_A^- \rangle / (\mathcal{R}_A^-)$.

From the very definition of $G_p^\circ(A)$ — see Definition 3.1.1 — it follows that

$$(3-2) \quad G_p^\circ(A) \cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) = \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times \cup \mathcal{R}_A^{[2]})$$

indeed, here above we are just taking larger sets of generators and of relations (with respect to Definition 3.1.1), but with enough redundancies as to find a different presentation of *the same* group.

From this we find a neat description of $G_p^\circ(A)$ by achieving the presentation (3-2) in a series of intermediate steps, namely adding only one bunch of relations at a time. As a first step, we have

$$(3-3) \quad \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^-) = \langle \Gamma_A^+ \rangle / (\mathcal{R}_A^+) * \langle \Gamma_A^- \rangle / (\mathcal{R}_A^-) \cong G_+(A) * G_p^{\circ,-}(A)$$

where $G_+(A) \cong \langle \Gamma_A^+ \rangle / (\mathcal{R}_A^+)$ by construction and $*$ denotes the free product (of two groups).

For the next two steps we can follow two different lines of action. On the one hand, (3-1) and (3-3) give $\langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times) \cong (G_+(A) * G_p^{\circ,-}(A)) / (\mathcal{R}_A^\times) \cong G_+(A) \rtimes G_p^{\circ,-}(A)$, where $G_+(A) \rtimes G_p^{\circ,-}(A)$ is the semidirect product of $G_+(A)$

with $G_{\mathcal{P}}^{\circ,-}(A)$ with respect to the “adjoint” action of the former on the latter. Then

$$\begin{aligned} G_{\mathcal{P}}^{\circ}(A) &\cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) \cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^{\times} \cup \mathcal{R}_A^{[2]}) \\ &\cong (G_+(A) \ltimes G_{\mathcal{P}}^{\circ,-}(A)) / (\overline{\mathcal{R}_A^{[2]}}) \cong (G_+(A) \ltimes G_{\mathcal{P}}^{\circ,-}(A)) / N_{[2]}(A), \end{aligned}$$

where $N_{[2]}(A)$ is the normal subgroup of $G_+(A) \ltimes G_{\mathcal{P}}^{\circ,-}(A)$ which is generated by $\{(g_{[2]}, g_{[2]}^{-1})\}_{g_{[2]} \in \Gamma_A^{[2]}}$. Thus $G_{\mathcal{P}}^{\circ}(A) = (G_+(A) \ltimes G_{\mathcal{P}}^{\circ,-}(A)) / N_{[2]}(A)$.

Similarly, we find also that

$$\langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^{[2]}) \cong (G_+(A) * G_{\mathcal{P}}^{\circ,-}(A)) / (\overline{\mathcal{R}_A^{[2]}}) \cong G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^{\circ,-}(A),$$

where

$$G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^{\circ,-}(A)$$

is the amalgamated product of $G_+(A)$ and $G_{\mathcal{P}}^{\circ,-}(A)$ over $G_+^{[2]}(A)$ with respect to natural monomorphisms $G_+^{[2]}(A) \hookrightarrow G_+(A)$ and $G_+^{[2]}(A) \hookrightarrow G_{\mathcal{P}}^{\circ,-}(A)$. Then

$$\begin{aligned} G_{\mathcal{P}}^{\circ}(A) &\cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) \cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^{[2]} \cup \mathcal{R}_A^{\times}) \\ &\cong (G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^{\circ,-}(A)) / (\overline{\mathcal{R}_A^{\times}}) \cong (G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^{\circ,-}(A)) / N_{\times}(A), \end{aligned}$$

where $N_{\times}(A)$ is the normal subgroup of

$$G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^{\circ,-}(A)$$

generated by

$$\begin{aligned} &\left\{ g_+(1 + \eta Y) g_+^{-1} (1 + \eta \operatorname{Ad}(g_+)(Y))^{-1} \right\}_{(\eta, Y) \in A_1 \times \mathfrak{g}_1}^{g_+ \in G_+(A)} \\ &\cup \left\{ g_+ g_{[2]} g_+ (g_+ \cdot_{G_+} g_{[2]} \cdot_{G_+} g_+)^{-1} \right\}_{g_+ \in G_+(A)}^{g_{[2]} \in \Gamma_A^{[2]}}. \end{aligned}$$

Thus

$$G_{\mathcal{P}}^{\circ}(A) = (G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^{\circ,-}(A)) / N_{\times}(A)$$

for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$; in functorial terms, this reads

$$G_{\mathcal{P}}^{\circ} = (G_+ \ltimes G_{\mathcal{P}}^{\circ,-}) / N_{[2]} \quad \text{and} \quad G_{\mathcal{P}}^{\circ} = (G_+ *_{G_+^{[2]}} G_{\mathcal{P}}^{\circ,-}) / N_{\times}, \quad \text{or} \quad G_{\mathcal{P}}^{\circ} = G_+ \ltimes_{G_+^{[2]}} G_{\mathcal{P}}^{\circ,-},$$

where the last, (hopefully) more suggestive notation

$$G_{\mathcal{P}}^{\circ} = G_+ \ltimes_{G_+^{[2]}} G_{\mathcal{P}}^{\circ,-}$$

tells us that $G_{\mathcal{P}}^{\circ}$ is the “amalgamate semidirect product” of G_+ and $G_{\mathcal{P}}^{\circ,-}$ over their common subgroup $G_+^{[2]}$.

3.2. The supergroup functor $G_{\mathcal{P}}^{\circ}$ as a Lie supergroup. We now aim to prove that the functor $G_{\mathcal{P}}^{\circ}$ is actually a Lie supergroup. We keep notations as before, in particular for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we denote by $G_{+}^{[2]}(A)$ the subgroup of $G_{+}(A)$ generated by

$$\{(1 + cX)\}_{X \in [\mathfrak{g}_1, \mathfrak{g}_1]}^{c \in A_1^{[2]}};$$

see Section 2.1.1 for the notation $A_1^{[2]}$. We begin with a “factorization result” for $G_{\mathcal{P}}^{\circ}$:

Proposition 3.2.1. *Let $\mathcal{P} := (G_{+}, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be a super Harish-Chandra pair over $\mathbb{K}$, let $\{Y_i\}_{i \in I}$ be a totally ordered $\mathbb{K}$ -basis of $\mathfrak{g}_1$ (for our fixed order in I) and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$. Then*

(a) *letting $\langle G_{-}^{\leq}(A) \rangle$ be the subgroup of $G_{\mathcal{P}}^{\circ}(A)$ generated by $G_{-}^{\leq}(A)$, we have $\langle G_{-}^{\leq}(A) \rangle = G_{\mathcal{P}}^{\circ, -}(A)$ and there exist set-theoretic factorizations (with respect to the group product “ $\cdot$ ”)*

$$G_{\mathcal{P}}^{\circ, -}(A) = G_{+}^{[2]}(A) \cdot G_{-}^{\leq}(A), \quad G_{\mathcal{P}}^{\circ, -}(A) = G_{-}^{\leq}(A) \cdot G_{+}^{[2]}(A),$$

(b) *there exist set-theoretic factorizations (with respect to the group product “ $\cdot$ ”)*

$$G_{\mathcal{P}}^{\circ}(A) = G_{+}(A) \cdot G_{-}^{\leq}(A), \quad G_{\mathcal{P}}^{\circ}(A) = G_{-}^{\leq}(A) \cdot G_{+}(A).$$

Proof. Claim (a) is the exact analogue of (2-5), and claim (b) the analogue of (2-7), in Proposition 2.4.8(b). In both cases the proof (up to trivialities) is identical, so we can skip it. $\square$

3.2.2. The representation $G_{\mathcal{P}}^{\circ} \rightarrow \mathrm{GL}(V)$. When discussing the structure of a Lie supergroup G , the factorization $G = G_0 \cdot G_{-}^{\leq}$ was just a intermediate step; Proposition 3.2.1 above gives us the parallel counterpart for $G_{\mathcal{P}}^{\circ}$. This factorization result for G is improved by the “global splitting theorem” — i.e., Theorem 2.4.10 — which, roughly speaking, states that for any $g \in G(A)$ the factorization pertaining to $G_0(A) \cdot G_{-}^{\leq}(A)$ has uniquely determined factors, and similarly any element in $G_{-}^{\leq}(A)$ has a unique factorization into an ordered product of factors of the form $(1 + \eta_i Y_i)$. Both results are proved by showing that two factorizations of the same object necessarily have identical factors; in other words, *distinct* choices of factors always give rise to *different* elements in $G(A)$ or in $G_{-}^{\leq}(A)$. This last fact was proved using the concrete realization of $G(A)$ as a special set of maps, namely $G(A) := \bigsqcup_{x \in |G|} \mathrm{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{|G|, x}, A)$; indeed, this algebra is rich enough to “separate” different elements of $G(A)$ itself just looking at their values as A -valued maps. When dealing with $G_{\mathcal{P}}^{\circ}(A)$ instead, that is defined abstractly, such a built-in realization is not available: our strategy then is to replace it with a suitable “partial linearization”, namely a representation of $G_{\mathcal{P}}^{\circ}(A)$ that, although not being faithful, is still “rich enough” to (almost) separate elements.

Let $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be given; as before, we fix a $\mathbb{K}$ -basis $\{Y_i\}_{i \in I}$ of $\mathfrak{g}$, where I is an index set in which we fix some total order, hence the basis itself is totally ordered as well.

Recall that the *universal enveloping algebra* $U(\mathfrak{g})$ is given by $U(\mathfrak{g}) := T(\mathfrak{g})/J$, where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ and J is the two-sided ideal in $T(\mathfrak{g})$ generated by the set (using notation $z^{(2)} := 2^{-1}[z, z]$)

$$\{xy - (-1)^{|x||y|}yx - [x, y], z^2 - z^{(2)} \mid x, y \in \mathfrak{g}_0 \cup \mathfrak{g}_1, z \in \mathfrak{g}_1\}$$

It is known then — see for instance [Varadarajan 2004, §7.2] — that one has splitting(s) of $\mathbb{K}$ -supercoalgebras

$$(3-4) \quad U(\mathfrak{g}) \cong U(\mathfrak{g}_0) \otimes_{\mathbb{K}} \bigwedge \mathfrak{g}_1 \cong \bigwedge \mathfrak{g}_1 \otimes_{\mathbb{K}} U(\mathfrak{g}_0)$$

and $\bigwedge \mathfrak{g}_1$ has $\mathbb{K}$ -basis $\{Y_{i_1} \wedge Y_{i_2} \wedge \cdots \wedge Y_{i_s} \mid s \leq |I|, i_1 < i_2 < \cdots < i_s\}$.

Now let $\mathbb{1}$ be the (one-dimensional) *trivial representation* of $\mathfrak{g}_0$. By *induction* from $\mathfrak{g}_0$ to $\mathfrak{g}$ we can consider the *induced representation* $V := \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(\mathbb{1})$, that is a $\mathfrak{g}$ -module. Looking at $\mathbb{1}$ and V respectively as a module over $U(\mathfrak{g}_0)$ and over $U(\mathfrak{g})$, taking (3-4) into account we get

$$(3-5) \quad V := \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(\mathbb{1}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} \mathbb{1} \cong \bigwedge \mathfrak{g}_1 \otimes_{\mathbb{K}} \mathbb{1} \cong \bigwedge \mathfrak{g}_1$$

The last one above is a natural isomorphism of $\mathbb{K}$ -superspaces, uniquely determined once a specific element $\underline{h} \in \mathbb{1}$ is fixed to form a $\mathbb{K}$ -basis of $\mathbb{1}$ itself: the isomorphism is $\omega \otimes \underline{h} \mapsto \omega$ for all $\omega \in \bigwedge \mathfrak{g}_1$.

This and its outcome give rise to similar functorial counterparts, for the Lie algebra valued $\mathbb{K}$ -superfunctors $\mathcal{L}_{\mathfrak{g}_0}$ and $\mathcal{L}_{\mathfrak{g}}$, as well as for the $\mathbb{K}$ -superfunctors associated with $U(\mathfrak{g}_0)$ and $U(\mathfrak{g})$, in the standard way, namely $A \mapsto A_0 \otimes_{\mathbb{K}} U(\mathfrak{g}_0)$ and $A \mapsto (A \otimes_{\mathbb{K}} U(\mathfrak{g}))_0 = (A_0 \otimes_{\mathbb{K}} U(\mathfrak{g}_0)) \oplus (A_1 \otimes_{\mathbb{K}} U(\mathfrak{g}_1))$ for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$.

On the other hand $\mathfrak{g}_0 = \text{Lie}(G_+)$, and $\mathbb{1}$ is also the trivial representation for G_+ , as a classical Lie group. Then the representation of $\mathfrak{g}$ on the space V also induces a representation of the super Harish-Chandra pair $\mathcal{P} = (G_+, \mathfrak{g})$ on the same V , in other words V bears also a structure of $(G_+, \mathfrak{g})$ -module — in the obvious sense: we have a morphism $(\mathbf{r}_+, \rho) : (G_+, \mathfrak{g}) \rightarrow (\text{GL}(V), \mathfrak{gl}(V))$ of super Harish-Chandra pairs. We will write again ρ for the representation map $\rho : U(\mathfrak{g}) \rightarrow \text{End}_{\mathbb{K}}(V)$ giving the $U(\mathfrak{g})$ -module structure on V .

Our key step now is that the above $(G_+, \mathfrak{g})$ -module structure on V “integrates” to a $G_{\mathcal{P}}^{\circ}$ -module structure.

Proposition 3.2.3. *Retain notation as above for the $(G_+, \mathfrak{g})$ -module V . There exists a unique structure of (left) $G_{\mathcal{P}}^{\circ}$ -module onto V which satisfies the following conditions: for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, the representation map $\mathbf{r}_{\mathcal{P}, A}^{\circ} : G_{\mathcal{P}}^{\circ}(A) \rightarrow \text{GL}(V)(A)$ is given on generators of $G_{\mathcal{P}}^{\circ}(A)$ — namely, all $g_+ \in G_+(A)$ and $(1 + \eta_i Y_i)$ for $i \in I$,*

$\eta_i \in A_1$ — by $\mathbf{r}_{\mathcal{P},A}^\circ(g_+) := \mathbf{r}_+(g_+)$, $\mathbf{r}_{\mathcal{P},A}^\circ(1 + \eta_i Y_i) := \rho(1 + \eta_i Y_i) = \text{id}_V + \eta_i \rho(Y_i)$, or in other words $g_+.v := \mathbf{r}_+(g_+)(v)$ and $(1 + \eta_i Y_i).v := \rho(1 + \eta_i Y_i)(v) = v + \eta_i \rho(Y_i)(v)$ for all $v \in V(A)$. Overall, this yields a morphism a $\mathbb{K}$ -supergroup functors $\mathbf{r}_{\mathcal{P}}^\circ : G_{\mathcal{P}}^\circ \rightarrow \text{GL}(V)$.

Proof. This follows from the whole construction, and from the very definition of $G_{\mathcal{P}}^\circ$. Indeed, by definition of representation for the super Harish-Chandra pair $\mathcal{P}$ we see that the operators $\mathbf{r}_{\mathcal{P},A}^\circ(g_+)$ and $\mathbf{r}_{\mathcal{P},A}^\circ(1 + \eta_i Y_i)$ on V — associated with the generators of $G_{\mathcal{P}}^\circ(A)$ — do satisfy all relations which, by Definition 3.1.1(a), are satisfied by the generators themselves. Thus they uniquely provide a well-defined group morphism $\mathbf{r}_{\mathcal{P},A}^\circ : G_{\mathcal{P}}^\circ(A) \rightarrow \text{GL}(V)(A)$ as required. The construction is clearly functorial in A , whence the claim. $\square$

The advantage of introducing the representation $\mathbf{r}_{\mathcal{P}}$ of $G_{\mathcal{P}}^\circ$ on V is that it lets us “separate” the odd points of $G_{\mathcal{P}}^\circ(A)$ from each other and from the even ones.

We are now ready to state and prove the “global splitting theorem” for $G_{\mathcal{P}}^\circ$ (see Theorem 2.4.10):

Proposition 3.2.4. (a) *The restriction of group multiplication in $G_{\mathcal{P}}^\circ$ provides isomorphisms of (set-valued) functors*

$$G_+ \times G_-^\leq \cong G_{\mathcal{P}}^\circ, \quad G_-^\leq \times G_+ \cong G_{\mathcal{P}}^\circ, \quad G_+^{[2]} \times G_-^\leq \cong G_{\mathcal{P}}^{\circ,-}, \quad G_-^\leq \times G_+^{[2]} \cong G_{\mathcal{P}}^{\circ,-}$$

(b) *There exists an isomorphism of (set-valued) functors $\mathbb{A}_{\mathbb{K}}^{0|d_1} \cong G_-^\leq$, with $d_1 := |I| = \dim_{\mathbb{K}}(\mathfrak{g}_1)$, given on A -points — for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — by*

$$\mathbb{A}_{\mathbb{K}}^{0|d_1}(A) = A_1^{d_1} \rightarrow G_-^\leq(A), \quad (\eta_i)_{i \in I} \mapsto \prod_{i \in I} (1 + \eta_i Y_i)$$

(c) *There exist isomorphisms of (set-valued) functors*

$$G_+ \times \mathbb{A}_{\mathbb{K}}^{0|d_1} \cong G_{\mathcal{P}}^\circ, \quad G_+^{[2]} \times \mathbb{A}_{\mathbb{K}}^{0|d_1} \cong G_{\mathcal{P}}^{\circ,-} \quad \text{and} \quad \mathbb{A}_{\mathbb{K}}^{0|d_1} \times G_+ \cong G_{\mathcal{P}}^\circ, \quad \mathbb{A}_{\mathbb{K}}^{0|d_1} \times G_+^{[2]} \cong G_{\mathcal{P}}^{\circ,-}$$

given on A -points — for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — respectively by

$$(g_+, (\eta_i)_{i \in I}) \mapsto g_+ \cdot \prod_{i \in I} (1 + \eta_i Y_i) \quad \text{and} \quad ((\eta_i)_{i \in I}, g_+) \mapsto \prod_{i \in I} (1 + \eta_i Y_i) \cdot g_+$$

Proof. The proof is quite close to (half of) that of Theorem 2.4.10, the difference being the use of the representation V of Section 3.2.2. For completeness we present it shortly, sketching (a) and (b), whence (c) follows.

(a) It is enough to prove the first identity for $G_{\mathcal{P}}^\circ$ (the others are similar). Thus our claim is: for any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, if $\hat{g}_+ \hat{g}_- = \check{g}_+ \check{g}_-$ for $\hat{g}_+, \check{g}_+ \in G_+(A)$ and $\hat{g}_-, \check{g}_- \in G_-^\leq(A)$, then $\hat{g}_+ = \check{g}_+$ and $\hat{g}_- = \check{g}_-$.

From $\hat{g}_+ \hat{g}_- = \check{g}_+ \check{g}_-$ we get $g := \hat{g}_- \check{g}_-^{-1} = \hat{g}_+^{-1} \check{g}_+ \in G_+(A)$, as $G_+(A)$ is a subgroup. Writing $\hat{g}_- = \prod_{i \in I} (1 + \hat{\eta}_i Y_i)$ and $\check{g}_- = \prod_{i \in I} (1 + \check{\eta}_i Y_i)$ we get $g := \hat{g}_- \check{g}_-^{-1} = \prod_{i \in I} (1 + \hat{\eta}_i Y_i) \prod_{i \in I} (1 - \check{\eta}_i Y_i) \in G_+(A)$.

Let $\mathfrak{a} := (\{\hat{\eta}_i, \check{\eta}_i\}_{i \in I})$ be the ideal of A generated by the $\hat{\eta}_i$ ’s and the $\check{\eta}_i$ ’s, set

$A \xrightarrow{\pi_n} A/\mathfrak{a}^n$ for the quotient map and $[a]_n := \pi_n(a)$ for $a \in A$, then $G_{\mathcal{P}}^{\circ}(A) \xrightarrow{G(\pi_n)} G_{\mathcal{P}}^{\circ}(A/\mathfrak{a}^n)$ for the associated group morphism and $[y]_n := G_{\mathcal{P}}^{\circ}(\pi_n)(y)$ for every $y \in G_{\mathcal{P}}^{\circ}(A)$. Now, the defining relations for $G_{-}^{\leq}(A/\mathfrak{a}^2)$ yield, setting $\alpha_i := \hat{\eta}_i - \check{\eta}_i$ for $i \in I$,

$$[g]_2 = \overrightarrow{\prod}_{i \in I} (1 + [\hat{\eta}_i]_2 Y_i) \cdot \overleftarrow{\prod}_{i \in I} (1 - [\check{\eta}_i]_2 Y_i) = \overrightarrow{\prod}_{i \in I} (1 + [\alpha_i]_2 Y_i) \in G_{-}^{\leq}(A/\mathfrak{a}^2).$$

The next step is to let $[g]_2$ act onto $\underline{b} \in V(A/\mathfrak{a}^2)$; we write $\bar{Y}$ for the image of $Y.b$ through the isomorphism $V = \bigwedge \mathfrak{g}_1.b \cong \bigwedge \mathfrak{g}_1$. Then $[g]_2.\underline{b} = \overrightarrow{\prod}_{i \in I} (1 + [\alpha_i]_2 Y_i).\underline{b} = \underline{b} + \sum_{i \in I} [\alpha_i]_2 \bar{Y}_i \in V(A/\mathfrak{a}^2)$ since $[\alpha_h]_2[\alpha_k]_2 = [0]_2 \in A/\mathfrak{a}^2$; on the other hand, $[g]_2.\underline{b} = \underline{b}$ as $[g]_2 \in G_{+}(A/\mathfrak{a}^2)$ and G_{+} acts trivially on V . Therefore, noting that $\{b\} \cup \{\bar{Y}_i\}_{i \in I}$ is part of the chosen basis of V , we get $[\alpha_i]_2 = [0]_2 \in A/\mathfrak{a}^2$, i.e., $\alpha_i \in \mathfrak{a}^2$, for all $i \in I$. By the same argument we find $[g]_3 = \overrightarrow{\prod}_{i \in I} (1 + [\alpha_i]_3 Y_i) \in G_{\mathcal{P}}^{\circ}(A/\mathfrak{a}^3)$ hence $[g]_3.\underline{b} = \underline{b} + \sum_{i \in I} [\alpha_i]_3 \bar{Y}_i \in V(A/\mathfrak{a}^3)$, which in turn implies $\alpha_i \in \mathfrak{a}^3$, for all $i \in I$. Iterating this process we find $\alpha_i \in \mathfrak{a}^n$ for all $n \in \mathbb{N}$, $i \in I$; as $\mathfrak{a}^n = \{0\}$ for $n \gg 0$ (since $\mathfrak{a}$ is generated by finitely many odd elements) we end up with $\alpha_i = 0$, i.e., $\hat{\eta}_i = \check{\eta}_i$, for all $i \in I$. This means $\hat{g}_{-} = \check{g}_{-}$, whence $\hat{g}_{+} = \check{g}_{+}$ too.

(b) By construction there exists a morphism $\Theta^{\leq} : \mathbb{A}_{\mathbb{K}}^{0|d_1} \rightarrow G_{-}^{\leq}$ of set-valued functors given on A -points — for $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — by the map

$$\Theta_A^{\leq} : \mathbb{A}_{\mathbb{K}}^{0|d_1}(A) := A_1^{\times d_1} \rightarrow G_{-}^{\leq}(A), (\eta_i)_{i \in I} \mapsto \overrightarrow{\prod}_{i \in I} (1 + \eta_i Y_i),$$

that is actually *surjective*, and we need to prove that is also *injective*. Now, let $(\hat{\eta}_i)_{i \in I}, (\check{\eta}_i)_{i \in I} \in A_1^{\times d_1}$ be such that $\Theta_A^{\leq}((\hat{\eta}_i)_{i \in I}) = \Theta_A^{\leq}((\check{\eta}_i)_{i \in I})$, i.e.,

$$\overrightarrow{\prod}_{i \in I} (1 + \hat{\eta}_i Y_i) = \overrightarrow{\prod}_{i \in I} (1 + \check{\eta}_i Y_i).$$

Then we repeat the argument in (a), now with $\hat{g}_{+} := 1 =: \check{g}_{+}$, which gives $\hat{\eta}_i = \check{\eta}_i$ for all $i \in I$, i.e., $(\hat{\eta}_i)_{i \in I} = (\check{\eta}_i)_{i \in I}$. $\square$

3.2.5. The Lie supergroup structure of $G_{\mathcal{P}}^{\circ}$. Given $\mathcal{P} \in (\mathbf{sHCp})_{\mathbb{K}}$ and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, Proposition 3.2.4(c) yields a bijection $\phi_A^{\circ} : G_{+}(A) \times \mathbb{A}_{\mathbb{K}}^{0|d_1}(A) \xrightarrow{\cong} G_{\mathcal{P}}^{\circ}(A)$ whose restriction to $G_{+}(A)$, identified with $G_{+}(A) \times \{(0)_{i \in I}\}$, is the identity — onto the copy of $G_{+}(A)$ naturally sitting inside $G_{\mathcal{P}}^{\circ}(A)$.

Now, $G_{+}(A)$ is by definition an A_0 -manifold (see [Balduzzi et al. 2013]), of the same type (real smooth, etc.) as the sHCp $\mathcal{P} := (G_{+}, \mathfrak{g})$ it pertains to; on the other hand, $\mathbb{A}_{\mathbb{K}}^{0|d_1}(A)$ carries natural, canonical structures of A_0 -manifold of any possible type (real smooth or real analytic if $\mathbb{K} = \mathbb{R}$, complex holomorphic if $\mathbb{K} = \mathbb{C}$), in particular then also of the type of $G_{+}(A)$. Then we know that there is also a canonical “product structure” of A_0 -manifold — of the same type of $G_{+}(A)$, i.e., of $\mathcal{P}$ — onto the Cartesian product $G_{+}(A) \times \mathbb{A}_{\mathbb{K}}^{0|d_1}(A)$. Using the bijection ϕ_A° above we push-forward this canonical A_0 -manifold structure of $G_{+}(A) \times \mathbb{A}_{\mathbb{K}}^{0|d_1}(A)$ onto $G_{\mathcal{P}}^{\circ}(A)$, which then is turned into an A_0 -manifold on its own, still of the same

type as $\mathcal{P}$. Strictly speaking, this structure of A_0 -manifold on $G_{\mathcal{P}}^{\circ}(A)$ formally depends on the choice of $G_{\leq}^{\circ}$, hence of a totally ordered $\mathbb{K}$ -basis of $\mathfrak{g}_1$, as this choice enters in the construction of ϕ_A° . However, thanks to the special form of the defining relations of $G_{\mathcal{P}}^{\circ}(A)$ it is straightforward to show that *changing such a basis amounts to changing local charts for the same, unique A_0 -manifold structure*; hence the structure is *independent* of such a choice.

Now, using the above structure of A_0 -manifold on $G_{\mathcal{P}}^{\circ}(A)$ for each $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, given a morphism $f : A' \rightarrow A''$ in $(\mathbf{Wsalg})_{\mathbb{K}}$ it is straightforward to check that the group morphism $G_{\mathcal{P}}^{\circ}(f) : G_{\mathcal{P}}^{\circ}(A') \rightarrow G_{\mathcal{P}}^{\circ}(A'')$ is a morphism of A'_0 -manifolds, hence it is a morphism of A_0 -manifolds (see [Balduzzi et al. 2013]). Thus $G_{\mathcal{P}}^{\circ}$ is also a functor from Weil $\mathbb{K}$ -superalgebras to A_0 -manifolds (real smooth, real analytic or complex holomorphic as $\mathcal{P}$ is).

At last, again looking at the commutation relations in $G_{\mathcal{P}}^{\circ}(A)$, we see that the group multiplication and the inverse map are “regular” (i.e., “real smooth”, “real analytic” or “complex holomorphic” depending on the type of $\mathcal{P}$); indeed, this is explicitly proved by calculations like those needed in the proof of Proposition 3.2.1(b) — that we skipped, so refer instead to the proof of Proposition 2.4.8(b). Thus they are morphisms of A_0 -manifolds, so $G_{\mathcal{P}}^{\circ}(A)$ is a group element among A_0 -manifolds, i.e., it is a Lie A_0 -group; hence (see Section 2.1.3) the functor $G_{\mathcal{P}}^{\circ}$ is a Lie supergroup (of real smooth, real analytic or complex holomorphic type as $\mathcal{P}$).

Eventually, the outcome of this discussion — and core result of the present section — is the following statement, which provides a “backward functor” from sHCps to Lie supergroups:

Theorem 3.2.6. *The recipe in Definition 3.1.1 provides functors*

$$\Psi^{\circ} : (\mathbf{sHCp})_{\mathbb{R}}^{\infty} \rightarrow (\mathbf{Lsgrp})_{\mathbb{R}}^{\infty}, \quad \Psi^{\circ} : (\mathbf{sHCp})_{\mathbb{R}}^{\omega} \rightarrow (\mathbf{Lsgrp})_{\mathbb{R}}^{\omega}, \quad \Psi^{\circ} : (\mathbf{sHCp})_{\mathbb{C}}^{\omega} \rightarrow (\mathbf{Lsgrp})_{\mathbb{C}}^{\omega}$$

given on objects by $\mathcal{P} \mapsto \Psi^{\circ}(\mathcal{P}) := G_{\mathcal{P}}^{\circ}$ and on morphisms by

$$((\phi_+, \varphi) : \mathcal{P}' \rightarrow \mathcal{P}'') \mapsto (\Psi^{\circ}((\phi_+, \varphi)) : \Psi^{\circ}(\mathcal{P}') := G_{\mathcal{P}'}^{\circ} \rightarrow G_{\mathcal{P}''}^{\circ} =: \Psi^{\circ}(\mathcal{P}''))$$

where the functor morphism $\Psi^{\circ}((\phi_+, \varphi)) : \Psi^{\circ}(\mathcal{P}') := G_{\mathcal{P}'}^{\circ} \rightarrow G_{\mathcal{P}''}^{\circ} =: \Psi^{\circ}(\mathcal{P}'')$ is defined by

$$(3-6) \quad \Psi^{\circ}((\phi_+, \varphi))_A : g'_+ \mapsto \phi_+(g'_+), \quad (1 + \eta Y') \mapsto (1 + \eta \varphi(Y'))$$

for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, $g'_+ \in G'_+(A)$, $\eta \in A_1$, $Y' \in \mathfrak{g}'$, with $\mathcal{P}' = (G'_+, \mathfrak{g}')$ and $\mathcal{P}'' = (G''_+, \mathfrak{g}'')$.

Proof. What is still left to prove is that the given definition for $\Psi^{\circ}((\phi_+, \varphi))$ actually makes sense. Now, (3-6) above fixes the values of our would-be morphism $\Psi^{\circ}((\phi_+, \varphi))_A$ on generators of $\Psi^{\circ}(\mathcal{P}')(A) := G_{\mathcal{P}'}^{\circ}(A)$: a direct check shows that all defining relations among such generators are respected, thus providing a group

morphism as required. However, we must still show that *this is a morphism of $\mathcal{A}_0$ -manifolds too*.

Let $\{Y'_i\}_{i \in I}$ and $\{Y''_j\}_{j \in J}$ be totally ordered $\mathbb{K}$ -bases of $\mathfrak{g}'_1$ and $\mathfrak{g}''_1$. Accordingly, both $G_{\mathcal{P}'}^\circ(A)$ and $G_{\mathcal{P}''}^\circ(A)$ admit factorizations as in Proposition 3.2.4(a), hence any $g' \in G_{\mathcal{P}'}^\circ(A)$ uniquely factors into $g' = g'_+ \cdot \prod_{i \in I} (1 + \eta_i Y'_i)$; then $\Psi^\circ((\phi_+, \varphi))_A(g') = \phi_+(g'_+) \cdot \prod_{i \in I} (1 + \eta_i \varphi(Y'_i))$ and from this, letting $\varphi(Y'_i) = \sum_{j \in J} c_{i,j} Z_j$ — with $c_{i,j} \in \mathbb{K}$ — we get

$$\Psi^\circ((\phi_+, \varphi))_A(g') = \phi_+(g'_+) \cdot \prod_{i \in I} \prod_{j \in J} (1 + \eta_i c_{i,j} Z_j).$$

Now we must reorder the result according to the factorization of $G_{\mathcal{P}''}^\circ(A)$ of the form $G_+ \times G_-$; in doing this, when we reorder the factor $\prod_{i \in I} \prod_{j \in J} (1 + \eta_i c_{i,j} Z_j)$ we find, via calculations as for Proposition 2.4.8(b), an outcome of the form $\prod_{r=1}^n (1 + a_r X_r)_{G_+''} \cdot \prod_{j \in J} (1 + \alpha_j Z_j)$ where the X_r 's belong to $\mathfrak{g}''_0$, the a_r 's are (even) polynomial expressions in the η_i 's, the α_j 's are (odd) polynomial expressions in the η_i 's. This implies that the map

$$\prod_{i \in I} (1 + \eta_i Y'_i) \mapsto \prod_{r=1}^n (1 + a_r X_r)_{G_+''} \cdot \prod_{j \in J} (1 + \alpha_j Z_j)$$

is a map of $\mathcal{A}_0$ -manifolds from $(G_{\mathcal{P}'}^\circ)_-^<(A)$ to $G_{\mathcal{P}''}^\circ(A)$. But $\phi_+ : G'_+(A) \rightarrow G''_+(A)$ is a map of $\mathcal{A}_0$ -manifolds too, by assumptions; this along with all the previous analysis eventually implies that $\Psi^\circ((\phi_+, \varphi))_A$ is a map of $\mathcal{A}_0$ -manifolds as claimed. $\square$

3.3. Supergroup functors out of super Harish-Chandra pairs: second recipe.

In this subsection we construct a second functor, denoted $G_{\mathcal{P}}^e$, which we later prove is a Lie supergroup: this is in fact a “sibling alternative” to the functor $G_{\mathcal{P}}^\circ$ considered in Section 3.1. As a matter of notation, recall that for any $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$, $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ and $\mathcal{X} \in A_0 \otimes_{\mathbb{K}} \mathfrak{g}_0$ there exists a well-defined $\exp_{G_+}(\mathcal{X}) \in G_+(A_0)$; furthermore, if in particular $\mathcal{X} \in A_1^2 \otimes_{\mathbb{K}} \mathfrak{g}_0$, then the formal series expansion of $\exp(\mathcal{X})$ can be actually realized as a finite sum. When no confusion is possible we shall drop the subscript G_+ and simply write $\exp(\mathcal{X})$ instead. Similarly, we shall presently introduce new formal elements of type “ $\exp(\mathcal{Y})$ ” with $\mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$. Finally, we extend the built-in G_+ -action onto $\mathfrak{g}_1$ to a (same-name) G_+ -action onto $A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$ by $\text{Ad}(g)(\sum_{s=1}^n \eta_s Y_s) := \sum_{s=1}^n \eta_s \text{Ad}(g)(Y_s)$ for all $\sum_{s=1}^n \eta_s Y_s := \sum_{s=1}^n \eta_s \otimes Y_s \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$.

Definition 3.3.1. Let $\mathcal{P} := (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be a super Harish-Chandra pair over $\mathbb{K}$.

(a) We introduce a functor $G_{\mathcal{P}}^e : (\mathbf{Wsalg})_{\mathbb{K}} \rightarrow (\mathbf{group})$ as follows. For any Weil superalgebra $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we define $G_{\mathcal{P}}^e(A)$ as being the group with generators the elements of the set

$$\Gamma_A := \{g_+, \exp(\mathcal{Y}) \mid g_+ \in G_+(A), \mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1\} = G_+(A) \cup \{\exp(\mathcal{Y})\}_{\mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1}$$

and relations (for $g_+, g'_+, g''_+ \in G_+(A)$, $\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'' \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$)

$$\begin{aligned} g'_+ \cdot g''_+ &= g'_+ \cdot_{G_+} g''_+, \quad \exp(0) = 1, \quad \exp(\mathcal{Y}) \cdot g_+ = g_+ \cdot \exp(\text{Ad}(g_+^{-1})(\mathcal{Y})), \\ \exp(\mathcal{Y}') \cdot \exp(\mathcal{Y}'') &= \exp(P_0^{(d_1)}(\mathcal{Y}', \mathcal{Y}'')) \cdot \exp(\mathcal{Y}' + \mathcal{Y}'' + P_1^{(d_1)}(\mathcal{Y}', \mathcal{Y}'')), \end{aligned}$$

with $P_0^{(d_1)}$ and $P_1^{(d_1)}$ as in Lemma 2.4.5(h). This yields the functor $G_{\mathcal{P}}^e$ on objects.

To define $G_{\mathcal{P}}^e$ on morphisms, for any morphism $f : A' \rightarrow A''$ in $(\mathbf{Wsalg})_{\mathbb{K}}$ we define the group morphism $G_{\mathcal{P}}^e(f) : G_{\mathcal{P}}^e(A') \rightarrow G_{\mathcal{P}}^e(A'')$ to be the unique one given — for $g'_+ \in G_+(A')$, $\eta \in A'_1$, $\mathcal{Y}' \in A'_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$ — by

$$G_{\mathcal{P}}^e(f)(g'_+) := G_+(f)(g'_+), \quad G_{\mathcal{P}}^e(f)(\exp(\mathcal{Y}')) := \exp(f(\mathcal{Y}')),$$

where $f(\mathcal{Y}') := \sum_{s=1}^n f(\eta'_s)Y_s$ for all $\mathcal{Y}' := \sum_{s=1}^n \eta'_s Y_s \in A'_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$.

(b) We define a functor $G_{\mathcal{P}}^{e,-} : (\mathbf{Wsalg})_{\mathbb{K}} \rightarrow (\mathbf{set})$ on any object $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ by

$$G_{\mathcal{P}}^{e,-}(A) := \langle \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \rangle \quad (\subseteq G_{\mathcal{P}}^e(A)) \quad (\subseteq G_{\mathcal{P}}^e(A))$$

— the subgroup of $G_{\mathcal{P}}^e(A)$ generated by $\exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) := \{\exp(\mathcal{Y})\}_{\mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1}$ — and on morphisms in the obvious way. By definition, $G_{\mathcal{P}}^{e,-}$ can be thought of as subfunctor of $G_{\mathcal{P}}^e$. $\diamond$

3.3.2. Another realization of $G_{\mathcal{P}}^e$. Given a super Harish-Chandra pair $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$, we present now another way of realizing the $\mathbb{K}$ -supergroup $G_{\mathcal{P}}^e$ introduced in Definition 3.3.1(a): this mimics what we did in Section 3.1.4, so we keep the same kind of notation and are a bit shorter.

For any fixed $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we denote by $G_+^{(2)}(A)$ the subgroup of $G_+(A)$ generated by the set $\{\exp(\mathcal{X}) \mid \mathcal{X} \in A_1^{[2]} \otimes_{\mathbb{K}} [\mathfrak{g}_1, \mathfrak{g}_1]\}$. Then one easily sees that $G_+^{(2)}(A)$ is *normal* in $G_+(A)$. Consider also

$$\Gamma_A^+ := G_+(A), \quad \Gamma_A^{(2)} := G_+^{(2)}(A), \quad \Gamma_A^- := \Gamma_A^{(2)} \cup \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$$

and the five sets of relations (for all $g_+, g'_+, g''_+ \in \Gamma_A^+$, $g_{(2)}, g'_{(2)}, g''_{(2)} \in \Gamma_A^{(2)}$, $\mathcal{Y}, \mathcal{Y}', \mathcal{Y}'' \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$)

$$\mathcal{R}_A^+ : g'_+ \cdot g''_+ = g'_+ \cdot_{G_+} g''_+, \quad \mathcal{R}_A^{(2)} : (g_{(2)})_{\Gamma_A^{(2)}} = (g_{(2)})_{\Gamma_A^+},$$

$$\mathcal{R}_A^{\times} : g_{(2)} \cdot g_+ = g_+ \cdot (g_+^{-1} \cdot_{G_+} g_{(2)} \cdot_{G_+} g_+), \quad \exp(\mathcal{Y}) \cdot g_+ = g_+ \cdot \exp(\text{Ad}(g_+^{-1})(\mathcal{Y}))$$

$$\mathcal{R}_A^- : \begin{cases} g'_{(2)} \cdot g''_{(2)} = g'_{(2)} \cdot_{G_+^{(2)}} g''_{(2)}, \quad \exp(\mathcal{Y}) \cdot g_{(2)} = g_{(2)} \cdot \exp(\text{Ad}(g_{(2)}^{-1})(\mathcal{Y})), \quad \exp(0) = 1 \\ \exp(\mathcal{Y}') \exp(\mathcal{Y}'') = \exp(P_0^{(d_1)}(\mathcal{Y}', \mathcal{Y}'')) \exp(\mathcal{Y}' + \mathcal{Y}'' + P_1^{(d_1)}(\mathcal{Y}', \mathcal{Y}'')) \\ \text{with } P_0^{(d_1)} \text{ and } P_1^{(d_1)} \text{ as given in Lemma 2.4.5(h)} \end{cases}$$

$$\mathcal{R}_A := \mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^{\times} \cup \mathcal{R}_A^{(2)}.$$

Then we *define* a new group, by generators and relations, namely $G_p^{e,-}(A) := \langle \Gamma_A^- \rangle / (\mathcal{R}_A^-)$.

Directly from Definition 3.3.1 it follows that

$$(3-7) \quad G_p^e(A) \cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) = \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times \cup \mathcal{R}_A^{(2)})$$

but we can also achieve the presentation (3-7) in a series of intermediate steps. As a first step, we have

$$(3-8) \quad \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^-) = \langle \Gamma_A^+ \rangle / (\mathcal{R}_A^+) * \langle \Gamma_A^- \rangle / (\mathcal{R}_A^-) \cong G_+(A) * G_p^{e,-}(A),$$

where $G_+(A) \cong \langle \Gamma_A^+ \rangle / (\mathcal{R}_A^+)$ by construction and $*$ denotes the free product (of two groups).

For the next two steps we can follow two different lines of action. The first gives

$$\langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times) \cong (G_+(A) * G_p^{e,-}(A)) / (\overline{\mathcal{R}_A^\times}) \cong G_+(A) \ltimes G_p^{e,-}(A)$$

because of (3-1) and (3-8), where $G_+(A) \ltimes G_p^{e,-}(A)$ is the semidirect product of $G_+(A)$ with $G_p^{e,-}(A)$. Then

$$\begin{aligned} \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) &\cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times \cup \mathcal{R}_A^{(2)}) \\ &\cong (G_+(A) \ltimes G_p^{e,-}(A)) / (\overline{\mathcal{R}_A^{(2)}}) \cong (G_+(A) \ltimes G_p^{\circ,-}(A)) / N_{(2)}(A), \end{aligned}$$

where $N_{(2)}(A)$ is the normal subgroup of $G_+(A) \ltimes G_p^{e,-}(A)$ generated by

$$\{(g_{(2)}, g_{(2)}^{-1})\}_{g_{(2)} \in \Gamma_A^{(2)}}.$$

This and (3-7) together yield $G_p^e(A) = (G_+(A) \ltimes G_-(A)) / N_{(2)}(A)$.

On the other hand, (3-1) and (3-8) jointly give

$$\langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^{(2)}) \cong (G_+(A) * G_p^{e,-}(A)) / (\overline{\mathcal{R}_A^{(2)}}) \cong G_+(A) \underset{G_+^{(2)}(A)}{*} G_p^{e,-}(A),$$

with the amalgamated product of $G_+(A)$ and $G_p^{e,-}(A)$ over $G_+^{(2)}(A)$ being denoted by $G_+(A) \underset{G_+^{(2)}(A)}{*} G_p^{e,-}(A)$. Then

$$\begin{aligned} \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) &\cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^{(2)} \cup \mathcal{R}_A^\times) \\ &\cong (G_+(A) \underset{G_+^{(2)}(A)}{*} G_p^{e,-}(A)) / (\overline{\mathcal{R}_A^\times}) \\ &\cong (G_+(A) \underset{G_+^{(2)}(A)}{*} G_p^{e,-}(A)) / N_\times(A), \end{aligned}$$

where $N_\times(A)$ is the normal subgroup of $G_+(A) \underset{G_+^{(2)}(A)}{*} G_p^{e,-}(A)$ generated by

$$\begin{aligned} &\{g_+ \exp(\mathcal{Y}) g_+^{-1} \exp(\text{Ad}(g_+)(\mathcal{Y}))^{-1}\}_{g_+ \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1}^{g_+ \in G_+(A)} \\ &\cup \{g_+ g_{(2)} g_+ (g_+ \cdot_{G_+} g_{(2)} \cdot_{G_+} g_+)^{-1}\}_{g_+ \in G_+(A)}^{g_{(2)} \in \Gamma_A^{(2)}} \end{aligned}$$

All this along with (3-7) eventually gives

$$G_{\mathcal{P}}^e(A) = (G_+(A) \underset{G_+^{(2)}(A)}{*} G_{\mathcal{P}}^{e,-}(A)) / N_{\times}(A)$$

for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$. In functorial terms this means

$$G_{\mathcal{P}}^e = (G_+ \ltimes G_{\mathcal{P}}^{e,-}) / N_{(2)} \quad \text{and} \quad G_{\mathcal{P}}^e = (G_+ \underset{G_+^{(2)}}{*} G_{\mathcal{P}}^{e,-}) / N_{\times}, \quad \text{or} \quad G_{\mathcal{P}}^e = G_+ \ltimes_{G_+^{(2)}} G_{\mathcal{P}}^{e,-}$$

so that $G_{\mathcal{P}}^e$ is the “amalgamate semidirect product” of G_+ and $G_{\mathcal{P}}^{e,-}$ over $G_+^{(2)}$.

3.4. The supergroup functor $G_{\mathcal{P}}^e$ as a Lie supergroup. We aim now to prove that the functor $G_{\mathcal{P}}^e$ is actually a Lie supergroup. We follow in the footsteps of in Section 3.2, so we can be somewhat shorter. We begin with the following “factorization result” for $G_{\mathcal{P}}^e$:

Proposition 3.4.1. *Let $\mathcal{P} := (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be a super Harish-Chandra pair over $\mathbb{K}$. Then there exist group-theoretic factorizations*

- (a) $G_{\mathcal{P}}^{e,-}(A) = G_+^{(2)}(A) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1), \quad G_{\mathcal{P}}^{e,-}(A) = \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \cdot G_+^{(2)}(A),$
- (b) $G_{\mathcal{P}}^e(A) = G_+(A) \cdot \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1), \quad G_{\mathcal{P}}^e(A) = \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \cdot G_+(A).$

Proof. Claim (a) is the exact analogue of (2-8), and claim (b) the analogue of (2-10), in Proposition 2.4.8(c). In both cases the proof (up to details) is the same, so we can skip it. $\square$

3.4.2. The representation $G_{\mathcal{P}}^e \rightarrow \mathrm{GL}(V)$. Let $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be any given super Harish-Chandra pair over $\mathbb{K}$. Just like we did for $G_{\mathcal{P}}^{\circ}$ in Section 3.2.2, we need for $G_{\mathcal{P}}^e$ as well a suitable linear representation V , which we now define along the same lines, keeping the same notation.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let

$$V := \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(\mathbb{1}) = U(\mathfrak{g}) \underset{U(\mathfrak{g}_0)}{\otimes} \mathbb{1} \cong \bigwedge \mathfrak{g}_1$$

be the $\mathfrak{g}$ -representation induced from the trivial representation $\mathbb{1}$ of $\mathfrak{g}_0$ — as in (3-5). As we saw in Section 3.2.2, there is a morphism $(r_+, \rho) : (G_+, \mathfrak{g}) \rightarrow (\mathrm{GL}(V), \mathfrak{gl}(V))$ of sHCps making V into a $(G_+, \mathfrak{g})$ -module; again by ρ we denote also the representation map $\rho : U(\mathfrak{g}) \rightarrow \mathrm{End}_{\mathbb{K}}(V)$ giving the $U(\mathfrak{g})$ -module structure on V , and similarly (in a functorial way) for the representation maps of $(A \otimes_{\mathbb{K}} \mathfrak{g})_0$ and $(A \otimes_{\mathbb{K}} U(\mathfrak{g}))_0$ onto $(A \otimes_{\mathbb{K}} V)_0$.

We will now show that the $(G_+, \mathfrak{g})$ -module structure on V can be “integrated” to a $G_{\mathcal{P}}^e$ -module structure.

Proposition 3.4.3. *Retain notation as above for the $(G_+, \mathfrak{g})$ -module V . There exists a unique structure of (left) $G_{\mathcal{P}}^e$ -module onto V which satisfies the following conditions: for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, the representation map $r_{\mathcal{P},A}^e : G_{\mathcal{P}}^e(A) \rightarrow \mathrm{GL}(V)(A)$*

is given on generators of $G_{\mathcal{P}}^e(A)$ — namely, all $g_+ \in G_+(A)$ and $\exp(\mathcal{Y})$ for $i \in I$, $\mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$ — by $\mathbf{r}_{\mathcal{P},A}^e(g_+) := \mathbf{r}_+(g_+)$, $\mathbf{r}_{\mathcal{P},A}^e(\exp(\mathcal{Y})) := \rho(\exp(\mathcal{Y})) = \exp(\rho(\mathcal{Y}))$, that is $g_+ \cdot v := \mathbf{r}_+(g_+)(v)$ and $\exp(\mathcal{Y}) \cdot v := \exp(\rho(\mathcal{Y}))(v)$ — with $\exp(\rho(\mathcal{Y}))$ being a finite sum — for all $v \in V(A)$. Overall, this yields a morphism of $\mathbb{K}$ -supergroup functors $\mathbf{r}_{\mathcal{P}}^e : G_{\mathcal{P}}^e \rightarrow \mathrm{GL}(V)$.

Proof. By definition of representation for the sHCp $\mathcal{P}$, the operators $\mathbf{r}_{\mathcal{P},A}^e(g_+)$ and $\mathbf{r}_{\mathcal{P},A}^e(\exp(\mathcal{Y}))$ on V satisfy all relations which, by Definition 3.3.1(a), are satisfied by the generators of $G_{\mathcal{P}}^e(A)$. Thus they define a unique group morphism $\mathbf{r}_{\mathcal{P},A}^e : G_{\mathcal{P}}^e(A) \rightarrow \mathrm{GL}(V)(A)$, functorial in A by construction, whence the claim. $\square$

We are ready to state the “global splitting theorem” for $G_{\mathcal{P}}^e$ (see Theorem 2.4.10):

Proposition 3.4.4. (a) *The restriction of group multiplication in $G_{\mathcal{P}}^e$ provides isomorphisms of (set-valued) functors*

$$\begin{aligned} G_+ \times \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) &\cong G_{\mathcal{P}}^e, & \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \times G_+ &\cong G_{\mathcal{P}}^e, \\ G_+^{(2)} \times \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) &\cong G_{\mathcal{P}}^{e,-}, & \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \times G_+^{(2)} &\cong G_{\mathcal{P}}^{e,-}. \end{aligned}$$

(b) *For any fixed $\mathbb{K}$ -basis $\{Y_i\}_{i \in I}$ of $\mathfrak{g}_1$, there exists an isomorphism of (set-valued) functors $\mathbb{A}_{\mathbb{K}}^{0|d_1} \cong \exp((-)_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$, with $d_1 := \dim_{\mathbb{K}}(\mathfrak{g}_1) = |I|$, given on A -points, for $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, by*

$$\mathbb{A}_{\mathbb{K}}^{0|d_1}(A) = A_1^{d_1} \rightarrow \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1), \quad (\eta_i)_{i \in I} \mapsto \exp(\sum_{i \in I} \eta_i Y_i)$$

(c) *There exist isomorphisms of (set-valued) functors*

$$G_+ \times \mathbb{A}_{\mathbb{K}}^{0|d_1} \cong G_{\mathcal{P}}^e, \quad G_+^{(2)} \times \mathbb{A}_{\mathbb{K}}^{0|d_1} \cong G_{\mathcal{P}}^{e,-} \quad \text{and} \quad \mathbb{A}_{\mathbb{K}}^{0|d_1} \times G_+ \cong G_{\mathcal{P}}^e, \quad \mathbb{A}_{\mathbb{K}}^{0|d_1} \times G_+^{(2)} \cong G_{\mathcal{P}}^{e,-}$$

given on A -points — for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — respectively by

$$(g_+, (\eta_i)_{i \in I}) \mapsto g_+ \cdot \exp(\sum_{i \in I} \eta_i Y_i) \quad \text{and} \quad ((\eta_i)_{i \in I}, g_+) \mapsto \exp(\sum_{i \in I} \eta_i Y_i) \cdot g_+$$

Proof. Like for Proposition 3.2.4, the proof is very close to (half of) that of Theorem 2.4.10, with a few, technical differences that involve the representation V of Section 3.4.2; the necessary changes can easily be dealt with much like in the proof of Proposition 3.2.4. Details are left to the reader. $\square$

3.4.5. The Lie supergroup structure of $G_{\mathcal{P}}^e$. For any given $\mathcal{P} \in (\mathbf{sHCp})_{\mathbb{K}}$ and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, by Proposition 3.4.4(c), we have a particular bijection

$$\phi_A^e : G_+(A) \times \mathbb{A}_{\mathbb{K}}^{0|d_1}(A) \xrightarrow{\cong} G_{\mathcal{P}}^e(A)$$

whose restriction to $G_+(A)$, identified with $G_+(A) \times \{(0)_{i \in I}\}$, is the identity — onto the copy of $G_+(A)$ inside $G_{\mathcal{P}}^e(A)$.

Now, $G_+(A)$ is by definition an A_0 -manifold (see [Balduzzi et al. 2013]), of the same type (real smooth, etc.) as $\mathcal{P} := (G_+, \mathfrak{g})$; on the other hand, $\mathbb{A}_{\mathbb{K}}^{0|d_1}(A)$

carries canonical structures of A_0 -manifold of any type (real smooth, etc.), then also of the type of $G_+(A)$. So there exists also a canonical “product structure” of A_0 -manifold — of the same type of $\mathcal{P}$ — onto $G_+(A) \times \mathbb{A}_{\mathbb{K}}^{0|d_1}(A)$. Then we push-forward — through ϕ_A^e — this canonical A_0 -manifold structure of $G_+(A) \times \mathbb{A}_{\mathbb{K}}^{0|d_1}(A)$ onto $G_{\mathcal{P}}^e(A)$, which then is an A_0 -manifold on its own.

Using the above mentioned structure of A_0 -manifold on $G_{\mathcal{P}}^e(A)$ for each $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, given a morphism $f : A' \rightarrow A''$ in $(\mathbf{Wsalg})_{\mathbb{K}}$ we can check that the corresponding group morphism $G_{\mathcal{P}}^e(f) : G_{\mathcal{P}}^e(A') \rightarrow G_{\mathcal{P}}^e(A'')$ is a morphism of A'_0 -manifolds, hence it is a morphism of A_0 -manifolds (see [Balduzzi et al. 2013]). Thus $G_{\mathcal{P}}^e$ is also a functor from Weil $\mathbb{K}$ -superalgebras to A_0 -manifolds (real smooth, real analytic or complex holomorphic as $\mathcal{P}$ is).

Finally, looking at the commutation relations in $G_{\mathcal{P}}^e(A)$ we find that the group multiplication and the inverse map are “regular” (i.e., “real smooth”, “real analytic” or “complex holomorphic”, according to the type of $\mathcal{P}$): this is proved via calculations like those used to prove Proposition 3.4.1(b) or Proposition 2.4.8(c). So $G_{\mathcal{P}}^e(A)$ is a group element among A_0 -manifolds, i.e., it is a Lie A_0 -group; thus (see Section 2.1.3), the functor $G_{\mathcal{P}}^e$ is a Lie supergroup (of real smooth, real analytic or complex holomorphic type as $\mathcal{P}$ is).

Eventually, the outcome of this discussion — the key result of this section — is the following statement, which provides a second “backward functor” from sHCps to Lie supergroups:

Theorem 3.4.6. *The recipe in Definition 3.3.1 provides functors*

$$\Psi^e : (\mathbf{sHCp})_{\mathbb{R}}^{\infty} \rightarrow (\mathbf{Lsgrp})_{\mathbb{R}}^{\infty}, \quad \Psi^e : (\mathbf{sHCp})_{\mathbb{R}}^{\omega} \rightarrow (\mathbf{Lsgrp})_{\mathbb{R}}^{\omega}, \quad \Psi^e : (\mathbf{sHCp})_{\mathbb{C}}^{\omega} \rightarrow (\mathbf{Lsgrp})_{\mathbb{C}}^{\omega}$$

given on objects by $\mathcal{P} \mapsto \Psi^e(\mathcal{P}) := G_{\mathcal{P}}^e$ and on morphisms by

$$((\phi_+, \varphi) : \mathcal{P}' \rightarrow \mathcal{P}'') \mapsto (\Psi^e((\phi_+, \varphi)) : \Psi^e(\mathcal{P}') := G_{\mathcal{P}'}^e \rightarrow G_{\mathcal{P}''}^e =: \Psi^e(\mathcal{P}'')),$$

where the functor morphism $\Psi^e((\phi_+, \varphi)) : \Psi^e(\mathcal{P}') := G_{\mathcal{P}'}^e \rightarrow G_{\mathcal{P}''}^e =: \Psi^e(\mathcal{P}'')$ is defined by

$$(3-9) \quad \Psi^e((\phi_+, \varphi))_A : g'_+ \mapsto \phi_+(g'_+), \quad \exp(\mathcal{Y}') \mapsto \exp(\varphi(\mathcal{Y}'))$$

for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, $g'_+ \in G'_+(A)$, $\mathcal{Y}' \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}'_+$, with $\mathcal{P}' = (G'_+, \mathfrak{g}'_+)$ and $\mathcal{P}'' = (G''_+, \mathfrak{g}''_+)$.

Proof. We are left to prove that the given definition for $\Psi^e((\phi_+, \varphi))$ makes sense: all the rest are either proved by our previous analysis or trivial. Now, (3-9) above fixes the values of our would-be morphism $\Psi^e((\phi_+, \varphi))_A$ on generators of $\Psi^e(\mathcal{P}')(A) := G_{\mathcal{P}'}^e(A)$: a direct check shows that all defining relations among such generators — in $G_{\mathcal{P}'}^e(A)$ — are mapped to corresponding relations in $G_{\mathcal{P}''}^e(A)$, thus yielding a well-defined group morphism. In particular, this follows from the special properties

of the Lie polynomials $P_0^{(d_1)}$ and $P_1^{(d_1)}$ and of their factors/summands $T_0^{(s)}$ and $T_1^{(s)}$ mentioned in Lemma 2.4.5(h).

However, we must still show that *each such* $\Psi^e((\phi_+, \varphi))_A$ *is a morphism of* $\mathcal{A}_0$ -*manifolds too.*

Both groups $G_{\mathcal{P}'}^e(A)$ and $G_{\mathcal{P}''}^e(A)$ admit factorizations of type $G_+ \times \exp(A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$, as in Proposition 3.4.4(a): so any $g' \in G_{\mathcal{P}'}^e(A)$ uniquely factors into $g' = g'_+ \cdot \exp(\mathcal{Y}')$; then $\Psi^e((\phi_+, \varphi))_A$ maps g' onto

$$(3-10) \quad \Psi^e((\phi_+, \varphi))_A(g') = \Psi^e((\phi_+, \varphi))_A(g'_+ \cdot \exp(\mathcal{Y}')) = \phi_+(g'_+) \cdot \exp(\varphi_A(\mathcal{Y}'))$$

where $\varphi_A(\mathcal{Y}')$ stands for the image of $\mathcal{Y}'$ for the map $\varphi_A : A_1 \otimes_{\mathbb{K}} \mathfrak{g}'_1 \rightarrow A_1 \otimes_{\mathbb{K}} \mathfrak{g}''_1$ obtained by scalar extension from $\varphi|_{\mathfrak{g}'_1} : \mathfrak{g}'_1 \rightarrow \mathfrak{g}''_1$. As both $\exp \circ \varphi_A \circ (\exp|_{A_1 \otimes \mathfrak{g}_1})^{-1}$ and ϕ_+ are maps of $\mathcal{A}_0$ -manifolds, from (3-10) we deduce that $\Psi^e((\phi_+, \varphi))_A$ is a map of $\mathcal{A}_0$ -manifolds too. $\square$

3.5. The new equivalences $(\mathbf{sHCp}) \cong (\mathbf{Lsgrp})$. In Section 3 we introduced two functors, denoted Ψ° and Ψ^e , from $\mathbf{sHCps}$ to Lie supergroups. We will now show that both these two functors are quasi-inverse to the “natural” functor Φ considered in Section 2.2, so that (together with Φ) they provide equivalences between the categories of $\mathbf{sHCps}$ and of Lie supergroups.

We begin with the first half of our task, namely proving that $\Phi \circ \Psi \cong \text{id}_{(\mathbf{sHCp})}$ for $\Psi \in \{\Psi^\circ, \Psi^e\}$.

Proposition 3.5.1. *Let Φ and $\Psi \in \{\Psi^\circ, \Psi^e\}$ be as in Section 2.2.2 and in Theorems 3.2.6 and 3.4.6.*

Then $\Phi \circ \Psi^\circ \cong \text{id}_{(\mathbf{sHCp})}$, where “ $(\mathbf{sHCp})$ ” must be read as either $(\mathbf{sHCp})_{\mathbb{R}}^\infty$, or $(\mathbf{sHCp})_{\mathbb{R}}^\omega$, or $(\mathbf{sHCp})_{\mathbb{C}}^\omega$, and Φ and Ψ must be taken as working onto the corresponding types of Lie supergroups or $\mathbf{sHCps}$

Proof. This follows almost directly from definitions: we sketch the case of Ψ° , that of Ψ^e being similar.

Consider a super Harish-Chandra pair $\mathcal{P} := (G_+, \mathfrak{g})$, and let $G_{\mathcal{P}}^\circ = \Psi^\circ(\mathcal{P})$, so that $(\Phi \circ \Psi^\circ)(\mathcal{P}) = \Phi(G_{\mathcal{P}}^\circ) = ((G_{\mathcal{P}}^\circ)_0, \text{Lie}(G_{\mathcal{P}}^\circ))$. Then $(G_{\mathcal{P}}^\circ)_0 = G_+$ just by construction. In addition, the definition of $\text{Lie}(G)$ and Proposition 3.2.4 together yield $\text{Lie}(G_{\mathcal{P}}^\circ) = \text{Lie}(G_+ \times G_-) = \text{Lie}(G_+) \oplus T_e(G_-) = \mathcal{L}_{\mathfrak{g}_0} \oplus \mathcal{L}_{\mathfrak{g}_1} = \mathcal{L}_{\mathfrak{g}}$ hence, identifying $\mathcal{L}_{\mathfrak{g}} = \mathfrak{g}$, simply $\text{Lie}(G_{\mathcal{P}}^\circ) = \mathfrak{g}$. Thus $(\Phi \circ \Psi^\circ)(\mathcal{P}) = \Phi(G_{\mathcal{P}}^\circ) = ((G_{\mathcal{P}}^\circ)_0, \text{Lie}(G_{\mathcal{P}}^\circ)) \cong (G_+, \mathfrak{g}) = \mathcal{P}$, i.e., $\Phi \circ \Psi^\circ$ acts on objects — up to natural isomorphisms — as the identity.

Now let $(\phi_+, \varphi) : \mathcal{P}' = (G'_+, \mathfrak{g}') \rightarrow (G''_+, \mathfrak{g}'') = \mathcal{P}''$ be a morphism of $\mathbf{sHCps}$ and $\phi := \Psi^\circ((\phi_+, \varphi))$ the corresponding (via Ψ°) morphism of supergroups; we aim to prove that $(\Phi \circ \Psi^\circ)((\phi_+, \varphi)) = \Phi(\phi)$ coincides, up to the natural isomorphisms $(\Phi \circ \Psi^\circ)(\mathcal{P}') \cong \mathcal{P}'$ and $(\Phi \circ \Psi^\circ)(\mathcal{P}'') \cong \mathcal{P}''$, with (ϕ_+, φ) itself.

By definition $\Phi(\phi) := (\phi|_{(G_{\mathcal{P}'})_0}, d\phi)$, and $\phi|_{(G_{\mathcal{P}'})_0} = \Psi^\circ((\phi_+, \varphi))|_{G'_+} = \phi_+$ by construction. Moreover, like in the proof of Theorem 3.2.6, using the factorizations $G_{\mathcal{P}'} = G'_+ \times G'^{<}_-$ and $G_{\mathcal{P}''} = G''_+ \times G''^{<}_-$ we find that onto $T_e(G_{\mathcal{P}'}) = T_e(G'_+) \oplus T_e(G'^{<}_-) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{g}$ our $d\phi$ acts by $d\phi|_{\mathfrak{g}_0} = d\phi_+ = \varphi|_{\mathfrak{g}_0}$ and $d\phi|_{\mathfrak{g}_1} = \varphi|_{\mathfrak{g}_1}$; this gives $d\phi = d\phi|_{\mathfrak{g}_0} \oplus d\phi|_{\mathfrak{g}_1} = \varphi_0 \oplus \varphi_1 = \varphi$, hence $\Phi(\phi) := (\phi|_{(G_{\mathcal{P}'})_0}, d\phi) = (\phi_+, \varphi)$. $\square$

The second half of our task is proving that $\Psi \circ \Phi$ is isomorphic to the identity functor on Lie supergroups:

Proposition 3.5.2. *Let Φ and $\Psi \in \{\Psi^\circ, \Psi^e\}$ be as in Section 2.2.2 and in Theorems 3.2.6 and 3.4.6.*

Then $\Psi^\circ \circ \Phi \cong \text{id}_{(\mathbf{Lsgrp})}$, where “ $(\mathbf{Lsgrp})$ ” must be read as either $(\mathbf{Lsgrp})_{\mathbb{R}}^\infty$, or $(\mathbf{Lsgrp})_{\mathbb{R}}^\omega$, or $(\mathbf{Lsgrp})_{\mathbb{C}}^\omega$, and Φ and Ψ must be taken as working onto the corresponding types of Lie supergroups or sHCps

Proof. We begin again by looking at Ψ° . Given a Lie supergroup G , set $\mathfrak{g} := \text{Lie}(G)$ and $\mathcal{P} := \Phi_g(G) = (G_0, \mathfrak{g})$. We look at $\Psi^\circ(\Phi(G)) = \Psi^\circ(\mathcal{P}) =: G_{\mathcal{P}}^\circ$, and prove that it is naturally isomorphic to G . For any $A \in (\mathbf{salg})_{\mathbb{k}}$, by abuse of notation we denote with the same symbol any $g_0 \in G_0(A)$ as belonging to $G(A)$ — via the embedding of $G_0(A)$ into $G(A)$ — and as an element of $G_{\mathcal{P}}^\circ(A)$. With this convention, we easily see that Lemma 2.4.5 yields the following: *there exists a unique group morphism $\phi_A : G_{\mathcal{P}}^\circ(A) \rightarrow G(A)$ such that $\phi_A(g_0) = g_0$ for all $g_0 \in G_0(A)$ and $\phi_A((1 + \eta Y)) = (1 + \eta Y)$ for all $\eta \in A_1, Y \in \mathfrak{g}_1$.*

Due to the factorization (2-7) in Proposition 2.4.8, we have also that *the morphism ϕ_A is surjective*. Even more, the *global splitting theorem* for G (Theorem 2.4.10) and for $G_{\mathcal{P}}^\circ$ (Proposition 3.2.4) together imply that *the morphism ϕ_A is also injective*, hence *it is a group isomorphism*. Finally, all these ϕ_A ’s are clearly natural in A , thus altogether they provide an isomorphism between $G_{\mathcal{P}}^\circ = \Psi^\circ(\Phi(G))$ and G .

The case of Ψ^e is dealt with similarly, using the parallel results for $G_{\mathcal{P}}^e$ to those applied for $G_{\mathcal{P}}^\circ$: namely, Lemma 2.4.5, formula (2-10) instead of (2-7), Proposition 3.4.4 instead of Proposition 3.2.4, etc. $\square$

To finish with, all the above jointly give the following, main result:

Theorem 3.5.3. *Let Φ and $\Psi \in \{\Psi^\circ, \Psi^e\}$ be as in Section 2.2.2 and in Theorems 3.2.6 and 3.4.6.*

Then Ψ° and Ψ^e are quasi-inverse to Φ , so they provide category equivalences

$$(\mathbf{sHCp})_{\mathbb{R}}^\infty \cong (\mathbf{Lsgrp})_{\mathbb{R}}^\infty, \quad (\mathbf{sHCp})_{\mathbb{R}}^\omega \cong (\mathbf{Lsgrp})_{\mathbb{R}}^\omega, \quad (\mathbf{sHCp})_{\mathbb{C}}^\omega \cong (\mathbf{Lsgrp})_{\mathbb{C}}^\omega$$

3.6. Linear case and representations. We shall now briefly discuss the fallout, in representation theory, of the existence of an equivalence between Lie supergroups and sHCps in particular when realized via the functors Ψ° and Ψ^e of Section 3.

3.6.1. The linear case. Assume that the Lie supergroup G is linear, i.e., it embeds into some $\mathrm{GL}(V)$, where V is a suitable superspace (in other words, there exists a faithful G -module V). Then G_0 and $\mathfrak{g} := \mathrm{Lie}(G)$ embed into $\mathrm{End}(V)$, and the relations linking them such that $(G_0, \mathfrak{g})$ is a super Harish-Chandra pair are relations among elements of the unital, associative superalgebra $\mathrm{End}(V)$. Conversely, one can formally define a “linear sHCp” as being any sHCp $(G_0, \mathfrak{g})$ such that both G_0 and $\mathfrak{g}$ embed into some $\mathrm{End}(V)$, and the compatibility relations linking G_0 and $\mathfrak{g}$ hold true as relations inside $\mathrm{End}(V)$ itself — mimicking what is done in [Gavarini 2016a, Definition 4.2.1(b)] in the algebraic setup. The above then tells us that, if G is linear, then its associated sHCp $\Phi(G) =: \mathcal{P}$ is linear too — both being linearized via their faithful representation onto V .

On the other hand, let us start with a linear sHCp, say $\mathcal{P} = (G_+, \mathfrak{g})$: so the latter is embedded (in the obvious sense) into the sHCp $(\mathrm{GL}_0(V), \mathfrak{gl}(V))$ for some V . Thus for $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ both $G_+(A)$ and $A \otimes_{\mathbb{K}} \mathfrak{g}$ are embedded into $(\mathrm{End}(V))(A)$, with relations among them — inside $(\mathrm{End}(V))(A)$ — induced by the very notion of linear sHCp. Now consider in $(\mathrm{End}(V))(A)$ all elements $\exp(\eta Y) = (1 + \eta Y)$ — with $\eta \in A_1$, $Y \in \mathfrak{g}_1$ — that actually belong to $(\mathrm{GL}(V))(A)$: moreover, $G_+(A) \subseteq (\mathrm{GL}(V))(A)$ too. Therefore, *we can take inside $(\mathrm{GL}(V))(A)$ the subgroup $G_{\mathcal{P}}^{\circ, V}(A)$ generated by $G_+(A)$ and by all the $(1 + \eta Y)$ ’s.*

A trivial check shows that the elements from $G_+(A)$ and the $(1 + \eta_i Y_i)$ ’s enjoy all relations that enter in the very definition $G_{\mathcal{P}}^{\circ}(A)$: thus, *there exists a (unique) group epimorphism $\phi_A^{\circ} : G_{\mathcal{P}}^{\circ}(A) \rightarrow G_{\mathcal{P}}^{\circ, V}(A)$ such that $\phi_A^{\circ}(g_+) = g_+$ and $\phi_A^{\circ}((1 + \eta Y)) = (1 + \eta Y)$ for all $g_+ \in G_+(A)$, $\eta \in A_1$, $Y \in \mathfrak{g}_1$.*

On the other hand, $G_{\mathcal{P}}^{\circ, V}(A)$ acts faithfully on V — indeed, it is “linearized by V ”. This allows one to show that *the like of the global splitting theorem* — see Theorem 2.4.10 and Proposition 3.2.4 — *does hold true for $G_{\mathcal{P}}^{\circ, V}(A)$* , via the same analysis and arguments used in the proofs of those results *but* for one change: the linearization of $G_{\mathcal{P}}^{\circ, V}(A)$ has to replace the following key ingredients: (1) in the proof of Theorem 2.4.10, one has that $G(A) := \coprod_{x \in |G|} \mathrm{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{|G|, x}, A)$, (2) in the proof of Proposition 3.2.4 (and the lemmas before), $G_{\mathcal{P}}^{\circ}(A)$ is acting onto $V := \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(\mathbb{1})$ — see (3-5).

In fact, in both cases — of either $G(A)$ or $G_{\mathcal{P}}^{\circ}(A)$ — the group under examination is realized as a group of maps and these are rich enough to “separate (enough) points” so as to guarantee the uniqueness of factorization(s) that is the core part of the global splitting theorem. In the case of $G_{\mathcal{P}}^{\circ, V}(A)$ instead, its built-in linearization provides a similar realization as “group of maps”, and this again allows to separate enough points to get global splitting(s) for $G_{\mathcal{P}}^{\circ, V}(A)$ too. Finally, thanks to the global splitting theorem for $G_{\mathcal{P}}^{\circ}(A)$ and for $G_{\mathcal{P}}^{\circ, V}(A)$, one can apply again the arguments used in the proof of Proposition 3.5.2 and successfully prove that *the above group (epi)morphism $\phi_A^{\circ} : G_{\mathcal{P}}^{\circ}(A) \rightarrow G_{\mathcal{P}}^{\circ, V}(A)$ is also injective, hence it is an isomorphism.*

By construction all these isomorphisms ϕ_A° are natural in A , hence they give altogether a functor isomorphism $\phi^\circ : G_{\mathcal{P}}^\circ \xrightarrow{\cong} G_{\mathcal{P}}^{\circ, V}$. Therefore $G_{\mathcal{P}}^\circ \cong G_{\mathcal{P}}^{\circ, V}$, which means that we found a different, concrete realization of $G_{\mathcal{P}}^\circ$, that is now constructed explicitly as the *linear* Lie supergroup $G_{\mathcal{P}}^{\circ, V}$.

In a parallel way, still starting with a linear sHCp $\mathcal{P} = (G_+, \mathfrak{g})$ embedded into $(\mathrm{GL}_0(V), \mathfrak{gl}(V))$, we consider in $(\mathrm{End}(V))(A)$ all elements of the form $\exp(\mathcal{Y})$ with $\mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$ and then take the subgroup $G_{\mathcal{P}}^{e, V}(A)$ of $(\mathrm{GL}(V))(A)$ generated by $G_+(A)$ and by all the $\exp(\mathcal{Y})$'s. Acting like above we find that there exists a group epimorphism $\phi_A^e : G_{\mathcal{P}}^e(A) \rightarrow G_{\mathcal{P}}^{e, V}(A)$ such that $\phi_A^e(g_+) = g_+$, $\phi_A^e(\exp(\mathcal{Y})) = \exp(\mathcal{Y})$, for all $g_+ \in G_+(A)$ and $\mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$. Even more, still by the same method as above we find that all these ϕ_A^e 's are in fact isomorphisms, natural in A , hence they define a functor isomorphism $\phi^e : G_{\mathcal{P}}^e \xrightarrow{\cong} G_{\mathcal{P}}^{e, V}$. This gives yet another concrete realization of $G_{\mathcal{P}}^e$, now explicitly realized as the *linear* Lie supergroup $G_{\mathcal{P}}^{e, V}$.

Finally, the construction also shows that, indeed, $G_{\mathcal{P}}^{\circ, V} = G_{\mathcal{P}}^{e, V}$.

3.6.2. Induction from G_0 to G . Let G be a supergroup (of any type), with classical subsupergroup G_0 , and V be any G_0 -module. We shall now present an explicit construction of the *induced G -module* $\mathrm{Ind}_{G_0}^G(V)$.

The G_0 -module V is also a $\mathfrak{g}_0$ -module, hence one has the induced $\mathfrak{g}$ -module $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$, that one can realize as $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V) = \mathrm{Ind}_{U(\mathfrak{g}_0)}^{U(\mathfrak{g})}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} V$. This bears also a unique structure of G_0 -module which is compatible with the $\mathfrak{g}$ -action and coincides with the original G_0 -action on $\mathbb{K} \otimes_{U(\mathfrak{g}_0)} V \cong V$ given from scratch. Indeed, we can describe explicitly this G_0 -action, as follows. First, definitions give $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} V = \bigwedge \mathfrak{g}_1 \otimes_{\mathbb{K}} V$ — see (3-4) — with the $\mathfrak{g}_0$ -action given by

$$x.(y \otimes v) = \mathrm{ad}(x)(y) \otimes v + y \otimes (x.v)$$

for $x \in \mathfrak{g}_0$, $y \in \bigwedge \mathfrak{g}_1$, $v \in V$, where by ad we denote the unique $\mathfrak{g}_0$ -action on $\bigwedge \mathfrak{g}_1$ by algebra derivations induced by the adjoint $\mathfrak{g}_0$ -action on $\mathfrak{g}_1$. Second, this action integrates to a (unique) G_0 -action given by $g_0.(y \otimes v) := \mathrm{Ad}(g_0)(y) \otimes (g_0.v)$ for $g_0 \in G_0$, $y \in \bigwedge \mathfrak{g}_1$, $v \in V$, where we write Ad for the G_0 -action on $\bigwedge \mathfrak{g}_1$ by algebra automorphisms induced by the adjoint G_0 -action on $\mathfrak{g}_1$.

The key point is that the above G_0 -action and the built-in $\mathfrak{g}$ -action on $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ are compatible, in that they make $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ into a module for the sHCp $\mathcal{P} := (G_0, \mathfrak{g}) = \Phi(G)$. Then for $\Psi \in \{\Psi^\circ, \Psi^e\}$, since $\Psi((G_0, \mathfrak{g})) \cong G$, clearly $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ bears a unique structure of G -module which corresponds to the previous $\mathcal{P}$ -action. In down-to-earth terms what happens is the following. The action of $\mathcal{P} := (G_0, \mathfrak{g}) = \Phi(G)$ onto $W := \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ is given by the G_0 -action (induced by that on V) and a compatible $\mathfrak{g}$ -action. Then for any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we have also that all of $A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$ “acts” onto $A \otimes_{\mathbb{K}} W$: thus well-defined operators $(1 + \eta Y)$ and $\exp(\mathcal{Y})$ — with $(\eta, Y) \in A_1 \times \mathfrak{g}_1$, $\mathcal{Y} \in A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1$ — exist in $(\mathrm{GL}(W))(A)$. One checks that these $(1 + \eta Y)$'s, or the $\exp(\mathcal{Y})$'s, altogether enjoy among themselves and with the

operators given by the G_0 -action all relations that enter in the definition of either $\Psi^\circ(\mathcal{P}) := G_\mathcal{P}^\circ$ or $\Psi^e(\mathcal{P}) = G_\mathcal{P}^e$ — thus in both cases we get an action of $\Psi(\mathcal{P})$ on W , extending the initial one by G_0 : but $\Psi(\mathcal{P}) = G$, so we are done.

Thus we define as $\text{Ind}_{G_0}^G(V)$ the space $W := \text{Ind}_{g_0}^g(V)$ endowed with this G -action: one checks that this construction is functorial in V and has the universal property making it adjoint of “restriction” (from G -modules to G_0 -modules), so it can be correctly called “induction” functor. In addition, if the G_0 -module V is faithful then the G -module $\text{Ind}_{G_0}^G(V)$ is faithful too: this means that if G_0 is linearizable, then G is linearizable too; in fact, from a linearization of G_0 we can construct (via induction) a linearization of G too.

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FABIO GAVARINI
 DIPARTIMENTO DI MATEMATICA
 UNIVERSITÀ DEGLI STUDI DI ROMA “TOR VERGATA”
 ROMA
 ITALY
 gavarini@mat.uniroma2.it

GENERALIZED MULLINEUX INVOLUTION AND PERVERSE EQUIVALENCES

THOMAS GERBER, NICOLAS JACON AND EMILY NORTON

We define a generalization of the Mullineux involution on multipartitions using the theory of crystals for higher-level Fock spaces. Our generalized Mullineux involution turns up in representation theory via two important derived functors on cyclotomic Cherednik category $\mathcal{O}$: Losev’s “ $\kappa = 0$ ” wall-crossing, and Ringel duality.

Introduction

It has been known since the foundational work of Frobenius that partitions of n naturally label the complex irreducible representations of the symmetric group $\mathfrak{S}_n$. If we take an irreducible representation labeled by a partition λ and tensor it with the sign representation, we obtain an irreducible representation labeled by the transpose of λ . The story in positive characteristic is more subtle: the irreducible representations of $\mathfrak{S}_n$ over a field of characteristic $p > 0$ are labeled by the p -regular partitions (partitions in which each nonzero part occurs at most $p - 1$ times). Tensoring such a representation with the sign representation still yields an irreducible representation, but the resulting involution on p -regular partitions lacks such a simple description as taking the transpose. Mullineux [1979] defined a combinatorial algorithm producing an involution on p -regular partitions (now called the Mullineux involution), and he conjectured that this involution describes the result of tensoring an irreducible representation with the sign representation in characteristic p .

Kleshchev [1996] came up with a surprising algorithm to compute the Mullineux involution. In fact, whereas Mullineux’s algorithm involved repeated operations with strips of boxes in the rim of the Young diagram, it was later understood that Kleshchev’s algorithm could be interpreted in terms of the Kashiwara crystal of an irreducible highest weight module of level 1 for the quantum group of affine type A_{p-1} [Lascoux et al. 1996]: the Mullineux involution is the automorphism of oriented $\mathbb{Z}/p\mathbb{Z}$ -colored graphs which switches the sign of each arrow. This algorithm led to Ford and Kleshchev’s proof of the Mullineux conjecture [1997]; a different proof was given later by Bessenrodt and Olsson [1998].

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The Mullineux involution can be generalized to various extents. First, one can look at the Hecke algebra of $\mathfrak{S}_n$ (which can be seen as a deformation of the group algebra) with parameter specialized to a primitive e -th root of 1, $e \in \mathbb{Z}_{\geq 2}$. An involution on the set of e -regular partitions (which parametrize the associated irreducible representations) can then be defined using crystals as above; see [Lascoux et al. 1996, Section 7]. Fayers [2008] defined a Mullineux involution for the Hecke algebra of the complex reflection group $G(\ell, 1, n)$ (the Ariki–Koike algebra). Fayers’ involution can also be computed using crystal graphs (now for irreducible highest weight modules of level ℓ) or via a combinatorial algorithm generalizing Mullineux’s original procedure [Jacon and Lecouvey 2009]. The Ariki–Koike algebra has cell modules labeled by all ℓ -partitions, but simples labeled only by Uglov ℓ -partitions (which coincide with e -regular partitions for $\ell = 1$). However, its module category is a quotient of a highest weight category $\mathcal{O}_{\kappa, \mathbf{s}}$ where every ℓ -partition labels a simple module, raising the question whether the Mullineux involution admits a further meaningful extension to that bigger category.

Namely, consider the category $\mathcal{O}_{\kappa, \mathbf{s}}(n)$ of the Cherednik algebra of $G(\ell, 1, n)$. This category depends on parameters $\kappa \in \mathbb{Q}^\times$ and $\mathbf{s} \in \mathbb{Q}^\ell$ [Ginzburg et al. 2003; Rouquier 2008; Losev 2017b] and its Grothendieck group has a basis consisting of ℓ -partitions of n . In order to relate categories depending on different parameters, Losev [2015] introduced derived equivalences called wall-crossing functors. Each wall-crossing can be thought of as a partial version of a duality functor called Ringel duality. The wall-crossing functors and Ringel duality are examples of a special kind of derived equivalence called a perverse equivalence [Chuang and Rouquier 2008; Losev 2017a], and consequently they effect a permutation of the set of simple objects, that is, a permutation of ℓ -partitions. It is natural to ask for an explicit formula for these combinatorial maps.

We now summarize the main results of this paper. In Theorem 2.10 we define a generalization of the Mullineux involution on all multipartitions. The proof uses the result of [Gerber 2018] that the $\widehat{\mathfrak{sl}_e}$ -, $\widehat{\mathfrak{sl}_\infty}$ -, and $\widehat{\mathfrak{sl}_\ell}$ -crystals on the level ℓ Fock space all commute. Our involution Φ is compatible with both Fayers’ and Losev’s involutions, recovering Fayers’ in the case of Uglov multipartitions. The next question is the representation-theoretic meaning of Φ . In Section 3 we study the combinatorics of perverse equivalences on module categories of Cherednik algebras. Theorems 3.5 and 3.7 give some formulas for the $\kappa = 0$ wall-crossing in terms of ℓ copies of the level 1 Mullineux involution; we recover [Losev 2015, Corollary 5.7] when $\ell = 1$. Next, we look for a duality functor which produces the involution Φ , and we find in Theorem 3.14 that Φ arises from Ringel duality. Here the perspective of diagrammatic Cherednik algebras [Webster 2017] is crucial, especially [Webster 2017, Corollary 5.11]. In Section 4, we define a refinement of Φ with a speculative eye towards the Alvis–Curtis duality, a perverse equivalence for finite groups of Lie

type which still lacks a combinatorial description outside type A . This generalizes Dudas and Jacon's [2018] definition of a generalized Mullineux involution in the case $\ell = 1$ by refining the $\mathfrak{sl}_\infty$ -crystal with respect to an integer parameter d .

1. The Mullineux involution for cyclotomic Hecke algebras

We here give a quick review of the definition of the Mullineux involution for cyclotomic Hecke algebras and its crystal interpretation [Fayers 2008; Jacon and Lecouvey 2009]. This generalizes the usual notion of Mullineux involution.

1A. Definition. Let $\ell \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$. Denote by $W_{\ell,n}$ the complex reflection group $G(\ell, 1, n) = \mathfrak{S}_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$. Let R be a field of arbitrary characteristic and let $v \in R^\times$ and let $(s_1, s_2, \dots, s_\ell)$ be an ℓ -tuple of integers.

The cyclotomic Hecke algebra (also called the Ariki–Koike algebra) $\mathcal{H}_{R,n}^s = \mathcal{H}(v; s_1, \dots, s_\ell)$ over R is the unital associative R -algebra with a presentation by

- generators: $T_0, T_1, \dots, T_{n-1}$,
- relations:

$$\begin{aligned} T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & (i = 1, \dots, n-2), \\ T_i T_j &= T_j T_i & (|j - i| > 1), \\ (T_0 - v^{s_1})(T_0 - v^{s_2}) \cdots (T_0 - v^{s_\ell}) &= 0, \\ (T_i - v)(T_i + 1) &= 0, & (i = 1, \dots, n-1). \end{aligned}$$

It can be seen as a deformation of the group algebra of $W_{\ell,n}$. In particular, if $\ell = 1$, it is the usual Hecke algebra of type A and if moreover $v = 1$, we obtain the group algebra $R\mathfrak{S}_n$ of the symmetric group. We denote by:

- Π^ℓ the set of all ℓ -partitions, that is, the set of all ℓ -tuples $(\lambda^1, \dots, \lambda^\ell)$ of partitions.
- $\Pi = \Pi^1$ the set of all partitions.

The unique ℓ -partition of size 0 is denoted by $\emptyset$. For any subset $\mathcal{E}$ of Π^ℓ and any $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{E}(n)$ the set of ℓ -partitions in $\mathcal{E}$ of total size $|\lambda^1| + \dots + |\lambda^\ell| = n$.

Let e be the multiplicative order of v in R . We assume that $v \neq 1$ so that we have $e \in \{2, 3, \dots\} \sqcup \{\infty\}$. We now recall several facts about the representation theory of cyclotomic Hecke algebras. We refer to [Geck and Jacon 2011, Chapter 5] for details. For each $\lambda \in \Pi^\ell(n)$, there is an $\mathcal{H}_{R,n}^s$ -module S^λ which is the Specht module associated to λ . There exists a natural bilinear form, $\mathcal{H}_{R,n}^s$ -invariant, on each of these modules and an associated radical such that the quotients $D^\lambda := S^\lambda / \text{rad}(S^\lambda)$ are either 0 or irreducible. The nonzero D^λ then give a complete set of nonisomorphic simple $\mathcal{H}_{R,n}^s$ -modules.

The set $\{\lambda \in \Pi^\ell(n) \mid D^\lambda \neq 0\}$ depends only on e and $\mathbf{s}$ and is known as the set of Kleshchev ℓ -partitions, denoted $\text{Kl}_{e,\mathbf{s}}(n)$. It was originally defined using the notion of crystal (see [Geck and Jacon 2011, Section 6.2.10]), but there is another independent description; see [Jacon 2018].

Remark 1.1. Let $e \geq 2$, $\mathbf{s} = (s_1, \dots, s_\ell)$ and $\mathbf{t} = (t_1, \dots, t_\ell)$ such that $t_i = s_i \bmod e$ for all $i = 1, \dots, \ell$. Observe that for any $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{H}_{R,n}^{\mathbf{s}} = \mathcal{H}_{R,n}^{\mathbf{t}}$, and the definition of Kleshchev ℓ -partitions gives that

$$\text{Kl}_{e,\mathbf{s}}(n) = \text{Kl}_{e,\mathbf{t}}(n).$$

Now if there exists $\sigma \in \mathfrak{S}_\ell$ such that $t_i = s_{\sigma(i)} \bmod e$ for all $i = 1, \dots, \ell$ then we still have that for any $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{H}_{R,n}^{\mathbf{s}} = \mathcal{H}_{R,n}^{\mathbf{t}}$ but $\text{Kl}_{e,\mathbf{s}}(n)$ is different from $\text{Kl}_{e,\mathbf{t}}(n)$ in general.

Set $\tilde{\mathcal{H}}_{R,n}^{\mathbf{s}} := \mathcal{H}_{R,n}(v^{-1}; s_\ell, \dots, s_1)$ and denote by $\tilde{T}_0, \dots, \tilde{T}_{\ell-1}$ the associated standard generators. For each $\lambda \in \Pi^\ell(n)$, denote by $\tilde{\mathcal{S}}^\lambda$ the associated Specht module of $\tilde{\mathcal{H}}_{R,n}^{\mathbf{s}}$. By [Fayers 2008], the simple modules of $\tilde{\mathcal{H}}_{R,n}^{\mathbf{s}}$ are labeled by the set $\text{Kl}_{e,-\mathbf{s}_{\text{rev}}}(n)$ where $-\mathbf{s}_{\text{rev}} = (-s_\ell, \dots, -s_1) \in (\mathbb{Z}/e\mathbb{Z})^\ell$. Thus, for each $\lambda \in \text{Kl}_{e,-\mathbf{s}_{\text{rev}}}(n)$, we have an associated simple $\tilde{\mathcal{H}}_{R,n}^{\mathbf{s}}$ -module $\tilde{D}^\lambda$. We have an involutive isomorphism $\theta : \mathcal{H}_{R,n}^{\mathbf{s}} \rightarrow \tilde{\mathcal{H}}_{R,n}^{\mathbf{s}}$ given by

$$T_0 \mapsto \tilde{T}_0, \quad T_i \mapsto -v\tilde{T}_i \quad (i = 1, \dots, n-1).$$

Then, θ induces a functor F from the category of $\tilde{\mathcal{H}}_{R,n}^{\mathbf{s}}$ -modules to the category of $\mathcal{H}_{R,n}^{\mathbf{s}}$ -modules. As a consequence, we obtain a bijective map

$$\mathfrak{m}_{e,\mathbf{s}} : \text{Kl}_{e,\mathbf{s}}(n) \rightarrow \text{Kl}_{e,-\mathbf{s}_{\text{rev}}}(n),$$

satisfying

$$F(\tilde{D}^{\mathfrak{m}_{e,\mathbf{s}}(\lambda)}) \simeq D^\lambda,$$

for all $\lambda \in \text{Kl}_{e,\mathbf{s}}$.

Remark 1.2. (1) By definition of θ , $\mathfrak{m}_{e,-\mathbf{s}_{\text{rev}}} \circ \mathfrak{m}_{e,\mathbf{s}} = \text{Id}_{\text{Kl}_{e,\mathbf{s}}}$ and $\mathfrak{m}_{e,\mathbf{s}} \circ \mathfrak{m}_{e,-\mathbf{s}_{\text{rev}}} = \text{Id}_{\text{Kl}_{e,-\mathbf{s}_{\text{rev}}}}$.

(2) Assume that $\ell = 1$. Then the map $\mathfrak{m}_{e,\mathbf{s}}$ is an involution and it does not depend on the choice of $\mathbf{s}$. In fact, we have $\mathfrak{m}_{e,\mathbf{s}} = m_e$, where m_e is the usual Mullineux involution defined in the introduction.

1B. The quantum algebra $\mathcal{U}_t(\widehat{\mathfrak{sl}}_e)$. We denote by $\Lambda_0, \dots, \Lambda_{e-1}$ (where the subscripts are understood modulo e) the fundamental weights attached to the Kac-Moody algebra $\widehat{\mathfrak{sl}}_e$. The simple roots are denoted by $\alpha_0, \dots, \alpha_{e-1}$ and $\delta := \alpha_0 + \dots + \alpha_{e-1}$ is the null root. The fundamental weights and the simple roots are related by the formula

$$\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} + \delta_{i,0}\delta \quad \text{for all } 0 \leq i \leq e-1$$

(where δ_{ij} denotes the Kronecker symbol). We denote by $\mathcal{P} = \bigoplus_{0 \leq i \leq e-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ the weight lattice and by $\mathcal{U}_t(\widehat{\mathfrak{sl}}_e)$ the quantum algebra associated to $\widehat{\mathfrak{sl}}_e$, where t is an indeterminate. This is an algebra over $\mathbb{C}(t)$ with generators $e_i, f_i, t_i^{\pm 1}$ ($0 \leq i \leq e-1$) and $\partial^{\pm 1}$ subject to standard relations which we do not recall. We refer to [Geck and Jacon 2011, Chapter 6] for details on this algebra and its representation theory.

1C. The level ℓ Fock space. Let us fix some notation. Fix $e, \ell \geq 2$ and $s \in \mathbb{Z}$. For $\mathbb{K} = \mathbb{Z}$ or $\mathbb{Q}$, we denote

$$\mathbb{K}^\ell(s) = \left\{ (s_1, \dots, s_\ell) \in \mathbb{K}^\ell \mid \sum_{i=1}^{\ell} s_i = s \right\}.$$

For $\mathbf{s} \in \mathbb{Z}^\ell(s)$, we denote by $\Pi_{\mathbf{s}}^\ell$ the set of all symbols of the form $|\lambda, \mathbf{s}\rangle$ with $\lambda \in \Pi^\ell$. Further, denote by Π_s^ℓ the sets of all elements in $\Pi_{\mathbf{s}}^\ell$ where $\mathbf{s} \in \mathbb{Z}^\ell(s)$. Let $\mathcal{F}_{e,s}$ be the $\mathbb{C}(t)$ -vector space with standard basis $\Pi_{\mathbf{s}}^\ell$, i.e., $\mathcal{F}_{e,s} = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{C}(t)|\lambda, \mathbf{s}\rangle$, called the Fock space of level ℓ and rank e (associated to the charge $\mathbf{s}$). This space can be endowed with a structure of an integrable $\mathcal{U}_t(\widehat{\mathfrak{sl}}_e)$ -module; see [Geck and Jacon 2011, Section 6.2].

One can decompose this module as a direct sum of remarkable vector spaces. Indeed, if $w := \sum_{0 \leq i \leq e-1} a_i \Lambda_i + d\delta \in \mathcal{P}$, define

$$\mathcal{F}_{e,s}[w] := \{m \in \mathcal{F}_{e,s} \mid \partial m = t^d m, t_i m = t^{a_i} m \text{ for all } i \in [0, e-1]\}.$$

If this space is nonzero, we say that w is a weight for $\mathcal{F}_{e,s}$ and $\mathcal{F}_{e,s}[w]$ is called the w -weight space. The elements of $\mathcal{F}_{e,s}[w]$ are called weight vectors. Importantly, each element of the standard basis $|\lambda, \mathbf{s}\rangle$ is a weight vector and the associated weight may be easily computed (see, for example, [Yvonne 2007, Corollary 2.5]). In particular, one can always write it as

$$d\delta + \sum_{0 \leq i \leq e-1} \Lambda_{s_i} - \sum_{0 \leq i \leq e-1} m_i \alpha_i$$

and the number $\sum_{0 \leq i \leq e-1} m_i$ corresponds to the size of λ . In particular, the weight of $|\emptyset, \mathbf{s}\rangle$ is $\sum_{0 \leq i \leq e-1} \Lambda_{s_i}$. Thus $\mathcal{F}_{e,s}$ is the direct sum of its weight spaces.

1D. The $\widehat{\mathfrak{sl}}_e$ -crystal of the Fock space. As mentioned in the introduction, an important part of the representation theory of cyclotomic Hecke algebras is controlled by the theory of crystals for Fock spaces. The $\widehat{\mathfrak{sl}}_e$ -crystal of the Fock space $\mathcal{F}_{e,s}$ is a combinatorial construction arising from the action of $\mathcal{U}_t(\widehat{\mathfrak{sl}}_e)$ on the Fock space (see the general definition in [Kashiwara 1991; Hong and Kang 2002]). Concretely, the $\widehat{\mathfrak{sl}}_e$ -crystal is a graph with

- vertices: the elements of $\Pi_{\mathbf{s}}^\ell$,

- arrows: $|\lambda, \mathbf{s}\rangle \xrightarrow{i} |\mu, \mathbf{s}\rangle$ for $\lambda, \mu \in \Pi^\ell$, $i \in \{0, \dots, e-1\}$ if and only if $|\mu, \mathbf{s}\rangle = \tilde{f}_i |\lambda, \mathbf{s}\rangle$, where $\tilde{f}_i$ is the i -th lowering Kashiwara operator of $\mathcal{U}_t(\widehat{\mathfrak{sl}}_e)$.

An explicit recursive formula for computing the $\widehat{\mathfrak{sl}}_e$ -crystal is given in [Jimbo et al. 1991] in terms of adding good boxes; see also [Foda et al. 1999]. It has infinitely many connected components, each of which is parametrized by its unique source vertex, called a highest weight vertex. We denote by $\text{Ug}_{e,\mathbf{s}}$ the ℓ -partitions appearing in the connected component parametrized by the highest weight vertex $\emptyset = (\emptyset, \dots, \emptyset)$, and call them the Uglov ℓ -partitions. When $\ell = 1$, this set is nothing but the set of e -regular partitions. The following is an easy consequence of the definition of Kleshchev and Uglov ℓ -partitions (see [Geck and Jacon 2011, Example 6.2.16]).

Proposition 1.3. *Fix $n \in \mathbb{Z}_{\geq 0}$. Let $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{t} = (t_1, \dots, t_\ell) \in \mathbb{Z}^\ell$ be such that $s_i = t_i \bmod e$ for all $i = 1, \dots, \ell$, and $t_i - t_{i-1} > n - 1$ for all $i = 2, \dots, \ell$. Then*

$$\text{Ug}_{e,\mathbf{t}}(n) = \text{Kl}_{e,\mathbf{t}}(n) = \text{Kl}_{e,\mathbf{s}}(n).$$

In other words, Kleshchev ℓ -partitions are a particular case of Uglov ℓ -partitions, i.e., we can index irreducible modules of cyclotomic Hecke algebras by certain vertices of Fock space crystals. The following result is due to Fayers [2008, Section 2] in the case of Kleshchev multipartitions (that is, under the condition of Proposition 1.3) and to [Jacon and Lecouvey 2009, Section 4] in general.

Theorem 1.4. *Let $n \in \mathbb{Z}_{\geq 0}$, $\mathbf{s} \in \mathbb{Z}^\ell(s)$ and $e \geq 2$. There exists a unique bijection*

$$\Phi_{e,\mathbf{s}} : \text{Ug}_{e,\mathbf{s}}(n) \rightarrow \text{Ug}_{e,-\mathbf{s}_{\text{rev}}}(n), \quad \lambda \mapsto \Phi_{e,\mathbf{s}}(\lambda)$$

such that

- $\Phi_{e,\mathbf{s}}(\emptyset) = \emptyset$,
- for all $0 \leq i \leq e-1$, we have $\Phi_{e,\mathbf{s}} \circ \tilde{f}_i = \tilde{f}_{-i} \circ \Phi_{e,\mathbf{s}}$.

This means that for all paths

$$|\emptyset, \mathbf{s}\rangle \xrightarrow{i_1} \cdot \xrightarrow{i_2} \cdot \xrightarrow{i_3} \dots \xrightarrow{i_n} |\lambda, \mathbf{s}\rangle$$

in the $\widehat{\mathfrak{sl}}_e$ -crystal on the Fock space $\mathcal{F}_{e,\mathbf{s}}$, there exists a corresponding path

$$|\emptyset, -\mathbf{s}_{\text{rev}}\rangle \xrightarrow{-i_1} \cdot \xrightarrow{-i_2} \cdot \xrightarrow{-i_3} \dots \xrightarrow{-i_n} |\mu, -\mathbf{s}_{\text{rev}}\rangle$$

in the $\widehat{\mathfrak{sl}}_e$ -crystal on the Fock space $\mathcal{F}_{e,-\mathbf{s}_{\text{rev}}}$ from the empty ℓ -partition to an ℓ -partition $\mu \in \text{Ug}_{e,-\mathbf{s}_{\text{rev}}}$. Then $\Phi_{e,\mathbf{s}}(\lambda) = \mu$. In [Jacon and Lecouvey 2009], it is explained how the map $\Phi_{e,\mathbf{s}}$ can be explicitly computed without constructing the $\widehat{\mathfrak{sl}}_e$ -crystal.

Example 1.5. Take $s = 4$, $e = 4$, $\ell = 3$, $\mathbf{s} = (5, -1, 0)$ (so that $-\mathbf{s}_{\text{rev}} = (0, 1, -5)$) and $\lambda = (1, 3.2, \emptyset)$.¹ One can write for instance $\lambda = \tilde{f}_1 \tilde{f}_1 \tilde{f}_3 \tilde{f}_0 \tilde{f}_2 \tilde{f}_3 \emptyset$, so that $\lambda \in \text{Ug}_{e,\mathbf{s}}(6)$. Therefore, in the crystal of the Fock space $\mathcal{F}_{e,-\mathbf{s}_{\text{rev}}}$, we get

$$\begin{aligned} \Phi_{e,\mathbf{s}} &= \tilde{f}_{-1} \tilde{f}_{-1} \tilde{f}_{-3} \tilde{f}_0 \tilde{f}_{-2} \tilde{f}_{-3} \emptyset \\ &= \tilde{f}_3 \tilde{f}_3 \tilde{f}_1 \tilde{f}_0 \tilde{f}_2 \tilde{f}_1 \emptyset \\ &= (2.1, 3, \emptyset). \end{aligned}$$

The following result by Fayers [2008] gives the desired crystal interpretation of the Mullineux involution for cyclotomic Hecke algebras.

Theorem 1.6 (Fayers). *Fix $n \in \mathbb{Z}_{\geq 0}$, $\mathbf{s} \in \mathbb{Z}^\ell$ and $e \geq 2$. For all $\lambda \in \text{Kl}_{e,\mathbf{s}}(n)$, we have*

$$\mathfrak{m}_{e,\mathbf{s}}(\lambda) = \Phi_{e,\mathbf{s}}(\lambda).$$

To summarize, starting with the usual Mullineux involution m_e for the symmetric group, we obtain

- a generalization of m_e : the involution $\mathfrak{m}_{e,\mathbf{s}}$ on the set of Kleshchev ℓ -partitions which label the irreducible representations of cyclotomic Hecke algebras. If $\ell = 1$, we have $\mathfrak{m}_{e,\mathbf{s}} = m_e$.
- a generalization of $\mathfrak{m}_{e,\mathbf{s}}$: the involution $\Phi_{e,\mathbf{s}}$ on the set of Uglov ℓ -partitions. If $\mathbf{s}$ is such that $s_i - s_{i-1} > n - 1$ for all $i = 2, \dots, \ell$, we have $\Phi_{e,\mathbf{s}} = \mathfrak{m}_{e,\mathbf{s}}$.

2. The generalized Mullineux involution

Recall from Section 1C that we have fixed $e, \ell \geq 2$ and $s \in \mathbb{Z}$. Let us denote $\mathcal{F}_{e,s} = \bigoplus_{\mathbf{s} \in \mathbb{Z}^\ell(s)} \mathcal{F}_{e,\mathbf{s}}$ and $\mathcal{F}_e = \bigoplus_{s \in \mathbb{Z}} \mathcal{F}_{e,s}$.

2A. Triple crystal structure. By Section 1C, the space $\mathcal{F}_{e,s}$ has a structure of integrable $\mathcal{U}_t(\widehat{\mathfrak{sl}_\ell})$ -module of level ℓ . This space can also be endowed with a structure of $\mathcal{U}_{-1/t}(\widehat{\mathfrak{sl}_\ell})$ -module of level e . Denote by $\dot{\Lambda}_0, \dots, \dot{\Lambda}_{\ell-1}$ the fundamental weights attached to the Kac–Moody algebra $\widehat{\mathfrak{sl}_\ell}$. The simple roots are denoted by $\dot{\alpha}_0, \dots, \dot{\alpha}_{\ell-1}$ and $\dot{\delta}$ is the null root. We denote by $\dot{\mathcal{P}} = \bigoplus_{0 \leq i \leq \ell-1} \mathbb{Z} \dot{\Lambda}_i \oplus \mathbb{Z} \dot{\delta}$ the corresponding weight lattice.

Following [Gerber 2018], there is a *level-rank duality* between ℓ -partitions and e -partitions. This is a map

$$k_s^{\ell,e} : \Pi_s^\ell \rightarrow \Pi_{-s}^e$$

inducing a linear map between the Fock spaces $\mathcal{F}_{e,s} \rightarrow \mathcal{F}_{\ell,-s}$. To avoid cumbersome

¹In the examples, we use the multiplicative notation for partitions and we forget the brackets around components of a multipartition.

notation, write k for $k_s^{\ell,e}$ and $\dot{k}$ for $k_{-s}^{e,\ell}$. From [Gerber 2018, Formula (3.8)], it is straightforward that $\dot{k} \circ k = \text{Id}_{\Pi_s^\ell}$ and $k \circ \dot{k} = \text{Id}_{\Pi_{-s}^e}$.

We can extend k linearly to $\mathcal{F}_{e,s}$, which endows it with the structure of a $\mathcal{U}_{-1/t}(\widehat{\mathfrak{sl}_\ell})$ -module, by considering the natural action on $\mathcal{F}_{\ell,-s}$ and composing with $\dot{k}$. This yields an $\widehat{\mathfrak{sl}_\ell}$ -crystal structure on $\mathcal{F}_{e,s}$. More precisely, if we denote by $\tilde{f}_j$, $j = 0, \dots, \ell - 1$, the lowering $\widehat{\mathfrak{sl}_\ell}$ -crystal operators, the action of $\tilde{f}_j$ on an ℓ -partition is defined by $\dot{k} \circ \tilde{f}_j \circ k$, as indicated by the following diagram:

$$(2.1) \quad \begin{array}{ccc} \Pi_s^\ell & \xrightarrow{k} & \Pi_{-s}^e \\ \downarrow & & \downarrow \tilde{f}_j \\ \Pi_s^\ell & \xleftarrow{\dot{k}} & \Pi_{-s}^e \end{array}$$

Remark 2.2. (1) As explained in [Gerber 2018, Section 7.1], the map k is, up to a twist by conjugation, categorified by *Koszul duality* between the corresponding Cherednik categories $\mathcal{O}$. This justifies the notation.

(2) The level-rank duality k used in our paper is not the same as the one used in Yvonne and Uglov's paper. However, our map can be recovered from Uglov and Yvonne's ones by composing with the map $|\lambda, \mathbf{s}\rangle \mapsto |\lambda_{\text{rev}}^{\text{tr}}, -\mathbf{s}_{\text{rev}}\rangle$, where for $\lambda := (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ we have

$$\lambda_{\text{rev}}^{\text{tr}} = ((\lambda^{(\ell)})^{\text{tr}}, \dots, (\lambda^{(1)})^{\text{tr}})$$

and λ^{tr} is the transpose of λ .

For $s \in \mathbb{Z}$, we denote

$$A(s) = \{(s_1, \dots, s_\ell) \in \mathbb{Z}^\ell(s) \mid s_1 \leq \dots \leq s_\ell \leq s_1 + e\}$$

and, in a dual fashion,

$$\dot{A}(s) = \{(t_1, \dots, t_\ell) \in \mathbb{Z}^\ell(s) \mid t_1 \leq \dots \leq t_\ell \leq t_1 + \ell\}.$$

Write $\dot{\emptyset} = (\emptyset, \dots, \emptyset) \in \Pi^e$. Note that for $\mathbf{s} \in A(s)$, the set $\text{Ug}_{e,s}$ has a convenient nonrecursive definition; see [Foda et al. 1999, Theorem 2.10]. By [Gerber 2018, Formula (3.8)], if $\mathbf{s} \in A(s)$, then $k|\dot{\emptyset}, \mathbf{s}\rangle = |\dot{\emptyset}, \dot{\mathbf{s}}\rangle$ for some $\dot{\mathbf{s}} \in \dot{A}(-s)$.

Finally, there is an $\mathfrak{sl}_\infty$ -crystal structure on Π_s^ℓ arising from the action of a Heisenberg algebra [Shan and Vasserot 2012; Losev 2015; Gerber 2019]. Its connected components are all isomorphic to the branching graph of the symmetric group in characteristic 0 and thus have vertices in bijection with Π . If λ_0 is a highest weight vertex for the $\mathfrak{sl}_\infty$ -crystal, then any ℓ -partition in the same crystal component as λ_0 is obtained as $\tilde{a}_\sigma(\lambda_0)$ for a unique $\sigma \in \Pi$, where $\tilde{a}_\sigma$ denotes the Heisenberg crystal operator associated to σ ; see [Losev 2015; Gerber 2019].

We will make repeated use of the following important theorem, proved in [Gerber 2018, Theorems 6.17 and 6.19], and its corollary.

Theorem 2.3. (1) *The three crystals pairwise commute.*

(2) *Every $|\lambda, \mathbf{s}\rangle \in \Pi_s^\ell$ decomposes as*

$$|\lambda, \mathbf{s}\rangle = \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle$$

for some $\mathbf{r} \in A(s)$, $\sigma \in \Pi$, $p, r \in \mathbb{Z}_{\geq 0}$ and some $i_p, \dots, i_1 \in \{0, 1, \dots, e-1\}$ and $j_r, \dots, j_1 \in \{0, 1, \dots, \ell-1\}$.

Corollary 2.4. *The elements $\mathbf{r}$, σ , p and r of Theorem 2.3 are uniquely determined by $|\lambda, \mathbf{s}\rangle$. This yields a bijection*

$$\beta : \Pi_s^\ell \rightarrow \bigsqcup_{\mathbf{r} \in A(s)} \text{Ug}_{e, \mathbf{r}} \times \Pi \times \text{Ug}_{\ell, \dot{\mathbf{r}}},$$

$$|\lambda, \mathbf{s}\rangle \mapsto (\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle, \sigma, \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} |\emptyset, \dot{\mathbf{r}}\rangle).$$

Proof. Let $\lambda \in \Pi^\ell$. By Theorem 2.3(2), there exist $\mathbf{r} \in A(s)$, $\sigma \in \Pi$, $p, r \in \mathbb{Z}_{\geq 0}$ and elements $i_p, \dots, i_1 \in \{0, 1, \dots, e-1\}$ and $j_r, \dots, j_1 \in \{0, 1, \dots, \ell-1\}$ such that

$$|\lambda, \mathbf{s}\rangle = \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle.$$

Assume that we have

$$\tilde{f}_{j'_r} \cdots \tilde{f}_{j'_1} \tilde{a}_{\sigma'} \tilde{f}_{i'_{p'}} \cdots \tilde{f}_{i'_1} |\emptyset, \mathbf{r}'\rangle = \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle$$

for $\mathbf{r}' \in A(s)$, $\sigma' \in \Pi$, $p', r' \in \mathbb{Z}_{\geq 0}$ and indices $i'_{p'}, \dots, i'_1 \in \{0, 1, \dots, e-1\}$ and $j'_r, \dots, j'_1 \in \{0, 1, \dots, \ell-1\}$. Then the elements

$$|\mu', \mathbf{t}'\rangle := \tilde{f}_{j'_r} \cdots \tilde{f}_{j'_1} \tilde{a}_{\sigma'} |\emptyset, \mathbf{r}'\rangle \quad \text{and} \quad |\mu, \mathbf{t}\rangle := \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} \tilde{a}_\sigma |\emptyset, \mathbf{r}\rangle$$

are both highest weight vertices in the $\widehat{\mathfrak{sl}}_e$ -crystal. As we have $\tilde{f}_{i'_{p'}} \cdots \tilde{f}_{i'_1} |\mu', \mathbf{t}'\rangle = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\mu, \mathbf{t}\rangle$, these two elements are in the same connected component of the $\widehat{\mathfrak{sl}}_e$ -crystal so they must be equal. From this equality, we deduce in the same way that the two $\widehat{\mathfrak{sl}}_e$ -highest weight vertices must be equal: $\tilde{a}_{\sigma'} |\emptyset, \mathbf{r}'\rangle = \tilde{a}_\sigma |\emptyset, \mathbf{r}\rangle$. By the description of the $\mathfrak{sl}_\infty$ -crystal operators [Losev 2015; Gerber 2018], we obtain $\sigma = \sigma'$ and $\mathbf{r}' = \mathbf{r}$. We deduce that $\tilde{f}_{j'_r} \cdots \tilde{f}_{j'_1} |\emptyset, \dot{\mathbf{r}}\rangle = \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} |\emptyset, \dot{\mathbf{r}}\rangle$, where $|\emptyset, \dot{\mathbf{r}}\rangle = \mathbf{k}|\emptyset, \mathbf{r}\rangle$. In particular, we have $r' = r$. Using the same argument but exchanging the roles of e and ℓ , we also get $\tilde{f}_{i'_{p'}} \cdots \tilde{f}_{i'_1} |\emptyset, \mathbf{r}\rangle = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle$ and $p' = p$. This proves uniqueness, and therefore β is well defined.

For $\mathbf{r} \in A(s)$ and $(\mathbf{v}, \pi) \in \text{Ug}_{e, \mathbf{r}} \times \text{Ug}_{\ell, \dot{\mathbf{r}}}$, by [Foda et al. 1999, Theorem 2.10], there exist indices $i_1, \dots, i_p \in \{0, 1, \dots, e-1\}$ and $j_1, \dots, j_r \in \{0, 1, \dots, \ell-1\}$ such that

$$\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle = |\mathbf{v}, \mathbf{r}\rangle \quad \text{and} \quad \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} |\emptyset, \dot{\mathbf{r}}\rangle = |\pi, \dot{\mathbf{r}}\rangle.$$

The map $\delta: \bigsqcup_{\mathbf{r} \in A(s)} \text{Ug}_{e,\mathbf{r}} \times \Pi \times \text{Ug}_{\ell,\mathbf{r}} \rightarrow \Pi_s^\ell, (\mathbf{v}, \sigma, \boldsymbol{\pi}) \mapsto \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle$ is well defined, since it does not depend on the choice of the indices and the three crystals commute. It is straightforward that β and δ are inverse to each other, which concludes the proof. $\square$

Example 2.5. Take $s = 1$, $e = 3$, $\ell = 4$, $\mathbf{s} = (-3, 2, 1, 1)$ and $\boldsymbol{\lambda} = (\emptyset, 3.2^2, \emptyset, 3)$. Then

$$\begin{aligned} \beta(|\boldsymbol{\lambda}, \mathbf{s}\rangle) &= (|(\emptyset, \emptyset, \emptyset, 3), (-1, 0, 0, 2)\rangle, (2), |(2^2, 2.1, \emptyset), (-1, -1, 1)\rangle) \\ &= (\tilde{f}_1 \tilde{f}_0 \tilde{f}_2 |(\emptyset, \emptyset, \emptyset, \emptyset), (-1, 0, 0, 2)\rangle, (2), \\ &\quad \tilde{f}_0 \tilde{f}_2 \tilde{f}_3 \tilde{f}_3 \tilde{f}_0 \tilde{f}_2 \tilde{f}_3 |(\emptyset, \emptyset, \emptyset), (-1, -1, 1)\rangle). \end{aligned}$$

Remark 2.6. Note that for $\ell = 1$, Corollary 2.4 reduces to a very simple bijection. Indeed, there is no $\widehat{\mathfrak{sl}}_\ell$ -crystal (and no level-rank duality) in this case, and the bijection associates to any partition λ a pair of partitions (ρ, σ) determined by the “euclidean division” of λ by e , as follows. Given two partitions μ and μ' , let $\mu \sqcup \mu'$ be the partition obtained by concatenating the two partitions and then reordering the parts to obtain a partition (see for instance [Dudas and Jacon 2018, Section 3.1]). Then we can uniquely write

$$\lambda = (\sigma)^e \sqcup \rho,$$

where ρ is an e -regular partition and $\sigma \in \Pi$.

Example 2.7. Choose $e = 3$ and $\lambda = (4^4.3^2.2.1^8)$. Then $\lambda = (4.1^2)^3 \sqcup (4.3^2.2.1^2)$.

2B. The generalized Mullineux map. In the following, we will need to go from one indexation by ℓ -partitions to the other by e -partitions using the map k . We will use the relationship between the weight spaces for the action of $\widehat{\mathfrak{sl}}_e$ and the weight spaces for the action of $\widehat{\mathfrak{sl}}_\ell$ of $\mathcal{F}_{s,e}$.

We start by defining a map $\theta_{\ell,e,s}$ by setting

$$\theta_{\ell,e,s} : \mathbb{Q}^\ell(s) \rightarrow \mathbb{Q}^\ell(e), \quad (s_1, \dots, s_\ell) \mapsto (e - s_1 + s_\ell, s_1 - s_2, \dots, s_{\ell-1} - s_\ell).$$

This is a bijection with inverse map

$$\theta_{\ell,e,s}^{-1} : \mathbb{Q}^\ell(e) \rightarrow \mathbb{Q}^\ell(s), \quad (a_1, \dots, a_\ell) \mapsto (s_1, \dots, s_\ell),$$

where we have for all $1 \leq i \leq \ell$,

$$s_i = \frac{1}{\ell} \left(s - \sum_{1 \leq j \leq \ell-1} j a_{j+1} \right) + \sum_{i+1 \leq j \leq \ell} a_j.$$

Lemma 2.8. Keeping the above notation, assume that $\mathbf{s} = \theta_{\ell,e,s}^{-1}(a_1, \dots, a_\ell)$; then we have $-\mathbf{s}_{\text{rev}} = \theta_{\ell,e,-s}^{-1}(a_1, a_\ell, \dots, a_2)$.

Proof. Write $\mathbf{s} = (s_1, \dots, s_\ell)$ and $\mathbf{v} = \theta_{\ell, e, -s}^{-1}(a_1, a_\ell, \dots, a_2)$. On the one hand, we have for all $i = 1, \dots, \ell$

$$s_i = \frac{1}{\ell} \left(s - \sum_{1 \leq j \leq \ell-1} j a_{j+1} \right) + \sum_{i+1 \leq j \leq \ell} a_j,$$

and on the other hand

$$\begin{aligned} v_{\ell-i+1} &= \frac{1}{\ell} \left(-s - \sum_{1 \leq j \leq \ell-1} j a_{\ell-j+1} \right) + \sum_{\ell-i+2 \leq j \leq \ell} a_{\ell-j+2} \\ &= \frac{1}{\ell} \left(-s - \sum_{1 \leq j \leq \ell-1} (\ell-k) a_{k+1} \right) + \sum_{2 \leq k \leq i} a_k. \end{aligned}$$

We obtain

$$s_j + v_{\ell-j+1} = 0$$

and the result follows. $\square$

We have the following result whose proof can be found in [Yvonne 2007, Proposition 2.12], taking into account Remark 2.2(2).

Proposition 2.9. *Let $\dot{\mathbf{s}} = (\dot{s}_1, \dots, \dot{s}_e) \in \mathbb{Z}^e(s)$ and let $\dot{w} \in \dot{\mathcal{P}}$ be a weight for $\mathcal{F}_{\ell, \dot{\mathbf{s}}}$. Then there exist a unique $\mathbf{s} \in \mathbb{Z}^\ell(s)$ and a unique $w \in \mathcal{P}$ such that $k(\mathcal{F}_{e, \mathbf{s}}[w]) = \mathcal{F}_{\ell, \dot{\mathbf{s}}}[\dot{w}]$. If we write $\dot{w} = d\dot{\delta} + \sum_{0 \leq i \leq \ell-1} a_i \dot{\Lambda}_{i-1}$ with $(a_1, \dots, a_\ell) \in \mathbb{Z}^\ell$ we have*

$$\mathbf{s} = \theta_{\ell, e, s}^{-1}(a_1, a_\ell, \dots, a_2).$$

Moreover the associated weight is

$$w = d\delta + \sum_{0 \leq i \leq e-1} (\dot{s}_i - \dot{s}_{i+1}) \Lambda_i,$$

where $\dot{s}_0 = \ell + \dot{s}_e$.

We are now ready to prove the first main result of this paper. Recall the generalized Mullineux map $\Phi_{e, \mathbf{s}}$ on Uglov ℓ -partitions of Theorem 1.4. By level-rank duality, we have a dual Mullineux map $\Phi_{\ell, \mathbf{t}}$ for all $\mathbf{t} \in \mathbb{Z}^e(-s)$ which acts on e -partitions.

Theorem 2.10.

(1) *There exists a unique bijection*

$$\Phi : \Pi_s^\ell \rightarrow \Pi_{-s}^\ell, \quad |\lambda, \mathbf{s}\rangle \mapsto |\mu, -\mathbf{s}_{\text{rev}}\rangle$$

such that for all $0 \leq i \leq e-1$, $\sigma \in \Pi$ and $0 \leq j \leq \ell-1$,

- (a) $\Phi \circ \tilde{f}_i = \tilde{f}_{-i} \circ \Phi$,
- (b) $\Phi \circ \tilde{a}_\sigma = \tilde{a}_{\sigma^{\text{tr}}} \circ \Phi$,

- (c) $\Phi \circ \tilde{f}_j = \tilde{f}_{-j} \circ \Phi$,
 (d) $\Phi(|\emptyset, \mathbf{s}\rangle) = |\emptyset, -\mathbf{s}_{\text{rev}}\rangle$.

(2) Using the notation of Corollary 2.4, we have

$$\Phi = \beta^{-1} \circ (\Phi_{e, \mathbf{r}}, (\cdot)^{\text{tr}}, \Phi_{\ell, \dot{\mathbf{r}}}) \circ \beta.$$

In other words, writing $|\lambda, \mathbf{s}\rangle = \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle$ with $\mathbf{r} \in A(s)$, we have $\Phi|\lambda, \mathbf{s}\rangle = \tilde{f}_{-j_r} \cdots \tilde{f}_{-j_1} \tilde{a}_{\sigma^{\text{tr}}} \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} |\emptyset, -\mathbf{r}_{\text{rev}}\rangle$.

(3) We have $|\lambda| = |\mu|$ if $|\mu, -\mathbf{s}_{\text{rev}}\rangle = \Phi(|\lambda, \mathbf{s}\rangle)$.

Proof. Let $\lambda \in \Pi^\ell$, $\mathbf{s} \in \mathbb{Z}^\ell(s)$ and write $|\lambda, \mathbf{s}\rangle = \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} \tilde{a}_\sigma \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{r}\rangle$ with $\mathbf{r} \in A(s)$ as in Theorem 2.3. If Φ satisfies the four assumptions of the theorem, we have that

$$\Phi(|\lambda, \mathbf{s}\rangle) = \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \tilde{a}_{\sigma^{\text{tr}}} \tilde{f}_{-j_r} \cdots \tilde{f}_{-j_1} |\emptyset, -\mathbf{r}_{\text{rev}}\rangle$$

and this shows uniqueness. Now, to prove (1) and (2), we need to show that there exists an ℓ -partition μ such that

$$|\mu, -\mathbf{s}_{\text{rev}}\rangle = \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \tilde{a}_{\sigma^{\text{tr}}} \tilde{f}_{-j_r} \cdots \tilde{f}_{-j_1} |\emptyset, -\mathbf{r}_{\text{rev}}\rangle.$$

First, note that $k|\emptyset, -\mathbf{r}_{\text{rev}}\rangle = |\dot{\emptyset}, -\dot{\mathbf{r}}_{\text{rev}}\rangle$ by [Gerber 2018, Formula (3.8)]. Consider the e -partition λ_2 such that

$$|\lambda_2, \dot{\mathbf{r}}\rangle = \tilde{f}_{j_r} \cdots \tilde{f}_{j_1} |\dot{\emptyset}, \dot{\mathbf{r}}\rangle$$

and the e -partition μ_2 such that

$$|\mu_2, -\dot{\mathbf{r}}_{\text{rev}}\rangle = \tilde{f}_{-j_r} \cdots \tilde{f}_{-j_1} |\dot{\emptyset}, -\dot{\mathbf{r}}_{\text{rev}}\rangle$$

defined thanks to Theorem 1.4, i.e.,

$$|\mu_2, -\dot{\mathbf{r}}_{\text{rev}}\rangle = \Phi_{\ell, \dot{\mathbf{r}}}(|\lambda_2, \dot{\mathbf{r}}\rangle).$$

Let $|\lambda_1, \mathbf{s}\rangle = k(|\lambda_2, \dot{\mathbf{r}}\rangle)$, so that $|\lambda, \mathbf{s}\rangle = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma |\lambda_1, \dot{\mathbf{s}}\rangle$. Let $\mathbf{v} \in \mathbb{Z}^\ell(-s)$ and μ_1 be the ℓ -partition such that $|\mu_1, \mathbf{v}\rangle = k(|\mu_2, -\dot{\mathbf{r}}_{\text{rev}}\rangle)$. Let us show that $\mathbf{v} = -\mathbf{s}_{\text{rev}}$. By hypothesis, the $\widehat{\mathfrak{sl}_\ell}$ -weight $\dot{w}$ of $|\lambda_2, \dot{\mathbf{r}}\rangle$ can be written

$$\dot{w} = d\dot{\delta} + \sum_{1 \leq i \leq \ell} \dot{\Lambda}_{r_i} - \sum_{1 \leq j \leq t} \alpha_{i_j}$$

and there exists a sequence of nonnegative integers $(a_1, \dots, a_\ell)$ such that

$$\dot{w} = d\dot{\delta} + \sum_{1 \leq i \leq \ell} a_i \dot{\Lambda}_{i-1}.$$

By Proposition 2.9, with this notation, we have $\mathbf{s} = \theta_{\ell, e, s}^{-1}(a_1, a_\ell, \dots, a_2)$. By hypothesis, the $\widehat{\mathfrak{sl}}_e$ -weight $\dot{w}'$ of $|\mu_2, -\dot{\mathbf{r}}_{\text{rev}}\rangle$ can be written

$$\dot{w}' = d'\dot{\delta} + \sum_{1 \leq i \leq \ell} \dot{\Lambda}_{-r_i} - \sum_{1 \leq j \leq t} \dot{\alpha}_{-i_j}$$

and thus we have

$$\dot{w}' = d'\dot{\delta} + \sum_{1 \leq i \leq \ell} a_i \dot{\Lambda}_{1-i}.$$

We thus have $\mathbf{v} = \theta_{\ell, e, -s}^{-1}(a_1, a_2, \dots, a_\ell)$. We conclude that $\mathbf{v} = -\mathbf{s}_{\text{rev}}$ using Lemma 2.8. Therefore, we can set μ to be the ℓ -partition such that $|\mu, -\mathbf{s}_{\text{rev}}\rangle = \beta^{-1}(\mu_1, \sigma, \mu_2)$, see Corollary 2.4, and this concludes the proof of (1) and (2).

It remains to prove (3). To do this, let us study the $\widehat{\mathfrak{sl}}_e$ -weight of $|\lambda_1, \mathbf{s}\rangle$ and $|\mu_1, -\mathbf{s}_{\text{rev}}\rangle$. Again by Proposition 2.9, the weight w of $|\lambda_1, \mathbf{s}\rangle$ is

$$w = d\delta + (e - r_1 + r_e)\Lambda_0 + (r_1 - r_2)\Lambda_1 + \dots + (r_{e-1} - r_e)\Lambda_{e-1}$$

which can be written as

$$w = d\delta + \sum_{1 \leq i \leq \ell} \Lambda_{t_i} - \sum_{0 \leq i \leq e-1} m_i \alpha_i$$

for a sequence of nonnegative integers $(m_i)_{i=0, \dots, e-1}$. Note that, with this notation, the number $N := \sum_{0 \leq i \leq e-1} m_i$ corresponds to the size of λ_1 (see Section 1C). Now again, the weight w' of $|\mu_1, -\mathbf{s}_{\text{rev}}\rangle$ is

$$w' = d\delta + (e - r_1 + r_e)\Lambda_0 + (r_{e-1} - r_e)\Lambda_1 + \dots + (r_1 - r_2)\Lambda_{e-1}$$

which thus can be written as

$$w' = d\delta + \sum_{1 \leq i \leq \ell} \Lambda_{-t_i} - \sum_{0 \leq i \leq e-1} (m_{-i}) \alpha_i.$$

We conclude that $|\lambda_1| = |\mu_1|$, that is, $N := \sum_{0 \leq i \leq e-1} m_i$. It follows that $|\mu| = |\lambda| = N + |\sigma|e + r$. $\square$

We may write $\Phi(\lambda)$ instead of $\Phi(|\lambda, \mathbf{s}\rangle)$ when the charge $\mathbf{s}$ is understood.

Example 2.11. Take the same values as in Example 2.5. Denote $\mathbf{r} = (-1, 0, 0, 2)$, so that $\dot{\mathbf{r}} = (-1, -1, 1)$. Then we have

$$\begin{aligned} \Phi(|\lambda, \mathbf{s}\rangle) &= \beta^{-1}(\Phi_{e, \mathbf{r}}(\tilde{f}_1 \tilde{f}_0 \tilde{f}_2 | \emptyset, \mathbf{r})), (2)^{\text{tr}}, \Phi_{\ell, \dot{\mathbf{r}}}(\tilde{f}_0 \tilde{f}_2 \tilde{f}_3 \tilde{f}_0 \tilde{f}_2 \tilde{f}_3 | \emptyset, \dot{\mathbf{r}})) \\ &= \beta^{-1}(\tilde{f}_2 \tilde{f}_0 \tilde{f}_1 | (\emptyset, \emptyset, \emptyset, \emptyset), (-2, 0, 0, 1)), (1^2), \\ &\quad \tilde{f}_0 \tilde{f}_2 \tilde{f}_1 \tilde{f}_1 \tilde{f}_0 \tilde{f}_2 \tilde{f}_1 | (\emptyset, \emptyset, \emptyset), (-1, 1, 1))) \\ &= \beta^{-1}(|(\emptyset, 1, \emptyset, 2), (-2, 0, 0, 1)\rangle, (1^2), |(\emptyset, 2^2, 2, 1), (-1, 1, 1)\rangle) \\ &= |(1, 2, 1, \emptyset, 2^3), (-1, -1, -2, 3)\rangle. \end{aligned}$$

The following corollary shows that Φ generalizes the map $\Phi_{e,s}$ of Section 1.

Corollary 2.12. *Let $e \geq 2$, $\mathbf{s} \in \mathbb{Z}^\ell(s)$, and $n \geq 0$. For all $\boldsymbol{\lambda} \in \text{Ug}_{e,s}(n)$, we have*

$$\Phi(\boldsymbol{\lambda}) = \Phi_{e,s}(\boldsymbol{\lambda}).$$

Proof. Let $\mathbf{s} \in \mathbb{Z}^\ell(s)$. By Property (3) of Theorem 2.10, we have

$$\Phi(|\emptyset, \mathbf{s}\rangle) = |\emptyset, -\mathbf{s}_{\text{rev}}\rangle.$$

Now if $\boldsymbol{\lambda} \in \text{Ug}_{e,s}(n)$ there exists a sequence of Kashiwara operators such that

$$\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{s}\rangle = |\boldsymbol{\lambda}, \mathbf{s}\rangle.$$

So we can use Property (1) of Theorem 2.10 to see that

$$\Phi(\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} |\emptyset, \mathbf{s}\rangle) = \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} |\emptyset, -\mathbf{s}_{\text{rev}}\rangle.$$

By definition of $\Phi_{e,s}$ in Theorem 1.4, we get the result. $\square$

2C. More on crystal isomorphisms. Let $P_\ell := \mathbb{Z}^\ell$ be the $\mathbb{Z}$ -module with standard basis $\{z_i \mid i = 1, \dots, \ell\}$. For $k = 1, \dots, \ell - 1$, we denote by σ_k the transposition $(k, k+1)$ of $\mathfrak{S}_\ell$. The extended affine symmetric group $\widehat{\mathfrak{S}}_\ell$ is the semidirect product $P_\ell \rtimes \mathfrak{S}_\ell$ with the relations given by $\sigma_i z_j = z_j \sigma_i$ for $j \neq i, i+1$ and $\sigma_i z_i \sigma_i = z_{i+1}$ for $i = 1, \dots, \ell - 1$ and $j = 1, \dots, \ell$. It acts faithfully on $\mathbb{Z}^\ell$ as follows: for any $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$,

$$\sigma_c \cdot \mathbf{s} = (s_1, \dots, s_{c-1}, s_{c+1}, s_c, s_{c+2}, \dots, s_\ell) \quad \text{for } c = 1, \dots, \ell - 1 \text{ and}$$

$$z_i \cdot \mathbf{s} = (s_1, s_2, \dots, s_i + e, \dots, s_\ell) \quad \text{for } i = 1, \dots, \ell.$$

If $\mathbf{s}$ and $\mathbf{s}'$ are in the same orbit modulo the action of $\widehat{\mathfrak{S}}_\ell$, then there is an $\widehat{\mathfrak{sl}}_e$ -crystal isomorphism $\Psi_{\mathbf{s} \rightarrow \mathbf{s}'}$ between the Fock spaces $\mathcal{F}_{e,\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{s}'}$; that is, a map

$$\Psi_{\mathbf{s} \rightarrow \mathbf{s}'} : \Pi_{\mathbf{s}}^\ell \rightarrow \Pi_{\mathbf{s}'}^\ell$$

such that

- $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is a highest weight vertex in $\mathcal{F}_{e,\mathbf{s}}$ if and only if $|\Psi_{\mathbf{s} \rightarrow \mathbf{s}'}(\boldsymbol{\lambda}), \mathbf{s}'\rangle$ is a highest weight vertex in $\mathcal{F}_{e,\mathbf{s}'}$,
- for all $\boldsymbol{\lambda} \in \Pi_{\mathbf{s}}^\ell$, we have $\Psi_{\mathbf{s} \rightarrow \mathbf{s}'}(\tilde{f}_i |\boldsymbol{\lambda}, \mathbf{s}\rangle) = \tilde{f}_i \Psi_{\mathbf{s} \rightarrow \mathbf{s}'}(|\boldsymbol{\lambda}, \mathbf{s}\rangle)$.

These crystal isomorphisms have been explicitly described in [Jacon and Lecouvey 2009]. Let us now come back to our situation. Assume that $\mathbf{s} \in \mathbb{Z}^\ell$ and choose any $\mathbf{s}' \in \mathbb{Z}^\ell$ in the orbit of $-\mathbf{s}$ modulo the action of the extended affine symmetric group. This is in particular the case for $-\mathbf{s}_{\text{rev}}$. There is an $\widehat{\mathfrak{sl}}_e$ -crystal isomorphism between the Fock spaces $\mathcal{F}_{e,-\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{s}'}$. Composing this map with Φ thus gives an isomorphism between $\mathcal{F}_{e,\mathbf{s}}$ and $\mathcal{F}_{e,\mathbf{s}'}$.

3. Combinatorics of perverse equivalences for cyclotomic Cherednik category $\mathcal{O}$

In Section 1, we studied the Mullineux involution in the context of representations of cyclotomic Hecke algebras. In this section, we use the results of Section 2 to study the Mullineux involution in the context of representations of cyclotomic rational Cherednik algebras. The goal is to realize the generalized Mullineux involution as the permutation of Π^ℓ induced by certain perverse equivalences. We follow Losev's approach [2015; 2017b].

3A. Representations of cyclotomic rational Cherednik algebras. We can deform $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rtimes \mathbb{C}W_{\ell,n}$ to obtain an algebra called the cyclotomic rational Cherednik algebra [Etingof and Ginzburg 2002]. This deformation depends on parameters $\kappa \in \mathbb{Q}^\times$ and $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{Q}^\ell$ (the charge). We denote it $H_{\kappa,\mathbf{s}}(n)$. The charge $\mathbf{s}$ is identified with $\mathbf{s} + \alpha(1, 1, \dots, 1)$ for any scalar α , thus the parameter space is ℓ -dimensional. As a $\mathbb{C}$ -vector space,

$$H_{\kappa,\mathbf{s}}(n) = \mathbb{C}[y_1, \dots, y_n] \otimes \mathbb{C}[W_{\ell,n}] \otimes \mathbb{C}[x_1, \dots, x_n].$$

This makes it possible to define a category $\mathcal{O}$ for $H_{\kappa,\mathbf{s}}(n)$ as the full subcategory of $H_{\kappa,\mathbf{s}}(n)$ -mod consisting of finitely generated $H_{\kappa,\mathbf{s}}(n)$ -modules which are locally nilpotent for the action of $\mathbb{C}[y_1, \dots, y_n]$ [Ginzburg et al. 2003]. Let $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ denote the category $\mathcal{O}$ of $H_{\kappa,\mathbf{s}}(n)$, and $\mathcal{O}_{\kappa,\mathbf{s}} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}_{\kappa,\mathbf{s}}(n)$. $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ is a highest weight category whose simple objects are indexed by $\text{Irr}_{\mathbb{C}} W_{\ell,n}$, the irreducible representations of the underlying group algebra $\mathbb{C}W_{\ell,n}$ [Ginzburg et al. 2003], and thus by $\Pi^\ell(n)$. Furthermore, the highest weight structure of $\mathcal{O}_{\kappa,\mathbf{s}}(n)$ depends on a partial order on $\Pi^\ell(n)$ which is determined by the parameter $(\kappa, \mathbf{s})$.

3B. Branching rules and crystals. In this subsection we restrict to the case of *integral parameters*, that is

$$\kappa = \pm 1/e \quad \text{for some } e \in \mathbb{Z}_{\geq 2} \quad \text{and} \quad \mathbf{s} \in \mathbb{Z}^\ell.$$

The (complexified) Grothendieck group of $\mathcal{O}_{\kappa,\mathbf{s}}$ is a level ℓ Fock space. Recall from Section 1C that the parameters $e, \mathbf{s}$ for the Fock space come from its $\mathcal{U}_t(\widehat{\mathfrak{sl}}_e)$ -module structure. Shan [2011] has shown that if $\kappa = 1/e$, there is a notion of branching rule arising from Bezrukavnikov and Etingof's parabolic induction functors [2009], which categorifies the $\widehat{\mathfrak{sl}}_e$ -crystal of the Fock space $\mathcal{F}_{e,\mathbf{s}}$ [Shan 2011, Theorem 6.3]. Moreover, there is a categorical Heisenberg action on $\mathcal{O}_{\kappa,\mathbf{s}}$ giving rise to the $\mathfrak{sl}_\infty$ -crystal on $\mathcal{F}_{e,\mathbf{s}}$ [Shan and Vasserot 2012; Losev 2015]. We have:

Theorem 3.1 [Shan and Vasserot 2012, Proposition 5.18]. *The following are equivalent:*

- (1) λ is a highest weight vertex for both the $\widehat{\mathfrak{sl}}_e$ - and $\mathfrak{sl}_\infty$ -crystals on $\mathcal{F}_{e,\mathbf{s}}$.

(2) $L(\lambda)$ is killed by the categorical Heisenberg and $\widehat{\mathfrak{sl}}_e$ annihilation operators.

(3) $L(\lambda)$ is finite-dimensional.

Thus finite-dimensional simples are labeled by the source vertices of the $\widehat{\mathfrak{sl}}_e$ - and $\mathfrak{sl}_\infty$ -crystals.

3C. Perverse equivalences. Perverse equivalences are a special kind of derived equivalence introduced by Chuang and Rouquier [2008; 2017] which are well-suited for combinatorial applications. Let $\mathcal{A}, \mathcal{A}'$ be abelian categories with finitely many simple objects and in which every object has finite length. Let S, S' be the sets of isomorphism classes of simple objects of $\mathcal{A}, \mathcal{A}'$ respectively. Let

$$0 \subset S_0 \subset S_1 \subset \cdots \subset S_r = S$$

be a filtration of S and $0 \subset S'_0 \subset S'_1 \subset \cdots \subset S'_r = S'$ a filtration of S' . Let $\mathcal{A}_i \subset \mathcal{A}$ and $\mathcal{A}'_i \subset \mathcal{A}'$ be the Serre subcategories generated by S_i and S'_i , respectively, and let $\pi : \{0, 1, \dots, r\} \rightarrow \mathbb{Z}$ be a function.

Definition 3.2 (Chuang–Rouquier). A derived equivalence $F : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A}')$ is *perverse* if for all $i \geq 0$ and all $s \in S_i \setminus S_{i-1}$, the complex $F(s)$ is such that

- (1) for $j \neq \pi(i)$, all composition factors of $H^j(F(s))$ are in S'_{i-1} ,
- (2) all the composition factors of $H^{\pi(i)}(F(s))$ are in S'_{i-1} except for a unique one in $S'_i \setminus S'_{i-1}$.

A perverse equivalence $F : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{A}')$ therefore gives rise to a canonical bijection $\mathfrak{f} : S \rightarrow S'$ sending $s \in S_i \setminus S_{i-1}$ to $\mathfrak{f}(s) := H^{\pi(i)}(F(s)) \bmod \mathcal{A}'_{i-1}$. In Sections 3D and 3E, we show how the generalized Mullineux involution arises from the perverse equivalences given by wall-crossing functors and Ringel duality.

3D. Wall-crossing functors. Let us recall Losev’s [2015] construction of wall-crossing functors. These are derived equivalences between category $\mathcal{O}_{\gamma, \dagger}(n)$ ’s for parameters which differ by a perturbation of the partial order on $\Pi^\ell(n)$. Denote by $\mathfrak{c}_\mathbb{Z} \subset \mathbb{Q}^\times \times \mathbb{Q}^\ell$ the ℓ -dimensional lattice in the parameter space $\mathbb{C}^\ell$ consisting of those parameters $c' = (\kappa', \mathbf{s}')$ that have *integral difference* with a fixed parameter $c = (\kappa, \mathbf{s})$, i.e., such that $\kappa' - \kappa \in \mathbb{Z}$ and $\kappa'(s'_i - s'_j) - \kappa(s_i - s_j) \in \mathbb{Z}$ for all $1 \leq i < j \leq \ell$. Note that replacing the ℓ -tuple $\mathbf{s} = (s_1, \dots, s_\ell)$ with $\mathbf{s} + (\alpha, \dots, \alpha)$ for $\alpha \in \mathbb{Z}$ does not affect the definition of the corresponding rational Cherednik algebra; see the formulas in [Shan and Vasserot 2012, §3.3, Theorem 6.10]. The lattice $\mathfrak{c}_\mathbb{Z}$ is the shift by c of the dual lattice to the lattice spanned by certain hyperplane elements; see [Losev 2015, Sections 2.1.4, 2.7, 4.1.3]. It is isomorphic to $\mathbb{Z}^\ell$. There is a finite set of hyperplanes in $\mathbb{C}^\ell$ called walls dividing $\mathfrak{c}_\mathbb{Z}$ into open cones called chambers. These hyperplanes are defined as follows. For each $\lambda = ((\lambda_1^1, \lambda_2^1, \dots), \dots, (\lambda_1^\ell, \lambda_2^\ell, \dots)) \in \Pi^\ell(n)$, let

$$[\lambda] := \{(a, b, j) \mid 1 \leq a, 1 \leq b \leq \lambda_a^j, 1 \leq j \leq \ell\}$$

be the Young diagram of λ . For each $(a, b, j) \in [\lambda]$, we define

$$co_c(\gamma) = \kappa \ell(b - a) + \ell h_j,$$

where $h_j = \kappa s_j - j/\ell$ and

$$c_\lambda := \sum_{\gamma \in [\lambda]} co_c(\gamma)$$

is called the c -function; the formula in our cyclotomic case was given in [Gordon and Losev 2014]. By definition, the walls are the hyperplanes $\Pi_{\lambda, \lambda'}$ given by $c_\lambda = c_{\lambda'}$. Among these walls, we will be interested in the so-called “essential walls” which are the following ones:

- (1) The wall $\kappa = 0$ between chambers containing parameters $(\kappa, \mathbf{s})$ such that the denominator e of κ satisfies $2 \leq e \leq n$, in terms of the above definition. It is of the form $\Pi_{\lambda, \lambda'}$ for some $[\lambda] = [\mu] \sqcup \{\gamma\}$ and $[\lambda'] = [\mu] \sqcup \{\gamma'\}$, for some multipartition μ and two boxes $\gamma = (a, b, j)$ and $\gamma' = (a', b', j')$ of μ with $j = j'$.
- (2) The walls $h_i - h_j = \kappa m$ with $i \neq j$, $m \in \mathbb{Z}$ and $|m| < n$ between chambers containing parameters such that $s_i - s_j - m \in \kappa^{-1}\mathbb{Z}$. In terms of the above definition they are of the form $\Pi_{\lambda, \lambda'}$ where

$$[\lambda] = [\mu] \sqcup \{\gamma\} \quad \text{and} \quad [\lambda'] = [\mu] \sqcup \{\gamma'\}$$

for a multipartition μ and two boxes $\gamma = (a, b, i)$ and $\gamma' = (a', b', j)$.

Two categories whose parameters lie in the same chamber are equivalent as highest weight categories [Losev 2015, Proposition 2.8]. On the other hand, the bounded derived categories of $\mathcal{O}_c(n)$ and $\mathcal{O}_{c'}(n)$ are derived equivalent when $c := (\kappa, \mathbf{s})$ is obtained from $c' := (\kappa', \mathbf{s}')$ by crossing a wall to an adjacent chamber. Two chambers are separated by the wall $\Pi_{\lambda, \lambda'}$ if and only if the sign of $c_\lambda - c_{\lambda'}$ in a chamber is opposite to $c_\lambda - c_{\lambda'}$ in the adjacent one. The derived equivalences $WC_{c \leftarrow c'} : D^b(\mathcal{O}_c) \rightarrow D^b(\mathcal{O}_{c'})$ are called wall-crossing functors. They are defined by taking the derived tensor product with a Harish-Chandra bimodule; see [Losev 2015, Section 2.8] or [Losev 2017a] for the construction of these functors.

Losev [2015, Proposition 2.12] proved that $WC_{c \leftarrow c'} : D^b(\mathcal{O}_c) \rightarrow D^b(\mathcal{O}_{c'})$ is a perverse equivalence with respect to the filtration of simple modules by their supports (the function π then picks out the dimension of the support). Therefore $WC_{c \leftarrow c'} : D^b(\mathcal{O}_c) \rightarrow D^b(\mathcal{O}_{c'})$ induces a canonical bijection $wc_{c \leftarrow c'}$ on Π^ℓ , called the combinatorial wall-crossing. For walls of type (b), the combinatorial wall-crossings have been studied in [Losev 2015; Jacon and Lecouvey 2018]. In particular, they are given by the crystal isomorphisms of Section 2C for the appropriate parameters [Jacon and Lecouvey 2018, Theorem 11]. We are here interested in the walls of type (a). First we need to see for which types of parameters they are defined.

We will denote by $w_{c \leftarrow +}$ the combinatorial wall-crossing from $\kappa > 0$ to $\kappa' < 0$ corresponding to the wall of type (a).

For the next proposition, we will use the following notation. For $c := (\kappa, \mathbf{s})$ and $\pi \in \mathfrak{S}_\ell$, we say that c is π -asymptotic if, for all $i = 1, \dots, \ell$,

$$\left(s_{\pi(i)} - \frac{\pi(i)}{\kappa \ell} \right) - \left(s_{\pi(i+1)} - \frac{\pi(i+1)}{\kappa \ell} \right) > n - 1.$$

Note that this is slightly more general than Losev's definition [2015], and that both agree for integral parameters. We say that c is *asymptotic* if it is π -asymptotic for some π .

Proposition 3.3. (1) *Let $c = (\kappa, \mathbf{s})$ and $c' = (\kappa', \mathbf{s}')$ in $c_{\mathbb{Z}}$ be such that κ and κ' have the same sign and such that c and c' are both π -asymptotic for some $\pi \in \mathfrak{S}_\ell$. Then there are no essential walls between c and c' .*

(2) *Assume that $c := (\kappa, \mathbf{s})$ and $c' := (\kappa - 1, \mathbf{s}')$ are two parameters with integral difference, with $\kappa > 0$ and $\kappa' := \kappa - 1 < 0$ lying in two different chambers separated by a unique wall of type (a) (and no wall of type (b)). Then there exists $\pi \in \mathfrak{S}_\ell$ such that c is π -asymptotic and c' is π' -asymptotic where $\pi' \in \mathfrak{S}_\ell$ is defined by $\pi'(i) = \pi(\ell - i + 1)$ for all $i = 1, \dots, \ell$. Conversely, if c and c' are as above, they are separated by a unique wall of type (a) and no wall of type (b).*

Proof. Let $(i_1, i_2) \in \{1, \dots, \ell\}^2$ be such that $i_1 \neq i_2$ and assume that

$$\left(s_{i_1} - \frac{i_1}{\kappa \ell} \right) - \left(s_{i_2} - \frac{i_2}{\kappa \ell} \right) > n - 1.$$

Let λ be an arbitrary ℓ -partition of n and let $\gamma_1 = (a_1, b_1, i_1)$ and $\gamma_2 = (a_2, b_2, i_2)$ be two boxes of the Young diagram. By [Geck and Jacon 2011, Example 5.5.18], we have

$$b_1 - a_1 - (b_2 - a_2) \geq 1 - n.$$

Thus we deduce

$$b_1 - a_1 + s_{i_1} - \frac{i_1}{\kappa \ell} > b_2 - a_2 + s_{i_2} - \frac{i_2}{\kappa \ell},$$

which implies that $co_c(\gamma_1) > co_c(\gamma_2)$. The same calculation holds for c' and we get $co_{c'}(\gamma_1) > co_{c'}(\gamma_2)$. Thus the order on boxes induced by the c -function is the same if we choose it with respect to c or to c' , and therefore there is no wall of type (b) between c and c' . Thus we cannot have any walls of type $\Pi_{\lambda, \lambda'}$ between the two parameters. This directly implies (1).

Now we prove (2). Assume thus that $c := (\kappa, \mathbf{s})$ and $c' := (\kappa - 1, \mathbf{s}')$ have integral difference, with $\kappa > 0$ and $\kappa' := \kappa - 1 < 0$ lying in two different chambers separated by a unique wall of type (a) (and no wall of type (b)). Let $(i_1, i_2) \in \{1, \dots, \ell\}^2$

be such that $i_1 \neq i_2$. Consider the ℓ -partition λ of $n-1$ such that $\lambda^k = \emptyset$ for all $k \in \{1, \dots, \ell\} \setminus \{i_1\}$ and $\lambda^{i_1} = (n-1)$. We then consider the ℓ -partition μ of n obtained from λ by adding $\gamma_1 := (1, n, i_1)$ and the ℓ -partition ν of n obtained from λ by adding $\gamma_2 := (1, 1, i_2)$. If $c := (\kappa, \mathbf{s})$ does not lie in an essential wall, we have two cases to consider:

- If $c_\mu < c_\nu$ then we have $co_c(\gamma_1) < co_c(\gamma_2)$. We thus obtain

$$\kappa \ell \left(n-1 + s_{i_1} - \frac{i_1}{\kappa \ell} \right) < \kappa \ell \left(s_{i_2} - \frac{i_2}{\kappa \ell} \right)$$

which implies that

$$\left(s_{i_2} - \frac{i_2}{\kappa \ell} \right) - \left(s_{i_1} - \frac{i_1}{\kappa \ell} \right) > n-1.$$

Now consider the ℓ -partition λ' of $n-1$ such that $\lambda'^k = \emptyset$ for all $k \in \{1, \dots, \ell\} \setminus \{i_1\}$ and $\lambda'^{i_1} = (1^{n-1})$. We then consider the ℓ -partition μ' of n obtained from λ by adding $\gamma_1 := (n, 1, i_1)$ and the ℓ -partition ν' of n obtained from λ by adding $\gamma_2 := (1, 1, i_2)$. We must have

$$\kappa \ell \left(1-n + s_{i_1} - \frac{i_1}{\kappa \ell} \right) < \kappa \ell \left(s_{i_2} - \frac{i_2}{\kappa \ell} \right)$$

but now as no walls of type (b) must be crossed to go to c' , we must also have

$$\kappa' \ell \left(1-n + s'_{i_1} - \frac{i_1}{\kappa' \ell} \right) < \kappa' \ell \left(s'_{i_2} - \frac{i_2}{\kappa' \ell} \right).$$

This implies that

$$\left(s'_{i_2} - \frac{i_2}{\kappa' \ell} \right) - \left(s'_{i_1} - \frac{i_1}{\kappa' \ell} \right) > n-1$$

because κ' is negative.

- If $c_\mu > c_\nu$, we now have $co_c(\gamma_1) > co_c(\gamma_2)$ and

$$\kappa \ell \left(n-1 + s_{i_1} - \frac{i_1}{\kappa \ell} \right) > \kappa \ell \left(s_{i_2} - \frac{i_2}{\kappa \ell} \right).$$

This implies that

$$\left(s'_{i_2} - \frac{i_2}{\kappa' \ell} \right) - \left(s'_{i_1} - \frac{i_1}{\kappa' \ell} \right) > n-1$$

and by considering the same λ' as above we now obtain

$$\left(s_{i_1} - \frac{i_1}{\kappa' \ell} \right) - \left(s_{i_2} - \frac{i_2}{\kappa' \ell} \right) > n-1.$$

We can thus conclude that c is π -asymptotic and c' is π' -asymptotic for a certain $\pi \in \mathfrak{S}_\ell$. Now let us take such a pair (c, c') and let $(i_1, i_2) \in \{1, \dots, \ell\}^2$ such that $i_1 \neq i_2$ and

$$\left(s_{i_1} - \frac{i_1}{\kappa\ell}\right) - \left(s_{i_2} - \frac{i_2}{\kappa\ell}\right) > n - 1$$

so that

$$\left(s'_{i_2} - \frac{i_2}{\kappa'\ell}\right) - \left(s'_{i_1} - \frac{i_1}{\kappa'\ell}\right) > n - 1.$$

We have already seen that for any ℓ -partition λ , and $\gamma_1 = (a_1, b_1, i_1)$ and $\gamma_2 = (a_2, b_2, i_2)$ two boxes of the Young diagram, we have $co_c(\gamma_1) > co_c(\gamma_2)$. But, the hypothesis also shows that $co_{c'}(\gamma_1) > co_{c'}(\gamma_2)$. This implies that no wall of type (b) is crossed between c and c' . If we now take an arbitrary ℓ -partition of $n - 1$ and consider two addable boxes γ_1 and γ_2 in the same component such that $co_c(\gamma_1) < co_c(\gamma_2)$, then we have $co_{c'}(\gamma_1) > co_{c'}(\gamma_2)$. This implies that a wall of type (a) is crossed. This concludes the proof. $\square$

Assume that $\mathbf{s}$ is π -asymptotic for some $\pi \in \mathfrak{S}_\ell$ and assume moreover that we are in the integral parameter case. We denote $\mathbf{s}_{\text{opp}} := \mathbf{s}'$ an associated multicharge which is π' -asymptotic with respect to the notation of the proposition. This is a slight abuse of notation because $\mathbf{s}'$ is of course not unique in general. However, two π -asymptotic parameters lie in the same chamber. Thus, the associated categories are equivalent as highest weight categories, and we can identify these two parameters. Moreover, note that one can assume that $\mathbf{s}'$ and $\mathbf{s}$ live in the same orbit modulo the action of the affine symmetric group. Indeed, if we denote $\mathbf{s} := (s_1, \dots, s_\ell)$. One can choose $(k_1, \dots, k_\ell) \in \mathbb{Z}^\ell$ so that $\mathbf{s}'' := (s_1 + k_1e, \dots, s_\ell + k_\ell e)$ is π' -asymptotic. In this case, $c'' = (\kappa', \mathbf{s}'')$ and c are in $c\mathbb{Z}$ and again c'' and c' lie in the same chamber.

Proposition 3.4 [Losev 2015, Proposition 5.6]. (1) *The $\widehat{\mathfrak{sl}}_e$ -crystal commutes with $w_{c \leftarrow +}$.*

(2) *The $\mathfrak{sl}_\infty$ -crystal commutes with $w_{c \leftarrow +}$ up to taking the transpose, that is, $w_{c \leftarrow +} \circ \tilde{a}_\sigma = \tilde{a}_{\sigma^{\text{tr}}} \circ w_{c \leftarrow +}$ for all $\sigma \in \Pi$.*

For the next definition, if λ is an arbitrary ℓ -partition and $\mathbf{s}$ an arbitrary ℓ -charge, we denote $|\lambda^{\text{tr}}, -\mathbf{s}\rangle$ by $(|\lambda, \mathbf{s}\rangle)^{\text{tr}}$, where λ^{tr} is the ℓ -partition $((\lambda^1)^{\text{tr}}, \dots, (\lambda^\ell)^{\text{tr}})$ and $-\mathbf{s} := (-s_1, \dots, -s_\ell)$. Now it follows from the definition of the crystal operators and [Losev 2015, Section 4.1.4] that when changing κ to $-\kappa$, $()^{\text{tr}}$ commutes with the $\widehat{\mathfrak{sl}}_e$ -crystal up to changing $\tilde{f}_i$ to $\tilde{f}_{-i}$ (see [Jacon and Lecouvey 2010, Section 3.2.3]), and commutes with the $\mathfrak{sl}_\infty$ -crystal up to taking the transpose (when $\kappa < 0$, the $\mathfrak{sl}_\infty$ -crystal adds horizontal e -strips instead of vertical e -strips; see [Losev 2015, Section 4.2.3]). We deduce the following result.

Theorem 3.5. *Assume that we are in the integral case, and that moreover $(\kappa, \mathbf{s})$ is asymptotic. Assume that $\lambda = (\lambda^1, \dots, \lambda^\ell)$ is in the connected component of the empty multipartition for the $\widehat{\mathfrak{sl}_e}$ -crystal and the $\mathfrak{sl}_\infty$ -crystal. Then we have*

$$\Psi_{-\mathbf{s}_{\text{rev}} \rightarrow -\mathbf{s}_{\text{opp}}} \circ \Phi(\lambda) = \text{wc}_{-\leftarrow+}(\lambda)^{\text{tr}}.$$

Proof. By hypothesis, there exist $\sigma \in \Pi$ and $(i_1, \dots, i_p) \in (\mathbb{Z}/e\mathbb{Z})^p$ such that

$$\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma |\emptyset, \mathbf{s}\rangle = |\lambda, \mathbf{s}\rangle.$$

From Theorem 2.10(1) together with the definition of the crystal isomorphism in Section 2C, we have

$$\begin{aligned} \Psi_{-\mathbf{s}_{\text{rev}} \rightarrow -\mathbf{s}_{\text{opp}}} \circ \Phi(\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma |\emptyset, \mathbf{s}\rangle) &= \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \Psi_{-\mathbf{s}_{\text{rev}} \rightarrow -\mathbf{s}_{\text{opp}}} \circ \Phi(\tilde{a}_\sigma |\emptyset, \mathbf{s}\rangle) \\ &= \tilde{f}_{-i_p} \cdots \tilde{f}_{-i_1} \tilde{a}_{\sigma^{\text{tr}}} |\emptyset, -\mathbf{s}_{\text{opp}}\rangle \end{aligned}$$

using the explicit formulae of the crystal isomorphism together with the explicit formula of the action of $\tilde{a}_\sigma$ [Gerber 2019, Section 5]. On the other hand, by Proposition 3.4, we also get

$$\text{wc}_{-\leftarrow+}(\tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_\sigma |\emptyset, \mathbf{s}\rangle) = \tilde{f}_{i_p} \cdots \tilde{f}_{i_1} \tilde{a}_{\sigma^{\text{tr}}} |\emptyset, \mathbf{s}_{\text{opp}}\rangle,$$

where the right-hand side is computed with respect to $\kappa' < 0$. The result then follows by sending κ' to $-\kappa'$ (the latter is positive) and applying our operator $()^{\text{tr}}$. $\square$

This Theorem is in fact a generalization of a result by Losev in level 1 [2015, Corollary 5.7], which we recover below. Using the notation of Remark 2.6, denote

$$M_e : \Pi \rightarrow \Pi, \quad \lambda = (\sigma^e) \sqcup \rho \mapsto (\sigma^{\text{tr}})^e \sqcup m_e(\rho).$$

Corollary 3.6 (Losev). *Suppose $\ell = 1$. Then for all $\lambda \in \Pi$,*

$$\text{wc}_{-\leftarrow+}(\lambda) = (M_e(\lambda))^{\text{tr}}.$$

Proof. In the case $\ell = 1$, the charge is irrelevant (see also Remark 1.2), thus so is $\Psi_{-\mathbf{s}_{\text{rev}} \rightarrow -\mathbf{s}_{\text{opp}}}$. By Remark 2.6, $\lambda = (\sigma^e) \sqcup \rho$ is the level 1 analogue of the decomposition of Corollary 2.4 used to define Φ , and we can identify $M_e(\lambda)$ with $(m_e(\rho), (\sigma^{\text{tr}})^e)$. Thus by Theorem 2.10(2), we have $\Phi(\lambda) = (m_e(\rho), (\sigma^{\text{tr}})^e) = M_e(\lambda)$, and we conclude using Theorem 3.5. $\square$

We are able to partially generalize the level 1 statement to level ℓ :

Theorem 3.7. *Assume that we are in the integral case, and that moreover $(\kappa, \mathbf{s})$ is asymptotic. Let $k \in \{1, \dots, \ell\}$ be such that*

$$s_k = \max\{s_i; i = 1, \dots, \ell\}.$$

Assume that $\lambda = (\lambda^1, \dots, \lambda^\ell)$ is such that λ^i is e -regular for all $i \in \{1, \dots, \ell-1\} \setminus \{k\}$ and λ^k is arbitrary. The combinatorial wall-crossing $\text{wc}_{-\leftarrow+}(\lambda)$ is then given by the formula

$$\text{wc}_{-\leftarrow+}(\lambda) = (m_e(\lambda^1), \dots, M_e(\lambda^k), \dots, m_e(\lambda^\ell))^{\text{tr}}.$$

Proof. Let us first assume that λ^j is e -regular for all $j = 1, \dots, \ell$. By [Losev 2015, Proposition 3.1], we know that the wall-crossing is independent of the choice of a Weil generic parameter. In our situation this means that we can assume that for each $\lambda \in \Pi^\ell(n)$, if two boxes have the same residues then they are in the same component. We thus have an action of a Kac–Moody algebra as a tensor product of ℓ copies of $\widehat{\mathfrak{sl}}_e$, one for each component of the multipartition. By Proposition 3.4, the associated Kashiwara operators commute with $\text{wc}_{-\leftarrow+}$. Moreover, we know that $\text{wc}_{-\leftarrow+}$ sends the empty multipartition to the empty multipartition and that for each e -regular partition λ , there exists a sequence of Kashiwara operators sending $\emptyset$ to λ . The result follows.

Next, assume that $\lambda = (\lambda^1, \dots, \lambda^\ell)$ is such that λ^j is e -regular for all $j \in \{1, \dots, \ell\} \setminus \{k\}$ and λ^k is arbitrary. By our assumption on k and the definition of the action of the Heisenberg crystal operators $\tilde{a}_\sigma$ in [Losev 2015, Proposition 5.3], we know that there exist $\sigma \in \Pi$ and $\mu' = (\mu^1, \dots, \mu^\ell)$ such that $\mu^i = \lambda^i$ if $i \neq k$, μ^k is e -regular, and $\tilde{a}_\sigma \mu = \lambda$. The result then follows from Proposition 3.4. $\square$

We obtain the following interesting corollary which was not immediate from the crystal graph perspective.

Corollary 3.8. *Assume that we are in the integral case and that $(\kappa, \mathbf{s})$ is an asymptotic parameter and assume that $\lambda = (\lambda^1, \dots, \lambda^\ell)$ is a highest weight vertex such that each λ^j is e -regular. Then*

$$\text{wc}_{-\leftarrow+}(\lambda) = (m_e(\lambda^1), \dots, m_e(\lambda^\ell))^{\text{tr}}$$

is a highest weight vertex for the opposite asymptotic parameter.

Example 3.9. Take $\ell = 2$ and $s \in \mathbb{Z}$ such that $s > n - 1$ so that $\mathbf{s} = (0, s)$ is an asymptotic 2-charge. The bipartitions (λ^1, λ^2) which are both highest weight vertex for the $\mathfrak{sl}_\infty$ -crystal and the $\widehat{\mathfrak{sl}}_e$ -crystal are exactly the ones satisfying $\lambda^2 = \emptyset$ and one of the following conditions:

- $\lambda^1 = \mu^e$ for a partition μ .
- λ^1 has exactly one good removable box of residue $s \bmod e$.

Let us consider the second case and assume that λ^1 is e -regular. Then our theorem asserts that $\text{wc}_{-\leftarrow+}(\lambda) = (m_e(\lambda^1)^{\text{tr}}, \emptyset)$. This is consistent with the fact that it must be both a highest weight vertex for the $\mathfrak{sl}_\infty$ -crystal and the $\widehat{\mathfrak{sl}}_e$ -crystal because $m_e(\lambda^1)^{\text{tr}}$ has exactly one removable box of residue $-s \bmod e$.

3E. Ringel duality for $\mathcal{O}_{\kappa, \mathbf{s}}$. Consider $\mathcal{O}_{\kappa, \mathbf{s}}(n)$, the cyclotomic Cherednik category $\mathcal{O}$ of the complex reflection group $G(\ell, 1, n)$ for parameters $\kappa = r/e$ such that $e \geq 2$ and $\gcd(r, e) = 1$, and $\mathbf{s} \in \mathbb{Q}^\ell$ satisfying $\kappa e s_i \in \mathbb{Z}$ for $i = 1, \dots, \ell$. The category $\mathcal{O}_{\kappa, \mathbf{s}}(n)$ is a highest weight category with standard objects $\Delta(\lambda)$, costandard objects $\nabla(\lambda)$, simple objects $L(\lambda)$, etc., indexed by $\{\lambda \in \Pi^\ell(n)\}$ [Ginzburg et al. 2003]. By general theory [Donkin 1998, Appendix] there is a corresponding Ringel duality functor

$$R : D^b(\mathcal{O}_{\kappa, \mathbf{s}}(n)) \xrightarrow{\sim} D^b({}^\vee \mathcal{O}_{\kappa, \mathbf{s}}(n))$$

induced by the derived Hom functor with respect to a full tilting module, and which was realized explicitly in [Ginzburg et al. 2003]. The category ${}^\vee \mathcal{O}_{\kappa, \mathbf{s}}(n)$ is called the Ringel dual of $\mathcal{O}_{\kappa, \mathbf{s}}(n)$. It is shown in [Ginzburg et al. 2003] that ${}^\vee \mathcal{O}_{\kappa, \mathbf{s}}(n)$ is again a cyclotomic Cherednik category $\mathcal{O}$. In particular, it holds that ${}^\vee \mathcal{O}_{\kappa, \mathbf{s}}(n) \simeq \mathcal{O}_{\kappa', \mathbf{s}'}(n)$ for some parameters $\kappa', \mathbf{s}'$ of the same rank and level, so that the Grothendieck group of $\bigoplus_n \mathcal{O}_{\kappa', \mathbf{s}'}(n)$ is again a level ℓ and rank e Fock space. Losev [2017a, Lemma 2.5] proved that Ringel duality is perverse with respect to the filtration by support of simple modules. As discussed in Section 3C, this means that we can pick out a unique composition factor $L(\mu)$ in the homology of the complex $R(L(\lambda))$ such that μ has maximal bidepth in the $\widehat{\mathfrak{sl}_e}$ and $\mathfrak{sl}_\infty$ -crystals. Set $r(\lambda) = \mu$. For the rest of this section, we assume n is fixed and write $\mathcal{O}_{\kappa, \mathbf{s}}$ as shorthand for $\mathcal{O}_{\kappa, \mathbf{s}}(n)$.

Note that in Section 3D, we have defined wall-crossing functors as crossing a single essential wall. However, there is a derived equivalence

$$\mathrm{WC}_{c' \leftarrow c} : D^b(\mathcal{O}_c) \xrightarrow{\sim} D^b(\mathcal{O}_{c'})$$

for any c' in the lattice $c\mathbb{Z}$ of parameters having integral difference with c , defined by taking the derived tensor product with a Harish-Chandra bimodule [Losev 2017a]. By abuse of language let us refer to such an equivalence as a wall-crossing functor whenever c and c' do not lie in the same chamber.

We are now going to recall Losev's comparison of the Ringel duality with the wall-crossing functor to a maximally distant chamber. The first part of the following proposition is from immediately after Theorem 6.1 in [Losev 2017a]. We add some details to the proof by justifying the assumption of the existence of a parameter c' in the same lattice as c such that the order on category $\mathcal{O}_{c'}$ is opposite to that on $\mathcal{O}_c$.

Lemma 3.10. *Assume the parameter $c = (\kappa, \mathbf{s})$ is as in the assumptions of [Losev 2017a, Theorem 4.1]. The following statements hold.*

- (1) *The Ringel duality $R : D^b(\mathcal{O}_{\kappa, \mathbf{s}}) \rightarrow D^b({}^\vee \mathcal{O}_{\kappa, \mathbf{s}})$ can be realized by the (inverse) wall-crossing functor to the opposite chamber.*
- (2) *The combinatorial bijection r induced by Ringel duality commutes with the $\widehat{\mathfrak{sl}_e}$ -crystal.*

Proof. (1) This is a special case of [Losev 2017a, Theorem 6.1], which follows from [Losev 2017a, Lemma 2.5 and Theorem 4.1]. We justify the existence of an opposite chamber and its intersection with the $\mathbb{Z}$ -lattice of parameters $\mathbf{c}_{\mathbb{Z}}$ in the case $W = G(\ell, 1, n)$.

The category $\mathcal{O}_{\kappa, \mathbf{s}}$ is a highest weight category with respect to the order on $\text{Irr } G(\ell, 1, n) = \Pi^\ell(n)$ given by the c -function c_λ (see Section 3D). It is a general fact about Ringel duality for highest weight categories that the Ringel dual category has the same poset with the opposite partial order [Donkin 1998, Appendix]. Thus the c -order on $\Pi^\ell(n)$ with respect to the parameters $(\kappa', \mathbf{s}')$ is opposite to the c -order on $\Pi^\ell(n)$ with respect to the parameters $(\kappa, \mathbf{s})$. Recall from [Losev 2015] that the c -order is constant on the open chambers given by the complement of the hyperplane arrangement of essential walls.

We claim that we can find an example of a parameter $c' = (\kappa', \mathbf{s}')$ with the property that $\mathcal{O}_{\kappa', \mathbf{s}'}$ realizes the Ringel dual of $\mathcal{O}_{\kappa, \mathbf{s}}$ and $(\kappa', \mathbf{s}')$ has integral difference with $(\kappa, \mathbf{s})$. Recall the definitions of walls and chambers from Section 3D. First, assume that $\kappa > 0$, and let $t \in \mathbb{Z}_{>0}$ such that $t\mathbf{s} \in \mathbb{Z}^\ell$ and $n \in \mathbb{Z}$ be such that $\kappa - n < 0$, t divides n and $n/(\ell\kappa) \in \mathbb{Z}$. Set $c' := (\kappa', \mathbf{s}') \in \mathbb{Q}_{<0}^\times \times \mathbb{Q}^\ell$ such that $\kappa' := \kappa - n$ and for all $j = 1, \dots, \ell$, we have $s'_j = s_j + jn/(\ell\kappa\kappa')$. Then for all $j = 1, \dots, \ell$, we have

$$\kappa' s'_j - \kappa s_j = (\kappa - n) \left(s_j + \frac{jn}{\ell\kappa(\kappa - n)} \right) - \kappa s_j = -ns_j + \frac{jn}{\ell\kappa} \in \mathbb{Z}.$$

Since we have in addition $\kappa - \kappa' \in \mathbb{Z}$, we deduce that $(\kappa, \mathbf{s})$ and $(\kappa', \mathbf{s}')$ are in the same lattice. Now, let $\gamma = (a, b, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0} \times \{1, \dots, \ell\}$ be a box of an ℓ -partition. We have

$$co_c(\gamma) = \kappa \ell \left(b - a + s_j - \frac{j}{\ell\kappa} \right) \quad \text{and} \quad co_{c'}(\gamma) = \kappa' \ell \left(b - a + s'_j - \frac{j}{\ell\kappa'} \right)$$

and for $j = 1, \dots, \ell$,

$$s'_j - \frac{j}{\ell\kappa'} - \left(s_j - \frac{j}{\ell\kappa} \right) = \frac{jn}{\ell\kappa\kappa'} - \frac{j}{\ell\kappa'} + \frac{j}{\ell\kappa} = \frac{jn - j\kappa + j(\kappa - n)}{\ell\kappa(\kappa - n)} = 0.$$

As κ and κ' have opposite sign, we conclude that c and c' induce opposite orders with respect to the c -function.

(2) The proof follows by exactly the same argument as the proof of [Losev 2015, Proposition 5.6(1)]. Indeed, the assumption of [Losev 2015, Proposition 5.6(1)] that the wall-crossing is through a single essential wall is only used in reference to [Losev 2015, Proposition 3.8], which states that wall-crossing functors across essential walls intertwine the parabolic restriction functors. The analogous statement that (inverse) Ringel duality intertwines parabolic induction and restriction functors is proved in [Losev 2017a, Lemma 4.7]. $\square$

The symmetrical statement to Lemma 3.10(2) for the $\widehat{\mathfrak{sl}}_\ell$ -crystal follows by level-rank duality which switches the roles of e and ℓ and commutes with Ringel duality. For this, we need to be in the integral parameter case again.

Lemma 3.11. *The map r commutes with the $\widehat{\mathfrak{sl}}_\ell$ -crystal.*

Proof. By [Rouquier et al. 2016, Theorem 7.4] and [Webster 2017], the category $\mathcal{O}_{\kappa, \mathbf{s}}$ is standard Koszul, which by [Mazorchuk 2010] implies that Ringel duality commutes with the Koszul duality. The Koszul duality K lifts the level-rank duality k , by which the $\widehat{\mathfrak{sl}}_\ell$ -crystal is defined, to the categorical level [Rouquier et al. 2016, Theorem 7.4]. Since we need to compare Ringel duality for level e with Ringel duality for level ℓ , write R for the former and $\dot{R}$ for the latter; likewise, write r for the former and $\dot{r}$ for the latter on the level of Grothendieck groups. We have $\dot{R}K = KR$ and thus $\dot{r}k = kr$ and $r\dot{k} = \dot{k}r$. Let $\tilde{f}_j$ be an $\widehat{\mathfrak{sl}}_\ell$ crystal operator. Recall from (2.1) that the action of $\tilde{f}_j$ in level ℓ and rank e is defined by $\dot{k}\tilde{f}_j k$. By the preceding lemma, in level e and rank ℓ we have $\tilde{f}_j \dot{r} = \dot{r} \tilde{f}_j$. Thus, $r(\dot{k}\tilde{f}_j k) = \dot{k}\tilde{f}_j k = \dot{k}\tilde{f}_j \dot{r}k = (\dot{k}\tilde{f}_j k)r$. $\square$

3F. The combinatorial Ringel duality. For this section, we again take parameters to be integral. The category $\mathcal{O}_{\kappa, \mathbf{s}}$ is equivalent as a highest weight category to the category $\mathcal{S}_{\kappa, \mathbf{s}}$ of finite-dimensional modules over the diagrammatic Cherednik algebra with Uglov weighting defined by Webster [2017, Theorem 4.8]. Webster [2017, Corollary 5.11] proved that for the category $\mathcal{S}_{\kappa, \mathbf{s}}$, taking the Ringel dual corresponds to sending κ to $-\kappa$ and keeping the charge $\mathbf{s}$ fixed. Moreover, there is a natural isomorphism of diagram algebras (which is given by replacing the label $i \in \mathbb{Z}/e\mathbb{Z}$ with $-i$ on black strands and the label $j \in \mathbb{Z}/\ell\mathbb{Z}$ with $-j$ on red strands; see [Webster 2017, Proposition 4.5]). This isomorphism induces an equivalence

$$* : \mathcal{S}_{\kappa, \mathbf{s}} \rightarrow \mathcal{S}_{-\kappa, -\mathbf{s}_{\text{rev}}}$$

which evidently satisfies $\tilde{f}_{-i} \circ * = * \circ \tilde{f}_i$ and $\tilde{f}_{-j} \circ * = * \circ \tilde{f}_j$. Composing $*$ with Ringel duality for the diagrammatic Cherednik algebra then gives an equivalence (via identification of $\mathcal{O}_{\kappa, \mathbf{s}}$ with $\mathcal{S}_{\kappa, \mathbf{s}}$),

$$* \circ R : D^b(\mathcal{O}_{\kappa, \mathbf{s}}) \simeq D^b(\mathcal{S}_{\kappa, \mathbf{s}}) \xrightarrow{\sim} D^b(\mathcal{S}_{-\kappa, -\mathbf{s}_{\text{rev}}}) \simeq D^b(\mathcal{O}_{-\kappa, -\mathbf{s}})$$

which is perverse, being the composition of abelian equivalences with a perverse equivalence [Chuang and Rouquier 2017].

Set $D = * \circ R : D^b(\mathcal{O}_{\kappa, \mathbf{s}}) \xrightarrow{\sim} D^b(\mathcal{O}_{-\kappa, -\mathbf{s}_{\text{rev}}})$ (see [Ginzburg et al. 2003, Proposition 4.10]). On the level of Grothendieck groups, D yields an involutive isomorphism of Fock spaces,

$$d : \mathcal{F}_{e, \mathbf{s}} \rightarrow \mathcal{F}_{e, -\mathbf{s}_{\text{rev}}}.$$

We call d the *combinatorial Ringel duality*.

Corollary 3.12. *For all $i = 0, \dots, e - 1$ and all $j = 0, \dots, \ell - 1$ we have*

$$d\tilde{f}_i = \tilde{f}_{-i}d \quad \text{and} \quad d\tilde{f}_j = \tilde{f}_{-j}d.$$

Proof. This is straightforward from Lemmas 3.10 and 3.11, because $d = * \circ r$ and $*$ verifies $\tilde{f}_{-i} \circ * = * \circ \tilde{f}_i$ and $\tilde{f}_{-j} \circ * = * \circ \tilde{f}_j$, as explained above. $\square$

Lemma 3.13. *For all $\sigma \in \Pi$, we have $d\tilde{a}_\sigma = \tilde{a}_{\sigma^{\text{tr}}}d$.*

Proof. Since D is perverse with respect to supports of simple modules, i.e., cuspidal depth, d is an involutive isomorphism of $\mathfrak{sl}_\infty$ -crystals $\mathcal{F}_{e,s} \rightarrow \mathcal{F}_{e,-s_{\text{rev}}}$. Thus either $d \circ \tilde{a}_\sigma = \tilde{a}_\sigma \circ d$ or $d \circ \tilde{a}_\sigma = \tilde{a}_{\sigma^{\text{tr}}} \circ d$. By Corollary 3.12, d commutes with the two kinds of Kashiwara crystal operators up to a change of sign in the indices. Therefore, by Theorem 2.3(2), it is enough to prove that the two maps agree on multipartitions of the form $|\emptyset, s\rangle$. Further, by [Jacon and Lecouvey 2018], $|\emptyset, s\rangle$ can be obtained by applying a sequence of combinatorial wall-crossings of type (b) to $|\emptyset, s'\rangle$ for some asymptotic parameter (κ, s') . By [Losev 2015, Proposition 5.6 (1) and (2)], the combinatorial wall-crossings of type (b) commute with the three kinds of crystal operators. Therefore, we can reduce the proof to the case where (κ, s) is asymptotic. Now, we have already seen that in this case, $\tilde{a}_\sigma$ acts only on one component of the empty multipartition, so we can reduce the proof to the case $\ell = 1$.

For the rest of the proof, consider the functors D and $* \circ \text{WC}_{-\leftarrow+}$ when $\ell = 1$ so that we are considering functors on the category $\mathcal{O}_\kappa(\mathfrak{S}_n)$ (as $G(1, 1, n) = \mathfrak{S}_n$). By [Rouquier 2008, Corollary 5.13], if $r > 0$ is coprime to e then $\mathcal{O}_{-r/e}(\mathfrak{S}_n) \simeq \mathcal{O}_{-1/e}(\mathfrak{S}_n)$ are equivalent as highest weight categories. Moreover, the equivalence sends $\Delta(\lambda)$ to $\Delta(\lambda)$, as follows from [Ginzburg et al. 2003, Corollary 6.10] together with the fact that the Hecke algebra at a primitive e' -th root of unity depends only on e , not on the root of unity [Brundan and Kleshchev 2009]. Thus, though abusing notation, the composition $* \circ \text{WC}_{-\leftarrow+}$ is well defined.

We claim that for $\ell = 1$, the combinatorial Ringel duality d coincides with the transpose of the combinatorial wall-crossing. If $\kappa > 0$ and $\kappa' < 0$ and $\text{WC}_{-\leftarrow+} : D^b(\mathcal{O}_\kappa(\mathfrak{S}_n)) \xrightarrow{\sim} D^b(\mathcal{O}_{\kappa'}(\mathfrak{S}_n))$ is the wall-crossing functor across the (unique) essential wall defined by the equation $\kappa = 0$, then the combinatorial wall-crossing $\text{wc}_{-\leftarrow+}$ is given by $\text{wc}_{-\leftarrow+}(\lambda) = (M_e(\lambda))^{\text{tr}}$ for all $\lambda \in \Pi(n)$ [Losev 2015, Corollary 5.7]. The functors R and $\text{WC}_{-\leftarrow+}$ are both perverse equivalences from $\mathcal{O}_\kappa(\mathfrak{S}_n)$ with the same perversity function, namely the cuspidal depth of $L(\lambda)$. Thus $D = * \circ R$ and $* \circ \text{WC}_{-\leftarrow+}$ are perverse self-equivalences of $\mathcal{O}_\kappa(\mathfrak{S}_n)$ with the same perversity function [Losev 2017a, Lemma 2.5 and Theorem 6.1]. By [Chuang and Rouquier 2017, Proposition 4.17], if $F : D^b(\mathcal{C}) \xrightarrow{\sim} D^b(\mathcal{D})$ and $G : D^b(\mathcal{C}) \xrightarrow{\sim} D^b(\mathcal{E})$ are two perverse equivalences with the same perversity function, then $G \circ F^{-1} : \mathcal{D} \xrightarrow{\sim} \mathcal{E}$ is an equivalence of abelian categories. In particular, this implies that $R(\mathcal{O}_\kappa(\mathfrak{S}_n)) \simeq \text{WC}_{-\leftarrow+}(\mathcal{O}_\kappa(\mathfrak{S}_n))$. Set $A := (* \circ \text{WC}_{-\leftarrow+}) \circ D^{-1} : \mathcal{O}_\kappa(\mathfrak{S}_n) \xrightarrow{\sim} \mathcal{O}_\kappa(\mathfrak{S}_n)$. On K_0 ,

it holds that $[D(\Delta(\lambda))] = [(* \circ R)(\Delta(\lambda))] = [(* \circ WC_{-\leftarrow+})(\Delta(\lambda))] = [\Delta(\lambda^{\text{tr}})]$ for any $\lambda \in \Pi(n)$ by [Ginzburg et al. 2003, Proposition 3.3, Corollary 4.8 and Proposition 4.10] and [Losev 2015, Proposition 3.2]. Thus $[A(\Delta(\lambda))] = [\Delta(\lambda)]$ for all $\lambda \in \Pi(n)$. Moreover, it follows from the definitions of R and $WC_{-\leftarrow+}$ that A sends standardly filtered objects to standardly filtered objects. Therefore $A(\Delta(\lambda)) = \Delta(\lambda)$, from which it follows that $A(L(\lambda)) = L(\lambda)$. We conclude that $d = (-)^{\text{tr}} \circ wc_{-\leftarrow+} = M_e$. This concludes the proof. $\square$

Now recall the generalized Mullineux involution $\Phi : \mathcal{F}_{e,s} \rightarrow \mathcal{F}_{e,-s_{\text{rev}}}$ from Theorem 2.10.

Theorem 3.14. *The map d is equal to Φ .*

Proof. Corollary 3.12 and Lemma 3.13 combined imply that d and Φ satisfy the same commutation relations with the operators for the $\widehat{\mathfrak{sl}}_{\infty-}$, $\widehat{\mathfrak{sl}}_{e-}$, and $\widehat{\mathfrak{sl}}_{\ell-}$ -crystals. Moreover, as seen in the proof of Lemma 3.13, d maps $|\emptyset, s\rangle$ to $|\emptyset, -s_{\text{rev}}\rangle$. By Theorem 2.10(1), Φ is the unique involution with these properties, so $d = \Phi$. $\square$

We recall for the reader the fact that the finite-dimensional modules of a rational Cherednik algebra $H_c(W)$ coincide with the cuspidal modules, i.e., those sent to 0 by every parabolic restriction functor with respect to a proper parabolic subgroup of W [Bezrukavnikov and Etingof 2009].

Corollary 3.15. *Suppose $L(\lambda)$ is a finite-dimensional irreducible representation of the rational Cherednik algebra of type $B_n = G(2, 1, n)$ for parameters corresponding to Fock space charge $s = (s_1, s_2) \in \mathbb{Z}^2$. Then $\Phi(\lambda) = \lambda$, and $D(L(\lambda)) = L(\lambda)$, where we identify $L(\lambda)$ with the complex concentrated in degree 0.*

Proof. First, we make some remarks which hold for arbitrary ℓ . By Theorem 3.1, $L(\lambda)$ is finite-dimensional if and only if $|\lambda, s\rangle$ has depth 0 in both the $\widehat{\mathfrak{sl}}_{\infty-}$ and $\widehat{\mathfrak{sl}}_{e-}$ -crystals. Recall that D is perverse with respect to filtration by the depth in the $\widehat{\mathfrak{sl}}_{e-}$ and $\widehat{\mathfrak{sl}}_{\infty-}$ -crystals, i.e., cuspidal depth. Since D is not just a derived equivalence but a perverse equivalence, by Definition 3.2 it holds that D induces an abelian equivalence on each associated graded layer of $\mathcal{O}_{\kappa,s}(n)$ with respect to the filtration by supports of simple modules. In particular, D restricts to an equivalence of abelian categories on the bottom filtration layer of $\mathcal{O}_{\kappa,s}(n)$, and this subcategory coincides with the subcategory of cuspidal, that is finite-dimensional, modules. Thus if λ has depth 0 in both the $\widehat{\mathfrak{sl}}_{e-}$ and $\widehat{\mathfrak{sl}}_{\infty-}$ -crystals, D sends $L(\lambda)$ to some $L(\mu)$ where μ again has depth 0 in both the $\widehat{\mathfrak{sl}}_{e-}$ and $\widehat{\mathfrak{sl}}_{\infty-}$ -crystals. Here, we identify $L(\lambda)$ with a complex concentrated in degree 0, and likewise for $L(\mu)$. In this case, $D(L(\lambda)) = L(\Phi(\lambda))$ by Theorem 3.14.

Now we specify to the case $\ell = 2$. When $\ell = 2$, the map Φ fixes the $\widehat{\mathfrak{sl}}_{\ell-}$ -crystal operators $\tilde{f}_j$ since $j = -j \bmod 2$. Moreover,

$$-s_{\text{rev}} = (-s_2, -s_1) = (s_1, s_2) - (s_2 + s_1, s_2 + s_1),$$

i.e., $-\mathbf{s}_{\text{rev}}$ is just an integer shift of $\mathbf{s}$. Recall the remark at the beginning of Section 3A that charges differing by an integer shift define the same rational Cherednik algebra. Thus we may identify the Cherednik algebras defined by $(\kappa, \mathbf{s})$ and $(\kappa, -\mathbf{s}_{\text{rev}})$, and hence we may identify their categories $\mathcal{O}$. Now, it is straightforward to see that the action of $\tilde{f}_j$ on an element of $\Pi_{\mathbf{s}}^{\ell}$ is not affected by an integer shift of the charge; more precisely,

$$\tilde{f}_j|\mathbf{v}, \mathbf{t}\rangle =: |\mathbf{v}', \mathbf{t}'\rangle \Rightarrow \tilde{f}_j|\mathbf{v}, \mathbf{t} + k\rangle = |\mathbf{v}', \mathbf{t}' + k\rangle$$

for all $\mathbf{v} \in \Pi^{\ell}$, $\mathbf{t} \in \mathbb{Z}^{\ell}(s)$ and $k \in \mathbb{Z}$. In particular, we get $\Phi(|\lambda, \mathbf{s}\rangle) = |\lambda, -\mathbf{s}_{\text{rev}}\rangle$, which proves the claim. $\square$

Remark 3.16. Corollary 3.15 implies that Ringel duality can be seen as a self-equivalence of $\mathcal{O}_{e,s}(B_n)$ which fixes cuspidal (i.e., finite-dimensional) modules, and we recover [Ginzburg et al. 2003, Remark 4.12].

Example 3.17. Take $e = 3$, $\mathbf{s} = (-1, 3)$ and $\lambda = (3.3, \emptyset)$. Then

$$\begin{aligned} \beta(|\lambda, \mathbf{s}\rangle) &= (|(\emptyset, \emptyset), (0, 2)\rangle, \emptyset, |(2, 2, 1), (-1, -1, 0)\rangle) \\ &= (|(\emptyset, \emptyset), (0, 2)\rangle, \emptyset, \tilde{f}_0\tilde{f}_0\tilde{f}_0\tilde{f}_1\tilde{f}_1|(\emptyset, \emptyset, \emptyset), (-1, -1, 0)\rangle), \end{aligned}$$

so that $L(\lambda)$ is finite-dimensional in $\mathcal{O}_{1/e,s}(6)$. We have

$$\begin{aligned} \Phi(|\lambda, \mathbf{s}\rangle) &= \beta^{-1}(|(\emptyset, \emptyset), (-2, 0)\rangle, \emptyset, \tilde{f}_0\tilde{f}_0\tilde{f}_0\tilde{f}_1\tilde{f}_1|(\emptyset, \emptyset, \emptyset), (0, 1, 1)\rangle) \\ &= \beta^{-1}(|(\emptyset, \emptyset), (-2, 0)\rangle, \emptyset, |(1, 2, 2), (0, 1, 1)\rangle) \\ &= |(3.3, \emptyset), (-3, 1)\rangle = |\lambda, -\mathbf{s}_{\text{rev}}\rangle. \end{aligned}$$

4. Other crystal structures and generalization of Mullineux involutions

We now explain a possible generalization of the Mullineux involution defined in [Dudas and Jacon 2018] that uses the results of Section 2. We already know that we have three different crystal structures on level ℓ Fock spaces which are pairwise commuting, namely the $\widehat{\mathfrak{sl}_{\ell-}}$, $\widehat{\mathfrak{sl}_{\infty}}$ - and $\widehat{\mathfrak{sl}_{\ell}}$ -crystals. We now slightly generalize this by introducing an additional parameter $d \in \mathbb{Z}_{\geq 0}$ (which plays the role of the parameter ℓ in [Dudas and Jacon 2018] in level one—that is, the type A situation). To do this, recall that if $\lambda \in \Pi$, we can uniquely write the decomposition

$$\lambda = \lambda_{(0)} + \lambda_{(1)}^d + \cdots + \lambda_{(n)}^{d^n}$$

for $n \in \mathbb{Z}_{\geq 0}$ and where each $\lambda_{(i)}$ is d -regular.

Let $j \in \mathbb{Z}_{>0}$. One can define an action of a Kashiwara operator $\tilde{f}_{k,j}$ (for $k = 0, \dots, d-1$) as

$$\tilde{f}_{k,j}.\lambda = \mu \quad \text{for all } \lambda \in \Pi,$$

where $\mu_{(t)} = \lambda_{(t)}$ when $t \neq d$ and $\mu_{(d)} = \tilde{f}_k\lambda_{(d)}$ (where $\tilde{f}_k$ is denoting the usual

Kashiwara operator acting on d -regular partitions).

Using the decomposition in Theorem 2.3, we get an action of Kashiwara operators on the whole Fock space as follows. Let $|\lambda, \mathbf{s}\rangle \in \Pi_s^\ell$ and write

$$\beta(|\lambda, \mathbf{s}\rangle) = (|\lambda_1, \mathbf{r}\rangle, \sigma, |\lambda_2, \dot{\mathbf{r}}\rangle).$$

Then, for all $j \in \mathbb{Z}_{>0}$,

$$\tilde{f}_{k,j} \cdot |\lambda, \mathbf{s}\rangle = \beta^{-1}(|\lambda_1, \mathbf{r}\rangle, \tilde{f}_{k,j} \cdot \sigma, |\lambda_2, \dot{\mathbf{r}}\rangle).$$

For each $j \in \mathbb{Z}_{\geq 0}$, we thus get an $\widehat{\mathfrak{sl}}_d$ -crystal.

It is immediate to see that these actions also commute with the $\widehat{\mathfrak{sl}}_e$ -crystal and the $\mathfrak{sl}_\ell$ -crystal (it just follows from the existence of the bijection β). Finally, there is an obvious analogue of the Mullineux involution for this decomposition, which depends on d . Namely, for $|\lambda, \mathbf{s}\rangle \in \Pi_s^\ell$ and $\beta(|\lambda, \mathbf{s}\rangle) = (|\lambda_1, \mathbf{r}\rangle, \sigma, |\lambda_2, \mathbf{r}\rangle)$, we define

$$\Phi^{(d)}(|\lambda, \mathbf{s}\rangle) = \beta^{-1}(m_{e,\mathbf{r}}(\lambda_1), m_e(\sigma_{(0)}) + m_e(\sigma_{(1)})^d + \cdots + m_e(\sigma_{(n)})^{d^n}, m_{\ell,\dot{\mathbf{r}}}(\lambda_2)),$$

so that $\Phi^{(d)}$ simultaneously generalizes Φ and the version of the Mullineux involution of [Dudas and Jacon 2018] (which we recover by taking $\ell = 1$). As in [Dudas and Jacon 2018], we believe it would be interesting to look at the case $\ell = 2$ and investigate the relationship between $\Phi^{(d)}$ on the one hand, and the Alvis–Curtis duality for unipotent representations of finite unitary groups in transverse characteristic d on the other hand (or more generally of finite groups of Lie types B and C).

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THOMAS GERBER
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
LAUSANNE
SWITZERLAND
thomas.gerber@epfl.ch

NICOLAS JACON
UFR SCIENCES EXACTES ET NATURELLES, LABORATOIRE DE MATHÉMATIQUES
UNIVERSITÉ DE REIMS CHAMPAGNE-ARDENNE CNRS UMR 9008
REIMS
FRANCE
nicolas.jacon@univ-reims.fr

EMILY NORTON
UNIVERSITY OF BONN
BONN
GERMANY
enorton@mpim-bonn.mpg.de

ISOTYPIC MULTI-HARMONIC POLYNOMIALS AND GELBART–HELGASON RECIPROCITY

ANTHONY C. KABLE

A result of Gelbart and Helgason that describes the harmonic analysis of the space of functions on certain Stiefel manifolds and a result of Folland that describes the harmonic analysis of the de Rham complex on a sphere are generalized to describe the harmonic analysis of the de Rham complex on certain Stiefel manifolds. Equivalently, the cobranching law from a fundamental representation of an orthogonal group to a larger orthogonal group is determined.

1. Introduction

Let $n = n_1 + n_2$. The group $\mathrm{SO}(n)$ contains $\mathrm{SO}(n_1)$ and $\mathrm{SO}(n_2)$ as subgroups embedded in the upper-right and lower-left corners, respectively. These subgroups commute with one another and so if (ρ, E_ρ) is a representation of $\mathrm{SO}(n)$ then the space $E_\rho^{\mathrm{SO}(n_2)}$ of $\mathrm{SO}(n_2)$ -fixed vectors in E_ρ affords a representation $\rho^{\mathrm{SO}(n_2)}$ of $\mathrm{SO}(n_1)$. Gelbart [1974] investigated the nature of $\rho^{\mathrm{SO}(n_2)}$ when ρ is irreducible and $n_1 < n_2$. He identified those ρ for which $\rho^{\mathrm{SO}(n_2)} \neq \{0\}$ and showed [Gelbart 1974, Proposition 2.2] that for these ρ the dimension of $\rho^{\mathrm{SO}(n_2)}$ is equal to the dimension of a certain irreducible representation of $\mathrm{GL}(n_1)$ that is determined by ρ .

Gelbart's remarkable observation was one starting point for a substantial amount of subsequent research. A high point in this literature is the work of Gross and Kunze [1984]. In that paper, the authors introduce the concept of a bitriangular structure in a Lie group, develop a general theory of such structures, and then apply this theory to reprove Gelbart's result, improve the coincidence of dimensions to what they call a spatial isomorphism between $\rho^{\mathrm{SO}(n_2)}$ and the restriction of the associated $\mathrm{GL}(n_1)$ representation to $\mathrm{SO}(n_1)$, and also to generalize to many other situations with similar features (such as that of $\mathrm{U}(n_1)$ and $\mathrm{U}(n_2)$ as subgroups of $\mathrm{U}(n)$). In an appendix to Gross' and Kunze's paper it is explained, based on comments made by the referee, how their results may be understood in the framework of Kostant's work [1963] on group representations on polynomial rings. Conceptually, one may think of Gross' and Kunze's results as giving a kind of algebraic Poisson

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transform from the space of the $GL(n_1)$ representation to the space of $\rho^{SO(n_2)}$. This is clearer in the subsequent work of Johnson [1987], who generalizes the result still further and gives a shorter proof of the generalization not using the machinery of bitriangular systems. In the same work, Johnson also explains that his main result is a generalization of a famous result of Helgason's [1970]. Knapp [2001] reviews the history of this circle of ideas, identifying Gelbart's observation and Helgason's results as the twin roots from which they grow.

Now suppose that (ν, E_ν) is an irreducible representation of $SO(n_2)$. Then one may seek to generalize the above-described results by considering the isotype $\rho^{SO(n_2), \nu}$ of ν in ρ . Since $SO(n_1)$ and $SO(n_2)$ commute, the isotype is a representation of $SO(n_1)$. A successful generalization would take the form

$$(1-1) \quad \text{Hom}_{SO(n_2)}(\nu, \text{res}_{SO(n_2)}^{SO(n)}(\rho)) \cong \text{res}_{SO(n_1)}^{GL(n_1)}(\Pi_{\rho, \nu})$$

as representations of $SO(n_1)$, where $\Pi_{\rho, \nu}$ is some representation of $GL(n_1)$ depending on ρ and ν . In light of the history sketched above, the author proposes to use the name Gelbart–Helgason reciprocity for results of this form and also for similar results associated with other pairs of subgroups such as those mentioned above.

The main results of the current work are Theorems 5.2 and 5.4. The former is a Gelbart–Helgason reciprocity result of the form

$$\text{Hom}_{O(n_2)}(\nu_a, \text{res}_{O(n_2)}^{O(n)}(\rho)) \cong \text{res}_{O(n_1)}^{GL(n_1)}(\Pi_{\rho, \nu_a}),$$

where ν_a is the a -th exterior power of the standard representation of $O(n_2)$ with $a \leq n_2/2$, ρ is any irreducible representation of $O(n)$, and Π_{ρ, ν_a} is described explicitly in terms of ρ and a . The latter is a similar result with special orthogonal groups in place of orthogonal groups. The original example of Gelbart–Helgason reciprocity that was described above follows from the $a = 0$ case of the second of these results. Just as with the original result, both of these results require restrictions on n_1 and n_2 : roughly, n_2 must be large enough relative to n_1 .

Gelbart's original observation arose in the context of analysis on Stiefel manifolds, where it was used to decompose the space of functions on the Stiefel manifold $SO(n)/SO(n_2)$ under the action of $SO(n)$. Theorem 5.4 may be interpreted in a similar way as giving the decomposition of the space of sections of the bundle of differential a -forms on the Stiefel manifold $SO(n)/SO(n_2)$. This interpretation is obtained by a standard argument using Frobenius reciprocity. In Gelbart's original approach to his result another theorem of Helgason's [1963] played an important role. This theorem ensures that a surface harmonic on $SO(n)/SO(n_2)$ (meaning a function belonging to the isotypic subspace of some irreducible representation of $SO(n)$) extends to a solid harmonic (meaning a multiharmonic polynomial on a suitable space of matrices). This allowed Gelbart to obtain his result by analyzing the space of solid harmonics. The corresponding fact does not appear explicitly in

our approach because we can take advantage of subsequent progress in the theory of harmonic tensors due to Howe and to Kashiwara and Vergne.

The case where $n_1 = 1$, so that the Stiefel manifold referred to in the previous paragraph is a sphere, was previously studied by Folland [1989], who gave the complete harmonic analysis of the de Rham complex on a sphere. In a similar way, Theorem 5.4 gives the complete harmonic analysis of the de Rham complex on the associated Stiefel manifold provided that $2n_1 \leq n_2$. In fact, Folland did more than just determine the abstract harmonic analysis of the de Rham complex on a sphere. Namely, he gave explicit embeddings of the abstract irreducible constituents into the space of differential forms. The motivation for the work reported here was to obtain similarly explicit models in the more general case in order to facilitate computations involving the $(\mathrm{SO}(n_2), \nu_a)$ -isotypes in representations of $\mathrm{SO}(n)$. Note that Gross and Kunze [1984] did give a model of the isotypic subspace in the case where $a = 0$, but we found their model unsuitable for the type of computation that is the present goal, whereas the models that follow from the method of proof used here are suitable. We chose not to emphasize this aspect of the problem in this work, as it might prove distracting to do so and the explicit models will be explained at length elsewhere.

It remains to describe the approach that we use to prove Theorems 5.2 and 5.4. We make extensive use of the classic results of Howe [1989] and Kashiwara and Vergne [1978], for which Howe's inspiring essay [1995] is a concise source. Specifically, from this essay, we use Theorem 2.1.2 (which Howe calls $(\mathrm{GL}(n), \mathrm{GL}(m))$ -duality), and Theorem 3.5.2 and Proposition 3.6.3 (which together constitute $(\mathrm{O}(n), \mathrm{GL}(m))$ -duality and which these days often go under the name of Howe duality). The necessary results are recalled in a suitable form in Section 3 after a brief discussion of multiharmonic polynomials in Section 2. Using these results, one could prove that it is always possible to find an explicit $\Pi_{\rho, \nu}$ such that the left-hand side of (1-1) is a quotient of the right-hand side. This statement by itself could be of some use, but the central issue is to show that the kernel of the surjection is zero under suitable hypotheses. The heart of the proof is Theorem 4.1, which expresses the $(\mathrm{O}(n_2), \nu_a)$ -isotype in a space of multiharmonic polynomials as a quotient of a simpler space and gives a numerical criterion for the kernel of the surjection to be zero. The method of proof is elementary, although it raises an interesting concrete problem in commutative algebra that we do not need to solve completely for the present application; a complete solution would be welcome. From this result, we deduce Theorem 5.2, and then Theorem 5.4 follows by purely representation-theoretic methods. Finally, we remark that this approach extends to all the other classical cases of Gelbart–Helgason reciprocity and results in similar generalizations of them. In fact, the proofs in other cases are somewhat simpler because we do not have to contend with a disconnected group such as $\mathrm{O}(n)$. A combined exposition proved awkward and so these results will appear elsewhere.

Finally, the author would like to thank the referee for helping him to improve the clarity of the introduction and for pointing out a helpful reference.

2. Background on multiharmonic polynomials

Let $V = \text{Mat}(n, m)$ denote the space of n -by- m matrices over a field F of characteristic zero. The group $G = \text{O}(n) \times \text{GL}(m)$ acts on this space by $(g, h)M = gMh^{-1}$. The induced action on the algebra $F[V]$ of polynomials on V is given by $((g, h)f)(M) = f((g, h)^{-1}M)$. For $l \in \mathbb{N}$, let $[l] = \{1, \dots, l\}$. The space V has a standard basis consisting of the matrix units E_{ir} with $i \in [n]$ and $r \in [m]$. Let $x_{ir} \in F[V]$ be the coordinate dual to E_{ir} and $x = [x_{ir}]$ be the matrix of these coordinates. The action of G on the space of linear polynomials on V may be summarized by the equation $[(g, h)x_{ir}] = g^\top x h$. Similarly, let ∂_{ir} be the partial derivative operator with respect to x_{ir} and $\partial = [\partial_{ir}]$. The algebra of constant-coefficient differential operators on V is naturally identified with the symmetric algebra $\text{sym}(V)$ and so has an induced action of G . This action satisfies $[(g, h)\partial_{ir}] = g\partial h^{-1}$.

We shall identify $\text{O}(n)$ and $\text{GL}(m)$ as subgroups of G in the standard way. Define polynomials φ_{rs} for $r, s \in [m]$ by $[\varphi_{rs}] = x^\top x$ and differential operators Δ_{rs} by $[\Delta_{rs}] = \partial^\top \partial$. Then $[(g, h)\varphi_{rs}] = h^\top [\varphi_{rs}] h$ and $[(g, h)\Delta_{rs}] = h^{-\top} [\Delta_{rs}] h^{-1}$. The polynomials φ_{rs} generate the algebra $F[V]^{\text{O}(n)}$ of $\text{O}(n)$ -invariant polynomials on V and the differential operators Δ_{rs} generate the algebra of $\text{O}(n)$ -invariant constant-coefficient differential operators on V . We call a polynomial $h \in F[V]$ *multiharmonic* if $\Delta_{rs}h = 0$ for all $r, s \in [m]$ and denote by $\mathcal{H} < F[V]$ the space of all multiharmonic polynomials. In addition, we denote by $\mathcal{I}$ the ideal of $F[V]$ generated by φ_{rs} for $r, s \in [m]$. It follows from the identities noted above for the transformation of φ_{rs} and Δ_{rs} under the action of G that both $\mathcal{H}$ and $\mathcal{I}$ are G -invariant.

If $p \in F[V]$ then we denote by $p(\partial)$ the operator that results from substituting ∂_{ir} for x_{ir} in p . We then introduce a bilinear form on $F[V]$ by

$$\langle p, q \rangle = \text{CT}(p(\partial)q(x)),$$

where CT denotes the constant term. If $(g, h) \in G$ then we have

$$\langle (g, h)p, q \rangle = \langle p, (g^\top, h^\top)q \rangle$$

and, in particular, the bilinear form is $\text{O}(n)$ -invariant. If $m = x^a$ (using multi-index notation) is a monomial then we have $\langle m, q \rangle = a!c_m(q)$, where $c_m(q)$ denotes the coefficient of the monomial m in q . It follows from this that the restriction of the bilinear form to the space of polynomials of given total degree is nondegenerate. Moreover, we have $\langle pr, q \rangle = \langle p, r(\partial)q \rangle$ for all $p, q, r \in F[V]$.

Lemma 2.1. *We have $\mathcal{H} = \mathcal{I}^\perp$ and $F[V] = \mathcal{H} \oplus \mathcal{I}$.*

Proof. The nondegeneracy of the bilinear form implies that $h \in \mathcal{H}$ if and only if $\langle p, \Delta_{ir} h \rangle = 0$ for all $p \in F[V]$. This is equivalent to $\langle p\varphi_{ir}, h \rangle = 0$ for all p , which in turn is equivalent to $\langle \varphi, h \rangle = 0$ for all $\varphi \in \mathcal{I}$. This establishes the first claim. For the second, we initially assume that $F = \mathbb{Q}$. Then the bilinear form is positive definite. Both φ_{ir} and Δ_{ir} are homogeneous with respect to total degree and it follows that both $\mathcal{I}$ and $\mathcal{H}$ are homogeneous. Using a subscript to denote total degree, we have $F[V]_k = \mathcal{I}_k^\perp \oplus^\perp \mathcal{I}_k = \mathcal{H}_k \oplus^\perp \mathcal{I}_k$, from which the second claim follows in this case. In general, $\varphi_{ir} \in \mathbb{Q}[V]$ and Δ_{ir} preserves $\mathbb{Q}[V]$ and it follows that $\mathcal{I} = F \otimes_{\mathbb{Q}} \mathcal{I}_{\mathbb{Q}}$ and $\mathcal{H} = F \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}}$ within $F[V] = F \otimes_{\mathbb{Q}} \mathbb{Q}[V]$. Thus the second claim follows in general by extension of scalars. $\square$

We denote by $P : F[V] \rightarrow \mathcal{H}$ the orthogonal projection map associated with the decomposition $F[V] = \mathcal{H} \oplus^\perp \mathcal{I}$. Note that the G -invariance of $\mathcal{H}$ and $\mathcal{I}$ implies that P is G -equivariant.

3. Background on representations

The purpose of this section is to recall some results about representations of $O(n)$ and $SO(n)$ in a form suitable for our use and to summarize some results of Howe and Kashiwara and Vergne that we shall require. As mentioned in the introduction, Howe's essay [1995] is a suitable source for much of this material. In particular, Theorem 2.1.2 of that work is recalled as (3-2), Proposition 3.6.3 is recalled as (3-1), and the description of the singular vectors given below is drawn from the discussion preceding the latter result. The other facts recalled here, such as the relationship between the representations of $SO(n)$ and those of $O(n)$, are common knowledge in representation theory. The terminology and notation introduced in this section will be used subsequently.

In this section we assume that F is algebraically closed. Let $l = \lfloor n/2 \rfloor$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{so}(n)$ and $\mathfrak{t} \oplus \mathfrak{u}$ be a Borel subalgebra of $\mathfrak{so}(n)$ with nilradical $\mathfrak{u}$. Let $T < O(n)$ be the intersection of the normalizers of $\mathfrak{t}$ and $\mathfrak{u}$. There is an element $\tau \in O(n)$ of order two such that $T = \{e, \tau\} \rtimes T^\circ$. By a representation we shall mean a finite-dimensional rational representation over F . If (ρ, E_ρ) is an irreducible representation of $SO(n)$ then the space $E_\rho^\mathfrak{u}$ of *singular vectors* in E_ρ has dimension one and the isomorphism class of (ρ, E_ρ) is determined by the action of T° on $E_\rho^\mathfrak{u}$. We may equally well consider the action of $\mathfrak{t}$ on $E_\rho^\mathfrak{u}$ and the functional corresponding to this action is called the *highest weight* of ρ . If (ρ, E_ρ) is an irreducible representation of $O(n)$ then the space $E_\rho^\mathfrak{u}$ has dimension either one or two and the isomorphism class of (ρ, E_ρ) is determined by the action of T on $E_\rho^\mathfrak{u}$. Moreover, $E_\rho^\mathfrak{u}$ is an irreducible representation of T with this action.

Let (ρ, E_ρ) be an irreducible representation of $SO(n)$ whose highest weight is stable under τ . Then (ρ, E_ρ) extends to a representation of $O(n)$ on the same space

in two different ways. We denote these extensions by (ρ_+, E_{ρ_+}) and (ρ_-, E_{ρ_-}) , where $E_{\rho_+} = E_{\rho_-} = E_\rho$, $\rho_+(\tau)|_{E_\rho^u} = 1$ and $\rho_-(\tau)|_{E_\rho^u} = -1$. Now let (ρ, E_ρ) be an irreducible representation of $\mathrm{SO}(n)$ whose highest weight is not stable under τ . Then there is a unique irreducible representation $(\hat{\rho}, E_{\hat{\rho}})$ of $\mathrm{O}(n)$ such that the restriction of $(\hat{\rho}, E_{\hat{\rho}})$ to $\mathrm{SO}(n)$ is isomorphic to the direct sum of (ρ, E_ρ) and $(\rho^\tau, E_{\rho^\tau})$, where ρ^τ denotes the composition of ρ with conjugation by τ and $E_{\rho^\tau} = E_\rho$. A model of this representation is obtained by inducing (ρ, E_ρ) from $\mathrm{SO}(n)$ to $\mathrm{O}(n)$. Every irreducible representation of $\mathrm{O}(n)$ arises by one of these two constructions from an irreducible representation of $\mathrm{SO}(n)$ or, more precisely, from the orbit under $\{e, \tau\}$ of an irreducible representation of $\mathrm{SO}(n)$.

Suppose that n is odd. Then τ lies in the center of $\mathrm{O}(n)$ and so T is abelian. Thus every irreducible representation of $\mathrm{O}(n)$ has the form $(\rho_\pm, E_{\rho_\pm})$ where (ρ, E_ρ) is an irreducible representation of $\mathrm{SO}(n)$. In the standard model of the root system of type B_l , the highest weight of an irreducible representation of $\mathrm{SO}(n)$ has the form $\lambda_1\varepsilon_1 + \cdots + \lambda_l\varepsilon_l$ with $\lambda_1, \dots, \lambda_l \in \mathbb{N}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. We associate this weight with the partition $[\lambda_1, \dots, \lambda_l]$. In this way, the highest weight of an irreducible representation of $\mathrm{SO}(n)$ is associated to a partition with at most l parts or, equivalently, to a Young diagram of depth at most l . If D is a Young diagram then we denote its depth by $\mathrm{dp}(D)$. Suppose that D is a Young diagram with $\mathrm{dp}(D) \leq l$. We define the diagram D' to be the result of changing the length of the first column in D from $\mathrm{dp}(D)$ to $n - \mathrm{dp}(D)$. Note that $\mathrm{dp}(D') = n - \mathrm{dp}(D)$ is at least $l + 1$ and at most n . The diagram D may be recovered from D' by repeating the operation that produced D' from D . We call a diagram D *special* if $\mathrm{dp}(D) \leq n$ and if $\mathrm{dp}(D) > l$ then the k -th row in D has length 1 for all $k > n - \mathrm{dp}(D)$. Let (ρ, E_ρ) be the irreducible representation of $\mathrm{SO}(n)$ associated to the diagram D . Define the *sign* of the diagram D to be

$$\mathrm{sgn}(D) = (-1)^{\lambda_1 + \cdots + \lambda_l}.$$

We associate the diagram D to the irreducible representation $(\rho_{\mathrm{sgn}(D)}, E_{\rho_{\mathrm{sgn}(D)}})$ of $\mathrm{O}(n)$ and the diagram D' to the irreducible representation $(\rho_{-\mathrm{sgn}(D)}, E_{\rho_{-\mathrm{sgn}(D)}})$ of $\mathrm{O}(n)$. In this way, we obtain a one-to-one correspondence between isomorphism classes of irreducible representations of $\mathrm{O}(n)$ and special diagrams.

Now suppose that n is even. Then T is not abelian and two-dimensional spaces of singular vectors are possible. In the standard model of the root system of type D_l , the highest weight of an irreducible representation of $\mathrm{SO}(n)$ has the form $\lambda_1\varepsilon_1 + \cdots + \lambda_l\varepsilon_l$ with $\lambda_1, \dots, \lambda_{l-1} \in \mathbb{N}$, $\lambda_l \in \mathbb{Z}$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l-1} \geq |\lambda_l|$. We associate this weight to the partition $[\lambda_1, \lambda_2, \dots, \lambda_{l-1}, |\lambda_l|]$. The action of τ changes the sign of λ_l and so in this way we associate the $\{e, \tau\}$ -orbit of the weight to a partition with at most l parts or, equivalently, to a Young diagram of depth at most l . The operation $D \mapsto D'$ on special Young diagrams has fixed points

in this case. We have $D = D'$ precisely when $\text{dp}(D) = l$. These diagrams are those corresponding to weights that are not fixed by τ and we associate such a diagram to $(\hat{\rho}, E_{\hat{\rho}})$ where (ρ, E_{ρ}) is an irreducible representation of $\text{SO}(n)$ whose highest weight is associated with the diagram. Those diagrams D with $\text{dp}(D) < l$ correspond to weights that are fixed by τ . If we have such a diagram D and (ρ, E_{ρ}) is an irreducible representation of $\text{SO}(n)$ whose highest weight corresponds to D then we associate the representation (ρ_+, E_{ρ_+}) of $\text{O}(n)$ to the diagram D and the representation (ρ_-, E_{ρ_-}) of $\text{O}(n)$ to the diagram D' . In this way, we obtain a one-to-one correspondence between isomorphism classes of irreducible representations of $\text{O}(n)$ and special diagrams, just as in the case where n is odd.

There is a similar, but simpler, correspondence between certain irreducible representations of $\text{GL}(m)$ and Young diagrams of depth at most m . We briefly recall this because we shall need the notation below. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{gl}(m)$ and $\mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{gl}(m)$ with nilradical $\mathfrak{n}$. If (π, E_{π}) is an irreducible representation of $\text{GL}(m)$ then the space $E_{\pi}^{\mathfrak{n}}$ of singular vectors is one-dimensional and the isomorphism class of (π, E_{π}) is determined by the action of $\mathfrak{h}$ on $E_{\pi}^{\mathfrak{n}}$. The functional associated with this action is the highest weight of (π, E_{π}) . In the standard model, this weight has the form $\lambda_1 \varepsilon_1 + \cdots + \lambda_m \varepsilon_m$ with $\lambda_1, \dots, \lambda_m \in \mathbb{Z}$ and $\lambda_1 \geq \cdots \geq \lambda_m$. The representation (π, E_{π}) is said to be *polynomial* if $\lambda_m \geq 0$. If (π, E_{π}) is an irreducible polynomial representation of $\text{GL}(m)$ then we associate (π, E_{π}) to the partition $[\lambda_1, \dots, \lambda_m]$ or equivalently to the associated Young diagram.

If D is a special Young diagram then we write σ_D for the irreducible representation of $\text{O}(n)$ that is associated to it. If D is a Young diagram of depth at most m then we write $\pi_{m,D}$ for the irreducible representation of $\text{GL}(m)$ that is associated to it. The representation of $G = \text{O}(n) \times \text{GL}(m)$ that is afforded by $\mathcal{H}$ decomposes as

$$(3-1) \quad \mathcal{H} \cong \bigoplus_D \sigma_D \boxtimes \pi_{m,D},$$

where the sum is over special diagrams of depth at most m . The action of G on $F[V]$ that we have been considering is the restriction to G of the action of $\text{GL}(n) \times \text{GL}(m)$ on $F[V]$ that is induced from the action $(h_1, h_2)M = h_1^{-\top} M h_2^{-1}$ of $\text{GL}(n) \times \text{GL}(m)$ on V . The representation of $\text{GL}(n) \times \text{GL}(m)$ that is afforded by $F[V]$ decomposes as

$$(3-2) \quad F[V] \cong \bigoplus_D \pi_{n,D} \boxtimes \pi_{m,D},$$

where the sum is over diagrams of depth at most $\min(n, m)$.

We shall need some explicit information about the summand corresponding to each diagram in (3-1) and (3-2). To this end, fix $i \in F$ such that $i^2 = -1$ and

introduce new coordinates y_{jr} , $j \in [n]$, $r \in [m]$, on V . If n is even then these coordinates are defined by

$$y_{jr} = \begin{cases} x_{2j-1,r} + ix_{2j,r} & \text{if } 1 \leq j \leq n/2, \\ x_{2n+1-2j,r} - ix_{2n+2-2j,r} & \text{if } n/2 + 1 \leq j \leq n. \end{cases}$$

If n is odd then they are instead defined by

$$y_{jr} = \begin{cases} x_{2j-1,r} + ix_{2j,r} & \text{if } 1 \leq j \leq (n-1)/2, \\ \sqrt{2}x_{nr} & \text{if } j = (n+1)/2, \\ x_{2n+1-2j,r} - ix_{2n+2-2j,r} & \text{if } (n+1)/2 + 1 \leq j \leq n. \end{cases}$$

Let $\tilde{\partial}_{jr}$ be the partial derivative operator with respect to y_{jr} . In its action on $F[V]$, the Lie algebra $\mathfrak{so}(n)$ of $O(n)$ is spanned by the operators

$$Z_{jk} = \sum_{r \in [m]} (y_{jr} \tilde{\partial}_{kr} - y_{n+1-k,r} \tilde{\partial}_{n+1-j,r})$$

for $j, k \in [n]$. Note that $Z_{n+1-k, n+1-j} = -Z_{jk}$. The subalgebra $\mathfrak{t}$ of $\mathfrak{so}(n)$ spanned by $Z_{11}, \dots, Z_{ll}$ is a Cartan subalgebra. If $\mathfrak{u}$ is the subalgebra spanned by Z_{jk} with $j \leq l$ and $j < k$ then $\mathfrak{t} \oplus \mathfrak{u}$ is a Borel subalgebra of $\mathfrak{so}(n)$ with nilradical $\mathfrak{u}$. With this choice of Borel subalgebra, the element $\tau \in O(n)$ may be taken to act by $\tau y_{jr} = -y_{jr}$ for $j \in [n]$ if n is odd and by

$$\tau y_{jr} = \begin{cases} y_{jr} & \text{if } j < n/2 \text{ or } j > n/2 + 1, \\ y_{n/2+1,r} & \text{if } j = n/2, \\ y_{n/2,r} & \text{if } j = n/2 + 1, \end{cases}$$

if n is even.

Similarly, in its action on $F[V]$, the Lie algebra $\mathfrak{gl}(m)$ of $GL(m)$ is spanned by the operators

$$W_{rs} = \sum_{j \in [n]} y_{jr} \tilde{\partial}_{js} = \sum_{j \in [n]} x_{jr} \partial_{js}$$

for $r, s \in [m]$. The subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(m)$ spanned by $W_{11}, \dots, W_{mm}$ is a Cartan subalgebra. If $\mathfrak{n}$ is the subalgebra spanned by W_{rs} with $r < s$ then $\mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{gl}(m)$ with nilradical $\mathfrak{n}$.

Let $y = [y_{jr}]$ and $\delta_k(y)$ be the k -by- k leading minor of the matrix y , with the convention that a minor that is larger than the matrix from which it is drawn is to be interpreted as 0. Thus $\delta_k(y) = 0$ if and only if $k > \min(n, m)$. With this convention in mind, we eschew saying “unless it is zero” repeatedly in the following discussion. If $a_1, \dots, a_l \in \mathbb{N}$ then the monomial $\delta^a(y) = \delta_1^{a_1}(y) \cdots \delta_l^{a_l}(y)$ is a singular vector in $\mathcal{H}$. If j is the largest index such that $a_j \neq 0$ then we define

$$\eta(\delta^a(y)) = \delta_1^{a_1}(y) \cdots \delta_{j-1}^{a_{j-1}}(y) \delta_j^{a_j-1}(y) \delta_{n-j}(y),$$

with the special case $\eta(1) = \delta_n$ when all the a_j are zero. The monomial $\eta(\delta^a(y))$ is a

singular vector in $\mathcal{H}$. The monomial $\tau(\delta^a(y))$ is also a singular vector in $\mathcal{H}$. Let D be the diagram associated with the partition $[a_1 + \cdots + a_l, a_2 + \cdots + a_l, \dots, a_{l-1} + a_l, a_l]$. If n is odd then $\delta^a(y)$ spans the space of singular vectors in the summand $\sigma_D \boxtimes \pi_{m,D}$ in (3-1) corresponding to D and $\eta(\delta^a(y))$ spans the space of singular vectors in the summand corresponding to D' . If n is even then the same is true if $a_l = 0$, but if $a_l \neq 0$ then $\delta^a(y)$ and $\tau(\delta^a(y))$ span the space of singular vectors in the summand corresponding to $D = D'$.

The description of the singular vectors in the summands in (3-2) is similar but simpler. Let $\delta_k(x)$ be the k -by- k leading minor of the matrix $x = [x_{jr}]$, with the same convention as above. Let $c = \min(m, n)$ and $a_1, \dots, a_c \in \mathbb{N}$. The monomial $\delta^a(x) = \delta_1^{a_1}(x) \cdots \delta_c^{a_c}(x)$ is a singular vector in $F[V]$ and spans the space of singular vectors in the summand $\pi_{n,D} \boxtimes \pi_{m,D}$ in (3-2), where D is the diagram associated to the partition $[a_1 + \cdots + a_c, a_2 + \cdots + a_c, \dots, a_{c-1} + a_c, a_c]$.

4. Isotypic multiharmonic polynomials

In this section we initially assume that F is algebraically closed. Let $n = n_1 + n_2$. Then $V = V_1 \oplus V_2$ where V_1 is the subspace of matrices of the form $\begin{pmatrix} M^{(1)} \\ 0 \end{pmatrix}$ with $M^{(1)} \in \text{Mat}(n_1, m)$ and V_2 is the subspace of matrices of the form $\begin{pmatrix} 0 \\ M^{(2)} \end{pmatrix}$ with $M^{(2)} \in \text{Mat}(n_2, m)$. We identify V_i with $\text{Mat}(n_i, m)$ and embed $O(n_1)$ and $O(n_2)$ into $O(n)$ in the upper-left and lower-right corners, respectively. We systematically use the superscripts (1) and (2) to denote the upper and lower block matrices in this way. When this notation is being used, we number the rows of the lower block matrix by the elements of $[n_2]$, so that $M_{jr}^{(2)} = M_{n_1+j,r}$. We require the alternate coordinates $y_{jr}^{(2)}$ that are related to the coordinates $x_{jr}^{(2)}$ as described in the previous section. Let $y^{(2)} = [y_{jr}^{(2)}]$ be the matrix of these coordinates. Note that we only make this change of coordinates in the lower block, leaving the upper block unchanged.

For $0 \leq a \leq n_2/2$, we denote by ν_a the a -th exterior power of the standard representation of $O(n_2)$. In particular, ν_0 is the trivial representation of $O(n_2)$. We denote by $\mathcal{H}^{O(n_2), \nu_a}$ the ν_a -isotypic subspace of $\mathcal{H}$ as a representation of $O(n_2)$.

Theorem 4.1. *Let*

$$\mathcal{M}_a = \bigoplus_{I \subset [n_2], J \subset [m], |I|=|J|=a} F[V_1] \det(y_{IJ}^{(2)}),$$

where $y_{IJ}^{(2)}$ denotes the minor matrix of $y^{(2)}$ drawn from the rows in I and the columns in J . Then we have $\mathbf{P}(\mathcal{M}_a) = \mathcal{H}^{O(n_2), \nu_a}$. If $\min(m, n_1) + a \leq n_2$ then we have $\mathcal{M}_a \cap \mathcal{I} = \{0\}$.

Proof. Let $\mathcal{H}^{(2)}$ be the space of multiharmonic polynomials in $F[V_2]$. The diagram D associated with the representation ν_a is a column of $a \leq n_2/2$ boxes and it follows from the results recalled in Section 3 that $(\mathcal{H}^{(2)})^{O(n_2), \nu_a}$ contains the singular vector $\delta_a(y)$ and that $(\mathcal{H}^{(2)})^{O(n_2), \nu_a}$ is generated by this vector under the actions

of $\mathfrak{so}(n_2) \oplus \mathfrak{gl}(m)$ and τ . The expressions for the action of the Lie algebra and of τ given in Section 3 make it clear that by applying them repeatedly to $\delta_a(y)$ we obtain linear combinations of a -by- a minors from $y^{(2)}$ and so

$$(\mathcal{H}^{(2)})^{O(n_2), v_a} \subset \sum_{I \subset [n_2], J \subset [m], |I|=|J|=a} F \det(y_{IJ}^{(2)}).$$

On the other hand, (3-1) implies that

$$\dim(\mathcal{H}^{(2)})^{O(n_2), v_a} = \dim(\sigma_D) \dim(\pi_{m,D}) = \binom{n_2}{a} \binom{m}{a},$$

which is equal to the number of elements in the spanning set just found. Thus

$$(\mathcal{H}^{(2)})^{O(n_2), v_a} = \bigoplus_{I \subset [n_2], J \subset [m], |I|=|J|=a} F \det(y_{IJ}^{(2)}).$$

It is a general fact that

$$F[V_2]^{O(n_2), v_a} = F[V_2]^{O(n_2)} (\mathcal{H}^{(2)})^{O(n_2), v_a}.$$

The map $F[V_1] \otimes F[V_2] \rightarrow F[V]$ that satisfies $f_1 \otimes f_2 \mapsto f_1 f_2$ is an isomorphism and this and the fact that $O(n_2)$ acts trivially on $F[V_1]$ now imply that

$$(4-1) \quad F[V]^{O(n_2), v_a} = \sum_{I \subset [n_2], J \subset [m], |I|=|J|=a} F[V_1] F[V_2]^{O(n_2)} \det(y_{IJ}^{(2)}).$$

Now suppose that $h \in \mathcal{H}^{O(n_2), v_a}$. For $r, s \in [m]$ we may write $\varphi_{rs} = \varphi_{rs}^{(1)} + \varphi_{rs}^{(2)}$ with $\varphi_{rs}^{(1)} \in F[V_1]$ and $\varphi_{rs}^{(2)} \in F[V_2]$. We have $F[V_2]^{O(n_2)} = F[\varphi_{rs}^{(2)}]$ and so (4-1) implies that we may write

$$h = \sum_{I \subset [n_2], J \subset [m], |I|=|J|=a} \sum_b p_{I,J,b} (\varphi^{(2)})^b \det(y_{IJ}^{(2)}),$$

where b is a multi-index and $p_{I,J,b} \in F[V_1]$. By replacing $\varphi^{(2)}$ by $\varphi - \varphi^{(1)}$ in this expression, we obtain

$$h \equiv \sum_{I \subset [n_2], J \subset [m], |I|=|J|=a} \sum_b p_{I,J,b} (-\varphi^{(1)})^b \det(y_{IJ}^{(2)}) \pmod{\mathcal{I}}$$

and so

$$h = \mathbf{P} \left(\sum_{I \subset [n_2], J \subset [m], |I|=|J|=a} \sum_b p_{I,J,b} (-\varphi^{(1)})^b \det(y_{IJ}^{(2)}) \right) \in \mathbf{P}(\mathcal{M}_a).$$

Note that the sum defining $\mathcal{M}_a$ is direct because if we choose $I_0 \subset [n_2]$ and $J_0 \subset [m]$ with $|I_0| = |J_0| = a$ and then specialize $y^{(2)}$ in such a way that the minor matrix $y_{I_0 J_0}^{(2)}$ is the a -by- a identity matrix and all entries not forced by this condition are 0 then we obtain $\det(y_{IJ}^{(2)}) = 1$ if $I = I_0$ and $J = J_0$ and $\det(y_{IJ}^{(2)}) = 0$ otherwise.

This observation implies that the minors $\det(y_{IJ}^{(2)})$ are independent over $F[V_1]$, as required. The argument to this point shows that $\mathcal{P}(\mathcal{M}_a) \supset \mathcal{H}^{O(n_2), v_a}$. The reverse inclusion is clear and so the first claim is proved.

The minors $\det(y_{IJ}^{(2)})$ lie in the F -span of the minors $\det(x_{IJ}^{(2)})$ and conversely. It follows that these two collections of minors are F -bases for the same space and so we have

$$\mathcal{M}_a = \bigoplus_{I \subset [n_2], J \subset [m], |I|=|J|=a} F[V_1] \det(x_{IJ}^{(2)}).$$

Now suppose that $\min(m, n_1) + a \leq n_2$. We wish to show that $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ and to do so, we shall make use of the commutative algebra

$$A = F[\varepsilon_1, \dots, \varepsilon_a] / (\varepsilon_1^2, \dots, \varepsilon_a^2).$$

Suppose first that $n_1 + a \leq n_2$. Fix $I_0 \subset [n_2]$ and $J_0 \subset [m]$ of cardinality a . Let $M^{(1)} \in V_1$. We shall construct a matrix $M \in V(A)$, the A -points of V , from this data. As implied by the notation, the upper block of M will be the given matrix $M^{(1)}$. Next we choose an injection $\iota : [n_1] \rightarrow [n_2] \setminus I_0$, which is possible since $n_1 \leq n_2 - a$, and partially define the matrix $M^{(2)}$ by

$$M_{jr}^{(2)} = \begin{cases} i M_{\iota^{-1}(j), r}^{(1)} & \text{if } j \in \text{im}(\iota), \\ 0 & \text{if } j \in [n_2] \setminus I_0 \text{ but } j \notin \text{im}(\iota). \end{cases}$$

This has the effect of inserting the matrix $i M^{(1)}$ into certain rows in the matrix $M^{(2)}$. To complete the definition of $M^{(2)}$ suppose that $I_0 = \{j_1 < \dots < j_a\}$ and $J_0 = \{r_1 < \dots < r_a\}$ and define

$$M_{jr}^{(2)} = \begin{cases} \varepsilon_k & \text{if } j = j_k \text{ and } r = r_k, \\ 0 & \text{otherwise.} \end{cases}$$

This has the effect of inserting the matrix $\text{diag}(\varepsilon_1, \dots, \varepsilon_k)$ into the rows and columns of $M^{(2)}$ indexed by I_0 and J_0 . Since no two ε_k lie in the same row as one another and $\varepsilon_k^2 = 0$ for all k , we have

$$M^\top M = (M^{(1)})^\top M^{(1)} + (i M^{(1)})^\top (i M^{(1)}) = 0.$$

That is, M is an A -point of the scheme defined by the ideal $\mathcal{I}$. Now suppose that

$$f = \sum_{I, J} f_{IJ}(x^{(1)}) \det(x_{IJ}^{(2)}) \in \mathcal{M}_a \cap \mathcal{I}.$$

Then $f(M) = 0$ and so

$$(4-2) \quad \sum_{I, J} f_{IJ}(M^{(1)}) \det(M_{IJ}^{(2)}) = 0.$$

Among the minors $\det(M_{IJ}^{(2)})$, only $\det(M_{I_0 J_0}^{(2)}) = \varepsilon_1 \cdots \varepsilon_a$ involves all the quantities ε_k . The minors $\det(M_{IJ}^{(2)})$ other than this one are linear combinations of products of fewer than a of the ε_k with coefficients drawn from F . Since the products $\varepsilon_S = \prod_{k \in S} \varepsilon_k$ with $S \subset [a]$ are an F -basis for A , it follows from (4-2) that $f_{I_0 J_0}(M^{(1)}) = 0$. Since $M^{(1)} \in V_1$ was arbitrary, this implies that $f_{I_0 J_0} = 0$, and since I_0 and J_0 were arbitrary, we conclude that $f = 0$, as required. This proves the second assertion when $n_1 + a \leq n_2$.

Now assume that $m + a \leq n_2$. We shall show that $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ by the same method that we just used in the case $n_1 + a \leq n_2$, albeit that the construction of the matrix M is more elaborate in the present case. Fix $I_0 \subset [n_2]$ and $J_0 \subset [m]$ of cardinality a and let $M^{(1)} \in V_1$. Let $E < F^{n_1}$ be the column space of $M^{(1)}$. Equip F^{n_1} with the bilinear form $(v, w) = v^\top w$ and write $E = E_0 \oplus^\perp E_1$ where the form is zero on E_0 and nondegenerate on E_1 . Let $w_1, \dots, w_d$ be an ordered basis for E_1 and define $Y = [w_1 | \cdots | w_d] \in \text{Mat}(n_1, d)$. The symmetric matrix $-Y^\top Y \in \text{Mat}(d)$ is invertible and hence the associated bilinear form on F^d is equivalent to the standard bilinear form. That is, there is an invertible matrix $Z \in \text{Mat}(d)$ such that $Z^\top Z = -Y^\top Y$. Let $Z = [z_1 | \cdots | z_d]$. Note that

$$d = \dim(E_1) \leq \dim(E) \leq m \leq n_2 - a,$$

so that we may choose an injection $\iota : [d] \rightarrow [n_2] \setminus I_0$. Let $T : E \rightarrow F^n$ be the linear map that satisfies

$$T(v) = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

for $v \in E_0$ and

$$T(w_j) = \begin{pmatrix} w_j \\ u_j \end{pmatrix}$$

where $u_j \in F^{n_2}$ is defined by

$$(u_j)_k = \begin{cases} (z_j)_{\iota^{-1}(k)} & \text{if } k \in \text{im}(\iota), \\ 0 & \text{if } k \notin \text{im}(\iota). \end{cases}$$

Note that if $j_1, j_2 \in [d]$ then

$$\begin{aligned} (T(w_{j_1}))^\top T(w_{j_2}) &= w_{j_1}^\top w_{j_2} + u_{j_1}^\top u_{j_2} = w_{j_1}^\top w_{j_2} + z_{j_1}^\top z_{j_2} \\ &= (Y^\top Y)_{j_1 j_2} + (Z^\top Z)_{j_1 j_2} \\ &= 0 \end{aligned}$$

and that by the choice of E_0 , we also have $(T(v))^\top T(w) = 0$ for any $v, w \in E_0$. Suppose that $M^{(1)} = [v_1 | \cdots | v_m]$ and define $M' = [T(v_1) | \cdots | T(v_m)] \in \text{Mat}(n, m)$, which is possible since $v_j \in E$ for all $j \in [m]$. The facts that we just noted imply that $(M')^\top M' = 0$. Suppose further that $I_0 = \{j_1 < \cdots < j_a\}$ and $J_0 = \{r_1 < \cdots < r_a\}$,

and define $M'' \in \text{Mat}(n, m)$ by

$$(M'')_{jr} = \begin{cases} \varepsilon_k & \text{if } j = n_1 + j_k \text{ and } r = r_k, \\ 0 & \text{otherwise.} \end{cases}$$

If a certain row in M' is nonzero then the corresponding row in M'' is zero and it follows that $(M')^\top M'' = 0$ and $(M'')^\top M' = 0$. Also, $(M'')^\top M'' = 0$ since $\varepsilon_k^2 = 0$ for all k . Let $M = M' + M'' \in \text{Mat}(n, m)$. Observe that the upper block in M is indeed $M^{(1)}$, so this notation is legitimate. We have $M^\top M = 0$; that is, M is an A -point of the scheme defined by the ideal $\mathcal{I}$. From this point, the same argument that we used in the previous case verifies that $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ as required. This completes the proof of the second assertion. $\square$

When F is not algebraically closed, Theorem 4.1 cannot be formulated in quite the same way because the coordinates y_{jr} are not necessarily available. However, as we saw in the course of the proof, the theorem can be reformulated with $\det(x_{IJ}^{(2)})$ in place of $\det(y_{IJ}^{(2)})$ and in this form it is intelligible over a general field F of characteristic 0. The reformulation is true in the same generality. The essential observation for this conclusion is that all the objects in the statement are defined over F , so that we may deduce it by extending scalars to an algebraic closure.

Note that $\mathcal{M}_0 = F[V_1]$. When $a = 0$ the numerical criterion given in Theorem 4.1 for it to be the case that $\mathcal{M}_0 \cap \mathcal{I} = \{0\}$ is sharp. Indeed, suppose that $n_2 < \min(m, n_1)$. We wish to show that $F[V_1] \cap \mathcal{I} \neq \{0\}$. Write

$$x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}.$$

The entries in the matrix $x^\top x = (x^{(1)})^\top x^{(1)} + (x^{(2)})^\top x^{(2)}$ lie in $\mathcal{I}$ and so $(x^{(1)})^\top x^{(1)} \equiv -(x^{(2)})^\top x^{(2)} \pmod{\mathcal{I}}$. Let $k = \min(m, n_1)$. Since $n_2 < k$, the k -by- k minors in the matrix $(x^{(2)})^\top x^{(2)}$ are identically zero. Thus the k -by- k minors in the matrix $(x^{(1)})^\top x^{(1)}$ lie in $\mathcal{I}$. They also lie in $F[V_1]$ and so it suffices to see that at least one of these minors is not identically zero. This may be done by specializing $x^{(1)}$ to either $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$ or $(I_k \ 0)$, depending on whether $m \leq n_1$ or $m > n_1$. On specialization, the leading k -by- k minor of $(x^{(1)})^\top x^{(1)}$ specializes to 1 and hence this polynomial is not identically zero.

If $\mathcal{M}_0 \cap \mathcal{I} \neq \{0\}$ then $\mathcal{M}_a \cap \mathcal{I} \neq \{0\}$ for all a and so this is always the case when $n_2 < \min(m, n_1)$. By Theorem 4.1, $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ if $n_2 \geq \min(m, n_1) + a$. It remains to decide whether or not $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ in the range $\min(m, n_1) \leq n_2 < \min(m, n_1) + a$. Note that this question makes sense even without imposing the condition $a \leq n_2/2$, which played no role in the proof of the third claim in Theorem 4.1. In this range, the answer to the question of whether or not $\mathcal{M}_a \cap \mathcal{I}$ is equal to $\{0\}$ varies depending on the values of n_1 and m . The following exemplifies this and its verification illustrates some useful techniques that might contribute to a complete resolution of the problem.

Example. When $n_1 = n_2 = 2$ we have $\mathcal{M}_1 \cap \mathcal{I} = \{0\}$ if $m = 2$ and $\mathcal{M}_1 \cap \mathcal{I} \neq \{0\}$ if $m = 3$.

To make the arguments easier to follow, we give the variables more memorable names. First consider $m = 2$ and let

$$x = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \\ u_4 & v_4 \end{pmatrix}.$$

The ideal $\mathcal{I}$ is generated by $\varphi_{11} = u_1^2 + \cdots + u_4^2$, $\varphi_{12} = u_1v_1 + \cdots + u_4v_4$, and $\varphi_{22} = v_1^2 + \cdots + v_4^2$. We first show that $\mathcal{I}$ is a radical ideal. It follows from Lemma 2.1 that $\sqrt{\mathcal{I}} = \mathcal{I} \oplus (\mathcal{H} \cap \sqrt{\mathcal{I}})$ and so it suffices to show that $\mathcal{H} \cap \sqrt{\mathcal{I}} = \{0\}$. Now $\mathcal{H} \cap \sqrt{\mathcal{I}}$ is G -invariant and so if it were nonzero then it would contain some singular vector. The description of the singular vectors given in Section 3 then implies that we would have $\delta_1^{a_1} \delta_2^{a_2} \in \sqrt{\mathcal{I}}$ for some $a_1, a_2 \in \mathbb{N}$, where $\delta_1 = u_1 + iu_2$ and

$$\delta_2 = \begin{vmatrix} u_1 + iu_2 & v_1 + iv_2 \\ u_3 + iu_4 & v_3 + iv_4 \end{vmatrix}.$$

The matrix

$$N = \begin{pmatrix} 1 & -1 \\ -i & -i \\ 1 & 1 \\ i & -i \end{pmatrix}$$

lies in the variety defined by $\sqrt{\mathcal{I}}$ and we have $\delta_1(N) = 2$, $\delta_2(N) = 4$. It follows that $\mathcal{H} \cap \sqrt{\mathcal{I}} = \{0\}$ and so $\mathcal{I}$ is a radical ideal. Let $\mathbb{X} = Z(\mathcal{I})$. It is a consequence of Witt's theorem that the set of matrices of rank two in $\mathbb{X}$ is a single G -orbit. It is not, however, a single orbit under $\mathrm{SO}(4) \times \mathrm{GL}(2)$, since a totally isotropic plane in F^4 gives rise to a volume form on F^4 that is preserved by $\mathrm{SO}(4) \times \mathrm{GL}(2)$ but not by G . Thus $\mathbb{X}$ decomposes into the union of two irreducible components that are interchanged by τ , say $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$. Correspondingly, we have $\mathcal{I} = \mathcal{P}_1 \cap \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct prime ideals that are interchanged by τ . Suppose now that

$$g = f_1u_3 + f_2v_3 + f_3u_4 + f_4v_4 \in \mathcal{M}_1 \cap \mathcal{I}$$

is not 0, with $f_1, \dots, f_4 \in F[u_1, u_2, v_1, v_2]$. Since $\mathcal{I}$ is a homogeneous ideal and u_3, v_3, u_4, v_4 are homogeneous, we may assume that g and $f_1, \dots, f_4$ are homogeneous. We may further suppose that g has been chosen to have the minimal possible degree for a nonzero homogeneous element of $\mathcal{M}_1 \cap \mathcal{I}$. Suppose that $M^{(1)} \in \mathrm{Mat}(2)$ has rank at most one. Then we have $M^{(1)} = \begin{pmatrix} t p & s p \\ t q & s q \end{pmatrix}$ for some $p, q, t, s \in F$. Working

over the algebra $A = F[\varepsilon]/(\varepsilon^2)$ we set

$$M = \begin{pmatrix} tp & sp \\ tq & sq \\ \varepsilon & 0 \\ it\sqrt{p^2 + q^2} & is\sqrt{p^2 + q^2} \end{pmatrix}$$

and note that M is an A -point of the scheme defined by $\mathcal{I}$ so that $g(M) = 0$. It follows that $f_1(M^{(1)}) = 0$. We similarly conclude that $f_2(M^{(1)}) = f_3(M^{(1)}) = f_4(M^{(1)}) = 0$. That is, $f_1, \dots, f_4$ lie in the vanishing ideal of the variety of matrices of rank at most one. This ideal is the principal ideal generated by the polynomial $\psi = u_1v_2 - u_2v_1$ and so $f_1, \dots, f_4$ are divisible by ψ . We have $\psi(N) = -2i$ and $\psi(\tau(N)) = 2$ and so ψ does not vanish on either component of $\mathbb{X}$. It follows that $g/\psi \in \mathcal{M}_1 \cap \mathcal{I}$. But the homogeneous degree of g/ψ is less than that of g , contrary to assumption. Thus $\mathcal{M}_1 \cap \mathcal{I} = \{0\}$, as claimed.

Now consider $m = 3$ and let

$$x = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{pmatrix}.$$

The ideal $\mathcal{I}$ is generated by the polynomials $\varphi_{11}, \varphi_{12}, \varphi_{13}, \varphi_{22}, \varphi_{23}$, and φ_{33} , where φ_{rs} is the dot product of the r -th column of x with the s -th column. Let

$$\delta = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

and note that δ is also the determinant of the transpose matrix. Thus

$$(4-3) \quad \delta^2 = \begin{vmatrix} \varphi_{11} - u_4^2 & \varphi_{12} - u_4v_4 & \varphi_{13} - u_4w_4 \\ \varphi_{12} - u_4v_4 & \varphi_{22} - v_4^2 & \varphi_{23} - v_4w_4 \\ \varphi_{13} - u_4w_4 & \varphi_{23} - v_4w_4 & \varphi_{33} - w_4^2 \end{vmatrix}.$$

Now the matrix

$$\begin{pmatrix} u_4^2 & u_4v_4 & u_4w_4 \\ u_4v_4 & v_4^2 & v_4w_4 \\ u_4w_4 & v_4w_4 & w_4^2 \end{pmatrix}$$

has rank one and so when we expand (4-3) we obtain

$$(4-4) \quad \delta^2 = \begin{vmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{12} & \varphi_{22} & \varphi_{23} \\ \varphi_{13} & \varphi_{23} & \varphi_{33} \end{vmatrix} - \begin{vmatrix} u_4^2 & \varphi_{12} & \varphi_{13} \\ u_4v_4 & \varphi_{22} & \varphi_{23} \\ u_4w_4 & \varphi_{23} & \varphi_{33} \end{vmatrix} - \begin{vmatrix} \varphi_{11} & u_4v_4 & \varphi_{13} \\ \varphi_{12} & v_4^2 & \varphi_{23} \\ \varphi_{13} & v_4w_4 & \varphi_{33} \end{vmatrix} - \begin{vmatrix} \varphi_{11} & \varphi_{12} & u_4w_4 \\ \varphi_{12} & \varphi_{22} & v_4w_4 \\ \varphi_{13} & \varphi_{23} & w_4^2 \end{vmatrix},$$

a combination of determinants each of which has at least two columns from $\mathcal{I}$. We

now differentiate both sides of (4-4) with respect to w_3 and divide by 2. On the left-hand side we obtain

$$g = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \cdot \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

and on the right-hand side we obtain a linear combination of nine determinants each of which has at least one column from $\mathcal{I}$. It follows that $g \in \mathcal{I}$. On the other hand, expanding δ along the third row shows that $g \in F[V_1]u_3 + F[V_1]v_3 + F[V_1]w_3 \subset \mathcal{M}_1$. Thus $g \in \mathcal{M}_1 \cap \mathcal{I}$ and so $\mathcal{M}_1 \cap \mathcal{I} \neq \{0\}$, as claimed. This completes the verification of the example. We remark that δ itself does not lie in $\mathcal{I}$ in this case, whereas δ^2 does, and so $\mathcal{I}$ is not a radical ideal.

5. Gelbart–Helgason reciprocity

In this section, we assume that F is algebraically closed. Given $\lambda_1, \dots, \lambda_m \in \mathbb{N}$, we will write $F[V](\lambda_1, \dots, \lambda_m)$ for the subspace of $F[V]$ consisting of polynomials that are column-multihomogeneous of multidegree $\lambda_1, \dots, \lambda_m$; that is, those polynomials φ that satisfy

$$\varphi(t_1 M_1 | \cdots | t_m M_m) = t_1^{\lambda_1} \cdots t_m^{\lambda_m} \varphi(M)$$

for all $M = [M_1 | \cdots | M_m] \in V$. Recall that in its action on $F[V]$, $\mathfrak{gl}(m)$ is spanned by the operators

$$W_{rs} = \sum_{j \in [n]} x_{jr} \partial_{js},$$

that $\mathfrak{h}$ denotes the span of $W_{11}, \dots, W_{mm}$ and that $\mathfrak{n}$ denotes the span of W_{rs} with $r < s$. The operator W_{rr} is the Euler operator associated with the r -th column and so $\varphi \in F[V](\lambda_1, \dots, \lambda_m)$ if and only if φ is an $\mathfrak{h}$ -weight vector with weight $\lambda_1 \varepsilon_1 + \cdots + \lambda_m \varepsilon_m$. We write $\mathcal{H}(\lambda_1, \dots, \lambda_m) = F[V](\lambda_1, \dots, \lambda_m) \cap \mathcal{H}$ and extend the notation similarly to other subspaces of $F[V]$.

Lemma 5.1. *Let D be a special diagram of depth m with associated partition $[\lambda_1, \dots, \lambda_m]$. Then the representation of $O(n)$ afforded by $\mathcal{H}^n(\lambda_1, \dots, \lambda_m)$ is isomorphic to σ_D .*

Proof. The elements of $\mathcal{H}^n(\lambda_1, \dots, \lambda_m)$ are singular vectors which have weight $\lambda_1 \varepsilon_1 + \cdots + \lambda_m \varepsilon_m$. The only irreducible representation $GL(m)$ that contains such vectors is $\pi_{m,D}$ and $\pi_{m,D}^n(\lambda_1, \dots, \lambda_m)$ is one-dimensional. Given this observation, the assertion follows from (3-1). $\square$

Theorem 5.2. *Suppose that $0 \leq a \leq n_2/2$ and that D is a special diagram. Let $\mathbb{S}_a(D)$ denote the set of all diagrams of depth at most n_1 that may be obtained*

from D by deleting a total of a boxes, no two in the same row. Then, as an $O(n_1)$ -representation,

$$\mathrm{Hom}_{O(n_2)}(\nu_a, \sigma_D^{O(n_2), \nu_a})$$

is isomorphic to a quotient of

$$(5-1) \quad \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}_{O(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E}).$$

If $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ then $\mathrm{Hom}_{O(n_2)}(\nu_a, \sigma_D^{O(n_2), \nu_a})$ is isomorphic to (5-1). In particular, this holds if $\min(\mathrm{dp}(D), n_1) + a \leq n_2$.

Proof. Let $m = \mathrm{dp}(D)$ and $[\lambda_1, \dots, \lambda_m]$ be the associated partition. We use the model of σ_D provided by Lemma 5.1 to compute the ν_a -isotype in σ_D under $O(n_2)$. The result is that

$$(5-2) \quad \sigma_D^{O(n_2), \nu_a} = (\mathcal{H}^n(\lambda_1, \dots, \lambda_m))^{O(n_2), \nu_a} = (\mathcal{H}^{O(n_2), \nu_a})^n(\lambda_1, \dots, \lambda_m),$$

because the actions of $O(n_2)$ and $\mathfrak{gl}(m)$ commute. By Theorem 4.1, this space is the image under the G -equivariant map $\mathbf{P}$ of the space $\mathcal{M}_a^n(\lambda_1, \dots, \lambda_m)$. The kernel of this map is the space $(\mathcal{M}_a \cap \mathcal{I})^n(\lambda_1, \dots, \lambda_m)$. As a representation of $\mathrm{GL}(n_1) \times \mathrm{GL}(m) \times O(n_2)$ we have

$$\begin{aligned} \mathcal{M}_a &\cong (F[V_1] \otimes \eta_a) \boxtimes \nu_a \\ &\cong \bigoplus_{E_1} \pi_{n_1, E_1} \boxtimes (\pi_{m, E_1} \otimes \eta_a) \boxtimes \nu_a \\ &\cong \bigoplus_{E_1} \bigoplus_{E_2} \pi_{n_1, E_1} \boxtimes \pi_{m, E_2} \boxtimes \nu_a, \end{aligned}$$

where η_a denotes the a -th exterior power of the standard representation of $\mathrm{GL}(n_2)$, E_1 runs over diagrams of depth at most $\min(m, n_1)$ and, for each E_1 , E_2 runs over diagrams that may be obtained from E_1 by adding a total of a boxes to E_1 , with no two in the same row. The first isomorphism relies on the observation that as a representation of $\mathrm{GL}(m) \times O(n_2)$ the space spanned by the minors $\det(y_{IJ}^{(2)})$ is isomorphic to $\eta_a \boxtimes \nu_a$. The second isomorphism follows from (3-2) applied to V_1 . The third isomorphism is a consequence of Pieri's rule. We now reverse the order of the sum in the above isomorphism to obtain

$$(5-3) \quad \mathcal{M}_a \cong \bigoplus_{E_2} \bigoplus_{E_1} \pi_{n_1, E_1} \boxtimes \pi_{m, E_2} \boxtimes \nu_a,$$

where E_2 runs over diagrams of depth at most m and, for each E_2 , E_1 runs over diagrams of depth at most n_1 that may be obtained from E_2 by removing a total of a boxes, with no two in the same row. It is important to note that a given diagram E_1 in this sum can arise from a given diagram E_2 in only one way; we simply decide

what set of rows to remove a box from. Similarly, a given diagram E_2 in the reversed sum can arise from a given diagram E_1 in only one way; we decide what set of rows to add a box to. This means that no multiplicity arises when we interchange the order of the sum. The isomorphism (5-3) implies that as a representation of $\mathrm{GL}(n_1) \times \mathrm{O}(n_2)$,

$$\mathcal{M}_a^n(\lambda_1, \dots, \lambda_m) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \pi_{n_1, E} \boxtimes v_a$$

since

$$(\pi_{m, E_2})^n(\lambda_1, \dots, \lambda_m) = \begin{cases} F & \text{if } E_2 = D, \\ \{0\} & \text{otherwise.} \end{cases}$$

Since $\mathbf{P}$ is $(\mathrm{O}(n_1) \times \mathrm{O}(n_2))$ -equivariant and the ideal $\mathcal{I}$ is $(\mathrm{O}(n_1) \times \mathrm{O}(n_2))$ -invariant, the preceding calculations imply that there is a short exact sequence

$$\{0\} \longrightarrow (\mathcal{M}_a \cap \mathcal{I})^n(\lambda_1, \dots, \lambda_m) \longrightarrow \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}_{\mathrm{O}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E}) \boxtimes v_a \longrightarrow \sigma_D^{\mathrm{O}(n_2), v_a} \longrightarrow \{0\}$$

of $(\mathrm{O}(n_1) \times \mathrm{O}(n_2))$ -representations. By applying the functor $\mathrm{Hom}_{\mathrm{O}(n_2)}(v_a, \cdot)$ to this short exact sequence we obtain the first and second conclusions. Given this, the third conclusion follows directly from Theorem 4.1. This completes the proof. $\square$

Lemma 5.3. *If ρ is a representation of $\mathrm{SO}(n)$ and v is a representation of $\mathrm{O}(n_2)$ then we have*

$$\mathrm{Hom}_{\mathrm{O}(n_2)}(v, \mathrm{res}_{\mathrm{O}(n_2)}^{\mathrm{O}(n)}(\mathrm{ind}_{\mathrm{SO}(n)}^{\mathrm{O}(n)}(\rho))) \cong \mathrm{Hom}_{\mathrm{SO}(n_2)}(\mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{O}(n_2)}(v), \mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{SO}(n)}(\rho)).$$

Proof. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}(\mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{SO}(n)}(\rho), \mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{O}(n_2)}(v)) \cong \mathrm{Hom}_{\mathrm{O}(n_2)}(\mathrm{ind}_{\mathrm{SO}(n_2)}^{\mathrm{O}(n_2)}(\mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{SO}(n)}(\rho)), v).$$

The double coset space $\mathrm{O}(n_2) \backslash \mathrm{O}(n) / \mathrm{SO}(n)$ is a singleton and $\mathrm{O}(n_2) \cap \mathrm{SO}(n) = \mathrm{SO}(n_2)$. Thus Mackey's induction-restriction theorem gives

$$\mathrm{ind}_{\mathrm{SO}(n_2)}^{\mathrm{O}(n_2)}(\mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{SO}(n)}(\rho)) \cong \mathrm{res}_{\mathrm{O}(n_2)}^{\mathrm{O}(n)}(\mathrm{ind}_{\mathrm{SO}(n)}^{\mathrm{O}(n)}(\rho)).$$

From this we obtain

$$\mathrm{Hom}_{\mathrm{O}(n_2)}(\mathrm{res}_{\mathrm{O}(n_2)}^{\mathrm{O}(n)}(\mathrm{ind}_{\mathrm{SO}(n)}^{\mathrm{O}(n)}(\rho)), v) \cong \mathrm{Hom}_{\mathrm{SO}(n_2)}(\mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{SO}(n)}(\rho), \mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{O}(n_2)}(v)).$$

Since all the representations we are considering are finite-dimensional, we obtain the required isomorphism by applying this to the contragredients of ρ and v . $\square$

In what follows, by abuse of notation, we shall denote the restriction of v_a to $\mathrm{SO}(n_2)$ by the same symbol. Recall that this restriction is irreducible if $a < n_2/2$ but decomposes as $v_a^+ \oplus v_a^-$ if $a = n_2/2$, where $v_{n_2/2}^+$ has highest weight $\varepsilon_1 + \dots + \varepsilon_{n_2/2}$ and $v_{n_2/2}^-$ has highest weight $\varepsilon_1 + \dots + \varepsilon_{n_2/2-1} - \varepsilon_{n_2/2}$.

Theorem 5.4. *Suppose that $0 \leq a \leq n_2/2$ and $n_1 + a \leq n_2$. Let (ρ, E_ρ) be an irreducible representation of $\mathrm{SO}(n)$ and D the associated diagram. If $\mathrm{dp}(D) < n/2$ then*

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}(v_a, \mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{SO}(n)}(\rho)) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}_{\mathrm{SO}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E}) \oplus \bigoplus_{E \in \mathbb{S}_a(D')} \mathrm{res}_{\mathrm{SO}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E})$$

as representations of $\mathrm{SO}(n_1)$. If $\mathrm{dp}(D) = n/2$ then

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}(v_a, \mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{SO}(n)}(\rho)) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}_{\mathrm{SO}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E})$$

as representations of $\mathrm{SO}(n_1)$.

Proof. First suppose that $\mathrm{dp}(D) < n/2$. Then $\rho^\tau \cong \rho$ and so

$$\mathrm{ind}_{\mathrm{SO}(n)}^{\mathrm{O}(n)}(\rho) \cong \sigma_D \oplus \sigma_{D'}.$$

Lemma 5.3 then implies that

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}(v_a, \mathrm{res}_{\mathrm{SO}(n_2)}^{\mathrm{SO}(n)}(\rho)) \cong \mathrm{Hom}_{\mathrm{O}(n_2)}(v_a, \mathrm{res}_{\mathrm{O}(n_2)}^{\mathrm{O}(n)}(\sigma_D \oplus \sigma_{D'})).$$

Moreover, the naturality of the isomorphisms used to prove Lemma 5.3 implies that this isomorphism respects the $\mathrm{SO}(n_1)$ actions. Since $n_1 + a \leq n_2$, Theorem 5.2 gives

$$\mathrm{Hom}_{\mathrm{O}(n_2)}(v_a, \mathrm{res}_{\mathrm{O}(n_2)}^{\mathrm{O}(n)}(\sigma_D \oplus \sigma_{D'})) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}_{\mathrm{O}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E}) \oplus \bigoplus_{E \in \mathbb{S}_a(D')} \mathrm{res}_{\mathrm{O}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E})$$

as representations of $\mathrm{O}(n_1)$ and hence

$$\mathrm{Hom}_{\mathrm{O}(n_2)}(v_a, \mathrm{res}_{\mathrm{O}(n_2)}^{\mathrm{O}(n)}(\sigma_D \oplus \sigma_{D'})) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}_{\mathrm{SO}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E}) \oplus \bigoplus_{E \in \mathbb{S}_a(D')} \mathrm{res}_{\mathrm{SO}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1, E})$$

as representations of $\mathrm{SO}(n_1)$, as required. Now suppose that $\mathrm{dp}(D) = n/2$, then $\rho^\tau \not\cong \rho$ and so

$$\mathrm{ind}_{\mathrm{SO}(n)}^{\mathrm{O}(n)}(\rho) \cong \sigma_D.$$

Given this, the proof is essentially the same as in the previous case. $\square$

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ANTHONY C. KABLE
DEPARTMENT OF MATHEMATICS
OKLAHOMA STATE UNIVERSITY
STILLWATER, OK
UNITED STATES
akable@math.okstate.edu

TWO APPLICATIONS OF THE INTEGRAL REGULATOR

MATT KERR AND MUXI LI

We review Li’s refinement of the KLM regulator map, and use it to detect torsion phenomena in higher Chow groups of number fields and families of elliptic curves.

1. Introduction

Let X be a smooth projective variety defined over a subfield of $\mathbb{C}$. The KLM formula is a morphism of complexes inducing the Bloch–Beilinson regulator map

$$\mathrm{Gr}_Y^p K_n^{\mathrm{alg}}(X)_{\mathbb{Q}} \cong H_{\mathcal{M}}^{2p-n}(X, \mathbb{Q}(p)) \rightarrow H_{\mathcal{D}}^{2p-n}(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}(p))$$

with rational coefficients, developed by the first author together with J. Lewis and S. Müller-Stach [Kerr 2003b; Kerr et al. 2006; Kerr and Lewis 2007] (see Section 3). It works by representing motivic cohomology by cycles in Bloch’s higher Chow complex (Section 2), and associating explicit currents on $X_{\mathbb{C}}^{\mathrm{an}}$ to them. The second author’s refinement now enables the direct computation of the *integral* regulator

$$H_{\mathcal{M}}^{2p-n}(X, \mathbb{Z}(p)) \rightarrow H_{\mathcal{D}}^{2p-n}(X, \mathbb{Z}(p))$$

on the level of higher Chow complexes [Li 2018].

The aim of this note is to offer a brief review of Li’s construction (Section 4) and give some first examples of how it may be used: First, we demonstrate how to find explicit torsion generators in higher Chow groups of number fields (Section 5); the results are summarized in Theorem 5.3. We also apply the formula (in Section 6) to *integrally* calculate a branch of the higher normal function arising from the mirror of local $\mathbb{P}^2$ (Proposition 6.4), along the way computing some integral regulator periods for curves of any genus (Theorem 6.2).

2. Higher Chow groups

Invented by Spencer Bloch [1986a; 1994] in the mid-1980s to geometrize Quillen’s higher algebraic K -theory, these generalize the usual Chow groups of cycles modulo

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rational equivalence (the $n = 0$ case). In particular, for X smooth quasiprojective over a field k , they satisfy¹

$$\mathrm{CH}^p(X, n)_{\mathbb{Q}} \cong \mathrm{Gr}_{\mathcal{V}}^p K_n^{\mathrm{alg}}(X)_{\mathbb{Q}}.$$

For such X (and no restriction on k), Voevodsky [2002] proved that they are *integrally* isomorphic to his motivic cohomology groups:

$$\mathrm{CH}^p(X, n) \cong H_{\mathcal{M}}^{2p-n}(X, \mathbb{Z}(p)).$$

Beyond their role in arithmetic geometry (e.g., Beilinson’s conjectures [1984]), they have recently shown up in several branches of physics (e.g., quantum field theory [Bloch et al. 2015] and topological string theory [del Ángel R. et al. 2019]) and mirror symmetry [Doran and Kerr 2014; Bloch et al. 2017]. We focus on the cubical presentation of $\mathrm{CH}^p(X, n)$ as the n -th homology of a complex of *higher Chow precycles* [Levine 1994]

$$\cdots \rightarrow Z^p(X, n+1) \xrightarrow{\partial} Z^p(X, n) \xrightarrow{\partial} Z^p(X, n-1) \rightarrow \cdots$$

or its (integrally quasi-isomorphic) subcomplex of *normalized precycles* [Bloch 2004]

$$\cdots \rightarrow N^p(X, n+1) \xrightarrow{\partial} N^p(X, n) \xrightarrow{\partial} N^p(X, n-1) \rightarrow \cdots.$$

A *higher Chow cycle* is an element of $\ker(\partial)$. Roughly speaking, these are relative codimension- p cycles on

$$(X \times \mathbb{A}^n, X \times \cup \mathbb{A}^{n-1}),$$

where the $\mathbb{A}^{n-1}$ ’s are inserted into $\mathbb{A}^n$ as a “cubical” configuration of hyperplanes. More precisely, writing $\{z_1, \dots, z_n\}$ for coordinates, $(z_i) = (z_i)_0 - (z_i)_{\infty}$ for their divisors and $|\cdot|$ for support, the *algebraic n -cube* is defined by

$$\square^n := (\mathbb{P}^1 \setminus \{1\})^n \supset \partial \square^n := \bigcup_i |(z_i)|.$$

Any component of $X \times \partial \square^n$ is called a *facet* of $X \times \square^n$, and a *face* is any intersection of facets. Now set²

$$\begin{aligned} c^p(X, n) &:= \{\text{cycles in } Z^p(X \times \square^n) \text{ meeting faces of } X \times \partial \square^n \text{ properly,} \\ &\quad \text{i.e., in the expected codimension}\}, \\ d^p(X, n) &:= \sum_i \{\text{cycles in } c^p(X, n) \text{ which are constant in } z_i\}, \end{aligned}$$

¹The subscript $\mathbb{Q}$ denotes $\otimes \mathbb{Q}$.

²Normalized precycles may be represented (in $Z^p(X, n)$) by Z satisfying $Z \cdot \{z_i = 0\} = 0$ ($\forall i$) and $Z \cdot \{z_i = \infty\} = 0$ ($i < n$) simply by adding an element of $d^p(X, n)$.

$$Z^P(X, n) := c^P(X, n)/d^P(X, n),$$

$$N^P(X, n) := \{Z \mid Z \cdot (z_i)_0 = 0 \ (\forall i) \text{ and } Z \cdot (z_i)_\infty = 0 \ (\forall i < n)\},$$

and for $Z \in Z^P(X, n)$ or $N^P(X, n)$,

$$\partial Z := \sum_{i=1}^n (-1)^i (Z \cdot (z_i)_\infty - Z \cdot (z_i)_0).$$

If $X = \text{Spec}(k)$, write $Z^P(k, n)$ etc. for short.

Example 2.1. For any positive integer ℓ , let ζ_ℓ denote a primitive ℓ -th root of unity. Parametrize a cycle in $N^2(\mathbb{Q}(\zeta_\ell), 3)$ by $t \in \mathbb{P}^1$:

$$Z_\ell^2 := \left(1 - \frac{\zeta_\ell}{t}, 1 - t, t^{-\ell}\right).$$

Intersections with facets $\{z_i = 0, \infty\}$ are given by $t = 0, 1, \zeta_\ell, \infty$. But all these intersections have some $z_j = 1$, so are trivial (as $1 \notin \square$). We also record the cycle

$$\mathcal{Z}_5^2 := Z_1^2 + \left(1 - \frac{\zeta_5}{t}, 1 - t, t^{-5}\right) + \left(1 - \frac{\bar{\zeta}_5}{t}, 1 - t, t^{-5}\right)$$

in $N^2(\mathbb{Q}(\sqrt{5}), 3)$ for later reference.

3. Abel–Jacobi maps

These simultaneously generalize two classical invariants:³

(1) Griffiths’ AJ map [1969]

$$\text{CH}^p(X, 0) \rightarrow H_{\mathcal{D}}^{2p}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}(p))$$

for X smooth projective over $\mathbb{C}$; and

(2) Borel’s regulator map [Borel 1977; Burgos Gil 2002]

$$\text{CH}^p(k, 2p - 1) \rightarrow \mathbb{C}/\mathbb{Z}(p)$$

for $k \subset \mathbb{C}$ a number field.

Defined abstractly by Bloch [1986b], they map higher Chow groups to Deligne cohomology:

$$\text{CH}^p(X, n) \xrightarrow{\text{AJ}_X^{p,n}} H_{\mathcal{D}}^{2p-n}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}(p)).$$

³The versions given here are, of course, not the original ones: (1) assembles Griffiths’ map (to the intermediate Jacobian $J^p(X)$) and the fundamental class map (to the Hodge classes $\text{Hg}^p(X)$) into a single invariant; while (2) is an integral lift of the composition $\text{CH}^p(k, 2p - 1) \rightarrow \text{CH}^p(k, 2p - 1)_{\mathbb{Q}} \cong \text{Gr}_Y^p K_{2p-1}(k)_{\mathbb{Q}} \xrightarrow{r_{\text{Bo}}} \mathbb{R} \cong \mathbb{C}/\mathbb{Z}(p)$.

Here we are taking X to be smooth projective, and defined over a subfield of $\mathbb{C}$. (In the smooth quasiprojective setting, the correct target is the absolute Hodge cohomology $H_{\mathcal{H}}^{2p-n}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}(p))$ [Kerr and Lewis 2007, §2].)

Kerr, Lewis, and Müller-Stach [Kerr et al. 2006] constructed a morphism of complexes, from a subcomplex of $Z^p(X, -\bullet)$ to a complex defining Deligne cohomology, which induces $\text{AJ}_X^{p,n}$. Writing C_{sing}^k for singular chains of real codimension k , D^k for currents of degree k , and $F^\bullet$ for the Hodge filtration, their morphism takes the form

$$\begin{aligned} \widetilde{\text{AJ}}_{\text{KLM}}^{p,-\bullet} : Z_{\mathbb{R}}^p(X, -\bullet) &\rightarrow C_{\mathcal{D}}^{2p+\bullet}(X, \mathbb{Z}(p)) \\ &:= C_{\text{sing}}^{2p+\bullet}(X; \mathbb{Z}(p)) \oplus F^p D^{2p+\bullet}(X) \oplus D^{2p-1+\bullet}(X), \end{aligned}$$

with differential $D_{\mathcal{D}}(\alpha, \beta, \gamma) = (-\partial\alpha, -d\beta, d\gamma - \beta + \alpha)$ on the right. For $Z \in Z_{\mathbb{R}}^p(X, n)$ with projections π_1 (to $\square^n$) and π_2 (to X), they set

$$\begin{aligned} \widetilde{\text{AJ}}_{\text{KLM}}^{p,n}(Z) &:= (2\pi i)^{p-n}((2\pi i)^n T_Z, \Omega_Z, R_Z) \\ &:= (2\pi i)^{p-n}(\pi_2)_*(\pi_1)^*((2\pi i)^n T_n, \Omega_n, R_n), \end{aligned}$$

where $T_n := \bigcap_{i=1}^n T_{z_i} = \mathbb{R}_{\leq 0}^{\times n}$, $\Omega_n := dz_1/z_1 \wedge \cdots \wedge dz_n/z_n$, and R_n is defined inductively by

$$R_n(z_1, \dots, z_n) := \log(z_1) \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n} - (-1)^n (2\pi i) R_{n-1}(z_2, \dots, z_n) \cdot \delta_{T_{z_1}}.$$

Here $\log(z_i)$ has a branch cut along $T_{z_i} = \{z_i \in \mathbb{R}_{<0}\}$, which is regarded as a 1-cochain on $\square^n$, oriented so that $\partial T_{z_i} = (z_i)_0 - (z_i)_\infty$; and $\delta_{T_{z_i}}$ is the current of integration along it. (Note that $\widetilde{\text{AJ}}_{\text{KLM}}^{p,n}$ vanishes identically on $d^p(X, n)$.)

The subcomplex $Z_{\mathbb{R}}^p(X, -\bullet) \subset Z^p(X, -\bullet)$ consists of cycles Z for which $Z_{\mathbb{C}}^{\text{an}}$ properly intersects certain combinations⁴ of $\{T_{z_i}\}$ and $\{|(z_j)|\}$; this is necessary in order for T_Z and R_Z to be well-defined. We call such precycles $\mathbb{R}$ -proper. Kerr and Lewis [2007] proved the inclusion is a *rational* quasi-isomorphism, by appealing to Kleiman transversality in K -theory. Unfortunately, the claimed integral moving lemma in [Kerr et al. 2006] (which would have made this quasi-isomorphism integral) was incorrect, and [Kerr and Lewis 2007] was only written after a prolonged and unsuccessful effort to repair the integral version.

Now suppose we have a cycle $Z \in \ker(\partial) \subset Z_{\mathbb{R}}^p(X, n)$ with

$$[\widetilde{\text{AJ}}_{\text{KLM}}^{p,n}(Z)] \in H_{\mathcal{D}}^{2p-n}(X, \mathbb{Z}(p))$$

torsion of order exactly M . This implies $[Z] \in H_n\{Z_{\mathbb{R}}^p(X, \bullet)\}$ is at least of this order. But for $[Z] \in H_n\{Z^p(X, \bullet)\} = \text{CH}^p(X, n)$, it implies no such thing: there could be

⁴Namely, for each $J \subset \{1, \dots, n\}$ and $k \in \{0, \dots, n\}$, set $I := \{1, \dots, k\} \cap (\{1, \dots, n\} \setminus J)$; then we require that $Z_{\mathbb{C}}^{\text{an}}$ meet $(\bigcap_{i \in I} \{T_{z_i}\}) \cap (\bigcap_{j \in J} |(z_j)|)$ properly.

a $W \in Z^p(X, n+1) \setminus Z_{\mathbb{R}}^p(X, n+1)$ with $\partial W = Z$. So the KLM map only induces a homomorphism

$$\mathrm{AJ}_{\mathbb{Q}}^{p,n} : \mathrm{CH}^p(X, n) \rightarrow H_{\mathcal{D}}^{2p-n}(X, \mathbb{Q}(p))$$

consistent with Bloch's $\mathrm{AJ}^{p,n}$. This is frustrating, as the KLM formulas appear to be well-adapted to detecting torsion!

For $X = \mathrm{Spec}(k)$ and $(p, n) = (2, 3)$, consider the portion

$$\begin{array}{ccccccc} \cdots & \rightarrow & Z_{\mathbb{R}}^2(k, 4) & \rightarrow & Z_{\mathbb{R}}^2(k, 3) & \rightarrow & Z_{\mathbb{R}}^2(k, 2) \rightarrow \cdots \\ & & \downarrow (2\pi i)^2 T_4 \cap (\cdot) & & \downarrow \frac{1}{2\pi i} \int_{(\cdot)} R_{2p-1} & & \downarrow 0 \\ \cdots & \longrightarrow & \mathbb{Z}(2) & \hookrightarrow & \mathbb{C} & \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

of the KLM map of complexes. We want to use the middle map to detect torsion. Denote its image on a cycle Z by $\mathcal{R}(Z) \in \mathbb{C}/\mathbb{Z}(2)$.

Example 3.1 [Petras 2009]. We calculate

$$\begin{aligned} \mathcal{R}(Z_{\ell}^2) &= \frac{1}{2\pi i} \int_{Z_{\ell}^2} R_3 \\ &= \frac{1}{2\pi i} \int_{Z_{\ell}^2} \left(\log(z_1) dz_2/z_2 \wedge dz_3/z_3 + (2\pi i) \log(z_2) dz_3/z_3 \cdot \delta_{T_{z_1}} \right. \\ &\quad \left. + (2\pi i)^2 \log(z_3) \delta_{T_{z_1} \cap T_{z_2}} \right) \\ &= \int_{Z_{\ell} \cap T_{z_1}} \log(z_2) \frac{dz_3}{z_3} = - \int_{T_{1-\zeta_{\ell}/t}} \log(1-t) \frac{dt}{t} \\ &= - \int_0^{\zeta_{\ell}} \log(1-t) \frac{dt}{t} = \mathrm{Li}_2(\zeta_{\ell}). \end{aligned}$$

For $\ell = 1$, this is $\pi^2/6 \in \mathbb{C}/\mathbb{Z}(2)$, which is 24-torsion, while (for the second cycle of Example 2.1) $\mathcal{R}(\mathcal{X}_5^2) = \mathrm{Li}_2(1) + \mathrm{Li}_2(\zeta_5) + \mathrm{Li}_2(\bar{\zeta}_5) = 7\pi^2/30$ is 120-torsion. To *deduce* that these orders of torsion exist in $\mathrm{CH}^2(\mathbb{Q}, 3)$ and $\mathrm{CH}^2(\mathbb{Q}(\sqrt{5}), 3)$, rather than just in $H_3\{Z_{\mathbb{R}}^2(k, \bullet)\}$, we need an improvement in technology.

4. The integral regulator

Turning to potential strategies for effecting this improvement, one can probably rule out:

(1) proving an integral moving lemma ($Z_{\mathbb{R}}^p(X, \bullet) \xrightarrow{\simeq} Z^p(X, \bullet)$),

and

(2) extending KLM to a map of complexes on $Z^p(X, \bullet)$,

as too naive (in view of the history recounted in Section 3); and while

- (3) extending KLM to an infinite family of homotopic maps on nested subcomplexes with union $Z^p(X, \bullet)$,

seemed promising, what ultimately worked was

- (4) extending KLM to an infinite family of homotopic maps on nested subcomplexes with union $N^p(X, \bullet)$.

The heuristic idea of (3) was to perturb the branch cuts $T_{z_i} = \{z_i \in \mathbb{R}_{<0}\}$ in $\log(z_i)$ to $T_{z_i}^\epsilon = \{z_i/e^{i\epsilon} \in \mathbb{R}_{<0}\}$ (for small $\epsilon \in \mathbb{R}_{>0}$) and take a limit as $\epsilon \rightarrow 0$, an approach that had been successfully applied in [Kerr 2003a, §9]. (We write $\log^\epsilon(z_i)$ for the branch of logarithm along $T_{z_i}^\epsilon$.) Unfortunately, there are cycles in $Z^2(\mathbb{C}, 3)$ whose intersection with $T_{z_1}^\epsilon \cap T_{z_2}^\epsilon \cap T_{z_3}^\epsilon$ is improper for every real ϵ near 0 [Li 2018, §3]. So we need to deform the branches by distinct $\{\epsilon_i\}$ (viz., $T_{z_1}^{\epsilon_1} \cap T_{z_2}^{\epsilon_2} \cap T_{z_3}^{\epsilon_3}$); but then we cannot expect a morphism of complexes (or “limit” thereof) on $Z^p(X, \bullet)$. This forces us into strategy (4), and working with normalized subcomplexes.

Let $\mathcal{B}_\varepsilon$ denote the set of infinite sequences $\underline{\varepsilon} = \{\varepsilon_i\}_{i>0}$, with

$$0 < \varepsilon_1 < \varepsilon, \quad 0 < \varepsilon_2 < e^{-1/\varepsilon_1}, \quad 0 < \varepsilon_3 < e^{-1/\varepsilon_2}, \quad \text{etc.},$$

so that when $\varepsilon \rightarrow 0$ its projection to any $(S^1)^n$ via $(e^{i\varepsilon_1}, \dots, e^{i\varepsilon_n})$ eventually avoids any given analytic subvariety. Let $N_\varepsilon^p(X, \bullet) \subset N^p(X, \bullet)$ denote the subcomplexes of cycles Z with Z^{an} properly intersecting (for each $\underline{\varepsilon} \in \mathcal{B}_\varepsilon$) certain⁵ combinations of $\{T_{z_i}^{\varepsilon_i}\}$ and $\{(z_j)_0, (z_j)_\infty\}$. Since for $\varepsilon' < \varepsilon$ we have $\mathcal{B}_{\varepsilon'} \subset \mathcal{B}_\varepsilon$, we also have $N_{\varepsilon'}^p(X, \bullet) \supset N_\varepsilon^p(X, \bullet)$.

Lemma 4.1 [Li 2018, Theorems 4.2 and 7.2]. *We have*

$$\bigcup_{\varepsilon>0} N_\varepsilon^p(X, n) = N^p(X, n) \quad \text{for all } n$$

and

$$\lim_{\varepsilon \rightarrow 0} H_n(N_\varepsilon^p(X, \bullet)) \cong H_n(N^p(X, \bullet)) \cong \text{CH}^p(X, n).$$

For any $\underline{\varepsilon} \in \mathcal{B}_\varepsilon$, replacing T_{z_i} by $T_{z_i}^{\varepsilon_i}$ and $\log(z_i)$ by $\log^{\varepsilon_i}(z_i)$ everywhere in the KLM formula yields a morphism of complexes

$$\widetilde{\text{AJ}}_{\varepsilon, \underline{\varepsilon}}^{p, -\bullet} : N_\varepsilon^p(X, -\bullet) \rightarrow C_{\mathcal{D}}^{2p+\bullet}(X, \mathbb{Z}(p)).$$

Lemma 4.2 [Li 2018, Theorem 6.1]. *Given $\underline{\varepsilon}, \underline{\varepsilon}' \in \mathcal{B}_\varepsilon$, $\widetilde{\text{AJ}}_{\varepsilon, \underline{\varepsilon}}^p$ and $\widetilde{\text{AJ}}_{\varepsilon, \underline{\varepsilon}'}^p$ are $(\mathbb{Z})$ -homotopic.*

Sketch of proof. Write $\bar{\square} := \mathbb{P}^1$ and fix an integer $N \gg 0$. For each multi-index $I \subset \{1, \dots, N\}$ and function $f : I \rightarrow \{0, \infty\}$, define $\partial_f^I \bar{\square}^N := \bigcap_{i \in I} \{z_i = f(i)\} \subset \bar{\square}^N$;

⁵as in the last footnote

and let $\rho_{i,I}^{f(i)} : \partial_f^I \bar{\square}^N \hookrightarrow \partial_f^{I \setminus i} \bar{\square}^N$ be the inclusions. Then we can define a double complex (zero outside $0 \leq a \leq N$)

$$E^{a,b} := \bigoplus_{|I|=N-a} \bigoplus_{f: I \rightarrow \{0, \infty\}} C_{\mathcal{D}}^{2a+b}(\partial_f^I \bar{\square}^N, \mathbb{Z}(a))$$

with $D_{\mathcal{D}}$ the vertical differential, $\delta := \sum_{I,f,i} (-1)^{\text{sgn}_I(i) + \text{sgn}(f(i))} (\rho_{i,I}^{f(i)})_*$ the horizontal differential, and $\mathbb{D} := D_{\mathcal{D}} + (-1)^b \delta$ the total differential.

Put $\epsilon_I := \{\epsilon_j\}_{j \notin I}$ and

$$\mathcal{R}_{I,f}^{\epsilon} := ((2\pi i)^a T_a^{\epsilon_I}, \Omega_a, R_a^{\epsilon_I}) \in C_{\mathcal{D}}^a(\partial_f^I \bar{\square}^N, \mathbb{Z}(a)) \quad (\cong C_{\mathcal{D}}^a((\mathbb{P}^1)^a, \mathbb{Z}(a))),$$

so that $\mathcal{R}_a^{\epsilon} := \{\mathcal{R}_{I,f}^{\epsilon}\}_{I,f}^{|I|=N-a}$ belongs to $E^{a,-a}$. Adapting [Kerr et al. 2006, (5.2)–(5.4)] to these perturbed currents, [Li 2018] deduces that $\mathcal{R}_{\square}^{\epsilon} := \{\mathcal{R}_a^{\epsilon}\}_{0 \leq a \leq N}$ is a 0-cocycle in $E^{\bullet,\bullet}$. The key point of [op. cit.] is then to construct a (-1) -cochain $\mathcal{S}_{\square}^{\epsilon, \epsilon'}$ in $E^{\bullet,\bullet}$ with $\mathbb{D}\mathcal{S}_{\square}^{\epsilon, \epsilon'} = \mathcal{R}_{\square}^{\epsilon} - \mathcal{R}_{\square}^{\epsilon'}$, and with respect to which the precycles in $N_{\epsilon}^p(X, \bullet)$ remain proper. This yields at once a homotopy formula in $C_{\mathcal{D}}^{\bullet}(X, \mathbb{Z}(\bullet))$ of the form

$$D_{\mathcal{D}}\mathcal{S}_{\square}^{\epsilon, \epsilon'} - \mathcal{S}_{\partial Z}^{\epsilon, \epsilon'} = \widetilde{\text{AJ}}_{\epsilon, \epsilon}^p(Z) - \widetilde{\text{AJ}}_{\epsilon, \epsilon'}^p(Z).$$

The construction of $\mathcal{S}_{\square}^{\epsilon, \epsilon'}$ proceeds by replacing each KLM triple

$$((2\pi i)^a T_a^{\epsilon}, \Omega_a, R_a^{\epsilon}) = \left(2\pi i T_{z_1}^{\epsilon_1}, \frac{dz_1}{z_1}, \log^{\epsilon_1}(z_1)\right) \cup \dots \cup \left(2\pi i T_{z_a}^{\epsilon_a}, \frac{dz_a}{z_a}, \log^{\epsilon_a}(z_a)\right)$$

by an alternating sum of cup-products in which $(-2\pi i \delta_{\{\arg(z_i) \in (\epsilon_i, \epsilon'_i)\}}, 0, 0)$ replaces the i -th term. See [op. cit.] for full details. $\square$

We therefore have well-defined, compatible maps

$$\text{AJ}_{\epsilon}^{p,n} : H_n(N_{\epsilon}^p(X, \bullet)) \rightarrow H_{\mathcal{D}}^{2p-n}(X, \mathbb{Z}(p))$$

for each $\epsilon > 0$, essentially given by $\lim_{\epsilon \rightarrow 0} \widetilde{\text{AJ}}_{\epsilon}^p$ (where the limit is taken so that $\epsilon > \epsilon_1 \gg \epsilon_2 \gg \epsilon_3 \gg \dots > 0$), and recovering $\widetilde{\text{AJ}}_{\text{KLM}}^p$ on

$$N_{\mathbb{R}}^p(X, \bullet) := Z_{\mathbb{R}}^p(X, \bullet) \cap N^p(X, \bullet).$$

More precisely:

Theorem 4.3 [Li 2018, §7]. *The $\{\widetilde{\text{AJ}}_{\epsilon, \epsilon}^{p, -\bullet}\}$ induce a homomorphism*

$$\text{AJ}_{\mathbb{Z}}^{p,n} : \text{CH}^p(X, n) \cong \lim_{\epsilon \rightarrow 0} H_n(N_{\epsilon}^p(X, \bullet)) \rightarrow H_{\mathcal{D}}^{2p-n}(X, \mathbb{Z}(p))$$

factoring $\text{AJ}_{\mathbb{Q}}^{p,n}$.

Corollary 4.4. *If a class $\xi \in \text{CH}^p(X, n)$ is represented by*

$$Z \in \ker(\partial) \subset N_{\mathbb{R}}^p(X, n) \left(\subset \bigcap_{\epsilon > 0} N_{\epsilon}^p(X, n) \right),$$

then

$$\mathrm{AJ}_{\mathbb{Z}}^{p,n}(\xi) = \lim_{\epsilon \rightarrow 0} \widetilde{\mathrm{AJ}}_{\epsilon}^{p,n}(Z) = \widetilde{\mathrm{AJ}}_{\mathrm{KLM}}^{p,n}(Z).$$

So the KLM formula holds verbatim on normalized, $\mathbb{R}$ -proper representatives, validating the deductions at the end of Example 3.1. It is this statement that we (primarily) use in the applications that follow.

5. Torsion generators

Let $\mu_{\infty} = \bigcup_{m \in \mathbb{N}} \mu_m \subset \mathbb{C}^*$ denote the roots of unity, and $w_r(k) := |(\mu_{\infty}^{\otimes r})^{\mathrm{Gal}(\bar{\mathbb{Q}}/k)}|$ for any number field $k \subset \mathbb{C}$. By the universal coefficient sequence for motivic cohomology

$$H_{\mathcal{M}}^0(k, \mathbb{Z}(r)) \rightarrow H_{\mathcal{M}}^0(k, \mathbb{Z}/m\mathbb{Z}(r)) \rightarrow H_{\mathcal{M}}^1(k, \mathbb{Z}(r)) \xrightarrow{\cdot m} H_{\mathcal{M}}^1(k, \mathbb{Z}(r))$$

and vanishing of $H_{\mathcal{M}}^0(k, \mathbb{Z}(r))$,⁶ we have

$$\mathrm{CH}^r(k, 2r-1)[m] \cong H_{\mathcal{M}}^1(k, \mathbb{Z}(r))[m] \cong H_{\mathcal{M}}^0(k, \mathbb{Z}/m\mathbb{Z}(r)).$$

Since the norm residue map

$$H_{\mathcal{M}}^0(k, \mathbb{Z}/m\mathbb{Z}(r)) \rightarrow H_{\mathrm{et}}^0(k, \mu_m^{\otimes r}) \cong (\mu_m^{\otimes r})^{\mathrm{Gal}(\bar{\mathbb{Q}}/k)}$$

is an isomorphism by a celebrated theorem of Rost-Voevodsky (see [Haesemeyer and Weibel 2019]), we conclude that $\mathrm{CH}^r(k, 2r-1)[m] \cong \mathbb{Z}/(m, w_r(k))\mathbb{Z}$, hence

$$\mathrm{CH}^r(k, 2r-1)_{\mathrm{tors}} \cong \mathbb{Z}/w_r(k)\mathbb{Z}.$$

Example 5.1. If $k = \mathbb{Q}$, one has⁷ $w_{2n}(k) = \text{denominator of } (|B_{2n}|)/4n$ (written in lowest terms) and $w_{2n+1} = 2$ for $n \geq 1$; so

$$\mathrm{CH}^2(\mathbb{Q}, 3) \cong \mathbb{Z}/24\mathbb{Z}, \quad \mathrm{CH}^3(\mathbb{Q}, 5)_{\mathrm{tors}} \cong \mathbb{Z}/2\mathbb{Z}, \quad \mathrm{CH}^4(\mathbb{Q}, 7) \cong \mathbb{Z}/240\mathbb{Z}.$$

For real quadratic fields $k = \mathbb{Q}(\sqrt{d})$, the situation is more complicated (see [Weibel 2013, §VI.2]); one computes for instance

d	2	3	5	7
$w_2(\mathbb{Q}(\sqrt{d}))$	48	24	120	24
$w_4(\mathbb{Q}(\sqrt{d}))$	480	240	240	240

⁶See [Weibel 2013, Exercise VI.4.6]: since $H_{\mathcal{M}}^{-1}(k, \mathbb{Z}/m\mathbb{Z}(r)) \cong H_{\mathrm{et}}^{-1}(k, \mu_m^{\otimes r}) = \{0\}$, $\cdot m$ is injective on $H_{\mathcal{M}}^0(k, \mathbb{Z}(r))$ (universal coefficient sequence), which is thus torsion-free; it has rank 0 since $H_{\mathcal{M}}^0(k, \mathbb{Q}(r)) \cong \mathrm{Gr}_r^{\vee} K_{2r}(k) \otimes \mathbb{Q} = \{0\}$ by Borel's theorem [1974].

⁷Bernoulli numbers: $|B_{2n}| = \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \dots$ for $n = 1, 2, 3, \dots$

so that only $\mathrm{CH}^2(\mathbb{Q}(\sqrt{2}), 3)$, $\mathrm{CH}^2(\mathbb{Q}(\sqrt{5}), 3)$ and $\mathrm{CH}^4(\mathbb{Q}(\sqrt{2}), 7)$ are different from the $k = \mathbb{Q}$ case. Finally, for cyclotomic $k = \mathbb{Q}(\zeta_p)$ ($p \geq 5$ prime) one can show that $w_3(\mathbb{Q}(\zeta_p)) = 2p$, while $w_3(\mathbb{Q}(\zeta_3)) = 18$.

For computing torsion orders of images under

$$\mathrm{AJ}_{\mathbb{Z}}^{r, 2r-1} : H_{2r-1}(N_{\mathbb{R}}^r(k, \bullet)) \rightarrow \mathbb{C}/(2\pi i)^r \mathbb{Z}, \quad Z \mapsto \frac{1}{(2\pi i)^{r-1}} \int_Z R_{2r-1} =: \mathcal{R}(Z)$$

we use the following basic calculation:

Proposition 5.2. *Suppose that for a given $r \in \mathbb{N}$ there exists a collection of closed precycles $Z_{\ell, a}^r \in N_{\mathbb{R}}^r(\mathbb{Q}(\zeta_\ell), 2r-1)$ with*

$$(5-1) \quad \mathcal{R}(Z_{\ell, a}^r) = (r-1)! \ell^{r-1} \mathrm{Li}_r(\zeta_\ell^a).$$

Then for $Z := \sum_{a=0}^{\ell-1} f(a) Z_{\ell, a}^r$ with $f(-a) = (-1)^r f(a)$, $\mathrm{AJ}(Z)$ is torsion of order given by the denominator of

$$(5-2) \quad \tau(Z) := |(2\pi i)^{-r} \mathcal{R}(Z)| = \pm \frac{\ell^{r-1}}{2r} \sum_{a=0}^{\ell-1} f(a) B_r(a/\ell),$$

where $B_r(\cdot)$ are the Bernoulli polynomials.⁸

Proof. Writing $\hat{f}(k) := \sum_{a=0}^{\ell-1} f(a) \zeta_\ell^{-ka}$ for the finite Fourier transform (for functions on $\mathbb{Z}/\ell\mathbb{Z}$), we have

$$\begin{aligned} \sum_{a=0}^{\ell-1} f(a) \mathrm{Li}_r(\zeta_\ell^a) &= \frac{1}{2} \sum_{a=0}^{\ell-1} f(a) \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\zeta_\ell^{ka}}{k^r} \\ &= \frac{(-1)^r}{2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{k^r} =: \frac{(-1)^r}{2} \tilde{L}(\hat{f}, r) \\ &= \frac{(2\pi i)^r}{2 \cdot r!} \sum_{a=0}^{\ell-1} f(a) B_r(a/\ell), \end{aligned}$$

where the last step is [Kerr 2018, Theorem 3.9]. □

In practice, the $\{Z_{\ell, a}^r\}$ will be obtained from a single cycle $Z_\ell^r = Z_{\ell, 1}^r$ by Galois conjugation. For $r = 2$, we already have this from Examples 2.1 and 3.1: namely,

$$(5-3) \quad Z_{\ell, a}^2 = \left(1 - \frac{\zeta_\ell^a}{t}, 1 - t, t^{-\ell}\right).$$

Now $\mathrm{CH}^r(k, 2r-1) = \mathrm{CH}^r(k, 2r-1)_{\mathrm{tors}} \Leftrightarrow r$ is even and k is totally real. In particular, assuming the hypothesis of Proposition 5.2, we obtain generators of

⁸ $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$, etc.

$\mathrm{CH}^{2n}(k, 4n-1)$ as follows:

$$\begin{aligned} k = \mathbb{Q}, \text{ any } n : \quad & \tau(Z_1^{2n}) = \frac{|B_{2n}|}{4n} \left(= \frac{1}{24}, \frac{1}{240}, \dots \right); \\ k = \mathbb{Q}(\sqrt{2}), \text{ } n = 1, 2 : \quad & \tau(Z_{8,1}^{2n} + Z_{8,7}^{2n}) = \frac{11}{48}, \frac{1313}{480}; \\ k = \mathbb{Q}(\sqrt{5}), \text{ } n = 1 : \quad & \tau(Z_1^2 + Z_{5,1}^2 + Z_{5,4}^2) = \frac{7}{120}. \end{aligned}$$

For $\mathrm{CH}^3(k, 5)_{\mathrm{tors}}$ one computes, for example,

$$\begin{aligned} k = \mathbb{Q}(\zeta_3) : \quad & \tau(Z_{3,1}^3 - Z_{3,2}^3) = \frac{1}{9}; \quad \text{and} \\ k = \mathbb{Q}(\zeta_5) : \quad & \tau(Z_{5,1}^3 - Z_{5,4}^3) = \frac{2}{5}, \end{aligned}$$

which miss only the 2-torsion element from $\mathrm{CH}^3(\mathbb{Q}, 5) \hookrightarrow \mathrm{CH}^3(k, 5)$. (So far we have no $N_{\mathbb{R}}^3(\mathbb{Q}, 5)$ representative for this element.)

It remains to construct the cycles of the Proposition for $r = 3, 4$. From [Kerr and Yang 2018, §4.2], for $r = 3$ we have⁹

$$(5-4) \quad Z_{\ell}^3 := -2 \left(\frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, 1 - \zeta_{\ell} t_1 t_2, t_1^{\ell}, t_2^{\ell} \right) \\ - \left(\frac{t}{t - 1}, \frac{1}{1 - \zeta_{\ell} t}, \frac{(u - t^{\ell})(u - t^{-\ell})}{(u - 1)^2}, t^{\ell} u, \frac{u}{t^{\ell}} \right)$$

which is normalized since all “boundaries” occur in the third coordinate. We have $\mathcal{R}(Z_{\ell}^3) = 2\mathrm{Li}_3(\zeta_{\ell})$ by [op. cit., Theorem 3.6], with only the first term contributing. (This gives in particular $\mathcal{R}(Z_1^3) = 2\mathrm{Li}_3(1) = 2\zeta(3)$.)

For $r = 4$, the first construction in [Kerr and Yang 2018, §4.3] would be in $N_{\mathbb{R}}^4(\mathbb{Q}(\zeta_{\ell}), 7)$, but there is an error in the computation of the boundary of the last component $\mathcal{W}_2$: in fact, it is degenerate,¹⁰ and the cycle $\mathcal{Z}$ is therefore not closed. A correct application of the strategy in [op. cit., §3.1], yields

$$(5-5) \quad Z_{\ell}^4 := 6 \left(\frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, \frac{t_3}{t_3 - 1}, 1 - \zeta_{\ell} t_1 t_2 t_3, t_1^{\ell}, t_2^{\ell}, t_3^{\ell} \right) \\ + \left(\frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, \frac{1}{1 - \zeta_{\ell} t_1 t_2}, \frac{(u - t_1^{\ell})(u - t_2^{\ell})(u - t_1^{-\ell} t_2^{-\ell})}{(u - 1)^3}, \frac{t_1^{\ell}}{u}, \frac{t_2^{\ell}}{u}, \frac{1}{u t_1^{\ell} t_2^{\ell}} \right) \\ + \left(\frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, \frac{1}{1 - \zeta_{\ell} t_1 t_2}, \frac{(u - t_1^{\ell})(u - t_2^{\ell})}{(u - t_1^{\ell} t_2^{\ell})(u - 1)}, \frac{t_1^{\ell}}{u}, \frac{t_2^{\ell}}{u}, \frac{u}{t_1^{\ell} t_2^{\ell}} \right) \\ + \left(\frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, \frac{1}{1 - \zeta_{\ell} t_1 t_2}, \frac{(u - t_1^{\ell})(u - t_1^{-\ell} t_2^{-\ell})}{(u - t_2^{-\ell})(u - 1)}, \frac{t_1^{\ell}}{u}, t_2^{\ell} u, \frac{1}{u t_1^{\ell} t_2^{\ell}} \right)$$

⁹The components are parametrized by (t_1, t_2) and (t, u) respectively.

¹⁰I.e., it belongs to $d^4(\mathbb{Q}(\zeta_{\ell}), 7)$, as can be seen by substituting $v = uw$.

$$\begin{aligned}
& + \left(\frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, \frac{1}{1 - \zeta_\ell t_1 t_2}, \frac{(u - t_2^\ell)(u - t_1^{-\ell} t_2^{-\ell})}{(u - t_1^{-\ell})(u - 1)}, t_1^\ell u, \frac{t_2^\ell}{u}, \frac{1}{t_1^\ell t_2^\ell u} \right) \\
& + \left(\frac{(v - u)(v - u^{-1})}{(v - 1)^2}, \frac{t}{t - 1}, \frac{1}{1 - \zeta t}, \frac{(u - t^\ell)(u - t^{-\ell})}{(u - 1)^2}, v, \frac{t^\ell}{u}, \frac{1}{ut^\ell} \right) \\
& + \left(\frac{t}{t - 1}, \frac{(v - u)(v - u^{-1})}{(v - 1)^2}, \frac{1}{1 - \zeta t}, \frac{(u - t^\ell)(u - t^{-\ell})}{(u - 1)^2}, \frac{t^\ell}{u}, v, \frac{1}{ut^\ell} \right) \\
& + \left(\frac{t}{t - 1}, \frac{1}{1 - \zeta t}, \frac{(v - u)(v - u^{-1})}{(v - 1)^2}, \frac{(u - t^\ell)(u - t^{-\ell})}{(u - 1)^2}, \frac{t^\ell}{u}, \frac{1}{ut^\ell}, v \right),
\end{aligned}$$

which belongs to $\ker(\partial) \cap N_{\mathbb{R}}^4(\mathbb{Q}(\zeta_\ell), 7)$. Only the first term contributes to $\mathcal{R}(Z_\ell^4) = -6\ell^3 \int_{[0,1]^3} \log(1 - \zeta_\ell t_1 t_2 t_3) dt_1/t_1 \wedge dt_2/t_2 \wedge dt_3/t_3 = 6\ell^3 \text{Li}_4(\zeta_\ell)$, see [op. cit., §3.2]. Again, cycles $Z_{\ell,a}^3$ and $Z_{\ell,a}^4$ are obtained by replacing ζ_ℓ by ζ_ℓ^a in (5-4) and (5-5), respectively.

Summarizing, we have

Theorem 5.3. *The cycles given in (5-3), (5-4), (5-5) generate the (cyclic, torsion) groups as shown in Table 1.*

Remark 5.4. An example of a cycle for which the log-branch perturbations *are* required for the integral regulator computation is

$$Z := Z_- - Z_+ := \left(\left(\frac{z - i}{z + i} \right)^{-2}, \left(\frac{z - 1}{z + 1} \right)^{-2}, z^{-2} \right) - \left(\left(\frac{z - i}{z + i} \right)^2, \left(\frac{z - 1}{z + 1} \right)^2, z^2 \right)$$

(parametrized by $z \in \mathbb{P}^1$) in $N^2(\mathbb{Q}(i), 3)$. Indeed, T_{z^2} has support on $i\mathbb{R}$ in the complex z -plane, and $T_{((z-1)/(z+1))^2}$ and $T_{((z-i)/(z+i))^2}$ have support on the unit circle S^1 in the complex z -plane, so that the triple intersection (essentially $i\mathbb{R} \cap S^1 \cap S^1$) is nonempty. Though this cycle is *non-torsion*, we briefly describe the computation. After making the deformation, $T_{((z-i)/(z+i))^2}$ and $T_{((z-1)/(z+1))^2}$ intersect twice with

Group	Cycle	Order
$\text{CH}^2(\mathbb{Q}, 3)$	Z_1^2	24
$\text{CH}^4(\mathbb{Q}, 7)$	Z_1^4	240
$\text{CH}^2(\mathbb{Q}(\sqrt{2}), 3)$	$Z_{8,1}^2 + Z_{8,7}^2$	48
$\text{CH}^4(\mathbb{Q}(\sqrt{2}), 7)$	$Z_{8,1}^4 + Z_{8,7}^4$	480
$\text{CH}^2(\mathbb{Q}(\sqrt{5}), 3)$	$Z_1^2 + Z_{5,1}^2 + Z_{5,4}^2$	120
$\frac{\text{CH}^3(\mathbb{Q}(\zeta_3), 5)_{\text{tors}}}{\text{CH}^3(\mathbb{Q}, 5)_{\text{tors}}}$	$Z_{3,1}^3 - Z_{3,2}^3$	9
$\frac{\text{CH}^3(\mathbb{Q}(\zeta_5), 5)_{\text{tors}}}{\text{CH}^3(\mathbb{Q}, 5)_{\text{tors}}}$	$Z_{5,1}^3 - Z_{5,4}^3$	5

Table 1. Some Generators (see Theorem 5.3).

opposite orientations, at points near $\mathbf{i}$ and $-\mathbf{i}$ with phase just greater than $\pi/2$ and $3\pi/2$, respectively. Since $\epsilon_3 \rightarrow 0$ much faster than ϵ_1 and ϵ_2 , in the limit the

$$2\pi \mathbf{i} \int_{Z_+} \log^{\epsilon_3}(z^2) \delta_{T^{\epsilon_1}_{((z-i)/(z+i))^2} \cap T^{\epsilon_2}_{((z-1)/(z+1))^2}}$$

term of $\frac{1}{2\pi \mathbf{i}} \int_{Z_+} R_3$ contributes $2\pi \mathbf{i}(\pi \mathbf{i} - \pi \mathbf{i}) = 0$. The remaining term yields

$$(5-6) \quad 2 \int_{Z_+} \log^{\epsilon_2} \left(\left(\frac{1-z}{1+z} \right)^2 \right) \frac{dz}{z} \delta_{T^{\epsilon_1}_{((z-i)/(z+i))^2}},$$

where $T^{\epsilon_1}_{((z-i)/(z+i))^2}$ consists of two paths from $-\mathbf{i}$ to $\mathbf{i}$, along which one checks that (in the limit)

$$\log^{\epsilon_2} \left(\left(\frac{1-z}{1+z} \right)^2 \right) = 2 \log^{\epsilon_2}(1-z) - 2 \log^{\epsilon_2}(1+z);$$

and so (5-6) becomes $8 \int_{-\mathbf{i}}^{\mathbf{i}} \log(1-z) \frac{dz}{z} - 8 \int_{-\mathbf{i}}^{\mathbf{i}} \log(1+z) \frac{dz}{z}$. Combining this with the portion from Z_- , we obtain

$$\mathcal{R}(Z) = 32\text{Li}_2(\mathbf{i}) - 32\text{Li}_2(-\mathbf{i}) = 64\mathbf{i}L(\chi_4, 2) \in \mathbb{C}/\mathbb{Z}(2).$$

6. Local $\mathbb{P}^2$ revisited

For a reflexive polytope $\Delta \subset \mathbb{R}^2$ with polar polytope Δ° , the mirror of $K_{\mathbb{P}_{\Delta^\circ}}$ (“local $\mathbb{P}_{\Delta^\circ}$ ”) can be identified with a family of $\text{CH}^2(\cdot, 2)$ -elements on a family of anticanonical (elliptic) curves in $\mathbb{P}_\Delta$ [Doran and Kerr 2011, §5]. As a second application of the integral regulator, we show how to apply it to compute the correct “torsion term” in the higher normal function associated to one of these families. That is, if $\mathcal{E} \xrightarrow{\pi} \mathbb{P}_t^1$ is smooth away from $\Sigma = \{0\} \cup \Sigma^*$, with fibers $E_t = \pi^{-1}(t)$, and $\Xi \in \text{CH}^2(\mathcal{E} \setminus E_0, 2)$ has fiberwise restrictions $\xi_t \in \text{CH}^2(E_t, 2)$ ($t \notin \Sigma$), we shall compute

$$\begin{aligned} \mathcal{R}_t &:= \text{AJ}^{2,2}(\xi_t) \in H_{\mathcal{O}}^2(E_t, \mathbb{C}/\mathbb{Z}(2)) \cong H^1(E_t, \mathbb{C}/\mathbb{Z}(2)) \\ &\cong \text{Hom}(H_1(E_t, \mathbb{Z}), \mathbb{C}/(2\pi \mathbf{i})^2 \mathbb{Z}) \end{aligned}$$

in a neighborhood of $t = 0$. Writing $\{\omega_t\}$ for a section of $\omega_{\mathcal{E}/\mathbb{P}^1}$ vanishing at ∞ , the constant term in (a branch of) the resulting truncated higher normal function

$$(6-1) \quad v(t) := \frac{1}{2\pi \mathbf{i}} \langle \omega_t, \mathcal{R}_t \rangle$$

will play a role in forthcoming work of the first author with C. Doran on quantum curves.

To begin in a somewhat more general scenario, let $\Delta \subset \mathbb{R}^2$ be any convex polytope with integer vertices $\{p_i = (a_i, b_i)\}_{i=1}^N$ and interior integer points $\{(v_j, w_j)\}_{j=1}^g$.

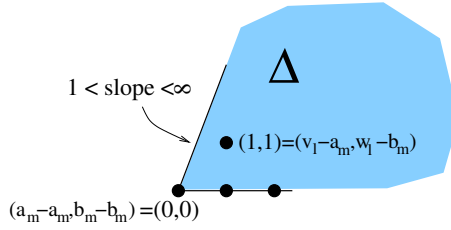


Figure 1. For the proof of Lemma 6.1.

Define a multiparameter family

$$\rho : \mathcal{C} \rightarrow \mathbb{C}^g, \quad C_{\underline{\lambda}} := \rho^{-1}(\underline{\lambda})$$

of (where smooth) genus g curves by taking the Zariski closure of

$$\mathcal{C}^* := \{(x, y; \lambda_1, \dots, \lambda_g) \mid 0 = \Phi_{\underline{\lambda}}(x, y) := \phi(x, y) - \sum_{j=1}^g \lambda_j x^{v_j} y^{w_j}\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^g$$

in $\mathbb{P}_{\Delta} \times \mathbb{C}^g$, where $\phi(x, y) := \sum_{(a,b) \in \partial \Delta \cap \mathbb{Z}^2} m_{a,b} x^a y^b$ is uniquely determined by requiring its edge polynomials to be powers of $(t+1)$. (If the edges of Δ have no interior points, then $\phi(x, y) = \sum_{i=1}^N x^{a_i} y^{b_i}$.) The symbol $\{-x, -y\}$ represents a closed precycle $\Xi^* \in Z^2(\mathcal{C}^*, 2)$ parametrized (in $\mathcal{C}^* \times \square^2$) by $(x, y, \underline{\lambda}, -x, -y)_{(x,y,\underline{\lambda}) \in \mathcal{C}^*}$.

Lemma 6.1. *The class of Ξ^* in $\text{CH}^2(\mathcal{C}^*, 2)$ is the restriction of a class $\Xi \in \text{CH}^2(\mathcal{C}, 2)$.*

Proof. We need only check that the Tame symbol of $\{-x, -y\}|_{C_{\underline{\lambda}}^*} \in K_2^M(\mathbb{C}(C_{\underline{\lambda}}^*))$ is zero for general $\underline{\lambda}$. The symbol $\{-x, -y\}$ is invariant under unimodular change of toric coordinates,¹¹ so we may assume that (after shifting Δ by $(-a_m, -b_m)$ for some m) we have a picture (Figure 1) where the bottom edge corresponds to the toric divisor at whose intersection with $C_{\underline{\lambda}}$ we wish to compute $\text{Tame}(\{-x, -y\}|_{C_{\underline{\lambda}}^*}) \in \mathbb{C}^*$. Since the edge polynomial is $(1+x)^c$, this intersection occurs at $(-1, 0)$, so the Tame symbol is 1. $\square$

Now set $\mathcal{R}_{\underline{\lambda}} := \text{AJ}^{2,2}(\Xi|_{C_{\underline{\lambda}}}) \in \text{Hom}(H_1(C_{\underline{\lambda}}), \mathbb{C}/\mathbb{Z}(2))$. Picking any vertex p_m of Δ , we can (via unimodular coordinate change) put it in the position indicated in Figure 1. In the new coordinates (still denoted (x, y)), $C_{\underline{\lambda}}$ is cut out by an equation of the form

$$\begin{aligned} 0 &= \tilde{\Phi}_{\underline{\lambda}} := x^{-a_m} y^{-b_m} \Phi_{\underline{\lambda}}(x, y) \\ &= (1+x)^{k_m} + y \left\{ \Psi_m(x, y) - \sum_{j=1}^g \lambda_j x^{v_j - a_m} y^{w_j - b_m - 1} \right\}, \end{aligned}$$

¹¹That is, replacing x, y by $x^a y^b, x^c y^d$ with $ad - bc = 1$; the a_i, b_i, v_i, w_i are changed accordingly.

and acquires a node at $(0, 0)$ as $\lambda_\ell \rightarrow \infty$. (Note that ℓ is determined by m .) The corresponding vanishing cycle α_m has image $|x| = |y| = \epsilon$ under

$$H_1(C_{\underline{\lambda}}), \mathbb{Z} \xrightarrow{\text{Tube}} H_2(\mathbb{P}_\Delta \setminus C_{\underline{\lambda}})$$

for large $|\lambda_\ell|$.

Theorem 6.2. *For $i\lambda_\ell \in \mathfrak{H}$ and $|\lambda_\ell| \gg 0$, and $\lambda_{j \neq \ell}$ sufficiently small,¹² we have*

$$\mathcal{R}_{\underline{\lambda}}(\alpha_m) = 2\pi i \left(-\log(\lambda_\ell) + \sum_{k \geq 1} \frac{1}{k} [\Psi_{\underline{\lambda}, \ell}^k]_0 \right) \in \mathbb{C}/\mathbb{Z}(2),$$

where $\Psi_{\underline{\lambda}, \ell} := (-1/\lambda_\ell)(x^{-v_\ell} y^{-w_\ell} \Phi_{\underline{\lambda}} + \lambda_\ell)$ and $[\cdot]_0$ takes the constant term in a Laurent polynomial.

Proof. We use the notation

$$\begin{aligned} R\{f_1, f_2\} &= \log(f_1) \frac{df_1}{f_1} - 2\pi i \log(f_2) \delta_{T_{f_1}}, \quad \text{and} \\ R\{f_1, f_2, f_3\} &= \log(f_1) \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} + 2\pi i \log(f_2) \frac{df_3}{f_3} \delta_{T_{f_1}} + (2\pi i)^2 \log(f_3) \delta_{T_{f_1} \cap T_{f_2}} \end{aligned}$$

for R_2 and R_3 with f_i replacing z_i . Writing D for the bottom-edge divisor in Figure 1, we have

$$\text{Tame}_D\{\tilde{\Phi}_{\underline{\lambda}}, -x, -y\} = \{\tilde{\Phi}_{\underline{\lambda}}(x, 0), -x\} = \{(1+x)^c, -x\} (= 1).$$

So writing $\Gamma = \{|x| = \epsilon \geq |y|\}$ ($\Rightarrow \alpha_m = \Gamma \cap C_{\underline{\lambda}}$) gives

$$\begin{aligned} \mathcal{R}_t(\alpha_m) &= \int_{\alpha_m} R\{-x, -y\} = \int_{\Gamma} R\{-x, -y\} \cdot \delta_{C_{\underline{\lambda}}} \\ &= \frac{-1}{2\pi i} \int_{\Gamma} d[R\{\tilde{\Phi}_{\underline{\lambda}}, -x, -y\}] - \int_{\Gamma} R\{(1+x)^c, -x\} \cdot \delta_D \\ &= \frac{-1}{2\pi i} \int_{\partial\Gamma} R\{\tilde{\Phi}_{\underline{\lambda}}, -x, -y\} - \int_{|x|=|y|=\epsilon} R\{(1+x)^c, -x\} \xrightarrow{0} \\ &= \frac{-1}{2\pi i} \int_{|x|=|y|=\epsilon} R\{x^{-v_\ell} y^{-w_\ell} \Phi_{\underline{\lambda}}, -x, -y\} \\ &= \frac{-1}{2\pi i} \int_{|x|=|y|=\epsilon} R\{\lambda(1 - \Psi_{\underline{\lambda}, \ell}), -x, -y\} \\ &= \frac{-1}{2\pi i} \int_{|x|=|y|=\epsilon} \{\log(\lambda) + \log(1 - \Psi_{\underline{\lambda}, \ell})\} \frac{dx}{x} \wedge \frac{dy}{y} \\ &= -2\pi i \log(\lambda) + 2\pi i \sum_{k \geq 1} \int_{|x|=|y|=\epsilon} \Psi_{\underline{\lambda}, \ell}^k \frac{dx}{x} \wedge \frac{dy}{y} \end{aligned}$$

¹²E.g., if $c := |\Delta \cap \mathbb{Z}^2|$, then $|\lambda_\ell| > c\epsilon^{-2}$ and $|\lambda_{j \neq \ell}| < 1/(c\epsilon^3)$ will do.

modulo $\mathbb{Z}(2)$. Here only the first term of R_3 enters since $T_{\lambda(1-\Psi)} \cap |x| = |y| = \epsilon$ is empty under the given assumptions. $\square$

Returning to the more specific scenario at the beginning of this section, if $g = 1$ and $\lambda_1 =: \lambda =: \frac{1}{t}$, then $\Phi_\lambda = \phi(x, y) - \lambda$ and $\Psi_{\lambda,1} = t\phi(x, y)$, so that (writing $\mathcal{R}_t$ instead of $\mathcal{R}_\lambda$), Theorem 6.2 yields:

Corollary 6.3. *If Δ is reflexive, then the α_m are all homologous ($=: \alpha$), and*

$$\mathcal{R}_t(\alpha) \equiv_{\mathbb{Z}(2)} 2\pi i \left(\log(t) + \sum_{k \geq 1} \frac{[\phi^k]_0}{k} \right)$$

for t small in the right-half-plane.

It remains to compute $\mathcal{R}_t(\beta)$ for a cycle β complementary to α (so that $\mathbb{Z}\langle \alpha, \beta \rangle = H_1(E_t, \mathbb{Z})$), which we shall do for the local $\mathbb{P}^2$ setting only: Δ the convex hull of $\{(1, 0), (0, 1), (-1, -1)\}$, and $\phi = x + y + x^{-1}y^{-1}$. (So $\mathbb{P}_{\Delta^\circ} \cong \mathbb{P}^2$, while the family of elliptic curves E_t lives in $\mathbb{P}_\Delta$.) Taking $t > 0$ small, write $(0 <) x_0(t) < x_-(t) < x_+(t) < \infty$ for the branch points of

$$E_{t(=\lambda^{-1})} : y^2 + (x - \lambda)y + x^{-1} = 0$$

over $\mathbb{P}_x^1$, and $y^\pm(x) = \frac{1}{2} \{ (\lambda - x) \pm \sqrt{(x - \lambda)^2 - 4x^{-1}} \}$. Then β (resp. α) is given by the difference of paths (on the two branches) between $x_0(t)$ and $x_-(t)$ (resp. $x_-(t)$ and $x_+(t)$).

Now $T_{-x} = \mathbb{R}_{>0} \subset \mathbb{P}_x^1$, so taking the y^+ -branch of β to run from x_0 to x_- in $\mathfrak{H}$, and the y^- -branch of β to run from x_- to x_0 in $-\mathfrak{H}$, we have $\beta \cap T_{-x} = (x_0, y_0) \cup (x_-, y_-)$; moreover, $\log(-x) = \log(x) \mp i\pi$ on the $y^\pm$ -branch of β . The upshot is that

$$\begin{aligned} \int_{\beta} R\{-x, -y\}|_{E_t} &= \int_{\beta} \log(-x) \frac{dy}{y} - 2\pi i \sum_{\beta \cap T_{-x}} \log(y) \\ &= - \int_{x_0(t)}^{x_-(t)} \log(x) d \log \left(\frac{y^+(x)}{y^-(x)} \right) = \int_{x_0(t)}^{x_-(t)} \log \left(\frac{y^+(x)}{y^-(x)} \right) \frac{dx}{x} \\ &= \int_{x_0(t)}^{x_-(t)} \log \left(\frac{1 + \sqrt{1 - \xi}}{1 - \sqrt{1 - \xi}} \right) \frac{dx}{x}, \end{aligned}$$

where $\xi = \frac{4t^2}{x(1-xt)^2}$. Writing for $\xi \in (0, 1)$

$$\log \left(\frac{1 + \sqrt{1 - \xi}}{1 - \sqrt{1 - \xi}} \right) + \log \left(\frac{\xi}{4} \right) =: - \sum_{m \geq 1} \alpha_m \xi^m,$$

the above integral decomposes into

$$-2 \log(t) \int_{x_0}^{x_-} \frac{dx}{x} + \int_{x_0}^{x_-} \log(x) \frac{dx}{x} + 2 \int_{x_0}^{x_-} \log(1 - xt) \frac{dx}{x} - \sum_{m \geq 1} \alpha_m \int_{x_0}^{x_-} \xi^m \frac{dx}{x}.$$

Using the approximations $x_0 \simeq 4t^2(1+8t^3)$ and $x_- \simeq t^{-1}(1-2t^{3/2}-2t^3)$, a lengthy direct computation gives that

$$\mathcal{R}_t(\beta) = \frac{9}{2} \log^2(t) - \frac{\pi^2}{2} + \mathcal{O}(t \log(t)).$$

Let $\delta_t := t \frac{d}{dt}$. By the general result [Doran and Kerr 2011, Corollary 4.1], one knows that $\nabla_{\delta_t} \mathcal{R}_t = [\omega_t]$, where

$$\omega_t := \text{Res}_{E_t} \left(\frac{\frac{dx}{x} \wedge \frac{dy}{y}}{1 - t\phi(x, y)} \right)$$

has its periods $\omega_t(\gamma) := \int_\gamma \omega_t$ annihilated by the Picard–Fuchs operator

$$\mathcal{L} = \delta_t^2 - 27t^3(\delta_t + 1)(\delta_t + 2).$$

The regulator periods $\mathcal{R}_t(\gamma)$ are therefore killed by $\mathcal{L} \circ \delta_t$. Since $\mathcal{L}(\cdot) = 0$ is known to have the basis of solutions

$$\begin{aligned} \pi_1 &= \sum_{n \geq 0} a_n t^{3n}, \\ \pi_2 &= 3 \log(t) \pi_1 + \sum_{n \geq 1} a_n b_n t^{3n}, \end{aligned}$$

with

$$a_n = \frac{(3n)!}{(n!)^3} \quad \text{and} \quad b_n = \sum_{k=0}^{n-1} \left(\frac{3}{3k+1} + \frac{3}{3k+2} - \frac{2}{k+1} \right),$$

it now follows that (writing $B_n = b_n - 1/n$)

$$\begin{aligned} \mathcal{R}_t(\alpha) &\equiv_{\mathbb{Z}(2)} 2\pi i \left(\log(t) + \sum_{n \geq 1} \frac{a_n}{3n} t^{3n} \right), \\ \mathcal{R}_t(\beta) &\equiv_{\mathbb{Z}(2)} \frac{9}{2} \log^2(t) + 3 \log(t) \sum_{n \geq 1} \frac{a_n}{n} t^{3n} + \sum_{n \geq 1} \frac{a_n B_n}{n} t^{3n} - \frac{\pi^2}{2}, \\ \omega_t(\alpha) &= 2\pi i \sum_{n \geq 0} a_n t^{3n}, \\ \omega_t(\beta) &= 9 \log(t) \sum_{n \geq 0} a_n t^{3n} + 3 \sum_{n \geq 1} a_n b_n t^{3n}, \end{aligned}$$

for $0 < |t| < \frac{1}{3}$. For (6-1), this yields

Proposition 6.4. *The truncated normal function mirror to local $\mathbb{P}^2$ is*

$$\begin{aligned} v(t) &= \left\langle \frac{\omega_t}{2\pi i}, \mathcal{R}_t \right\rangle = \frac{1}{2\pi i} (\mathcal{R}_t(\alpha) \omega_t(\beta) - \mathcal{R}_t(\beta) \omega_t(\alpha)) \\ &\equiv \frac{9}{2} \log^2(t)(1+6t^3) + 3 \log(t)(9t^3) + \frac{\pi^2}{2} + (3\pi^2 - 9)t^3 + \mathcal{O}(t^6 \log^2 t), \end{aligned}$$

modulo the period lattice $\mathbb{Z}(2)\langle\omega_t(\alpha), \omega_t(\beta)\rangle$.

Remark 6.5. This is closely related to computations in [Hosono 2006; Mohri et al. 2001]; the main difference — and the salient result here — is the identification of $\pi^2/2$ as the correct torsion offset for our motivically defined ν .

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MATT KERR
DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY IN ST. LOUIS
ST. LOUIS, MO
UNITED STATES
matkerr@math.wustl.edu

MUXI LI
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
HEFEI
CHINA
limuxi@ustc.edu.cn

DEFINABILITY AND APPROXIMATIONS IN TRIANGULATED CATEGORIES

ROSANNA LAKING AND JORGE VITÓRIA

We give criteria for subcategories of a compactly generated algebraic triangulated category to be precovering or preenveloping. These criteria are formulated in terms of closure conditions involving products, coproducts, directed homotopy colimits and further conditions involving the notion of purity. In particular, we provide sufficient closure conditions for a subcategory of a compactly generated algebraic triangulated category to be a torsion class. Finally we explore applications of the previous results to the theory of recollements.

1. Introduction

Given a subcategory $\mathcal{X}$ of a given category $\mathcal{C}$, it is natural to ask whether every object of $\mathcal{C}$ admits morphisms from $\mathcal{X}$ satisfying a universal property (being terminal) among all morphisms from $\mathcal{X}$. This is the notion of $\mathcal{X}$ being precovering and the notion of preenveloping is dual. For example, modules over a ring always admit projective precovers and injective envelopes; these are universal maps to/from a given module from/to the distinguished classes of projective/injective modules. Approximation theory studies the subcategories $\mathcal{X}$ that can provide such approximation maps to all objects in the category. Some subcategories providing good approximations arise through the notion of torsion pairs and definability.

Torsion pairs in abelian or triangulated categories are special pairs of subcategories, one of them being precovering and the other preenveloping. In triangulated categories such pairs assume particular relevance when they “behave well” with respect to the shift functor: these torsion pairs are called t-structures or co-t-structures. Areas where these torsion pairs play a particularly important role include silting/tilting theory (e.g., [Keller and Vossieck 1988; Keller and Nicolás 2013; Koenig and Yang 2014]); the study of derived equivalences in algebra and geometry

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(e.g., [Psaroudakis and Vitória 2018]); and the study of Bridgeland's stability conditions [2007].

Definable subcategories of a module category [Crawley-Boevey 1998] have their origins in the model theory of modules and are well known to have good approximation-theoretic properties. Indeed, they are both precovering and preenveloping [Crivei et al. 2010]. In fact, a subcategory is precovering if it is closed under coproducts and pure quotients [Holm and Jørgensen 2008] and it is preenveloping if it is closed under products and pure subobjects [Rada and Saorín 1998]. The notion of definable subcategories of (compactly generated) triangulated categories has appeared in [Krause 2002], and they are known to be preenveloping [Angeleri Hügel et al. 2017], but they are generally less well-understood.

For categories of modules over a ring, a subcategory is a torsion class if and only if it is closed under coproducts, quotients and extensions [Dickson 1966], and a subcategory is definable if and only if it is closed under products, pure subobjects and pure quotients (see [Crawley-Boevey 1998] and Theorem 2.4(1)). In this paper we provide analogous closure conditions for subcategories of triangulated categories that yield good approximation-theoretic properties. However, in triangulated categories all monomorphisms and epimorphisms split, and so we are lacking a useful notion of subobject and quotient object. We approach this deficit by considering the class of compactly generated algebraic triangulated categories, which include, in particular, derived categories of rings. The assumption that $\mathcal{T}$ is compactly generated allows us to deal with *pure subobjects* and *pure quotient objects*. We summarise our main results as follows.

Theorem. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category and $\mathcal{X}$ a subcategory of $\mathcal{T}$. Then the following statements hold.*

- (1) *If $\mathcal{X}$ is closed under coproducts and pure quotients, then $\mathcal{X}$ is precovering. If $\mathcal{X}$ is closed under products and pure subobjects, then $\mathcal{X}$ is preenveloping.*
- (2) *The subcategory $\mathcal{X}$ is definable in $\mathcal{T}$ if and only if it is closed under pure subobjects, products and pure quotients. In particular, definable subcategories are both precovering and preenveloping.*
- (3) *If $\mathcal{X}$ is closed under coproducts, extensions and pure quotients, then $\mathcal{X}$ is a torsion class in $\mathcal{T}$.*

In order to obtain (2) we make reference to a recent characterisation of definable subcategories when $\mathcal{T}$ is the underlying category of a strong and stable derivator [Laking 2019]: a subcategory is definable if and only if it is closed under products, pure subobjects and directed homotopy colimits. The assumption that $\mathcal{T}$ is both compactly generated and algebraic means that $\mathcal{T}$ is equivalent to $\mathcal{D}(\mathcal{A})$ for a small differential graded (dg, for short) category $\mathcal{A}$ [Keller 1994a]. In particular, this

means that $\mathcal{T}$ is equivalent to the homotopy category of a nice model structure and so we may apply results from the setting of strong and stable derivators [Rahn 2013]. However, it is not clear that the notion of “closure under directed homotopy colimits” is independent of the choice of $\mathcal{A}$. Thus the second part of the theorem above should be seen as a simplification of [Laking 2019] in the case where $\mathcal{T}$ is algebraic, which also relieves us of the choice of derivator.

The results summarised above have direct implications in other lines of research. Indeed, for example, our techniques make explicit a reformulation (already implicit in [Krause 2005]) of the telescope conjecture for a compactly generated triangulated category in the case where the category is also algebraic.

The structure of the paper is as follows. Section 2 consists of some preliminaries concerning approximation theory, compactly generated triangulated categories and derived categories of small dg categories. In Section 3 we consider compactly generated algebraic triangulated categories, which can be realised both as the homotopy category of an abelian model category and the stable category of a Frobenius exact category. We recall the background that we require on these topics and then use it to prove some preliminary results on closure conditions for subcategories of such categories. In Section 4 we use those closure conditions to obtain approximation-theoretic properties for certain subcategories. In Section 5 we turn to the topic of torsion pairs in triangulated categories and we formulate sufficient conditions for a subcategory to be a torsion class. Finally, Section 6 discusses applications of the previous sections to recollements and the telescope conjecture.

Notation and conventions. By “subcategory” of a given category, we mean a “full and strict subcategory”. Therefore, all subcategories considered are determined by the objects lying in them, and we often refer to the subcategory as the class of its objects. Given a category $\mathcal{X}$ and a subcategory $\mathcal{S}$ of $\mathcal{X}$, we denote by $\mathcal{S}^\perp$ the subcategory of all X in $\mathcal{X}$ such that $\mathrm{Hom}_{\mathcal{X}}(S, X) = 0$, for all S in $\mathcal{S}$. If a subcategory $\mathcal{Y}$ is contained in $\mathcal{S}^\perp$ then we write $\mathrm{Hom}_{\mathcal{X}}(\mathcal{S}, \mathcal{Y}) = 0$. Given a small preadditive category $\mathcal{X}$, we denote by $\mathrm{Mod}(\mathcal{X})$ the abelian category of contravariant additive functors from $\mathcal{X}$ to the category of abelian groups $\mathrm{Mod}(\mathbb{Z})$. Finally, given $\mathcal{C}$ a subcategory of an additive category $\mathcal{X}$, we denote by $\mathrm{Add}(\mathcal{C})$ the subcategory of $\mathcal{X}$ whose objects are summands of direct sums of objects in $\mathcal{C}$.

2. Preliminaries

In this section we begin with some definitions and statements that we will need later. In Section 2A we briefly recall some known results about approximation theory in locally finitely presented categories. Following this, in Section 2B we define compactly generated triangulated categories and concentrate in particular

on the theory of purity therein. Section 2C consists of a short overview of derived categories of small dg categories.

2A. Approximation theory. We begin with the following definitions from approximation theory.

Definition 2.1. A subcategory $\mathcal{X}$ of an additive category $\mathcal{A}$ is said to be:

- *Precovering* if for every object A in $\mathcal{A}$ there is X in $\mathcal{X}$ and a morphism $f : X \rightarrow A$ such that $\text{Hom}_{\mathcal{A}}(X', f)$ is surjective for all X' in $\mathcal{X}$. The map f is then called an $\mathcal{X}$ -precover of A .
- *Preenveloping* if for every object A in $\mathcal{A}$ there is X in $\mathcal{X}$ and a morphism $f : A \rightarrow X$ such that $\text{Hom}_{\mathcal{A}}(f, X')$ is surjective for all X' in $\mathcal{X}$. The map f is then called an $\mathcal{X}$ -preenvelope of A .
- *Covering* if every object A in $\mathcal{A}$ admits a right minimal $\mathcal{X}$ -precover (i.e., an $\mathcal{X}$ -precover $f : X \rightarrow A$ such that any endomorphism g of X with the property that $fg = f$ must be an automorphism).
- *Enveloping* if every object A in $\mathcal{A}$ admits a left minimal $\mathcal{X}$ -preenvelope (i.e., an $\mathcal{X}$ -preenvelope $f : A \rightarrow X$ such that any endomorphism g of X with the property that $gf = f$ must be an automorphism).
- *Coreflective* if the inclusion functor of $\mathcal{X}$ into $\mathcal{A}$ admits a right adjoint.
- *Reflective* if the inclusion functor of $\mathcal{X}$ into $\mathcal{A}$ admits a left adjoint.
- *Bireflective* if the inclusion functor of $\mathcal{X}$ into $\mathcal{A}$ admits both a left and a right adjoint.

Applying the counit of the adjunction we have that any coreflective subcategory is covering and, dually, the unit of the adjunction shows that any reflective subcategory is enveloping.

We will consider sufficiently nice additive categories in which certain closure conditions on a subcategory imply that it is precovering and/or preenveloping.

Definition 2.2. Let $\mathcal{A}$ be a cocomplete additive category (i.e., an additive category with all colimits). An object A in $\mathcal{A}$ is *finitely presented* if $\text{Hom}_{\mathcal{A}}(A, -)$ commutes with directed colimits. The category $\mathcal{A}$ is said to be *locally finitely presented* if there is a set $\mathcal{S}$ of finitely presented objects such that every object of $\mathcal{A}$ is a directed colimit of objects in $\mathcal{S}$.

As $\mathcal{A}$ is a cocomplete additive category, we can also define the notions of pure subobject and pure quotient. This definition will be used to express closure conditions in Theorem 2.4.

Definition 2.3. Let $\mathcal{A}$ be a cocomplete additive category. A morphism $f : X \rightarrow Y$ is said to be a *pure monomorphism* (respectively, a *pure epimorphism*) in $\mathcal{A}$ if there

is a directed system of split monomorphisms (respectively, split epimorphisms) $(f_i : X_i \rightarrow Y_i)_{i \in I}$ such that $f = \varinjlim_I f_i$. An object X is said to be a *pure subobject* (respectively, a *pure quotient*) of Y if there is a pure monomorphism $X \rightarrow Y$ (respectively, a pure epimorphism $X \rightarrow Y$).

The following theorem holds more generally. However, for simplicity, we restrict its scope to the setting in Definition 2.2.

Theorem 2.4. *Let $\mathcal{X}$ be a subcategory of a locally finitely presented additive category $\mathcal{A}$.*

- (1) *If $\mathcal{X}$ is closed under pure quotients, then it is closed under directed colimits if and only if it is closed under coproducts.*
- (2) [Krause 2012, Theorem 4] *If $\mathcal{X}$ is closed under pure quotients and coproducts (or, equivalently, under pure quotients and directed colimits), then $\mathcal{X}$ is precovering.*
- (3) [Krause 2012, Proposition 5] *If $\mathcal{A}$ admits products and if $\mathcal{X}$ is closed under pure subobjects and products, then $\mathcal{X}$ is preenveloping.*

Proof. We discuss only item (1). Every coproduct is a directed colimit and, in a locally finitely presented category, it is easy to see that every directed colimit is a pure quotient of a coproduct. $\square$

Remark 2.5. As observed in [Krause 2012; Adámek and Rosický 1994], in locally finitely presented additive categories, any subcategory closed under directed colimits and pure subobjects is also closed for pure quotients. Hence, if $\mathcal{X}$ is a subcategory of a locally finitely presented category $\mathcal{A}$ closed under directed colimits and pure subobjects, then $\mathcal{X}$ is precovering.

In nice enough categories, we also get that a precovering class closed under directed colimits is, in fact, covering. Once again, taking into account our purposes, we simplify the setting in which the following theorem holds. Recall that a category is said to be *well-powered* if the subobjects of any given object form a set.

Theorem 2.6 [Enochs 1981, Section 7]. *Let $\mathcal{A}$ be a cocomplete and well-powered abelian category and $\mathcal{X}$ a subcategory. If $\mathcal{X}$ is precovering in $\mathcal{A}$ and $\mathcal{X}$ is closed under directed colimits, then $\mathcal{X}$ is covering in $\mathcal{A}$.*

Examples of cocomplete well-powered abelian categories abound. In particular, any Grothendieck category satisfies these properties (see, for example, [Stenström 1975, Chapter IV, Proposition 6.6]).

Corollary 2.7. *Let $\mathcal{A}$ be a locally finitely presented abelian category and $\mathcal{X}$ a subcategory. If $\mathcal{X}$ is closed under pure quotients and coproducts, then it is covering.*

Proof. By [Crawley-Boevey 1994, Section 2.4], any locally finitely presented abelian category is a Grothendieck category and therefore well-powered. This is then a direct combination of Theorems 2.4 and 2.6. $\square$

If $\mathcal{T}$ is a triangulated category, there are also some sufficient conditions guaranteeing that certain precovering (respectively, preenveloping) subcategories are covering (respectively, enveloping).

Definition 2.8. A subcategory $\mathcal{U}$ of $\mathcal{T}$ is *suspended* (respectively, *cosuspended*) if it is closed under summands and extensions, and $\mathcal{U}[1]$ is contained in $\mathcal{U}$ (respectively, $\mathcal{U}[-1]$ is contained in $\mathcal{U}$). Moreover, a subcategory $\mathcal{U}$ is *triangulated* if it is both suspended and cosuspended.

The following theorem is a direct consequence of the arguments presented in [Neeman 2010, Proposition 1.4] (see also Theorem 5.9).

Theorem 2.9 [Neeman 2010, Proposition 1.4]. *Let $\mathcal{U}$ be a subcategory of a triangulated category $\mathcal{T}$.*

- (1) *If $\mathcal{U}$ is suspended, then $\mathcal{U}$ is precovering if and only if $\mathcal{U}$ is covering, and if and only if $\mathcal{U}$ is coreflective.*
- (2) *If $\mathcal{U}$ is cosuspended, then $\mathcal{U}$ is preenveloping if and only if $\mathcal{U}$ is enveloping, and if and only if $\mathcal{U}$ is reflective.*
- (3) *If $\mathcal{U}$ is a triangulated subcategory, then $\mathcal{U}$ is both precovering and preenveloping if and only if $\mathcal{U}$ is both covering and enveloping, and if and only if $\mathcal{U}$ is bireflective.*

2B. Compactly generated triangulated categories. We will consider triangulated categories that have a nice set of generators.

Definition 2.10. If $\mathcal{T}$ is a triangulated category with coproducts, then

- an object X in $\mathcal{T}$ is *compact* if $\mathrm{Hom}_{\mathcal{T}}(X, -)$ commutes with direct sums;
- $\mathcal{T}$ is said to be *compactly generated* if the subcategory of compact objects, denoted by $\mathcal{T}^c$, is skeletally small and generates $\mathcal{T}$ (i.e., if $\mathrm{Hom}_{\mathcal{T}}(C, X) = 0$ for all C compact, then $X \cong 0$).

Compactly generated triangulated categories admit a *pure-exact structure*, which we discuss in the following. We consider the category $\mathrm{Mod}(\mathcal{T}^c)$ of additive functors $(\mathcal{T}^c)^{op} \rightarrow \mathrm{Mod}(\mathbb{Z})$. This is an abelian category with directed colimits and, in fact, it is a locally finitely presented (even locally coherent) Grothendieck category. We denote by $y : \mathcal{T} \rightarrow \mathrm{Mod}(\mathcal{T}^c)$ the functor sending an object X in $\mathcal{T}$ to the functor $yX := \mathrm{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{T}^c}$.

Definition 2.11. A triangle in a compactly generated triangulated category $\mathcal{T}$

$$\Delta : L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow L[1]$$

is said to be a *pure triangle* if the sequence

$$\mathbf{y}(\Delta) : \mathbf{y}L \xrightarrow{\mathbf{y}f} \mathbf{y}M \xrightarrow{\mathbf{y}g} \mathbf{y}N$$

is a short exact sequence in $\text{Mod}(\mathcal{T}^c)$. In that case we say that f is a *pure monomorphism*, g is a *pure epimorphism*, L is a *pure subobject* of M and N is a *pure quotient* of M . Moreover, an object L is *pure-injective* if every pure triangle of the form Δ splits. An object N is *pure-projective* if every pure triangle of the form Δ splits.

It is well known (see, for example, [Beligiannis 2000, Sections 8 and 11]) that the subcategory of pure-projective objects in a compactly generated triangulated category coincides with the additive closure of the compact objects $\text{Add}(\mathcal{T}^c)$. The following lemma provides a useful method to construct pure triangles.

Lemma 2.12 [Krause 2002, Lemma 2.8]. *Let $\mathcal{T}$ be compactly generated. A triangle Δ in $\mathcal{T}$ is pure if and only if there is a directed system of split triangles $(\Delta_i)_{i \in I}$ such that the short exact sequence $\mathbf{y}\Delta$ coincides with the short exact sequence $\varinjlim_I \mathbf{y}\Delta_i$.*

This lemma shows that indeed the notions of pure triangle in $\mathcal{T}$ and pure-exact sequence in the locally finitely presented category $\text{Mod}(\mathcal{T}^c)$ are naturally compatible. Note that, as a consequence of the lemma above, a morphism f in $\mathcal{T}$ is a pure monomorphism if and only if $\mathbf{y}f$ is a pure monomorphism in $\text{Mod}(\mathcal{T}^c)$. Dually, it also follows that a morphism g in $\mathcal{T}$ is a pure epimorphism if and only if $\mathbf{y}g$ is a pure epimorphism in $\text{Mod}(\mathcal{T}^c)$.

2C. Derived categories of dg categories. Among all compactly generated triangulated categories, we will be particularly interested in those that are equivalent to the derived category $\mathcal{D}(\mathcal{A})$ of a small dg category $\mathcal{A}$. If $\mathbb{K}$ is a commutative ring, then a $\mathbb{K}$ -linear category $\mathcal{A}$ is a *dg category* if it is enriched over the category $\mathcal{C}(\mathbb{K})$ of chain complexes over $\mathbb{K}$. Since we will not directly make use of this enriched structure, we refer the reader to the following example for an intuitive idea of what this means and to sources such as [Krause 2007, Section 6] and [Keller 1994a] for a detailed account.

Example 2.13. Let $\mathbb{K}$ be a commutative ring. We will describe a dg category $\mathcal{C}_{\text{dg}}(\mathbb{K})$ with the same objects as $\mathcal{C}(\mathbb{K})$. For any pair X, Y of objects in $\mathcal{C}(\mathbb{K})$ we must define a chain complex $\mathcal{H}om(X, Y)$. For $n \in \mathbb{Z}$, we take

$$\mathcal{H}om(X, Y)^n := \prod_{p \in \mathbb{Z}} \text{Hom}_{\mathbb{K}}(X^p, Y^{p+n})$$

and the differential is defined to be $d^n(f^p) := d_Y \circ f^p - (-1)^n f^{p+1} \circ d_X$ where f^p is in $\text{Hom}_{\mathbb{K}}(X^p, Y^{p+n})$.

The derived category $\mathcal{D}(\mathcal{A})$ is constructed as a localisation of an abelian category $\mathcal{C}(\mathcal{A})$, which is a generalisation of the category of chain complexes over a ring. For the details of the definition of $\mathcal{C}(\mathcal{A})$ we refer to [Keller 1994a, Section 2.1]. The category $\mathcal{D}(\mathcal{A})$ is obtained from $\mathcal{C}(\mathcal{A})$ by formally inverting a class $\mathcal{W}$ of morphisms in $\mathcal{C}(\mathcal{A})$ called quasi-isomorphisms. As one might expect, the morphisms in $\mathcal{W}$ generalise the quasi-isomorphisms in the category of chain complexes over a ring; an exact definition may be found in [Keller 1994a, Section 4.1].

3. Compactly generated algebraic triangulated categories

A famous theorem of Keller [1994a, Subsection 4.1 and Theorem 4.3] states that the class of derived categories of small dg categories coincides with the class of stable categories of Frobenius exact categories that are compactly generated. These categories are called compactly generated algebraic triangulated categories and, in this section, we explore their properties from these two distinct points of view.

The process of formally inverting the quasi-isomorphisms in $\mathcal{C}(\mathcal{A})$ for a small dg category $\mathcal{A}$ may be done within the framework of model categories and we present this point of view in Section 3A. The category $\mathcal{D}(\mathcal{A})$ may also be realised as the stable category of a Frobenius exact category and there is a close relationship between the torsion pairs in $\mathcal{D}(\mathcal{A})$ and the cotorsion pairs in the Frobenius exact category. More details of this construction are given in Section 3B. It is the interplay between these two structures underlying the category $\mathcal{D}(\mathcal{A})$ that will allow us to develop approximation theory in the later sections.

3A. Compactly generated algebraic triangulated categories as homotopy categories. One way of seeing that the localisation of $\mathcal{C}(\mathcal{A})$ at $\mathcal{W}$ exists is to observe that there is a model structure on $\mathcal{C}(\mathcal{A})$ such that the quasi-isomorphisms coincide with the weak equivalences. In particular, we will consider the so-called *projective model structure* on $\mathcal{C}(\mathcal{A})$ described in the following proposition. We refer the reader to [Hirschhorn 2003; Hovey 1999] for more details on the theory of model categories. Given a model category $\mathcal{M}$ we will denote its homotopy category by $\mathrm{Ho}(\mathcal{M})$. For a proof of the following well-known proposition, see [Becker 2014, Proposition 1.3.5].

Proposition 3.1. *Let $\mathcal{A}$ be a small dg category. There exists a model structure on $\mathcal{C}(\mathcal{A})$ such that the weak equivalences are the quasi-isomorphisms and every object is fibrant and it is uniquely determined by these properties. The homotopy category of this model structure is equivalent to $\mathcal{D}(\mathcal{A})$.*

Notation 3.2. We will fix the following notation with respect to the projective model structure.

- We denote the associated localisation functor by $\pi : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$.
- We will denote the class of weak equivalences (i.e., quasi-isomorphisms) by $\mathcal{W}$.

- We will denote the class of cofibrant objects by $\mathcal{C}$ (Keller [1994a] refers to the objects in $\mathcal{C}$ as *objects with property (P)*).

By [Šťovíček and Pospíšil 2016, Lemma 2.12], the category $\mathcal{C}(\mathcal{A})$ is a locally finitely presented Grothendieck category with enough projective objects. In Section 4 we will prove an analogue of Theorem 2.4 for $\mathcal{D}(\mathcal{A})$. The strategy will be to determine which subcategories $\mathcal{X}$ of $\mathcal{D}(\mathcal{A})$ have a preimage $\pi^{-1}(\mathcal{X})$ in $\mathcal{C}(\mathcal{A})$ satisfying the closure conditions specified in Theorem 2.4. We, therefore, need to discuss the products, coproducts and directed colimits (or suitable replacements of these) in both $\mathcal{C}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$.

It is well known that the model structure in $\mathcal{C}(\mathcal{A})$ described in the proposition above induces model structures in the category of functors from a small category S to $\mathcal{C}(\mathcal{A})$ [Lurie 2009, Proposition A.2.8.2]. We denote this category, usually called the category of S -diagrams in $\mathcal{C}(\mathcal{A})$, by $\mathcal{C}(\mathcal{A})^S$ and the corresponding localisation functor by $\pi_S : \mathcal{C}(\mathcal{A})^S \rightarrow \mathrm{Ho}(\mathcal{C}(\mathcal{A})^S)$.

Remark 3.3. It is easy to see that if S is a discrete category (i.e., if the only morphisms of S are identities on objects), then $\mathcal{C}(\mathcal{A})^S$ is nothing but the product, indexed by S , of copies of the category $\mathcal{C}(\mathcal{A})$. In that case, it then follows that the homotopy category for the induced model structure is nothing but the product of the respective homotopy categories, i.e., $\mathrm{Ho}(\mathcal{C}(\mathcal{A})^S) \simeq \mathrm{Ho}(\mathcal{C}(\mathcal{A}))^S \simeq \mathcal{D}(\mathcal{A})^S$. However, this equivalence does not generally hold for small categories which are not discrete.

We recall that if $F : \mathcal{M} \rightarrow \mathcal{N}$ is a functor between model categories, then the total left derived functor (respectively, the total right derived functor) is a functor $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightarrow \mathrm{Ho}(\mathcal{N})$ (respectively, $\mathbb{R}F : \mathrm{Ho}(\mathcal{M}) \rightarrow \mathrm{Ho}(\mathcal{N})$) — see [Hirschhorn 2003, Section 8.4] for the definition of total derived functors. We consider the following total derived functors:

- For each directed small category S (i.e., the set of objects of S is preordered by its morphisms and every finite subset has an upper bound), we consider the total left derived functor of the directed colimit functor $\varinjlim_S : \mathcal{C}(\mathcal{A})^S \rightarrow \mathcal{C}(\mathcal{A})$, and we denote it by

$$\mathbb{L} \varinjlim_S : \mathrm{Ho}(\mathcal{C}(\mathcal{A})^S) \rightarrow \mathrm{Ho}(\mathcal{C}(\mathcal{A})) \simeq \mathcal{D}(\mathcal{A}).$$

This derived functor turns out to be left adjoint to the natural diagonal (or constant) functor $\delta_S : \mathrm{Ho}(\mathcal{C}(\mathcal{A})) \simeq \mathcal{D}(\mathcal{A}) \rightarrow \mathrm{Ho}(\mathcal{C}(\mathcal{A})^S)$ and we will, therefore, refer to $\mathbb{L} \varinjlim_S$ as the *directed homotopy colimit functor*.

- For each discrete category S , we consider the total right derived functor of the product functor $\prod : \mathcal{C}(\mathcal{A})^S \rightarrow \mathcal{C}(\mathcal{A})$, which yields a functor $\mathbb{R} \prod : \mathcal{D}(\mathcal{A})^S \rightarrow \mathcal{D}(\mathcal{A})$. This derived functor is right adjoint to the natural diagonal functor $\delta_S : \mathrm{Ho}(\mathcal{C}(\mathcal{A})) \simeq \mathcal{D}(\mathcal{A}) \rightarrow \mathrm{Ho}(\mathcal{C}(\mathcal{A})^S) \cong \mathcal{D}(\mathcal{A})^S$ and, therefore, it coincides with the product functor in $\mathcal{D}(\mathcal{A})$ [Hovey 1999, Example 1.3.11].

- For each discrete category S , we consider the total left derived functor of the coproduct functor $\coprod : \mathcal{C}(\mathcal{A})^S \rightarrow \mathcal{C}(\mathcal{A})$, which yields a functor $\mathbb{L} \coprod : \mathcal{D}(\mathcal{A})^S \rightarrow \mathcal{D}(\mathcal{A})$. This derived functor is left adjoint to the natural diagonal functor $\delta_S : \mathrm{Ho}(\mathcal{C}(\mathcal{A})) \simeq \mathcal{D}(\mathcal{A}) \rightarrow \mathrm{Ho}(\mathcal{C}(\mathcal{A})^S) \cong \mathcal{D}(\mathcal{A})^S$ and, therefore, it coincides with the coproduct functor in $\mathcal{D}(\mathcal{A})$ [Hovey 1999, Example 1.3.11].

In the same vein as [Saorín et al. 2017, Lemma 7.1(4)] we observe that the exactness of (co)products and directed colimits in $\mathcal{C}(\mathcal{A})$ allow for an easy computation of (co)products and directed homotopy colimits in $\mathcal{D}(\mathcal{A})$.

Proposition 3.4. *Let $\mathcal{A}$ be a small dg category and let S be a small category.*

- (1) *If S is directed and X is an object of $\mathcal{C}(\mathcal{A})^S$, then $\mathbb{L} \varinjlim_S \pi_S X \cong \pi(\varinjlim_S X)$.*
- (2) *If S is discrete and $X = (X_s)_{s \in S}$ is an object of $\mathcal{C}(\mathcal{A})^S$, then $\prod_{s \in S} \pi X_s \cong \pi(\prod_{s \in S} X_s)$.*
- (3) *If S is discrete and $X = (X_s)_{s \in S}$ is an object of $\mathcal{C}(\mathcal{A})^S$, then $\coprod_{s \in S} \pi X_s \cong \pi(\coprod_{s \in S} X_s)$.*

Proof. This result is well known, but we sketch an argument along the lines of [Toën and Vaquié 2007, Proposition 2.2(2)]. We focus on statement (1), since the proof of statements (2) and (3) are analogous. Let S be a directed small category. First we observe that, since $\varinjlim_S$ is exact in $\mathrm{Mod}(\mathbb{Z})$, it commutes with the cohomology functor in $\mathcal{C}(\mathcal{A})$. This implies that weak equivalences (i.e., quasi-isomorphisms) are preserved under directed colimits. The statement then follows from [Rahn $\geq$ 2020, Corollary 7.2.5]. $\square$

By [Saorín et al. 2017, Proposition 5.4], we know that $\mathbf{y}$ sends directed homotopy colimits in $\mathcal{T} := \mathcal{D}(\mathcal{A})$ to directed colimits in $\mathrm{Mod}(\mathcal{T}^c)$. We therefore obtain the following corollary of Proposition 3.4. In fact, it was proved directly in [Nicolás and Saorín 2009, Lemma 6.3(2)].

Corollary 3.5. *Let $\mathcal{A}$ be a small dg category. Denote $\mathcal{T} := \mathcal{D}(\mathcal{A})$ and let $\mathbf{y} : \mathcal{T} \rightarrow \mathrm{Mod}(\mathcal{T}^c)$ be the functor defined in Section 2B. Then the composition $\mathbf{y} \circ \pi : \mathcal{C}(\mathcal{A}) \rightarrow \mathrm{Mod}(\mathcal{T}^c)$ preserves directed colimits.*

The upshot of Proposition 3.4 is that we can transfer closure conditions between $\mathcal{C}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$. In order to make this precise, we will need the following definitions.

Definition 3.6. Let $\mathcal{A}$ be a small dg category and $\mathcal{X}$ a subcategory of $\mathcal{D}(\mathcal{A})$.

- (1) The *preimage* of $\mathcal{X}$ in $\mathcal{C}(\mathcal{A})$ is the subcategory

$$\pi^{-1}(\mathcal{X}) := \{X \in \mathcal{C}(\mathcal{A}) \mid \pi(X) \in \mathcal{X}\}.$$

- (2) The subcategory $\mathcal{X}$ is *closed under directed homotopy colimits* if for every directed small category S and every object $X = (X_s)_{s \in S}$ in $\mathcal{C}(\mathcal{A})^S$ such that $\pi(X_s)$ is in $\mathcal{X}$ for every s in S , we have that $\mathbb{L} \varinjlim_S \pi_S X$ is in $\mathcal{X}$.

Corollary 3.7. *Let $\mathcal{A}$ be a small dg category and let $\mathcal{X}$ be a subcategory of $\mathcal{D}(\mathcal{A})$. The following statements hold.*

- (1) $\mathcal{X}$ is closed under products if and only if $\pi^{-1}(\mathcal{X})$ is closed under products.
- (2) $\mathcal{X}$ is closed under coproducts if and only if $\pi^{-1}(\mathcal{X})$ is closed under coproducts.
- (3) $\mathcal{X}$ is closed under directed homotopy colimits if and only if $\pi^{-1}(\mathcal{X})$ is closed under directed colimits.

The following lemma indicates that the property of being closed under pure subobjects or pure quotients can also be transferred from $\mathcal{D}(\mathcal{A})$ to $\mathcal{C}(\mathcal{A})$.

Lemma 3.8. *Let $\mathcal{A}$ be a small dg category and let $\mathcal{X}$ be a subcategory of $\mathcal{D}(\mathcal{A})$. If $\mathcal{X}$ is closed under pure quotients (respectively, pure subobjects) in $\mathcal{D}(\mathcal{A})$, then so is its preimage $\pi^{-1}(\mathcal{X})$.*

Proof. We prove the statement for pure quotients, since the other statement can be proved analogously. Let M be an object in $\pi^{-1}(\mathcal{X})$ and suppose that there is a pure epimorphism $f : M \rightarrow N$ in the locally finitely presented category $\mathcal{C}(\mathcal{A})$. Then we have that $f = \varinjlim_I f_i$ for some directed system of split epimorphisms $(f_i)_{i \in I}$ in $\mathcal{C}(\mathcal{A})$. Since, by Corollary 3.5 the composition $y \circ \pi$ commutes with directed colimits, it follows that $y\pi(f) = \varinjlim_I y\pi(f_i)$. Since $\pi(f_i)$ are split epimorphisms in $\mathcal{D}(\mathcal{A})$, from Lemma 2.12 we have that $\pi(f)$ is a pure epimorphism in $\mathcal{T}$ and, thus, $\pi(N)$ must also lie in $\mathcal{X}$. Hence N lies in $\pi^{-1}(\mathcal{X})$ as wanted. $\square$

Now we can prove the analogous statement to Theorem 2.4(1).

Proposition 3.9. *Let $\mathcal{A}$ be a small dg category and let $\mathcal{X}$ be a subcategory of $\mathcal{D}(\mathcal{A})$ that is closed under pure quotients. Then the subcategory $\mathcal{X}$ is closed under directed homotopy colimits if and only if it is closed under coproducts.*

Proof. Since every coproduct is a directed homotopy colimit, it remains to show that if $\mathcal{X}$ that is closed under coproducts then it is also closed under directed homotopy colimits. By Lemma 3.8, the subcategory $\pi^{-1}(\mathcal{X})$ of $\mathcal{C}(\mathcal{A})$ is closed under pure quotients and it is also clearly closed under coproducts. By Theorem 2.4(1), it follows that $\pi^{-1}(\mathcal{X})$ is closed under directed colimits and, therefore, by Corollary 3.7, we conclude that $\mathcal{X}$ is closed under directed homotopy colimits. $\square$

3B. Compactly generated algebraic triangulated categories as stable categories.

The stable category $\underline{\mathcal{F}}$ of a Frobenius exact category $\mathcal{F}$ is always triangulated [Happel 1988, Theorem 2.6] and many, if not all, triangulated categories arising in algebra are of this form. This therefore motivates the next definition.

Definition 3.10. A triangulated category is called *algebraic* if it is equivalent to the stable category of a Frobenius exact category.

We will make use of the following Frobenius exact category yielding $\mathcal{D}(\mathcal{A})$ as its stable category.

Proposition 3.11 (see, e.g., [Beligiannis and Reiten 2007, Theorem VIII.4.2]). *Let $\mathcal{A}$ be a small dg category and let $\mathcal{C}$ be the subcategory of $\mathcal{C}(\mathcal{A})$ consisting of cofibrant objects in the projective model structure. The following statements hold.*

- (1) *The subcategory $\mathcal{C}$ with the exact structure inherited from the abelian structure on $\mathcal{C}(\mathcal{A})$ is a Frobenius exact category.*
- (2) *The restriction $\pi|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{D}(\mathcal{A})$ induces an equivalence of categories between $\underline{\mathcal{C}}$ and $\mathcal{D}(\mathcal{A})$.*

We finish this section with a brief account of the connection between the torsion pairs in $\underline{\mathcal{C}}$ and the complete cotorsion pairs in $\mathcal{C}$. This will prove useful in Section 5.

Definition 3.12. A pair of subcategories $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{T}$ is called a *torsion pair* if

- (1) $\mathcal{U}$ and $\mathcal{V}$ are closed under summands;
- (2) $\text{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$;
- (3) for any object X in $\mathcal{T}$, there are objects U in $\mathcal{U}$, V in $\mathcal{V}$ and a triangle

$$U \rightarrow X \rightarrow V \rightarrow U[1].$$

If $(\mathcal{U}, \mathcal{V})$ is a torsion pair in $\mathcal{T}$, then we say that $\mathcal{U}$ is a *torsion class* and $\mathcal{V}$ is a *torsionfree class*.

It follows easily from the definition that $\mathcal{V} = \mathcal{U}^{\perp}$ and $\mathcal{U} = {}^{\perp}\mathcal{V}$. If $\mathcal{T}$ is an algebraic triangulated category, then torsion pairs in $\mathcal{T}$ are in bijection with certain pairs of subcategories of $\mathcal{F}$. Let us recall the definition of a complete cotorsion pair in an exact category.

Definition 3.13. Let $\mathcal{F}$ be an exact category. We say that a pair of subcategories $(\mathcal{M}, \mathcal{N})$ of $\mathcal{F}$ is a *complete cotorsion pair* if the following statements hold.

- $\mathcal{N} = \text{Ker Ext}_{\mathcal{F}}^1(\mathcal{M}, -)$ and $\mathcal{M} = \text{Ker Ext}_{\mathcal{F}}^1(-, \mathcal{N})$.
- $\mathcal{M}$ is *special precovering*, i.e., for every object X in $\mathcal{F}$ there are objects M_1 in $\mathcal{M}$, N_1 in $\mathcal{N}$ and a conflation

$$0 \longrightarrow N_1 \longrightarrow M_1 \xrightarrow{f} X \longrightarrow 0.$$

- $\mathcal{N}$ is *special preenveloping*, i.e., for every object X in $\mathcal{F}$ there are objects N_2 in $\mathcal{N}$, M_2 in $\mathcal{M}$ and a conflation

$$0 \longrightarrow X \xrightarrow{g} N_2 \longrightarrow M_2 \longrightarrow 0.$$

From the above conflations, it is easy to see that, indeed, f is an $\mathcal{M}$ -precover and g is an $\mathcal{N}$ -preenvelope.

Example 3.14. The well-known Wakamatsu’s lemma states that, given an extension-closed precovering subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ such that $\mathcal{X}$ -precovers are epimorphisms, if $\mathcal{X}$ is covering then it is special precovering. In particular, if $\mathcal{A}$ is a locally finitely presented abelian category with enough projectives and $\mathcal{X}$ is a subcategory of $\mathcal{A}$ containing the projective objects and closed under extensions, directed colimits and pure quotients, then by Corollary 2.7 and Wakamatsu’s lemma, $\mathcal{X}$ is special precovering in $\mathcal{A}$.

Example 3.15. Abelian model structures are intimately related with cotorsion pairs (see [Hovey 2002]). In particular, the projective model structure in $\mathcal{C}(\mathcal{A})$ from Proposition 3.1 guarantees the existence of a cotorsion pair $(\mathcal{C}, \mathcal{W}_0)$, where $\mathcal{C}$ is the subcategory of cofibrant objects and $\mathcal{W}_0$ is the category of trivial objects (i.e., those which are quasi-isomorphic to zero or, in other words, acyclic). Moreover, this cotorsion pair is hereditary, i.e., $\text{Ext}_{\mathcal{C}(\mathcal{A})}^n(C, W) = 0$ for all $n > 0$ and for all C in $\mathcal{C}$ and W in $\mathcal{W}_0$ (see, for example, [Becker 2014, Corollary 1.1.12]).

Theorem 3.16 [Saorín and Šťovíček 2011, Proposition 3.3]. *Let $\mathcal{T}$ be the stable category of a Frobenius exact category $\mathcal{F}$ and let $\phi : \mathcal{F} \rightarrow \underline{\mathcal{F}} = \mathcal{T}$ denote the canonical functor. There is a bijection between complete cotorsion pairs in $\mathcal{F}$ and torsion pairs in $\mathcal{T}$ given by the assignment $(\mathcal{M}, \mathcal{N}) \mapsto (\phi\mathcal{M}, (\phi\mathcal{N})[1])$.*

4. Approximation theory in algebraic triangulated categories

In this section we discuss sufficient closure conditions for a subcategory of a compactly generated algebraic triangulated category to be precovering or preenveloping. The trick is to lift those closure conditions to a locally finitely presented category and use the results on approximation theory available there. All the results in this section will be stated for a compactly generated algebraic triangulated category $\mathcal{T}$ but we wish to make reference to some notion of homotopy colimit in $\mathcal{T}$. We therefore make the following definition.

Definition 4.1. Let $\mathcal{T}$ be a compactly generated algebraic triangulated category. We will say that a subcategory $\mathcal{X}$ is *closed under homotopy colimits* if there exists a small dg category $\mathcal{A}$ and an equivalence $F : \mathcal{T} \rightarrow \mathcal{D}(\mathcal{A})$ such that $F(\mathcal{X})$ is closed under homotopy colimits in $\mathcal{D}(\mathcal{A})$.

The following theorem is a triangulated analogue of Theorem 2.4.

Theorem 4.2. *Let $\mathcal{X}$ be a subcategory of a compactly generated algebraic triangulated category $\mathcal{T}$.*

- (1) *If $\mathcal{X}$ is closed under pure quotients and coproducts, then $\mathcal{X}$ is precovering.*
- (2) *If $\mathcal{X}$ is closed under pure subobjects and products, then $\mathcal{X}$ is preenveloping.*

Proof. (1) Since there exists a small dg category $\mathcal{A}$ such that $\mathcal{T} \simeq \mathcal{D}(\mathcal{A})$, it suffices to prove (1) for $\mathcal{D}(\mathcal{A})$. So let $\mathcal{X}$ be a subcategory of $\mathcal{D}(\mathcal{A})$ that is closed under pure quotients and coproducts. Let $\pi^{-1}(\mathcal{X})$ denote the preimage of $\mathcal{X}$ in $\mathcal{C}(\mathcal{A})$ (as in Definition 3.6). Then, by Corollary 3.7 and Lemma 3.8, the subcategory $\pi^{-1}(\mathcal{X})$ is closed under coproducts and pure quotients. Thus, it follows from Theorem 2.4(2) that $\pi^{-1}(\mathcal{X})$ is precovering in $\mathcal{C}(\mathcal{A})$. We use this fact to show that $\mathcal{X}$ is precovering in $\mathcal{D}(\mathcal{A})$.

Let $\pi(M)$ be an arbitrary object in $\mathcal{D}(\mathcal{A})$ and consider a $\pi^{-1}(\mathcal{X})$ -precover $p : X \rightarrow M$ of M in $\mathcal{C}(\mathcal{A})$. We show that $\pi(p) : \pi(D) \rightarrow \pi(M)$ is an $\mathcal{X}$ -precover. Let $f : \pi(E) \rightarrow \pi(M)$ be any morphism from an object in $\mathcal{X}$ to $\pi(M)$ and let $c : C_E \rightarrow E$ be a cofibrant replacement of E . Note that $\pi(c)$ is an isomorphism and so f factors through $\pi(p)$ if and only if $f \circ \pi(c)$ factors through $\pi(p)$. Moreover, the morphism $f \circ \pi(c)$ is of the form $\pi(g)$ for some $g : C_E \rightarrow M$ because C_E is cofibrant. Finally, we observe that C_E is contained in $\pi^{-1}(\mathcal{X})$ and so g factors through p . Thus, $\pi(g) = f \circ \pi(c)$ (and therefore f) factors through $\pi(p)$ as required.

(2) The proof is very similar to (1), the only significant difference being that to build a $\mathcal{X}$ -preenvelope of an object $\pi(M)$ in $\mathcal{D}(\mathcal{A})$ we should consider a $\pi^{-1}(\mathcal{X})$ -preenvelope of a cofibrant replacement C_M of M . Indeed, if $p : C_M \rightarrow M$ is such a cofibrant replacement and $f : C_M \rightarrow X$ is a $\pi^{-1}(\mathcal{X})$ -preenvelope of C_M , then the composition $\pi(f) \circ \pi(p)^{-1}$ can be shown as above to be a $\mathcal{X}$ -preenvelope of $\pi(M)$. $\square$

Corollary 4.3. *If $\mathcal{X}$ is closed under pure quotients and directed homotopy colimits, then $\mathcal{X}$ is precovering.*

Proof. This is immediate from Theorem 4.2 and Proposition 3.9. $\square$

Remark 4.4. Using Remark 2.5, it can be seen that a subcategory $\mathcal{X}$ of a compactly generated algebraic triangulated category $\mathcal{T}$ that is closed under directed homotopy colimits and pure subobjects is also closed under pure quotients. Hence, such a subcategory $\mathcal{X}$ is also precovering.

Example 4.5. Consider an object X in a compactly generated algebraic triangulated category $\mathcal{T}$. Let $\text{PGen}(X)$ denote the subcategory of $\mathcal{T}$ whose objects are pure quotients of coproducts of X . Dually, let $\text{PCogen}(X)$ denote the subcategory of $\mathcal{T}$ whose objects are pure subobjects of products of X . Using the fact that products and coproducts preserve pure epimorphisms and pure monomorphisms, it then follows from Theorem 4.2 that $\text{PGen}(X)$ is precovering and that $\text{PCogen}(X)$ is preenveloping.

Note that, in view of Remark 4.4, any subcategory closed under pure subobjects, directed homotopy colimits and products will be simultaneously precovering and preenveloping. It turns out that these closure conditions actually characterise an important type of subcategories.

Definition 4.6. Let $\mathcal{T}$ be a compactly generated triangulated category. An additive (covariant) functor $F : \mathcal{T} \rightarrow \text{Mod}(\mathbb{Z})$ is said to be *coherent* if there is an exact sequence

$$\text{Hom}_{\mathcal{T}}(Y, -) \xrightarrow{\text{Hom}_{\mathcal{T}}(f, -)} \text{Hom}_{\mathcal{T}}(X, -) \longrightarrow F \longrightarrow 0$$

with $f : X \rightarrow Y$ a map in $\mathcal{T}^c$. A subcategory $\mathcal{X}$ of $\mathcal{T}$ is said to be *definable* if there is a set of coherent functors $(F_i)_{i \in I}$ such that $F_i(X) = 0$ for all i in I if and only if X lies in $\mathcal{X}$.

The following theorem was originally stated in the more general setting of stable derivators [Laking 2019, Theorem 3.11]. Derived categories of small dg categories lie at the base of a stable derivator since they are homotopy categories of nice model categories [Rahn 2013, Proposition 1.30]. We may, therefore, restate the theorem in our setting and furthermore, we use Proposition 3.9 to “simplify” the original statement.

Theorem 4.7. *The following are equivalent for a subcategory $\mathcal{X}$ of a compactly generated algebraic triangulated category.*

- (1) $\mathcal{X}$ is definable.
- (2) $\mathcal{X}$ is closed under pure subobjects, products and directed homotopy colimits.
- (3) $\mathcal{X}$ is closed under pure subobjects, products and pure quotients.

Proof. (1) $\Leftrightarrow$ (2): This statement holds in our setting by [Laking 2019, Theorem 3.11].

(2) $\Rightarrow$ (3): It follows from Remark 4.4 that if $\mathcal{X}$ is closed under pure subobjects and directed homotopy colimits, then it is also closed under pure quotients.

(3) $\Rightarrow$ (2): Since any coproduct is a pure subobject of the corresponding product, if $\mathcal{X}$ is closed under pure subobjects and products it is then closed under coproducts as well. If additionally $\mathcal{X}$ is closed under pure quotients, then by Proposition 3.9 it is also closed under directed homotopy colimits. $\square$

As expected, definable subcategories therefore have nice approximation-theoretic properties.

Corollary 4.8. *If $\mathcal{X}$ is a definable subcategory of a compactly generated algebraic triangulated category, then $\mathcal{X}$ is precovering and preenveloping.*

Proof. This is an immediate consequence of Theorems 4.7 and 4.2. $\square$

Remark 4.9. Note that it was also shown in [Angeleri Hgel et al. 2017, Proposition 4.5] that definable subcategories are preenveloping in any compactly generated triangulated category.

Using Theorem 2.9, we provide sufficient conditions for a subcategory to be covering or enveloping.

Corollary 4.10. *Let $\mathcal{X}$ be a subcategory of a compactly generated algebraic triangulated category.*

- (1) *If $\mathcal{X}$ is suspended and closed under coproducts and pure quotients, it is covering.*
- (2) *If $\mathcal{X}$ is cosuspended and closed under products and pure subobjects, it is enveloping.*

In particular, if $\mathcal{X}$ is triangulated and definable, it is both covering and enveloping.

Proof. Statements (1) and (2) follow from Theorems 2.9 and 4.2. The last statement follows from Theorem 4.7, combining (1) and (2) (or, alternatively, from Theorem 2.9 and Corollary 4.8). $\square$

Remark 4.11. Suspended subcategories closed under coproducts are also closed under directed homotopy colimits. This is a conjugation of [Šťovíček and Pospíšil 2016, Proposition 2.4] and [Ponto and Shulman 2016, Theorem 7.13]. It then follows Remark 4.4 and part (1) of the corollary above that if $\mathcal{X}$ is suspended and closed under coproducts and pure subobjects, then $\mathcal{X}$ is covering.

Also by Theorem 2.9, subcategories satisfying the conditions of Corollary 4.10 are part of certain torsion pairs (namely t-structures). That is the topic of Section 5.

5. Torsion pairs in algebraic triangulated categories

Typical examples of (pre)covering and (pre)enveloping subcategories of triangulated categories occur in torsion pairs (see Definition 3.12). Clearly, every torsion class is precovering and every torsionfree class is preenveloping. In the next theorem we will make use of the results of Section 4, as well as the correspondence given by Theorem 3.16 in order to establish sufficient conditions, via closure properties, for a subcategory to be a torsion class. We will need the following lemma.

Lemma 5.1. *Let $\mathcal{F}$ be an exact category with enough injectives and $\mathcal{M}$ a subcategory of $\mathcal{F}$. Then there is a complete cotorsion pair $(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M}$ is a special precovering and idempotent complete subcategory of $\mathcal{F}$.*

Proof. It is clear from the definition that a subcategory $\mathcal{M}$ which is part of a complete cotorsion pair $(\mathcal{M}, \mathcal{N})$ must be special precovering and idempotent complete. We prove the converse statement. Indeed, let $\mathcal{M}$ be special precovering and idempotent complete, and consider the class $\mathcal{N} := \text{Ker Ext}_{\mathcal{F}}^1(\mathcal{M}, -)$. We must show that $\mathcal{M} = \text{Ker Ext}_{\mathcal{F}}^1(-, \mathcal{N})$. Clearly $\mathcal{M}$ is contained in $\text{Ker Ext}_{\mathcal{F}}^1(-, \mathcal{N})$. Suppose X is in $\text{Ker Ext}_{\mathcal{F}}^1(-, \mathcal{N})$ and let

$$0 \longrightarrow N_1 \longrightarrow M_1 \xrightarrow{f} X \longrightarrow 0$$

be a conflation of $\mathcal{F}$, where f is a special precover of X . Since $\text{Ext}_{\mathcal{F}}^1(X, N_1) = 0$, the sequence is then split and so X lies in $\mathcal{M}$.

Finally we must obtain a special $\mathcal{N}$ -preenvelope for each object Y in $\mathcal{F}$. Since $\mathcal{F}$ has enough injectives, we may apply Salce's lemma (see, for example, [Göbel and Trlifaj 2006, Lemma 2.2.6]) to obtain a conflation

$$0 \longrightarrow Y \xrightarrow{g} N_2 \longrightarrow M_2 \longrightarrow 0$$

with M_2 in $\mathcal{M}$ and N_2 in $\mathcal{N}$. $\square$

We can now prove the main theorem of this section.

Theorem 5.2. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category. Then every subcategory $\mathcal{X}$ that is closed under extensions, pure quotients and coproducts is a torsion class. In particular, every definable extension-closed subcategory is a torsion class.*

Proof. Let $\mathcal{A}$ be a small dg category such that $\mathcal{T} = \mathcal{D}(\mathcal{A})$ for which the subcategory $\mathcal{X}$ of $\mathcal{T}$ satisfies the closure conditions above. Let $\mathcal{C}$ be the Frobenius exact subcategory of $\mathcal{C}(\mathcal{A})$ as in Proposition 3.11. By Theorem 3.16, it is sufficient to show that there is a cotorsion pair $(\mathcal{M}, \mathcal{N})$ in $\mathcal{C}$ such that $\pi(\mathcal{M}) = \mathcal{X}$. So let $\pi^{-1}(\mathcal{X})$ be the preimage of $\mathcal{X}$ (see Definition 3.6) and let $\mathcal{M} = \mathcal{C} \cap \pi^{-1}(\mathcal{X})$. By [Becker 2014, Proposition 1.1.15], we have that $\pi(\mathcal{M}) = \mathcal{X}$.

We will show that $\mathcal{M}$ is an idempotent complete and special precovering subcategory of $\mathcal{C}$, and the result then follows from Lemma 5.1. First, since $\mathcal{X}$ is closed under pure quotients, then so is $\pi^{-1}(\mathcal{X})$ by Lemma 3.8. In particular, $\pi^{-1}(\mathcal{X})$ is closed under direct summands and, since $\mathcal{C}$ also has this property, it follows that indeed $\mathcal{M}$ is idempotent complete. So it only remains to show that $\mathcal{M}$ is special precovering in $\mathcal{C}$. We first prove that it is, in fact, a special precovering class in $\mathcal{C}(\mathcal{A})$. Note that, since $\pi^{-1}(\mathcal{X})$ is also closed under coproducts, it follows from Corollary 2.7 that it is covering in $\mathcal{C}(\mathcal{A})$ and, moreover, it contains all the projective objects of $\mathcal{C}(\mathcal{A})$ (i.e., the trivial cofibrant objects in $\mathcal{C}(\mathcal{A})$). Since $\pi^{-1}(\mathcal{X})$ is also extension-closed it then follows from Wakamatsu's lemma (see Example 3.14) that for each Z in $\mathcal{C}(\mathcal{A})$ there is a short exact sequence

$$0 \longrightarrow K \longrightarrow X \xrightarrow{f} Z \longrightarrow 0$$

with K in $\text{Ker Ext}_{\mathcal{C}(\mathcal{A})}^1(\pi^{-1}(\mathcal{X}), -)$ and X in $\pi^{-1}(\mathcal{X})$. Moreover, the subcategory of cofibrant objects $\mathcal{C}$ is part of a complete cotorsion pair $(\mathcal{C}, \mathcal{W}_0)$, where $\mathcal{W}_0$ is the subcategory of $\mathcal{C}(\mathcal{A})$ formed by the acyclic dg modules (see Example 3.15). Therefore there exists a short exact sequence

$$0 \longrightarrow W \longrightarrow C \xrightarrow{g} X \longrightarrow 0$$

with W in $\mathcal{W}_0$ and C in $\mathcal{C}$. We claim that $f \circ g : C \rightarrow Z$ is a special $\mathcal{M}$ -precover in $\mathcal{C}(\mathcal{A})$. First observe that, indeed, C lies in $\mathcal{M}$ since $\pi(C) \cong \pi(X)$. It now suffices

to check that $\text{Ext}_{\mathcal{C}(\mathcal{A})}^1(\mathcal{M}, \text{Ker}(f \circ g)) = 0$. The snake lemma guarantees that there is a short exact sequence

$$0 \longrightarrow W \longrightarrow \text{Ker}(f \circ g) \longrightarrow K \longrightarrow 0.$$

Applying $\text{Ext}_{\mathcal{C}(\mathcal{A})}^1(M, -)$ to the sequence, with M in $\mathcal{M}$, yields

$$\text{Ext}_{\mathcal{C}(\mathcal{A})}^1(M, \text{Ker}(f \circ g)) = 0$$

as wanted. Finally, since $(\mathcal{C}, \mathcal{W}_0)$ is a hereditary cotorsion pair (see Example 3.15), $\mathcal{C}$ is closed under kernels of deflations and $\text{Ker}(f \circ g)$ lies in $\mathcal{C}$. Therefore, we have that $\mathcal{M}$ is special precovering in $\mathcal{C}$. $\square$

Remark 5.3. It is well known that in a well-powered cocomplete abelian category, torsion classes are precisely those which are closed under extensions, coproducts and quotients. The theorem above provides us with a triangulated analogue for the sufficiency of these closure conditions. Note, however, that while torsion classes in abelian categories are covering, this is not necessarily the case in triangulated categories.

This theorem allows us to construct a series of torsion pairs.

Corollary 5.4. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category. Then the following classes are torsion classes in $\mathcal{T}$.*

- (1) $P^\perp$ for any pure-projective object P .
- (2) ${}^\perp E$ for any pure-injective object E .
- (3) $\text{PGen}(P)$ for a pure-projective object P satisfying $\text{Hom}_{\mathcal{T}}(P, P^{(I)}[1]) = 0$ for any set I .

Proof. (1) If P is pure-projective, then it is easy to see that $P^\perp$ is closed under pure subobjects, pure quotients and products. Therefore, by Theorem 4.7, $P^\perp$ is definable. In particular it is also closed under coproducts. Clearly $P^\perp$ is extension-closed and, therefore, it satisfies the assumptions of Theorem 5.2.

(2) If E is pure-injective, then it is easy to see that ${}^\perp E$ satisfies the assumptions of Theorem 5.2.

(3) By Example 4.5, the subcategory $\text{PGen}(P)$ is closed under coproducts and pure quotients. In view of Theorem 5.2 it remains to show that it is extension-closed. First we observe that, since $\text{Hom}_{\mathcal{T}}(P, P^{(I)}[1]) = 0$, we in fact have that $\text{Hom}_{\mathcal{T}}(P, M[1]) = 0$ for any M in $\text{PGen}(P)$. Indeed, if $f : P^{(I)} \rightarrow M$ is a pure epimorphism, then so is $f[1]$ and, therefore, $\text{Hom}_{\mathcal{T}}(P, f[1])$ is surjective (since P is pure-projective). This proves the claim. Finally, consider a triangle

$$X \xrightarrow{r} Y \xrightarrow{s} Z \xrightarrow{t} X[1]$$

with X and Z in $\text{PGen}(P)$. Given a pure epimorphism $q : P^{(J)} \rightarrow Z$, since $t \circ q = 0$,

there is a morphism $\tilde{q} : P^{(J)} \rightarrow Y$ such that $t \circ \tilde{q} = q$. Finally, given also a pure epimorphism $p : P^{(I)} \rightarrow X$, it follows that the morphism

$$(r \circ p, -\tilde{q}) : P^{(I)} \oplus P^{(J)} \rightarrow Y$$

is a pure epimorphism, proving that Y lies in $\text{PGen}(P)$. □

Remark 5.5. We observe that the torsion classes considered above relate to the following three examples.

(1) It is shown in [Aihara and Iyama 2012] that, in a compactly generated triangulated category $\mathcal{T}$, the orthogonal $S^\perp$ to a set S of compact objects (or, equivalently, the orthogonal $S^\perp$ to the pure-projective object S obtained as the coproduct of all objects in S) is a torsionfree class. In [Šťovíček and Pospíšil 2016] it is then shown that the subcategory $S^\perp$ is also a torsion class if the category $\mathcal{T}$ is furthermore assumed to be algebraic. The statement (1) of the Corollary above is a slight generalisation of this second fact.

(2) Statement (2) of the above Corollary is, to some extent, dual to that of [Aihara and Iyama 2012], where it is shown that compact objects generate torsion pairs. Here we show that pure-injective objects cogenerate torsion pairs in triangulated categories, just as they cogenerate cotorsion pairs in module categories (see, for example, [Göbel and Trlifaj 2006]). Certain objects cogenerating torsion pairs were studied in [Oppermann et al. 2019] — these are called 0-cocompact. All known examples of 0-cocompact objects are pure-injective. On the other hand, it follows from the definition that a 0-cocompact object X has the property that ${}^\perp X$ is closed under countable products. There are examples of pure-injective objects that do not satisfy this property (see [Angeleri Hügel and Hrbek 2019, Example 4.10]) and that, therefore, are not 0-cocompact. In general, however, the relation between the notions of pure-injectivity and 0-cocompactness remains rather mysterious.

(3) Statement (3) of the above Corollary is analogous to a statement well known for module categories, namely that given a module M over a ring R such that $\text{Ext}_R^1(M, \text{Gen}(M)) = 0$, then $\text{Gen}(M)$ is a torsion class — see for example [Angeleri Hügel et al. 2016, Lemma 2.3].

Theorem 5.2 allows us to give a criterion for a torsionfree class to be also a torsion class.

Definition 5.6. If a subcategory $\mathcal{Z}$ of $\mathcal{T}$ is simultaneously a torsion and a torsionfree class, then we call $\mathcal{Z}$ a *TTF class* and $({}^\perp \mathcal{Z}, \mathcal{Z}, \mathcal{Z}^\perp)$ a *TTF triple*. Moreover, we say that the torsion pairs $({}^\perp \mathcal{Z}, \mathcal{Z})$ and $(\mathcal{Z}, \mathcal{Z}^\perp)$ are *adjacent*.

Proposition 5.7. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category. Every torsionfree class closed under coproducts and pure quotients is a TTF class. In particular, every definable torsionfree class is a TTF class.*

Proof. This follows from Theorem 5.2 using the fact that torsionfree classes are extension-closed. $\square$

The question of when torsion pairs admit adjacent torsion pairs is extensively studied in [Bondarko 2016, Section 3]. TTF classes are interesting from many different points of view. Examples of suspended or cosuspended TTF classes arise naturally in silting theory (see, for example, [Marks and Vitória 2018]). In this case, the torsion pairs involved are t-structures or co-t-structures.

Definition 5.8. If $(\mathcal{U}, \mathcal{V})$ is a torsion pair, then we say that it is

- a *t-structure* if $\mathcal{U}$ is suspended (or, equivalently, if $\mathcal{V}$ is cosuspended);
- a *co-t-structure* if $\mathcal{U}$ is cosuspended (or, equivalently, if $\mathcal{V}$ is suspended);
- *left nondegenerate* if $\bigcap_{i \in \mathbb{Z}} \mathcal{U}[i] = 0$;
- *right nondegenerate* if $\bigcap_{i \in \mathbb{Z}} \mathcal{V}[i] = 0$.

If $(\mathcal{U}, \mathcal{V})$ is a t-structure, we say that $\mathcal{U}$ is an *aisle*, $\mathcal{V}$ is a *coaisle* and $\mathcal{U}[-1] \cap \mathcal{V}$ is the *heart* of $(\mathcal{U}, \mathcal{V})$.

It is well known (see [Beilinson et al. 1982]) that the heart $\mathcal{H}$ of a t-structure in a triangulated category $\mathcal{T}$ is an abelian category and that there is a naturally defined cohomological functor $H^0 : \mathcal{T} \rightarrow \mathcal{H}$. Moreover, in this case, torsion precovers and torsionfree preenvelopes can be chosen functorially (and we will denote the approximation triangle for an object X in $\mathcal{T}$ by $U_X \rightarrow X \rightarrow V_X \rightarrow U_X[1]$). Although this statement goes back to the definition of t-structure in [Beilinson et al. 1982], it can also be obtained as a consequence of Theorem 2.9. In fact, the following is a well-known result due to Keller and Vossieck.

Theorem 5.9 [Keller and Vossieck 1988, Proposition 1.1]. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ a suspended subcategory of $\mathcal{T}$. The following statements are equivalent.*

- (1) *There is a t-structure $(\mathcal{U}, \mathcal{U}^\perp)$ in $\mathcal{T}$.*
- (2) *The inclusion of $\mathcal{U}$ in $\mathcal{T}$ admits a right adjoint.*
- (3) *The inclusion of $\mathcal{U}^\perp$ in $\mathcal{T}$ admits a left adjoint.*

Theorem 5.9 together with Theorem 2.9 allows us to restate Corollary 4.10 in the following way.

Proposition 5.10. *Let $\mathcal{X}$ be a subcategory of a compactly generated algebraic triangulated category.*

- (1) *If $\mathcal{X}$ is suspended and closed under coproducts and pure quotients, then $(\mathcal{X}, \mathcal{X}^\perp)$ is a t-structure.*

(2) If $\mathcal{X}$ is cosuspended and closed under products and pure subobjects, then $({}^\perp \mathcal{X}, \mathcal{X})$ is a t-structure.

In particular, if $\mathcal{X}$ is triangulated and definable, then both $({}^\perp \mathcal{X}, \mathcal{X})$ and $(\mathcal{X}, \mathcal{X}^\perp)$ are t-structures, i.e., $\mathcal{X}$ is a triangulated TTF class.

For a subcategory $\mathcal{X}$ of a triangulated category $\mathcal{T}$ and I a subset of $\mathbb{Z}$, we write

$$\mathcal{X}^{\perp_I} := \bigcap_{i \in I} (\mathcal{X}[-i])^\perp, \quad {}^\perp_I \mathcal{X} := \bigcap_{i \in I} {}^\perp (\mathcal{X}[i]).$$

For each i in $\mathbb{Z}$, we will denote the set $\{j \in \mathbb{Z} \mid j > i\}$ by $> i$, and similarly for $< i$, $\leq i$, $\geq i$ and i . The following corollary uses the proposition above to build t-structures out of pure-injective objects in a minimal way.

Corollary 5.11. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category and E a pure-injective object in $\mathcal{T}$. Then there is a t-structure $({}^{\perp \leq 0} E, \mathcal{V}_E)$ and $\mathcal{V}_E$ is the smallest coaisle containing E .*

Proof. By Proposition 5.10(1), there is a t-structure of the form $({}^{\perp \leq 0} E, \mathcal{V}_E)$ and, clearly, E lies in $\mathcal{V}_E$. Suppose now that $\mathcal{V}$ is a coaisle containing E . Then $E[k]$ lies in $\mathcal{V}$ for all $k \leq 0$ and, therefore, ${}^\perp \mathcal{V}$ is contained in ${}^{\perp \leq 0} E$. Hence, we have $\mathcal{V}_E \subseteq \mathcal{V}$. $\square$

6. Definability and recollements

In this section we consider the case of triangulated TTF classes, i.e., triangulated subcategories which are both torsion and torsionfree classes. We know from Proposition 5.10 that triangulated TTF classes are bireflective subcategories. These turn out to be related to certain diagrams of functors called recollements. Let us recall and relate all these concepts.

Definition 6.1. A diagram of triangulated categories and triangle functors of the form

$$(6-1) \quad \begin{array}{ccccc} & i^* & & j_! & \\ & \curvearrowright & & \curvearrowright & \\ \mathcal{B} & \xrightarrow{i_*} & \mathcal{T} & \xrightarrow{j^*} & \mathcal{Y} \\ & \curvearrowleft & & \curvearrowleft & \\ & i^! & & j_* & \end{array}$$

is said to be a *recollement* of $\mathcal{T}$ if

- (1) $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples;
- (2) i_* , $j_!$ and j_* are fully faithful; and
- (3) $\text{Im } i_* = \text{Ker } j^*$.

Two recollements given by triples $(i^*, i_*, i^!)$, $(j_!, j^*, j_*)$ and $(i'^*, i'_*, i'^!)$, $(j'_!, j'^*, j'_*)$, respectively are *equivalent* if $\text{Im } i_* = \text{Im } i'_*$, $\text{Im } j_! = \text{Im } j'_!$ and $\text{Im } j_* = \text{Im } j'_*$.

Recall that given a recollement as above, for each object X in the triangulated $\mathcal{T}$, the units and counits of the adjunctions yield triangles as follows:

$$\begin{aligned} j_! j^* X &\rightarrow X \rightarrow i_* i^* X \rightarrow j_! j^* X[1], \\ i_* i^! X &\rightarrow X \rightarrow j_* j^* X \rightarrow i_* i^! X[1]. \end{aligned}$$

Triangulated TTF triples also turn out to be related to the notion of (co)smashing subcategory.

Definition 6.2. A triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ with products and coproducts is said to be *smashing* if $\mathcal{S}$ is coreflective and $\mathcal{S}^\perp$ is closed under coproducts. Dually, we say that a triangulated subcategory $\mathcal{C}$ is *cosmashing* if it is reflective and ${}^\perp\mathcal{C}$ is closed under products.

We now state the following equivalent descriptions of triangulated TTF classes. Most of the conditions are well known, however we provide an argument for the sake of completion.

Proposition 6.3. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category. The following are equivalent for a triangulated subcategory $\mathcal{B}$ of $\mathcal{T}$.*

- (1) $\mathcal{B}$ is a TTF class in $\mathcal{T}$.
- (2) $\mathcal{B} = \mathcal{S}^\perp$ for a smashing subcategory $\mathcal{S}$.
- (3) $\mathcal{B} = {}^\perp\mathcal{C}$ for a cosmashing subcategory $\mathcal{C}$.
- (4) $\mathcal{B}$ is bireflective.
- (5) There is a recollement of $\mathcal{T}$ of the form (6-1).
- (6) $\mathcal{B}$ is definable.
- (7) $\mathcal{B}$ is closed under products and pure subobjects.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): We refer to [Nicolás and Saorín 2018, Corollary 3.15].

(1) $\Leftrightarrow$ (4): This follows, for example, from Theorem 5.9.

(4) $\Rightarrow$ (5): Define $i_* : \mathcal{B} \rightarrow \mathcal{T}$ to be the inclusion functor, $\mathcal{Y} := \mathcal{T}/\mathcal{B}$ to be the Verdier quotient and let $j^* : \mathcal{T} \rightarrow \mathcal{Y}$ be the corresponding Verdier quotient functor. Since $\mathcal{B}$ is bireflective, it then follows from [Cline et al. 1988, Theorem 2.1] that j^* has both left and right adjoints and that these are fully faithful.

(5) $\Rightarrow$ (4): This is clear by definition of bireflective subcategory.

(1) $\Rightarrow$ (6): Let $(\mathcal{S}, \mathcal{B}, \mathcal{C})$ be a triangulated TTF triple. Consider the ideal $\mathcal{I}$ of morphisms in $\mathcal{T}^c$ consisting of morphisms factoring through the smashing subcategory $\mathcal{S}$. By [Krause 2005, Theorem 12.1], the class

$$\mathcal{I}^\perp := \{B \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(g, B) = 0 \text{ for all } g \in \mathcal{I}\}$$

coincides with the class $\mathcal{B}$. For each g in $\mathcal{I}$, we may complete to a triangle

$$L \xrightarrow{g} M \xrightarrow{f} N \rightarrow L[1]$$

and note that $\mathrm{Hom}_{\mathcal{T}}(g, B) = 0$ if and only if $\mathrm{Hom}_{\mathcal{T}}(f, B)$ is surjective. That is, if and only if the coherent functor $F_f := \mathrm{Coker}(\mathrm{Hom}_{\mathcal{T}}(f, -))$ vanishes on B . Thus, the class $\mathcal{B}$ is definable.

(6) $\Rightarrow$ (1): This follows from Proposition 5.10.

(6) $\Rightarrow$ (7): This follows from Theorem 4.7.

(7) $\Rightarrow$ (6): Since $\mathcal{B}$ closed under pure subobjects and products, it is also closed under coproducts. Since it is furthermore triangulated, it is then closed under directed homotopy colimits (see Remark 4.11), and the result then follows from Theorem 4.7. $\square$

Remark 6.4. The bijective correspondence between triangulated TTF classes, smashing subcategories and equivalence classes of recollements is well-understood for a large class of triangulated categories (see, for example, [Nicolás 2007]). Moreover, in a compactly generated triangulated category, the equivalence between (2) and (6) can be found in [Krause 2005]. It boils down to the fact that smashing subcategories of compactly generated triangulated categories are determined by certain ideals of morphisms between compact objects. We use those arguments to prove (1) $\Rightarrow$ (6), and provide new arguments for the converse implication.

Remark 6.5. The telescope conjecture for compactly generated triangulated categories asserts that every smashing subcategory is compactly generated, i.e., that given a smashing subcategory $\mathcal{S}$ of a compactly generated triangulated category $\mathcal{T}$, there is a set of compact objects $\mathcal{K}$ such that $\mathcal{S}^{\perp} = \mathcal{K}^{\perp}$. If $\mathcal{T}$ is furthermore algebraic, Proposition 6.3 allows us to restate the telescope conjecture in an equivalent way: for every triangulated subcategory $\mathcal{B}$ closed under products and pure subobjects, there is a set of compact objects $\mathcal{K}$ such that $\mathcal{B} = \mathcal{K}^{\perp}$. The telescope conjecture is, however, known to be false for general compactly generated algebraic triangulated categories (see [Keller 1994b] and [Bazzoni and Šťovíček 2017] for counterexamples). Still, it remains a difficult problem to identify which triangulated categories satisfy the telescope conjecture.

Next we observe that given an extension-closed definable subcategory of a compactly generated algebraic triangulated category (which by Theorem 5.2 is a torsion class), we can build a subcategory satisfying the equivalent conditions of the proposition above.

Corollary 6.6. *Let $\mathcal{X}$ be a subcategory of a compactly generated algebraic triangulated category $\mathcal{T}$. If $\mathcal{X}$ is closed under extensions, products and pure subobjects, then $\bigcap_{i \in \mathbb{Z}} \mathcal{X}[i]$ is a triangulated TTF class.*

Proof. It is clear that the intersection is closed under extensions, products and pure subobjects. Now observe that if X lies in $\mathcal{X}[i]$ for all i in $\mathbb{Z}$, then $X[k]$ lies in $\mathcal{X}[k+i]$ for all i in $\mathbb{Z}$, thus showing that $\bigcap_{i \in \mathbb{Z}} \mathcal{X}[i]$ is triangulated. The assertion then follows directly from Proposition 6.3. $\square$

Recollements of triangulated categories are used to glue torsion pairs. We recall the following well-known theorem (whose proof is the same as the original statement made for t-structures in [Beilinson et al. 1982]). Given two subcategories $\mathcal{W}$ and $\mathcal{Z}$ of $\mathcal{T}$, we denote by $\mathcal{W} * \mathcal{Z}$ the subcategory of $\mathcal{T}$ whose objects are those X in $\mathcal{T}$ for which there are objects W in $\mathcal{W}$, Z in $\mathcal{Z}$ and a triangle

$$W \rightarrow X \rightarrow Z \rightarrow W[1].$$

Theorem 6.7 [Beilinson et al. 1982, Théorème 1.4.10, Proposition 1.4.12]. *Let $\mathcal{T}$ be a triangulated category and consider a recollement of $\mathcal{T}$ of the form (6-1), and denote by $(\mathcal{S}, \mathcal{B}, \mathcal{C})$ its associated triangulated TTF triple.*

- (1) *If $(\mathcal{W}, \mathcal{Z})$ and $(\mathcal{U}, \mathcal{V})$ are torsion pairs in $\mathcal{B}$ and $\mathcal{Y}$, respectively, then there is a torsion pair in $\mathcal{T}$ of the form $(j_! \mathcal{U} * i_* \mathcal{W}, i_* \mathcal{Z} * j_* \mathcal{V})$.*
- (2) *A torsion pair $(\mathcal{M}, \mathcal{N})$ in $\mathcal{T}$ is obtained by the construction in (1) if and only if $j_! j^* \mathcal{M} \subseteq \mathcal{M}$ and if and only if $j_* j^* \mathcal{N} \subseteq \mathcal{N}$, in which case the torsion pairs that give rise to it are $(j^* \mathcal{M}, j^* \mathcal{N})$ in $\mathcal{Y}$ and $(i^* \mathcal{M}, i^! \mathcal{N})$ in $\mathcal{B}$. Moreover, in this case, we have that $j_! j^* \mathcal{M} = \mathcal{M} \cap \mathcal{S}$ and $j_* j^* \mathcal{N} = \mathcal{N} \cap \mathcal{C}$.*

The torsion pair in $\mathcal{T}$ obtained in the theorem above is said to be *glued* from the torsion pair $(\mathcal{W}, \mathcal{Z})$ in $\mathcal{B}$ and the torsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{Y}$ along the given recollement.

The following well-known lemma will be useful in what follows.

Lemma 6.8. *Let $\mathcal{C}$ be a triangulated subcategory of a triangulated category $\mathcal{T}$. Let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in $\mathcal{T}$ and suppose $\mathcal{U} \subseteq \mathcal{C}$, then $(\mathcal{U}, \mathcal{V} \cap \mathcal{C})$ is a torsion pair in $\mathcal{C}$. Similarly, if $\mathcal{V} \subseteq \mathcal{C}$, then $(\mathcal{U} \cap \mathcal{C}, \mathcal{V})$ is a torsion pair in $\mathcal{C}$.*

Proof. Clearly the first two conditions of Definition 3.12 will restrict to $\mathcal{C}$. Moreover, if $\mathcal{U}$ is contained in $\mathcal{C}$ then, for any X in $\mathcal{C}$, the first two terms of the triangle given by Definition 3.12 are contained in $\mathcal{C}$. Since $\mathcal{C}$ is closed under extensions and shifts, it follows that the third term will be contained in $\mathcal{V} \cap \mathcal{C}$ and so $(\mathcal{U}, \mathcal{V} \cap \mathcal{C})$ is a torsion pair in $\mathcal{C}$. A similar argument yields the second claim. $\square$

We finish with a result that states that any torsion pair with a definable torsion or torsionfree class can be obtained by glueing along a recollement of a trivial torsion pair with a torsion pair with certain nondegeneracy conditions. Moreover, we show that there is a natural operation generating a new torsion pair from an old one (see also [Psaroudakis 2018, Section 10] for an analogous construction).

Proposition 6.9. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category and let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in $\mathcal{T}$ with $\mathcal{U}$ a definable subcategory. Consider the bireflective subcategory $\mathcal{B} = \bigcap_{i \in \mathbb{Z}} \mathcal{U}[i] = {}^{\perp \mathbb{Z}} \mathcal{V}$ given by Corollary 6.6 and the associated recollement*

$$\begin{array}{ccccc} & i^* & & j_! & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{B} & \xrightarrow{i_*} & \mathcal{T} & \xleftarrow{j^*} & \mathcal{Y} \\ & \curvearrowleft & & \curvearrowright & \\ & i^! & & j_* & \end{array}$$

where i_* is the corresponding inclusion functor into $\mathcal{T}$. Denote by $\mathcal{S}$ and $\mathcal{C}$ the associated smashing and cosmashing subcategories, respectively. Then the following statements hold.

- (1) *The torsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{T}$ is obtained by glueing*
 - *the left nondegenerate torsion pair $(j^*\mathcal{U}, j^*\mathcal{V})$ which, under the equivalence between $\mathcal{Y}$ and $\mathcal{C}$ induced by j_* , corresponds to the torsion pair $(\mathcal{U} \cap \mathcal{C}, \mathcal{V})$ in $\mathcal{C}$;*
 - *the trivial torsion pair $(\mathcal{B}, 0)$ in $\mathcal{B}$.*
- (2) *The pair $(\mathcal{U} \cap \mathcal{S}, \mathcal{B} * \mathcal{V})$ is a left nondegenerate torsion pair in $\mathcal{T}$ obtained by glueing*
 - *the left nondegenerate torsion pair $(j^*\mathcal{U}, j^*\mathcal{V})$;*
 - *the trivial torsion pair $(0, \mathcal{B})$ in $\mathcal{B}$.*

Proof. Firstly, note that indeed the equality $\bigcap_{i \in \mathbb{Z}} \mathcal{U}[i] = {}^{\perp \mathbb{Z}} \mathcal{V}$ follows immediately from the fact that, for each i in $\mathbb{Z}$, we have $\mathcal{U}[i] = {}^{\perp} \mathcal{V}[i]$.

(1) First observe that, since $\mathcal{V}$ is contained in $\mathcal{C}$, we have that $j_* j^* \mathcal{V} = \mathcal{V}$. Therefore, by Theorem 6.7, the torsion pair $(\mathcal{U}, \mathcal{V})$ is obtained by glueing $(j^*\mathcal{U}, j^*\mathcal{V})$ and $(i^*\mathcal{U}, i^!\mathcal{V})$. Moreover, by Lemma 6.8, $(\mathcal{U} \cap \mathcal{C}, \mathcal{V})$ is a torsion pair in $\mathcal{C}$ with torsionfree class $j_* j^* \mathcal{V} = \mathcal{V}$. Therefore, it must be the one corresponding under the equivalence j_* between $\mathcal{Y}$ and $\mathcal{C}$ to the torsion pair $(j^*\mathcal{U}, j^*\mathcal{V})$ (i.e., we get that $j_* j^* \mathcal{U} = \mathcal{U} \cap \mathcal{C}$). Note also that $\bigcap_{i \in \mathbb{Z}} (\mathcal{U} \cap \mathcal{C})[i] = \mathcal{B} \cap \mathcal{C} = 0$ and, thus, $(j^*\mathcal{U}, j^*\mathcal{V})$ is indeed left nondegenerate. Finally, observe that $i^!\mathcal{V} = 0$ since $\text{Ker}(i^!) = \mathcal{C}$ and $\mathcal{V}$ is contained in $\mathcal{C}$. Therefore we also have $i^*\mathcal{U} = \mathcal{B}$, completing the proof.

(2) We glue the torsion pairs $(j^*\mathcal{U}, j^*\mathcal{V})$ in $\mathcal{Y}$ and $(0, \mathcal{B})$ in $\mathcal{B}$. By Theorem 6.7(1), the resulting torsion pair is $(j_! j^* \mathcal{U} * 0, \mathcal{B} * j_* j^* \mathcal{V})$. Since, as seen in Theorem 6.7(2), we have $j_! j^* \mathcal{U} = \mathcal{U} \cap \mathcal{S}$ and $j_* j^* \mathcal{V} = \mathcal{V} \cap \mathcal{C} = \mathcal{V}$, we get the desired torsion pair. Finally, observe that, as in (1), it is easy to see that the resulting torsion pair is left nondegenerate. $\square$

Next we will consider the dual case of a definable torsionfree class. It follows from Proposition 5.7 that such a class is in fact a TTF class and, therefore, we obtain the following stronger result.

Proposition 6.10. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category and let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in $\mathcal{T}$ with $\mathcal{V}$ a definable subcategory. Consider the bireflective subcategory $\mathcal{B} = \bigcap_{i \in \mathbb{Z}} \mathcal{V}[i] = \mathcal{U}^{\perp_{\mathbb{Z}}}$ given by Corollary 6.6 and the associated recollement*

$$(6-2) \quad \begin{array}{ccccc} & i^* & & j_! & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{B} & \xrightarrow{i_*} & \mathcal{T} & \xleftarrow{j^*} & \mathcal{Y} \\ & \curvearrowleft & & \curvearrowright & \\ & i^! & & j_* & \end{array}$$

where i_* is the corresponding inclusion functor into $\mathcal{T}$. Denote by $\mathcal{S}$ and $\mathcal{C}$ the associated smashing and cosmashing subcategories, respectively. Then the following statements hold.

- (1) *The torsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{T}$ is obtained by glueing*
 - *the right nondegenerate torsion pair $(j^*\mathcal{U}, j^*\mathcal{V})$ which, under the equivalence between $\mathcal{Y}$ and $\mathcal{S}$ induced by $j_!$, corresponds to the torsion pair $(\mathcal{U}, \mathcal{V} \cap \mathcal{S})$ in $\mathcal{S}$;*
 - *the trivial torsion pair $(0, \mathcal{B})$ in $\mathcal{B}$.*
- (2) *The pair $(\mathcal{U} * \mathcal{B}, \mathcal{V} \cap \mathcal{C})$ is a right nondegenerate torsion pair in $\mathcal{T}$ obtained by glueing*
 - *the right nondegenerate torsion pair $(j^*\mathcal{U}, j^*\mathcal{V})$;*
 - *the trivial torsion pair $(\mathcal{B}, 0)$ in $\mathcal{B}$.*
- (3) *The class $\mathcal{V}$ can be expressed as $\mathcal{B} * (\mathcal{V} \cap \mathcal{C})$ and also as $(\mathcal{V} \cap \mathcal{S}) * \mathcal{B}$.*

Proof. Statements (1) and (2) are dual to Proposition 6.9. For statement (3) we use the fact that, by Proposition 5.7, we have a torsion pair $(\mathcal{V}, \mathcal{V}^{\perp})$. Indeed, this allows us to conclude, by Proposition 6.9(1), that $j_*j^*\mathcal{V} = \mathcal{V} \cap \mathcal{C}$. Using item (1) above, we then obtain that $\mathcal{V} = \mathcal{B} * (\mathcal{V} \cap \mathcal{C})$ by Theorem 6.7. On the other hand, by applying Proposition 6.9(1) to the torsion pair $(\mathcal{V}, \mathcal{V}^{\perp})$ we have that it is obtained by glueing $(j^*\mathcal{V}, j^*(\mathcal{V}^{\perp}))$ and $(\mathcal{B}, 0)$. By item (1) above, we have that $j_!j^*\mathcal{V} = \mathcal{V} \cap \mathcal{S}$ and so we conclude that $\mathcal{V} = (\mathcal{V} \cap \mathcal{S}) * \mathcal{B}$ by Theorem 6.7. $\square$

We end this paper by considering a special case of the above proposition where $\mathcal{V}$ is cosuspended. This allows us to provide a structural description of t-structures with a definable coaisle.

Proposition 6.11. *Let $\mathcal{T}$ be a compactly generated algebraic triangulated category and let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{T}$ with $\mathcal{V}$ a definable subcategory. Consider the bireflective subcategory $\mathcal{B} = \bigcap_{i \in \mathbb{Z}} \mathcal{V}[i]$ given by Corollary 6.6 and denote by $\mathcal{S} = {}^{\perp}\mathcal{B}$ and $\mathcal{C} = \mathcal{B}^{\perp}$ the associated smashing and cosmashing subcategories, respectively. Then there is a pure-injective object C such that:*

- (1) *The smallest coaisle containing C is $\mathcal{V}_C = \mathcal{V} \cap \mathcal{C}$.*

(2) The definable coaisle $\mathcal{V}$ can be expressed as $\mathcal{B} * \mathcal{V}_C$.

(3) The heart $\mathcal{H}$ of $(\mathcal{U}, \mathcal{V})$ is equivalent to the heart of the t-structure $({}^{\perp \leq 0} C, \mathcal{V}_C)$.

Proof. Since $\mathcal{V}$ is definable we have that, by [Saorín et al. 2017, Theorem C] and [Laking 2019, Theorem 3.11], the heart $\mathcal{H} = \mathcal{U}[-1] \cap \mathcal{V}$ of $(\mathcal{U}, \mathcal{V})$ is a Grothendieck abelian category. In particular, $\mathcal{H}$ has an injective cogenerator E . Since $\mathcal{V}$ is closed under coproducts, it follows that the cohomological functor $H^0 : \mathcal{T} \rightarrow \mathcal{H}$ commutes with coproducts [Angeleri Hügel et al. 2017, Lemma 3.3]. Using Brown representability as in [Nicolás et al. 2019], we then conclude that there is an object C representing the functor $\text{Hom}_{\mathcal{H}}(H^0(-), E)$ i.e.,

$$\text{Hom}_{\mathcal{T}}(-, C) \cong \text{Hom}_{\mathcal{H}}(H^0(-), E).$$

Moreover, by the proof of [Angeleri Hügel et al. 2017, Theorem 3.6], we have that C is pure-injective.

(1) By Proposition 6.10(2), we have a t-structure $(\mathcal{U} * \mathcal{B}, \mathcal{V} \cap \mathcal{C})$ in $\mathcal{T}$. We will show that this t-structure coincides with the t-structure $({}^{\perp \leq 0} C, \mathcal{V}_C)$ given by Corollary 5.11. To show that $\mathcal{V}_C$ is contained in $\mathcal{V} \cap \mathcal{C}$, it suffices to show that C lies in $\mathcal{V} \cap \mathcal{C}$. By construction of the functor H^0 (see, for example, [Beilinson et al. 1982]), it follows that $H^0(\mathcal{U}) = 0$ and $H^0(\mathcal{V}[k]) = 0$ for all $k < 0$. In particular, we have that $H^0(\mathcal{B}) = 0$ and, since H^0 is a cohomological functor, it follows that $H^0(\mathcal{U} * \mathcal{B}) = 0$. This implies that, by construction of C , we have that C lies in $(\mathcal{U} * \mathcal{B})^{\perp} = \mathcal{V} \cap \mathcal{C}$.

For the reverse inclusion, let Z be an object in ${}^{\perp \leq 0} C$, and consider the triangle arising from the recollement (6-2):

$$j_! j^* Z \rightarrow Z \rightarrow i_* i^! Z \rightarrow j_! j^* Z[1].$$

Applying the cohomological functor H^0 to the triangle, we conclude that

$$H^0(j_! j^* Z[k]) \cong H^0(Z[k])$$

for all $k \in \mathbb{Z}$ because $H^0(\mathcal{B}) = 0$. Moreover, we have that $H^0(Z[k]) = 0$ for all $k \geq 0$ by definition of C and choice of Z . Finally, note that by Proposition 6.10(1), the functor $j_!$ identifies the right nondegenerate t-structure $(j^* \mathcal{U}, j^* \mathcal{V})$ with $(\mathcal{U}, \mathcal{V} \cap \mathcal{S})$ in $\mathcal{S}$. Since $j_! j^* Z$ lies in $\mathcal{S}$, the nondegeneracy guarantees that $j_! j^* Z$ lies in $\mathcal{U}$ if and only if $H^0(j_! j^* Z[k]) = 0$ for all $k \geq 0$ (see, for example, [Beilinson et al. 1982]). Thus we have that, indeed, $j_! j^* Z$ lies in $\mathcal{U}$ and Z lies in $\mathcal{U} * \mathcal{B}$. This proves that ${}^{\perp \leq 0} C \subseteq \mathcal{U} * \mathcal{B}$ and, hence, $\mathcal{V}_C \subseteq \mathcal{V} \cap \mathcal{C}$.

(2) This follows by using (1) together with Proposition 6.10(3).

(3) Recall that, by Proposition 6.10(2), the t-structure $(\mathcal{U}, \mathcal{V})$ can be obtained by glueing the t-structures $(0, \mathcal{B})$ and $(j^* \mathcal{U}, j^* \mathcal{V})$. It then follows from [Beilinson et al. 1982] (see also [Psaroudakis 2018, Section 4] for details) that the functor $j^*|_{\mathcal{H}} : \mathcal{H} \rightarrow j^* \mathcal{H}$ is a Serre quotient functor inducing a recollement of abelian

categories and that, moreover, $\text{Ker}(j^*|_{\mathcal{H}})$ is the heart of the t-structure $(0, \mathcal{B})$ (which is zero). Therefore, $j^*|_{\mathcal{H}}$ is indeed an equivalence of categories and since $(j_*\mathcal{U}, j^*\mathcal{V})$ identifies with $(\mathcal{U}*\mathcal{B}, \mathcal{V}\cap\mathcal{C})$ via the functor j_* (see Proposition 6.10(3)), the result follows by (1) above. $\square$

Remark 6.12. Note that, in [Nicolás et al. 2019], the pure-injective object C obtained in the above proposition is known as a partial cosilting object.

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ROSANNA LAKING
 DIPARTIMENTO DI INFORMATICA
 UNIVERSITÀ DEGLI STUDI DI VERONA
 VERONA
 ITALY
rosanna.laking@univr.it

JORGE VITÓRIA
 DIPARTIMENTO DI MATEMATICA E INFORMATICA
 UNIVERSITÀ DEGLI STUDI DI CAGLIARI
 CAGLIARI
 ITALY
jorge.vitoria@unica.it

REMARKS ON THE THETA CORRESPONDENCE OVER FINITE FIELDS

DONGWEN LIU AND ZHICHENG WANG

S.-Y. Pan decomposed the uniform projection of the Weil representation of a finite symplectic-odd orthogonal dual pair, in terms of Deligne–Lusztig virtual characters, assuming that the order of the finite field is large enough. We use Pan’s decomposition to study the theta correspondence for dual pairs of this kind, following the approach of Adams and Moy and Aubert, Michel and Rouquier. Our results give the theta correspondence between unipotent representations and certain quadratic unipotent representations.

1. Introduction

The theory of Weil representations and theta correspondence over finite fields has been studied extensively (see, e.g., [Adams and Moy 1993; Aubert et al. 1996; 2016; Epequin Chavez 2019; Gérardin 1977; Gurevich and Howe 2017; Howe 1973; Pan 2016; 2019a; 2019b; Srinivasan 1968; 1979b]) and has many applications to the representation theory of p -adic groups, but there are still a few important open problems unsolved in the finite field case. In particular, there is a complete understanding for the theta correspondence of unipotent representations for finite reductive dual pairs, with only the exception of symplectic-odd orthogonal dual pairs. The aim of this paper is to fill this gap and give some answers for this missing case. In contrast to other dual pairs, it is well known that the theta lifting of unipotent representations may not be unipotent in general for symplectic-odd orthogonal dual pairs. It turns out that new phenomena occur and the correspondence involves certain quadratic unipotent representations, which are called θ -representations in this paper.

To give some details, let us start from the Deligne–Lusztig characters [1976], which are a major tool for the study of representations of finite Lie groups. Let F be a finite field of q elements, G be a connected reductive group defined over F , and F be the corresponding Frobenius endomorphism. Let $G = G^F$ be the group of rational points of G . For an F -stable maximal torus T of G and a character θ

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of $T = T^F$, let $R_{T,\theta}^G$ denote the virtual character of G defined by P. Deligne and G. Lusztig in the seminal work [Deligne and Lusztig 1976].

Let T_0 be a maximally split F -stable maximal torus of G , and $W_G = N(T_0)/T_0$ be the Weyl group of G . For each $w \in W_G$, we can associate an F -stable maximal torus T_w of G and a specific character of $T_w = T_w^F$ of order 2, denoted by θ_w [Pan 2016]. If $G = \mathrm{SO}_{2n+1}$ where n is a positive integer, then

$$\frac{1}{|W_{\mathrm{SO}_{2n+1}}|} \sum_{w \in W_{\mathrm{SO}_{2n+1}}} R_{T_w, \theta_w}^{\mathrm{SO}_{2n+1}}$$

is the unique linear character of $\mathrm{SO}_{2n+1}(q)$ of order 2, which will be denoted by $\chi_{\mathrm{SO}_{2n+1}}$. For $n = 0$, by convention $\chi_{\mathrm{SO}_{2n+1}}$ is the trivial character. An irreducible representation π of G is called a θ -representation if it occurs in some R_{T_w, θ_w}^G . The θ -representations can be interpreted as certain quadratic unipotent representations of finite reductive Lie groups (see, e.g., [Mœglin 1995] for the notion of quadratic unipotent representations for p -adic groups). For SO_{2n+1} , $\pi \rightarrow \chi_{\mathrm{SO}_{2n+1}} \pi$ gives a bijection between unipotent representations and θ -representations.

Let $\omega_{\mathrm{Sp}_{2n}}$ be the Weil representation of the finite symplectic group $\mathrm{Sp}_{2n}(q)$ with q odd, which depends on a choice of a nontrivial additive character ψ of F (see [Gérardin 1977]). We will denote by $\omega_{\mathrm{Sp}_{2n}}^\#$ the uniform projection of $\omega_{\mathrm{Sp}_{2n}}$, i.e., the projection onto the subspace of virtual characters spanned by all the Deligne–Lusztig characters. A pair (G, G') of mutually centralized reductive subgroups of Sp_{2n} is called a reductive dual pair. Then the Weil representation $\omega_{\mathrm{Sp}_{2n}}$ restricts to a representation $\omega_{G, G'}$ of $G \times G'$, whose character will be denoted by the same notation.

J. Adams and A. Moy [1993] show that for unitary or symplectic-even orthogonal dual pairs, if $\pi \otimes \pi'$ occurs in $\omega_{G, G'}$, then π is unipotent if and only if π' is unipotent, in which case they are in correspondence under the decomposition of the uniform projection $\omega_{G, G'}^\#$. They also study the first occurrence of cuspidal representations and give the explicit correspondence of cuspidal unipotent representations.

As we mentioned above, it is known that for symplectic-odd orthogonal dual pairs, theta correspondence no longer preserves unipotent representations in general. On the other hand, Pan [2016] gives the decomposition of $\omega_{G, G'}^\#$ for such dual pairs. It turns out that based on Pan's work we can obtain the following analog of the result in [Adams and Moy 1993], which fills the gap in the symplectic-odd orthogonal case. See Theorem 3.7 for details, and also Theorem 3.3 for a general statement.

Theorem 1.1. *Assume that q is odd and large enough such that the main result in [Srinivasan 1979b] holds. Let λ_k be the unique cuspidal unipotent representation of Sp_{2n} , $n = k(k+1)$. Then its theta lifting to O_{2n+1}^ϵ , $\epsilon = \pm$, is the unique cuspidal θ -representation of O_{2n+1}^ϵ with trivial central character, and is the first occurrence in each Witt tower of odd orthogonal groups.*

The proof of this theorem invokes some ingredients, among which we would like to mention the so-called conservation relation (see, e.g., [Pan 2002; Sun and Zhu 2015]). Such a relation provides a link between the first occurrence of the theta lifting of a representation towards two different Witt towers. For finite dual pairs it is known only for cuspidal representations, which is sufficient for our purpose.

Applying Lusztig's bijection [1988] we are able to classify the cuspidal θ -representations of symplectic groups (Theorem 3.11). Consequently we can switch the roles of symplectic and odd orthogonal groups in Theorem 1.1 and obtain an analogous result Theorem 3.12, the proof of which uses the conservation relations in an inductive manner.

The cuspidal representations are the building blocks of the so-called Harish-Chandra series. As a second step, A.-M. Aubert, J. Michel and R. Rouquier [Aubert et al. 1996, Théorème 3.7] study the theta correspondence of Harish-Chandra series of unipotent cuspidal representations for unitary and symplectic-even orthogonal dual pairs. According to Lusztig's bijection and [Pan 2002, Theorem 3.10], using their results we should be able to handle the theta correspondence of any irreducible representations. Based on Theorem 1.1, we are able to extend their results to the symplectic-odd orthogonal case using similar calculations to those in [Aubert et al. 1996].

Our second main result Theorem 4.2 establishes the theta correspondence between the Harish-Chandra series of cuspidal (quadratic) unipotent representations of symplectic-odd orthogonal dual pairs. We briefly describe its content as follows. Denote T_l the split torus over F of rank l , and let θ_l , $\theta_{k,l}$ and $\theta'_{k,l}$ be the quadratic characters of T_l defined in Section 4. Let $(G_n, G'_{n'})$ be a dual pair of symplectic and odd orthogonal groups over F , and λ be an irreducible cuspidal representation of G_n such that its theta lifting λ' to $G'_{n'}$ is irreducible cuspidal as well. Let γ be an irreducible constituent of the parabolic induction $R_{G_n \times T_{m-n}}^{G_m}(\lambda \times \theta)$, where θ is some quadratic character specified below, and $\Theta_{G'_{m'}}(\gamma)$ be the theta lifting of γ to $G'_{m'}$. Then Theorem 4.2 states that $\Theta_{G'_{m'}}(\gamma) = 0$ if $m' < n'$, and is spanned by irreducible constituents of $R_{G'_{n'} \times T_{m'-n'}}^{G'_{m'}}(\lambda' \otimes \theta')$ otherwise, where the pair (θ, θ') of quadratic characters is as follows:

- If G_n is symplectic and $G'_{n'}$ is odd orthogonal, then (θ, θ') is equal to $(\theta_{m-n}, \mathbf{1})$ or $(\mathbf{1}, \theta'_{m-n, m'-n'})$.
- If G_n is odd orthogonal and $G'_{n'}$ is symplectic, then (θ, θ') is equal to $(\mathbf{1}, \theta_{m'-n'})$ or $(\theta_{m-n}, \theta_{m-n, m'-n'})$.

We expect that our results may have some applications towards the theta lifting of quadratic unipotent representations of p -adic groups.

From Mackey theory one knows that constituents of such Harish-Chandra series can be parametrized by irreducible representations of certain Weyl groups. Thus the

theta correspondence reduces to a correspondence of Weyl group representations. In [Aubert et al. 1996], explicit formulas have been proved for the unitary case and conjectured for the symplectic-even orthogonal case, and the latter has been proved recently in [Pan 2019a]. We expect that similar conjectures should also hold for the symplectic-odd orthogonal case. We also remark that from the explicit correspondence we can observe that the conservation relations in general do not hold for noncuspidal representations of finite dual pairs. See [Pan 2019b, Section 9] for more results on the (non-)conservation relations.

The paper is organized as follows. In Section 2 we briefly recall the theory of Deligne–Lusztig characters as well as Pan’s decomposition formula [2016]. In Section 3 we prove the theta correspondence between cuspidal unipotent representations and cuspidal θ -representations, following [Adams and Moy 1993] and using the conservation relations. Based on this result, in Section 4 we establish the theta correspondence of certain Harish-Chandra series, using some calculations of Jacquet and induction functors as in [Aubert et al. 1996] and [Mœglin et al. 1987]. We explain briefly the correspondence of Weyl group representations mentioned above at the end of Section 4.

Notation. In this paper the cardinality q of F is assumed to be odd and large enough (as in Theorem 3.1). The finite Lie group $G = G^F$ is also written as $G(q)$. Representations are realized on vector spaces over $\overline{\mathcal{Q}}_\ell$ where $\ell \neq p$ is a prime. We do not distinguish a (virtual) representation and its character. Thus $R(G)$ stands for the set of virtual representations of G as well as the set of integral combinations of irreducible characters. For $\pi \in R(G)$, let $\text{Irr}(\pi)$ be the set of constituents of π . For $\rho \in \text{Irr}(G)$, we simply write $\rho \in \pi$ if $\rho \in \text{Irr}(\pi)$. For $\pi_1, \pi_2 \in R(G)$, let

$$(\pi_1, \pi_2)_G = \frac{1}{|G|} \sum_{g \in G} \pi_1(g) \overline{\pi_2(g)},$$

where the conjugate is the restriction of the usual complex conjugate to $\overline{\mathcal{Q}} \subset \overline{\mathcal{Q}}_\ell$.

2. Deligne–Lusztig theory

Let F be a finite field of q elements, G be a connected reductive group defined over F and F be the corresponding Frobenius endomorphism. The group of rational points of G is denoted by $G = G^F$.

2A. Conjugacy classes of maximal tori. For an F -stable maximal torus T of G , let $W_G(T) = N_G(T)/T$ be its Weyl group, where $N_G(T)$ is the normalizer of T in G , and let $T = T^F$ be the group of rational points of T .

Fix a maximally split F -stable maximal torus T_0 of G , and denote its Weyl group by W_G . A conjugate ${}^g T_0 = gT_0g^{-1}$ of T_0 by an element $g \in G$ is F -stable if and only if $g^{-1}F(g) \in N_G(T_0)$. By the Lang–Steinberg theorem, for each $w \in W_G$ we may

choose an element $g \in G$ such that $g^{-1}F(g)$ is in $N_G(T_0)$ whose image in W_G is equal to w . The F -stable maximal torus ${}^sT_0$ will be denoted by T_w . Two elements $w, w' \in W_G$ are called F -conjugate if there exists $x \in W_G$ such that $w' = x^{-1}wF(x)$. The map ${}^sT_0 \rightarrow g^{-1}F(g)$ gives a bijection between the G -conjugacy classes of F -stable maximal tori of G and the F -conjugacy classes of W_G .

2B. Deligne–Lusztig characters. In their celebrated paper, Deligne and Lusztig [1976] defined a virtual representation $R_{T,\theta}^G$ of G , associated to a character θ of T . The construction of Deligne–Lusztig characters makes use of the deep theory of ℓ -adic cohomology. Here we only review some standard facts on these characters which will be used in this paper (see [Carter 1985, Chapter 7]).

If $y = su$ is the Jordan decomposition of an element $y \in G$, then

$$(2-1) \quad R_{T,\theta}^G(y) = \frac{1}{|C^0(s)|} \sum_{g \in G, s^g \in T} \theta(s^g) Q_{sT}^{C^0(s)}(u)$$

where $C^0(s) = C_G^0(s)$ is the connected component of the centralizer of s in G , and $Q_{sT}^{C^0(s)} = R_{sT,1}^{C^0(s)}(u)$ is the Green function of $C^0(s)$ associated to ${}^sT$. Note that $s^g = g^{-1}sg \in T$ if and only if ${}^sT = gTg^{-1} \subset C^0(s)$.

It is known that

$$(2-2) \quad 1 = \frac{1}{|W_G|} \sum_{w \in W_G} R_{T_w,1}^G = \sum_{(T) \subset G} \frac{1}{|W_G(T)|} R_{T,1}^G$$

where $(T) \subset G$ means that the summation is taken over the G -conjugacy classes of F -stable maximal tori in G .

Two virtual representations $R_{T,\theta}^G$ and $R_{T',\theta'}^G$ are disjoint if (T, θ) and (T', θ') are not geometrically conjugate. The elements in the subspace of class functions on G spanned by all the $R_{T,\theta}^G$'s are called uniform functions of G . For a class function f on G , denote by $f^\#$ its orthogonal projection to the subspace of uniform functions.

2C. Weyl group. By [Srinivasan 1979b, Lemma 3.1], there is a natural bijection between conjugacy classes of maximal tori in $\mathrm{Sp}_{2n}(q)$ and the disjoint union of the $O_{2n}^\epsilon(q)$ -conjugacy classes of maximal tori in $\mathrm{SO}_{2n}^\epsilon(q)$ for $\epsilon = \pm$. For $w \in W_{\mathrm{Sp}_{2n}}$, we define $\epsilon_w = \epsilon$ if the F -stable maximal torus T_w of $\mathrm{Sp}_{2n}(q)$ corresponds to an F -stable maximal torus in $\mathrm{SO}_{2n}^\epsilon(q)$, and define

$$W_{\mathrm{Sp}_{2n}}^\epsilon = \{w \in W_{\mathrm{Sp}_{2n}} \mid \epsilon_w = \epsilon\}.$$

We shall identify the Weyl groups of Sp_{2n} and SO_{2n+1} , which will be denoted by W_n . Thus we also write $W_n^\epsilon = W_{\mathrm{Sp}_{2n}}^\epsilon$. The Weyl group $W_{\mathrm{SO}_{2n}^+}$ of SO_{2n}^+ can be regarded as a subgroup of W_n of index 2.

Clearly, $\epsilon_w = +$ if $T_w \simeq \mathrm{GL}_1(q)$ in $\mathrm{Sp}_2(q)$, and $\epsilon_w = -$ if $T_w \simeq U_1(q)$ in $\mathrm{Sp}_2(q)$. Since $\mathrm{Sp}_{2n}(q^t)$ can be regarded as a subgroup of $\mathrm{Sp}_{2nt}(q)$ for $t \in \mathbb{N}$, an F -stable

maximal torus of $\mathrm{Sp}_{2n}(q^t)$ is also an F -stable maximal torus of $\mathrm{Sp}_{2nt}(q)$. Hence $W_{\mathrm{Sp}_{2n}(q^t)}$ can be embedded as a subset of $W_{\mathrm{Sp}_{2nt}(q)}$. For $w \in W_{\mathrm{Sp}_{2n}(q^t)}$, the value of ϵ_w is the same no matter whether w is regarded as an element of $W_{\mathrm{Sp}_{2n}(q^t)}$ or $W_{\mathrm{Sp}_{2nt}(q)}$, and therefore $\epsilon_w = +$ (resp. $\epsilon_w = -$) if $T \cong \mathrm{GL}_1(q^t)$ (resp. $T_w \cong U_1(q^t)$) in $\mathrm{Sp}_{2t}(q)$.

2D. The character θ_w . If $T \cong \mathrm{GL}_1(q^t)$ or $U_1(q^t)$, let θ_T denote the unique character of T of order 2, i.e., $\theta_T(a) = \nu(a)^{(q^t-1)/2}$ (resp. $\theta_T(a) = \nu(a)^{(q^t+1)/2}$) for $a \in T$ if $T \cong \mathrm{GL}_1(q^t)$ (resp. $T \cong U_1(q^t)$), where ν is an isomorphism of T onto a subgroup of $\overline{\mathbf{Q}}_\ell^\times$. Define $\theta_T = \theta_{T_1} \otimes \cdots \otimes \theta_{T_r}$ if $T \cong T_1 \times \cdots \times T_r$ is an F -stable maximal torus of a connected reductive group G , where $T_i \cong \mathrm{GL}_1(q^{t_i})$ or $U_1(q^{t_i})$ for some positive integer t_i , $i = 1, \dots, r$. It is clear that ${}^g\theta_T = \theta_{sT}$ for $g \in G$. If $T = T_w$ for some $w \in W_G$, the character θ_T will be denoted by θ_w .

We can check that $\theta_T = \chi_G|_T$ if G is equal to GL_n , U_n or SO_n (n a positive integer), where χ_G is the unique linear character of G of order 2. It is known as the spinor norm character in the case of orthogonal groups. For a representation or a character π of G , we abbreviate $\chi_G \otimes \pi$ by $\chi_G \pi$.

The following identity can be found in [Pan 2016].

Lemma 2.1. *If $G = \mathrm{GL}_n$, U_n or SO_n , then*

$$\chi_G = \frac{1}{|W_G|} \sum_{w \in W_G} R_{T_w, \theta_w}^G.$$

Lemma 2.2. *If $u \in G$ is a unipotent element, then $\chi_G(u) = 1$.*

Proof. By (2-1), (2-2) and Lemma 2.1,

$$\begin{aligned} \chi_G(u) &= \frac{1}{|W_G|} \sum_{w \in W_G} R_{T_w, \theta_w}^G(u) \\ &= \frac{1}{|W_G|} \sum_{w \in W_G} \frac{1}{|G|} \sum_{g \in G} \theta_w(1) Q_{sT}^G(u) \\ &= \frac{1}{|W_G|} \sum_{w \in W_G} \frac{1}{|G|} \sum_{g \in G} Q_{sT}^G(u) = 1(u) = 1. \end{aligned} \quad \square$$

Recall that an irreducible representation π is called a unipotent representation if it occurs in some $R_{T,1}^G$. We have the following:

Lemma 2.3. *If $G = \mathrm{GL}_n$, U_n or SO_n , then*

- (i) π is a unipotent representation if and only if $\chi_G \pi$ occurs in some R_{T_w, θ_w}^G ,
- (ii) R_{T_w, θ_w}^G only consists of $\chi_G \pi$ with π unipotent.

Proof. If $y = su$ is the Jordan decomposition of an element $y \in G$, then

$$\begin{aligned}\chi_G R_{T_w,1}^G(y) &= \chi_G(y) \frac{1}{|C^0(s)|} \sum_{g \in G, s^G \in T_w} 1(s^g) \mathcal{Q}_{sT_w}^{C^0(s)}(u) \\ &= \frac{1}{|C^0(s)|} \sum_{g \in G, s^G \in T_w} \chi_G(s^g) \chi_G(u) \mathcal{Q}_{sT_w}^{C^0(s)}(u) \\ &= \frac{1}{|C^0(s)|} \sum_{g \in G, s^G \in T_w} \theta_w(s^g) \mathcal{Q}_{sT_w}^{C^0(s)}(u) = R_{T_w, \theta_w}^G(y). \quad \square\end{aligned}$$

Similar to the notion of unipotent representations, a representation π is called a θ -representation if it occurs in some R_{T_w, θ_w}^G . For $G = \mathrm{Sp}_{2n}$, we do not have nontrivial characters of G , but we can still talk about θ -representations.

3. Weil representations and cuspidal unipotent representations

Adams and Moy [1993] (see also [Aubert et al. 1996; 2016; Epequin Chavez 2019]) consider the question of how the unipotent representations behave under theta correspondence. In particular they show that theta correspondence sends a cuspidal unipotent representation of G to a cuspidal unipotent representation of G' if $(G, G') = (\mathrm{Sp}_{2n}, O_{2n'}^\epsilon)$ or $(U_n, U_{n'})$. In the proof they use the decomposition of the Weil representation of $(\mathrm{Sp}_{2n}, O_{2n'}^\epsilon)$ and $(U_n, U_{n'})$ given in [Srinivasan 1979b]. Pan [2016] gives a decomposition of the Weil representation of a finite symplectic-odd orthogonal dual pair, which we have recalled as Theorem 3.1. Thus we can give the correspondence of cuspidal unipotent representations of such dual pairs based on Pan's result, together with another ingredient called conservation relation.

3A. Weil representation and theta lifting. Let $\omega_{\mathrm{Sp}_{2n}}$ be the character of the Weil representation (see [Gérardin 1977]) of the symplectic group Sp_{2n} which depends on a nontrivial additive character ψ of F , and let $\omega_{\mathrm{Sp}_{2n}}^\#$ denote the uniform projection of $\omega_{\mathrm{Sp}_{2n}}$, i.e., the projection onto the subspace of virtual characters spanned by all the Deligne–Lusztig characters. Let (G, G') be a reductive dual pair in $\mathrm{Sp}_{2n}(q)$. Write $\omega_{G, G'}$ for the restriction of $\omega_{\mathrm{Sp}_{2n}}$ to $G \times G'$. Then it decomposes into a direct sum

$$\omega_{G, G'} = \bigoplus_{\pi, \pi'} m_{\pi, \pi'} \pi \otimes \pi',$$

where π and π' run over irreducible representations of G and G' , respectively. We can rearrange this sum to get

$$\omega_{G, G'} = \bigoplus_{\pi} \pi \otimes \Theta_{G'}(\pi),$$

where $\Theta_{G'}(\pi) = \bigoplus_{\pi'} m_{\pi, \pi'} \pi'$ is a (not necessarily irreducible) representation of G' , called the (big) theta lifting of π to G' . The theta lifting from G to G' will be written

as $\pi \mapsto \Theta_{G'}(\pi)$, and we have $\pi' \in \Theta_{G'}(\pi)$ if and only if $\pi \otimes \pi' \in \omega_{G,G'}$. We should mention that it is an important open problem to find some “good” constituents from $\Theta_{G'}(\pi)$ (see [Epequin Chavez 2019; Gurevich and Howe 2017] for different approaches and results in this direction). By abuse of notation we also write $\omega_{G,G'}$ for its character, and write $\omega_{G,G'}^\#$ for the uniform projection of $\omega_{G,G'}$.

It is often more convenient to work with the families of dual pairs associated to Witt towers instead of a single dual pair. Thus let us introduce systematically some notation for Witt towers. We will consider two Witt towers $\mathcal{T}$ and $\mathcal{T}'$ such that $(G_n, G'_{n'})$ form a reductive dual pair of type I for any $G_n \in \mathcal{T}$ and $G'_{n'} \in \mathcal{T}'$. There are the following types of Witt towers:

- For unitary groups there are two Witt towers $U^+ = \{U_{2n}\}_{n \geq 0}$ and $U^- = \{U_{2n+1}\}_{n \geq 0}$.
- For symplectic groups there is only one Witt tower $\mathrm{Sp} = \{\mathrm{Sp}_{2n}\}_{n \geq 0}$.
- For even orthogonal groups there are two Witt towers $O_{\mathrm{even}}^+ = \{O_{2n}^+\}_{n \geq 0}$ and $O_{\mathrm{even}}^- = \{O_{2n}^-\}_{n \geq 1}$.
- For odd orthogonal groups there are two Witt towers $O_{\mathrm{odd}}^\epsilon = \{O_{2n+1}^\epsilon\}_{n \geq 0}$, $\epsilon = \pm$.

For the dual pair $(\mathrm{Sp}_{2n}, O_{2n'}^\epsilon)$ where $\epsilon = \pm$, we write $\omega_{n,n'}^\epsilon$ instead of $\omega_{G,G'}$. Since there are also two different nondegenerate symmetric bilinear forms on an odd-dimensional vector space over F (though their orthogonal groups are physically equal), we have two different types of symplectic-odd orthogonal dual pairs. Similarly we denote them by $(\mathrm{Sp}_{2n}, O_{2n'+1}^\epsilon)$ and the corresponding Weil representations by $\omega_{n,n'}^\epsilon$, where $\epsilon = \pm$. For unitary groups write $G_n^\epsilon = U_{2n}$ for $\epsilon = +$ or U_{2n+1} for $\epsilon = -$. Then we denote the Weil representation of $(G_n^\epsilon, G_{n'}^{\epsilon'})$ by $\omega_{n,n'}^{\epsilon,\epsilon'}$. We will specify the dual pair we are working with when necessary, so it should cause no confusion when we use the notation $\omega_{n,n'}^\epsilon$ or $\omega_{n,n'}^{\epsilon,\epsilon'}$, and occasionally we suppress the signs to ease notation.

Now we recall the main result of [Pan 2016], which will be a crucial tool used in this paper.

Theorem 3.1. *Suppose that q is sufficiently large such that the main theorem in [Srinivasan 1979b] holds. Then for a positive integer n and a nonnegative integer n' ,*

$$\omega_{\mathrm{Sp}_{2n}, \mathrm{SO}_{2n'+1}}^\# \cdot (1 \otimes \chi_{\mathrm{SO}_{2n'+1}}) = \sum_{k=0}^{\min(n,n')} \frac{1}{|W_k|} \frac{1}{|W_{\mathrm{Sp}_{2(n-k)}}|} \frac{1}{|W_{\mathrm{SO}_{2(n'-k)+1}}|} \\ \sum_{v \in W_k} \sum_{\theta \in \mathrm{Irr}(T_v)} \sum_{w \in W_{\mathrm{Sp}_{2(n-k)}}} \sum_{w' \in W_{\mathrm{SO}_{2(n'-k)+1}}} \epsilon_w R_{T_v \times T_w, \theta \otimes \theta_w}^{\mathrm{Sp}_{2n}} \\ \otimes R_{T_v \times T_{w'}, \theta \otimes \theta_{w'}}^{\mathrm{SO}_{2n'+1}}$$

where $\mathrm{Irr}(T_v)$ denotes the set of irreducible characters of the torus T_v .

We should point out that the above formula for the uniform projection is independent of the choice of $\mathrm{SO}_{2n'+1} = \mathrm{SO}_{2n'+1}^\epsilon$. In the rest of this paper the finite field F is assumed to be large enough as in Theorem 3.1.

3B. Correspondence of unipotent and θ -representations. For finite unitary and symplectic-even orthogonal dual pairs, Adams and Moy showed that if $\pi \otimes \pi'$ occurs in $\omega_{n,n'}$, then π is unipotent if and only if π' is unipotent. This does not hold for finite symplectic-odd orthogonal dual pairs, in which case one of the results we will prove says that π is unipotent if and only if π' is a θ -representation when $n = n'$. Much as in [Adams and Moy 1993], our proof makes use of the decomposition of the Weil representation given by Theorem 3.1 in the symplectic-odd orthogonal case.

The following well-known result can be found in [Adams and Moy 1993; Mœglin et al. 1987], which says that the first occurrence of theta lifting of an irreducible cuspidal representation in a given Witt tower is also irreducible cuspidal, while the later occurrences are not cuspidal.

Theorem 3.2. *Let $(G_n, G'_{n'})$ be a dual pair of finite classical groups. Assume that π is an irreducible cuspidal representation of G_n , and that*

- (i) $\pi \mapsto \pi'$ in the theta correspondence for $G_n \times G'_{n'}$,
- (ii) π does not occur in the theta correspondence for $G_n \times G'_k$ for any $k < n'$.

Then π' is irreducible cuspidal, and $\Theta_{G'_k}(\pi)$ has no cuspidal constituents for $k > n'$.

Recall that by Lemma 2.3, the map $\pi \mapsto \chi_G \pi$ ($G \neq \mathrm{Sp}_{2n}$) gives a bijection between the unipotent representations and θ -representations of G , and by Lemma 2.2 it preserves the set of cuspidal representations.

We have the following basic result.

Theorem 3.3. *Suppose that an irreducible representation $\pi \otimes \pi'$ of $(\mathrm{Sp}_{2n}, \mathrm{SO}_{2n'+1}^\epsilon)$ occurs in $\omega_{n,n'}^\epsilon$. Then the following hold.*

- (i) *If π is a unipotent representation of Sp_{2n} , then $n' \geq n$ and $\pi' \in R_{T_v \times T_{w'}, \theta_v \otimes 1}^{\mathrm{SO}_{2n'+1}^\epsilon}$ for some $v \in W_n$ and $w' \in W_{n'-n}$.*
- (ii) *If π' is a θ -representation of $\mathrm{SO}_{2n'+1}^\epsilon$, then $n \geq n'$ and $\pi \in R_{T_v \times T_w, 1 \otimes \theta_w}^{\mathrm{Sp}_{2n}}$ for some $v \in W_{n'}$ and $w \in W_{n-n'}$.*
- (iii) *π is a θ -representation of Sp_{2n} if and only if π' is a unipotent representation of $\mathrm{SO}_{2n'+1}^\epsilon$.*

Proof. The proof is similar in spirit to that of [Adams and Moy 1993, Theorem 3.5]. First of all, for $G = \mathrm{SO}_{2n'+1}$ from $\theta_T = \chi_G|_T$ it follows that all the pairs (T_w, θ_w) , $w \in W_n$, are in the same geometric conjugacy class. More generally, two pairs of the form $(T_{v_i} \times T_{w_i}, 1 \times \theta_{w_i})$, $v_i \in W_{n_i}$, $w_i \in W_{n'-n_i}$, $i = 1, 2$, are geometrically conjugate if and only if $n_1 = n_2$.

We need to recall some basic facts. Denote $G \cdot G'$ by H , and the character $R_T^G(\theta) \times R_{T'}^{G'}(\theta')$ by $R_{T \cdot T'}^H(\theta \cdot \theta')$. These characters form an orthogonal basis for the space of uniform class functions of H . The set of irreducible characters $\mathcal{E}(H)$ of H can be partitioned by geometric conjugacy classes (see, e.g., [Lusztig 1984])

$$\mathcal{E}(H) = \coprod_s \mathcal{E}(H, (s)),$$

where $s = s_{G^*} \cdot s_{G'^*}$ runs over the semisimple conjugacy classes of the dual group $H^* = G^* \cdot G'^*$. The irreducible constituents of a single $R_{T \cdot T'}^H(\theta \times \theta')$ are all associated to one geometric conjugacy class (s) . Let π_H be a representation of H . Then we have

$$\pi_H = \bigoplus_s \pi_H(s),$$

where s runs over the semisimple conjugacy classes in H^* and $\pi_H(s) \in \mathcal{E}(H, (s))$ is a subrepresentation of H . It is known that if $\pi \in \mathcal{E}(H, (s))$ and $\pi' \in \mathcal{E}(H, (s'))$ with $(s) \neq (s')$, then $(\pi, \pi')_H = 0$. We also have

$$\pi_H^\# = \bigoplus_s \pi_H(s)^\#.$$

If $\pi_H(s) \neq 0$, then it is a positive combination of representations in $\mathcal{E}(H, (s))$. It follows from the fact that the regular representation is uniform that $\pi_H(s)^\#$ is nonzero.

Now let $\pi_H = \pi \otimes \pi'$, which occurs in $\omega_{n,n'}^\epsilon$ by assumption. To prove (i), we note that if π is unipotent then $\pi_H \in \mathcal{E}(H, (s))$ with $s_{G^*} = 1$. By the above argument, $\pi_H^\# = \pi_H(s)^\# \neq 0$; hence it occurs in some

$$R_{T,1}^{\mathrm{Sp}_{2n}} \otimes R_{T',\theta}^{\mathrm{SO}_{2n'+1}^\epsilon},$$

which has to appear in a summand of $\omega_{n,n'}^\#$. Then the conclusion follows from Theorem 3.1 and the first paragraph of the proof. The proof of (ii) and (iii) is similar and will be omitted. $\square$

In particular, by Theorem 3.3 (i) and (ii), if $n = n'$ then π is a unipotent representation of Sp_{2n} if and only if π' is a θ -representation of $\mathrm{SO}_{2n+1}^\epsilon$. Applying Theorems 3.2 and 3.3, we shall describe the theta correspondence between cuspidal unipotent representations of Sp_{2n} and cuspidal θ -representations of $\mathrm{SO}_{2n'+1}$. We remark that the latter notion is not vacuous thanks to the following simple lemma.

Lemma 3.4. *If π is a cuspidal unipotent representation of a finite classical group G which is not symplectic, then $\chi_G \pi$ is a cuspidal θ -representation.*

Proof. For any unipotent element u , by Lemma 2.2 we have

$$\chi_G \pi(u) = \chi_G(u) \pi(u) = \pi(u).$$

Hence $\chi_G \pi$ is cuspidal. It is a θ -representation by Lemma 2.3. $\square$

3C. First occurrence for symplectic-odd orthogonal pairs. For an irreducible representation π of G_n , the smallest integer $n'(\pi)$ such that π occurs in $\omega_{n,n'(\pi)}$ is called the *first occurrence index* of π in the Witt tower $\{G'_{n'}\}_{n' \geq 0}$. By [Mœglin et al. 1987, Chapitre 3, Lemme IV.2], there exists n' such that $\Theta_{G'_{n'}}(\pi) \neq 0$, hence $n'(\pi)$ is well-defined. In this section we compute the first occurrence of cuspidal unipotent and θ -representations for symplectic-odd orthogonal dual pairs. We begin by reviewing Lusztig's results [1977] on the cuspidal unipotent representations of symplectic and orthogonal groups.

Theorem 3.5. *The groups*

- (i) U_n , $n = k(k+1)/2$,
- (ii) Sp_{2n} , $n = k(k+1)$,
- (iii) SO_{2n+1} , $n = k(k+1)$,
- (iv) $\mathrm{SO}_{2n}^\epsilon$, $n = k^2$, $\epsilon = \mathrm{sgn}(-1)^k$

are the only groups in their respective Lie families which possess a cuspidal unipotent representation. In each case, the specified group G has a unique cuspidal unipotent representation.

By Lemma 3.4, we have the following immediate consequence for cuspidal θ -representations.

Corollary 3.6. *The groups*

- (1) U_n , $n = k(k+1)/2$,
- (2) SO_{2n+1} , $n = k(k+1)$,
- (3) $\mathrm{SO}_{2n}^\epsilon$, $n = k^2$, $\epsilon = \mathrm{sgn}(-1)^k$

are the only groups in their respective Lie families which possess a cuspidal θ -representation. In each case, the specified group G has a unique cuspidal θ -representation.

We define a θ -representation of O_{2n+1} , which is nonconnected, to be a constituent of $\mathrm{Ind}_{\mathrm{SO}_{2n+1}}^{O_{2n+1}} \pi$ for a θ -representation π of SO_{2n+1} . Let λ'_k stand for the unique cuspidal unipotent representation of SO_{2n+1} , $n = k(k+1)$. Then

$$\lambda'_{k,\chi} := \chi_{\mathrm{SO}_{2n+1}} \lambda'_k$$

is the unique cuspidal θ -representation of SO_{2n+1} , and we have

$$\mathrm{Ind}_{\mathrm{SO}_{2n+1}}^{O_{2n+1}} \lambda'_{k,\chi} = \lambda'^+_{k,\chi} \oplus \lambda'^-_{k,\chi},$$

where $\lambda'^+_{k,\chi} := \lambda'_{k,\chi} \otimes 1$ and $\lambda'^-_{k,\chi} := \lambda'_{k,\chi} \otimes \mathrm{sgn}$ are cuspidal θ -representations of $O_{2n+1} \cong \mathrm{SO}_{2n+1} \times \{\pm 1\}$. Now we can give the first main result of this section, which is stated as Theorem 1.1 in the Introduction.

Theorem 3.7. *Let λ_k be the unique cuspidal unipotent representation of Sp_{2n} , $n = k(k+1)$. Then the first occurrence index $n'^\epsilon(\lambda_k)$ in the Witt tower $\{O_{2n'+1}^\epsilon\}_{n' \geq 0}$ is equal to n , and the theta lifting $\Theta_{O_{2n+1}^\epsilon}(\lambda_k)$ equals $\lambda_{k,\chi}^+$.*

The proof of this theorem involves three ingredients. The first one is the following information which can be read from [Howe 1973].

Proposition 3.8. *Every irreducible representation of Sp_{2n} is contained in $\omega_{n,n}^+ \oplus \omega_{n,n}^-$, where $\omega_{n,n'}^\epsilon = \omega_{\mathrm{Sp}_{2n}, O_{2n'+1}^\epsilon}$.*

The second ingredient is the following so-called conservation relation for cuspidal representations given in [Pan 2002].

Proposition 3.9. (i) *Let π be an irreducible cuspidal representation of Sp_{2n} and $n'^\epsilon(\pi)$ be the first occurrence index of π in the Witt tower $O_{\mathrm{odd}}^\epsilon = \{O_{2n'+1}^\epsilon\}_{n' \geq 0}$. Then*

$$n'^+(\pi) + n'^-(\pi) = 2n.$$

(ii) *Let π' be an irreducible cuspidal representation of $O_{2n'+1}^\epsilon$ and $n(\pi')$ be the first occurrence index of π' in the Witt tower $\mathrm{Sp} = \{\mathrm{Sp}_{2n}\}_{n \geq 0}$. Then*

$$n(\pi') + n(\pi' \otimes \mathrm{sgn}) = 2n' + 1.$$

A few remarks are in order. In the literature, the conservation relations are usually formulated in terms of the dimensions of vector spaces, but in this paper we will use the above form for convenience. Here we only recalled the statement for symplectic-odd orthogonal dual pairs and we refer the readers to [Pan 2002] for other cases. In the world of p -adic local fields, the conservation relations have been established in full generality in [Sun and Zhu 2015]. We should also mention that the standard formulation of conservation relations does not seem to hold in general for noncuspidal representations of finite dual pairs, and we will give some explanations at the end of the paper.

The last ingredient we need is that the action of $-I$ under a unipotent representation is trivial. This fact might be known but we could not find a reference, so we include a detailed proof for completeness. It can be used to determine whether the theta lifting of a unipotent representation takes place in the even or odd part of the Weil representation, but we will not need this result.

Proposition 3.10. (i) *If π is a unipotent representation of GL_n , U_n , Sp_{2n} or $\mathrm{SO}_{2n}^\epsilon$, then $\pi(-I)$ is trivial.*

(ii) *If π is a unipotent representation of Sp_{2n} and $\pi \otimes \pi'$ occurs in the Weil representation $\omega_{n,n'}^\epsilon$ of $(\mathrm{Sp}_{2n}, O_{2n'+1}^\epsilon)$, then $\pi'(-I)$ is trivial.*

Proof. (ii) clearly follows from (i) because $\pi(-I) = \pi'(-I)$ through the Weil representation $\omega_{n,n'}^\epsilon$. We shall prove (i) for all the classical groups (except odd

orthogonal groups) simultaneously by induction on n . We first observe that by (2-1),

$$(3-1) \quad R_{T,1}^G(-I) = R_{T,1}^G(I).$$

If $n = 1$, then G has only two unipotent representations 1 and π . Hence the conclusion follows from (3-1). Assume that $n > 1$ and $\pi \in R_{T,1}^G$ for some T . If T is not anisotropic, then it is contained in a Levi subgroup L of a proper parabolic subgroup P of G , i.e., $P = P^F$ for some F -stable proper parabolic subgroup P of G . By Frobenius reciprocity,

$$(\pi, R_{T,1}^G)_G = (\pi, \text{Ind}_P^G R_{T,1}^L)_G = (J_P^G(\pi), R_{T,1}^L)_L \neq 0,$$

where J_P^G is the Jacquet functor. We may write $L = \text{GL}_{n_1} \times \text{GL}_{n_2} \times \cdots \times \text{GL}_{n_r} \times G_{n_0}$, where $n_i < n$, $i = 0, \dots, r$, and G_{n_0} is a classical group of the same type as G . There exists $\pi_0 \in J_P^G(\pi)$ such that $(\pi_0, R_{T,1}^L)_L \neq 0$. Then by induction we have that $\pi(-I) = \pi_0(-I)$ is trivial. By [Srinivasan 1979a, Theorem 6.25], if $\pi \notin R_{T,1}^G$ for any T which is not anisotropic, then π is a cuspidal representation. If $G = \text{GL}_n$ then we are done because it does not have cuspidal unipotent representations when $n > 1$. If $G \neq \text{GL}_n$, then it remains to consider the case that π is the unique cuspidal unipotent representation of G (if it exists). The uniqueness of π enables us to write

$$(\pi, R_{T,1}^G)\pi = R_{T,1}^G - \sum_{\pi_i \in R_{T,1}^G, \pi_i \neq \pi} (\pi_i, R_{T,1}^G)_G \pi_i,$$

where the π_i 's are all unipotent representations but not cuspidal. Then the conclusion follows from (3-1) and that $\pi_i(-I)$ is trivial for each i . $\square$

Proof of Theorem 3.7. By Proposition 3.8, the cuspidal unipotent representation λ_k of Sp_{2n} is contained in $\omega_{n,n}^+ \oplus \omega_{n,n}^-$. By Theorem 3.3, λ_k is not contained in any $\omega_{n,n'}^\epsilon$ when $n' < n$, and for some $\epsilon_0 \in \{+, -\}$ the theta lifting $\Theta_{O_{2n+1}^{\epsilon_0}}(\lambda_k)$ to $O_{2n+1}^{\epsilon_0}$ is the first occurrence in the Witt tower $\mathcal{O}_{\text{odd}}^{\epsilon_0}$; hence it is a cuspidal θ -representation. Then it follows from Corollary 3.6 and Proposition 3.10 that $\Theta_{O_{2n+1}^{\epsilon_0}}(\lambda_k) = \lambda_{k,\chi}^+$. The conservation relation Proposition 3.9(i) implies that $n'^{-\epsilon_0}(\lambda_k) = 2n - n'^{\epsilon_0}(\lambda_k) = n$, and one can similarly deduce that $\Theta_{O_{2n+1}^{-\epsilon_0}}(\lambda_k) = \lambda_{k,\chi}^+$ as well. $\square$

We also study the first occurrence of the theta lifting of cuspidal θ -representations of Sp_{2n} in $\mathcal{O}_{\text{odd}}^\epsilon$, which by Theorem 3.3 will be cuspidal unipotent representations. By definition, a unipotent representation of O_{2n+1} is a constituent of $\text{Ind}_{\text{SO}_{2n+1}}^{O_{2n+1}}(\pi)$ for a unipotent representation π of SO_{2n+1} . Let λ'_k be the unique cuspidal unipotent representation of SO_{2n+1} , $n = k(k+1)$. Then we have

$$(3-2) \quad \text{Ind}_{\text{SO}_{2n+1}}^{O_{2n+1}}(\lambda'_k) = \lambda_k^+ \oplus \lambda_k^-,$$

where $\lambda_k^+ := \lambda'_k \otimes 1$ and $\lambda_k^- := \lambda'_k \otimes \text{sgn}$ are cuspidal unipotent representations of $O_{2n+1} \cong \text{SO}_{2n+1} \times \{\pm 1\}$.

Applying a result of Lusztig, we first obtain the following classification of cuspidal θ -representations of finite symplectic groups.

Theorem 3.11. *The groups Sp_{2n} , $n = k^2$, are the only symplectic groups which possess cuspidal θ -representations. Each Sp_{2k^2} , $k \neq 0$, has two cuspidal θ -representations $\lambda_{k,\alpha}$ and $\lambda_{k,\beta}$, which satisfy $\lambda_{k,i}(-I) = (-1)^k$, $i = \alpha, \beta$.*

Proof. We first recall some general results. Assume that G is a simple simply connected linear algebraic group over F , and T is an F -stable maximal torus of G . Let G^* be the dual group of G , and T^* be the F -stable maximal torus of G^* which is the dual of T . Write $G^* = G^{*F}$, $T^* = T^{*F}$. Then a character θ of T corresponds to some element $s \in T^*$, which is well-defined up to conjugacy by $W_{G^*}(T^*)^F$. In this case we write $R_{T^*,s}^G$ instead of $R_{T,\theta}^G$. Let $\mathcal{E}(G, (s))$ be the set of irreducible representations which occur in $R_{T^*,s}^G$ for some T^* such that $s \in T^*$. By [Lusztig 1988], there is a bijection

$$\mathcal{L} : \mathcal{E}(G, (s)) \rightarrow \mathcal{E}(C_{G^*}(s), (1)),$$

where $\mathcal{E}(C_{G^*}(s), (1))$ is the set of irreducible representations of $C_{G^*}(s)$ whose restrictions to $C_{G^*}(s)^\circ$ are sums of unipotent representations of $C_{G^*}(s)^\circ$. Note that $C_{G^*}(s)$ may not be connected in general. Moreover if the identity components of the centers of G and $C_{G^*}(s)$ have the same F -rank, then $\pi \in \mathcal{E}(G, (s))$ is cuspidal if and only if $\mathcal{L}(\pi) \in \mathcal{E}(C_{G^*}(s), (1))$ is cuspidal (see, e.g., [Lusztig 1977, Chapter 9] and [Malle 2017, Lemma 2.7]).

For $G = \mathrm{Sp}_{2n}$ we have $G^* = \mathrm{SO}_{2n+1}$. The pair (T_w, θ_w) corresponds to $s \in T_w$, where $s = (-I, 1)$ with I being the identity in $\mathrm{SO}_{2n}^{\epsilon_w} \subset \mathrm{SO}_{2n+1}$. Then

$$C_{G^*}(s) \cong O_{2n}^{\epsilon_w} \subset \mathrm{SO}_{2n+1};$$

hence by the above results we have a bijection between cuspidal θ -representations of Sp_{2n} and the union of cuspidal unipotent representations of O_{2n}^ϵ over $\epsilon = \pm$. By Theorem 3.5, each $O_{2k^2}^\epsilon$ with $\epsilon := \mathrm{sgn}(-1)^k$ has two cuspidal unipotent representations, which are the two constituents induced from the unique cuspidal unipotent representation of $\mathrm{SO}_{2k^2}^\epsilon$, and these are the only even orthogonal groups which possess cuspidal unipotent representations. Hence, each Sp_{2n} , $n = k^2$, has two cuspidal θ -representations $\lambda_{k,1}$ and $\lambda_{k,2}$, and these are the only possibilities.

From [Howe 1973] we also know that every irreducible representation of Sp_{2n} is contained in $\omega_{\mathrm{Sp}_{2n}, O_{2n}^+} \oplus \omega_{\mathrm{Sp}_{2n}, O_{2n}^-}$. Each $\lambda_{k,i}$, $i = \alpha, \beta$, occurs in some R_{T_w, θ_w}^G such that $\epsilon_w = \epsilon = \mathrm{sgn}(-1)^k$. Similar to the proof of Theorems 3.3 and 3.7, applying the main result of [Srinivasan 1979b] as well as conservation relations we can show that $\lambda_{k,i}$ occurs in $\omega_{\mathrm{Sp}_{2n}, O_{2n}^\epsilon}$ and $\Theta_{O_{2n}^\epsilon}(\lambda_{k_i})$, $i = \alpha, \beta$, are the two cuspidal θ -representations of O_{2n}^ϵ . Since $\chi_{\mathrm{SO}_{2n}^\pm}(-I) = \pm 1$, we obtain that $\lambda_{k,i}(-I) = \chi_{\mathrm{SO}^\epsilon}(-I) = (-1)^k$. $\square$

Finally, we can give the second main result of this section.

Theorem 3.12. *Let $\lambda_{k,i}$, $i = \alpha, \beta$, be the cuspidal θ -representations of Sp_{2k^2} , and let $n^{\epsilon}(\lambda_{k,i})$ be the first occurrence index of $\lambda_{k,i}$ in the Witt tower $\mathbf{O}_{\mathrm{odd}}^{\epsilon}$. Then for some $\epsilon_0 \in \{+, -\}$ one has $n^{\epsilon_0}(\lambda_{k,\alpha}) = n'^{-\epsilon_0}(\lambda_{k,\beta}) = k(k-1)$ and $n'^{-\epsilon_0}(\lambda_{k,\alpha}) = n^{\epsilon_0}(\lambda_{k,\beta}) = k(k+1)$. Write $\epsilon(k) = \mathrm{sgn}(-1)^k$. Then the theta liftings are given by*

$$\begin{aligned} \Theta_{O_{2k(k-1)+1}^{\epsilon_0}}(\lambda_{k,\alpha}) &= \lambda_{k-1}^{\prime\epsilon(k)}, & \Theta_{O_{2k(k+1)+1}^{-\epsilon_0}}(\lambda_{k,\alpha}) &= \lambda_k^{\prime\epsilon(k)}, \\ \Theta_{O_{2k(k+1)+1}^{\epsilon_0}}(\lambda_{k,\beta}) &= \lambda_k^{\prime\epsilon(k)}, & \Theta_{O_{2k(k-1)+1}^{-\epsilon_0}}(\lambda_{k,\beta}) &= \lambda_{k-1}^{\prime\epsilon(k)}, \end{aligned}$$

where $\lambda_k^{\prime\epsilon}$ is defined by (3-2).

We remark that the ϵ_0 above depends on the choice of the nontrivial character ψ of $\mathbf{F}$ which we used to define the Weil representations. If ψ is replaced by $\psi_t(x) := \psi(tx)$ for some $t \in \mathbf{F}^{\times} \setminus \mathbf{F}^{\times 2}$, then ϵ_0 should be replaced by $-\epsilon_0$.

Proof. We use an inductive argument which is parallel to the proof in [Adams and Moy 1993]. By convention we let λ_0 be the trivial representation of Sp_0 .

For Sp_2 , $\lambda_{1,\alpha}$ is contained in $\omega_{1,1}^+ \oplus \omega_{1,1}^-$. Since by Theorem 3.3(iii) the first occurrence of the theta lifting of $\lambda_{1,\alpha}$ is unipotent cuspidal, we see that $\lambda_{1,\alpha}$ occurs in $\omega_{1,0}^{\epsilon_0}$ for some ϵ_0 . By conservation relation, $\lambda_{1,\alpha}$ occurs in $\omega_{1,2}^{-\epsilon_0}$ as the first occurrence in $\mathbf{O}_{\mathrm{odd}}^{-\epsilon_0}$. By Theorem 3.11, we have the theta liftings $\Theta_{O_1^{\epsilon_0}}(\lambda_{1,\alpha}) = \lambda_0^{\prime-}$ and $\Theta_{O_5^{-\epsilon_0}}(\lambda_{1,\alpha}) = \lambda_2^{\prime-}$. Similarly $\lambda_{1,\beta}$ is contained in $\omega_{1,0}^+ \oplus \omega_{1,0}^-$. It cannot happen that $\Theta_{O_1^{\epsilon_0}}(\lambda_{1,\alpha}) = \Theta_{O_1^{\epsilon_0}}(\lambda_{1,\beta}) = \lambda_0^{\prime-}$, hence $\lambda_{1,\beta}$ must be contained in $\omega_{1,0}^{-\epsilon_0}$. By conservation again, $\lambda_{1,\beta}$ occurs in $\omega_{1,2}^{\epsilon_0}$ as the first occurrence in $\mathbf{O}_{\mathrm{odd}}^{\epsilon_0}$. The theta liftings of $\lambda_{1,\beta}$ to $O_1^{-\epsilon_0}$ and $O_5^{\epsilon_0}$ are therefore $\lambda_0^{\prime-}$ and $\lambda_2^{\prime-}$, respectively. We may exhaust the theta correspondence inductively through the following diagram:

$$\begin{array}{ccccccc} \mathrm{Sp}_0 & \lambda_0 & \longrightarrow & \lambda_0^{'+} = 1 & \omega_{0,0}^{\epsilon_0} \\ \mathrm{Sp}_0 & \lambda_0 & \longrightarrow & \lambda_0^{'+} = 1 & \omega_{0,0}^{-\epsilon_0} \\ \mathrm{Sp}_2 & \lambda_{1,\alpha} & \longrightarrow & \lambda_0^{\prime-} = \mathrm{sgn} & \omega_{1,0}^{\epsilon_0} \\ \mathrm{Sp}_2 & \lambda_{1,\beta} & \longrightarrow & \lambda_0^{\prime-} = \mathrm{sgn} & \omega_{1,0}^{-\epsilon_0} \\ \mathrm{Sp}_2 & \lambda_{1,\alpha} & \longrightarrow & \lambda_1^{\prime-} & \omega_{1,2}^{-\epsilon_0} \\ \mathrm{Sp}_2 & \lambda_{1,\beta} & \longrightarrow & \lambda_1^{\prime-} & \omega_{1,2}^{\epsilon_0} \\ & & & \vdots & \\ \mathrm{Sp}_{2k^2} & \lambda_{k,\alpha} & \longrightarrow & \lambda_{k-1}^{\prime\epsilon(k)} & \omega_{k^2,k(k-1)}^{\epsilon_0} \\ \mathrm{Sp}_{2k^2} & \lambda_{k,\beta} & \longrightarrow & \lambda_{k-1}^{\prime\epsilon(k)} & \omega_{k^2,k(k-1)}^{-\epsilon_0} \\ \mathrm{Sp}_{2k^2} & \lambda_{k,\alpha} & \longrightarrow & \lambda_k^{\prime\epsilon(k)} & \omega_{k^2,k(k+1)}^{-\epsilon_0} \\ \mathrm{Sp}_{2k^2} & \lambda_{k,\beta} & \longrightarrow & \lambda_k^{\prime\epsilon(k)} & \omega_{k^2,k(k+1)}^{\epsilon_0} \end{array}$$

In general for Sp_{2k^2} , by Proposition 3.8, $\lambda_{k,i}$, $i = \alpha, \beta$, are contained in $\omega_{k^2,k^2}^+ \oplus \omega_{k^2,k^2}^-$. Since the first occurrences of their theta liftings are cuspidal unipotent, they are contained in $\omega_{k^2,k(k-1)}^+ \oplus \omega_{k^2,k(k-1)}^-$. By induction we have

$$\Theta_{O_{2k(k-1)+1}^{\epsilon_0}}(\lambda_{k-1,\beta}) = \lambda_{k-1}^{\epsilon(k-1)}.$$

Then by the conservation relation given by Proposition 3.9(ii) we may arrange the indices such that

$$\Theta_{O_{2k(k-1)+1}^{\epsilon_0}}(\lambda_{k,\alpha}) = \lambda_{k-1}^{\epsilon(k-1)} \otimes \mathrm{sgn} = \lambda_{k-1}^{\epsilon(k)}.$$

By conservation again, $\lambda_{k,\alpha}$ occurs in $\omega_{k^2,k(k+1)}^{-\epsilon_0}$. In a similar way, $\lambda_{k,\beta}$ occurs in $\omega_{k^2,k(k-1)}^{-\epsilon_0}$ and $\omega_{k^2,k(k+1)}^{\epsilon_0}$. The remaining assertions about the theta liftings are clear. $\square$

4. Theta correspondence and Harish-Chandra series

In this section we study the theta correspondence of representations in certain Harish-Chandra series in a way similar to [Aubert et al. 1996].

Let T_l be the split torus over F of rank l , and $\theta_l := \theta_{T_l}$ the order 2 character of T_l defined in Section 2D. In particular $(T_1, \theta_1) = (\mathrm{GL}_1, \chi_{\mathrm{GL}_1})$. For later use, for $k, l \geq 0$ we introduce a pair of quadratic characters $\theta_{k,l}$ and $\theta'_{k,l}$ of T_l by

$$(4-1) \quad \theta_{k,l} = \begin{cases} 1_k \otimes \theta_{l-k} & \text{if } k \leq l, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$(4-2) \quad \theta'_{k,l} = \theta_{k,l} \theta_l = \begin{cases} \theta_k \otimes 1_{l-k} & \text{if } k \leq l, \\ \theta_l & \text{otherwise,} \end{cases}$$

where 1_k and 1_{l-k} are the trivial characters of T_k and T_{l-k} , respectively.

For two groups $G_n \subset G_m$ in the same Witt tower, we can embed G_n into the Levi subgroup $G_n \times T_{m-n}$ of a parabolic subgroup of G_m . Let λ be a cuspidal representation of G_n , and denote by $R(G_m)_\lambda$ the subset of $R(G_m)$ whose elements are spanned by

$$\mathrm{Irr}(G_m)_\lambda := \mathrm{Irr}(R_{G_n \times T_{m-n}}^{G_m}(\lambda \otimes 1)).$$

Similarly introduce the subset $R(G_m)_{\lambda,\theta}$ of $R(G_m)$ whose elements are spanned by

$$\mathrm{Irr}(G_m)_{\lambda,\theta} := \mathrm{Irr}(R_{G_n \times T_{m-n}}^{G_m}(\lambda \otimes \theta_{m-n})).$$

Note that in this context the functor R above stands for the usual parabolic induction, which coincides with the generalized Deligne–Lusztig induction.

The standard Levi subgroups L of G_m are of the form

$$L = \mathrm{GL}_{n_1}(q) \times \cdots \times \mathrm{GL}_{n_r}(q) \times G_l(q)$$

such that $m = n_1 + \cdots + n_r + l$. A representation ρ of L is a cuspidal, unipotent

or θ -representation if it is of the form $\rho = \rho_1 \otimes \cdots \otimes \rho_r \otimes \sigma$, where ρ_i and σ are cuspidal, unipotent or θ -representations of $\mathrm{GL}_{n_i}(q)$ and $G_l(q)$, respectively.

For a finite (reductive) group G , let $(R^G, \overline{\mathcal{Q}}_\ell[G])$ be the regular representation of $G \times G$ on the space of $\overline{\mathcal{Q}}_\ell$ -valued functions f on G defined by

$$(R^G(g_1, g_2)f)(g) = f(g_1^{-1}gg_2), \quad \text{for all } g_1, g, g_2 \in G.$$

Then R^G has the well-known Peter–Weyl decomposition

$$R^G = \sum_{\pi \in \mathrm{Irr}(G)} \pi \otimes \pi^\vee,$$

where $\pi^\vee$ is the contragredient of π . For a quadratic character χ of G , let χR^G be the representation of $G \times G$ defined by

$$\chi R^G = \sum_{\pi \in \mathrm{Irr}(G)} \chi \pi \otimes \pi^\vee = \sum_{\pi \in \mathrm{Irr}(G)} \pi \otimes \chi \pi^\vee.$$

The following result given in [Aubert et al. 1996, Théorème 3.7] shows that the theta correspondence is compatible with the Harish-Chandra decomposition and that it preserves cuspidal unipotent representations for the dual pairs $(U_n(q), U_{n'}(q))$ and $(\mathrm{Sp}_{2n}(q), O_{2n'}^\epsilon(q))$.

Theorem 4.1. *Assume $(\mathcal{T}, \mathcal{T}') = (U^\epsilon, U^{\epsilon'})$, $(\mathrm{Sp}, O_{\mathrm{even}}^\epsilon)$ or $(O_{\mathrm{even}}^\epsilon, \mathrm{Sp})$, $\epsilon, \epsilon' = \pm$. Let λ be an irreducible cuspidal representation of $G_n \in \mathcal{T}$ and let $n' = n'(\lambda)$ be its first occurrence index in $\mathcal{T}'$, so that $\lambda' := \Theta_{G_{n'}}^\epsilon(\lambda)$ is an irreducible cuspidal representation of $G_{n'}' \in \mathcal{T}'$. Then for $\gamma \in \mathrm{Irr}(G_m)_\lambda$, $\Theta_{G_{m'}}^{\epsilon'}(\gamma) = 0$ whenever $m' < n'$ and $\Theta_{G_{m'}}^{\epsilon'}(\gamma) \in R(G_{m'}')_{\lambda'}$ otherwise.*

We are going to extend this theorem to the Witt tower $(\mathrm{Sp}, O_{\mathrm{odd}}^\epsilon)$. Since $O_{2n'+1}^+$ and $O_{2n'+1}^-$ are isomorphic, we may simply write $\mathrm{Irr}(O_{2n'+1}^\epsilon)$ instead of $\mathrm{Irr}(O_{2n'+1}^+)$ and $\mathrm{Irr}(O_{2n'+1}^-)$. The main result of this section is the following theorem, which is compatible with Pan's decomposition formula (Theorem 3.1).

Theorem 4.2. *Let λ be an irreducible cuspidal representation of Sp_{2n} and let $n' = n'(\lambda)$ be its first occurrence index in $O_{\mathrm{odd}}^\epsilon$, so that $\lambda' := \Theta_{O_{2n'+1}^\epsilon}^\epsilon(\lambda)$ is an irreducible cuspidal representation of $O_{2n'+1}^\epsilon$. Then the following holds.*

- (i) *For $\gamma \in \mathrm{Irr}(\mathrm{Sp}_{2m})_{\lambda, \theta}$, $\Theta_{O_{2m'+1}^\epsilon}^\epsilon(\gamma) = 0$ if $m' < n'$ and $\Theta_{O_{2m'+1}^\epsilon}^\epsilon(\gamma) \in R(O_{2m'+1}^\epsilon)_{\lambda'}$ otherwise.*
- (ii) *For $\gamma' \in \mathrm{Irr}(O_{2m'+1}^\epsilon)_{\lambda'}$, $\Theta_{\mathrm{Sp}_{2m}}^\epsilon(\gamma') = 0$ if $m < n$ and $\Theta_{\mathrm{Sp}_{2m}}^\epsilon(\gamma') \in R(\mathrm{Sp}_{2m})_{\lambda, \theta}$ otherwise.*
- (iii) *For $\gamma \in \mathrm{Irr}(\mathrm{Sp}_{2m})_\lambda$, $\Theta_{O_{2m'+1}^\epsilon}^\epsilon(\gamma) = 0$ if $m' < n'$ and $\Theta_{O_{2m'+1}^\epsilon}^\epsilon(\gamma) \in R(O_{2m'+1}^\epsilon)_{\lambda', \theta'}$ otherwise, where $\theta' := \theta'_{m-n, m'-n'}$ is defined by (4-2).*

- (iv) For $\gamma' \in \text{Irr}(O_{2m'+1}^\epsilon)_{\lambda', \theta}$, $\Theta_{\text{Sp}_{2m}}(\gamma') = 0$ if $m < n$ and $\Theta_{\text{Sp}_{2m}}(\gamma') \in R(\text{Sp}_{2m})_{\lambda, \theta'}$, where $\theta' := \theta_{m'-n', m-n}$ is defined by (4-1).

Proof. We will only prove (i) and (iii). The proofs of (ii) and (iv) are similar and will be left to the reader. The proofs are parallel to Theorem 4.1 and we shall follow the calculations in [Aubert et al. 1996] and [Mœglin et al. 1987] closely.

We first prove (i), by induction on m .

- Suppose that $m = n$ (i.e., $\gamma = \lambda$ is cuspidal) and $m' > n'$. It is known (see [Mœglin et al. 1987, Chapitre 3]) that each constituent $\gamma' \in \Theta_{O_{2m'+1}^\epsilon}(\gamma)$ is noncuspidal. Let j be the maximum integer such that

$$\gamma \in R_{O_{2(m'-j)+1} \times \text{GL}_j}^{O_{2m'+1}}(\lambda'_1 \otimes \rho')$$

with $\lambda'_1 \in \text{Irr}(O_{2(m'-j)+1})$ cuspidal and $\rho' \in \text{Irr}(\text{GL}_j)$. Since $\gamma' \in \Theta_{O_{2m'+1}^\epsilon}(\gamma)$,

$$\begin{aligned} 0 &< (\omega_{m,m'}^\epsilon, \gamma \otimes \gamma')_{\text{Sp}_{2m} \times O_{2m'+1}} \\ &\leq (\omega_{m,m'}^\epsilon, \gamma \otimes R_{O_{2(m'-j)+1} \times \text{GL}_j}^{O_{2m'+1}}(\lambda'_1 \otimes \rho'))_{\text{Sp}_{2m} \times O_{2m'+1}} \\ &= (*R_{O_{2(m'-j)+1} \times \text{GL}_j}^{O_{2m'+1}}(\omega_{m,m'}^\epsilon), \gamma \otimes \lambda'_1 \otimes \rho')_{\text{Sp}_{2m} \times O_{2(m'-j)+1} \times \text{GL}_j}. \end{aligned}$$

Here $*R$ stands for the Jacquet functor, which is adjoint to the induction functor R . We have the following decomposition (see [Mœglin et al. 1987, Chapitre 3, IV Théorème 5]):

$$\begin{aligned} &*R_{O_{2(m'-j)+1} \times \text{GL}_j}^{O_{2m'+1}}(\omega_{m,m'}^\epsilon) \\ &= \bigoplus_{i=0}^{\min(m,j)} R_{\text{Sp}_{2(m-i)} \times \text{GL}_i \times (\text{GL}_{j-i} \times \text{GL}_i) \times O_{2(m'-j)+1}}^{\text{Sp}_{2m} \times \text{GL}_j \times O_{2(m'-j)+1}}(\omega_{m-i,m'-j}^\epsilon \otimes 1_{\text{GL}_{j-i}} \otimes \chi_{\text{GL}_i} R^{\text{GL}_i}). \end{aligned}$$

Hence $(\omega_{m,m'}^\epsilon, \gamma \otimes \gamma')$ is bounded by

$$\sum_{i=0}^{\min(m,j)} \left(\omega_{m-i,m'-j}^\epsilon \otimes 1_{\text{GL}_{j-i}} \otimes \chi_{\text{GL}_i} R^{\text{GL}_i}, *R_{\text{Sp}_{2(m-i)} \times \text{GL}_i \times (\text{GL}_{j-i} \times \text{GL}_i) \times O_{2(m'-j)+1}}^{\text{Sp}_{2m} \times \text{GL}_j \times O_{2(m'-j)+1}}(\gamma \otimes \lambda'_1 \otimes \rho') \right),$$

where the scalar product in the i -th summand is taken over the group

$$\text{Sp}_{2(m-i)} \times O_{2(m'-j)+1} \times \text{GL}_{j-i} \times \text{GL}_i \times \text{GL}_i.$$

Since $\gamma = \lambda$ is cuspidal, the only nonzero term corresponds to $i = 0$, which implies

$$(\omega_{m,m'-j}^\epsilon \otimes 1_{\text{GL}_j}, \gamma \otimes \lambda'_1 \otimes \rho') > 0.$$

It follows that $\rho' = 1_{\text{GL}_j}$ and $\lambda'_1 \in \Theta_{O_{2(m'-j)+1}^\epsilon}(\gamma)$. Because λ'_1 is cuspidal, we must have $m' - j = n'$ and $\lambda'_1 = \lambda'$, i.e., $\gamma' \in \text{Irr}(O_{2m'+1})_{\lambda'}$.

- Suppose that $m > n$. If $\gamma \in \text{Irr}(\text{Sp}_{2m})_{\lambda, \theta}$, then there exists $\gamma_1 \in \text{Irr}(\text{Sp}_{2(m-1)})_{\lambda, \theta}$ such that $\gamma \in R_{\text{Sp}_{2(m-1)} \times \text{GL}_1}^{\text{Sp}_{2m}}(\gamma_1 \otimes \chi_{\text{GL}_1})$. For $\gamma' \in \Theta_{O_{2m'+1}}^\epsilon(\gamma)$ we have

$$\begin{aligned} 0 &< (\omega_{m,m'}^\epsilon, \gamma \otimes \gamma')_{\text{Sp}_{2m} \times O_{2m'+1}} \\ &\leq (\omega_{m,m'}^\epsilon, R_{\text{Sp}_{2(m-1)} \times \text{GL}_1}^{\text{Sp}_{2m}}(\gamma_1 \otimes \chi_{\text{GL}_1}) \otimes \gamma')_{\text{Sp}_{2m} \times O_{2m'+1}} \\ &= (*R_{\text{Sp}_{2(m-1)} \times \text{GL}_1}^{\text{Sp}_{2m}}(\omega_{m,m'}^\epsilon), \gamma_1 \otimes \chi_{\text{GL}_1} \otimes \gamma')_{\text{Sp}_{2(m-1)} \times \text{GL}_1 \times O_{2m'+1}} \end{aligned}$$

We have the decomposition

$$\begin{aligned} &*R_{\text{Sp}_{2(m-1)} \times \text{GL}_1}^{\text{Sp}_{2m}}(\omega_{m,m'}^\epsilon) \\ &= \omega_{m-1,m'}^\epsilon \otimes \chi_{\text{GL}_1} + R_{\text{Sp}_{2(m-1)} \times O_{2m'-1} \times \text{GL}_1 \times \text{GL}_1}^{\text{Sp}_{2(m-1)} \times O_{2m'+1} \times \text{GL}_1}(\omega_{m-1,m'-1}^\epsilon \otimes \chi_{\text{GL}_1} R^{\text{GL}_1}). \end{aligned}$$

Hence $(\omega_{m,m'}^\epsilon, \gamma \otimes \gamma')$ is bounded by

$$\begin{aligned} &(\omega_{m-1,m'}^\epsilon \otimes \chi_{\text{GL}_1}, \gamma_1 \otimes \gamma' \otimes \chi_{\text{GL}_1})_{\text{Sp}_{2(m-1)} \times O_{2m'+1} \times \text{GL}_1} \\ &+ (\omega_{m-1,m'-1}^\epsilon \otimes \chi_{\text{GL}_1} R^{\text{GL}_1}, \gamma_1 \otimes *R_{O_{2m'-1} \times \text{GL}_1}^{O_{2m'+1}}(\gamma') \otimes \chi_{\text{GL}_1})_{\text{Sp}_{2(m-1)} \times O_{2m'-1} \times \text{GL}_1 \times \text{GL}_1}. \end{aligned}$$

If the first term is nonzero, then $\gamma' \in \Theta_{O_{2m'+1}}^\epsilon(\gamma_1)$. By induction, $\gamma' \in R(O_{2m'+1})_{\lambda'}$. If the second term is nonzero, then there exist $\gamma'_1 \in \text{Irr}(O_{2m'-1})$ and $\rho' \in \text{Irr}(\text{GL}_1)$ such that $\gamma'_1 \otimes \rho' \in *R_{O_{2m'-1} \times \text{GL}_1}^{O_{2m'+1}}(\gamma')$ and

$$(\omega_{m-1,m'-1}^\epsilon \otimes \chi_{\text{GL}_1} R^{\text{GL}_1}, \gamma_1 \otimes \gamma'_1 \otimes \rho' \otimes \chi_{\text{GL}_1})_{\text{Sp}_{2(m-1)} \times O_{2m'-1} \times \text{GL}_1 \times \text{GL}_1} \neq 0.$$

It follows that $\rho' = 1$ and $\gamma'_1 \in \Theta_{O_{2m'-1}}^\epsilon(\gamma_1)$. By induction, $\gamma'_1 \in \text{Irr}(O_{2m'-1})_{\lambda'}$ and therefore $\gamma' \in \text{Irr}(O_{2m'+1})_{\lambda'}$ by Frobenius reciprocity. This proves (i).

Next we prove (iii), again by induction on m . Assume that $\gamma \in \text{Irr}(\text{Sp}_{2m})_\lambda$ and $\gamma' \in \Theta_{O_{2m'+1}}^\epsilon(\gamma)$. The case $m = n$ is covered by (i), so we may assume that $m > n$. Then there exists $\gamma_1 \in \text{Irr}(\text{Sp}_{2(m-1)})_\lambda$ such that $\gamma \in R_{\text{Sp}_{2(m-1)} \times \text{GL}_1}^{\text{Sp}_{2m}}(\gamma_1 \otimes 1)$. Since $\gamma' \in \Theta_{O_{2m'+1}}^\epsilon(\gamma)$, we have

$$\begin{aligned} 0 &< (\omega_{m,m'}^\epsilon, \gamma \otimes \gamma')_{\text{Sp}_{2m} \times O_{2m'+1}} \leq (\omega_{m,m'}^\epsilon, R_{\text{Sp}_{2(m-1)} \times \text{GL}_1}^{\text{Sp}_{2m}}(\gamma_1 \otimes 1) \otimes \gamma')_{\text{Sp}_{2m} \times O_{2m'+1}} \\ &= (*R_{\text{Sp}_{2(m-1)} \times \text{GL}_1}^{\text{Sp}_{2m}}(\omega_{m,m'}^\epsilon), \gamma_1 \otimes 1 \otimes \gamma')_{\text{Sp}_{2(m-1)} \times \text{GL}_1 \times O_{2m'+1}}. \end{aligned}$$

Similar to the proof of (i), $(\omega_{m,m'}^\epsilon, \gamma \otimes \gamma')$ is bounded by

$$\begin{aligned} &(\omega_{m-1,m'}^\epsilon \otimes \chi_{\text{GL}_1}, \gamma_1 \otimes \gamma' \otimes 1)_{\text{Sp}_{2(m-1)} \times O_{2m'+1} \times \text{GL}_1} \\ &+ (\omega_{m-1,m'-1}^\epsilon \otimes \chi_{\text{GL}_1} R^{\text{GL}_1}, \gamma_1 \otimes *R_{O_{2m'-1} \times \text{GL}_1}^{O_{2m'+1}}(\gamma') \otimes 1)_{\text{Sp}_{2(m-1)} \times O_{2m'-1} \times \text{GL}_1 \times \text{GL}_1}. \end{aligned}$$

The first term is zero. If the second term is nonzero, then there exists $\gamma'_1 \in \text{Irr}(O_{2m'-1})$

and $\rho' \in \text{Irr}(\text{GL}_1)$ such that $\gamma'_1 \otimes \rho' \in {}^*R_{O_{2m'-1} \times \text{GL}_1}^{O_{2m'+1}}(\gamma')$ and

$$(\omega_{m-1, m'-1}^\epsilon \otimes \chi_{\text{GL}_1} R^{\text{GL}_1}, \gamma_1 \otimes \gamma'_1 \otimes \rho' \otimes 1)_{\text{Sp}_{2(m-1)} \times O_{2m'-1} \times \text{GL}_1 \times \text{GL}_1} \neq 0.$$

It follows that $\rho' = \chi_{\text{GL}_1}$ and $\gamma'_1 \in \Theta_{O_{2m'-1}}^\epsilon(\gamma_1)$. By induction,

$$\gamma'_1 \in R_{O_{2n'+1} \times T_{m'-n'-1}}^{O_{2m'-1}}(\lambda' \otimes \theta'_{m-n-1, m'-n'-1}),$$

hence by Frobenius reciprocity

$$\gamma' \in R_{O_{2n'+1} \times T_{m'-n'}}^{O_{2m'+1}}(\lambda' \otimes \theta'_{m-n, m'-n'}).$$

This finishes the proof of (iii). $\square$

In particular, if we consider the Harish-Chandra series of cuspidal unipotent representations, then combining Theorem 3.3 we obtain the following immediate consequence.

Corollary 4.3. *Let $\lambda := \lambda_k$ be the unique cuspidal unipotent representation of Sp_{2n} , $n = k(k+1)$, and $\lambda' := \lambda_{k, \chi}^{'+}$ be its theta lifting to O_{2n+1}^ϵ . Then the following holds.*

(i) *For $\gamma \in \text{Irr}(\text{Sp}_{2m})_\lambda$, $\Theta_{O_{2m'+1}}^\epsilon(\gamma) = 0$ if $m' < m$ and*

$$\text{Irr}(\Theta_{O_{2m'+1}}^\epsilon(\gamma)) \subset \text{Irr}(R_{O_{2n+1} \times T_{m-n} \times T_{m'-m}}^{O_{2m'+1}}(\lambda' \otimes \theta_{m-n} \otimes 1))$$

otherwise.

(ii) *For $\gamma' \in \text{Irr}(O_{2m'+1}^\epsilon)_{\lambda', \theta}$, $\Theta_{\text{Sp}_{2m}}(\gamma') = 0$ if $m < m'$ and*

$$\text{Irr}(\Theta_{\text{Sp}_{2m}}(\gamma')) \subset \text{Irr}(R_{\text{Sp}_{2n} \times T_{m'-n} \times T_{m-m'}}^{\text{Sp}_{2m}}(\lambda \otimes 1 \otimes \theta_{m-m'}))$$

otherwise.

We end this paper by some remarks on the open problem of finding the explicit theta correspondence of Harish-Chandra series, which can be reformulated as a correspondence of Weyl group representations. Explicit descriptions have been proved for unitary dual pairs ([Aubert et al. 1996, Théorème 3.10]) and symplectic-even orthogonal dual pairs ([Aubert et al. 1996, Conjecture 3.11], proved recently in [Pan 2019a]). These results are also developed recently in [Epequin Chavez 2019] in order to pick up extremal components from the big theta lifting. We expect that a similar conjecture should hold for symplectic-odd orthogonal case as well according to the results in this section.

Let us give a few details. From Mackey theory we know that irreducible constituents in the Harish-Chandra series of a cuspidal unipotent representation are parametrized by irreducible representations of a Weyl group. This observation leads to a connection with the Springer correspondence [Aubert et al. 2016; Epequin Chavez 2019]. More precisely, let λ_k be the unique cuspidal unipotent

representation of a classical group G_{n_k} , where n_k is defined in Theorem 3.5. Namely, we have the table

n_k	type of G	sign of G
$\lfloor k(k+1)/4 \rfloor$	unitary	$\text{sgn}(-1)^{k(k+1)/2}$
$k(k+1)$	symplectic	NA
$k(k+1)$	odd orthogonal	$\pm$
k^2	even orthogonal	$\text{sgn}(-1)^k$

For $G_{n_k} \subset G_m$ in the same Witt tower, the set $\text{Irr}(G_m)_{\lambda_k}$ is parametrized by irreducible representations of the group

$$W_{G_m}(\lambda_k) = \{x \in N_{G_m}(L)/L \mid \lambda_k^x \cong \lambda_k\},$$

where $L = G_{n_k} \times T_{m-n_k}$. The uniqueness of λ_k implies that the defining condition of $W_{G_m}(\lambda_k)$ is trivial, hence

$$W_{G_m}(\lambda_k) = N_{G_m}(L)/L \cong W_{m-n_k}$$

which is a Weyl group of type B_{m-n_k} .

By [Adams and Moy 1993], the theta correspondence takes λ_k to $\lambda'_{k'}$, where $k' = k$ for the symplectic-even orthogonal case and $k' = k \pm 1$ for the unitary case. Hence in these cases the theta correspondence between Harish-Chandra series of cuspidal unipotent representations can be characterized as a correspondence between representations of a pair of Weyl groups $(W_{m-n_k}, W_{m'-n_{k'}})$. As we mentioned above, explicit correspondence of representations of such a pair $(W_l, W_{l'})$ has been proved for unitary dual pairs and symplectic-even orthogonal dual pairs in [Aubert et al. 1996] and [Pan 2019a], respectively. The set $\text{Irr}(W_l)$ has a well-known parametrization by bipartitions of l (see, e.g., [Geck and Pfeiffer 2000, Theorem 5.5.6]). J. Epequin Chavez [2019] further develops the above correspondence using bipartitions. Among other applications, an observation we make from the results in [Epequin Chavez 2019] is that in contrast to the p -adic case [Sun and Zhu 2015], the conservation relations do not hold in general for noncuspidal representations of finite dual pairs. It should be very interesting to understand the conservation type relations for finite dual pairs beyond the cuspidal case.

For the symplectic-odd orthogonal case, recall from Theorem 3.7 that λ_k is the unique cuspidal unipotent representation of Sp_{2n_k} , and $\lambda_{k,\chi}^{'+}$ is the unique cuspidal θ -representation of O_{2n_k+1} with trivial central character, $n_k = k(k+1)$. In a similar manner as in the unipotent case, the Harish-Chandra series $\text{Irr}(\text{Sp}_{2m})_{\lambda_k, \theta}$ and $\text{Irr}(O_{2m'+1})_{\lambda_{k,\chi}^{'+}}$ are in bijection with $\text{Irr}(W_{m-n_k})$ and $\text{Irr}(W_{m'-n_k})$, respectively. By Theorem 4.2 (i) and (ii), we expect that a conjecture analogous to [Aubert et al. 1996, Conjecture 3.11] should hold for the theta correspondence between $\text{Irr}(\text{Sp}_{2m})_{\lambda_k, \theta}$ and $\text{Irr}(O_{2m'+1})_{\lambda_{k,\chi}^{'+}}$.

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DONGWEN LIU
SCHOOL OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU
CHINA
malu@zju.edu.cn

ZHICHENG WANG
SCHOOL OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU
CHINA
11735009@zju.edu.cn

ON THE CONFIGURATIONS OF CENTERS OF PLANAR HAMILTONIAN KOLMOGOROV CUBIC POLYNOMIAL DIFFERENTIAL SYSTEMS

JAUME LLIBRE AND DONGMEI XIAO

We study the kind of centers that Hamiltonian Kolmogorov cubic polynomial differential systems can exhibit. Moreover, we analyze the possible configurations of these centers with respect to the invariant coordinate axes, and obtain that the real algebraic curve $xy(a + bx + cy + dx^2 + exy + fy^2) = h$ has at most four families of level ovals in $\mathbb{R}^2$ for all real parameters a, b, c, d, e, f and h .

1. Introduction and statement of the main results

An equilibrium p of a differential system in $\mathbb{R}^2$ is a *center* if there is a neighborhood U of p such that p is the unique equilibrium in U and $U \setminus \{p\}$ is filled by periodic orbits surrounding p . The equilibrium p is a *focus* if there is a neighborhood U of p such that all the orbits in $U \setminus \{p\}$ spiral tending to p either in forward, or in backward time. These notions of center and focus go back to Poincaré [1881] and Dulac [1908].

In the qualitative theory of planar polynomial differential systems, the problem of distinguishing between a focus or a center (known simply as the *center-focus problem*), and the problem of knowing the possible configurations of centers are two very important topics, which are related to the Hilbert's 16th problem; see [Hilbert 1900; Ilyashenko 2002; Li 2003].

For the quadratic polynomial differential systems, the center-focus problem and the possible configurations of their centers were solved by Bautin [1952], Kapteyn [1911; 1912], Schlomiuk [1993], Vulpe [1983] and Żołądek [1994c]. However, these two problems are unsolved for cubic polynomial differential systems. There are many works on the centers for some different subclasses of cubic differential systems. For example, the centers of the cubic polynomial differential systems without quadratic terms have been determined by Malkin [1964], Vulpe and Sibirskii [1988],

Xiao is the corresponding author.

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and Żołądek [1994b]. The classification of reversible cubic polynomial differential systems with a center has been done by Żołądek [1994a], and Buzzi et al. [2009]. The Hamiltonian linear type centers and the nilpotent ones of cubic polynomial differential systems without quadratic terms have been classified by Colak et al. [2014a; 2014b; 2015; 2017].

In this paper we study the centers and their configurations for the Hamiltonian cubic polynomial differential systems having two invariant straight lines which intersect. Since with an affine transformation these two invariant straight lines go through the axes of coordinates, these systems become Kolmogorov systems [Kolmogorov 1936; Llibre and Xiao 2017] which allows us to study dynamics of species in competition [Hofbauer and Sigmund 1988; Smale 1976]. So it is a very interesting topic to understand their global dynamics from the view of mathematician and biologist.

Hamiltonian systems come originally from physics, but also play an important role in dynamical system theory [Arnold 1978]. Hamiltonian systems in the plane are the easiest differential systems having a first integral. The Kolmogorov polynomial differential systems [1936] in the plane generalize the Lotka–Volterra polynomial differential systems [Lotka 1925; Volterra 1926] of degree two to higher degree. In this paper we focus our attention on subclasses of cubic polynomial differential systems which are Hamiltonian and Kolmogorov systems, and we shall read them as a Hamiltonian–Kolmogorov system.

Let (x, y) be the coordinates of $\mathbb{R}^2$. We consider Kolmogorov systems of the form

$$(1) \quad \begin{aligned} \dot{x} &= x P_1(x, y), \\ \dot{y} &= y P_2(x, y), \end{aligned}$$

in $\mathbb{R}^2$, where the dot throughout this paper means the derivative with respect to the independent variable t , and the $P_i(x, y)$ for $i = 1, 2$ are polynomials in the variables x and y such that the maximum of their degrees is two. So system (1) is a cubic polynomial differential system. Next, we characterize the Kolmogorov systems (1) which are Hamiltonian.

Theorem 1. *Assume that the maximum degree of the polynomials $P_i(x, y)$ is two for $i = 1, 2$. Then system (1) is a Hamiltonian system if and only if its Hamiltonian is of the form*

$$(2) \quad H(x, y) = xy(a + bx + cy + dx^2 + exy + fy^2),$$

where a, b, c, d, e and f are real parameters.

Theorem 1 is proved in Section 2. Therefore the Hamiltonian–Kolmogorov polynomial systems of degree three are

$$(3) \quad \begin{aligned} \dot{x} &= -x(a + bx + 2cy + dx^2 + 2exy + 3fy^2), \\ \dot{y} &= y(a + 2bx + cy + 3dx^2 + 2exy + fy^2), \end{aligned}$$

where a, b, c, d, e and f are real parameters. For brevity we denote these systems as *HK-systems* in what follows.

A polynomial differential system in the plane having a center at the origin of coordinates, after a linear change of variables and a rescaling of the time variable (if necessary), can be written in one of the following three forms:

$$(4) \quad \begin{aligned} \dot{x} &= -y + X_2(x, y), \\ \dot{y} &= x + Y_2(x, y), \end{aligned}$$

called a *linear type center*;

$$(5) \quad \begin{aligned} \dot{x} &= y + X_2(x, y), \\ \dot{y} &= Y_2(x, y), \end{aligned}$$

called a *nilpotent center*;

$$(6) \quad \begin{aligned} \dot{x} &= X_2(x, y), \\ \dot{y} &= Y_2(x, y), \end{aligned}$$

called a *degenerate center*, where $X_2(x, y)$ and $Y_2(x, y)$ are polynomials starting at least with terms of second order. For additional information about these three classes of centers, see, for instance, [Llibre 2016].

The objective of this paper is to study the centers and their configurations of the HK-systems (3). In other words, we study the existence and configurations of level ovals for planar real algebraic curves $H(x, y) = h$, $h \in \mathbb{R}$.

Our main result is the following.

Theorem 2. *Consider the HK-systems (3) defined by the Hamiltonian (2). Then a system (3) has at most four centers and the following statements hold.*

- (i) *If a HK-system (3) has nonisolated equilibria, i.e., there is an infinite number of equilibria, then this system has at most one center, which is of linear type. Hence there exist values of a, b, c, d, e and f for which $H(x, y) = h$ has no level ovals for all real h , and there exist other values of a, b, c, d, e and f for which $H(x, y) = h$ has only one family of level ovals for all real h .*
- (ii) *A HK-system (3) has neither degenerate centers nor two nilpotent centers for any real a, b, c, d, e and f .*
- (iii) *If a HK-system (3) has a unique center, then this center can be of linear type or nilpotent. That is, there is only a family of level ovals of $H(x, y) = h$ for all real h ; see Figure 1.*
- (iv) *If a HK-system (3) has two centers, then these two centers are either two linear type centers, or one linear type center and one nilpotent center. And there are HK-systems with two centers realizing all possible configurations with respect to the*

coordinate axes; i.e., two centers in the same quadrant (see Figure 2), two centers in different opposite quadrants (see Figure 3), and two centers in different consecutive quadrants (see Figure 4). Hence, the level ovals of $H(x, y) = h$ have two families for all real h .

(v) If a HK-system (3) has three centers, then they are of linear type and there are HK-systems with three centers realizing all possible configurations with respect to the coordinate axes; i.e., three centers in the same quadrant (see Figure 5), two centers in one quadrant and the other in a consecutive quadrant (see Figure 6), two centers in one quadrant and the other in the opposite quadrant (see Figure 7), and three centers in three different quadrants (see Figure 8). Moreover in this case, the level ovals of $H(x, y) = h$ have three families for all real h .

(vi) If a HK-system (3) has four centers, then they are of linear type. The level ovals of the corresponding Hamiltonian function, $H(x, y) = h$, have exactly four families for all real numbers h . Moreover, there are HK-systems with four centers in the four different quadrants (see Figure 9), and with four centers with two in the same quadrant, and each one of the other two centers is located in a different adjacent quadrant (see Figure 10).

Theorem 2 is proved in Section 3. As a direct application of Theorem 2, we have:

Corollary 3. *The real algebraic curve*

$$xy(a + bx + cy + dx^2 + exy + fy^2) = h$$

has at most four families of level ovals in $\mathbb{R}^2$ for all real parameters a, b, c, d, e, f and h .

2. Preliminaries

In this section we first give the proof of Theorem 1, then discuss the equilibria of the systems (3) and recall some definitions of equilibria and related results.

Proof of Theorem 1. Consider an arbitrary polynomial in the variables x and y of degree four,

$$H = \sum_{i+j=1}^4 a_{ij} x^i y^j.$$

We let H define a Hamiltonian system of the form (1). Then,

$$(7) \quad \dot{x} = -H_y, \quad \dot{y} = H_x.$$

In order for H_y to appear x as a common factor we take $a_{01} = a_{02} = a_{03} = a_{04} = 0$, and y appears as a common factor in H_x if and only if $a_{10} = a_{20} = a_{30} = a_{40} = 0$.

Therefore the Hamiltonian system (7) becomes a Kolmogorov system (1) if and only if

$$H = xy(a_{11} + a_{21}x + a_{12}y + a_{31}x^2 + a_{22}xy + a_{13}y^2).$$

Renaming the coefficients of H , the theorem follows. $\square$

We now discuss the equilibria of a system (3) in $\mathbb{R}^2$. It is clear that the number of equilibria of system (3) is finite in $\mathbb{R}^2$ if and only if this polynomial vector field

$$(p_1(x, y), p_2(x, y)) = (-x(a + bx + 2cy + dx^2 + 2exy + 3fy^2), \\ y(a + 2bx + cy + 3dx^2 + 2exy + fy^2))$$

does not have a common nonconstant component by Bézout's theorem (see for instance [Fulton 1969]). Hence, if the two polynomials $p_1(x, y)$ and $p_2(x, y)$ have a nonconstant common component (or *common divisor*), then system (3) has nonisolated equilibria in $\mathbb{R}^2$. The following lemma gives the necessary and sufficient conditions for the existence of a nonconstant common divisor of $p_1(x, y)$ and $p_2(x, y)$.

Lemma 4. *Polynomials $p_1(x, y)$ and $p_2(x, y)$ have a nonconstant common divisor if and only if one of the conditions in Table 1 holds, where conditions C9 and C12 are equivalent.*

Proof. It is clear that polynomials $p_1(x, y)$ and $p_2(x, y)$ do not have a common divisor which is a polynomial of (x, y) with degree three. Hence, we only look for the common divisor which is a polynomial of (x, y) with degree two or one.

By straightforward calculation, polynomials $p_1(x, y)$ and $p_2(x, y)$ have a common divisor with degree two if and only if the common divisor has one of the four forms x^2 , y^2 , xy and $a + 2exy$ for any real parameters a, b, c, d, e and f . Hence, the conditions C1–C4 for the corresponding common divisor with degree two can be given in the first four rows in Table 1.

We now look for the common divisor of polynomials $p_1(x, y)$ and $p_2(x, y)$ with degree only one. Note that

$$p_1(x, y) = -x(a + bx + 2cy + dx^2 + 2exy + 3fy^2), \\ p_2(x, y) = y(a + 2bx + cy + 3dx^2 + 2exy + fy^2).$$

Suppose that $y - s - tx$ is a common divisor of $a + bx + 2cy + dx^2 + 2exy + 3fy^2$ and $a + 2bx + cy + 3dx^2 + 2exy + fy^2$, where s and t are undetermined real parameters. Then for any real x ,

$$(8) \quad \begin{aligned} a + bx + 2c(s + tx) + dx^2 + 2ex(s + tx) + 3f(s + tx)^2 &\equiv 0, \\ a + 2bx + c(s + tx) + 3dx^2 + 2ex(s + tx) + f(s + tx)^2 &\equiv 0. \end{aligned}$$

<i>list</i>	<i>conditions</i>	<i>nonconstant common divisors</i>
C1	$a = b = c = e = f = 0, d \neq 0$	x^2
C2	$a = b = c = d = e = 0, f \neq 0$	y^2
C3	$a = b = c = d = f = 0, e \neq 0$	xy
C4	$b = c = d = f = 0, ae \neq 0$	$a + 2exy$
C5	$a = c = f = 0, b \neq 0$ (or $de \neq 0$)	x
C6	$a = b = d = 0, c \neq 0$ (or $ef \neq 0$)	y
C7	$c = e = f = 0, bd \neq 0, a = \frac{b^2}{4d}$	$x + \frac{b}{2d}$
C8	$a = b = c = 0, ef \neq 0, d = \frac{e^2}{4f}$	$x + \frac{2f}{e}y$
C9	$bef \neq 0, a = \frac{b^2f}{e^2}, c = \frac{2bf}{e}, d = \frac{e^2}{4f}$	$x + \frac{2f}{e}y + \frac{2bf}{e^2}$
C10	$b = d = e = 0, ac \neq 0, f = \frac{c^2}{4a}$	$y + \frac{2a}{c}$
C11	$a = b = c = 0, de \neq 0, f = \frac{e^2}{4d}$	$y + \frac{2d}{e}x$
C12	$ace \neq 0, b = \frac{2ae}{c}, d = \frac{ae^2}{c^2}, f = \frac{c^2}{4a}$	$y + \frac{2ae}{c^2}x + \frac{2a}{c}$

Table 1. The necessary and sufficient conditions for nonconstant common divisor.

Thus,

$$\begin{aligned}
 (9) \quad & a + 2cs + 3fs^2 = 0, & a + cs + fs^2 &= 0, \\
 & b + 2ct + 2es + 6fst = 0, & 2b + ct + 2es + 2fst &= 0, \\
 & d + 2et + 3ft^2 = 0, & 3d + 2et + ft^2 &= 0.
 \end{aligned}$$

We divide four cases of (s, t) to study the solutions of system (9).

Case 1: $s = t = 0$. From (9), $a = b = d = 0$. Hence, $a + bx + 2cy + dx^2 + 2exy + 3fy^2$ and $a + 2bx + cy + 3dx^2 + 2exy + fy^2$ have a common divisor y if $a = b = d = 0$ and $c^2 + e^2 + f^2 \neq 0$. However, if we require this common divisor to be the only polynomial with degree one, then $a = b = d = 0$ and $c \neq 0$, or $a = b = d = 0$ and $ef \neq 0$, which is the condition C6 in Table 1.

Case 2: $s \neq 0$ and $t = 0$. The system (9) becomes

$$\begin{aligned}
 & a + 2cs + 3fs^2 = 0, \\
 & a + cs + fs^2 = 0, \\
 & b = d = e = 0.
 \end{aligned}$$

Therefore, when $b = d = e = 0, ac \neq 0$ and $f = c^2/(4a)$, we have that $y + 2a/c$ is the common divisor of

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 \quad \text{and} \quad a + 2bx + cy + 3dx^2 + 2exy + fy^2.$$

In fact, now,

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 = \frac{c^2}{4a} \left(y + \frac{2a}{c} \right) \left(3y + \frac{2a}{c} \right)$$

and

$$a + 2bx + cy + 3dx^2 + 2exy + fy^2 = \frac{c^2}{4a} \left(y + \frac{2a}{c} \right)^2.$$

Case 3: $s = 0$ and $t \neq 0$. The system (9) becomes

$$\begin{aligned} a &= b = c = 0, \\ d + 2et + 3ft^2 &= 0, \\ 3d + 2et + ft^2 &= 0. \end{aligned}$$

Therefore, when $a = b = c = 0$, $de \neq 0$ and $f = e^2/(4d)$, we have that $y + 2d/ex$ is the common divisor of

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 \quad \text{and} \quad a + 2bx + cy + 3dx^2 + 2exy + fy^2.$$

In fact, now we have

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 = \frac{3e^2}{4d} \left(y + \frac{2d}{e}x \right) \left(y + \frac{2d}{3e}x \right)$$

and

$$a + 2bx + cy + 3dx^2 + 2exy + fy^2 = \frac{3e^2}{4d} \left(y + \frac{2d}{e}x \right) \left(y + \frac{6d}{e}x \right).$$

Case 4: $s \neq 0$ and $t \neq 0$. From the first two equations in (9) it follows that

$$(10) \quad s = -\frac{2a}{c} \quad \text{with } ac \neq 0.$$

From the last two equations in (9) it follows that

$$(11) \quad t = -\frac{2d}{e} \quad \text{with } de \neq 0.$$

Plugging (10) and (11) into (9), we obtain

$$b = \frac{2ae}{c}, \quad d = \frac{ae^2}{c^2}, \quad f = \frac{c^2}{4a}.$$

Hence when $ace \neq 0$, $b = 2ae/c$, $d = ae^2/c^2$ and $f = c^2/(4a)$, we have that $y + 2ae/c^2x + 2a/c$ is a common divisor of $a + bx + 2cy + dx^2 + 2exy + 3fy^2$ and $a + 2bx + cy + 3dx^2 + 2exy + fy^2$.

In fact, now we have

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 = \frac{c^2}{4a} \left(y + \frac{2ae}{c^2}x + \frac{2a}{c} \right) \left(3y + \frac{2ae}{c^2}x + \frac{2a}{c} \right),$$

and

$$a + 2bx + cy + 3dx^2 + 2exy + fy^2 = \frac{c^2}{4a} \left(y + \frac{2ae}{c^2}x + \frac{2a}{c} \right) \left(y + \frac{6ae}{c^2}x + \frac{2a}{c} \right).$$

Summarizing the above analysis, we obtain the necessary and sufficient conditions that $p_1(x, y)$ and $p_2(x, y)$ have common divisors y , $y - s$, $y - tx$ and $y - s - tx$, respectively. This leads to the conditions C6 and C10–C12 in Table 1.

Using similar arguments, one can discuss the common divisor of

$$a + bx + 2cy + dx^2 + 2exy + 3fy^2 \quad \text{and} \quad a + 2bx + cy + 3dx^2 + 2exy + fy^2$$

with form $x - s - ty$. Hence, we obtain the conditions shown in the condition C5 and C7–C9 in Table 1. $\square$

Remark 5. The eleven conditions from C1 to C11 in Table 1 give the necessary and sufficient conditions so that the polynomial $H(x, y)$ can be factorized in real number field. This gives all possible real branch curves of the real algebraic curve $H(x, y) = 0$.

From Bézout's theorem and a straightforward calculation, we have:

Theorem 6. *The following statements hold on the equilibria of HK-system (3).*

(I) *If polynomials $p_1(x, y)$ and $p_2(x, y)$ have a nonconstant common divisor $g(x, y)$, then a HK-system (3) has nonisolated equilibria, and each point on the curve $g(x, y) = 0$ is an equilibrium of the HK-system (3). Hence a HK-system (3) has infinitely many equilibria.*

(II) *If polynomials $p_1(x, y)$ and $p_2(x, y)$ do not have nonconstant common divisors, then a HK-system (3) has at most nine finite equilibria with five of them located at the coordinate axes and the other four are in the interior of the quadrants in $\mathbb{R}^2$.*

In the two-dimensional phase plane $\mathbb{R}^2$, we say that an equilibrium is *nonelementary* if both of the eigenvalues of the linear part of the vector field at that point are zero, and *elementary* otherwise. A nonelementary equilibrium is called *degenerate* if the linear part is identically zero, otherwise it is called *nilpotent*. An elementary equilibrium is *hyperbolic* if both of the eigenvalues of the linear part at the equilibrium have nonzero real part, and it is called *semihyperbolic* if one and only one of the two eigenvalues is zero. The unique elementary equilibrium which is neither hyperbolic nor semihyperbolic is the one with purely imaginary eigenvalues, which only can be the focus or center; for more details, see [Dumortier et al. 2006; Zhang et al. 1992].

The local phase portraits at the nilpotent equilibrium points were classified by Andreev [1958]; see also Theorem 3.5 of [Dumortier et al. 2006]. We summarize the conditions which characterize whether a nilpotent equilibrium is either a focus or a center as follows.

Lemma 7. *Let $(0, 0)$ be an isolated equilibrium point of the analytic differential system*

$$(12) \quad \dot{x} = y + A(x, y), \quad \dot{y} = B(x, y),$$

defined in a neighborhood of the point $(0, 0)$, and assume that $A(x, y)$ and $B(x, y)$ start at least with terms of degree two in x and y . Let $y = f(x)$ be the solution of the equation $y + A(x, y) = 0$ in a neighborhood of the point $(0, 0)$, and consider $F(x) = B(x, f(x))$ and $G(x) = (\partial A / \partial x + \partial B / \partial y)(x, f(x))$. Then the origin of system (12) is a center or a focus if and only if one of the following conditions holds.

- (i) $G(x) \equiv 0$, $F(x) = ax^m + o(x^m)$, $m > 1$ is an odd integer and $a < 0$.
- (ii) $G(x) = bx^n + o(x^n)$, $F(x) = ax^m + o(x^m)$, $m > 1$ is an odd integer, $a < 0$, and either $m < 2n + 1$ or both $m = 2n + 1$ and $b^2 + 4a(n + 1) < 0$.

For convenience, prior to applying Lemma 7, we recall some characteristics of a nilpotent equilibrium. As it is known that a planar analytic differential system

$$(13) \quad \dot{u} = f(u, v), \quad \dot{v} = g(u, v),$$

has a nilpotent equilibrium at (u_0, v_0) , the Jacobian matrix M of system (13) at (u_0, v_0) must be

$$M = \begin{pmatrix} \frac{\partial f(u_0, v_0)}{\partial u} & \frac{\partial f(u_0, v_0)}{\partial v} \\ \frac{\partial g(u_0, v_0)}{\partial u} & \frac{\partial g(u_0, v_0)}{\partial v} \end{pmatrix} \triangleq \begin{pmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{pmatrix},$$

where $m_1^2 + m_2m_3 = 0$ and $m_1^2 + m_2^2 + m_3^2 \neq 0$.

By an affine transformation

$$(14) \quad \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix},$$

system (13) can be transformed into system (12), where the matrix P depends on m_1 , m_2 and m_3 , and has one of the following three forms:

- (i) If $m_1 \neq 0$, then $m_2 \neq 0$ and $m_3 \neq 0$. Hence,

$$P = \begin{pmatrix} 0 & -\frac{m_2}{m_1^2} \\ 1 & \frac{m_2}{m_1} \end{pmatrix}.$$

(ii) If $m_1 = 0$ and $m_2 \neq 0$, then $m_3 = 0$. Hence,

$$P = \begin{pmatrix} \frac{1}{m_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

(iii) If $m_1 = 0$ and $m_3 \neq 0$, then $m_2 = 0$. Hence,

$$P = \begin{pmatrix} 0 & -\frac{1}{m_3} \\ -1 & 0 \end{pmatrix}.$$

Theorem 8. *Suppose that system (13) is a Hamiltonian system with a nilpotent equilibrium at (u_0, v_0) , then this nilpotent equilibrium is a center if and only if the case (i) in Lemma 7 holds, that is, $G(x) \equiv 0$ and $F(x) = ax^m + o(x^m)$, where $m > 1$ is an odd integer and $a < 0$.*

Proof. Since system (13) is a Hamiltonian system, there exists an analytic function $H(u, v)$ such that

$$\frac{\partial H(u, v)}{\partial v} \triangleq H_v(u, v) = f(u, v), \quad \frac{\partial H(u, v)}{\partial u} \triangleq H_u(u, v) = -g(u, v).$$

Thus, system (13) can be written as

$$(15) \quad \dot{u} = H_v(u, v), \quad \dot{v} = -H_u(u, v)$$

with $H_v(u_0, v_0) = -H_u(u_0, v_0) = 0$.

Using the affine transformation (14) and its inverse transformation

$$(16) \quad \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

system (15) can be transformed to

$$(17) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = P \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^T \begin{pmatrix} \tilde{H}_x(x, y) \\ \tilde{H}_y(x, y) \end{pmatrix},$$

where $\tilde{H}(x, y) = H(u(x, y), v(x, y))$ and P^T is the transpose of matrix P .

Let

$$Q = P \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^T.$$

Depending on the Jacobian matrix M of system (13) at nilpotent equilibrium (u_0, v_0) , by direct computation we obtain three forms of Q as follows.

(i) If $m_1 \neq 0$,

$$Q = \begin{pmatrix} 0 & \frac{m_2}{m_1^2} \\ -\frac{m_2}{m_1^2} & 0 \end{pmatrix}.$$

(ii) If $m_1 = 0$ and $m_2 \neq 0$,

$$Q = \begin{pmatrix} 0 & \frac{1}{m_2} \\ -\frac{1}{m_2} & 0 \end{pmatrix}.$$

(iii) If $m_1 = 0$ and $m_3 \neq 0$,

$$Q = \begin{pmatrix} 0 & -\frac{1}{m_3} \\ \frac{1}{m_3} & 0 \end{pmatrix}.$$

This gives that system (17) is a Hamiltonian system even though P is not a symplectic matrix if nonzero $m_i \neq 1$ for some $i \in \{1, 2, 3\}$. Following the notation in Lemma 7, we have

$$G(x) = c \left(\frac{\partial^2 \tilde{H}(x, y)}{\partial x \partial y} - \frac{\partial^2 \tilde{H}(x, y)}{\partial y \partial x} \right) \equiv 0,$$

where c is one of m_2/m_1^2 , $1/m_2$ and $-1/m_3$.

Therefore, we obtain the conclusion by Lemma 7. $\square$

3. Proof of Theorem 2

We shall prove Theorem 2 statement by statement in this section.

Proof of statement (i) of Theorem 2. Since the x -axis and y -axis are invariant by the flow of a HK-system (3), either the equilibrium on the coordinate axes or the nonisolated equilibrium cannot become a center by the definition of center. If a HK-system (3) does not have a center, then the algebraic curve $H(x, y) = h$ does not have level ovals in $\mathbb{R}^2$.

From Theorem 6, a HK-system (3) has at most four isolated equilibria outside the coordinate axes. Hence, a HK-system (3) has at most four centers in the interior of quadrants of $\mathbb{R}^2$.

We now prove statement (i) of Theorem 2: *if a HK-system (3) has nonisolated equilibria, then this system has at most one center, which is of linear type.*

From Lemma 4, we know that a HK-system (3) has nonisolated equilibria if and only if one of the conditions listed in Table 1 holds. By straightforward calculation, a HK-system (3) has no isolated equilibrium outside the coordinate axes for conditions C1–C4, C7, C8, C10 and C11 in Table 1. Hence, under these conditions a HK-system (3) has no center equilibrium, which implies that $H(x, y) = h$ has no level ovals for all real h .

When condition C5 holds (i.e., $a = c = f = 0$, $b \neq 0$ (or $de \neq 0$)), a HK-system (3) has an isolated equilibrium outside the coordinate axes if and only if $bde \neq 0$. In the case $a = c = f = 0$, $bde \neq 0$, a HK-system (3) has a unique isolated equilibrium at $(-b/(2d), -b/(4e))$ with eigenvalues $\pm b^2/(2\sqrt{2}|d|)i$, which is a linear type center.

<i>conditions</i>	<i>center</i>	<i>level ovals of $H(x, y) = h$</i>
C1	none	$dx^3y = h$, none
C2	none	$fx^3y = h$, none
C3	none	$ex^2y^2 = h$, none
C4	none	$xy(a+exy) = h$, none
C5 $\begin{cases} b=0, de \neq 0 \\ bde \neq 0 \end{cases}$	none	$x^2y(dx+ey) = h$, none
	$(-\frac{b}{2d}, -\frac{b}{4e})$	$x^2y(b+dx+ey) = h$, a family
C6 $\begin{cases} c=0, ef \neq 0 \\ cef \neq 0 \end{cases}$	none	$xy^2(ex+fy) = h$, none
	$(-\frac{c}{4e}, -\frac{c}{2f})$	$xy^2(c+ex+fy) = h$, a family
C7	none	$xy(b+2dx)^2y/(4d) = h$, none
C8	none	$xy(ex+2fy)^2/(4f) = h$, none
C9	$(-\frac{bf}{2e^2}, -\frac{b}{4e})$	$xy(2bf+e^2x+2efy)^2/(4e^2f) = h$, a family
C10	none	$xy(2a+cy)^2/(4a) = h$, none
C11	none	$xy(2dx+ey)^2/(4d) = h$, none

Table 2. Center of a HK-system (3) with nonisolated equilibria.

So there is only one family of level ovals for $H(x, y) = x^2y(b+dx+ey) = h$, $h \in \mathbb{R}$.

Similarly, we can discuss the cases C6 and C9 (or C12). For clarity, we list them in Table 2. Thus, statement (i) is proved. $\square$

It is clear that a HK-system (3) has no center if it does not have any isolated equilibria outside the coordinate axes. In the proof of the remaining statements of Theorem 2, we always assume that a HK-system (3) has an isolated equilibrium outside the coordinate axes. Without loss of generality, it is in the positive quadrant, and consequently we can assume that it is $(x, y) = (\alpha, \beta)$ with α and β positive real numbers. Then, scaling the variables x and y if necessary, we can suppose that $\alpha = \beta = 1$. In short, in the rest of the proofs of the statements of Theorem 2 we always assume that the point $(1, 1)$ will be an equilibrium of a HK-system (3). Hence, two parameters of a, b, c, d, e and f can be determined by the other four parameters, e.g.,

$$e = -\frac{1}{4}(2a + 5b + c + 8d), \quad f = \frac{1}{2}(b - c + 2d).$$

Therefore a HK-system (3) with equilibrium $(1, 1)$ becomes

$$(18) \quad \begin{aligned} \dot{x} &= -x(a+bx+dx^2+2cy-axy-\frac{5}{2}bxy-\frac{1}{2}cxy-4dxy+\frac{3}{2}by^2-\frac{3}{2}cy^2+3dy^2), \\ \dot{y} &= y(a+2bx+3dx^2+cy-axy-\frac{5}{2}bxy-\frac{1}{2}cxy-4dxy+by^2/2-cy^2/2+dy^2). \end{aligned}$$

The Jacobi matrix of a HK-system (18) at $(1, 1)$ is

$$M = \begin{pmatrix} (2a + 3b + c + 4d)/2 & (2a - b + 3c - 4d)/2 \\ -(2a + b + c - 4d)/2 & -(2a + 3b + c + 4d)/2 \end{pmatrix},$$

and the determinant of M is denoted by Δ :

$$\Delta = ac - bc - 3ab - \frac{5b^2 - c^2}{2} - 2(4a + 3b + 3c)d.$$

Proof of statement (ii) of Theorem 2. To prove that a HK-system (3) has neither degenerate centers nor two nilpotent centers for any real a, b, c, d, e and f , we first prove that a HK-system (3) does not have a degenerate center.

Assume that a HK-system (3) has a degenerate center at $(1, 1)$ without loss of generality. Then the HK-system (3) is system (18), $\Delta = 0$ and the matrix M must be identically zero, that is, the parameters a, b, c and d must satisfy the equations

$$\begin{aligned} 2a + 3b + c + 4d &= 0, \\ 2a - b + 3c - 4d &= 0, \\ 2a + b + c - 4d &= 0. \end{aligned} \tag{19}$$

The solutions of (19) are

$$b = c = -4d, \quad a = 6d.$$

In the following we claim that the equilibrium $(1, 1)$ is not a degenerate center of system (18) if $b = c = -4d$ and $a = 6d$.

When $b = c = -4d$ and $a = 6d$, system (18) becomes

$$\begin{aligned} \dot{x} &= -dx(6 - 4x + x^2 - 8y + 2xy + 3y^2), \\ \dot{y} &= dy(6 - 8x + 3x^2 - 4y + 2xy + y^2). \end{aligned} \tag{20}$$

Of course, $d \neq 0$, otherwise this system has no isolated equilibria.

Moving the equilibrium $(1, 1)$ of system (20) to the origin of the coordinates, we obtain

$$\begin{aligned} \dot{x} &= -d(1 + x)(x^2 + 2xy + 3y^2), \\ \dot{y} &= d(1 + y)(3x^2 + 2xy + y^2). \end{aligned} \tag{21}$$

Doing blow ups of the origin of system (21), letting $x = r \cos \theta$, $y = r \sin \theta$, it can be seen that the local phase portrait of the origin is formed by two hyperbolic sectors, which implies that the origin of system (21) is not a center. So the equilibrium $(1, 1)$ of system (20) is not a center. For more information on the changes of variables called blow ups for studying the local phase portraits of the equilibrium points of a planar analytic differential system, see, for instance, Section 3.1 of [Dumortier et al. 2006].

Therefore, we obtain that a HK-system (3) does not have a degenerate center.

To complete the proof of statement (ii) of Theorem 2, it only remains to prove that there do not exist values of the parameters a, b, c, d, e and f such that a HK-system (3) has two nilpotent centers.

It is well known that if an equilibrium is a nilpotent center, then the two eigenvalues of this equilibrium are zero, but the Jacobi matrix at this equilibrium is not identically zero.

Assume that $(1, 1)$ is a nilpotent center of HK-system (3), then eigenvalues of the Jacobi matrix M at equilibrium $(1, 1)$ are

$$(22) \quad \pm \sqrt{\frac{1}{2}(5b^2 + 2cb + 12db - c^2 + 12cd + 2a(3b - c + 8d))},$$

which must be zero, but $M \neq 0$. We separate the rest of the proof into two cases.

Case I : $3b - c + 8d = 0$.

Case II: $3b - c + 8d \neq 0$.

First, we consider Case I. Here, $c = 3b + 8d$, which leads to the eigenvalues being $\pm|b + 4d|i$. Therefore $b = -4d$. Plugging the expressions of c and b into M , the matrix M becomes

$$\begin{pmatrix} a - 6d & a - 6d \\ -a + 6d & -a + 6d \end{pmatrix}.$$

Since $M \neq 0$, $a - 6d \neq 0$.

Now system (18) becomes

$$(23) \quad \begin{aligned} \dot{x} &= -ax + 4dx^2 + 8dxy - dx^3 + (a - 8d)x^2y - 3dxy^2, \\ \dot{y} &= ay - 8dxy - 4dy^2 + 3dx^2y - (a - 8d)xy^2 + dy^3. \end{aligned}$$

If $a - 12d \neq 0$, then system (23) has only two equilibria outside the coordinate axes, one is $(1, 1)$ and the other is $P = (-a/(a - 12d), -a/(a - 12d))$.

The determinant of the Jacobi matrix M_P of system (23) at equilibrium P is denoted by Δ_P :

$$\Delta_P = -\frac{16a^2(a - 6d)^2d}{(a - 12d)^3}.$$

We distinguish three subcases to discuss Δ_P :

(I.1) $ad \neq 0$. Here, $\Delta_P \neq 0$, which implies that equilibrium P is either a linear type center or a saddle depending on $\Delta_P > 0$ or $\Delta_P < 0$, respectively, i.e., P is not a nilpotent center. Hence, system (23) cannot have two nilpotent centers as $c = 3b + 8d$, $a - 12d \neq 0$ and $ad \neq 0$.

(I.2) $a = 0$ and $d \neq 0$. Here, $\Delta_P = 0$ and $M_P = 0$. This gives that P is a degenerated equilibrium. We have proved that the degenerated equilibrium of a

HK-system (3) cannot be a center; hence, system (23) cannot have two nilpotent centers as $c = 3b + 8d$, $a = 0$ and $d \neq 0$.

(I.3) $d = 0$ and $a \neq 0$. Here, $\Delta_p = 0$ and system (23) becomes

$$\begin{aligned}\dot{x} &= -ax(1 - xy), \\ \dot{y} &= ay(1 - xy),\end{aligned}$$

which has nonisolated equilibria. Statement (i) (see C4 in Table 2) has shown that this system has no center. Therefore, system (23) cannot have two nilpotent centers if $c = 3b + 8d$ and $a - 12d \neq 0$.

If $a - 12d = 0$, then system (23) has a unique equilibrium at $(1, 1)$ outside the coordinate axes. Hence, system (23) cannot have two nilpotent centers as $c = 3b + 8d$ and $a - 12d = 0$.

In summary, in Case I we prove that if a HK-system (3) has a nilpotent equilibrium at $(1, 1)$, then this system does not have another nilpotent center which is different from $(1, 1)$. Hence a HK-system (3) cannot have two nilpotent centers if $3b - c + 8d = 0$.

We now discuss whether a HK-system (3) has two nilpotent equilibria in Case II: $3b - c + 8d \neq 0$.

Since $3b - c + 8d \neq 0$ and the eigenvalues (22) must be zero in order for equilibrium $(1, 1)$ to be a nilpotent center,

$$a = \frac{c^2 - 5b^2 - 2bc - 12bd - 12cd}{2(3b - c + 8d)}.$$

Then the Jacobi matrix of system (18) at the equilibrium $(1, 1)$ is

$$\begin{pmatrix} \frac{(b+4d)(2b-c+4d)}{3b-c+8d} & -\frac{(2b-c+4d)^2}{3b-c+8d} \\ \frac{(b+4d)^2}{3b-c+8d} & -\frac{(b+4d)(2b-c+4d)}{3b-c+8d} \end{pmatrix},$$

and this equilibrium is nilpotent if and only if $(b + 4d)^2 + (2b - c + 4d)^2 \neq 0$.

We distinguish three subcases to study whether the equilibrium $(1, 1)$ is a nilpotent center: (II.1) $b + 4d = 0$ and $2b - c + 4d \neq 0$, (II.2) $b + 4d \neq 0$ and $2b - c + 4d = 0$, (II.3) $(b + 4d)(2b - c + 4d) \neq 0$.

(II.1) If $b + 4d = 0$ and $2b - c + 4d \neq 0$, then $b = -4d$. By moving the equilibrium $(1, 1)$ to the origin, system (18) becomes

$$\begin{aligned}(24) \quad \dot{x} &= (c+4d)y - dx^2 + (c+2d)xy + \frac{3}{2}(c+2d)y^2 - dx^3 - 2dx^2y + \frac{3}{2}(c+2d)xy^2, \\ \dot{y} &= 3dx^2 + 2dxy - \frac{1}{2}(c+2d)y^2 + 3dx^2y + 2dxy^2 - \frac{1}{2}(c+2d)y^3.\end{aligned}$$

We must assume that $c + 4d \neq 0$, otherwise the equilibrium $(0, 0)$ of system (24)

is not nilpotent. Rescaling the independent variable by $c + 4d$ we have system (24) in the normal form for applying Lemma 7.

$$\begin{aligned}
 \dot{x} &= y - \frac{d}{c+4d}x^2 + \frac{c+2d}{c+4d}xy + \frac{3(c+2d)}{2(c+4d)}y^2 \\
 &\quad - \frac{d}{c+4d}x^3 - \frac{2d}{c+4d}x^2y + \frac{3(c+2d)}{2(c+4d)}xy^2, \\
 \dot{y} &= \frac{3d}{c+4d}x^2 + \frac{2d}{c+4d}xy - \frac{c+2d}{2(c+4d)}y^2 \\
 &\quad + \frac{3d}{c+4d}x^2y + \frac{2d}{c+4d}xy^2 - \frac{c+2d}{2(c+4d)}y^3.
 \end{aligned}
 \tag{25}$$

Using the notation of Lemma 7,

$$\begin{aligned}
 G(x) &\equiv 0, \\
 F(x) &= \frac{3d}{c+4d}x^2 + \frac{2d^2}{(c+4d)^2}x^3 + O(x^4).
 \end{aligned}$$

Therefore, we must take $d = 0$ if we want the equilibrium $(0, 0)$ to be a nilpotent center by Theorem 8. However, when $d = 0$, system (24) becomes

$$\begin{aligned}
 \dot{x} &= \frac{1}{2}cx(y-1)(-1+3y), \\
 \dot{y} &= -\frac{1}{2}cy(y-1)^2,
 \end{aligned}
 \tag{26}$$

which has nonisolated equilibria in the line $y = 1$ and the x -axis is invariant for the flow of this system. Hence, the origin $(0, 0)$ cannot become center of system (26), which implies that equilibrium $(1, 1)$ is not a nilpotent center of system (18) as $b + 4d = 0$ and $2b - c + 4d \neq 0$.

(II.2) If $b + 4d \neq 0$ and $2b - c + 4d = 0$, then $c = 2b + 4d$.

Let $x = Y + 1$, $y = X + 1$. Then system (18) becomes

$$\begin{aligned}
 \dot{X} &= (b+4d)Y + 3dY^2 + 2dXY - \frac{1}{2}(b+2d)X^2 + 3dXY^2 \\
 &\quad - (b+2d)X^2Y - \frac{1}{2}(b+2d)X^3, \\
 \dot{Y} &= -dY^2 + (b+2d)XY + \frac{3}{2}(b+2d)X^2 - dY^3 \\
 &\quad + (b+2d)XY^2 + \frac{3}{2}(b+2d)X^2Y.
 \end{aligned}
 \tag{27}$$

Since $b + 4d \neq 0$, we have system (27) in the normal form for applying Lemma 7 by rescaling the independent variable $b + 4d$. Again using the notation of that lemma, we have $G(X) \equiv 0$ and

$$F(X) = \frac{3(b+2d)}{2(b+4d)}X^2 + \frac{(b+2d)^2}{2(b+4d)^2}X^3 + O(X^4).$$

Therefore we must take $b = -2d$ if the origin of system (27) is a nilpotent center. However, when $b = -2d$, system (27) becomes

$$(28) \quad \begin{aligned} \dot{X} &= dY(1+X)(2+3Y), \\ \dot{Y} &= -dY^2(1+Y), \end{aligned}$$

which has nonisolated equilibria which fill up the X -axis. Thus, the equilibrium $(0, 0)$ cannot be a center of system (28), which gives that the equilibrium $(1, 1)$ of system (18) is not a nilpotent center as $b + 4d \neq 0$ and $2b - c + 4d = 0$.

(II.3) If $(b + 4d)(2b - c + 4d) \neq 0$, we shall apply Theorem 8 to prove that the nilpotent equilibrium $(1, 1)$ is not a center.

To apply this theorem, we first need to translate the equilibrium $(1, 1)$ of system (18) to the origin of coordinates. For this, we make the change of variables $x = X + 1$ and $y = Y + 1$ in system (18), and then write the matrix of the linear part at the origin of this system into its real Jordan normal form with the change of variables

$$X = -\frac{(c - 2b - 4d)^2 u}{3b - c + 8d} - \frac{(2b - c + 4d)v}{b + 4d}, \quad Y = -\frac{(b + 4d)(2b - c + 4d)u}{3b - c + 8d}.$$

Finally, a HK-system (3) in the variables (u, v) can be written as

$$(29) \quad \begin{aligned} \dot{u} &= v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 + a_{21}u^2v + a_{12}uv^2, \\ \dot{v} &= b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3, \end{aligned}$$

where

$$\begin{aligned} a_{20} &= \frac{(2b - c + 4d)(b^2 + 2(b + c)d)(17b^2 - 14cb + 80db + 3c^2 + 96d^2 - 32cd)}{2(-3b + c - 8d)^2(b + 4d)}, \\ a_{11} &= \frac{2(2b - c + 4d)(2b^3 - (c + d)b^2 + 8(c - 5d)db + d(-3c^2 + 24dc - 64d^2))}{(b + 4d)^2(3b - c + 8d)}, \\ a_{02} &= \frac{3d(2b - c + 4d)(-3b + c - 8d)}{(b + 4d)^3}, \\ a_{30} &= -\frac{(-2b + c - 4d)^2(b^2 + 2(b + c)d)(17b^2 - 14cb + 80db + 3c^2 + 96d^2 - 32cd)}{2(3b - c + 8d)^3}, \\ a_{21} &= -\frac{(-2b + c - 4d)^2(5b^3 - 2(c - 5d)b^2 + 16(c - 2d)db - 2d(3c^2 - 24dc + 32d^2))}{(-3b + c - 8d)^2(b + 4d)}, \\ a_{12} &= \frac{3d(-2b + c - 4d)^2}{(b + 4d)^2}, \end{aligned}$$

$$\begin{aligned}
b_{20} &= -\frac{3(2b-c+4d)(b^2+2(b+c)d)(5b^2-4(c-6d)b+c^2+32d^2-8cd)}{2(-3b+c-8d)^2}, \\
b_{11} &= -\frac{(2b-c+4d)(b^2+2(b+c)d)(17b^2-14cb+80db+3c^2+96d^2-32cd)}{(-3b+c-8d)^2(b+4d)}, \\
b_{02} &= -\frac{(2b-c+4d)(2b^3-(c+d)b^2+8(c-5d)db+d(-3c^2+24dc-64d^2))}{(b+4d)^2(3b-c+8d)}, \\
b_{30} &= \frac{2(b+4d)(2b-c+4d)^3(b^2+2(b+c)d)(7b^2-5cb+36db+c^2+48d^2-12cd)}{(-3b+c-8d)^4}, \\
b_{21} &= \frac{3(-2b+c-4d)^2(b^2+2(b+c)d)(17b^2-14cb+80db+3c^2+96d^2-32cd)}{2(3b-c+8d)^3}, \\
b_{12} &= \frac{(-2b+c-4d)^2(5b^3-2(c-5d)b^2+16(c-2d)db-2d(3c^2-24dc+32d^2))}{(-3b+c-8d)^2(b+4d)}, \\
b_{03} &= -\frac{(-2b+c-4d)^2d}{(b+4d)^2}.
\end{aligned}$$

Using the notation introduced in the statement of Lemma 7, we have

$$\begin{aligned}
v &= f(u) = -a_{20}u^2 + (a_{11}a_{20} - a_{30})u^3 + O(u^4), \\
G(u) &\equiv 0, \\
F(u) &= b_{20}u^2 + (b_{30} - a_{20}b_{11})u^3 + O(u^4).
\end{aligned}$$

In order for the equilibrium $(0, 0)$ of system (29) to be a nilpotent center, from Theorem 8 we require

$$b_{20} = -\frac{3(2b-c+4d)(b^2+2(b+c)d)(5b^2-4(c-6d)b+c^2+32d^2-8cd)}{2(-3b+c-8d)^2} = 0.$$

Since $2b - c + 4d \neq 0$, we must have

$$(b^2 + 2(b+c)d)(5b^2 - 4(c-6d)b + c^2 + 32d^2 - 8cd) = 0.$$

Then either $b^2 + 2(b+c)d = 0$ or $5b^2 - 4(c-6d)b + c^2 + 32d^2 - 8cd = 0$.

We now divide into three subcases to discuss whether the equilibrium $(1, 1)$ of system (18) is a nilpotent center:

$$(II.a) \quad b^2 + 2(b+c)d = 0 \text{ and } b+c \neq 0.$$

$$(II.b) \quad b^2 + 2(b+c)d = 0 \text{ and } b+c = 0.$$

$$(II.c) \quad b^2 + 2(b+c)d \neq 0 \text{ and } 5b^2 - 4(c-6d)b + c^2 + 32d^2 - 8cd = 0.$$

For (II.a), $d = -b^2/(2(b+c))$. Plugging in the expressions of a and d , system (18) becomes

$$\begin{aligned}\dot{x} &= \frac{x(bx + cy - b - c)(bx + 3cy - b - c)}{2(b+c)}, \\ \dot{y} &= -\frac{y(bx + cy - b - c)(3bx + cy - b - c)}{2(b+c)},\end{aligned}$$

which has a nonisolated equilibrium $(1, 1)$ on the straight line $bx + cy - b - c = 0$ filled with equilibria, consequently it cannot be a nilpotent center.

For (II.b), $b = c = 0$ and system (18) becomes

$$\begin{aligned}\dot{x} &= -dx(x - 3y)(x - y), \\ \dot{y} &= -dy(-3x + y)(x - y),\end{aligned}$$

which again has a nonisolated equilibrium $(1, 1)$ on a straight line $x - y = 0$ filled with equilibria.

For (II.c), $d = (-3b + c \pm \sqrt{-(b-c)^2})/8$. Since d is real number, we must have $b = c$ and $d = (-3b + c)/8$, which implies $3b - c + 8d = 0$. This is not possible because we are in Case II: $3b - c + 8d \neq 0$.

In summary, in Case II we prove that a HK-system (3) cannot have any a nilpotent equilibrium. Hence a HK-system (3) cannot have two nilpotent centers. This completes the proof of statement (ii) of Theorem 2. $\square$

From the proof of nonexistence of two nilpotent centers, we can see that a HK-system (3) can have a nilpotent center. And a HK-system (3) has at most two isolated equilibria outside the coordinate axes if there is a nilpotent center. Therefore when a HK-system (3) has three or four centers, all of them must be linear type centers. Moreover, if a HK-system (3) has only two isolated equilibria outside the coordinate axes, both of them are centers: one is nilpotent and the other is linear type, and they are either in the same quadrant or in the opposite quadrant. Hence, if one of two centers is nilpotent, they cannot be in consecutive quadrants.

In the proofs of statements (iii)–(vi) of Theorem 2, we will give examples showing the existence of HK-systems with centers realizing all possible configurations with respect to the coordinate axes.

Proof of statement (iii) of Theorem 2. The eigenvalues of the linear part of a HK-system (3) at the equilibrium $(1, 1)$ are $\pm\sqrt{\Delta}/2$ where

$$\Delta = 6ab + 5b^2 - 2ac + 2bc - c^2 + 16ad + 12bd + 12cd.$$

So if $\Delta < 0$ the equilibrium $(1, 1)$ is a linear type center, because the HK-system is a Hamiltonian system and it cannot have a focus at the equilibrium $(1, 1)$. Hence we have proved that there are HK-systems having at least one linear type center.

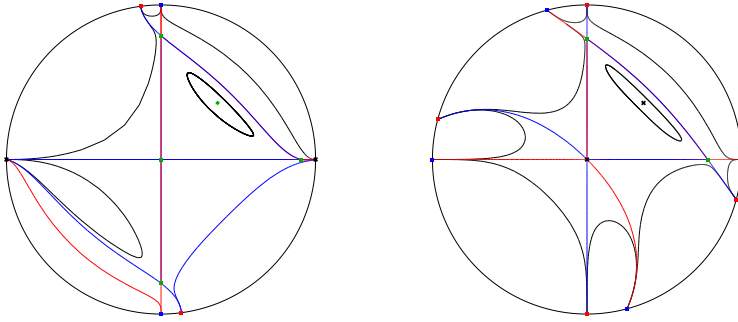


Figure 1. Left: A HK-system (30) with a unique linear type center.
Right: A HK-system (31) with a unique nilpotent center.

Now we shall show that there are HK-systems with a unique center of linear type. Consider the HK-system defined by the Hamiltonian

$$(30) \quad H = xy(-40 + 4x + 15xy + 2y^2).$$

The equilibria of this HK-system are

$$(0, 0), \quad (0, \pm 2\sqrt{5}), \quad (10, 0), \quad (1, 1).$$

So the only one of these equilibria which can be a center is the equilibrium $(1, 1)$, which is a linear type center because the eigenvalues of its linear part are $\pm\sqrt{11/40}i$. See its global phase portrait in the Poincaré disc in Figure 1(left). For a definition of the Poincaré disc, see, for instance, Chapter 5 of [Dumortier et al. 2006].

The real algebraic curve

$$xy(-40 + 4x + 15xy + 2y^2) = h$$

is a family of level ovals in $\mathbb{R}^2$ if and only if $-19 < h < 0$.

Next, we provide a HK-system with a unique nilpotent center. Consider the HK-system defined by the Hamiltonian

$$(31) \quad H = xy \left(x + y - \frac{x^2}{4} - xy - \frac{y^2}{4} \right),$$

i.e., the HK-system is

$$(32) \quad \begin{aligned} \dot{x} &= -x \left(x + 2y - \frac{x^2}{4} - 2xy - \frac{3y^2}{4} \right), \\ \dot{y} &= y \left(2x + y - \frac{3x^2}{4} - 2xy - \frac{y^2}{4} \right). \end{aligned}$$

The equilibria of this HK-system are

$$(0, 0), \quad (4, 0), \quad (0, 4), \quad (1, 1).$$

Therefore again the only one of these equilibria which can be a center is the equilibrium $(1, 1)$. The two eigenvalues of this equilibrium are zero, and since its linear part is not identically zero, it is a nilpotent equilibrium. We shall use Lemma 7 for proving that it is a nilpotent center.

First, using the translations $x = X + 1$ and $y = Y + 1$ we translate the equilibrium $(1, 1)$ to the origin of coordinates, and we get the differential system

$$(33) \quad \begin{aligned} \dot{X} &= \frac{1}{4}(6X + 6Y + 7X^2 + 14XY + 3Y^2 + X^3 + 8X^2Y + 3XY^2), \\ \dot{Y} &= -\frac{1}{4}(6X + 6Y + 3X^2 + 14XY + 7Y^2 + 3X^2Y + 8XY^2 + Y^3). \end{aligned}$$

In order to apply Lemma 7 we shall write the linear part at the origin of this system in the real Jordan normal form. So we use the change of variables $X = 3u/2$ and $Y = v - 3u/2$, and in the new variables u and v , the differential system (33) becomes

$$(34) \quad \begin{aligned} \dot{u} &= v - \frac{3}{2}u^2 + 2uv + \frac{1}{2}v^2 - \frac{9}{4}u^3 + \frac{3}{4}u^2v + \frac{3}{4}uv^2, \\ \dot{v} &= -v^2 + 3uv - \frac{27}{4}u^3 + \frac{27}{4}u^2v - \frac{3}{4}uv^2 - \frac{1}{4}v^3. \end{aligned}$$

Now we can apply Lemma 7 to system (34), and using its notation we have that $v = f(u) = 3u^2/2 + O(u^3)$ and $F(u) = -9u^3/4 + O(u^4)$. Then, $G(u) \equiv 0$, $a = -9/3 < 0$ and $m = 3$, so the origin of system (34) is a center by Theorem 8. Consequently the equilibrium $(1, 1)$ of system (32) is a nilpotent center. See its global phase portrait in the Poincaré disc in Figure 1(right).

The real algebraic curve

$$xy \left(x + y - \frac{x^2}{4} - xy - \frac{y^2}{4} \right) = h$$

is a family of level ovals in $\mathbb{R}^2$ if and only if $0 < h < \frac{1}{2}$.

□

Proof of statement (iv) of Theorem 2. First we prove the existence of a HK-system having two linear type centers in the positive quadrant, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(35) \quad H = xy(200 - 60x - 240y + 6x^2 + 23xy + 96y^2).$$

The corresponding HK-system has four equilibria outside the axes of coordinates, namely,

$$p_1 = (1, 1), \quad p_2 = \left(4, \frac{1}{4}\right), \quad p_{3,4} = \left(\frac{20(9 \pm \sqrt{10})}{71}, \frac{5(9 \pm \sqrt{10})}{71}\right).$$

The equilibria p_1 and p_2 are hyperbolic saddles because the determinant of their linear parts in both is -1350 , and the equilibria $p_{3,4}$ are linear type centers because

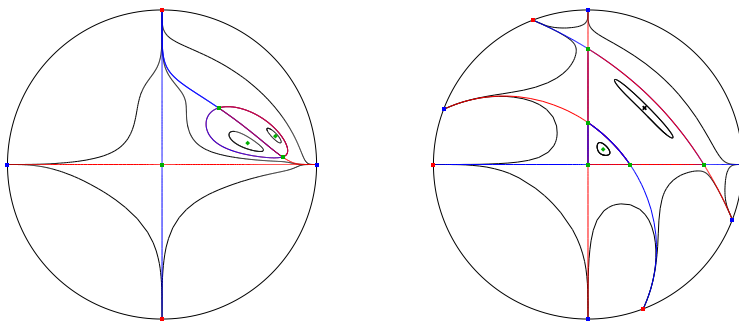


Figure 2. Left: A HK-system (35) with two linear type centers in the same quadrant. Right: A HK-system (36) with one linear type center and one nilpotent center, both in the same quadrant.

the determinant of their linear parts is

$$\frac{120000(7460 \pm 1531\sqrt{10})}{357911},$$

and the system is Hamiltonian. The phase portrait of the HK-system defined by the Hamiltonian (35) is given in Figure 2(left). The real algebraic curve

$$xy(200 - 60x - 240y + 6x^2 + 23xy + 96y^2) = h$$

is a family of level ovals in $\mathbb{R}^2$ if and only if either $10000(467 - 240\sqrt{10})/357911 < h < 25$ or $25 < h < 10000(467 + 240\sqrt{10})/357911$. Taken together, this real algebraic curve has two families of level ovals in $\mathbb{R}^2$ for all real h .

Now we provide a HK-system having one linear type center and one nilpotent center in the positive quadrant, and no other centers. The HK-system defined by the Hamiltonian

$$(36) \quad H = xy(2 - 4x + x^2 - 4y + 3xy + y^2)$$

has only the following two equilibria outside the axes of coordinates:

$$p_1 = (1, 1) \quad p_2 = \left(\frac{1}{5}, \frac{1}{5}\right).$$

The equilibrium p_2 has eigenvalues $\pm\sqrt{128/25}i$, so it is a linear type center.

We shall show that the equilibrium p_1 is a nilpotent center. The two eigenvalues of this equilibrium are zero, and since its linear part is not identically zero, it is a nilpotent equilibrium. In order to study its local phase portrait we translate the equilibrium p_1 to the origin of coordinates using the translations $x = X + 1$ and $y = Y + 1$, and we obtain the differential system

$$(37) \quad \begin{aligned} \dot{X} &= 4X - 4Y - 5X^2 - 10XY - 3Y^2 - X^3 - 6X^2Y - 3XY^2, \\ \dot{Y} &= 4X + 4Y + 3X^2 + 10XY + 5Y^2 + 3X^2Y + 6XY^2 + Y^3. \end{aligned}$$

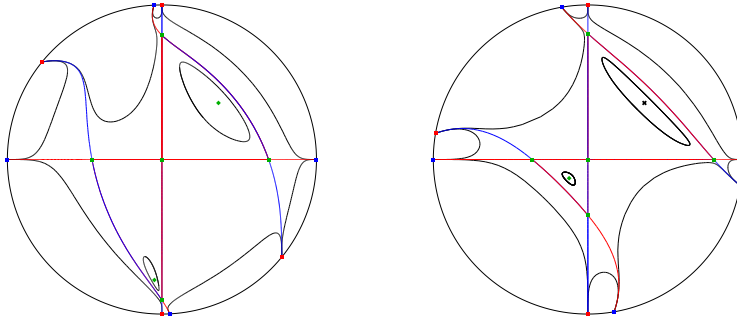


Figure 3. Left: A HK-system (39) with two linear type centers in different opposite quadrants. Right: A HK-system (40) with one linear type center and one nilpotent center in different opposite quadrants.

Now we shall write the linear part at the origin of this system in its real Jordan normal form; for this we use the change of variables $X = -4u$ and $Y = 4u + v$. In the new variables u and v the differential system (37) becomes

$$(38) \quad \begin{aligned} \dot{u} &= v - 8u^2 - 4uv + \frac{3}{4}v^2 + 32u^3 - 3uv^2, \\ \dot{v} &= 16uv + 2v^2 - 256u^3 - 96u^2v + v^3. \end{aligned}$$

We can apply Lemma 7 to system (38), and using the notation of that lemma, we have $v = f(u) = 8u^2 + O(u^3)$ and $F(u) = -128u^3 + O(u^4)$. Then, $G(u) \equiv 0$, $a = -128 < 0$ and $m = 3$, so the origin of system (38) is a center by Theorem 8. Consequently the equilibrium p_1 of the Hamiltonian system defined by (36) is a nilpotent center. See its global phase portrait in the Poincaré disc in Figure 2(right). The real algebraic curve

$$xy(2 - 4x + x^2 - 4y + 3xy + y^2) = h$$

is a family of level ovals in $\mathbb{R}^2$ if and only if either $-1 < h < 0$ or $0 < h < \frac{3}{125}$. In total, this real algebraic curve has two families of level ovals in $\mathbb{R}^2$ for all real h .

Now we prove the existence of a HK-system having two linear type centers in different opposite quadrants, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(39) \quad H = xy(96 + 48x - 12y - 32x^2 - 41xy - 2y^2).$$

The corresponding HK-system has four equilibria outside the axes of coordinates, namely,

$$(1, 1), \quad p_2 = \left(-\frac{1}{4}, -4\right).$$

These two equilibria are linear type centers because the determinant of their linear parts is $475/384$ and the system is Hamiltonian. The phase portrait of the HK-system defined by the Hamiltonian (39) is given in Figure 3(left). The real algebraic curve

$$xy(96 + 48x - 12y - 32x^2 - 41xy - 2y^2) = h$$

has two families of level ovals in $\mathbb{R}^2$ if and only if $0 < h < 57$.

Now we exhibit a HK-system having one linear type center and one nilpotent center in different opposite quadrants, and no other centers. Thus the HK-system defined by the Hamiltonian

$$(40) \quad H = xy(4 + 4x + 4y - x^2 - 6xy - y^2),$$

has only the following two equilibria outside the axes of coordinates:

$$p_1 = (1, 1) \quad p_2 = \left(-\frac{1}{4}, -\frac{1}{4}\right).$$

The linear part of the equilibrium p_2 has determinant equal to $225/64$, so it is a linear type center.

We shall prove that the equilibrium p_1 is a nilpotent center. Since the two eigenvalues of this equilibrium are zero, and its linear part is not identically zero, it is a nilpotent equilibrium. For studying its local phase portrait we translate the equilibrium p_1 to the origin of coordinates using the translations $x = X + 1$ and $y = Y + 1$, and we obtain the differential system

$$(41) \quad \begin{aligned} \dot{X} &= \frac{3}{4}(10X + 10Y + 11X^2 + 22XY + 3Y^2 + X^3 + 12X^2Y + 3XY^2), \\ \dot{Y} &= -\frac{3}{4}(10X + 10Y + 3X^2 + 22XY + 11Y^2 + 3X^2Y + 12XY^2 + Y^3). \end{aligned}$$

We write the linear part at the origin of this system in its real Jordan normal form using the change of variables $X = 15u/2$ and $Y = v - 15u/2$. In the new variables u and v , the differential system (41) becomes

$$(42) \quad \begin{aligned} \dot{u} &= v + \frac{1}{20}(-900u^2 + 240uv + 6v^2 - 6750u^3 + 675u^2v + 45uv^2), \\ \dot{v} &= -\frac{3}{4}(-120uv + 8v^2 + 6750u^3 - 1350u^2v + 45uv^2 + v^3). \end{aligned}$$

Applying Lemma 7 to system (42), and using the notation of that theorem, we have $v = f(u) = 45u^2 + O(u^3)$ and $F(u) = -101258u^3/2 + O(u^4)$. Then, $G(u) \equiv 0$, $a = -10125/2 < 0$ and $m = 3$, so the origin of system (42) is a center by Theorem 8. Consequently the equilibrium p_1 of the Hamiltonian system defined by (40) is a nilpotent center. See its global phase portrait in the Poincaré disc in Figure 3(right). The real algebraic curve

$$xy(4 + 4x + 4y - x^2 - 6xy - y^2) = h$$

has two families of level ovals in $\mathbb{R}^2$ if and only if $0 < h < \frac{3}{32}$. And when $\frac{3}{32} \leq h < 4$,

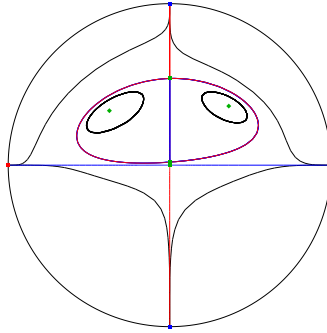


Figure 4. HK-systems with two centers in different consecutive quadrants.

then $xy(4 + 4x + 4y - x^2 - 6xy - y^2) = h$ has a family of level ovals in $\mathbb{R}^2$. In summary, for all $h \in \mathbb{R}$, the real algebraic curve $xy(4 + 4x + 4y - x^2 - 6xy - y^2) = h$ has two families of level ovals in $\mathbb{R}^2$.

We provide a HK-system having two linear type centers in different consecutive quadrants, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(43) \quad H = xy(178074 + 356148x - 4989116y + 558568x^2 - 404079xy + 3231200y^2).$$

The corresponding HK-system has two equilibria outside the axes of coordinates, namely,

$$(1, 1), \quad \left(-1, \frac{361}{400}\right).$$

These two equilibria are linear type centers because the determinants of their linear parts are

$$6889484999096 \quad \text{and} \quad \frac{461859088701689761}{80000},$$

respectively. The phase portrait of the HK-system defined by the Hamiltonian (43) is given in Figure 4. This completes the proof of statement (iv) of Theorem 2. The real algebraic curve

$$xy(178074 + 356148x - 4989116y + 558568x^2 - 404079xy + 3231200y^2) = h$$

has a family of level ovals in $\mathbb{R}^2$ if and only if either $-1069205 < h < 0$ or $0 < h < 162546646559/160000$. In summary, for all $h \in \mathbb{R}$, the real algebraic curve $xy(178074 + 356148x - 4989116y + 558568x^2 - 404079xy + 3231200y^2) = h$ has two families of level ovals in $\mathbb{R}^2$. $\square$

Proof of statement (v) of Theorem 2. We show a HK-system having three linear type centers in the same quadrant, and no other centers. Let the HK-system be defined by the Hamiltonian

$$(44) \quad H = xy(20 - 20x - 80y + 2x^2 + 31xy + 32y^2).$$

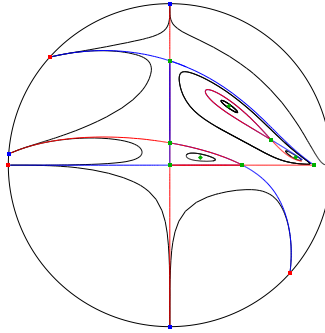


Figure 5. HK-systems with three linear type centers in the same quadrant.

The associated HK-system has four equilibria outside the axes of coordinates, namely,

$$(45) \quad (1, 1), \quad (4, \tfrac{1}{4}), \quad (\tfrac{2}{47}(30 \pm \sqrt{430}), \tfrac{1}{94}(30 \pm \sqrt{430})).$$

The determinants of their linear parts are

$$\frac{27}{10}, \quad \frac{27}{10} \quad \text{and} \quad \frac{4(\pm 995\sqrt{430} - 13588)}{103823},$$

respectively. Therefore the three first equilibria of (45) are linear type centers and the fourth one is a hyperbolic saddle.

The real algebraic curve

$$xy(20 - 20x - 80y + 2x^2 + 31xy + 32y^2) = h$$

has two families of level ovals in $\mathbb{R}^2$ if and only if

$$-15 < h < -\frac{100(6889 + 344\sqrt{430})}{103823},$$

and it has a family of level ovals in $\mathbb{R}^2$ if and only if either

$$-\frac{100(6889 + 344\sqrt{430})}{103823} \leq h < 0 \quad \text{or} \quad 0 < h < \frac{100(-6889 + 344\sqrt{430})}{103823}.$$

In summary, for all $h \in \mathbb{R}$, the real algebraic curve

$$xy(20 - 20x - 80y + 2x^2 + 31xy + 32y^2) = h$$

has four families of level ovals in $\mathbb{R}^2$. In Figure 5 we show the phase portrait of the HK-system defined by the Hamiltonian (44).

Now we provide a HK-system having two linear type centers in the same quadrant and the other in a consecutive quadrant, and no other centers. Let the HK-system be defined by the Hamiltonian

$$(46) \quad H = xy(-20 - 20x + 104y + 2x^2 + 5xy - 60y^2).$$

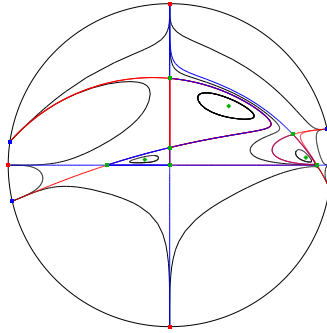


Figure 6. HK-systems with two linear type centers in the same quadrant and the other in a consecutive quadrant.

The associated HK-system has four equilibria outside the axes of coordinates, namely,

$$(47) \quad (4, 1), \quad (1, 1), \quad \left(\frac{2(56071 \mp 829\sqrt{6157})}{101(485 \pm \sqrt{6157})}, \frac{1}{606}(121 \mp \sqrt{6157}) \right).$$

The determinants of their linear parts are

$$-\frac{108}{5}, \quad \frac{63}{10} \quad \text{and} \quad \frac{658(731635 \pm 9034\sqrt{6157})}{46363545},$$

respectively. Therefore the first equilibrium of (47) is a hyperbolic saddle and the other three equilibria are linear type centers. In Figure 6 we show the phase portrait of the HK-system defined by the Hamiltonian (46). The phase portrait shows that the real algebraic curve

$$xy(-20 - 20x + 104y + 2x^2 + 5xy - 60y^2) = h$$

has three families of level ovals in $\mathbb{R}^2$ for all $h \in \mathbb{R}$, in which this real algebraic curve has two families of level ovals if and only if

$$0 < h < \frac{94(-2956253 + 37859\sqrt{6157})}{9272709},$$

and it has a family of level ovals if and only if either

$$\frac{94(-2956253 + 37859\sqrt{6157})}{9272709} \leq h < 11 \quad \text{or} \quad -\frac{94(2956253 + 37859\sqrt{6157})}{9272709} < h < -16.$$

We give a HK-system having three linear type centers; two in the same quadrant and the other in the opposite quadrant, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(48) \quad H = xy(-40 - 20x - 80y + 2x^2 + 61xy + 32y^2).$$

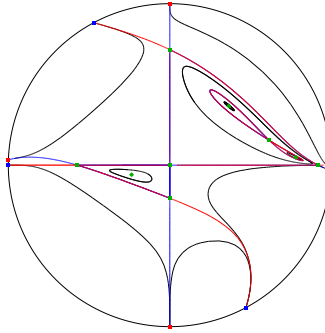


Figure 7. HK-systems with three linear type centers; two in the same quadrant and the other in the opposite quadrant.

The corresponding HK-system has four equilibria outside the axes of coordinates, namely,

$$(49) \quad (1, 1), \quad (4, \frac{1}{4}), \quad (\frac{4}{77}(15 \mp \sqrt{610}), \frac{1}{77}(15 \mp \sqrt{610})).$$

The determinants of their linear parts are

$$\frac{81}{40}, \quad \frac{81}{40} \quad \text{and} \quad \frac{4(\pm 2995\sqrt{610} - 30988)}{456533},$$

respectively. Therefore the three first equilibria of (49) are linear type centers and the fourth one is a hyperbolic saddle. The phase portrait of the HK-system defined by the Hamiltonian (48) is given in Figure 7. The phase portrait shows that the real algebraic curve

$$xy(-40 - 20x - 80y + 2x^2 + 61xy + 32y^2) = h$$

has four families of level ovals in $\mathbb{R}^2$ for all $h \in \mathbb{R}$, in which this real algebraic curve has two families of level ovals if and only if either

$$-45 < h < -\frac{400(25189 + 976\sqrt{610})}{456533} \quad \text{or} \quad \frac{400(-25189 + 976\sqrt{610})}{456533} < h < 0,$$

and it has a family of level ovals if and only if

$$-\frac{400(25189 + 976\sqrt{610})}{456533} \leq h \leq \frac{400(-25189 + 976\sqrt{610})}{456533}.$$

We show a HK-system having three linear type centers in three different quadrants, and no other centers. Consider the HK-system defined by the Hamiltonian

$$(50) \quad H = xy(-108 - 108x + 32y + 72x^2 + 37xy + 2y^2).$$

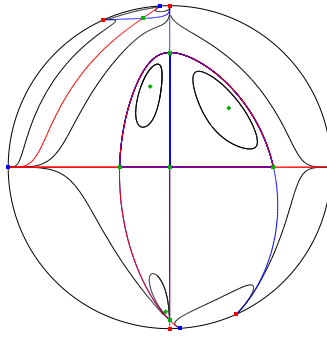


Figure 8. HK-systems with three linear type centers in three different quadrants.

The corresponding HK-system has four equilibria outside the axes of coordinates, namely,

$$(51) \quad (1, 1), \quad \left(-\frac{1}{4}, -9\right), \quad \left(\frac{2}{793}(-606 \pm 275\sqrt{3}), \frac{27}{793}(239 \mp 115\sqrt{3})\right).$$

The determinants of their linear parts are

$$\frac{2425}{288}, \quad \frac{3925}{1458} \quad \text{and} \quad -\frac{47375215450}{498677257} \pm \frac{27501846750\sqrt{3}}{498677257},$$

respectively. Therefore the three first equilibria of (51) are linear type centers and the fourth one is a hyperbolic saddle. In Figure 8 we show the phase portrait of the HK-system defined by the Hamiltonian (50). The real algebraic curve

$$xy(-108 - 108x + 32y + 72x^2 + 37xy + 2y^2) = h$$

has three families of level ovals in $\mathbb{R}^2$ for all $h \in \mathbb{R}$, in which this real algebraic curve has two families of level ovals if and only if $-73 < h < 0$, and it has a family of level ovals if and only if either

$$-\frac{4293}{16} < h \leq -73 \quad \text{or} \quad 0 < h < \frac{5832(-241463148 + 140160125\sqrt{3})}{498677257}.$$

This completes the proof of statement (v) of Theorem 2. $\square$

Proof of statement (vi) of Theorem 2. From the proof of the nonexistence of two nilpotent centers, we can see that the four centers must be linear if a HK-system (3) has four centers. And we claim that this system has exactly four families of periodic orbits, one of which contains a linear type center in its interior.

In fact, if a HK-system (3) has four centers, then this system has four families of periodic orbits, one of which contains a linear type center in its interior by the definition of center.

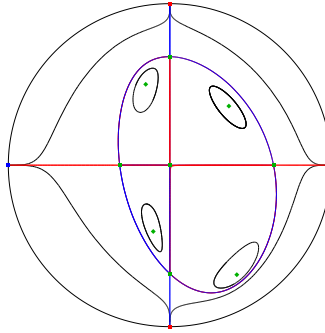


Figure 9. HK-systems with four linear type centers in four different quadrants.

On the other hand, if a HK-system (3) has four centers, then all other equilibria of this system should be on the x -axis or the y -axis, i.e., this system cannot have the other equilibria in the interior of quadrants of $\mathbb{R}^2$ by Theorem 6 and statement (i) of Theorem 2. Note that the x -axis and y -axis are invariant for the flow of a HK-system (3). The periodic orbit of a HK-system (3) cannot intersect with the x -axis or the y -axis. Hence, there does not exist a periodic orbit of this system such that the periodic orbit contains more than two equilibria in its interior due to the sum of indices of these equilibria being one and the index of a linear type center being one. Therefore a HK-system having four linear type centers has exactly four families of periodic orbits, one of which contains a linear type center in its interior. This gives that the level set of the corresponding Hamiltonian

$$xy(a + bx + cy + dx^2 + exy + fy^2) = h$$

has four families of level ovals in $\mathbb{R}^2$.

We now face an interesting problem: what are the configurations of four centers with respect to the coordinate axes if a HK-system (3) has four centers? In the following, we provide a HK-system having four linear type centers in four different quadrants. Let the HK-system be defined by the Hamiltonian

$$(52) \quad H = xy(-12 - 12x + 8x^2 + 5xy + 2y^2).$$

The associated HK-system has four equilibria outside the axes of coordinates, namely,

$$(53) \quad (1, 1), \quad \left(-\frac{1}{4}, -1\right), \quad \left(\frac{2}{13}(6 \pm 5\sqrt{3}), \frac{1}{13}(-15 \mp 19\sqrt{3})\right),$$

with one equilibrium in each quadrant. The determinants of their linear parts are

$$\frac{23}{6}, \quad \frac{83}{96} \quad \text{and} \quad \frac{2(30191 \pm 16525\sqrt{3})}{2197},$$

respectively. All these determinants are positive. Hence the four equilibria of (53)

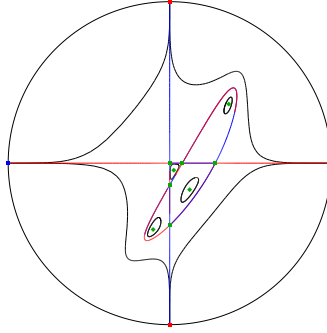


Figure 10. HK-systems with four linear type centers; two are in the same quadrant, and each one of the other two centers is located in a different adjacent quadrant.

are linear type centers. In Figure 9 there is the phase portrait of the HK-system defined by the Hamiltonian (52). The real algebraic curve

$$xy(-12 - 12x + 8x^2 + 5xy + 2y^2) = h$$

has four families of level ovals in $\mathbb{R}^2$ for all $h \in \mathbb{R}$, in which this real algebraic curve has two families of level ovals if and only if either

$$-\frac{21}{16} < h < 0 \quad \text{or} \quad 0 < h < \frac{72(1660 - 911\sqrt{3})}{2197},$$

and it has a family of level ovals if and only if either

$$-9 < h \leq -\frac{21}{16} \quad \text{or} \quad \frac{72(1660 - 911\sqrt{3})}{2197} \leq h < \frac{72(1660 + 911\sqrt{3})}{2197}.$$

We show a HK-system having four linear type centers, two in the same quadrant, and each one of the other two centers located in a different adjacent quadrant. Let the HK-system be defined by the Hamiltonian

$$(54) \quad H = xy(57 - 456x + 270y + 592x^2 - 710xy + 229y^2).$$

The associated HK-system has four equilibria outside the axes of coordinates, namely,

$$(55) \quad (1, 1), \quad \left(-\frac{1}{4}, -1\right), \quad \left(\frac{72732 \pm 5\sqrt{95939265}}{470788}, \frac{19(-16455 \pm \sqrt{95939265})}{1456898}\right).$$

The determinants of their linear parts are

$$\frac{5159}{8664}, \quad \frac{367}{4332} \quad \text{and} \quad \frac{5(109357753909491323 \pm 8468817930191\sqrt{95939265})}{36973060615773331024},$$

respectively. All these determinants are positive. Therefore the four equilibria

of (55) are linear type centers. In Figure 10 there is the phase portrait of the HK-system defined by the Hamiltonian (54). The real algebraic curve

$$xy(57 - 456x + 270y + 592x^2 - 710xy + 229y^2) = h$$

has four families of level ovals in $\mathbb{R}^2$ for all $h \in \mathbb{R}$, in which this real algebraic curve has three families of level ovals if and only if

$$\frac{16245(2340735907022315 - 256172312597\sqrt{95939265})}{36973060615773331024} < h < 0,$$

it has two families of level ovals if and only if

$$-\frac{21}{8} < h \leq \frac{16245(2340735907022315 - 256172312597\sqrt{95939265})}{36973060615773331024},$$

and it has a family of level ovals if and only if either $-18 < h \leq -\frac{21}{8}$ or

$$0 < h < \frac{16245(2340735907022315 + 256172312597\sqrt{95939265})}{36973060615773331024}. \quad \square$$

From our results remains the following open question.

Open question. Are the two configurations of four centers with respect to the coordinate axes described in statement (vi) of Theorem 2 the only ones which are realizable with four centers by the HK-systems?

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JAUME LLIBRE
 DEPARTAMENT DE MATEMÀTIQUES
 UNIVERSITAT AUTÒNOMA DE BARCELONA
 BARCELONA
 SPAIN
 jllibre@mat.uab.cat

DONGMEI XIAO
 SCHOOL OF MATHEMATICAL SCIENCES
 SHANGHAI JIAO TONG UNIVERSITY
 SHANGHAI
 CHINA
 xiaodm@sjtu.edu.cn

2-CATEGORIES OF SYMMETRIC BIMODULES AND THEIR 2-REPRESENTATIONS

VOLODYMYR MAZORCHUK, VANESSA MIEMIETZ AND XIAOTING ZHANG

In this article we analyze the structure of 2-categories of symmetric projective bimodules over a finite dimensional algebra with respect to the action of a finite abelian group. We determine under which condition the resulting 2-category is fiat (in the sense of Mazorchuk and Miemietz (2011)) and classify simple transitive 2-representations of this 2-category (under some mild technical assumption). We also study several classes of examples in detail.

1. Introduction and description of the results

In the last 20 years, many exciting breakthroughs in representation theory, see, e.g., [Khovanov 2000; Chuang and Rouquier 2008; Brundan and Stroppel 2012; Elias and Williamson 2014; Williamson 2017], have originated from the idea of categorification. This has inspired the subject of 2-representation theory which studies 2-categories with suitable finiteness conditions. An appropriate 2-analogue of a finite dimensional algebra was defined in [Mazorchuk and Miemietz 2011] and called finitary 2-category. Various aspects of the structure and 2-representation theory of finitary and, more specifically, fiat 2-categories have been studied in [Mazorchuk and Miemietz 2011; 2014; 2016a; 2016b; 2016c; 2016d; Mackaay et al. 2019; 2018; Chan and Mazorchuk 2019; Chan and Miemietz 2020]. In particular, [Mazorchuk and Miemietz 2016d] introduces the notion of simple transitive 2-representation which is an appropriate categorification of the concept of an irreducible representation. A natural and interesting problem is the classification of simple transitive 2-representations for various classes of 2-categories, see [Mazorchuk 2017].

One interesting example of a fiat 2-category is the 2-category of Soergel bimodules associated to the coinvariant algebra of a finite Coxeter system, see [Soergel 1992; 2007; Elias and Williamson 2014]. For these 2-categories, simple transitive 2-representations have been classified in several special cases including type A , the dihedral types and some small classical types, see [Mazorchuk and Miemietz 2016d; Kildetoft et al. 2019; Mackaay and Tubbenhauer 2019; Mackaay et al. 2018].

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The article [Mackaay et al. 2018] develops a reduction technique that reduces the classification problem to smaller 2-categories which, in practice, are often given by “symmetric bimodules” as defined in a special case in [Kildetoft et al. 2019].

Inspired by this, in the present article we formalize the concept of symmetric bimodules under the action of an abelian group. While defining symmetric bimodules with respect to a nonabelian group action is possible, decompositions will no longer simply rely on the Pontryagin dual group and the techniques of this paper would need to be changed significantly. We study the resulting 2-categories of projective symmetric bimodules and their 2-representations. We show that these 2-categories are weakly fiat provided that the underlying algebra is self-injective and fiat if the underlying algebra is weakly symmetric, mirroring the situation for the 2-category of all projective bimodules from [Mazorchuk and Miemietz 2011, Subsection 7.3] and [Mazorchuk and Miemietz 2016b, Subsection 2.8]. Using [Mackaay et al. 2018], we reduce the classification of simple transitive 2-representations in the weakly symmetric case to the classification of module categories over the 2-category $\text{Rep}(G)$ from [Ostrik 2003].

One of the main results of [Mazorchuk and Miemietz 2016a] classifies a class of “simple” 2-categories with a particularly nice combinatorial structure. Here we study one of the smallest families of 2-categories which do not fit into the setup of [Mazorchuk and Miemietz 2016a]. We show that these can always be realized inside a 2-category of symmetric bimodules.

The paper is organized as follows. In Section 2 we collect the necessary preliminaries. In Section 3 we introduce symmetric bimodules and study their structure and simple transitive 2-representations. The latter are classified in Theorem 17. In Section 4 we study 2-categories with one object which, apart from the identity 1-morphism, have precisely two indecomposable 1-morphisms up to isomorphism and these two form a biadjoint pair and their own left/right/two-sided cell. In Theorem 25 we realize such 2-categories as 2-subcategories of certain symmetric bimodules.

2. Preliminaries

2.1. Setup. We work over an algebraically closed field $\mathbb{k}$.

2.2. Finitary 2-categories. A $\mathbb{k}$ -linear category is called *finitary* if it is small and equivalent to the category of finitely generated projective modules over some finite dimensional associative $\mathbb{k}$ -algebra.

We call a 2-category $\mathcal{C}$ *finitary* (over $\mathbb{k}$) if it has finitely many objects, each morphism category $\mathcal{C}(i, j)$ is a finitary $\mathbb{k}$ -linear category, all compositions are (bi)additive and $\mathbb{k}$ -linear and all identity 1-morphisms are indecomposable (see [Mazorchuk and Miemietz 2011, Subsection 2.2]).

We say that $\mathcal{C}$ is *weakly fiat* if it is finitary and has a weak antiautomorphism $(\cdot)^*$ of finite order and adjunction morphisms, see [Mazorchuk and Miemietz 2016b, Subsection 2.5]. If $(\cdot)^*$ is involutive, we say that $\mathcal{C}$ is *fiat*, see [Mazorchuk and Miemietz 2011, Subsection 2.4].

2.3. 2-representations. Let $\mathcal{C}$ be a finitary 2-category. A *finitary 2-representation* of $\mathcal{C}$ is a 2-functor from $\mathcal{C}$ to the 2-category $\mathfrak{A}_{\mathbb{k}}^f$ whose

- objects are finitary $\mathbb{k}$ -linear categories;
- 1-morphisms are additive $\mathbb{k}$ -linear functors;
- 2-morphisms are natural transformations of functors.

Such 2-representations form a 2-category denoted $\mathcal{C}\text{-afmod}$, see [Mazorchuk and Miemietz 2016a, Subsection 2.3].

Similarly we define an *abelian 2-representation* of $\mathcal{C}$ as a 2-functor from $\mathcal{C}$ to the 2-category whose

- objects are categories equivalent to categories of finitely generated modules over finite dimensional $\mathbb{k}$ -algebras;
- 1-morphisms are right exact $\mathbb{k}$ -linear functors;
- 2-morphisms are natural transformations of functors.

Such 2-representations form a 2-category denoted $\mathcal{C}\text{-mod}$, see [Mazorchuk and Miemietz 2016a, Subsection 2.3]. Following [Mazorchuk and Miemietz 2014, Subsection 4.2], we denote by $\overline{\cdot}$ the *abelianization* 2-functor from $\mathcal{C}\text{-afmod}$ to $\mathcal{C}\text{-mod}$.

A finitary 2-representation $\mathbf{M}$ is called *simple transitive* provided that $\coprod_{i \in \mathcal{C}} \mathbf{M}(i)$ has no nontrivial $\mathcal{C}$ -invariant ideals, see [Mazorchuk and Miemietz 2016d, Subsection 3.5].

2.4. Cells and cell 2-representations. Given two indecomposable 1-morphisms F and G in $\mathcal{C}$, we define $F \geq_L G$ if F is isomorphic to a direct summand of $H \circ G$, for some 1-morphism H . This produces the *left* preorder $\geq_L$, of which the equivalence classes are called *left cells*. Similarly one obtains the *right* preorder $\geq_R$ and the corresponding *right cells*, and the *two-sided* preorder $\geq_J$ and the corresponding *two-sided cells*.

For each simple transitive 2-representation $\mathbf{M}$, there is a unique two-sided cell which is maximal, with respect to $\geq_J$, among those two-sided cells whose 1-morphisms are not annihilated by $\mathbf{M}$. This two-sided cell is called the *apex* of $\mathbf{M}$, see [Chan and Mazorchuk 2019, Subsection 3.2].

If $\mathcal{J}$ is a two-sided cell of $\mathcal{C}$, we say that $\mathcal{C}$ is *$\mathcal{J}$ -simple* if any nonzero 2-ideal of $\mathcal{C}$ contains the identity 2-morphisms of all 1-morphisms in $\mathcal{J}$.

Each left cell of a fiat 2-category contains a so-called *Duflo involution*, see [Mazorchuk and Miemietz 2011, Subsection 4.5].

For a left cell $\mathcal{L}$ in $\mathcal{C}$, there exists $i \in \mathcal{C}$ such that all 1-morphisms in $\mathcal{L}$ starts at i . Denote by $N_{\mathcal{L}}$ the 2-subrepresentation of P_i which is defined as for each $j \in \mathcal{C}$ the category $N_{\mathcal{L}}(j)$ is the additive closure of $P_i(j)$ consisting of all 1-morphisms F with $F \geq_{\mathcal{L}} \mathcal{L}$. From [Mazorchuk and Miemietz 2016d, Lemma 3], we know that the 2-representation $N_{\mathcal{L}}$ contains a unique maximal ideal which does not contain any id_F for $F \in \mathcal{L}$, denoted $\mathcal{I}_{\mathcal{L}}$. The quotient $C_{\mathcal{L}} := N_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}$ is called the *cell 2-representation* associated to $\mathcal{L}$.

3. Symmetric bimodules and their simple transitive 2-representations

3.1. Symmetric bimodules. Let A be a finite dimensional, unital, associative $\mathbb{k}$ -algebra. We assume that A is basic and that $\{e_1, e_2, \dots, e_k\}$ is a complete set of pairwise orthogonal primitive idempotents in A .

Let G be a finite abelian subgroup of the group of automorphisms of A . Assume that $\text{char}(\mathbb{k})$ does not divide $|G|$. The action of G on A induces an action of G on the category of A - A -bimodules via $M \mapsto {}^\varphi M^\varphi$, where the action of A on ${}^\varphi M^\varphi$ is given by

$$a \cdot m \cdot b := \varphi(a)m\varphi(b), \quad \text{for all } a, b \in A \text{ and } m \in M.$$

We will write ${}^\varphi f^\varphi$ for the translate of a morphism f under the action of $\varphi \in G$.

Let $\mathcal{X}$ denote the category whose objects are A - A -bimodules and morphisms between A - A -bimodules M and N are defined by

$$\text{Hom}_{\mathcal{X}}(M, N) := \bigoplus_{\varphi \in G} \text{Hom}_{A-A}(M, {}^\varphi N^\varphi).$$

An element $f \in \text{Hom}_{\mathcal{X}}(M, N)$ is thus represented by a tuple $(f_\varphi)_{\varphi \in G}$, where the component f_φ is in $\text{Hom}_{A-A}(M, {}^\varphi N^\varphi)$. For any $f \in \text{Hom}_{\mathcal{X}}(M, N)$ and $g \in \text{Hom}_{\mathcal{X}}(N, K)$, considering

$$\begin{aligned} \text{Hom}_{A-A}(N, {}^\psi K^\psi) \otimes \text{Hom}_{A-A}(M, {}^\varphi N^\varphi) &\rightarrow \text{Hom}_{A-A}(M, {}^{\varphi\psi} K^{\varphi\psi}), \\ g_\psi \otimes f_\varphi &\mapsto {}^\varphi(g_\psi)^\varphi \circ f_\varphi, \end{aligned}$$

where we use $\varphi\psi = \psi\varphi$ on the right-hand side, the composition $g \circ f$ is given by

$$(1) \quad \begin{aligned} \text{Hom}_{\mathcal{X}}(N, K) \otimes \text{Hom}_{\mathcal{X}}(M, N) &\rightarrow \text{Hom}_{\mathcal{X}}(M, K), \\ (g_\psi)_{\psi \in G} \otimes (f_\varphi)_{\varphi \in G} &\mapsto \left(\sum_{\varphi \in G} {}^\varphi(g_{\sigma\varphi^{-1}})^\varphi \circ f_\varphi \right)_{\sigma \in G}. \end{aligned}$$

This composition can be depicted by the diagram

$$M \xrightarrow{(f_\varphi)_{\varphi \in G}} \bigoplus_{\varphi \in G} {}^\varphi N^\varphi \xrightarrow{({}^\varphi(g_{\sigma\varphi^{-1}})^\varphi)_{\varphi, \sigma \in G}} \bigoplus_{\sigma \in G} {}^\sigma K^\sigma.$$

We refer to [Cibils and Marcos 2006] for details. In $\mathcal{X}$, we have an isomorphism

$$M \cong {}^\varphi M^\varphi,$$

for all $\varphi \in G$, since id_M can appear in the φ^{-1} -component of $\text{Hom}_{\mathcal{X}}(M, {}^\varphi M^\varphi)$. Furthermore, there is a faithful embedding of $\mathcal{X}$ into the category of all A - A -bimodules by sending an object $M \in \mathcal{X}$ to the A - A -bimodule $\bigoplus_{\varphi \in G} {}^\varphi M^\varphi$ and each morphism $f = (f_\sigma)_{\sigma \in G} \in \text{Hom}_{\mathcal{X}}(M, N)$ to the A - A -bimodule homomorphism $({}^\varphi(f_{\varphi^{-1}\psi})^\varphi)_{\varphi, \psi \in G}$ in $\text{Hom}_{A-A}(\bigoplus_{\varphi \in G} {}^\varphi M^\varphi, \bigoplus_{\psi \in G} {}^\psi N^\psi)$.

We denote by $\tilde{\mathcal{X}}$ the idempotent completion of $\mathcal{X}$, i.e., an object of $\tilde{\mathcal{X}}$ is given by a pair (M, e) , where M is an A - A -bimodule and e is an idempotent in $\text{End}_{\mathcal{X}}(M)$. For an A - A -bimodule M , set

$$G_M := \{\varphi \in G \mid M \cong {}^\varphi M^\varphi\}$$

which is a subgroup of G .

Remark 1. As we will often encounter and use in this article, computation of homomorphism in $\tilde{\mathcal{X}}$ using homomorphisms in $\mathcal{X}$ requires care. Given M and M' in $\mathcal{X}$ and idempotents e and e' in $\text{End}_{\mathcal{X}}(M)$ and $\text{End}_{\mathcal{X}}(M')$, respectively, in the computation of $\text{Hom}_{\tilde{\mathcal{X}}}((M, e), (M', e'))$ using $\text{Hom}_{\mathcal{X}}(M, M')$ it is very important to make sure that the elements from $\text{Hom}_{\mathcal{X}}(M, M')$ one works with do belong to $e' \circ \text{Hom}_{\mathcal{X}}(M, M') \circ e$. This is usually achieved by pre- and postcomposing the elements one works with e and e' , respectively. Moreover, for any element f in $\text{Hom}_{\tilde{\mathcal{X}}}((M, e), (M', e'))$, we have

$$(2) \quad f \circ \text{id}_{(M, e)} = f = \text{id}_{(M', e')} \circ f,$$

where, in fact, $\text{id}_{(M, e)} = e$ and $\text{id}_{(M', e')} = e'$.

As usual, we denote by $\widehat{G}$ the *Pontryagin dual* of G whose elements are all group homomorphisms from G to $\mathbb{k}^*$ with respect to pointwise multiplication. As G is finite and abelian, the group $\widehat{G}$ is (noncanonically) isomorphic to G and $\widehat{\widehat{G}}$ is canonically isomorphic to the group of isomorphism classes of simple G -modules with respect to taking tensor products.

The group algebra $\mathbb{k}[G]$ is commutative and semisimple and admits a unique decomposition into a product of $|G|$ copies of $\mathbb{k}$. Let $\{\pi_\chi \mid \chi \in \widehat{G}\}$ be the corresponding primitive idempotents. Each π_χ has the form $\frac{1}{|G|} \sum_{\alpha \in G} \chi(\alpha) \alpha$ and hence defines an idempotent $\tilde{\pi}_\chi$ in $\text{End}_{\mathcal{X}}(A)$ given by the tuple $((\chi(\alpha)/|G|)\alpha)_{\alpha \in G}$.

For an arbitrary subgroup H of G , we have a natural surjection $\widehat{G} \rightarrow \widehat{H}$ given by restriction. For $\zeta \in \widehat{H}$ and $\chi \in \widehat{H}$, we define $\chi\zeta \in \widehat{H}$ via $\chi\zeta(\alpha) := \chi(\alpha)\zeta(\alpha)$, for $\alpha \in H$.

Lemma 2. (i) *Let M be an indecomposable A - A -bimodule. Then there is an isomorphism of algebras*

$$\text{End}_{\mathcal{X}}(M)/\text{Rad}(\text{End}_{\mathcal{X}}(M)) \cong \mathbb{k}[G_M]/\text{Rad}(\mathbb{k}[G_M]) \cong \mathbb{k}[G_M].$$

(ii) *Indecomposable objects of $\tilde{\mathcal{X}}$ are of the form $M_{\varepsilon_\chi} := (M, \varepsilon_\chi)$, where M is an indecomposable A - A -bimodule and $\chi \in \widehat{G}_M$. Here, for $\alpha \in G$, the α -component of ε_χ is $(\chi(\alpha)/|G_M|)\alpha$, if $\alpha \in G_M$, and zero otherwise.*

Proof. Note that, for $\alpha \in G$, if ${}^\alpha M^\alpha$ is not isomorphic to M , then the α -component of any endomorphism of M belongs to the radical of $\text{End}_{\mathcal{X}}(M)$. Therefore claim (i) follows from (1). Claim (ii) follows from (1) and the definitions. $\square$

The category of all A - A -bimodules has a natural monoidal structure given by the tensor product over A . We define a tensor product on $\mathcal{X}$ by

$$M \otimes_{\mathcal{X}} N := \bigoplus_{\varphi \in G} (M \otimes_A {}^\varphi N^\varphi), \quad \text{for any } A\text{-}A\text{-bimodules } M \text{ and } N,$$

$$f \otimes_{\mathcal{X}} g := (f_\alpha \otimes {}^\gamma (g_{\beta\gamma^{-1}})^\gamma)_{\alpha, \beta, \gamma \in G},$$

$$\text{where } f_\alpha \otimes {}^\gamma (g_{\beta\gamma^{-1}})^\gamma : M \otimes_A {}^\gamma N^\gamma \rightarrow {}^\alpha (M')^\alpha \otimes_A {}^\beta (N')^\beta,$$

for any A - A -bimodules M, M', N and N' and morphisms

$$f = (f_\alpha)_{\alpha \in G} \in \text{Hom}_{\mathcal{X}}(M, M'), \quad g = (g_\beta)_{\beta \in G} \in \text{Hom}_{\mathcal{X}}(N, N').$$

Note that there is no identity object with respect to the tensor product $\otimes_{\mathcal{X}}$ unless G is trivial. In general, $\otimes_{\mathcal{X}}$ does not define a monoidal structure on $\mathcal{X}$. However, we will see in Propositions 6 and 7 that $\tilde{\mathcal{X}}$ has an identity given by tensoring with $(A, \tilde{\pi}_{1_{\widehat{G}}})$. The asymmetry of the above definition is only notational as the following lemma shows.

Lemma 3. *In the category $\mathcal{X}$, there is an isomorphism*

$$\bigoplus_{\varphi \in G} (M \otimes_A {}^\varphi N^\varphi) \cong \bigoplus_{\varphi \in G} ({}^{\varphi^{-1}} M^{\varphi^{-1}} \otimes_A N).$$

Proof. We first note that the map $m \otimes n \mapsto m \otimes n$ gives rise to an isomorphism of A - A -bimodules from $M \otimes_A {}^\varphi N$ to $M^{\varphi^{-1}} \otimes_A N$. Thus we have an isomorphism

$$(3) \quad {}^{\varphi^{-1}} M^{\varphi^{-1}} \otimes_A N \cong {}^{\varphi^{-1}} (M \otimes_A {}^\varphi N^\varphi) {}^{\varphi^{-1}}$$

of A - A -bimodules. We hence have an isomorphism

$${}^{\varphi^{-1}} M^{\varphi^{-1}} \otimes_A N \xrightarrow{(f_\psi)_{\psi \in G}} M \otimes_A {}^\varphi N^\varphi,$$

where $f_{\varphi^{-1}}$ is given by (3) and the remaining components are zero. $\square$

Remark 4. Under the isomorphism provided by Lemma 3, the morphism $f \otimes_{\mathcal{X}} g$ in $\text{Hom}_{\mathcal{X}}(M \otimes_{\mathcal{X}} N, M' \otimes_{\mathcal{X}} N')$ has components of the form

$${}^{\gamma}(f_{\alpha\gamma^{-1}})^{\gamma} \otimes g_{\beta} : {}^{\gamma}M^{\gamma} \otimes_A N \rightarrow {}^{\alpha}(M')^{\alpha} \otimes_A {}^{\beta}(N')^{\beta}.$$

Lemma 5. (i) *The operation $\otimes_{\mathcal{X}}$ is bifunctorial.*

(ii) *If e and f are idempotents in $\mathcal{X}$, then so is $e \otimes_{\mathcal{X}} f$. Hence $\otimes_{\mathcal{X}}$ extends to a bifunctor $\otimes_{\tilde{\mathcal{X}}} : \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{X}}$ given by $(M, e) \otimes_{\tilde{\mathcal{X}}} (N, f) = (M \otimes_{\mathcal{X}} N, e \otimes_{\mathcal{X}} f)$.*

Proof. Let M, N, K, L, X and Y be objects in $\mathcal{X}$. Let $f : M \rightarrow K$, $g : N \rightarrow L$, $h : K \rightarrow X$ and $l : L \rightarrow Y$ be morphisms in $\mathcal{X}$ with their only nonzero components being f_{α} , $g_{\gamma\varphi^{-1}}$, $h_{\beta\alpha^{-1}}$ and $l_{\delta\gamma^{-1}}$, respectively. Consider the diagram

$$\begin{array}{ccc} M & \otimes_A & {}^{\varphi}N^{\varphi} \\ f_{\alpha} \downarrow & & \downarrow {}^{\varphi}(g_{\gamma\varphi^{-1}})^{\varphi} \\ {}^{\alpha}K^{\alpha} & \otimes_A & {}^{\gamma}L^{\gamma} \\ {}^{\alpha}(h_{\beta\alpha^{-1}})^{\alpha} \downarrow & & \downarrow {}^{\gamma}(l_{\delta\gamma^{-1}})^{\gamma} \\ {}^{\beta}X^{\beta} & \otimes_A & {}^{\delta}Y^{\delta} \end{array}$$

Bifunctoriality of $\otimes_A$ yields

$$\begin{aligned} (4) \quad & ({}^{\alpha}(h_{\beta\alpha^{-1}})^{\alpha} \circ f_{\alpha}) \otimes ({}^{\gamma}(l_{\delta\gamma^{-1}})^{\gamma} \circ {}^{\varphi}(g_{\gamma\varphi^{-1}})^{\varphi}) \\ & = ({}^{\alpha}(h_{\beta\alpha^{-1}})^{\alpha} \otimes {}^{\gamma}(l_{\delta\gamma^{-1}})^{\gamma}) \circ (f_{\alpha} \otimes {}^{\varphi}(g_{\gamma\varphi^{-1}})^{\varphi}). \end{aligned}$$

The left and the right side of (4) coincide with

$$(h \circ f)_{\beta} \otimes {}^{\varphi}(l \circ g)_{\delta\varphi^{-1}}^{\varphi} \quad \text{and} \quad {}^{\alpha}(h \otimes l)_{\beta\alpha^{-1}}^{\alpha} \circ (f \otimes g)_{\alpha},$$

respectively. As these are the only nonzero components in $(h \circ f) \otimes_{\mathcal{X}} (l \circ g)$ and $(h \otimes_{\mathcal{X}} l) \circ (f \otimes_{\mathcal{X}} g)$, respectively, and have the same source and target, we obtain

$$(5) \quad (h \circ f) \otimes_{\mathcal{X}} (l \circ g) = (h \otimes_{\mathcal{X}} l) \circ (f \otimes_{\mathcal{X}} g)$$

in this case. The general case of equality (5) follows by linearity, implying claim (i).

Claim (ii) follows from equality (5). $\square$

Proposition 6. *For any $\chi, \zeta \in \widehat{G}$, we have*

$$(6) \quad (A, \tilde{\pi}_{\chi}) \otimes_{\tilde{\mathcal{X}}} (A, \tilde{\pi}_{\zeta}) \cong (A, \tilde{\pi}_{\chi\zeta}).$$

Proof. We start by constructing a morphism f from the right-hand side of (6) to the left-hand side, which is defined as follows:

$$f := (f_{\sigma, \tau})_{\sigma, \tau \in G} : A \rightarrow \bigoplus_{\sigma, \tau \in G} {}^{\sigma}A^{\sigma} \otimes_A {}^{\tau}A^{\tau},$$

where $f_{\sigma,\tau}$ is given by

$$(7) \quad 1 \mapsto \frac{1}{|G|^2} \chi(\sigma) \zeta(\tau) (1 \otimes 1) \in {}^\sigma A^\sigma \otimes_A {}^\tau A^\tau.$$

Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\pi}_{\chi\zeta}} & \bigoplus_{\beta \in G} {}^\beta A^\beta \\ \downarrow (f_{\sigma,\tau})_{\sigma,\tau \in G} & & \downarrow ({}^\beta(f_{\alpha\beta^{-1},\delta\beta^{-1}})_{\alpha\beta^{-1},\delta\beta^{-1} \in G}^\beta)_{\beta \in G} \\ \bigoplus_{\sigma,\tau \in G} {}^\sigma A^\sigma \otimes_A {}^\tau A^\tau & \xrightarrow{({}^\sigma(\tilde{\pi}_\chi \otimes_\chi \tilde{\pi}_\zeta)^\sigma)_{\sigma \in G}} & \bigoplus_{\alpha,\delta \in G} {}^\alpha A^\alpha \otimes_A {}^\delta A^\delta \end{array}$$

By definition, the $\tau\sigma^{-1}$, $\alpha\sigma^{-1}$, $\delta\sigma^{-1}$ -component of $\tilde{\pi}_\chi \otimes_\chi \tilde{\pi}_\zeta$ is multiplication by the scalar

$$(8) \quad \frac{1}{|G|^2} \chi(\alpha\sigma^{-1}) \zeta(\delta\tau^{-1}) = \frac{1}{|G|^2} \chi(\alpha) \chi(\sigma^{-1}) \zeta(\delta) \zeta(\tau^{-1}).$$

Now we compute the components of the two compositions $(\tilde{\pi}_\chi \otimes_\chi \tilde{\pi}_\zeta) \circ f$ (corresponding to the path going down and then right) and $f \circ \tilde{\pi}_{\chi\zeta}$ (corresponding to the path going right and then down) in our diagram which end up in a specific ${}^\alpha A^\alpha \otimes_A {}^\delta A^\delta$. One way around, using (7) and (8), we obtain that 1 is sent to

$$\begin{aligned} \frac{1}{|G|^4} \sum_{\sigma,\tau \in G} \chi(\sigma) \chi(\alpha) \chi(\sigma^{-1}) \zeta(\tau) \zeta(\delta) \zeta(\tau^{-1}) (1 \otimes 1) &= \frac{|G|^2}{|G|^4} \chi(\alpha) \zeta(\delta) (1 \otimes 1) \\ &= \frac{1}{|G|^2} \chi(\alpha) \zeta(\delta) (1 \otimes 1). \end{aligned}$$

The other way around, using (7) we obtain that 1 is sent to

$$\frac{1}{|G|^3} \sum_{\beta \in G} \chi \zeta(\beta) \chi(\alpha\beta^{-1}) \zeta(\delta\beta^{-1}) (1 \otimes 1) = \frac{1}{|G|^2} \chi(\alpha) \zeta(\delta) (1 \otimes 1).$$

Hence the diagram commutes and, moreover, (2) is satisfied. Thus f represents a morphism from the right-hand side of (6) to the left-hand side.

We proceed by constructing a morphism g from the left-hand side of (6) to the right-hand side. Consider the diagram

$$\begin{array}{ccc} \bigoplus_{\alpha \in G} (A \otimes_A {}^\alpha A^\alpha) & \xrightarrow{\tilde{\pi}_\chi \otimes_\chi \tilde{\pi}_\zeta} & \bigoplus_{\gamma,\delta \in G} {}^\gamma A^\gamma \otimes_A {}^\delta A^\delta \\ \downarrow \frac{1}{|G|} (\chi(\beta) \zeta(\beta\alpha^{-1}))_{\alpha,\beta \in G} & & \downarrow \frac{1}{|G|} (\chi(\sigma\gamma^{-1}) \zeta(\sigma\delta^{-1}))_{\delta\gamma^{-1},\sigma\gamma^{-1} \in G}^\gamma \\ \bigoplus_{\beta \in G} {}^\beta A^\beta & \xrightarrow{({}^\beta \tilde{\pi}_{\chi\zeta})_{\beta \in G}} & \bigoplus_{\sigma \in G} {}^\sigma A^\sigma \end{array}$$

whose vertical part defines $\mathbf{g}$, whose α, β -component sends $1 \otimes 1$ to $\frac{1}{|G|} \chi(\beta) \zeta(\beta \alpha^{-1}) 1$. For fixed $\alpha, \sigma \in G$, going one way around, using (8) we obtain

$$\sum_{\gamma, \delta \in G} \frac{1}{|G|^3} \chi(\gamma) \zeta(\delta \alpha^{-1}) \chi(\sigma \gamma^{-1}) \zeta(\sigma \delta^{-1}) = \frac{1}{|G|} \chi(\sigma) \zeta(\sigma \alpha^{-1}).$$

The other way around yields

$$\frac{1}{|G|^2} \sum_{\beta \in G} \chi(\beta) \zeta(\beta \alpha^{-1}) \chi \zeta(\sigma \beta^{-1}) = \frac{1}{|G|} \chi(\sigma) \zeta(\sigma \alpha^{-1})$$

which implies that the diagram commutes and, moreover,

$$\mathbf{g} \circ (\tilde{\pi}_\chi \otimes_\chi \tilde{\pi}_\zeta) = \tilde{\pi}_{\chi\zeta} \circ \mathbf{g} = \mathbf{g}.$$

Now we claim that both compositions $\mathbf{f} \circ \mathbf{g}$ and $\mathbf{g} \circ \mathbf{f}$ are the identities, i.e., of the respective idempotents. The φ -component of the composition $\mathbf{g} \circ \mathbf{f}$ sends 1 to

$$\frac{1}{|G|^3} \sum_{\sigma, \tau \in G} \chi(\sigma) \zeta(\tau) \chi(\varphi \sigma^{-1}) \zeta(\varphi \tau^{-1}) = \frac{1}{|G|} \chi \zeta(\varphi).$$

The α, σ, τ -component of the composition $\mathbf{f} \circ \mathbf{g}$ sends $1 \otimes 1$ to

$$\frac{1}{|G|^3} \sum_{\beta \in G} \chi(\beta) \zeta(\beta \alpha^{-1}) \chi(\sigma \beta^{-1}) \zeta(\tau \beta^{-1}) (1 \otimes 1) = \frac{1}{|G|^2} \chi(\sigma) \zeta(\tau \alpha^{-1}) (1 \otimes 1).$$

The claim follows. $\square$

Proposition 7. *Let $i, j \in \{1, 2, \dots, k\}$ and $M = Ae_i \otimes_{\mathbb{k}} e_j A$. Let further $\chi \in \widehat{G}_M$ and $\zeta \in \widehat{G}$. Then*

$$(9) \quad (M, \varepsilon_\chi) \otimes_{\tilde{\chi}} (A, \tilde{\pi}_\zeta) \cong (M, \varepsilon_{\chi\zeta}).$$

Proof. We follow the proof of Proposition 6. We start by constructing a morphism $\mathbf{f}$ from the right-hand side of (9) to the left-hand side. Consider the morphism

$$\mathbf{f} := (f_{\sigma, \tau})_{\sigma, \tau \in G} : M \rightarrow \bigoplus_{\sigma, \tau \in G} {}^\sigma M^\sigma \otimes_A {}^\tau A^\tau,$$

where $f_{\sigma, \tau}$ is given by

$$e_i \otimes e_j \mapsto \frac{1}{|G_M||G|} \chi(\sigma) \zeta(\tau) (\sigma(e_i) \otimes \sigma(e_j) \otimes 1) \in {}^\sigma M^\sigma \otimes_A {}^\tau A^\tau.$$

if $\sigma \in G_M$ and zero otherwise. Consider the diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\varepsilon_{\chi\zeta}} & \bigoplus_{\beta \in G} {}^\beta M^\beta \\
 \downarrow (f_{\sigma,\tau})_{\sigma,\tau \in G} & & \downarrow ({}^\beta(f_{\alpha\beta^{-1},\delta\beta^{-1}})_{\alpha\beta^{-1},\delta\beta^{-1} \in G}^\beta)_{\beta \in G} \\
 \bigoplus_{\sigma,\tau \in G} {}^\sigma M^\sigma \otimes_A {}^\tau A^\tau & \xrightarrow{(\sigma(\varepsilon_\chi \otimes_\chi \tilde{\pi}_\zeta)^\sigma)_{\sigma \in G}} & \bigoplus_{\alpha,\delta \in G} {}^\alpha M^\alpha \otimes_A {}^\delta A^\delta
 \end{array}$$

By definition, the $\tau\sigma^{-1}$, $\alpha\sigma^{-1}$, $\delta\sigma^{-1}$ -component of $\varepsilon_\chi \otimes_\chi \tilde{\pi}_\zeta$ sends $e_i \otimes e_j \otimes 1$ to

$$\frac{1}{|G_M||G|} \chi(\alpha) \chi(\sigma^{-1}) \zeta(\delta) \zeta(\tau^{-1}) (\alpha\sigma^{-1}(e_i) \otimes \alpha\sigma^{-1}(e_j) \otimes 1)$$

if $\alpha\sigma^{-1} \in G_M$ and zero otherwise. Now, going to the right and then down, the α , δ -component of the composition $f \circ \varepsilon_{\chi\zeta}$ sends $e_i \otimes e_j$ to

$$\frac{1}{|G_M||G|} \chi(\alpha) \zeta(\delta) (\alpha(e_i) \otimes \alpha(e_j) \otimes 1),$$

if α is in G_M , and zero otherwise. Going down and then to the right, the α , δ -component of $(\varepsilon_\chi \otimes_\chi \tilde{\pi}_\zeta) \circ f$ gives the same result, which also equals $f_{\alpha,\delta}$.

To construct a morphism g from the left-hand side of (9) to the right-hand side, consider the diagram

$$\begin{array}{ccc}
 \bigoplus_{\alpha \in G} (M \otimes_A {}^\alpha A^\alpha) & \xrightarrow{\varepsilon_\chi \otimes_\chi \tilde{\pi}_\zeta} & \bigoplus_{\gamma,\delta \in G} {}^\gamma M^\gamma \otimes_A {}^\delta A^\delta \\
 \downarrow (g_{\beta,\alpha})_{\alpha,\beta \in G} & & \downarrow ({}^\gamma(g_{\sigma\gamma^{-1},\delta\gamma^{-1}})_{\delta\gamma^{-1},\sigma\gamma^{-1} \in G}^\gamma)_{\delta \in G} \\
 \bigoplus_{\beta \in G} {}^\beta M^\beta & \xrightarrow{({}^\beta \varepsilon_{\chi\zeta})_{\beta \in G}} & \bigoplus_{\sigma \in G} {}^\sigma M^\sigma
 \end{array}$$

where $g_{\beta,\alpha}$ sends $e_i \otimes e_j \otimes 1 \in M \otimes_A {}^\alpha A^\alpha$ to $\frac{1}{|G_M|} \chi(\beta) \zeta(\beta\alpha^{-1}) (\beta(e_i) \otimes \beta(e_j)) \in {}^\beta M^\beta$, if $\beta \in G_M$, and to zero otherwise.

For fixed $\alpha, \sigma \in G$, going one way around we obtain the map which sends $e_i \otimes e_j \otimes 1$ to

$$\frac{1}{|G_M|} \chi(\sigma) \zeta(\sigma\alpha^{-1}) (\sigma(e_i) \otimes \sigma(e_j)),$$

if $\sigma \in G_M$, and to zero otherwise, which coincides with $g_{\sigma,\alpha}$. The other way around gives the same result.

Checking that both compositions $f \circ g$ and $g \circ f$ are the respective idempotents is similar to the proof of Proposition 6. $\square$

3.2. Tensoring symmetric bimodules with A -modules.

Proposition 8. (i) *There is a bifunctor $\otimes^{(r)} : \text{mod-}A \times \mathcal{X} \rightarrow \text{mod-}A$ defined by*

$$(V, M) \mapsto V \otimes^{(r)} M := \bigoplus_{\varphi \in G} V \otimes_A {}^\varphi M^\varphi,$$

$$(f, g) \mapsto f \otimes^{(r)} g := \left(f \otimes {}^\varphi (g_{\alpha\varphi^{-1}})^\varphi \right)_{\varphi, \alpha \in G}.$$

(ii) *There is a bifunctor $\otimes^{(l)} : \mathcal{X} \times A\text{-mod} \rightarrow A\text{-mod}$ defined by*

$$(M, V) \mapsto M \otimes^{(l)} V := \bigoplus_{\varphi \in G} {}^\varphi M^\varphi \otimes_A V,$$

$$(g, f) \mapsto g \otimes^{(l)} f := \left({}^\varphi (g_{\alpha\varphi^{-1}})^\varphi \otimes f \right)_{\alpha, \varphi \in G}.$$

Proof. We note that we use Lemma 3 for the formulation of claim (ii). The proof of both claims is similar to the proof of Lemma 5(i). $\square$

Proposition 9. (i) *The bifunctor $\otimes^{(r)}$ induces a bifunctor $\text{mod-}A \times \tilde{\mathcal{X}} \rightarrow \text{mod-}A$ (which we will denote by the same symbol, abusing notation).*

(ii) *The bifunctor $\otimes^{(l)}$ induces a bifunctor $\tilde{\mathcal{X}} \times A\text{-mod} \rightarrow A\text{-mod}$ (which we will denote by the same symbol, abusing notation).*

Proof. Let $(M, e) \in \tilde{\mathcal{X}}$. Then, for any $V \in \text{mod-}A$, the endomorphism $\text{id}_V \otimes^{(r)} e$ is an idempotent endomorphism of $V \otimes^{(r)} M$, so we can define $V \otimes^{(r)} (M, e)$ as the image of this idempotent. It is easy to check that this does the job for claim (i). Claim (ii) is similar. $\square$

3.3. The 2-category $\mathcal{G}_A$ of projective symmetric bimodules. Assume that we are in the setup of Section 3.1. Let

$$A = A_1 \times A_2 \times \cdots \times A_n$$

be the (unique up to permutation of factors) decomposition of A into a direct product of indecomposable algebras. Assume that the action of each $\varphi \in G$ preserves each A_i . Also assume that none of the A_i is simple. We also consider the algebra $B := A \times \mathbb{k}$ which will play a crucial role in the proof of Theorem 17.

For each $i \in \{1, 2, \dots, n\}$, fix a small category $\mathcal{C}_i$ equivalent to $A_i\text{-proj}$. Define the 2-category $\mathcal{G}_A$ to have

- objects $1, 2, \dots, n$, where we identify $\mathbf{i}$ with $\mathcal{C}_i$;
- 1-morphisms are endofunctors of $\mathcal{C} := \coprod_{\mathbf{i}} \mathcal{C}_i$ isomorphic to functors $X \otimes^{(l)} \_$, where X is in the additive closure of $(A \oplus (A \otimes_{\mathbb{k}} A), \text{id}_{A \oplus (A \otimes_{\mathbb{k}} A)})$ inside $\tilde{\mathcal{X}}$;
- 2-morphisms are given by morphisms between X and X' in $\tilde{\mathcal{X}}$;
- horizontal composition is just composition of functors;

- vertical composition is inherited from $\tilde{\mathcal{X}}$;
- the identity 1-morphism in $\mathcal{G}_A(\mathbf{i}, \mathbf{i})$ is isomorphic to $(A_i, \tilde{\pi}_{1_{\widehat{G}}}) \otimes^{(l)} \_$.

Note that the restriction on $\text{char}(\mathbb{k})$ as not dividing the order of G is necessary to have identity 1-morphisms. Observe further that $A \oplus (A \otimes_{\mathbb{k}} A)$ is invariant, up to isomorphism, under the functor $M \mapsto {}^\varphi M^\varphi$, for any $\varphi \in G$. The fact that this defines a 2-category is justified by Proposition 7, showing that $(A_i, \tilde{\pi}_{1_{\widehat{G}}}) \otimes^{(l)} \_$ is indeed an identity, and the following lemma.

Lemma 10. *Let X and Y be in the additive closure of $(A \oplus (A \otimes_{\mathbb{k}} A), \text{id}_{A \oplus (A \otimes_{\mathbb{k}} A)})$ inside $\tilde{\mathcal{X}}$. Then there is an isomorphism*

$$(X \otimes^{(l)} \_) \circ (Y \otimes^{(l)} \_) \cong (X \otimes_{\tilde{\mathcal{X}}} Y) \otimes^{(l)} \_$$

of endofunctors of $\mathcal{C}$.

Proof. First we assume that X and Y are in $\mathcal{X}$. Then, for any $P \in \mathcal{C}$, we have

$$(X \otimes_{\mathcal{X}} Y) \otimes^{(l)} P = \left(\bigoplus_{\varphi \in G} X \otimes_A {}^\varphi Y^\varphi \right) \otimes^{(l)} P = \bigoplus_{\varphi, \psi \in G} {}^\psi (X \otimes_A {}^\varphi Y^\varphi) {}^\psi \otimes_A P$$

and

$$X \otimes^{(l)} (Y \otimes^{(l)} P) = X \otimes^{(l)} \left(\bigoplus_{\varphi \in G} {}^\varphi Y^\varphi \otimes_A P \right) = \bigoplus_{\varphi, \psi \in G} {}^\psi X {}^\psi \otimes_A {}^\varphi Y^\varphi \otimes_A P.$$

Choosing an isomorphism

$$\bigoplus_{\varphi, \psi \in G} {}^\psi (X \otimes_A {}^\varphi Y^\varphi) {}^\psi \cong \bigoplus_{\varphi, \psi \in G} {}^\psi X {}^\psi \otimes_A {}^\varphi Y^\varphi$$

of A - A -bimodules yields the desired isomorphism of functors.

Now, let e and f be idempotents in $\text{End}_{\mathcal{X}}(X)$ and $\text{End}_{\mathcal{X}}(Y)$, respectively. Consider

$$\begin{aligned} (e \otimes_{\mathcal{X}} f) \otimes^{(l)} \text{id}_P &= ({}^\gamma (e_{\alpha\gamma^{-1}})^\gamma \otimes f_\beta)_{\alpha, \beta, \gamma \in G} \otimes^{(l)} \text{id}_P \\ &= ({}^\delta ({}^\gamma (e_{\alpha\gamma^{-1}})^\gamma \otimes f_\beta)_{\beta\delta^{-1}}^\delta \otimes \text{id}_P)_{\alpha, \beta, \gamma, \delta \in G} \\ &= ({}^\gamma (e_{\alpha\gamma^{-1}})^\gamma \otimes {}^\delta (f_{\beta\delta^{-1}})^\delta \otimes \text{id}_P)_{\alpha, \beta, \gamma, \delta \in G} \end{aligned}$$

and

$$\begin{aligned} e \otimes^{(l)} (f \otimes^{(l)} \text{id}_P) &= e \otimes^{(l)} ({}^\psi (f_{\varphi\psi^{-1}})^\psi \otimes \text{id}_P)_{\varphi, \psi \in G} \\ &= ({}^\tau (e_{\sigma\tau^{-1}})^\tau \otimes {}^\psi (f_{\varphi\psi^{-1}})^\psi \otimes \text{id}_P)_{\sigma, \tau, \varphi, \psi \in G}, \end{aligned}$$

we see that

$$(e \otimes_{\mathcal{X}} f) \otimes^{(l)} \text{id}_P = e \otimes^{(l)} (f \otimes^{(l)} \text{id}_P)$$

and hence the isomorphism in the previous paragraph descends to the summands (X, e) and (Y, f) . $\square$

The 2-category $\mathcal{G}_B$ is defined similarly. To distinguish the underlying categories of bimodules, we use the notation $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ for the corresponding categories of symmetric B - B -bimodules.

3.4. Two-sided cells in $\mathcal{G}_A$. We recall the notation introduced just before Lemma 2.

Proposition 11. *The 2-category $\mathcal{G}_A$ has $n + 1$ two-sided cells, namely*

- (a) *for $i = 1, 2, \dots, n$, the two-sided cell $\mathcal{J}_i$ consisting of $|G|$ elements $(\mathbb{1}_i, \tilde{\pi}_\varphi)$, where $\varphi \in G$,*
- (b) *the two-sided cell $\mathcal{J}_0$ consisting of all isomorphism classes of indecomposable 1-morphisms in the additive closure of $(A \otimes_{\mathbb{k}} A, \text{id}_{A \otimes_{\mathbb{k}} A})$ inside $\tilde{\mathcal{X}}$.*

Proof. Since tensor products in which one of the factors is a projective bimodule never contain a copy of the regular bimodule as a direct summand, the existence of two-sided cells as claimed in part (i) follows from Proposition 6. To complete the proof of the proposition, it remains to show that all isomorphism classes of indecomposable 1-morphisms in the additive closure of $(A \otimes_{\mathbb{k}} A, \text{id}_{A \otimes_{\mathbb{k}} A})$ inside $\tilde{\mathcal{X}}$ belong to the same two-sided cell. Ignoring idempotents in $\mathcal{X}$, the claim follows directly from [Mazorchuk and Miemietz 2016d, Subsection 5.1]. In full generality, the statement is then proved using Proposition 7. $\square$

3.5. Adjunctions. In this subsection, we study adjunctions in the 2-category $\mathcal{G}_A$ under the assumption that A is self-injective. We assume that A is basic and that there is a fixed complete G -invariant set E of primitive idempotents. We denote by ν the bijection on E which is induced by the Nakayama automorphism of A given by

$$\text{Hom}_{\mathbb{k}}(eA, \mathbb{k}) \cong A\nu(e), \quad \text{for } e \in E.$$

For a primitive idempotent $e \in A$, we denote by ε_e the idempotent in $\text{End}_{\mathcal{Y}}(Ae)$ or $\text{End}_{\mathcal{Y}}(eA)$ corresponding to the trivial character of $G_{A\nu(e)} = G_{Ae} = G_{eA} =: G_e$. We denote by $\mathbf{m}$ the multiplication map in B .

Proposition 12. *We have adjunctions*

- (a) $((Ae, \varepsilon_e), (eA, \varepsilon_e))$ in $\tilde{\mathcal{Y}}$;
- (b) $((eA, \varepsilon_e), (A\nu(e), \varepsilon_{\nu(e)}))$ in $\tilde{\mathcal{Y}}$.

Proof. We first define the counit

$$\epsilon : (Ae \otimes_{\mathcal{Y}} eA, \varepsilon_e \otimes_{\mathcal{Y}} \varepsilon_e) \rightarrow (A, \tilde{\pi}_{1_{\hat{G}}}).$$

This is defined by the vertical part of the diagram

$$\begin{array}{ccc}
 \bigoplus_{\varphi} Ae \otimes_{\mathbb{k}} {}^{\varphi}eA^{\varphi} & \xrightarrow{\frac{1}{|G_e|^2}(\alpha \otimes \beta \varphi^{-1})_{\alpha, \beta \varphi^{-1} \in G_e}} & \bigoplus_{\alpha, \beta} {}^{\alpha}Ae^{\alpha} \otimes_{\mathbb{k}} {}^{\beta}eA^{\beta} \\
 \downarrow \frac{1}{|G_e|}(\gamma \circ \mathbf{m})_{\gamma, \varphi \in G} & & \downarrow \frac{1}{|G_e|}(\delta \circ \mathbf{m} \circ (\alpha \otimes \alpha)^{-1})_{\delta, \alpha, \beta \in G} \\
 \bigoplus_{\gamma} {}^{\gamma}A^{\gamma} & \xrightarrow{\frac{1}{|G|}(\delta \gamma^{-1})_{\delta, \gamma \in G}} & \bigoplus_{\delta} {}^{\delta}A^{\delta}
 \end{array}$$

where here and in the rest of the proof all elements indexing direct sums run through G , and the notation $(\alpha \otimes \beta \varphi^{-1})_{\alpha, \beta \varphi^{-1} \in G_e}$ should be read as the (α, β, φ) -component of the map being defined as zero if the conditions $\alpha, \beta \varphi^{-1} \in G_e$ are not satisfied.

To check that the vertical arrows define a morphism $(Ae \otimes_{\mathbb{k}} {}^{\varphi}eA, \varepsilon_e \otimes_{\mathbb{k}} {}^{\varphi}\varepsilon_e) \rightarrow (A, \tilde{\pi}_{1_{\widehat{G}}})$, we need to verify that the diagram commutes and the result coincides with the original map, namely, satisfying (2). We consider the (δ, φ) -component of both compositions. First going to the right and then down, $e \otimes \varphi(e)$ is mapped to

$$\frac{1}{|G_e|^3} \sum_{\alpha \in G_e, \beta \in \varphi G_e} \delta(e \alpha^{-1} \beta(e)) = \begin{cases} \frac{1}{|G_e|} \delta(e) & \text{if } \varphi \in G_e, \\ 0 & \text{otherwise.} \end{cases}$$

The other way around, $e \otimes \varphi(e)$ is sent to

$$\frac{1}{|G_e| |G|} \sum_{\gamma \in G} \delta(e \varphi(e)) = \begin{cases} \frac{1}{|G_e|} \delta(e) & \text{if } \varphi \in G_e, \\ 0 & \text{otherwise,} \end{cases}$$

which coincides with the image of $(1/|G_e|)(\delta \circ \mathbf{m})_{\delta, \varphi \in G}$ on $e \otimes \varphi(e)$. So our counit ϵ is, indeed, well-defined.

We now define the unit

$$\eta : (\mathbb{k}, \tilde{\pi}_{1_{\widehat{G}}}) \rightarrow (eA \otimes_{\mathbb{k}} {}^{\varphi}eA, \varepsilon_e \otimes_{\mathbb{k}} {}^{\varphi}\varepsilon_e)$$

by the vertical part of the diagram

$$\begin{array}{ccc}
 \mathbb{k} & \xrightarrow{\frac{1}{|G|}(\varphi)_{\varphi \in G}} & \bigoplus_{\varphi} {}^{\varphi}\mathbb{k}^{\varphi} \\
 \downarrow (\eta_1^{\delta, \alpha})_{\delta \alpha^{-1} \in G_e} & & \downarrow (\eta_{\varphi}^{\beta, \gamma})_{\beta \gamma^{-1} \in G_e, \varphi \in G} \\
 \bigoplus_{\delta, \alpha} {}^{\delta}eA^{\delta} \otimes_A {}^{\alpha}Ae^{\alpha} & \xrightarrow{\frac{1}{|G_e|^2}(\beta \delta^{-1} \otimes \gamma \alpha^{-1})_{\beta \delta^{-1}, \gamma \alpha^{-1} \in G_e}} & \bigoplus_{\beta, \gamma} {}^{\beta}eA^{\beta} \otimes_A {}^{\gamma}Ae^{\gamma}
 \end{array}$$

where

$$(10) \quad \eta_{\varphi}^{\beta, \gamma}(\varphi(1)) = \begin{cases} \frac{1}{|G_e|} \beta(e) \otimes \gamma(e) & \text{if } \beta\gamma^{-1} \in G_e, \\ 0 & \text{otherwise.} \end{cases}$$

We again check that this defines a morphism $(\mathbb{k}, \tilde{\pi}_{1_{\tilde{G}}}) \rightarrow (eA \otimes_{\mathcal{Y}} Ae, \varepsilon_e \otimes_{\mathcal{Y}} \varepsilon_e)$ by verifying (2). Computing the (β, γ) -component of the path first going to the right and then down, we see that

$$1 \mapsto \begin{cases} \frac{1}{|G_e|} \beta(e) \otimes \gamma(e) & \text{if } \beta\gamma^{-1} \in G_e, \\ 0 & \text{otherwise,} \end{cases}$$

that is, $(\eta_1^{\beta, \gamma})_{\beta\gamma^{-1} \in G_e}$. The other way around,

$$1 \mapsto \sum_{\delta \in \beta G_e, \alpha \in \gamma G_e} \frac{1}{|G_e|^3} \beta(e) \otimes \gamma(e) = \begin{cases} \frac{1}{|G_e|} \beta(e) \otimes \gamma(e) & \text{if } \beta\gamma^{-1} \in G_e, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the condition $\delta\alpha^{-1} \in G_e$ is automatically satisfied for $\beta\gamma^{-1} \in G_e$ and $\delta \in \beta G_e, \alpha \in \gamma G_e$. Thus our unit η is well-defined as well.

Now we need to check the adjunction axioms. Denoting (Ae, ε_e) by F and (eA, ε_e) by G , we first verify

$$F \rightarrow F\mathbb{1}_j \rightarrow FGF \rightarrow \mathbb{1}_i F \rightarrow F$$

is the identity, for appropriate i and j . To this end, consider the commutative diagram

$$\begin{array}{ccc} Ae & \xrightarrow{\frac{1}{|G_e|}(\iota)_{\iota \in G_e}} & \bigoplus_{\iota} {}^{\iota}Ae^{\iota} \\ \downarrow \frac{1}{|G_e||G|}(\kappa \otimes \varphi(1))_{\kappa \in G_e, \varphi \in G} & & \downarrow \frac{1}{|G_e||G|}(\alpha \iota^{-1} \otimes \psi(1))_{\alpha \iota^{-1} \in G_e, \psi \in G} \\ \bigoplus_{\kappa, \varphi} {}^{\kappa}Ae^{\kappa} \otimes_{\mathbb{k}} \varphi \mathbb{k} \varphi & \xrightarrow{\frac{1}{|G_e||G|}(\alpha \kappa^{-1} \otimes \psi \varphi^{-1})_{\alpha \kappa^{-1} \in G_e, \psi, \varphi \in G}} & \bigoplus_{\alpha, \psi} {}^{\alpha}Ae^{\alpha} \otimes_{\mathbb{k}} \psi \mathbb{k} \psi \\ \downarrow \frac{1}{|G_e|}(\beta \kappa^{-1} \otimes \eta_{\varphi}^{\gamma, \delta})_{\beta \kappa^{-1}, \gamma \delta^{-1} \in G_e, \varphi \in G} & & \downarrow \frac{1}{|G_e|}(\lambda \alpha^{-1} \otimes \eta_{\psi}^{\mu, \nu})_{\lambda \alpha^{-1}, \mu \nu^{-1} \in G_e, \psi \in G} \\ \bigoplus_{\beta, \gamma, \delta} {}^{\beta}Ae^{\beta} \otimes_{\mathbb{k}} {}^{\gamma}eA^{\gamma} \otimes_A {}^{\delta}Ae^{\delta} & \xrightarrow{\frac{1}{|G_e|^3}(\lambda \beta^{-1} \otimes \mu \gamma^{-1} \otimes \nu \delta^{-1})} & \bigoplus_{\lambda, \mu, \nu} {}^{\lambda}Ae^{\lambda} \otimes_{\mathbb{k}} {}^{\mu}eA^{\mu} \otimes_A {}^{\nu}Ae^{\nu} \\ \downarrow \frac{1}{|G_e|^2}((\xi \beta^{-1} \circ m) \otimes \theta \delta^{-1})_{\theta \delta^{-1} \in G_e, \xi, \beta, \gamma \in G} & & \downarrow \frac{1}{|G_e|^2}((\sigma \lambda^{-1} \circ m) \otimes \tau \nu^{-1})_{\tau \nu^{-1} \in G_e, \sigma, \lambda, \mu \in G} \\ \bigoplus_{\xi, \theta} {}^{\xi}A^{\xi} \otimes_A {}^{\theta}Ae^{\theta} & \xrightarrow{\frac{1}{|G_e||G|}(\sigma \xi^{-1} \otimes \tau \theta^{-1})_{\tau \theta^{-1} \in G_e, \sigma, \xi \in G}} & \bigoplus_{\sigma, \tau} {}^{\sigma}A^{\sigma} \otimes_A {}^{\tau}Ae^{\tau} \\ \downarrow \frac{1}{|G_e||G|}(m \circ (\rho \xi^{-1} \otimes \rho \theta^{-1}))_{\rho \theta^{-1} \in G_e, \xi \in G} & & \downarrow \frac{1}{|G_e||G|}(m \circ (\omega \sigma^{-1} \otimes \omega \tau^{-1}))_{\omega \tau^{-1} \in G_e, \sigma \in G} \\ \bigoplus_{\rho} {}^{\rho}Ae^{\rho} & \xrightarrow{\frac{1}{|G_e|}(\omega \rho^{-1})_{\omega \rho^{-1} \in G_e}} & \bigoplus_{\omega} {}^{\omega}Ae^{\omega} \end{array}$$

where in the third horizontal arrow the conditions are $\lambda\beta^{-1}, \mu\gamma^{-1}, \nu\delta^{-1} \in G_e$.

We want the ρ -component of the composition on the left-hand side of the diagram to be given by $(1/|G_e|)\rho$, if $\rho \in G_e$, and by zero otherwise. To see this, first notice that multiplication $\mathbf{m}$ in the third map will give something nonzero only if $\beta\gamma^{-1} \in G_e$. Taking into account all conditions specified in the diagram, this forces $\kappa, \beta, \gamma, \delta, \theta, \rho \in G_e$ in order for the ρ -component to be nonzero. Each choice of $\kappa, \beta, \gamma, \delta, \theta \in G_e$ and $\varphi, \xi \in G$ yields a summand $(1/(|G|^2|G_e|^6))\rho$ in the composition (recall the factor $1/|G_e|$ in (10)). Summing over all these possibilities hence produces the desired result.

The fact that the composition

$$G \rightarrow \mathbb{1}_j G \rightarrow GFG \rightarrow G\mathbb{1}_i \rightarrow G$$

is the identity follows as above by flipping all tensor factors and replacing Ae by eA in appropriate places. This proves part (a).

Assume that $E = \{e_1, e_2, \dots, e_k\}$. For any $1 \leq i \leq k$, we choose a Jordan–Hölder series of each Ae_i by

$$Ae_i = X_{i,0} \supsetneq X_{i,1} \supsetneq X_{i,2} \supsetneq \dots \supsetneq X_{i,m_i} \supsetneq X_{i,m_i+1} = 0.$$

As our algebra A is basic, each $\mathbb{k}$ -space $X_{i,j}/X_{i,j+1}$, where $0 \leq j \leq m_i$, is of dimension one. For each i , we fix some basis

$$E_i := \{e_i, x_{i,j} \mid 1 \leq j \leq m_i\}$$

of Ae_i such that we have $x_{i,j} \in X_{i,j} \setminus X_{i,j+1}$, for every j . Then

$$A := \bigcup_{i=1}^k E_i \supset E$$

is a basis of A . Let $\mathbf{t} : A \rightarrow \mathbb{k}$ be the unique linear map such that, for all $a \in A$, we have

$$\mathbf{t}(a) = \begin{cases} 1, & a = x_{i,m_i}, \text{ for some } i, \\ 0, & \text{otherwise.} \end{cases}$$

For $a \in A$, we denote by a^* the unique element in A which satisfies

$$\mathbf{t}(ba^*) = \begin{cases} 1, & b = a, \\ 0, & b \in A \setminus \{a\}. \end{cases}$$

We now define the unit

$$\tilde{\eta} : (A, \tilde{\pi}_{1_{\tilde{G}}}) \rightarrow (Av(e) \otimes_{\mathcal{Y}} eA, \varepsilon_{v(e)} \otimes_{\mathcal{Y}} \varepsilon_e)$$

by the vertical part of the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\frac{1}{|G|}(\varphi)_{\varphi \in G}} & \bigoplus_{\varphi} {}^{\varphi}A^{\varphi} \\
 \downarrow (\tilde{\eta}_1^{\alpha, \beta})_{\alpha\beta^{-1} \in G_e} & & \downarrow (\tilde{\eta}_{\varphi}^{\gamma, \delta})_{\gamma\delta^{-1} \in G_e, \varphi \in G} \\
 \bigoplus_{\alpha, \beta} {}^{\alpha}Av(e)^{\alpha} \otimes_{\mathbb{k}} {}^{\beta}eA^{\beta} & \xrightarrow{\frac{1}{|G_e|^2}(\gamma\alpha^{-1} \otimes \delta\beta^{-1})_{\gamma\alpha^{-1}, \delta\beta^{-1} \in G_e}} & \bigoplus_{\gamma, \delta} {}^{\gamma}Av(e)^{\gamma} \otimes_{\mathbb{k}} {}^{\delta}eA^{\delta}
 \end{array}$$

where

$$(11) \quad \tilde{\eta}_{\varphi}^{\gamma, \delta}(\varphi(1)) = \begin{cases} \frac{1}{|G_e|} \sum_{a \in A} \gamma(a^*v(e)) \otimes \delta(ea) & \text{if } \gamma\delta^{-1} \in G_e, \\ 0 & \text{otherwise.} \end{cases}$$

Going right and then down, summing over $\varphi \in G$ cancels the scalar $1/|G|$ and hence the image of 1 in the (γ, δ) -component is given by the right-hand side of (11) and equals the (γ, δ) -component of $(\tilde{\eta}_1^{\gamma, \delta}(1))_{\gamma\delta^{-1} \in G_e}$, which implies the first equality of (2). Going down and then right, to obtain a nonzero contribution, we need $\alpha\beta^{-1}, \gamma\alpha^{-1}, \delta\beta^{-1} \in G_e$, which yields $\delta\gamma^{-1} \in G_e$. Summing over all such choices of α and β , the image of 1 in the (γ, δ) -component is again given by the right-hand side of (11), implying the second equality of (2). Hence $\tilde{\eta}$ is well-defined.

Now we define the counit

$$\tilde{\epsilon} : (eA \otimes_{\mathcal{Y}} Av(e), \varepsilon_e \otimes_{\mathcal{Y}} \varepsilon_{v(e)}) \rightarrow (\mathbb{k}, \tilde{\pi}_{1_{\tilde{G}}})$$

by the vertical part of the diagram

$$\begin{array}{ccc}
 \bigoplus_{\varphi} eA \otimes_A {}^{\varphi}Av(e)^{\varphi} & \xrightarrow{\frac{1}{|G_e|^2}(\rho \otimes \psi \varphi^{-1})_{\rho, \psi \varphi^{-1} \in G_e}} & \bigoplus_{\rho, \psi} {}^{\rho}eA^{\rho} \otimes_{\mathbb{k}} {}^{\psi}Av(e)^{\psi} \\
 \downarrow \frac{1}{|G_e|}(\alpha \circ t \circ m)_{\alpha, \varphi \in G} & & \downarrow \frac{1}{|G_e|}(\beta \circ t \circ m \circ (\rho \otimes \rho)^{-1})_{\beta, \psi, \rho \in G} \\
 \bigoplus_{\alpha} {}^{\alpha}\mathbb{k}^{\alpha} & \xrightarrow{\frac{1}{|G|}(\beta\alpha^{-1})_{\alpha, \beta \in G}} & \bigoplus_{\beta} {}^{\beta}\mathbb{k}^{\beta}.
 \end{array}$$

Consider the (β, φ) -component of both compositions. First going down and then to the right, the first map is zero, for $\varphi \notin G_e$, as $t(eAf) = 0$ unless $f = v(e)$. If $\varphi \in G_e$, then each α contributes $(1/(|G||G_e|))\beta \circ t \circ m$. Hence the resulting map is $(1/|G_e|)\beta \circ t \circ m$ and the second equality of (2) is satisfied. The right vertical map is zero unless $\psi\rho^{-1} \in G_e$ which, together with the conditions on the upper horizontal map, forces $\varphi, \rho, \psi \in G_e$. Summing over the choices for ρ and ψ , we obtain the same resulting map and thus the diagram commutes, in which case the first equality of (2) holds. Hence $\tilde{\epsilon}$ is well-defined.

We now verify the adjunction axioms. Denoting (eA, ε_e) by $\tilde{F}$ and $(Av(e), \varepsilon_{v(e)})$ by $\tilde{G}$, we need to show that

$$\tilde{F} \rightarrow \tilde{F}\mathbb{1}_i \rightarrow \tilde{F}\tilde{G}\tilde{F} \rightarrow \mathbb{1}_j\tilde{F} \rightarrow \tilde{F}$$

is the identity. To this end, we assemble our maps in a large commutative diagram as in part (a) and compute the left-hand side. This is given by the composition

$$\begin{array}{ccc}
 eA & \xrightarrow{\frac{1}{|G||G_e|}(\delta \otimes \varphi(1))_{\delta \in G_e, \varphi \in G}} & \bigoplus_{\delta, \varphi} {}^\delta eA^\delta \otimes_A {}^\varphi A^\varphi \\
 & & \downarrow \frac{1}{|G_e|}(\alpha \delta^{-1} \otimes \tilde{\eta}_\varphi^{\beta, \gamma})_{\alpha \delta^{-1}, \beta \gamma^{-1} \in G_e, \varphi \in G} \\
 \bigoplus_{\xi, \theta} {}^\xi \mathbb{k}^\xi \otimes_{\mathbb{k}} {}^\theta eA^\theta & \xleftarrow{\frac{1}{|G_e|^2}((\xi \circ t \circ m \circ (\alpha \otimes \alpha)^{-1}) \otimes \theta \gamma^{-1})} & \bigoplus_{\alpha, \beta, \gamma} {}^\alpha eA^\alpha \otimes_A {}^\beta Av(e)^\beta \otimes_{\mathbb{k}} {}^\gamma eA^\gamma \\
 & & \downarrow \frac{1}{|G_e||G|}(m \circ (\omega \xi^{-1} \otimes \omega \theta^{-1}))_{\omega \theta^{-1} \in G_e, \xi \in G} \\
 & & \bigoplus_{\omega} {}^\omega eA^\omega
 \end{array}$$

where in the third map the conditions are $\theta \gamma^{-1} \in G_e$ and $\alpha, \beta, \xi \in G$. However, the third map is zero unless $\alpha \beta^{-1} \in G_e$ (for the same reason involving t as used above), which, together with the other conditions in the diagram, shows that we have a nonzero contribution to the ω -component only if $\delta, \alpha, \beta, \gamma, \theta, \omega \in G_e$. For $\omega \in G_e$, the contribution of a fixed choice of $\delta, \varphi, \alpha, \beta, \gamma, \xi, \theta$ to the image of e is

$$\frac{1}{|G|^2|G_e|^6} \omega \left(\sum_{a \in A} t(e\beta\alpha^{-1}(a^*v(e)))ea \right).$$

Observing that, by our choice of A , we have

$$t(e\beta\alpha^{-1}(a^*v(e))) = \begin{cases} 1 & \text{if } a = e, \\ 0 & \text{otherwise,} \end{cases}$$

and summing over all choices of $\delta, \varphi, \alpha, \beta, \gamma, \xi, \theta$, we obtain that the image of e is $\frac{1}{|G_e|}\omega(e)$, as desired.

Now we verify the other axiom. The composition

$$\tilde{G} \rightarrow \mathbb{1}_i\tilde{G} \rightarrow \tilde{G}\tilde{F}\tilde{G} \rightarrow \tilde{G}\mathbb{1}_j \rightarrow \tilde{G}$$

is given by the diagram, which consists of the left-hand side of a large commutative diagram as in part (a),

$$\begin{array}{ccc}
 Av(e) & \xrightarrow{\frac{1}{|G||G_e|}(\alpha(1)\otimes\varphi)_{\alpha\in G, \varphi\in G_e}} & \bigoplus_{\alpha, \varphi} {}^\alpha A^\alpha \otimes_A {}^\varphi Av(e)^\varphi \\
 & & \downarrow \frac{1}{|G_e|}(\tilde{\eta}_\alpha^{\delta, \beta} \otimes \gamma \varphi^{-1})_{\delta\beta^{-1}, \gamma\varphi^{-1}\in G_e, \alpha\in G} \\
 \bigoplus_{\xi, \theta} {}^\theta Av(e)^\theta \otimes_{\mathbb{k}} {}^\xi \mathbb{k}^\xi & \xleftarrow{\frac{1}{|G_e|^2}(\theta\delta^{-1}\otimes(\xi\circ t\circ m\circ(\beta\otimes\beta)^{-1}))} & \bigoplus_{\delta, \beta, \gamma} {}^\delta Av(e)^\delta \otimes_{\mathbb{k}} {}^\beta eA^\beta \otimes_A {}^\gamma Av(e)^\gamma \\
 \downarrow \frac{1}{|G_e||G|}(m\circ(\omega\theta^{-1}\otimes\omega\xi^{-1}))_{\omega\theta^{-1}\in G_e, \xi\in G} & & \\
 \bigoplus_{\omega} {}^\omega Av(e)^\omega & &
 \end{array}$$

where in the third map the conditions are $\theta\delta^{-1} \in G_e$ and $\beta, \gamma, \xi \in G$. Note that the third map is zero unless $\beta\gamma^{-1} \in G_e$. Taking into account all conditions in the diagram, this shows that we have a nonzero contribution to the ω -component only if $\varphi, \gamma, \beta, \delta, \theta, \omega \in G_e$. For $\omega \in G_e$, the contribution of a fixed choice of $\alpha, \varphi, \gamma, \beta, \delta, \theta, \xi$ to the image of $v(e)$ is

$$\frac{1}{|G|^2|G_e|^6} \omega \left(\sum_{a \in A} a^* v(e) t(ea\beta^{-1}\gamma(v(e))) \right).$$

By our choice of A , we have

$$t(ea\beta^{-1}\gamma(v(e))) = \begin{cases} 1 & \text{if } a^* = v(e), \\ 0 & \text{otherwise.} \end{cases}$$

Summing over all choices of $\alpha, \varphi, \gamma, \beta, \delta, \theta, \xi$, we obtain that the image of $v(e)$ is $(1/|G_e|)\omega(v(e))$, as desired. This completes the proof. $\square$

We now consider tensor products of indecomposable projective symmetric B - B -bimodules with simple quotients of projective A - $\mathbb{k}$ -bimodules. To this end, we extend our notation to $G_{fe} := G_{Af \otimes_{\mathbb{k}} eA} (= G_e \cap G_f)$, for $e, f \in E$, and denote the simple quotient of (Ae, ε_e) by (L_e, ε_e) . As each $\varphi \in G$ is an automorphism of A , we have the induced action of φ on $\{(L_e, \varepsilon_e) \mid e \in E\}$ which maps each vector space L_e to the vector space $L_{\varphi(e)}$.

Lemma 13. *In $\tilde{\mathcal{Y}}$, there is an isomorphism*

$$(Af \otimes_{\mathbb{k}} eA, \varepsilon_{fe}) \otimes_{\tilde{\mathcal{Y}}} (L_e, \varepsilon_e) \cong \left(Af, \frac{1}{|G_{fe}|}(\gamma)_{\gamma \in G_{fe}} \right) \cong \bigoplus_{\xi} (Af, \varepsilon_{\xi}),$$

where ξ runs over all characters appearing in the induction of the trivial G_{fe} -module to G_f .

Proof. We first construct an isomorphism between $(Af \otimes_{\mathbb{k}} eA, \varepsilon_{fe}) \otimes_{\tilde{Y}} (L_e, \varepsilon_e)$ and $(Af, (1/|G_{fe}|)(\gamma)_{\gamma \in G_{fe}})$. In one direction, the morphism $\mathbf{g}$ is given by the diagram

$$\begin{array}{ccc}
 Af & \xrightarrow{\frac{1}{|G_{fe}|}(\gamma)_{\gamma \in G_{fe}}} & \bigoplus_{\gamma} {}^{\gamma}Af^{\gamma} \\
 \downarrow \frac{1}{|G_{fe}||G_e|}(\psi \otimes \psi(e) \otimes \varphi(l))_{\psi \in G_{fe}, \varphi \in G_e} & & \downarrow \frac{1}{|G_{fe}||G_e|}(\alpha \gamma^{-1} \otimes \alpha(e) \otimes \beta(l))_{\alpha \gamma^{-1} \in G_{fe}, \beta \in G_e} \\
 \bigoplus_{\varphi, \psi} {}^{\psi}Af \otimes_{\mathbb{k}} eA^{\psi} \otimes_A {}^{\varphi}(L_e)^{\varphi} & \xrightarrow{\frac{1}{|G_{fe}||G_e|}(\alpha \psi^{-1} \otimes \alpha \psi^{-1} \otimes \beta \varphi^{-1})} & \bigoplus_{\alpha, \beta} {}^{\alpha}Af \otimes_{\mathbb{k}} eA^{\alpha} \otimes_A {}^{\beta}(L_e)^{\beta}
 \end{array}$$

where the lower horizontal map is indexed by $\alpha \psi^{-1} \in G_{fe}$, $\beta \varphi^{-1} \in G_e$, and l denotes the canonical generator of the one-dimensional module L_e (the image of e in L_e). To see that the diagram commutes, first notice that the (α, β) -component of the object in the lower right-hand corner is nonzero if and only if $\alpha \beta^{-1} \in G_e$. If this is the case, then the (α, β) -component of the map going first to the right and then down is given by

$$f \mapsto \begin{cases} \frac{1}{|G_{fe}||G_e|} \alpha(f) \otimes \alpha(e) \otimes \beta(l) & \text{if } \alpha \in G_{fe}, \\ 0 & \text{otherwise.} \end{cases}$$

First going down and then to the right, we notice that $\varphi \in G_e$, $\psi \in G_{fe}$ forces $\beta \in G_e$, $\alpha \in G_{fe}$, so we obtain the same result, which verifies (2).

A morphism $\mathbf{h}$ in the other direction is given by

$$\begin{array}{ccc}
 \bigoplus_{\varphi} {}^{\varphi}Af \otimes_{\mathbb{k}} eA \otimes_A {}^{\varphi}(L_e)^{\varphi} & \xrightarrow{\frac{1}{|G_{fe}||G_e|}(\alpha \otimes \alpha \otimes \beta \varphi^{-1})_{\alpha \in G_{fe}, \beta \varphi^{-1} \in G_e}} & \bigoplus_{\alpha, \beta} {}^{\alpha}Af \otimes_{\mathbb{k}} eA^{\alpha} \otimes_A {}^{\beta}(L_e)^{\beta} \\
 \downarrow \frac{1}{|G_{fe}|}(\delta \otimes \mathbf{m})_{\delta \in G_{fe}, \varphi \in G} & & \downarrow \frac{1}{|G_{fe}|}(\gamma \alpha^{-1} \otimes \mathbf{m} \circ (\alpha \otimes \alpha)^{-1})_{\gamma \alpha^{-1} \in G_{fe}, \beta \in G} \\
 \bigoplus {}^{\delta}Af^{\delta} & \xrightarrow{\frac{1}{|G_{fe}|}(\gamma \delta^{-1})_{\gamma \delta^{-1} \in G_{fe}}} & \bigoplus_{\gamma} {}^{\gamma}Af^{\gamma}
 \end{array}$$

A nonzero contribution to the (γ, φ) -component, when going first down and then to the right can only happen for $\varphi \in G_e$ and $\gamma \in G_{fe}$, in which case the generator $f \otimes e \otimes \varphi(l)$ (as an A - $\mathbb{k}$ -bimodule) gets sent to $(1/|G_{fe}|)\gamma(f)$. Similarly, going first to the right and then down, a nonzero contribution only occurs for $\alpha \beta^{-1} \in G_e$, which, together with the conditions in the diagram, again forces $\beta, \varphi \in G_e$, $\alpha, \gamma \in G_{fe}$. Hence, summing over such α and β , we obtain the same result which verifies (2).

To check that both compositions of $\mathbf{g}$ and $\mathbf{h}$ are the respective identities (that is, the correct idempotents), it suffices to consider the compositions of the left-hand side of the diagrams.

Starting with $\mathbf{g} \circ \mathbf{h}$ and considering

$$\bigoplus_{\varphi} Af \otimes_{\mathbb{K}} eA \otimes_A {}^{\varphi}(L_e)^{\varphi} \rightarrow \bigoplus_{\delta} {}^{\delta}Af^{\delta} \rightarrow \bigoplus_{\alpha, \beta} {}^{\alpha}Af \otimes_{\mathbb{K}} eA^{\alpha} \otimes_A {}^{\beta}(L_e)^{\beta},$$

the (α, β, φ) -component of the composition is zero unless $\varphi, \beta \in G_e$, $\alpha \in G_{fe}$, in which case the generator $f \otimes e \otimes \varphi(l)$ is mapped to $(1/(|G_{fe}||G_e|))\alpha(f) \otimes \alpha(e) \otimes \beta(l)$, as desired.

For $\mathbf{h} \circ \mathbf{g}$, we consider

$$Af \rightarrow \bigoplus_{\varphi, \psi} {}^{\psi}Af \otimes_{\mathbb{K}} eA^{\psi} \otimes_A {}^{\varphi}(L_e)^{\varphi} \rightarrow \bigoplus_{\gamma} {}^{\gamma}Af^{\gamma}$$

and verify that, in the γ -component, f is indeed sent to $(1/|G_{fe}|)\gamma(f)$, if $\gamma \in G_{fe}$, and zero otherwise, as claimed.

Hence we have an isomorphism

$$(Af \otimes_{\mathbb{K}} eA, \varepsilon_{fe}) \otimes \tilde{\mathcal{Y}}(L_e, \varepsilon_e) \cong \left(Af, \frac{1}{|G_{fe}|}(\gamma)_{\gamma \in G_{fe}} \right),$$

as stated.

Now notice that $(1/|G_{fe}|) \sum_{\gamma \in G_{fe}} \gamma$ is a trivial idempotent on G_{fe} . When viewed as an idempotent of the larger group G_f , it decomposes into precisely the (multiplicity-free) sum of those idempotents affording characters ξ of G_f which appear in the induction of the trivial character from G_{fe} to G_f . This proves the proposition. $\square$

Proposition 14. *We have adjunctions $((Af \otimes_{\mathbb{K}} eA, \varepsilon_{fe}), (Av(e) \otimes_{\mathbb{K}} fA, \varepsilon_{v(e)f}))$, for idempotents $e, f \in A$.*

Proof. From the defining action of $\mathcal{G}_B$ on B -mod we have that $(Af \otimes_{\mathbb{K}} eA, \varepsilon_{fe})$ is left adjoint to $(Av(e) \otimes_{\mathbb{K}} fA, \varepsilon_{\chi})$, for some $\chi \in \widehat{G}_{v(e)f}$. We thus have an isomorphism of nonzero spaces of homomorphisms,

$$\begin{aligned} \text{Hom}_{\tilde{\mathcal{Y}}} \left(\left(Af, \frac{1}{|G_{fe}|}(\beta)_{\beta \in G_{fe}} \right), (L_f, \varepsilon_f) \right) \\ \cong \text{Hom}_{\tilde{\mathcal{Y}}}((Af \otimes_{\mathbb{K}} eA, \varepsilon_{fe}) \otimes \tilde{\mathcal{Y}}(L_e, \varepsilon_e), (L_f, \varepsilon_f)) \\ \cong \text{Hom}_{\tilde{\mathcal{Y}}}((L_e, \varepsilon_e), (Av(e) \otimes_{\mathbb{K}} fA, \varepsilon_{\chi}) \otimes \tilde{\mathcal{Y}}(L_f, \varepsilon_f)), \end{aligned}$$

where the first isomorphism follows from Lemma 13.

By (the opposite of) Proposition 7, noting that $G_{fe} = G_{v(e)f}$ and $G_{v(e)} = G_e$, there are isomorphisms

$$\begin{aligned}
 ((Av(e) \otimes_{\mathbb{k}} fA, \varepsilon_\chi) \otimes_{\tilde{\mathcal{Y}}} (L_f, \varepsilon_f)) \\
 &\cong (A, \tilde{\pi}_\chi) \otimes_{\tilde{\mathcal{Y}}} (Av(e) \otimes_{\mathbb{k}} fA, \varepsilon_{v(e)f}) \otimes_{\tilde{\mathcal{Y}}} (L_f, \varepsilon_f) \\
 &\cong (A, \tilde{\pi}_\chi) \otimes_{\tilde{\mathcal{Y}}} \left(Av(e), \frac{1}{|G_{fe}|} (\gamma)_{\gamma \in G_{fe}} \right) \\
 &\cong \left(Av(e), \frac{1}{|G_{fe}|} (\chi(\gamma)\gamma)_{\gamma \in G_{fe}} \right),
 \end{aligned}$$

yielding the isomorphism

$$\begin{aligned}
 \text{Hom}_{\tilde{\mathcal{Y}}} \left(\left(Af, \frac{1}{|G_{fe}|} (\beta)_{\beta \in G_{fe}} \right), (L_f, \varepsilon_f) \right) \\
 \cong \text{Hom}_{\tilde{\mathcal{Y}}} \left((L_e, \varepsilon_e), \left(Av(e), \frac{1}{|G_{fe}|} (\chi(\gamma)\gamma)_{\gamma \in G_{fe}} \right) \right).
 \end{aligned}$$

Denote the left-hand side by U and the right-hand side by V .

Now we claim that $\dim_{\mathbb{k}} U = 1$. By (2), for any morphism $\mathbf{g} \in U$, we have

$$(12) \quad \text{id}_{(L_f, \varepsilon_f)} \circ \mathbf{g} = \mathbf{g} = \mathbf{g} \circ \text{id}_{(Af, (1/|G_{fe}|)(\beta)_{\beta \in G_{fe}})}.$$

Assume that $\mathbf{g} = (k_\alpha \alpha)_{\alpha \in G_f}$, where $k_\alpha \in \mathbb{k}$, and consider the diagram

$$\begin{array}{ccc}
 Af & \xrightarrow{\frac{1}{|G_{fe}|} (\beta)_{\beta \in G_{fe}}} & \bigoplus_{\beta} {}^{\beta} Af^{\beta} \\
 \downarrow (k_\alpha \alpha)_{\alpha \in G_f} & & \downarrow (k_{\gamma\beta^{-1}} \gamma \beta^{-1})_{\gamma\beta^{-1} \in G_f} \\
 \bigoplus_{\alpha} {}^{\alpha} (L_f)^{\alpha} & \xrightarrow{\frac{1}{|G_f|} (\gamma \alpha^{-1})_{\gamma \alpha^{-1} \in G_f}} & \bigoplus_{\gamma} {}^{\gamma} (L_f)^{\gamma}
 \end{array}$$

The morphism $\text{id}_{(L_f, \varepsilon_f)} \circ \mathbf{g}$ is exactly the path going down and then right, taking into account the fact that $\alpha \in G_f$ forces $\gamma \in G_f$, and the γ -component of this morphism is given by

$$f \mapsto \begin{cases} \frac{1}{|G_f|} (\sum_{\alpha \in G_f} k_\alpha) \gamma(f) & \text{if } \gamma \in G_f, \\ 0 & \text{otherwise.} \end{cases}$$

The first equality of (12) shows that $(1/|G_f|) \sum_{\alpha \in G_f} k_\alpha = k_\gamma$, for all $\gamma \in G_f$, and hence $k_\gamma = k_{\gamma'}$ for all $\gamma, \gamma' \in G_f$. Then we obtain $\mathbf{g} = (k_\alpha)_{\alpha \in G_f}$, where $k \in \mathbb{k}$, and the first equality is automatically satisfied. Going right and then down and using the fact that $\beta \in G_{fe}$, $\gamma\beta^{-1} \in G_f$ implies $\gamma \in G_f$, the second equality of (12) is easily verified. The claim follows.

As $U \cong V$, we have $\dim_{\mathbb{k}} V = 1$. Using (2), for any morphism $\mathbf{h} \in V$, we have

$$(13) \quad \text{id}_{(Av(e), (1/|G_{fe}|)(\chi(\gamma)\gamma)_{\gamma \in G_{fe}})} \circ \mathbf{h} = \mathbf{h} = \mathbf{h} \circ \text{id}_{(L_e, \varepsilon_e)}.$$

Assume that $\mathbf{h} = (l_\alpha \alpha)_{\alpha \in G_e}$, where $l_\alpha \in \mathbb{k}$, and consider the diagram

$$\begin{array}{ccc} L_e & \xrightarrow{\frac{1}{|G_e|}(\beta)_{\beta \in G_e}} & \bigoplus_{\beta}^{\beta} (L_e)^{\beta} \\ \downarrow (l_\alpha \alpha)_{\alpha \in G_e} & & \downarrow (l_{\delta\beta^{-1}} \delta\beta^{-1})_{\delta\beta^{-1} \in G_e} \\ \bigoplus_{\alpha}^{\alpha} Av(e)^{\alpha} & \xrightarrow{\frac{1}{|G_{fe}|}(\chi(\delta\alpha^{-1})\delta\alpha^{-1})_{\delta\alpha^{-1} \in G_{fe}}} & \bigoplus_{\delta}^{\delta} Av(e)^{\delta} \end{array}$$

The morphism $\mathbf{h} \circ \text{id}_{(L_e, \varepsilon_e)}$ coincides with the path going to the right and then down. Note that $\beta \in G_e$, $\delta\beta^{-1} \in G_e$ forces $\delta \in G_e$. Then the δ -component of this composition is given by

$$e \mapsto \begin{cases} \frac{1}{|G_e|} \left(\sum_{\beta \in G_e} l_{\delta\beta^{-1}} \right) \delta(e) & \text{if } \delta \in G_e, \\ 0 & \text{otherwise.} \end{cases}$$

By reindexing, the second equality of (13) shows that, for all $\delta \in G_e$, we have $(1/|G_e|) \sum_{\sigma \in G_e} l_\sigma = l_\delta$, and thus $l_\delta = l_{\delta'}$, for all $\delta, \delta' \in G_e$. Therefore we have $\mathbf{h} = (l_\alpha)_{\alpha \in G_e}$, where $l \in \mathbb{k}$, and the second equality holds. Due to $\dim_{\mathbb{k}} V = 1$, the first equality of (13) should also hold for any $l \in \mathbb{k}^*$. Going down and then to the right, the δ -component of $\text{id}_{(Av(e), (1/|G_{fe}|)(\chi(\gamma)\gamma)_{\gamma \in G_{fe}})} \circ \mathbf{h}$ is zero unless $\delta \in G_e$, in which case e is sent to

$$\frac{1}{|G_{fe}|} \left(\sum_{\alpha \in \delta G_{fe}} \chi(\delta\alpha^{-1}) \right) l \delta(e) = \frac{1}{|G_{fe}|} \left(\sum_{\varphi \in G_{fe}} \chi(\varphi) \right) l \delta(e).$$

The first equality implies that $(1/|G_{fe}|)(\sum_{\varphi \in G_{fe}} \chi(\varphi)) = 1$. By multiplying any $\chi(\gamma)$, where $\gamma \in G_{fe}$, to both side of the latter, we obtain

$$\chi(\gamma) = \frac{1}{|G_{fe}|} \left(\sum_{\varphi \in G_{fe}} \chi(\gamma\varphi) \right) = \frac{1}{|G_{fe}|} \left(\sum_{\psi \in G_{fe}} \chi(\psi) \right) = 1.$$

Therefore $\chi = \varepsilon_{v(e)f}$ and the proof is complete. $\square$

Proposition 15. *In $\mathcal{G}_B$, we have adjunctions $((A_i, \tilde{\pi}_\chi), (A_i, \tilde{\pi}_{\chi^{-1}}))$, for each $\chi \in \widehat{G}$ and $i = 1, \dots, n$. Similarly, we have adjunction $((\mathbb{k}, \tilde{\pi}_\chi), (\mathbb{k}, \tilde{\pi}_{\chi^{-1}}))$, for each $\chi \in \widehat{G}$.*

Proof. By Proposition 6, we have

$$(A_i, \tilde{\pi}_\chi) \otimes_{\tilde{Y}} (A_i, \tilde{\pi}_{\chi^{-1}}) \cong (A_i, \tilde{\pi}_{1_{\widehat{G}}}) \cong (A_i, \tilde{\pi}_{\chi^{-1}}) \otimes_{\tilde{Y}} (A_i, \tilde{\pi}_\chi)$$

and similarly for $\mathbb{k}$. Both unit and counit are then just identities and the claim is immediate. $\square$

Proposition 16. *If A is self-injective, then the 2-categories $\mathcal{G}_A$ and $\mathcal{G}_B$ are weakly fiat. If A is weakly symmetric, then both $\mathcal{G}_A$ and $\mathcal{G}_B$ are fiat.*

Proof. Assume A is self-injective. Proposition 7 shows that any indecomposable 1-morphism can be written as a product of those treated in Propositions 12, 14 and 15. This implies that $\mathcal{G}_A$ and $\mathcal{G}_B$ are weakly fiat. If A is weakly symmetric, then ν is the identity, and all adjunctions given in Propositions 12, 14 and 15 become (weakly) involutive, proving fiatness. $\square$

3.6. Simple transitive 2-representations of $\mathcal{G}_A$. Now we can formulate our first main result. We assume that A is weakly symmetric, basic and there is a fixed complete G -invariant set E of primitive idempotents, so that $\mathcal{G}_A$ and $\mathcal{G}_B$ are fiat.

Theorem 17. *Under the above assumptions, for every two-sided cell $\mathcal{J}$ in $\mathcal{G}_A$, there is a natural bijection between equivalence classes of simple transitive 2-representations of $\mathcal{G}_A$ with apex $\mathcal{J}$ and pairs (K, ω) , where K is a subgroup of G and $\omega \in H^2(K, \mathbb{k}^*)$.*

Proof. For $\mathcal{J} = \mathcal{J}_i$, where $i = 1, 2, \dots, n$, Proposition 6 shows that the $\mathcal{J}$ -simple quotient of $\mathcal{G}_A$ is biequivalent to the 2-category $\text{Rep}(G)$ from [Ostrik 2003]. Therefore the statement follows from [Ostrik 2003, Theorem 2].

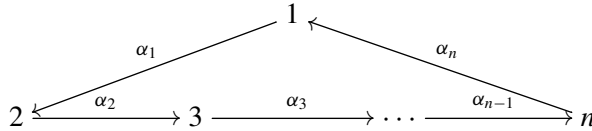
For $\mathcal{J} = \mathcal{J}_0$, consider $B = A \times \mathbb{k}$. Then we can realize $\mathcal{G}_A$ as a both 1- and 2-full subcategory of $\mathcal{G}_B$ in the obvious way. Let j denote the object corresponding to the additional factor $\mathbb{k}$. Let $\mathcal{H}_1$ be the $\mathcal{H}$ -class in $\mathcal{G}_B$ containing the identity 1-morphism on $\mathbb{k}$. By Lemma 2, $\mathcal{H}_1$ contains $|G|$ indecomposable 1-morphisms, moreover, $\mathcal{H}_1$ is contained in $\mathcal{J}_0^{(B)}$, the two-sided cell of projective bimodules in $\mathcal{G}_B$. Note that the 1- and 2-full 2-subcategory $\mathcal{A}_{\mathcal{H}_1}$ of $\mathcal{G}_B$ with object j is biequivalent to the 2-category $\text{Rep}(G)$ as above. Hence, by [Ostrik 2003, Theorem 2], there is a natural bijection between equivalence classes of simple transitive 2-representations of $\mathcal{A}_{\mathcal{H}_1}$ with apex $\mathcal{H}_1$ and pairs (K, ω) as in the theorem. By [Mackaay et al. 2018, Theorem 15], there is a bijection between equivalence classes of simple transitive 2-representations of $\mathcal{A}_{\mathcal{H}_1}$ and equivalence classes of simple transitive 2-representations of $\mathcal{G}_B$ with apex $\mathcal{J}_0^{(B)}$.

Let $\mathcal{H}_2$ be any self-dual $\mathcal{H}$ -class in $\mathcal{G}_A$ contained in $\mathcal{J}_0$. Notice that this is also a self-dual $\mathcal{H}$ -class in $\mathcal{G}_B$ contained in $\mathcal{J}_0^{(B)}$. Let $\mathcal{A}_{\mathcal{H}_2}$ be the corresponding 1- and 2-full 2-subcategory of $\mathcal{G}_A$ (and of $\mathcal{G}_B$), see [Mackaay et al. 2018, Subsection 4.2]. By [Mackaay et al. 2018, Theorem 15], there is a bijection between equivalence classes of simple transitive 2-representations of $\mathcal{A}_{\mathcal{H}_2}$ and equivalence classes of simple transitive 2-representations of $\mathcal{G}_B$ with apex $\mathcal{J}_0^{(B)}$, and also of $\mathcal{G}_A$ with apex $\mathcal{J}_0$. The claim follows. $\square$

To prove Theorem 17, one could alternatively use [Mackaay et al. 2019, Corollary 12].

Remark 18. An analogue of Theorem 17 is also true in the weakly fiat case, that is when A is just self-injective but not necessarily weakly symmetric. However, the proof requires an adjustment of the results of [Mackaay et al. 2018, Theorem 15] to the case when instead of one diagonal $\mathcal{H}$ -cell one considers a diagonal block which is stable under $\star$. One could carefully go through the proof [Mackaay et al. 2018, Theorem 15] and check that everything works.

3.7. A class of examples. Fix a positive integer $n > 1$ and let A be the quotient of the path algebra of the cyclic quiver



modulo the ideal generated by all paths of length n . Now we let G be the cyclic group of order n whose generator φ acts on A by sending e_i to e_{i+1} and α_i to α_{i+1} (where we compute indices modulo n).

For $i = 1, 2, \dots, n$, we denote by F_i the indecomposable 1-morphism in $\mathcal{G}_A$ corresponding to tensoring with $Ae_1 \otimes_{\mathbb{k}} e_i A$ (we omit the idempotents since the action of G is free). Then (F_i, F_{n+1-i}) is an adjoint pair (and, indeed, biadjoint), for each i . Hence $\mathcal{G}_A$ is fiat.

Note that every subgroup of G is cyclic and $H^2(\mathbb{Z}/k\mathbb{Z}, \mathbb{k}^*) \cong \mathbb{k}^*/(\mathbb{k}^*)^k \cong \{e\}$ since $\mathbb{k}$ is algebraically closed. Therefore, simple transitive 2-representations of $\mathcal{G}_A$ are in bijection with divisors of n . For $d \mid n$, the algebra underlying the simple transitive 2-representations of $\mathcal{G}_A$ corresponding to d is the algebra $A^{\langle \varphi^{n/d} \rangle}$ with the obvious action of $\mathcal{G}_A$. Here $A^{\langle \varphi^{n/d} \rangle}$ denotes the invariant subalgebra of A under the action of the subgroup $\langle \varphi^{n/d} \rangle$ of G , which is generated by $\varphi^{n/d}$.

4. Two-element $\mathcal{H}$ -cells with no self-adjoint elements

4.1. Basic combinatorics.

Proposition 19. Let $\mathcal{C}$ be a fiat 2-category such that

- $\mathcal{C}$ has one object $\mathfrak{i}$;
- $\mathcal{C}$ has two two-sided cells, each of which is also a right cell and a left cell, one being $\{\mathbb{1}_{\mathfrak{i}}\}$ and the other one given by $\{F, G\}$ with $F \not\cong G$;
- $F^* \cong G$.

Then there exists $n \in \mathbb{Z}_{>0}$ such that

$$(14) \quad FF \cong FG \cong GF \cong GG \cong (F \oplus G)^{\oplus n}.$$

Proof. We have

$$\begin{aligned} FF &\cong F^{\oplus a_1} \oplus G^{\oplus a_2}, & FG &\cong F^{\oplus b_1} \oplus G^{\oplus b_2}, \\ GF &\cong F^{\oplus c_1} \oplus G^{\oplus c_2}, & GG &\cong F^{\oplus d_1} \oplus G^{\oplus d_2}, \end{aligned}$$

for some $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{Z}_{\geq 0}$.

From $F^* \cong G$, we see that $(FG)^* \cong FG$ and $(GF)^* \cong GF$. This implies $b_1 = b_2 =: b$ and $c_1 = c_2 =: c$. Furthermore, $(FF)^* \cong GG$, which implies $a_1 = d_2 =: x$ and $a_2 = d_1 =: y$.

As G is in the same left cell as F , we obtain $y + c > 0$ and $y + b > 0$.

Case 1. $y = 0$. In this case we have $c, b > 0$ by the above, and

$$FF \cong F^{\oplus x}, \quad FG \cong F^{\oplus b} \oplus G^{\oplus b}, \quad GF \cong F^{\oplus c} \oplus G^{\oplus c}, \quad GG \cong G^{\oplus x}.$$

We use this to compute both sides of the isomorphism $(FG)G \cong F(GG)$. This yields $b^2 = xb$ (by comparing the coefficients as F) and $b^2 + xb = xb$ (by comparing the coefficients as G). Hence $b = 0$, a contradiction. Therefore this case cannot occur.

Case 2. $y > 0$. In this case we have

$$FF \cong F^{\oplus x} \oplus G^{\oplus y}, \quad FG \cong F^{\oplus b} \oplus G^{\oplus b}, \quad GF \cong F^{\oplus c} \oplus G^{\oplus c}, \quad GG \cong F^{\oplus y} \oplus G^{\oplus x}.$$

We use this to compute both sides of the isomorphism $(FG)F \cong F(GF)$, and obtain $xc = xb$ (by comparing the coefficients as F) and $yc = yb$ (by comparing the coefficients as G). As $y > 0$, we have $c = b$.

Finally, we compute both sides of the isomorphism $(FG)G \cong F(GG)$. This implies $b^2 + by = xy + bx$ (by comparing the coefficients as F) and $b^2 + bx = y^2 + bx$ (by comparing the coefficients as G). As $y > 0$ and $b \geq 0$, from the second equation we deduce $b = y$. Using $y > 0$ and $b = y$, the first equation yields $b = x$. The claim follows. $\square$

4.2. The algebra of the cell 2-representation. Let $\mathcal{C}$ be a fiat 2-category as in Proposition 19. Consider the cell 2-representation $\mathbf{C}_{\mathcal{H}}$ of $\mathcal{C}$, where $\mathcal{H} = \{F, G\}$. Denote by A its underlying basic algebra with a fixed decomposition $1_A = e_F + e_G$ of the identity into primitive orthogonal idempotents. Let P_F and P_G denote the corresponding indecomposable projective A -modules and L_F and L_G their respective simple tops. Note that fiatness of $\mathcal{C}$ implies self-injectivity of A (see [Kildetoft et al. 2019, Theorem 2]).

From (14) we obtain that the matrix describing the action of both F and G in the cell 2-representation (in the basis of indecomposable projective modules) is

$$(15) \quad \begin{pmatrix} n & n \\ n & n \end{pmatrix}.$$

By [Mazorchuk and Miemietz 2016d, Lemma 10], the same matrix describes the action of both F and G in the abelianization of the cell 2-representation in the basis of simple modules. Without loss of generality, assume that G is the Duflo involution of the left cell $\mathcal{H}$. In the cell 2-representation, we then have $GL_G \cong P_G$ and $FL_G \cong P_F$. This, together with the description of the matrix of the action in (15), shows that

$$(16) \quad [P_G : L_F] = [P_F : L_F] = [P_G : L_G] = [P_F : L_G] = n.$$

Therefore, the Cartan matrix of A is given by (15).

As the bimodules X and Y , representing F and G , respectively, are projective, see [Kildetoft et al. 2019, Theorem 2; Mazorchuk and Miemietz 2016d, Lemma 13] for details, we deduce that $Ae_F \otimes_{\mathbb{k}} e_G A$ appears as a direct summand of X and $Ae_G \otimes_{\mathbb{k}} e_G A$ appears as a direct summand of Y . Due to $G^* \cong F$, and

$$0 \neq \text{Hom}_{C_{\mathcal{H}}}(GL_G, L_G) \cong \text{Hom}_{C_{\mathcal{H}}}(L_G, FL_G) \cong \text{Hom}_{C_{\mathcal{H}}}(L_G, P_F),$$

the algebra A is not weakly symmetric. Furthermore, we have

$$0 = \text{Hom}_A(P_G, L_F) \cong \text{Hom}_A(GL_G, L_F) \cong \text{Hom}_A(L_G, FL_F),$$

so FL_F is a direct sum of copies of P_G . Comparing the Cartan matrix of A with the matrix of the action of F in the basis of simple modules (both given by (15)), we see that $FL_F \cong P_G$. Similarly we deduce $GL_F \cong P_F$. Hence, we have

$$(17) \quad X \cong Ae_F \otimes_{\mathbb{k}} e_G A \oplus Ae_G \otimes_{\mathbb{k}} e_F A \quad \text{and} \quad Y \cong Ae_F \otimes_{\mathbb{k}} e_F A \oplus Ae_G \otimes_{\mathbb{k}} e_G A.$$

4.3. Functors isomorphic to the identity endomorphism of 2-representations. In this subsection, we will formulate a general result for an arbitrary finitary 2-category $\mathcal{C}$. This result will be needed for Section 4.5. For simplicity, we assume that $\mathcal{C}$ has only one object $\mathbf{i}$. Let $\mathbf{M}$ be a finitary 2-representation of $\mathcal{C}$. Let $(\text{Id}_{\mathbf{M}}, \eta) : \mathbf{M} \rightarrow \mathbf{M}$ be the identity endomorphism of $\mathbf{M}$. Here η is given by the family $\{\eta_F \mid F \in \mathcal{C}(\mathbf{i}, \mathbf{i})\}$ of natural transformations where each η_F is the identity natural transformation of $\mathbf{M}(F)$.

Lemma 20. *Let $\Phi : \mathbf{M}(\mathbf{i}) \rightarrow \mathbf{M}(\mathbf{i})$ be a functor isomorphic to the identity functor $\text{Id}_{\mathbf{M}(\mathbf{i})}$. Then there exists a family of natural isomorphisms $\{\zeta_F \mid F \in \mathcal{C}(\mathbf{i}, \mathbf{i})\} =: \zeta$ such that (Φ, ζ) is an endomorphism of the 2-representation $\mathbf{M}$.*

Proof. Note that $\Phi \cong \text{Id}_{\mathbf{M}(\mathbf{i})}$ as a functor. Let $\theta : \text{Id}_{\mathbf{M}(\mathbf{i})} \rightarrow \Phi$ be a fixed natural isomorphism and set $\nu := \theta^{-1}$. For any 1-morphisms F, G and 2-morphism $\alpha : F \rightarrow G$, consider the diagram

$$(18) \quad \begin{array}{ccccc} \Phi \circ \mathbf{M}(F) & \xrightarrow{\nu \circ_h \text{id}_{\mathbf{M}(F)}} & \mathbf{M}(F) & \xrightarrow{\text{id}_{\mathbf{M}(F)} \circ_h \theta} & \mathbf{M}(F) \circ \Phi \\ \downarrow \text{id}_\Phi \circ_h \mathbf{M}(\alpha) & & \downarrow \mathbf{M}(\alpha) & & \downarrow \mathbf{M}(\alpha) \circ_h \text{id}_\Phi \\ \Phi \circ \mathbf{M}(G) & \xrightarrow{\nu \circ_h \text{id}_{\mathbf{M}(G)}} & \mathbf{M}(G) & \xrightarrow{\text{id}_{\mathbf{M}(G)} \circ_h \theta} & \mathbf{M}(G) \circ \Phi \end{array}$$

Here in the middle column we use the fact that, for any 1-morphism H , we have

$$\text{Id}_{\mathbf{M}(\mathbf{i})} \circ_h \mathbf{M}(H) = \mathbf{M}(H) = \mathbf{M}(H) \circ_h \text{Id}_{\mathbf{M}(\mathbf{i})}.$$

Diagram (18) commutes thanks to the interchange law, indeed, both paths in the left square are equal to $\nu \circ_h \mathbf{M}(\alpha)$ and both paths in the right square are equal to $\mathbf{M}(\alpha) \circ_h \theta$.

For each 1-morphism F , define

$$(19) \quad \zeta_F := (\text{id}_{\mathbf{M}(F)} \circ_h \theta) \circ_v (\nu \circ_h \text{id}_{\mathbf{M}(F)}) : \Phi \circ \mathbf{M}(F) \rightarrow \mathbf{M}(F) \circ \Phi.$$

Now we claim that (Φ, ζ) is an endomorphism of the 2-representation $\mathbf{M}$. Commutativity of (18) gives $(\mathbf{M}(\alpha) \circ_h \text{id}_\Phi) \circ_v \zeta_F = \zeta_G \circ_v (\text{id}_\Phi \circ_h \mathbf{M}(\alpha))$. We are hence left to check the equality

$$(20) \quad \zeta_{F \circ G} = (\text{id}_{\mathbf{M}(F)} \circ_h \zeta_G) \circ_v (\zeta_F \circ_h \text{id}_{\mathbf{M}(G)}).$$

Here, by definition, we have

$$\begin{aligned} \text{id}_{\mathbf{M}(F)} \circ_h \zeta_G &= (\text{id}_{\mathbf{M}(F)} \circ_h \text{id}_{\mathbf{M}(G)} \circ_h \theta) \circ_v (\text{id}_{\mathbf{M}(F)} \circ_h \nu \circ_h \text{id}_{\mathbf{M}(G)}) \\ &= (\text{id}_{\mathbf{M}(F \circ G)} \circ_h \theta) \circ_v (\text{id}_{\mathbf{M}(F)} \circ_h \nu \circ_h \text{id}_{\mathbf{M}(G)}), \end{aligned}$$

and

$$\begin{aligned} \zeta_F \circ_h \text{id}_{\mathbf{M}(G)} &= (\text{id}_{\mathbf{M}(F)} \circ_h \theta \circ_h \text{id}_{\mathbf{M}(G)}) \circ_v (\nu \circ_h \text{id}_{\mathbf{M}(F)} \circ_h \text{id}_{\mathbf{M}(G)}) \\ &= (\text{id}_{\mathbf{M}(F)} \circ_h \theta \circ_h \text{id}_{\mathbf{M}(G)}) \circ_v (\nu \circ_h \text{id}_{\mathbf{M}(F \circ G)}). \end{aligned}$$

Now (20) follows from the fact that $\nu\theta = \text{Id}_{\mathbf{M}(\mathbf{i})}$. The proof is complete. $\square$

Remark 21. (i) The natural isomorphism $\theta : \text{Id}_{\mathbf{M}(\mathbf{i})} \rightarrow \Phi$ defines a modification from $(\text{Id}_{\mathbf{M}}, \eta)$ to (Φ, ζ) whose inverse is given by $\nu : \Phi \rightarrow \text{Id}_{\mathbf{M}(\mathbf{i})}$. Indeed, for

any 1-morphisms F, G and any 2-morphism $\alpha : F \rightarrow G$, we have

$$\begin{aligned}
 (\mathbf{M}(\alpha) \circ_h \theta) \circ_v \eta_F &= \mathbf{M}(\alpha) \circ_h \theta \\
 &= (\text{id}_{\mathbf{M}(G)} \circ_h \theta) \circ_v (\mathbf{M}(\alpha) \circ_h \text{id}_{\mathbf{M}(i)}) \\
 &= (\text{id}_{\mathbf{M}(G)} \circ_h \theta) \circ_v (\text{id}_{\mathbf{M}(i)} \circ_h \mathbf{M}(\alpha)) \\
 &= (\text{id}_{\mathbf{M}(G)} \circ_h \theta) \circ_v (\nu \circ_h \text{id}_{\mathbf{M}(G)}) \circ_v (\theta \circ_h \mathbf{M}(\alpha)) \\
 &= \zeta_G \circ_v (\theta \circ_h \mathbf{M}(\alpha)),
 \end{aligned}$$

- (ii) Any invertible modification θ from $(\text{Id}_{\mathbf{M}}, \eta)$ to some $(\Phi, \zeta) \in \text{End}_{\mathcal{C}\text{-afmod}}(\mathbf{M})$ defines a natural isomorphism from $\text{Id}_{\mathbf{M}(i)}$ to Φ . Moreover, from the fact that $(\mathbf{M}(\text{id}_F) \circ_h \theta) \circ_v \eta_F = \zeta_F \circ_v (\theta \circ_h \mathbf{M}(\text{id}_F))$, it follows that each ζ_F is uniquely defined by (19) with $\nu := \theta^{-1}$.

4.4. Inductive limit construction for 2-representations. Assume that we are in the same setup as in Section 4.3. For any finitary 2-representation $\mathbf{M}$ of $\mathcal{C}$, we denote by $\overline{\mathbf{M}}^{\text{pr}}$ the 2-subrepresentation of $\overline{\mathbf{M}}$ with the action of $\mathcal{C}$ restricted to the category $\overline{\mathbf{M}}^{\text{pr}}(i)$ consisting of projective objects in $\overline{\mathbf{M}}(i)$. There exists a strict 2-natural transformation $\Upsilon : \mathbf{M} \rightarrow \overline{\mathbf{M}}^{\text{pr}}$ given by sending an object X to the diagram $0 \rightarrow X$ with the obvious assignment on morphisms. Similarly to [Mackaay and Mazorchuk 2017, Subsection 5.8], we have a direct system

$$(21) \quad \mathbf{M} \rightarrow \overline{\mathbf{M}}^{\text{pr}} \rightarrow (\overline{\mathbf{M}}^{\text{pr}})^{\text{pr}} \rightarrow \dots,$$

where each arrow is given by Υ with $\mathbf{M}$ replaced by the starting point corresponding to this arrow. We denote by $\underline{\mathbf{M}}$ the inductive limit of (21). This is a 2-representation of $\mathcal{C}$ and the natural embedding of $\mathbf{M}$ into $\underline{\mathbf{M}}$ is an equivalence. Let $\mathcal{L}$ be a left cell of $\mathcal{C}$ and $\mathbf{C}_{\mathcal{L}} := \mathbf{N}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}$ the corresponding cell 2-representation. By the Yoneda lemma, see [Mazorchuk and Miemietz 2014, Lemma 9], for any object X in $\mathbf{M}(i)$ there exists a strict 2-natural transformation $\Lambda_X : \mathbf{P}_i \rightarrow \mathbf{M}$ which sends $\mathbb{1}_i$ to X and, moreover, any morphism $f : X \rightarrow Y$ in $\mathbf{M}(i)$ extends to a modification $\theta_f : \Lambda_X \rightarrow \Lambda_Y$. If $\mathcal{I}_{\mathcal{L}}$ annihilates X , then Λ_X induces a strict 2-natural transformation Λ'_X from $\mathbf{C}_{\mathcal{L}}$ to $\mathbf{M}$. Indeed, we have the following commutative diagram:

$$\begin{array}{ccccc}
 \mathbf{N}_{\mathcal{L}} & \xhookrightarrow{\quad \Xi \quad} & \mathbf{P}_i & \xrightarrow{\quad \Lambda_X \quad} & \mathbf{M} \\
 \Pi \downarrow & & & \nearrow \Lambda'_X & \\
 \mathbf{C}_{\mathcal{L}} & & & &
 \end{array}$$

If $\mathcal{I}_{\mathcal{L}}$ also annihilates Y , then Λ_Y gives rise to a strict 2-natural transformation Λ'_Y from $\mathbf{C}_{\mathcal{L}}$ to $\mathbf{M}$ such that $\Lambda_Y \Xi = \Lambda'_Y \Pi$. Due to surjectivity of Π , the modification $\theta_f \circ_h \text{id}_{\Xi}$ from $\Lambda_X \Xi = \Lambda'_X \Pi$ to $\Lambda_Y \Xi = \Lambda'_Y \Pi$ induces a modification θ'_f from

Λ'_X to Λ'_Y in $\text{Hom}_{\mathcal{C}\text{-afmod}}(\underline{C}_{\mathcal{L}}, \underline{M})$. By functoriality of the abelianization, via the limiting construction (21) we thus obtain two 2-natural transformations

$$\Lambda'_X, \Lambda'_Y \in \text{Hom}_{\mathcal{C}\text{-afmod}}(\underline{C}_{\mathcal{L}}, \underline{M})$$

and the modification $\theta' : \Lambda'_X \rightarrow \Lambda'_Y$.

4.5. Symmetries of the cell 2-representation.

Lemma 22. *The annihilators in $\mathcal{C}$ of L_F and L_G coincide.*

Proof. Since G is the Duflo involution in $\mathcal{H}$, it follows from [Mazorchuk and Miemietz 2014, Subsection 6.5] that the annihilator of L_F is contained in the annihilator of L_G (as the latter is a certain unique maximal left ideal by [Mazorchuk and Miemietz 2014, Proposition 21]). Furthermore, the evaluation at L_G , inside the abelianized cell 2-representation, of $\text{Hom}_{\mathcal{C}}(H_1, H_2)$ is full for all $H_1, H_2 \in \{F, G\}$ by [Mazorchuk and Miemietz 2014, Subsection 6.5].

Hence, if the annihilator of L_F were strictly contained in the annihilator of L_G , the dimension of the endomorphism space (in the cell 2-representation) of $(F \oplus G)L_F$ would be strictly bigger than the dimension of the endomorphism space of $(F \oplus G)L_G$. However, from Section 4.2 we know that $(F \oplus G)L_F \cong (F \oplus G)L_G$. The claim follows. $\square$

On the one hand, by [Mazorchuk and Miemietz 2014, Lemma 9], sending $\mathbb{1}_i$ to L_F extends to a strict 2-natural transformation $\Phi : P_i \rightarrow \overline{C}_{\mathcal{H}}$. By Lemma 22, we know that $\Phi \Xi$ factors through $C_{\mathcal{H}}$ and obtain a strict 2-natural transformation Φ' from $C_{\mathcal{H}}$ to $\overline{C}_{\mathcal{H}}$. Note that Φ sends both F and G to projective objects in $\overline{C}_{\mathcal{H}}(i)$. Therefore Φ' is also a strict 2-natural transformation from $C_{\mathcal{H}}$ to $\overline{C}_{\mathcal{H}}^{\text{pr}}$ and we have the following commutative diagram:

$$(22) \quad \begin{array}{ccccccc} N_{\mathcal{L}} & \xrightarrow{\Xi} & P_i & \xrightarrow{\Phi} & \overline{C}_{\mathcal{H}} & \xrightarrow{\overline{\Phi}'} & \overline{(C_{\mathcal{H}}^{\text{pr}})} \\ & \searrow \Phi \Xi & & \nearrow & \nearrow & \nearrow & \\ & & \overline{C}_{\mathcal{H}}^{\text{pr}} & & \overline{(C_{\mathcal{H}}^{\text{pr}})}^{\text{pr}} & & \\ \Pi \downarrow & \nearrow \Phi' & \nearrow \Phi' & \nearrow \Phi' & \nearrow \Phi' & \nearrow \Phi' & \\ C_{\mathcal{H}} & & & & & & \end{array}$$

Applying the procedure in Section 4.4 to $\Phi' : C_{\mathcal{H}} \rightarrow \overline{C}_{\mathcal{H}}^{\text{pr}}$, we obtain a strict 2-natural transformation $\underline{\Phi}'$ in $\text{End}_{\mathcal{C}\text{-afmod}}(\underline{C}_{\mathcal{H}})$ which swaps the isomorphism classes of indecomposable projectives.

On the other hand, sending $\mathbb{1}_i$ to L_G extends to a strict 2-natural transformation $\Psi : P_i \rightarrow \overline{C}_{\mathcal{H}}$, and the latter induces a strict 2-natural transformation Ψ' from $C_{\mathcal{H}}$

to $\overline{C_{\mathcal{H}}}$ (which factors through $\overline{C_{\mathcal{H}}}^{\text{pr}}$). For Ψ , we have a diagram similar to the one in (22). Note that $\overline{\Psi'}\Psi(\mathbb{1}_i) = L_G$ and $\overline{\Phi'}(L_F) \cong L_G$. For a fixed isomorphism $\alpha : L_G \rightarrow \overline{\Phi'}(L_F)$, by Section 4.4 there exists an invertible modification ϑ from $\tilde{\Psi} = \overline{\Psi'}\Psi'$ to $\tilde{\Phi} = \overline{\Phi'}\Phi'$ (here both equalities are in $\text{Hom}_{\mathcal{C}\text{-afmod}}(C_{\mathcal{H}}, (\overline{C_{\mathcal{H}}}^{\text{pr}})^{\text{pr}})$). Note that the limiting construction (21) applied to $\tilde{\Psi}$ gives a functor isomorphic to $\text{Id}_{C_{\mathcal{H}}}$. Using Subsections 4.3 and 4.4, we thus get an invertible modification ϑ from $\text{Id}_{C_{\mathcal{H}}}$ to $(\Phi')^2$. Following [Mackaay and Mazorchuk 2017, Lemma 18] and the proof of [Mackaay and Mazorchuk 2017, Proposition 19], we obtain that

- (a) for any $v \in \text{Hom}_{\mathcal{C}\text{-afmod}}(\text{Id}_{C_{\mathcal{H}}}, (\Phi')^2)$, we have $\text{id}_{(\Phi')^2} \circ_h v = v \circ_h \text{id}_{(\Phi')^2}$;
- (b) there exists an invertible modification $v \in \text{Hom}_{\mathcal{C}\text{-afmod}}(\text{Id}_{C_{\mathcal{H}}}, (\Phi')^2)$ such that we have either $\text{id}_{\Phi'} \circ_h v = v \circ_h \text{id}_{\Phi'}$ or $\text{id}_{\Phi'} \circ_h v = -v \circ_h \text{id}_{\Phi'}$.

Note that $(\Phi')^2$ preserves the isomorphism classes of projectives and hence defines an autoequivalence of $C_{\mathcal{H}}$ which is isomorphic to the identity. Therefore Φ' induces an automorphism φ of A and such that φ^2 , corresponding to $(\Phi')^2$, is an inner automorphism of A , see [Zimmermann 2014, Lemma 1.10.9]. Assume that the inner automorphism φ^2 is of the form $x \mapsto axa^{-1}$, where $x \in A$, for some fixed invertible element $a \in A$. Similarly to the paragraph before [Kildetoft et al. 2019, Proposition 39], there exists an element $b \in A$ which is a polynomial in a^{-1} and such that $b^2 = a^{-1}$. Let σ be the inner automorphism of A given by $x \mapsto bxb^{-1}$, for $x \in A$.

Lemma 23. $(\sigma\varphi)^4 = \text{id}_A$.

Proof. The obvious fact that φ and φ^2 commute is equivalent to the requirement that $t := \varphi(a^{-1})a$ belongs to the center of A . We have

$$\varphi(t) = \varphi^2(a^{-1})\varphi(a) = aa^{-1}a^{-1}\varphi(a) = a^{-1}\varphi(a) = t^{-1}.$$

Therefore $\varphi^2(t) = t = a\varphi(a^{-1})aa^{-1} = a\varphi(a^{-1})$, which implies that $\varphi(a^{-1})$ and a commute. Consequently, $\varphi(a^{-1})$ and a^{-1} commute. This implies that any polynomial in $\varphi(a^{-1})$ commutes with any polynomial in a^{-1} . Therefore $\varphi(b)$ and b commute and thus $\varphi(b^{-1})$ and b commute as well. Hence the elements $a, a^{-1}, b, b^{-1}, \varphi(a), \varphi(a^{-1}), \varphi(b), \varphi(b^{-1})$ all commute.

A direct computation shows that the action of $(\sigma\varphi)^4$ on A is given by conjugation with

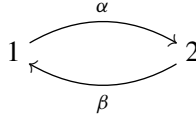
$$b\varphi(b)aba^{-1}\varphi(a)\varphi(b)\varphi(a^{-1})a^2.$$

Using commutativity of the factors, this reduces to $a\varphi(a^{-1})$ which is central. The claim follows. $\square$

The functor of twisting A -modules by σ is isomorphic to the identity functor as σ is inner. By Lemma 20, the functor of twisting by σ gives rise to an endomorphism Σ of $\underline{C_{\mathcal{H}}}$ which preserves the isomorphism classes of projectives. Then the 2-natural

transformation $\Omega := \Sigma \xrightarrow{\Phi'} \in \text{End}_{\mathcal{C}\text{-afmod}}(\underline{\mathcal{C}}_{\mathcal{H}})$ induces an automorphism on A given by $\sigma\varphi$. We denote this automorphism by ι .

Example 24. Let A be the quotient of the path algebra of the quiver



modulo the relations $\alpha\beta = \beta\alpha = 0$. Let φ be the automorphism of A defined by $\varphi(e_1) = e_2$, $\varphi(e_2) = e_1$, $\varphi(\alpha) = -\beta$ and $\varphi(\beta) = \alpha$. Then $\varphi^4 = \text{id}_A$ but $\varphi^2 \neq \text{id}_A$. In fact, φ^2 is conjugation by $a = e_1 - e_2$. Note that the element $\varphi(a^{-1})a = -e_1 - e_2$ is central. This example shows that $\xrightarrow{\Phi'}$ does not necessarily correspond to an automorphism of order 2.

4.6. Connection to $\mathcal{G}_A$. Set G to be the cyclic group generated by ι (note that $|G| = 2$ or $|G| = 4$) and consider the fiat 2-category $\mathcal{G}_A$, where A is the underlying algebra of $\mathcal{C}_{\mathcal{H}}$. Let $\mathcal{H}_A$ denote the full and faithful 2-subcategory of $\mathcal{G}_A$ generated by $(A, \tilde{\pi}_{1_{\hat{G}}})$ and 1-morphisms in the two-sided cell $\mathcal{J}_0$, referring to Section 3.3 and Section 3.4 for notation.

Theorem 25. *If $\mathcal{C}$ is $\mathcal{H}$ -simple, then $\mathcal{C}$ is biequivalent to a 2-subcategory of $\mathcal{H}_A$.*

Proof. As mentioned above, $\mathcal{C}_{\mathcal{H}}$ is equivalent to the cell 2-representation $\mathcal{C}_{\mathcal{H}}$. As $\mathcal{C}$ is $\mathcal{H}$ -simple, the 2-representation $\xrightarrow{\mathcal{C}_{\mathcal{H}}}$ gives a faithful 2-functor from $\mathcal{C}$ to $\mathcal{C}_A$. Note that the 1-morphisms F and G are represented, respectively, by X, Y in (17) under the 2-functor $\xrightarrow{\mathcal{C}_{\mathcal{H}}}$. Assume that the family of natural isomorphisms

$$\eta := \left\{ \eta_H : \Omega \circ \xrightarrow{\mathcal{C}_{\mathcal{H}}} (H) \rightarrow \xrightarrow{\mathcal{C}_{\mathcal{H}}} (H) \circ \Omega \mid H \in \mathcal{C}(\mathbf{i}, \mathbf{i}) \right\}$$

is the data associated to the 2-natural transformation $\Omega \in \text{End}_{\mathcal{C}\text{-afmod}}(\xrightarrow{\mathcal{C}_{\mathcal{H}}})$ constructed above. Thus, for any 2-morphism $\alpha : H \rightarrow K$ in $\mathcal{C}$, we have

$$(23) \quad (\xrightarrow{\mathcal{C}_{\mathcal{H}}}(\alpha) \circ_h \Omega) \circ_v \eta_H = \eta_K \circ_v (\Omega \circ_h \xrightarrow{\mathcal{C}_{\mathcal{H}}}(\alpha)).$$

Due to the fact that Ω swaps the isomorphism classes of projectives in $\xrightarrow{\mathcal{C}_{\mathcal{H}}}(\mathbf{i})$, all A - A -bimodule homomorphisms corresponding to nonzero 2-morphisms in $\mathcal{C}$ are symmetric in the sense that they are uniquely determined by their images on the representatives of distinct G -orbits of indecomposable direct summands of the source and the images on the remaining indecomposable summands of the source can be obtained by (23). $\square$

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VOLODYMYR MAZORCHUK
 DEPARTMENT OF MATHEMATICS
 UPPSALA UNIVERSITY
 UPPSALA
 SWEDEN
 mazor@math.uu.se

VANESSA MIEMIETZ
 SCHOOL OF MATHEMATICS
 UNIVERSITY OF EAST ANGLIA
 NORWICH
 UNITED KINGDOM
 v.miemietz@uea.ac.uk

XIAOTING ZHANG
 ACADEMY FOR MULTIDISCIPLINARY STUDIES
 CAPITAL NORMAL UNIVERSITY
 BEIJING
 CHINA
 xiaoting.zhang09@hotmail.com

THE HOMOTOPY GROUPS OF THE η -PERIODIC MOTIVIC SPHERE SPECTRUM

KYLE ORMSBY AND OLIVER RÖNDIGS

We compute the homotopy groups of the η -periodic motivic sphere spectrum over a field k of finite cohomological dimension with characteristic not 2 and in which -1 is a sum of four squares. We also study the general characteristic 0 case and show that the α_1 -periodic slice spectral sequence over $\mathbb{Q}$ determines the α_1 -periodic slice spectral sequence over all extensions $k/\mathbb{Q}$. This leads to a speculation on the role of a “connective Witt-theoretic J -spectrum” in η -periodic motivic homotopy theory.

1. Introduction

The motivic sphere spectrum S is the unit object in the tensor triangulated stable homotopy category of motivic spectra $(SH^{\mathbb{A}^1}(k), \wedge)$ over a field k . In this category, both the simplicial circle S^1 and the punctured affine line $\mathbb{A}^1 \setminus 0$ are $\wedge$ -invertible, so it is crucial that we understand the bigraded homotopy groups $\pi_{\star} S := \bigoplus_{m,n \in \mathbb{Z}} \pi_{m+n\alpha} S$ where $\pi_{m+n\alpha} S := [(S^1)^{\wedge m} \wedge (\mathbb{A}^1 \setminus 0)^{\wedge n}, S]_{SH^{\mathbb{A}^1}(k)}$. See the introduction to [Ormsby et al. 2018] for a more complete discussion of the importance of this ring.

The motivic Hopf map $\eta \in \pi_{\alpha} S$ which is represented by the canonical $\mathbb{G}_m$ -torsor $\mathbb{A}^2 \setminus 0 \rightarrow \mathbb{P}^1$ plays an especially important role in $\pi_{\star} S$. This class is nonnilpotent over all fields [Morel 2004] and thus represents a first example of exotic behavior in $\pi_{\star} S$, differentiating it from the classical stable stems. (Recall that $\eta^4 = 0$ classically, and that the Nishida nilpotence theorem [1973] tells us that all classes of nonzero degree in the classical stable stems are nilpotent.) Let

$$\eta^{-1} S := \operatorname{hocolim}(S \xrightarrow{\eta} \Sigma^{-\alpha} S \xrightarrow{\eta} \Sigma^{-2\alpha} S \xrightarrow{\eta} \Sigma^{-3\alpha} S \xrightarrow{\eta} \dots)$$

denote the η -periodic sphere spectrum.¹ We have $\pi_{\star} \eta^{-1} S \cong \eta^{-1} \pi_{\star} S$ (where the latter term represents the localization of the ring $\pi_{\star} S$ at the multiplicative set $\{\eta, \eta^2, \dots\}$),

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¹Other authors have referred to this object as the η -local or η -inverted sphere. We have chosen our terminology to match the language of classical v_n -periodic homotopy theory, which seems appropriate given the emerging role of η in motivic nilpotence and periodicity [Andrews 2018; Gheorghe 2017].

so inverting η annihilates $\Gamma_\eta := \{x \in \pi_\star S \mid x\eta^N = 0 \text{ for some } N\}$ and induces an injection $\pi_\star S / \Gamma_\eta \rightarrow \pi_\star \eta^{-1} S$.

A number of authors have studied $\pi_\star \eta^{-1} S$ over particular fields, including M. Andrews and H. Miller [2017] over $\mathbb{C}$, B. Guillou and D. Isaksen [2015; 2016] over $\mathbb{C}$ and $\mathbb{R}$, and G. Wilson [2018] over finite fields, local fields, and $\mathbb{Q}$. Over $\mathbb{C}$, $\pi_{m+n\alpha} S \cong W(\mathbb{C}) \cong \mathbb{Z}/2$ for nonnegative m congruent to 0 or 3 mod 4, whereas more complicated “image of J ”-style patterns occur in $\pi_\star \eta^{-1} S_2^\wedge$ (the bigraded homotopy groups of the η -periodic 2-complete sphere) over $\mathbb{R}$ and $\mathbb{Q}$. These authors work with either the motivic Adams–Novikov or motivic Adams spectral sequence in order to produce their results. In addition to these results, Röndigs [2018] has shown that $\pi_1 \eta^{-1} S = \pi_2 \eta^{-1} S = 0$ over all fields of characteristic different from 2.

In this note, we use the α_1 -periodic slice spectral sequence to completely determine $\pi_\star \eta^{-1} S$ over finite-cohomological dimension fields with characteristic different from 2 in which -1 is a sum of four squares.² Let $W(k)$ denote the Witt ring of quadratic forms over k modulo the hyperbolic plane.

Main Theorem (see Theorem 4.8). *Let k be a field of finite cohomological dimension with characteristic not 2. If -1 is a sum of four squares in k , then*

$$\pi_\star \eta^{-1} S \cong W(k)[\eta^{\pm 1}, \sigma, \mu]/(\sigma^2),$$

where $|\sigma| = 3 + 4\alpha$ and $|\mu| = 4 + 5\alpha$. In particular, the bigraded homotopy groups of $\eta^{-1} S$ are

$$\pi_{m+n\alpha} \eta^{-1} S \cong \begin{cases} W(k) & \text{if } m \geq 0 \text{ and } m \equiv 0 \text{ or } 3 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$$

In Corollary 4.9 (see also [Ormsby et al. 2018, Theorem 5.5]), we see that for fields satisfying the same hypotheses,

$$\pi_{m+n\alpha} S \cong \pi_{m+n\alpha} \eta^{-1} S$$

for $2n \geq \max\{3m+5, 4m\}$, so we have also computed a bi-infinite range of homotopy groups of the motivic sphere spectrum.

The picture is less clear for fields in which -1 is not a sum of four squares, but we are able to produce some partial results in Section 4. Let $\alpha_1^{-1} \mathcal{S}(k)$ denote the α_1 -periodic slice spectral sequence over k . In Theorem 4.5, we show that $\alpha_1^{-1} \mathcal{S}(\mathbb{Q})$ determines $\alpha_1^{-1} \mathcal{S}(k)$ for any field extension $k/\mathbb{Q}$. This leads to a conjecture on the differentials in $\alpha_1^{-1} \mathcal{S}(k)$ and some speculations regarding the structure of $\pi_\star \eta^{-1} S$ in general.

²The smallest r such that -1 is a sum of r squares in k is called the *level* of k and is often denoted $s(k)$. (The s is for *Stufe*.) By a theorem of Pfister [1965], $s(k)$ is always a power of 2. See [Lam 2005, Examples XI.2.4] for examples of fields of various levels.

2. The α_1 -periodic slice spectral sequence

In this section, we set up the α_1 -periodic slice spectral sequence and discuss its convergence properties and first two pages over a general field of characteristic different from 2.

We refer to [Röndigs et al. 2019] for the setup of the slice spectral sequence, and only briefly recall some pertinent facts here. The sphere spectrum S is effective and thus has a slice tower of the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & f_3 S & \longrightarrow & f_2 S & \longrightarrow & f_1 S & \longrightarrow & f_0 S = S \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & s_3 S & & s_2 S & & s_1 S & & s_0 S \end{array}$$

where $f_q S$ is the q -th effective cover of S and $s_q S$ is the q -th slice of S . Associated with this tower is the slice spectral sequence $\mathcal{S}$ with E_1 -page $\mathcal{S}_1^{q,m+n\alpha} = \pi_{m+n\alpha} s_q S$. (If we need to refer to the base field, then we will denote this spectral sequence by $\mathcal{S}(k)$.) The differentials take the form $d_r : \mathcal{S}_r^{q,m+n\alpha} \rightarrow \mathcal{S}_r^{q+r,m-1+n\alpha}$.

The following theorem states the basic convergence properties of $\mathcal{S}$; it is a concatenation of [Röndigs et al. 2019, Theorem 3.50] and [Ormsby et al. 2018, Theorem 1.3]. Recall that the η -complete sphere spectrum is $S_\eta^\wedge := \operatorname{holim} S/\eta^n$ where S/η^n is the cofiber of $\eta^n : \Sigma^{n\alpha} S \rightarrow S$.

Theorem 2.1. *The slice spectral sequence for S conditionally converges to $\pi_\star S_\eta^\wedge$. Moreover, if $\operatorname{cd} k < \infty$, then $S_\eta^\wedge \simeq S$ and the slice spectral sequence for S converges conditionally to $\pi_\star S$.*

We also have control over $\mathcal{S}_1$ via the following slice computation. We let $\operatorname{Ext}_{\operatorname{MU}_* \operatorname{MU}}^{s,t}$ denote the cohomology of the MU-Hopf algebroid in cohomological degree s and internal degree t . (Recall that this is the E_2 -page of the Novikov, i.e., MU-Adams, spectral sequence from topology.)

Theorem 2.2 [Röndigs et al. 2019, Theorem 2.2]. *The q -th slice of the motivic sphere spectrum is*

$$s_q S \simeq \bigvee_{s \geq 0} \Sigma^{q-s+q\alpha} H \operatorname{Ext}_{\operatorname{MU}_* \operatorname{MU}}^{s,2q}$$

at least after inverting the exponential characteristic of the base field.

We refer to [Ravenel 1986] for basic facts about $\operatorname{Ext}_{\operatorname{MU}_* \operatorname{MU}}$, and we use its naming conventions for elements. Importantly, there is a single nonzero class $\alpha_1 \in \operatorname{Ext}_{\operatorname{MU}_* \operatorname{MU}}^{1,2}$ that represents η . Multiplication by α_1 induces a map of spectral sequences $\alpha_1 : \mathcal{S}^{q,m+n\alpha} \rightarrow \mathcal{S}^{q+1,m+(n+1)\alpha}$. Taking the colimit of the tower given by iterating this map produces the α_1 -periodic slice spectral sequence, $\alpha_1^{-1} \mathcal{S}$. We will analyze the target and convergence properties of $\alpha_1^{-1} \mathcal{S}$ momentarily, but it

certainly appears that this construction ought to say something about $\eta^{-1}S$ or a related object.

There is another obvious spectral sequence we could consider, namely the slice spectral sequence for $\eta^{-1}S$, but it turns out that the two spectral sequences are the same. For a motivic spectrum X , recall from [Röndigs et al. 2019, Definition 3.1] that $\mathrm{sc}(X)$ is the *slice completion* of X .

Theorem 2.3. *The slice spectral sequence for $\eta^{-1}S$ and $\alpha_1^{-1}\mathcal{S}$ are isomorphic as spectral sequences. They both have first page additively isomorphic to*

$$\pi_{\star}H\mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3, \alpha_4]/(\alpha_4^2)$$

and conditionally converge to $\pi_{\star}\mathrm{sc}(\eta^{-1}S)$ in the sense of [Boardman 1999]. If $\mathrm{cd} k < \infty$, then $\mathrm{sc}(\eta^{-1}S) \simeq \eta^{-1}S$ and the isomorphic spectral sequences conditionally converge to $\pi_{\star}\eta^{-1}S$.

Remark 2.4. In Corollary 2.9 and Section 3, we will see that convergence is in fact strong if k has odd characteristic or $\mathrm{cd}_2 k < \infty$. In Section 4, we will see that convergence over characteristic 0 fields to $\pi_{\star}\mathrm{sc}(\eta^{-1}S)$ is strong.

Remark 2.5. As a ring object, $s_{\star}\eta^{-1}S$ is not an $H\mathbb{F}_2$ -algebra [Röndigs et al. 2019, Remark 2.33], and our identification of $\alpha_1^{-1}\mathcal{S}_1$ in Theorem 2.3 is not multiplicative. By a bidegree argument and the general properties of slice multiplicativity given in [Röndigs et al. 2019, Section 2.4], the multiplication on $\alpha_1^{-1}\mathcal{S}_1$ agrees with the “naive” multiplication up to addition of some terms involving Sq^1 . Our determination of $\alpha_1^{-1}\mathcal{S}_2$ does not depend on the precise multiplicative structure, and we will see in Theorem 2.6 that the multiplication on $\alpha_1^{-1}\mathcal{S}_2$ is fairly simple.

Proof of Theorem 2.3. Let E denote the slice spectral sequence for $\eta^{-1}S$. Then

$$E_1 \cong \pi_{\star}H\mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3, \alpha_4]/(\alpha_4^2)$$

by [Röndigs et al. 2019, Theorem 2.35].³ In particular, the canonical map $\mathcal{S} \rightarrow E$ takes α_1 to a unit and hence induces a map $\alpha_1^{-1}\mathcal{S} \rightarrow E$. By [Andrews and Miller 2017, Corollary 6.2.3], $\alpha_1^{-1}\mathrm{Ext}_{\mathrm{MU}_{\star}\mathrm{MU}} \cong \mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3, \alpha_4]/(\alpha_4^2)$. Given this result and the form of $\mathcal{S}_1$ in Theorem 2.2, we conclude that $\alpha_1^{-1}\mathcal{S}_1 \rightarrow E_1$ is an isomorphism, and it follows that $\alpha_1^{-1}\mathcal{S} \cong E$.

The first convergence statement in Theorem 2.1 is formal given the construction of the slice spectral sequence (see [Röndigs et al. 2019, §3.1]). For the second convergence statement, Theorem 2.2 tells us that $\mathrm{sc}(S) \simeq S$ when $\mathrm{cd} k < \infty$. Given the conditional convergence conditions of [Boardman 1999, Definition 5.10], our result follows as long as the sequential colimit that inverts α_1 commutes with

³Although Theorem 2.2 holds after inverting the exponential characteristic, the slices $s_{\star}\eta^{-1}S$ are known without inverting the exponential characteristic. The reason is that if k is a field of odd characteristic p , multiplication with p is an isomorphism on the Witt ring of k , and hence on $\eta^{-1}S$.

the limit defining slice completion. Our assumption on cohomological dimension implies a vanishing line parallel to α_1 -multiplication, and hence the limit in question is finite and commutes with sequential colimits. $\square$

This leads us to the main theorem of this section, a determination of the first slice differentials and $\alpha_1^{-1}\mathcal{S}_2$:

Theorem 2.6. *Over any field k of characteristic different from 2, the first slice differential for $\eta^{-1}S$ is represented by the $(4, 4)$ -periodic matrix*

$$\begin{pmatrix} Sq^2 & 0 & \tau & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ Sq^3Sq^1 & 0 & Sq^2 + \rho Sq^1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & Sq^2 & 0 & \tau & 0 & \cdots \\ 0 & 0 & Sq^3Sq^1 & Sq^2Sq^1 + Sq^3 & Sq^2 & \rho & \tau & \cdots \\ 0 & 0 & 0 & Sq^3Sq^1 & 0 & Sq^2 + \rho Sq^1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & Sq^3Sq^1 & Sq^2Sq^1 + Sq^3 & Sq^2 + \rho Sq^1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & Sq^3Sq^1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & Sq^3Sq^1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here the i -th column, $i \geq 0$, gives the first slice differential restricted to the summand $\Sigma^{i+q\alpha} H\mathbb{F}_2$ of $s_q\eta^{-1}S$ (the summand is 0 if $i = 1$). The j -th row, $j \geq 0$, describes the incoming first slice differential for the summand $\Sigma^{j+1+(q+1)\alpha} H\mathbb{F}_2$ of $\Sigma s_{q+1}\eta^{-1}S$ (the summand is 0 if $j = 1$).

This results in an isomorphism of k_*^M -algebras

$$\alpha_1^{-1}\mathcal{S}_2 \cong k_*^M[\alpha_1^{\pm 1}, \alpha_4, \alpha_5]/(\alpha_4^2),$$

where $|\alpha_4| = (4, 3 + 4\alpha)$ and $\alpha_5 = \alpha_3^2\alpha_1^{-1}$ has degree $(5, 4 + 5\alpha)$.

Remark 2.7. The factor of α_1^{-1} in the definition of α_5 is not strictly necessary, but is there so that $\bar{\alpha}_5 \mapsto \alpha_5$ under the localization map $\mathcal{S} \rightarrow \alpha_1^{-1}\mathcal{S}$.

Remark 2.8. The determination of the first slice differential for $\eta^{-1}S$ complements the occurrences of multiplications with τ , which were used in [Ormsby et al. 2018, page 11] to deduce vanishing columns in the Andrews–Miller range of the unlocalized slice spectral sequence.

The pattern of differentials indicated by Theorem 2.6 is represented graphically in Figure 1. The form of $\alpha_1^{-1}\mathcal{S}_2$ also implies an important convergence result, which we state presently.

Corollary 2.9. *If $\text{cd}_2(k) = r < \infty$, then $\alpha_1^{-1}\mathcal{S}(k)$ collapses at its $(r+1)$ -th page and converges strongly to $\pi_*(\eta^{-1}S)$.*

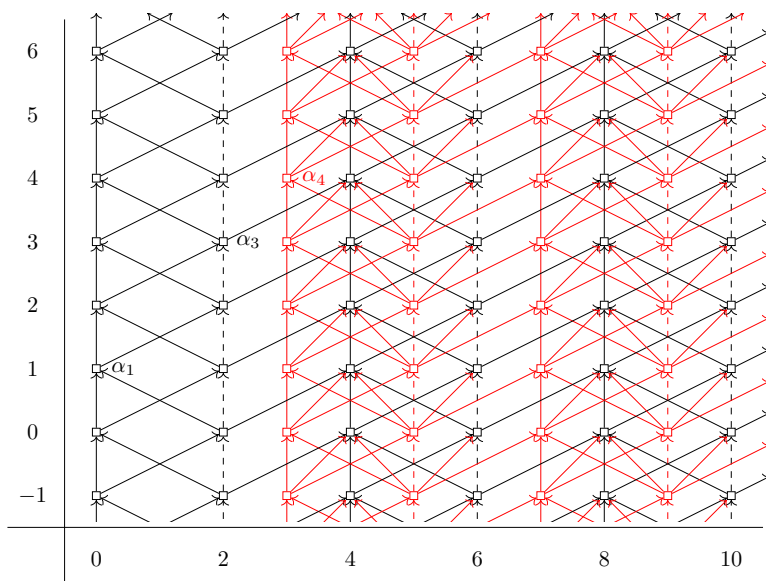


Figure 1. The first page of the α_1 -periodic slice spectral sequence with its differentials. A $\square$ in position (m, n) represents a copy of $\pi_* \Sigma^{m+n\alpha} H\mathbb{F}_2 \cong \Sigma^{m+n\alpha} k_*^M[\tau]$ in slice degree n . The black portions of the diagram are detected by the unit map $\eta^{-1}S \rightarrow kw$, and the red portions are induced by $\sigma_\infty : \Sigma^3 kw \rightarrow \eta^{-1}S$. Arrows with slope $-\frac{1}{2}$ represent τ , arrows with slope -1 represent ρ , solid vertical arrows represent Sq^2 , dashed vertical arrows represent $Sq^2 + \rho Sq^1$, arrows with slope 1 represent $Sq^2 Sq^1 + Sq^3$, and arrows with slope $\frac{1}{2}$ represent $Sq^3 Sq^1$.

Proof. The form of $\alpha_1^{-1}\mathcal{S}_2$ (which is presented graphically in Figure 2) and the fact that $k_{>r}^M(k) = 0$ imply that $d_{>r} = 0$. This collapse along with the conditional convergence of Theorem 2.1 imply the strong convergence portion of the corollary. $\square$

Our proof of Theorem 2.6 requires a fascinating detour through connective Witt K -theory. Let KQ denote the motivic spectrum representing Hermitian K -theory,⁴ let $KW = \eta^{-1}KQ$ denote the η -periodization of KQ , which is the motivic spectrum representing Balmer's higher Witt groups, and let kw denote the connective cover of KW (in the sense of Morel's homotopy t -structure).

The slices of kw , the d_1 -differentials of its slice spectral sequence, and the effect of s_* on the unit map $\eta^{-1}S \rightarrow kw$ are known.

Theorem 2.10 [Ananyevskiy et al. 2020, Theorem 19]. *Suppose the base scheme S is a scheme over $\text{Spec}(\mathbb{Z}[1/2])$. Then*

$$s_* kw \simeq H\mathbb{Z}[\eta^{\pm 1}, \sqrt{\beta}]/(2\eta, 2\sqrt{\beta}, \eta^2 \xrightarrow{Sq^1} \sqrt{\beta}),$$

⁴The Q is for Quadratic.

where $|\eta| = \alpha$, $|\sqrt{\beta}| = 2 + 2\alpha$, and the first slice differential takes the form

$$\begin{pmatrix} \text{Sq}^2 & 0 & \tau & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \text{Sq}^3\text{Sq}^1 & 0 & \text{Sq}^2 + \rho\text{Sq}^1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \text{Sq}^3\text{Sq}^1 & 0 & \text{Sq}^2 & 0 & \tau & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \text{Sq}^3\text{Sq}^1 & 0 & \text{Sq}^2 + \rho\text{Sq}^1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{Sq}^3\text{Sq}^1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(with the same conventions as Theorem 2.6). Moreover, there is a splitting of $s_0\eta^{-1}\mathbf{S}$ such that the unit $\eta^{-1}\mathbf{S} \rightarrow \mathbf{k}w$ induces an inclusion on every even summand, and Sq^1 on every odd summand.

Proof. The description of the slices, as well as their multiplicative structure, is given in [Ananyevskiy et al. 2020, Theorem 19]. The behavior of the unit map follows from [Röndigs et al. 2019, Lemmas 2.28 and 2.29]. $\square$

$n + 6$	k_6^M	k_6^M	k_6^M	k_6^M	k_6^M	k_6^M	k_6^M
$n + 5$	k_5^M	k_5^M	k_5^M	k_5^M	k_5^M	k_5^M	k_5^M
$n + 4$	k_4^M	k_4^M	k_4^M	k_4^M	k_4^M	k_4^M	k_4^M
$n + 3$	k_3^M	k_3^M	k_3^M	k_3^M	k_3^M	k_3^M	k_3^M
$n + 2$	k_2^M	k_2^M	k_2^M	k_2^M	k_2^M	k_2^M	k_2^M
$n + 1$	k_1^M	k_1^M	k_1^M	k_1^M	k_1^M	k_1^M	k_1^M
n	k_0^M	k_0^M	k_0^M	k_0^M	k_0^M	k_0^M	k_0^M
	0	2	4	6	8	10	12

Figure 2. The weight n portion of $\alpha_1^{-1}\mathcal{S}_2$ with simplicial degree on the horizontal axis and slice degree on the vertical axis. This page is 0 below slice degree n and the k_*^M -towers extend upwards. Differentials respect weight, and d_r has degree $(-1, r)$ in this representation. If the cohomological dimension of $\mathbf{k}$ is at most 1, then this page is concentrated in slice degrees n and $n + 1$ and there are no nonzero targets for d_r , $r \geq 2$.

Note that s_*kw and the pattern of d_1 differentials are precisely the black portion of Figure 1. The remaining portion of $s_*\eta^{-1}S$ (the red part of Figure 1) is handled by the following theorem. Following the standard convention, we write Π_n for $\bigoplus_{m \in \mathbb{Z}} \pi_{n+m\alpha}$.

Theorem 2.11. *Over $k = \mathbb{C}$, there is a unique homotopy class $\sigma_\infty : \Sigma^3kw \rightarrow \eta^{-1}S$ inducing an isomorphism on Π_3 . This map induces $(1, \text{Sq}^1)$ on every summand of a slice.*

Proof. Fix $k = \mathbb{C}$. Observe first that $kw = \eta^{-1}kq$ is a cellular spectrum, where kq is the connective cover of KQ . In fact, kq is slice-finitary by [Ananyevskiy et al. 2020, Theorem 17]; hence its slice completion $\text{sc}(kq)$ is cellular by [Röndigs et al. 2019, Proposition 3.45]. It follows that the slice completion $\text{sc}(kw)$ is cellular also. On the level of homotopy sheaves, [Levine 2011, Theorem 1] implies that the slice filtration on kw induces the fundamental ideal filtration on Witt sheaves. The latter is finite over the complex numbers, whence $\text{sc}(kw) \simeq kw$ is cellular. Recall from [Röndigs 2018, Section 4] that there is a cell presentation of kw over $\mathbb{C}$ of the following form. Namely, there is a sequence of cellular motivic spectra factoring the unit of kw as

$$\eta^{-1}S = D_1 \xrightarrow{i_1} D_2 \xrightarrow{i_2} \cdots \rightarrow D_n \xrightarrow{i_n} \cdots \rightarrow kw$$

such that for every n the map $D_n \rightarrow kw$ is $(4n-1)$ -connective and the composition $\eta^{-1}S \rightarrow D_n \rightarrow kw$ induces isomorphisms on Π_{4k} . For every $n \geq 1$, there is a unique nontrivial class $a_n : \Sigma^{4n-1}\eta^{-1}S \rightarrow D_n$ in $\pi_{4n-1}D_n \cong \pi_{4n-1}\eta^{-1}S$ such that

$$\Sigma^{4n-1}\eta^{-1}S \xrightarrow{a_n} D_n \xrightarrow{i_n} D_{n+1} \xrightarrow{c_n} \Sigma^{4n}\eta^{-1}S$$

is a homotopy cofiber sequence with c_n inducing an isomorphism on Π_{4k+3} whenever $k \geq n$. Taking the colimit as $n \rightarrow \infty$ gives a cell presentation of kw .

We now construct σ_∞ . Consider the map $\sigma_1 = \eta^{-4}\sigma : \Sigma^3\eta^{-1}S = \Sigma^3D_1 \rightarrow \eta^{-1}S$. Assume for induction that for some $n \geq 1$ a map $\sigma_n : \Sigma^3D_n \rightarrow \eta^{-1}S$ is given such that

- (1) $\sigma_n i_{n-1} = \sigma_{n-1}$ and
- (2) $[\Sigma^4 a_{n-1}, \eta^{-1}S] : [\Sigma^4 D_{n-1}, \eta^{-1}S] \rightarrow [\Sigma^{4n-1} \eta^{-1}S, \eta^{-1}S]$ is an isomorphism.

Then the cofiber sequence above induces a long exact sequence

$$\begin{aligned} [\Sigma^{4n+2}\eta^{-1}S, \eta^{-1}S] &\xleftarrow{a_n} [\Sigma^3D_n, \eta^{-1}S] \xleftarrow{i_n} [\Sigma^3D_{n+1}, \eta^{-1}S] \\ &\xleftarrow{c_n} [\Sigma^{4n+3}\eta^{-1}S, \eta^{-1}S] \xleftarrow{a_n} [\Sigma^4D_n, \eta^{-1}S] \leftarrow \cdots \end{aligned}$$

The Andrews–Miller theorem [2017] on $\pi_*\eta^{-1}S$ implies

$$0 = [\Sigma^{4n+2}S, \eta^{-1}S] = [\Sigma^{4n+2}\eta^{-1}S, \eta^{-1}S],$$

showing that σ_n lifts to a map σ_{n+1} such that $\sigma_{n+1}i_n = \sigma_n$.

Now note that assumption (2) implies that

$$[\Sigma^4 c_{n-1}, \eta^{-1}S] : [\Sigma^{4n} \eta^{-1}S, \eta^{-1}S] \rightarrow [\Sigma^4 D_n, \eta^{-1}S]$$

is surjective; furthermore, the composition

$$[\Sigma^{4n} \eta^{-1}S, \eta^{-1}S] \xrightarrow{[\Sigma^4 c_{n-1}, \eta^{-1}S]} [\Sigma^4 D_n, \eta^{-1}S] \xrightarrow{[\Sigma^4 a_n, \eta^{-1}S]} [\Sigma^{4n+3} \eta^{-1}S, \eta^{-1}S]$$

is the map sending $\eta^{-5n} \mu^n$ to $\eta^{-5n-4} \sigma \mu^n$, and hence an isomorphism. It follows that the map $[\Sigma^4 a_n, \eta^{-1}S]$ is an isomorphism, as desired. (In fact, we also get that there is a *unique* $\sigma_{n+1} : \Sigma^3 D_{n+1} \rightarrow \eta^{-1}S$ such that $\sigma_{n+1} i_n = \sigma_n$.)

Induction and the universal property of colimits now produce a map

$$\sigma_\infty : \Sigma^3 D_\infty \simeq \Sigma^3 \text{kw} \rightarrow \eta^{-1}S$$

sending $1 \in \pi_3 \Sigma^3 \text{kw}$ to $\eta^{-4} \sigma \in \pi_3 \eta^{-1}S$. The uniqueness of σ_∞ follows from the Milnor exact sequence and the vanishing of $\lim_n^1 [\Sigma^4 D_n, \eta^{-1}S]$ (every group $[\Sigma^4 D_n, \eta^{-1}S]$ being finite of order 2).

Since σ_∞ is a map of $\eta^{-1}S$ -modules, it induces isomorphisms on Π_{4m+3} for every integer m . The statement on slices follows from the behavior of the unit map $\eta^{-1}S \rightarrow \text{kw}$ on slices given in Theorem 2.10. $\square$

Proof of Theorem 2.6. As $\eta^{-1}S$ and $s_* \eta^{-1}S$ are invariant under base change, it suffices to determine the first slice differential d_1 over $\mathbb{Z}[1/2]$. On a summand $\Sigma^n H\mathbb{F}_2$, it is of the form

$$(x_n \tau, a_n \tau \text{Sq}^1 + \beta_n, b_n \text{Sq}^2 + \gamma_n, c_n \text{Sq}^2 \text{Sq}^1 + d_n \text{Sq}^3, e_n \text{Sq}^3 \text{Sq}^1)$$

with $x_n, a_n, b_n, c_n, d_n, e_n$ elements in $\mathbb{Z}/2$, and β_n, γ_n square classes of units in $\mathbb{Z}[1/2]$. The behavior of the unit map $\eta^{-1}S \rightarrow \text{kw}$ on slices from Theorem 2.10 provides immediate restrictions:

$$\begin{aligned} x_{4n} &= 0, & a_{4n} \rho + \beta_{4n} &= \gamma_{4n}, & b_{4n} &= 1, & e_{4n} &= 1 + c_{4n}, \\ x_{4n+1} + a_{4n+1} &= 1, & (a_{4n+1} + 1) \rho &= \beta_{4n+1}, & c_{4n+1} &= 1, & b_{4n+1} &= d_{4n+1}, \\ x_{4n+2} &= 1, & (a_{4n+2} + 1) \rho + \beta_{4n+2} &= \gamma_{4n+2}, & b_{4n+2} &= 1, & e_{4n+2} &= 1 + c_{4n+2}, \\ x_{4n+3} &= a_{4n+3}, & a_{4n+3} \rho &= \beta_{4n+3}, & c_{4n+3} &= 1, & b_{4n+3} &= d_{4n+3}. \end{aligned}$$

The map $\sigma_\infty : \Sigma^3 \text{kw}_{\mathbb{C}} \rightarrow \eta^{-1}S_{\mathbb{C}}$ on slices from Theorem 2.11 imposes further restrictions:

$$\begin{aligned} x_{4n+1} &= 1, & a_{4n+1} &= 0, & e_{4n+1} &= 1 + d_{4n+2}, \\ b_{4n+1} &= c_{4n+1} = d_{4n+1} = 1, & b_{4n+3} &= c_{4n+3} = d_{4n+3} = 1, \\ x_{4n+3} &= 0 = a_{4n+3}, & e_{4n+3} &= 1 + d_{4n+4}. \end{aligned}$$

Base change and the previous equations then provide the following equations:

$$\beta_{4n+1} = \rho, \quad \gamma_{4n+1} + \beta_{4n+2} = \rho, \quad \beta_{4n+3} = \gamma_{4n+3} = \gamma_{4n} = \beta_{4n} = \beta_{4n+2} = 0.$$

Since the composition $d_1 \circ d_1$ is equal to 0, Adem relations imply further coefficients. Considering the component

$$d_1^2 : \Sigma^{4n+2} H\mathbb{F}_2 \rightarrow \Sigma^{4n+1+2\alpha} H\mathbb{F}_2$$

implies that $a_{4n+2} = a_{4n} = 0$. Considering the component

$$d_1^2 : \Sigma^{4n} H\mathbb{F}_2 \rightarrow \Sigma^{4n+2+2\alpha} H\mathbb{F}_2$$

provides that $c_{4n} = d_{4n} = 0$, and hence $e_{4n} = 1 = e_{4n+3}$. The similar component

$$d_1^2 : \Sigma^{4n+2} H\mathbb{F}_2 \rightarrow \Sigma^{4n+4+2\alpha} H\mathbb{F}_2$$

gives only $c_{4n+2} = d_{4n+2}$, and hence $e_{4n+1} = e_{4n+2}$. Resorting to s_*S provides the solution $c_{4n+2} = d_{4n+2} = 0$ and $e_{4n+1} = e_{4n+2} = 1$ as follows. Consider the summand $\Sigma^{2n+(2n+1)\alpha} H\mathbb{F}_2$ in $s_{2n+1}S$ generated by α_{2n+1} . The first slice differential maps it via $\text{inc}_{y_{2n+2}}^2 \text{Sq}^2 \text{Sq}^1$ to the top degree summand $\Sigma^{2n+1+(2n+2)\alpha} H\mathbb{Z}/(y_{2n+2})$ in $s_{2n+2}S$ by [Röndigs et al. 2019, Lemma 4.1]; here y_{2n+2} is the order of a cyclic group and divisible by four, and $\text{inc}_{y_{2n+2}}^2$ is the map $H\mathbb{F}_2 \rightarrow H\mathbb{Z}/(y_{2n+2})$ induced by the canonical inclusion $\mathbb{Z}/(2) \rightarrow \mathbb{Z}/(y_{2n+2})$. The map

$$\Sigma^\alpha \Sigma^{2n+1+(2n+2)\alpha} H\mathbb{Z}/(y_{2n+2}) \rightarrow \Sigma^{2n+1+(2n+3)\alpha} H\mathbb{F}_2$$

induced by η is the projection $\text{pr}_2^{y_{2n+2}}$. Hence already after one multiplication with η , the degree $2 + \alpha$ part of the first differential is zero on that summand. It follows that $c_{2n} = 0$.

Given the form of the differentials, the additive calculation of $\alpha_1^{-1}\mathcal{S}_2$ is nearly the same as the proof of [Röndigs and Østvær 2016, Theorem 6.3]. The exotic multiplication on $\alpha_1^{-1}\mathcal{S}_1$ mentioned in Remark 2.5 reduces to $k_*^M[\alpha_1^{\pm 1}, \alpha_4, \alpha_5]/(\alpha_4^2)$ in the subquotient $\alpha_1^{-1}\mathcal{S}_2$ since Sq^1 is only potentially nonzero on terms involving an odd power of τ , and there are no τ 's in $\alpha_1^{-1}\mathcal{S}_2$. $\square$

3. Computations for fields with odd characteristic or cohomological dimension at most 1

Given the form of $\alpha_1^{-1}\mathcal{S}_2$ and the spectral sequence's convergence properties determined in the previous section, we can now make short work of the following computations.

Proposition 3.1. *If $\text{cd}_2 k \leq 1$, then the α_1 -periodic slice spectral sequence for S collapses with $\alpha_1^{-1}\mathcal{S}_2 = \alpha_1^{-1}\mathcal{S}_\infty$ and converges strongly to $\pi_*\eta^{-1}S$.*

Proof. This is a specialization of Corollary 2.9. $\square$

Theorem 3.2. *If k has odd characteristic, then the α_1 -periodic slice spectral sequence for S collapses with $\alpha_1^{-1}\mathcal{S}_2 = \alpha_1^{-1}\mathcal{S}_\infty$.*

Proof. Suppose k has characteristic p and write i for the map $\mathrm{Spec} k \rightarrow \mathrm{Spec} \mathbb{F}_p$. The essentially smooth base change functor i^* induces a map of spectral sequences $i^*: \alpha_1^{-1}\mathcal{S}_2(\mathbb{F}_p) \rightarrow \alpha_1^{-1}\mathcal{S}_2(k)$ which is given by the extension of scalars map on k_*^M and the identity on α_i for $i = 1, 4, 5$. Given the form of $\alpha_1^{-1}\mathcal{S}_2(k)$, it suffices to show $d_r^k \alpha_5 = 0$ for all $r \geq 2$, but $d_r^k \alpha_5 = i^* d_r^{\mathbb{F}_p} \alpha_5$, and $d_r^{\mathbb{F}_p} \alpha_5 = 0$ by Proposition 3.1. $\square$

At this point, we know that if k has odd characteristic or if $\mathrm{cd}_2 k \leq 1$, then the α_1 -periodic slice spectral sequence collapses with

$$\alpha_1^{-1}\mathcal{S}_2 = \alpha_1^{-1}\mathcal{S}_\infty \cong k_*^M[\alpha_1^{\pm 1}, \alpha_4, \alpha_5]/\alpha_4^2.$$

Paired with the conditional convergence portion of Theorem 2.3, this implies that the spectral sequence in fact converges strongly to $\pi_*\eta^{-1}S$. In order to completely determine $\pi_*\eta^{-1}S$ for such k , we must resolve extension problems and understand the multiplicative structure.

Suppose that $s \equiv 0$ or $3 \pmod{4}$, and consider the short exact sequences

$$(3-1) \quad 0 \rightarrow f_{q+1}\pi_s\eta^{-1}S \rightarrow f_q\pi_s\eta^{-1}S \rightarrow k_q^M \rightarrow 0$$

obtained from the slice filtration and the determination of $\alpha_1^{-1}\mathcal{S}_\infty$ for a field of odd characteristic. Choose a lift $g_s \in f_0\pi_s\eta^{-1}S = \pi_s\eta^{-1}S$ of the nontrivial element in k_0^M , compatible with field extensions from the prime field. If $s = 0$, $\pi_0\eta^{-1}S$ is known to be the Witt ring by Morel's theorem, and g_0 should be chosen as the unit. The slice filtration on $\pi_0\eta^{-1}S$ coincides with the filtration by powers of the fundamental ideal I , as one deduces for example from [Levine 2011]. The multiplicative structure on the slice filtration then supplies a natural transformation to the sequence (3-1) from the short exact sequence

$$0 \rightarrow I^{q+1} \rightarrow I^q \rightarrow h^{q,q} \rightarrow 0$$

solving Milnor's conjecture on quadratic forms [Orlov et al. 2007]. The convergence statement Corollary 2.9 shows that this natural transformation is an isomorphism for fields of finite cohomological dimension. Since the constructions involved commute with filtered colimits of fields, it is thus an isomorphism for any field of odd characteristic. In particular, the slice filtration is Hausdorff by the main result of [Arason and Pfister 1971].

Proposition 3.3. *If k has odd characteristic or if $\mathrm{cd}_2 k \leq 1$, then, as a ring,*

$$\pi_*\mathrm{sc}(\eta^{-1}S) \cong W(k)[\eta^{\pm 1}, \sigma, \mu]/(\sigma^2),$$

where $|\eta| = \alpha$, $|\sigma| = 3 + 4\alpha$, and $|\mu| = 4 + 5\alpha$. If, additionally, $\mathrm{cd} k < \infty$,

then $\mathrm{sc}(\eta^{-1}S) \simeq \eta^{-1}S$ and this is a computation of the η -periodic homotopy groups of the motivic sphere spectrum.

Proof. The additive structure (which is simply a copy of $W(k)$ in nonnegative simplicial degrees congruent to 0 or 3 mod 4) follows from the above filtration considerations. There is no room for hidden extensions, so the result follows. $\square$

4. Characteristic 0 fields

We now consider the α_1 -periodic slice spectral sequence over a general field k of characteristic 0. We prove that for any k this spectral sequence converges strongly to $\pi_* \mathrm{sc}(\eta^{-1}S)$. Moreover, the spectral sequence over $\mathbb{Q}$ completely determines the spectral sequence over k in a manner that we make precise in Theorem 4.5. This allows us to extend the conclusion of Proposition 3.3 to fields with $\mathrm{cd}_2 k \leq 2$ and to extensions of $\mathbb{Q}(\sqrt{-1})$, resulting in Theorem 4.8. We conclude with a conjectural description of the differentials which we hope will inspire further work on this problem.

The structure of our argument is somewhat surprising. After proving that $d_2^{\mathbb{Q}} = 0$, we are able to put strong restrictions on the form of the differentials which may appear in $\alpha_1^{-1}\mathcal{S}(\mathbb{Q})$. We then employ a theorem of Orlov, Vishik and Voevodsky [Orlov et al. 2007] to show that for arbitrary $k/\mathbb{Q}$, the differentials in $\alpha_1^{-1}\mathcal{S}(k)$ are of the same form. The proscribed form of the differentials guarantees that Boardman's $RE_\infty = 0$, whence strong convergence follows. The primary obstruction to computing the differentials seems to be the lack of a good description of $\mathrm{sc}(\eta^{-1}S)$.

We make some preliminary definitions in order to start our arguments. Recall that $\alpha_1^{-1}\mathcal{S}_1 \cong \pi_* H\mathbb{F}_2[\alpha_1^{\pm 1}, \alpha_3, \alpha_4]/\alpha_4^2$. For $k \geq 0$, set $\alpha_{2k+1} := \alpha_3^k \alpha_1^{1-k}$, and for $k \geq 2$ set $\alpha_{2k} := \alpha_4 \alpha_3^{k-2} \alpha_1^{2-k}$. These classes are chosen so that $\bar{\alpha}_\ell \mapsto \alpha_\ell$ under the localization map $\mathcal{S}_1 \rightarrow \alpha_1^{-1}\mathcal{S}_1$ for all $\ell \neq 2$. Note that as a $\pi_* H\mathbb{F}_2[\alpha_1^{\pm 1}]$ -module, $\alpha_1^{-1}\mathcal{S}_1$ is generated by $1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \dots$. Also note (for the purposes of applying the Leibniz rule) that, up to multiplication by a unit, α_{4k+1} is the square of α_{2k+1} .

Lemma 4.1. *The d_2 differential in $\alpha_1^{-1}\mathcal{S}(\mathbb{Q})$ is trivial.*

Proof. It suffices to prove that $d_2^{\mathbb{Q}}\alpha_5 = 0$. We know that $d_2^{\mathbb{Q}}\alpha_5 \in k_2^M(\mathbb{Q})\{\alpha_3^3\alpha_4\}$. Base change to $\mathbb{Q}_p$ provides a comparison map $\alpha_1^{-1}\mathcal{S}(\mathbb{Q}) \rightarrow \alpha_1^{-1}\mathcal{S}(\mathbb{Q}_p)$. Since $\mathrm{cd}_2(\mathbb{Q}_p) = 2$, Corollary 2.9 implies that $\alpha_1^{-1}\mathcal{S}(\mathbb{Q}_p)$ converges strongly to $\pi_* \eta^{-1}S$. Furthermore, every class in $k_2^M(\mathbb{Q})$ is detected in some $k_2^M(\mathbb{Q}_p)$.⁵ As such, the computations of Wilson [2018] over $\mathbb{Q}_p$ imply that $d_2^{\mathbb{Q}}\alpha_5 = 0$. $\square$

⁵Indeed, [Milnor 1970, Lemma A.1] tells us that the map $k_2^M(\mathbb{Q}) \rightarrow k_2^M(\mathbb{R}) \oplus \bigoplus_p k_2^M(\mathbb{Q}_p)$ is injective and computed on components by quadratic Hilbert symbols. Hilbert reciprocity then implies that $k_2^M(\mathbb{Q}) \rightarrow \bigoplus_p k_2^M(\mathbb{Q}_p)$ is injective as well.

Theorem 4.2. *There is a nondecreasing⁶ sequence of extended integers $r_k \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$ for $k \geq 2$ such that if $r_k < \infty$ then $d_{r_k}^{\mathbb{Q}} \alpha_{2^k+1} = \rho^{r_k} \alpha_{2^k} \alpha_1^{r_k+1}$, and if $r_k = \infty$ then α_{2^k+1} is a permanent cycle in $\alpha_1^{-1} \mathcal{S}(\mathbb{Q})$. The rest of the differentials in $\alpha_1^{-1} \mathcal{S}(\mathbb{Q})$ are determined by the Leibniz rule.*

Remark 4.3. The above theorem may be thought of in the following terms. In the weight n α_1 -periodic slice spectral sequence, the $4k$ -column of $\alpha_1^{-1} \mathcal{S}_2$ is, up to multiplication by some power of the unit α_1 , generated by α_{4k+1} , and the $4k-1$ -column is generated by α_{4k} in the same sense. These columns are connected by $d_{r_{v_2(4k)}} = \cdot \rho^{r_{v_2(4k)}}$ on k_*^M , where v_2 is 2-adic valuation.

Proof. By Lemma 4.1 and Theorem 2.6,

$$\alpha_1^{-1} \mathcal{S}_3 \cong k_*^M[\alpha_1^{\pm 1}, \alpha_4, \alpha_5]/\alpha_4^2.$$

If the spectral sequence does not collapse, then the first nonzero differential is necessarily of the form $d_r \alpha_5 = x \alpha_4 \alpha_1^{r+1}$ for some $r \geq 3$ and $x \in k_r^M(\mathbb{Q})$. Since $k_r^M(\mathbb{Q}) = \mathbb{Z}/2\{\rho^r\}$ for $r \geq 3$, we in fact have $d_r \alpha_5 = \rho^r \alpha_4 \alpha_1^{r+1}$. Set r_2 equal to this r . The $\alpha_1^{-1} \mathcal{S}_{r_2+1}$ -page then has $k_*^M[\alpha_1^{\pm 1}]/\rho^{r_2}$ in positive stems congruent to 3 mod 8 and $\rho^{r_2} k_*^M[\alpha_1^{\pm 1}]$ in positive stems congruent to 4 mod 8 (where $x k_*^M = \{y \in k_*^M \mid xy = 0\}$ is the x -torsion in k_*^M); the (r_2+1) -page also continues to have $k_*^M[\alpha_1^{\pm 1}]$ in nonnegative stems congruent to 0 or 7 mod 8, and is 0 otherwise.

The potential targets of the $\rho^{r_2} k_*^M[\alpha_1^{\pm 1}]$ terms are all 0; hence these classes are permanent. Thus the next nonzero differential in the spectral sequence (if one exists) is necessarily of the form $d_{r_3} \alpha_9 = \rho^{r_3} \alpha_8 \alpha_1^{r_3+1}$. The ρ^{r_3} -torsion terms in the (r_3+1) -page are again permanent, and the next differential is of the form

$$d_{r_4} \alpha_{2^4+1} = \rho^{r_4} \alpha_{2^4} \alpha_1^{r_4+1}.$$

Proceeding inductively proves the theorem. $\square$

We now abstract the behavior observed in Theorem 4.2 and show that it is in fact generic.

Definition 4.4. For a given field k , suppose that there is a nondecreasing sequence of extended integers $r_k \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$ for $k \geq 2$ such that the differentials $d_{r_k} \alpha_{2^k+1} = \rho^{r_k} \alpha_{2^k} \alpha_1^{r_k+1}$ and the Leibniz rule determine $\alpha_1^{-1} \mathcal{S}(k)$. In this case, we call $\{r_2, r_3, \dots\}$ the *profile* of $\alpha_1^{-1} \mathcal{S}(k)$ and say that $\alpha_1^{-1} \mathcal{S}(k)$ is *determined by the profile* $\{r_k\}$.

Theorem 4.5. *Let $\{r_k\}$ denote the profile of $\alpha_1^{-1} \mathcal{S}(\mathbb{Q})$ (guaranteed to exist by Theorem 4.2). Then for any characteristic 0 field k , $\alpha_1^{-1} \mathcal{S}(k)$ is also determined by the profile $\{r_k\}$.*

⁶In fact, the sequence is strictly increasing unless it is eventually constant at ∞ .

Proof. Consider the map of spectral sequences $i^* : \alpha_1^{-1} \mathcal{S}(\mathbb{Q}) \rightarrow \alpha_1^{-1} \mathcal{S}(k)$ induced by essentially smooth base change along $\mathrm{Spec} k \rightarrow \mathrm{Spec} \mathbb{Q}$. We have

$$d_{r_k}^k \alpha_{2k+1} = i^* d_{r_k}^{\mathbb{Q}} \alpha_{2k+1} = i^* \rho^{r_k} \alpha_{2k} \alpha_1^{r_k+1} = \rho^{r_k} \alpha_{2k} \alpha_1^{r_k+1}.$$

It remains to show that ${}_{\rho^{r_k}} k_*^M[\alpha_1^{\pm 1}]\{\alpha_{2k+1}\}$ supports no higher differentials. Invoking [Orlov et al. 2007, Theorem 3.3], we see that ${}_{\rho^{r_k}} k_*^M$ is generated in degree 1 as a k_*^M -module, so it suffices to show that $d_r^k[u]\alpha_{2k+1} = 0$ for all $r > r_k$ and $u \in k^\times$ such that $[u]\rho^{r_k} = 0 \in k_{r_k+1}^M$. Fix such a u and consider the subextension $k/\mathbb{Q}(u)/\mathbb{Q}$. Let $j : \mathrm{Spec} \mathbb{Q}(u) \rightarrow \mathrm{Spec} \mathbb{Q}$ denote the corresponding map with associated map of spectral sequences $j^* : \alpha_1^{-1} \mathcal{S}(\mathbb{Q}(u)) \rightarrow \alpha_1^{-1} \mathcal{S}(k)$. Our argument now splits into two cases: u algebraic, and u transcendental.

First suppose that u is algebraic, in which case $\mathbb{Q}(u)$ is a number field. Tate's theorem [Milnor 1970, Theorem A.2] implies that $k_n^M(\mathbb{Q}(u)) = \mathbb{Z}/2\{\rho^n\}$ or 0 for $n \geq 3$ and we have already seen that $d_{r_k}^{\mathbb{Q}(u)} \alpha_{2k+1} = \rho^{r_k} \alpha_{2k} \alpha_1^{r_k+1}$. Recall that $r_k \geq 3$, so this differential kills k_*^M classes at and above degree r_k . In particular, for $r > r_k$ the target group for $d_r^{\mathbb{Q}(u)}[u]\alpha_{2k+1}$ is 0 and hence the differential is 0. Finally, we see that

$$d_r^k[u]\alpha_{2k+1} = j^* d_r^{\mathbb{Q}(u)}[u]\alpha_{2k+1} = 0$$

as well, as desired.

Now suppose that u is transcendental, in which case [Milnor 1970, Theorem 2.3] implies that there is a split short exact sequence

$$0 \rightarrow k_*^M \mathbb{Q} \rightarrow k_*^M \mathbb{Q}(u) \xrightarrow{\oplus \partial_\pi} \bigoplus_{\pi} k_{*-1}^M \mathbb{Q}[u]/(\pi) \rightarrow 0,$$

where π ranges over monic irreducible polynomials in $\mathbb{Q}[u]$ and

$$\partial_\pi : k_*^M \mathbb{Q} \rightarrow k_{*-1}^M \mathbb{Q}[u]/(\pi)$$

is the residue map taking $[\pi, u_2, u_3, \dots, u_n]$ to $[u_2, \dots, u_n]$. In particular, for $n \geq 4$, $k_n^M \mathbb{Q}(u)$ has $\mathbb{F}_2$ -basis consisting of ρ^n and $[\pi]\rho^{n-1}$ for $\pi \in \mathbb{Q}[u]$ monic irreducible. Thus the differential

$$d_{r_k}^{\mathbb{Q}(u)} \alpha_{2k+1} = \rho^{r_k} \alpha_{2k} \alpha_1^{r_k+1}$$

kills $k_*^M(\mathbb{Q}(u))[\alpha_1^{\pm 1}]\{\alpha_{2k}\}$ in Milnor-degree $r_k + 1$ and above. It follows that $d_r^{\mathbb{Q}(u)}[u]\alpha_{2k+1} = 0$ for $r > r_k$ and the same base change trick as in the previous paragraph implies that $d_r^k[u]\alpha_{2k+1} = 0$. We conclude that $\alpha_1^{-1} \mathcal{S}(k)$ is determined by the profile $\{r_k\}$. $\square$

Theorem 4.6. *Let k be any field of characteristic different from 2. Then $\alpha_1^{-1} \mathcal{S}(k)$ converges strongly to $\pi_* \mathrm{sc}(\eta^{-1} S)$. If $\mathrm{cd} k < \infty$, this target is isomorphic to $\pi_* \eta^{-1} S$.*

Proof. We have already verified this result for odd characteristic fields and fields with finite cohomological dimension. It remains to check characteristic 0 fields of

arbitrary cohomological dimension. By Theorem 2.3, we have weak convergence to $\mathrm{sc}(\eta^{-1}\mathbf{S})$, so it suffices to check vanishing of Boardman's RE_∞ term (for $E = \alpha_1^{-1}\mathcal{S}(\mathbf{k})$). By [Boardman 1999, Remark after Theorem 7.1], it in turn suffices to check that for each tridegree $(s, m + n\alpha)$ there are at most finitely many nonzero differentials $d_r : \alpha_1^{-1}\mathcal{S}_r^{s, m+n\alpha} \rightarrow \alpha_1^{-1}\mathcal{S}_r^{s+r, m-1+n\alpha}$. By Theorem 4.5, $\alpha_1^{-1}\mathcal{S}(\mathbf{k})$ has profile $\{r_k\}$ where $\{r_k\}$ is the profile of $\alpha_1^{-1}\mathcal{S}(\mathbb{Q})$. In particular, the finiteness condition on nonzero differentials is met, and we may conclude that we indeed have strong convergence. $\square$

Theorem 4.7. *Suppose $\mathbf{k}$ is a field of characteristic 0 which has profile $\{r_k\}$. Let v_2 denote 2-adic valuation. If $r_k < \infty$ for all $\mathbf{k}$, then*

$$\pi_m \mathrm{sc}(\eta^{-1}\mathbf{S}) \cong \begin{cases} W(\mathbf{k}) & \text{if } m = 0; \\ W(\mathbf{k})/2^{r_k} & \text{if } m > 0, m = 4\ell - 1, \text{ and } k = v_2(4\ell); \\ 2^{r_k} W(\mathbf{k}) & \text{if } m > 0, m = 4\ell, \text{ and } k = v_2(4\ell); \\ 0 & \text{otherwise.} \end{cases}$$

If $\{r_k\}$ eventually takes the value ∞ with first instance $r_K = \infty$, then

$$\pi_m \mathrm{sc}(\eta^{-1}\mathbf{S}) \cong \begin{cases} W(\mathbf{k}) & \text{if } m = 0; \\ W(\mathbf{k})/2^{r_k} & \text{if } m > 0, m = 4\ell - 1, \text{ and } k = v_2(4\ell) < K; \\ 2^{r_k} W(\mathbf{k}) & \text{if } m > 0, m = 4\ell, \text{ and } k = v_2(4\ell) < K; \\ W(\mathbf{k}) & \text{if } m > 0, m = 4\ell - 1 \text{ or } 4\ell, \text{ and } v_2(4\ell) \geq K; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This all follows from the slice filtration being the I -adic filtration, ρ representing 2 in $W(\mathbf{k})$, the structure of the differentials in Theorem 4.5, and strong convergence in Theorem 4.6. $\square$

Theorem 4.8. *Suppose that $\mathbf{k}$ is not of characteristic 2 and that -1 is a sum of four squares in $\mathbf{k}$. Then, as a ring,*

$$\pi_\star \mathrm{sc}(\eta^{-1}\mathbf{S}) \cong W(\mathbf{k})[\eta^{\pm 1}, \sigma, \mu]/(\sigma^2)$$

where $|\eta| = \alpha$, $|\sigma| = 3 + 4\alpha$, and $|\mu| = 4 + 5\alpha$. If additionally $\mathrm{cd} \mathbf{k} < \infty$, then $\mathrm{sc}(\eta^{-1}\mathbf{S}) \simeq \eta^{-1}\mathbf{S}$ and this is a computation of the η -periodic homotopy groups of the motivic sphere spectrum.

Proof. By Proposition 3.3, we may additionally assume that $\mathrm{char} \mathbf{k} = 0$. It is standard that the latter condition is equivalent to $\rho^3 = 0 \in k_3^M(\mathbf{k})$ (see [Lam 2005, Corollary X.6.20]). By Theorem 4.5, we see that the spectral sequence collapses (regardless of the profile of $\mathbb{Q}$). By Theorem 4.6, this proves the theorem. $\square$

Corollary 4.9. *If $\text{cd } k < \infty$ and $n \geq \max\{3m + 5, 4m\}$, then $\pi_{m+n\alpha} S \cong \pi_{m+n\alpha} \eta^{-1} S$; if, additionally, k is not of characteristic 2 and -1 is a sum of four squares in k , then these groups are 0 or $W(k)$ according to whether $m \equiv 1, 2 \pmod{4}$ or $m \equiv 0, 3 \pmod{4}$, respectively.*

Proof. All but the final statement was already observed in [Ormsby et al. 2018, Theorem 5.5]. $\square$

We certainly do *not* expect that the α_1 -periodic slice spectral sequence collapses at its E_2 page in general. Indeed, inspired by the computations of $\pi_{\star} \eta^{-1} S_2^{\wedge}$ by Guillou and Isaksen [2016] over $\mathbb{R}$ and Wilson [2018] over $\mathbb{Q}$, we make the following conjecture.

Conjecture 4.10. *The α_1 -periodic slice spectral sequence over $\mathbb{Q}$ has profile $\{3, 4, 5, \dots\}$, i.e., $r_k = k + 1$ for all k .*

If Conjecture 4.10 holds, then over k of characteristic 0,

$$\pi_m \text{sc}(\eta^{-1} S) \cong \begin{cases} W(k) & \text{if } m = 0, \\ W(k)/2^{\nu_2(4\ell)+1} & \text{if } m = 4\ell - 1 > 0, \\ 2^{\nu_2(4\ell)+1} W(k) & \text{if } m = 4\ell > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Curiously, this makes it appear as if $\text{sc}(\eta^{-1} S)$ might fit into a “connective image of J fiber sequence” of the form $\text{sc}(\eta^{-1} S) \rightarrow kw \rightarrow \Sigma^4 kw$ where kw is the connective cover of the 2-complete Witt K -theory spectrum. Over $k = \mathbb{C}$, one may show that the cone on the map $\sigma_{\infty} : \Sigma^3 kw \rightarrow \eta^{-1} S$ of Theorem 2.11 coincides with kw . In fact, the composition $\Sigma^3 kw \rightarrow \eta^{-1} S \rightarrow kw$ is zero, as one may deduce inductively, starting with the triviality of

$$\Sigma^3 \eta^{-1} S \rightarrow \Sigma^3 kw \rightarrow \eta^{-1} S \rightarrow kw$$

and continuing along the cell presentation of kw given in the proof of Theorem 2.11. Hence there is an induced map from the cone of $\Sigma^3 kw \rightarrow \eta^{-1} S$ to kw . This map induces an isomorphism on homotopy groups, and hence is an equivalence by cellularity. In particular, one may express $\eta^{-1} S$ over the complex numbers as the fiber of a map $kw \rightarrow \Sigma^4 kw$.

The Adams operations on the 2-complete algebraic K -theory spectrum $\text{KGL}_2^{\wedge}$ constitute an action of $\mathbb{Z}_2^{\times}$, the units in the 2-adic integers. When k has finite virtual cohomological dimension, the results of [Hu et al. 2011] imply that $\text{KQ}_2^{\wedge} \simeq (\text{KGL}_2^{\wedge})^{h\{\pm 1\}}$ inherits an action of $\mathbb{Z}_2^{\times}/\{\pm 1\} \cong \mathbb{Z}_2$ by Adams operations. Inverting η and taking the connective cover results in Adams operations on $kw_2^{\wedge}$. For any such ψ^g , the difference of ring maps $\psi^g - 1 : kw_2^{\wedge} \rightarrow kw_2^{\wedge}$ lifts to a map $\psi^g - 1 : kw_2^{\wedge} \rightarrow \Sigma^4 kw_2^{\wedge}$. (This can be seen by observing that $\Sigma^4 kw_2^{\wedge}$ is the 4-connective

cover of $\mathrm{kw}_2^\wedge$ and the cofiber of $\Sigma^4 \mathrm{kw}_2^\wedge \rightarrow \mathrm{kw}_2^\wedge$ is the Eilenberg–MacLane spectrum associated with the homotopy module $\underline{W}_2^\wedge[\eta^{\pm 1}]$.)

The η -periodic unit $\eta^{-1}S_2^\wedge \rightarrow \mathrm{kw}_2^\wedge$ factors through the fiber jw_g of $\psi^g - 1 : \mathrm{kw}_2^\wedge \rightarrow \Sigma^4 \mathrm{kw}$ because $\psi^g - 1$ is a difference of ring maps. This leads to the following conjecture, which is similar in spirit to Mahowald’s presentation of the v_1 -periodic sphere in topology.

Conjecture 4.11. *The map $\eta^{-1}S_2^\wedge \rightarrow \mathrm{jw}_3$ induced by the η -periodic unit $\eta^{-1}S_2^\wedge \rightarrow \mathrm{kw}_2^\wedge$ is an equivalence.*

Let HW denote the Eilenberg–MacLane spectrum associated with the homotopy module $\underline{W}[\eta^{\pm 1}]$. Work in progress of Tom Bachmann and Mike Hopkins suggests that the action of ψ^3 on $HW \wedge \widehat{\mathrm{kw}}_2$ is such that the unit map smashed with $HW_2^\wedge$ induces an equivalence $HW_2^\wedge \rightarrow HW \wedge \mathrm{jw}_3$. Since $\eta^{-1}S$ is HW -complete, this would immediately prove that $\eta^{-1}S_2^\wedge \simeq \mathrm{jw}_3$. It is presumably also the case that $(\psi^3 - 1)(\beta^k) = (9^k - 1)\beta^k$ on $\pi_\star \mathrm{kw}_2^\wedge$, in which case a comparison of slice spectral sequences would prove Conjecture 4.10.

Remark 4.12. The equivalence $\eta^{-1}S_2^\wedge \simeq \mathrm{jw}_3$ would also lead to a complete determination of the homotopy type and groups of $\eta^{-1}S$. Let X_k denote the Harrison space of orderings of k . Then $\pi_\star\{\eta^{-1}, 1/2\}S \cong H^0(X_k; \pi_m^{\mathrm{top}}S[1/2])$, which can be seen by the results of [Bachmann 2018], a descent spectral sequence, and the fact that X_k is a Stone space. The 2-primary arithmetic fracture square would then imply that

$$\pi_m \eta^{-1}S \cong \begin{cases} W(k) & \text{if } m = 0, \\ H^0(X_k; \pi_m^{\mathrm{top}}S[1/2]) & \text{if } m > 0 \text{ and } m \equiv 1 \text{ or } 2 \pmod{4}, \\ W(k)/2^{v_2(4\ell)+1} \oplus H^0(X_k; \pi_m^{\mathrm{top}}S[1/2]) & \text{if } m = 4\ell - 1 > 0, \\ 2^{v_2(4\ell)+1} W(k) \oplus H^0(X_k; \pi_m^{\mathrm{top}}S[1/2]) & \text{if } m = 4\ell > 0, \\ 0 & \text{otherwise.} \end{cases}$$

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KYLE ORMSBY
DEPARTMENT OF MATHEMATICS
REED COLLEGE
PORTLAND, OR
UNITED STATES
ormsbyk@reed.edu

OLIVER RÖNDIGS
INSTITUT FÜR MATHEMATIK
UNIVERSITÄT OSNABRÜCK
OSNABRÜCK
GERMANY
oroendig@uni-osnabrueck.de

ON THE NOETHER PROBLEM FOR TORSION SUBGROUPS OF TORI

FEDERICO SCAVIA

We consider the Noether Problem for stable and retract rationality for the sequence of d -torsion subgroups $T[d]$ of a torus T , $d \geq 1$. We show that the answer to these questions only depends on $d \pmod{e(T)}$, where $e(T)$ is the period of the generic T -torsor. When T is the norm one torus associated to a finite Galois extension, we find all d such that the Noether Problem for retract rationality has a positive solution for $T[d]$. We also give an application to the Grothendieck ring of stacks.

1. Introduction

Let k be a field, G be a finite group (i.e., a finite constant group scheme over k) and V be a faithful k -linear G -representation. The Noether Problem asks whether the quotient variety V/G is rational, that is, birational to some affine space over k . This question originated in Noether's work [1917] on the inverse Galois problem. If $k = \mathbb{Q}$, and V/G is rational over G for some V , then Hilbert's irreducibility theorem implies that G arises as a Galois group over $\mathbb{Q}$; see [Jensen et al. 2002, §3.3, §5.1].

It turns out that the Noether Problem does not always have a positive solution. Swan [1969] and Voskresenski [1970] gave counterexamples to the Noether Problem over $\mathbb{Q}$ for some cyclic groups of prime order ($\mathbb{Z}/47\mathbb{Z}$, $\mathbb{Z}/113\mathbb{Z}$, $\mathbb{Z}/223\mathbb{Z}$, ...).

Subsequent work naturally led to several variants of the original Noether Problem. For example, one may ask if the variety V/G is stably rational, or retract rational; see [Colliot-Thélène and Sansuc 2007, §1] for the definitions. The examples of Swan and Voskresenski give a negative answer to the Noether Problem for stable rationality and, by a result of Saltman [1984, Theorem 4.12], a positive answer for retract rationality.

One may also consider the Noether Problem for an arbitrary linear algebraic group scheme G over k (not necessarily smooth). Let V be a finite-dimensional generically free representation of G . By [Berhuy and Favi 2003, Theorem 4.7], there exists a dense G -invariant open subset of V such that a geometric quotient U/G

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exists and $U \rightarrow U/G$ is a G -torsor. The variety U/G may be regarded as a birational approximation of the classifying stack BG . Indeed, by the no-name lemma [Reichstein and Vistoli 2006, Lemma 2.1], the stable (retract) rationality of U/G does not depend on the representation V but only on G . We say that BG is stably (retract) rational if U/G is, that is, if the Noether Problem for stable (retract) rationality for G has an affirmative answer; see also [Florence and Reichstein 2018, §3].

The stable (retract) rationality of BG in the case when G is a group of multiplicative type is an interesting problem. Lenstra [1974] gave a complete classification of abelian groups for which the Noether Problem fails. In [Lenstra 1974, (7.2)], he showed that the smallest group for which the Noether Problem over $\mathbb{Q}$ fails is the cyclic group $\mathbb{Z}/8\mathbb{Z}$.

If $G = T$ is a k -torus then, by an observation of Colliot-Thélène and Merkurjev [Scavia 2020, Proposition 6.1], BT is stably (retract) rational if and only if the dual torus of T is. If the group of multiplicative type G is not necessarily a torus, we embed G in a quasisplit torus S . By construction, S is a G -stable open subset of a generically free G -representation. The quotient S/G is a torus, and BG is stably (retract) rational if and only if S/G is.

The stable (retract) rationality of a k -torus T may be understood in terms of the action of the absolute Galois group of k on the character lattice of T ; see [Voskresenski 1998, §4.7] and [Colliot-Thélène and Sansuc 1977]. A complete classification of stably (retract) rational k -tori, in the spirit of [Lenstra 1974], does not exist. The birational geometry of tori is an interesting and subtle topic, and has been studied in [Voskresenski 1965; 1971; 1973; Miyata 1971; Endô and Miyata 1973; 1975; Colliot-Thélène and Sansuc 1977; 1987; Kunyavski 1978; 1987], among many others.

Of particular interest are the birational properties of norm one tori. Let L/k be a finite separable field extension, and let $T = R_{L/k}^{(1)}(\mathbb{G}_m)$ be the associated norm one torus. It fits into a short exact sequence

$$(1-1) \quad 1 \rightarrow T \rightarrow R_{L/k}(\mathbb{G}_m) \xrightarrow{N} \mathbb{G}_m \rightarrow 1,$$

where $R_{L/k}$ denotes Weil restriction and N is the norm map. For the case when L/k is Galois, the retract rationality of T has been studied in [Colliot-Thélène and Sansuc 1977]. For the case when L/k is not necessarily Galois, the rationality problem for norm one tori has been considered in special cases from a computational perspective by Hoshi and Yamasaki [2017; 2018] and by Hasegawa, Hoshi and Yamasaki [Hasegawa et al. 2020].

Since the Noether Problem for groups of multiplicative type in full generality is out of the reach of currently known techniques, we restrict ourselves to the following class of groups. Let k be a field and T a k -torus. For every $d \geq 1$, we consider the Noether Problem for stable rationality and retract rationality for the torsion

subgroup $T[d]$. Note that these questions are not covered by the aforementioned results of Lenstra [1974]. Lenstra solved the Noether Problem for finite abelian constant groups. The torsion subgroup schemes $T[d]$ we will consider are finite and abelian, but not necessarily constant.

In Proposition 3.3 we show that the answer to each of these versions of the Noether Problem for the torsion subgroups $T[d]$ (of a particular torus T) is periodic in d . We investigate more closely the retract rationality of $BT[d]$ in the case where $T = R_{L/k}^{(1)}(\mathbb{G}_m)$ is a norm one torus and L/k is a finite Galois extension. Note that sequence (1-1) implies that BT is stably rational, but as we will see, this need not be true for $BT[d]$. By Lemma 5.2 it is enough to consider the case when G is a p -group. For every L/k and every $d \geq 1$, we determine whether $BT[d]$ is retract rational.

Theorem 1.1. *Let L/k be a finite Galois extension such that $G := \text{Gal}(L/k)$ has order p^n for some prime number p , and let $T := R_{L/k}^{(1)}(\mathbb{G}_m)$. Assume that p is odd.*

- (a) *If G is cyclic, then $BT[d]$ is retract rational for every d .*
- (b) *If G is not cyclic, then $BT[d]$ is retract rational if and only if $p \nmid d$.*

Assume that $p = 2$.

- (a') *If G is cyclic, then $BT[d]$ is retract rational for every d .*
- (b') *If G is a dihedral group, then $BT[d]$ is retract rational if and only if $4 \nmid d$.*
- (c') *Otherwise, $BT[d]$ is retract rational if and only if d is odd.*

Assume that $\text{char } k = p$ is positive. Then it is well known that for every finite constant p -group Γ , $B\Gamma$ is stably rational; see [Jensen et al. 2002, §5.6]. By contrast, Theorem 1.1 gives examples of finite group schemes A of order a power of p for which BA is not even retract rational. Note that such A are nonreduced.

In Section 8 we give an application of Proposition 3.3 to the Grothendieck ring of k -stacks $K_0(\text{Stacks}_k)$.

2. Preliminaries

Let k be a field, and let G be a linear algebraic group over k . If V is a generically free representation of G , there exists a nonempty open G -invariant subset $U \subseteq V$ together with a G -torsor $U \rightarrow Z$, where Z is a k -variety. The G -torsor $U \rightarrow Z$ is called a standard G -torsor in [Merkurjev 2018]. We say that BG is stably birational to a variety X if Z is birational to X . We say that BG is stably rational or retract rational if Z is. Different choices of V and U yield stably birational Z , hence the definitions are independent of the choice of V and U ; see [Merkurjev 2018, §5].

Let T be a k -torus. We will denote by $\hat{T}$ the character lattice of T . Recall that this is $\text{Hom}_{k_s}(T_{k_s}, \mathbb{G}_{m,k_s})$, where k_s denotes a separable closure of k . The association $T \mapsto \hat{T}$ establishes an antiequivalence between the categories of k -tori and the

category of $\mathbb{Z}$ -free continuous $\mathrm{Gal}(k_s/k)$ -modules of finite rank; see [Voskresenski 1998, §3.4]. We refer the reader to [Voskresenski 1998, p. 27] for the definitions of splitting group and minimal splitting field of T . We write T° for the dual torus of T , that is, the k -torus whose character lattice is dual to $\hat{T}$.

Consider a short exact sequence

$$1 \rightarrow T \rightarrow S \xrightarrow{\varphi} Q \rightarrow 1$$

where S is quasisplit. Then S is an open subset of an affine space V , and the multiplication action of T on S extends to a linear action on V . It follows that BT is stably birational to Q .

The generic fiber of φ is a T -torsor over $k(Q)$, and is called a generic T -torsor. If $E \rightarrow \mathrm{Spec} K$ is a T -torsor, the period of $E \rightarrow \mathrm{Spec} K$ is by definition its order in the group $H^1(K, T_K)$. We denote by $e(T)$ the period of a generic T -torsor. By [Merkurjev 2010, Proposition 1.1], $e(T)$ is divisible by the period of every T -torsor. In particular, it does not depend on the choice of the resolution (3-1).

Lemma 2.1. *Let T be a k -torus, with splitting group G . Then:*

- (a) $e(T)$ divides the order of G .
- (b) $e(T) = e(T^\circ)$.

Proof. (a) Let l/k be the minimal splitting field of T . The extension l/k is Galois, with Galois group G . Let $E \rightarrow \mathrm{Spec} K$ be a generic T -torsor, where K is a field containing k , and denote by L the compositum of K and l inside some fixed algebraic closure of K containing l . Then L splits T_K and so $H^1(L, T_L) = H^1(L, \mathbb{G}_m^{\mathrm{rk} T}) = 0$. The restriction-corestriction sequence

$$H^1(K, T_K) \rightarrow H^1(L, T_L) \rightarrow H^1(K, T_K)$$

shows that $H^1(K, T_K)$ is $[L : K]$ -torsion, hence $e(T) \mid [L : K]$. By basic Galois theory $[L : K] \mid [l : k] = |G|$, so $e(T) \mid |G|$, as desired.

(b) Set $M := \hat{T}$ and fix a coflasque resolution $0 \rightarrow N \rightarrow U \rightarrow M \rightarrow 0$, corresponding to a class $\alpha \in \mathrm{Ext}_G^1(M, N)$. Let M' and N' be the dual lattices of M and N , respectively. There is a G -equivariant isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(N', M')$$

given by taking the transpose. By [Brown 1982, §III, Proposition 2.2] we obtain a commutative diagram of group isomorphisms

$$(2-1) \quad \begin{array}{ccc} H^1(G, \mathrm{Hom}_{\mathbb{Z}}(M, N)) & \xrightarrow{\sim} & \mathrm{Ext}_G^1(M, N) \\ \downarrow \wr & & \downarrow \wr \eta \\ H^1(G, \mathrm{Hom}_{\mathbb{Z}}(N', M')) & \xrightarrow{\sim} & \mathrm{Ext}_G^1(N', M'), \end{array}$$

where η is defined by sending an extension $0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0$ to its

dual $0 \rightarrow M' \rightarrow P' \rightarrow N' \rightarrow 0$. By [Merkurjev 2010, Theorem 3.1], $e(T)$ is the order of α in $\text{Ext}_G^1(M, N)$. By [Merkurjev 2010, Theorem 3.2], $e(T^\circ)$ is equal to the order of the class α' of the dual sequence $0 \rightarrow M' \rightarrow U' \rightarrow N' \rightarrow 0$ in $\text{Ext}_G^1(N', M')$. Now (2-1) shows α and α' have the same order, hence $e(T) = e(T^\circ)$. $\square$

3. Periodicity

The purpose of this section is to prove Proposition 3.3. Let T be a k -torus. Fix a coflasque resolution of T ,

$$(3-1) \quad 1 \rightarrow T \rightarrow S \xrightarrow{\varphi} Q \rightarrow 1,$$

and the corresponding short exact sequence of character lattices

$$(3-2) \quad 0 \rightarrow \hat{Q} \rightarrow \hat{S} \rightarrow \hat{T} \rightarrow 0.$$

For every $d \geq 1$, we may consider the following commutative diagram with exact rows and columns:

$$(3-3) \quad \begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & \downarrow \\ & & & & & & T \\ & & & & & & \downarrow \\ & & & & & & T_d \\ 1 & \longrightarrow & T[d] & \longrightarrow & S & \xrightarrow{\psi} & T_d \longrightarrow 1 \\ & & \downarrow & & \parallel & & \downarrow \\ 1 & \longrightarrow & T & \longrightarrow & S & \xrightarrow{\varphi} & Q \longrightarrow 1 \\ & & \downarrow d & & & & \downarrow \\ & & T & & & & 1 \\ & & \downarrow & & & & \\ & & 1 & & & & \end{array}$$

Here, the second row is (3-1), the torus T_d is defined so as to make the first row exact, and the copy of T in the upper right corner is identified with the copy of T on the lower left corner via the connecting homomorphism given by the snake lemma. Since S is quasisplit, we see that $BT[d]$ is stably birational to T_d .

The main ingredient for the proof of Proposition 3.3 is the following observation.

Proposition 3.1. *Let $\alpha \in \text{Ext}_G^1(\hat{T}, \hat{Q})$ be the class of (3-2), and α_d be the class of the sequence*

$$0 \rightarrow \hat{Q} \rightarrow \hat{T}_d \rightarrow \hat{T} \rightarrow 0$$

in $\text{Ext}_G^1(\hat{T}, \hat{Q})$, for every $d \geq 1$. Then $\alpha_d = d \cdot \alpha$.

In particular, $T_m \cong T_n$ when $m \equiv \pm n \pmod{e(T)}$, and if $e(T) \mid d$, then $T_d \cong T \times Q$.

Proof. From (3-3), we obtain the following diagram with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & T & \longrightarrow & S & \xrightarrow{\varphi} & Q \longrightarrow 1 \\ & & \downarrow d & & \downarrow \psi & & \parallel \\ 1 & \longrightarrow & T & \longrightarrow & T_d & \longrightarrow & Q \longrightarrow 1 \end{array}$$

Passing to character lattices, we get

$$(3-4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \hat{Q} & \longrightarrow & \hat{T}_d & \longrightarrow & \hat{T} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow d \\ 0 & \longrightarrow & \hat{Q} & \longrightarrow & \hat{S} & \longrightarrow & \hat{T} \longrightarrow 0 \end{array}$$

By [Merkurjev 2010, Theorem 3.1], $e(T)$ is equal to the order of the class $\alpha \in \text{Ext}_G^1(\hat{T}, \hat{Q})$ of the sequence

$$0 \rightarrow \hat{Q} \rightarrow \hat{S} \rightarrow \hat{T} \rightarrow 0.$$

The long exact sequence for the functor $\text{Hom}_G(\hat{T}, -)$ associated to (3-4) gives a commutative square

$$\begin{array}{ccc} \text{Hom}_G(\hat{T}, \hat{T}) & \xrightarrow{\partial_1} & \text{Ext}_G^1(\hat{T}, \hat{Q}) \\ \downarrow d & & \parallel \\ \text{Hom}_G(\hat{T}, \hat{T}) & \xrightarrow{\partial_2} & \text{Ext}_G^1(\hat{T}, \hat{Q}). \end{array}$$

We have $\alpha_d = \partial_1(\text{id}_{\hat{T}})$ and $\alpha = \partial_2(\text{id}_{\hat{T}})$, hence $\alpha_d = d \cdot \alpha$.

Since α has order $e(T)$, if $m \equiv n \pmod{e(T)}$ then $T_m \cong T_n$. Recall that if M, N are G -lattices and $\gamma \in \text{Ext}_G^1(M, N)$ is the class of $0 \rightarrow N \rightarrow P \xrightarrow{\eta} M \rightarrow 0$, then $-\gamma$ is the class of $0 \rightarrow N \rightarrow P \xrightarrow{-\eta} M \rightarrow 0$. Hence $T_m \cong T_n$ when $m \equiv \pm n \pmod{e(T)}$. Finally, if $e(T) \mid d$, then $\alpha_d = 0$, so $\hat{T}_d \cong \hat{T} \oplus \hat{Q}$. $\square$

Before proving Proposition 3.3, we need the following lemma.

Lemma 3.2. *Let T_1 and T_2 be two k -tori split by a finite Galois extension with Galois group G . Let $f : T_1 \rightarrow T_2$ be a surjective homomorphism of tori of finite degree prime to $|G|$. Then T_1 is retract rational if and only if T_2 is retract rational.*

Proof. Let

$$1 \rightarrow F_i \rightarrow S_i \rightarrow T_i \rightarrow 1$$

be a flasque resolution of T_i , for $i = 1, 2$. By [Saltman 1984, Theorem 3.14(a)], T_i is retract rational if and only if $\hat{F}_i$ is invertible, that is, $\hat{F}_i$ is a direct summand of a permutation G -lattice. It thus suffices to show that $\hat{F}_1$ is invertible if and only if $\hat{F}_2$ is.

Set $a := \deg(f)$. If p is a prime that does not divide a , the map $f : T_1 \rightarrow T_2$

induces an isomorphism $(\hat{T}_1)_{(p)} \cong (\hat{T}_2)_{(p)}$ of $\mathbb{Z}_{(p)}[G]$ -modules. Construct the pushout diagram

$$(3-5) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ & & (\hat{F}_2)_{(p)} & = & (\hat{F}_2)_{(p)} & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & (\hat{S}_2)_{(p)} & \longrightarrow & X_p & \longrightarrow & (\hat{F}_1)_{(p)} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & (\hat{T}_1)_{(p)} & \longrightarrow & (\hat{S}_1)_{(p)} & \longrightarrow & (\hat{F}_1)_{(p)} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

Since the $\hat{F}_i$ are flasque and the $\hat{S}_i$ are permutation lattices for $i = 1, 2$, by [Colliot-Thélène and Sansuc 1977, Lemme 1(vii)] we have

$$\mathrm{Ext}_{\mathbb{Z}_{(p)}[G]}^1((\hat{F}_i)_{(p)}, (\hat{S}_{3-i})_{(p)}) = \mathrm{Ext}_G^1(\hat{F}_i, \hat{S}_{3-i})_{(p)} = 0.$$

It follows from the diagram that $(\hat{F}_1 \oplus \hat{S}_2)_{(p)} \cong X_p \cong (\hat{F}_2 \oplus \hat{S}_1)_{(p)}$ for every p not dividing a . Since $|G|$ and a are coprime, this holds for every p which divides $|G|$. In the terminology of [Scavia 2020, Definition 2.1], this means that $\hat{F}_1$ is p -invertible if and only if $\hat{F}_2$ is, for every prime p which divides $|G|$. By [Scavia 2020, Lemma 2.3], it follows that $\hat{F}_1$ is invertible if and only if $\hat{F}_2$ is, as desired. $\square$

Proposition 3.3. *Let T be a k -torus and let $m, n \geq 1$.*

- (a) *If $m \equiv \pm n \pmod{e(T)}$, then $BT[m]$ and $BT[n]$ are stably birational.*
- (b) *If $e(T) \mid n$, then $BT[n]$ is stably birational to $T \times BT$.*
- (c) *If $n \equiv \pm 1 \pmod{e(T)}$, then $BT[n]$ is stably rational.*
- (d) *If $\langle m \rangle = \langle n \rangle$ in $\mathbb{Z}/e(T)\mathbb{Z}$, then $BT[m]$ is retract rational if and only if $BT[n]$ is. In particular, $BT[m]$ is retract rational when m is invertible in $\mathbb{Z}/e(T)\mathbb{Z}$.*

Proof of Proposition 3.3. (a) By construction $BT[d]$ is stably birational to T_d for every $d \geq 1$, so the conclusion follows from Proposition 3.1.

(b) If $e(T) \mid n$, by Proposition 3.1 the sequence

$$1 \rightarrow T \rightarrow T_n \rightarrow Q \rightarrow 1$$

splits, so $T_n \cong T \times Q$. Since BT is stably birational to Q and $BT[n]$ is stably birational to T_n , we conclude that $BT[n]$ is stably birational to $T \times BT$.

(c) Since $T[1]$ is trivial, $T_1 = S$ is rational. Now (c) follows from (a).

(d) Let $m, n \geq 1$ be such that $\langle m \rangle = \langle n \rangle$ in $\mathbb{Z}/e(T)\mathbb{Z}$. There exists $a \geq 1$ invertible modulo $e(T)$ such that $m \equiv an \pmod{e(T)}$. By Proposition 3.1, $T_m \cong T_{an}$. By Lemma 3.2, T_m is retract rational if and only if T_{am} is. The second statement is now a consequence of (b) and (c). $\square$

Remark 3.4. Using Lemma 2.1(a), we see that Proposition 3.3 remains valid if we replace $e(T)$ by $|G|$.

It would be interesting to know if the hypotheses of Proposition 3.3(d) imply that $BT[m]$ and $BT[n]$ are stably birational to each other. Statements of this sort are in the spirit of Amitsur's conjecture; see [Amitsur 1955] or [Auel et al. 2011, Conjecture 10.2]. We thank the referee for this comment.

4. Norm one tori

Let $T := R_{L/k}^{(1)}(\mathbb{G}_m)$, where L/k is a finite Galois extension with Galois group G . By [Merkurjev 2010, Example 4.1], $e(T^\circ) = [L : k] = |G|$. Using Lemma 2.1(b), we deduce that $e(T) = |G|$.

Define T_d via the diagram (3-3), where $S = R_{L/k}(\mathbb{G}_m)$ and T is embedded in S as the kernel of the norm map, so that $Q = \mathbb{G}_m$. By construction we have a short exact sequence

$$1 \rightarrow T[d] \rightarrow R_{L/k}(\mathbb{G}_m) \rightarrow T_d \rightarrow 1.$$

Let $\{e_g\}_{g \in G}$ be the standard basis of $\mathbb{Z}[G]$, and for every subgroup G' of G let $\sigma_{G'} := \sum_{g \in G'} e_g \in \mathbb{Z}[G]$. We have $\hat{T} = \mathbb{Z}[G]/\langle \sigma_G \rangle$, so the sequence

$$1 \rightarrow T[d] \rightarrow T \xrightarrow{d} T \rightarrow 1$$

shows that the character module of $T[d]$ is $(\mathbb{Z}/d\mathbb{Z}[G])/\langle \sigma_G \rangle$. It follows that

$$\hat{T}_d = \langle d\mathbb{Z}[G], \sigma_G \rangle = \langle dg, \sigma_G \rangle_{g \in G}.$$

We may thus construct the exact sequence

$$(4-1) \quad 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[G] \oplus \mathbb{Z} \rightarrow \hat{T}_d \rightarrow 0,$$

where the first homomorphism sends 1 to $(1, \dots, 1; -d)$ and the second map sends $(g, 0) \mapsto dg$ and $(0, 1) \mapsto \sigma_G$.

If G is a finite group, M is a G -module and $i \in \mathbb{Z}$, we write $H^i(G, M)$ for the i -th group of Tate cohomology, group denoted by $\hat{H}^0(G, M)$ if i is explicitly set equal to 0.

Lemma 4.1. *Let T and G be as above. For every $i \in \mathbb{Z}$ and $d \geq 1$, there is an isomorphism*

$$H^i(G, \hat{T}_d) \xrightarrow{\sim} H^i(G, \mathbb{Z}/d\mathbb{Z}),$$

which is compatible with restriction to subgroups of G .

Proof. Let $d \geq 1$, and let U_d be the character module of $T[d]$. We have a short exact sequence of G -modules

$$0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \mathbb{Z}/d\mathbb{Z}[G] \rightarrow U_d \rightarrow 0,$$

where the first homomorphism is induced by $1 \mapsto \sigma_G$. We also have the sequence

$$0 \rightarrow \hat{T}_d \rightarrow \mathbb{Z}[G] \rightarrow U_d \rightarrow 0,$$

corresponding to the first row of (3-3). By Shapiro's lemma $H^i(G, \mathbb{Z}[G]) = 0$ and $H^i(G, \mathbb{Z}/d\mathbb{Z}[G]) = 0$ for every integer i . Looking at the associated cohomology long exact sequences, we deduce

$$H^i(G, \hat{T}_d) \cong H^{i-1}(G, U_d) \cong H^i(G, \mathbb{Z}/d\mathbb{Z}).$$

The compatibility of this isomorphism with restrictions is clear from the construction. $\square$

Recall that if M is a G -module and $i \in \mathbb{Z}$, then

$$\text{III}^i(G, M) := \text{Ker}(H^i(G, M) \rightarrow \oplus_{g \in G} H^i(\langle g \rangle, M)).$$

If M is a G -lattice, by [Lorenz 2005, Proposition 2.9.2(a)] we have

$$(4-2) \quad \text{III}^2(G, M) \cong H^1(G, [M]^{\text{fl}}).$$

Lemma 4.2. *Let T and G be as above, and $d \geq 1$. Let G act trivially on $\mathbb{Z}/d\mathbb{Z}$, and assume that $\text{III}^2(G, \mathbb{Z}/d\mathbb{Z}) \neq 0$. Then $BT[d]$ is not retract rational.*

Proof. By Lemma 4.1 we have a commutative square with horizontal isomorphisms,

$$(4-3) \quad \begin{array}{ccc} H^2(G, \hat{T}_d) & \xrightarrow{\sim} & H^2(G, \mathbb{Z}/d\mathbb{Z}) \\ \downarrow & & \downarrow \\ \oplus_{g \in G} H^2(\langle g \rangle, \hat{T}_d) & \xrightarrow{\sim} & \oplus_{g \in G} H^2(\langle g \rangle, \mathbb{Z}/d\mathbb{Z}). \end{array}$$

The diagram shows that $\text{III}^2(G, \hat{T}_d) \cong \text{III}^2(G, \mathbb{Z}/d\mathbb{Z})$, so $\text{III}^2(G, \hat{T}_d) \neq 0$. By (4-2), we deduce that $H^1(G, [\hat{T}_d]^{\text{fl}}) \neq 0$, hence $[\hat{T}_d]^{\text{fl}}$ is not invertible. It follows that T_d is not retract rational, so $BT[d]$ is not retract rational. $\square$

The following group-theoretic lemma will be used in the proof of Theorem 1.1.

Lemma 4.3. *Let p be a prime, and let G be a finite p -group.*

- (a) *Assume G does not contain a proper subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Then either G is cyclic, or $p = 2$ and $G = Q_{2^n}$ for some $n \geq 3$.*
- (b) *Assume that every proper subgroup of G is cyclic. Then either G is cyclic or isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, or $p = 2$ and $G = Q_8$.*
- (c) *Assume that $p = 2$ and that every proper subgroup of G is cyclic or dihedral. Then G is cyclic, dihedral, or $|G| = 8$.*

Proof. (a) Assume that G contains at least two distinct subgroups of order p . Since the center $Z(G)$ of G is not trivial, there exists a subgroup H of $Z(G)$ order p . If H' is another subgroup of order p , then $\langle H, H' \rangle \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. It follows from the assumption that $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Assume now that G contains exactly one subgroup of order p . By [Berkovich 2008, Proposition 1.3], either G is cyclic or $p = 2$ and G is a generalized quaternion group:

$$Q_{2^{n+1}} := \langle a, b \mid a^{2^{n-1}} = b^2, a^{2^n} = e, b^{-1}ab = a^{-1} \rangle.$$

(b) If p is odd, the statement immediately follows from (a), so assume $p = 2$. If $n \geq 4$, Q_{2^n} contains Q_8 as a proper subgroup. It follows from (a) that if $p = 2$ and G is not cyclic, then $G = Q_8$.

(c) The conclusion is obvious when G is abelian, so assume that G is not abelian. We may also suppose that $|G| = 2^n$ for some $n \geq 4$. If G contains at least one nonabelian proper subgroup, it is dihedral [Miller 1907]. Assume now that every proper subgroup of G is abelian. By [Rédei 1947] or [Kang and Reichstein 2002, Lemma 4.5], G is isomorphic to

$$A_{u,v} := \langle a, b \mid a^{2^u} = b^{2^v} = 1, ba = a^{2^{u-1}+1}b \rangle \cong \mathbb{Z}/2^u\mathbb{Z} \rtimes \mathbb{Z}/2^v\mathbb{Z},$$

where $u \geq 2$, $v \geq 1$ and $n = u + v$, or to

$$B_{u,v} := \langle a, b, c \mid a^{2^u} = b^{2^v} = c^2 = 1, ba = cab \rangle \cong (\mathbb{Z}/2^u\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/2^v\mathbb{Z},$$

where $u, v \geq 1$ and $n = u + v + 1$. Since $n \geq 4$, both groups contain a subgroup of the form $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which is neither cyclic nor dihedral, hence no such G exists. $\square$

5. Proof of Theorem 1.1 for p odd

Lemma 5.1. *Let G be a finite group, H be a subgroup of G and $d \geq 1$ be an integer. Let L/k be a Galois extension with Galois group G , $k' := L^H$, $T := R_{L/k}^{(1)}(\mathbb{G}_m)$ and $T' := R_{L/k'}^{(1)}(\mathbb{G}_m)$. Then $BT_{k'}[d]$ is stably birational to $BT'[d]$ over k' .*

In particular, if $BT'[d]$ is not retract rational over k' , then $BT[d]$ is not retract rational over k .

Proof. Let $r := [k' : k]$. We have an isomorphism

$$T_{k'} \cong T' \times R_{L/k'}(\mathbb{G}_m)^{r-1},$$

and so $T_{k'}[d]$ is isomorphic to $T'[d] \times R_{L/k'}(\mathbb{G}_m)[d]^{r-1}$. This shows that

$$BT_{k'}[d] \cong BT'[d] \times BR_{L/k'}(\mathbb{G}_m)[d]^{r-1}.$$

Since $BR_{L/k'}(\mathbb{G}_m)[d]$ is stably rational over k' , we conclude that $BT_{k'}[d]$ is stably birational to $BT'[d]$ over k' . $\square$

The next lemma allows us to reduce the problem of the retract rationality of $BT[d]$ to the case when G is a p -group.

Lemma 5.2. *Let T be a k -torus split by a group G . Assume that $d \mid |G|$ and let $d = p_1^{a_1} \cdots p_r^{a_r}$ be the prime factorization of d . For every $i = 1, \dots, r$, let G_i be a p_i -Sylow subgroup of G , set $k_i := L^{G_i}$, and let $T_i := R_{L/k_i}^{(1)}(\mathbb{G}_m)$. Then $BT[d]$ is retract rational if and only if $BT_i[p_i^{a_i}]$ is retract rational over k_i for every $i = 1, \dots, r$.*

Proof. If k is finite, every k -torus is retract rational, and there is nothing to prove. Assume now that k is infinite. We have $BT[d] \cong BT[p_1^{a_1}] \times \cdots \times BT[p_r^{a_r}]$, so $BT[d]$ is retract rational if and only if $BT[p_i^{a_i}]$ is retract rational for $i = 1, \dots, r$, that is, if and only if $T_{p_i^{a_i}}$ is retract rational for every i . By [Colliot-Thélène and Sansuc 1977, Lemme 9], $T_{p_i^{a_i}}$ is retract rational if and only if $(T_{k_j})_{p_i^{a_i}} = (T_{p_i^{a_i}})_{k_j}$ is retract rational over k_j for every $j = 1, \dots, r$. Thus $BT[d]$ is retract rational if and only if $BT_{k_j}[p_i^{a_i}]$ is retract rational over k_j for every i, j .

If $i \neq j$, $p_i^{a_i}$ is invertible modulo $|G_j|$, so by Proposition 3.3(d) $BT_{k_j}[p_i^{a_i}]$ is retract rational over k_j . It follows that $BT[d]$ is retract rational if and only if $BT_{k_i}[p_i^{a_i}]$ is retract rational over k_i for every i . The conclusion now follows from Lemma 5.1. $\square$

Let L/k be a finite Galois extension with Galois group G , and let $T := R_{L/k}^{(1)}(\mathbb{G}_m)$.

Lemma 5.3. *Let p be an odd prime, $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. If p is odd, $BT[p]$ is not retract rational.*

Proof. By Lemma 4.2 it suffices to prove that $\text{III}^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \neq 0$. Let H_p be the Heisenberg group of order p^3 . The group H_p fits in a central short exact sequence,

$$1 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow H_p \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \rightarrow 1.$$

Let $\alpha \in H^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ be the class corresponding to this extension. Since p is odd, H_p is not commutative, so $\alpha \neq 0$. If $g \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, the image of α in $H^2(\langle g \rangle, \mathbb{Z}/p\mathbb{Z})$ is the class of the extension

$$1 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \pi^{-1}(\langle g \rangle) \rightarrow \langle g \rangle \rightarrow 1.$$

Since every nontrivial element $g \in H_p$ has order p , $\pi^{-1}(\langle g \rangle) \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, hence α restricts to the trivial class in $H^2(\langle g \rangle, \mathbb{Z}/p\mathbb{Z})$. Therefore we have that $0 \neq \alpha \in \text{III}^2(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, as desired. $\square$

Note that Lemma 5.3 fails when $p = 2$; see Lemma 7.1(a) below.

Proof of Theorem 1.1 for p odd. Note that if $p \nmid d$, then $BT[d]$ is retract rational by Proposition 3.3(d). From now on, we assume that $p \mid d$.

(a) When G is cyclic, every torus split by G is retract rational by [Colliot-Thélène and Sansuc 1977, Proposition 2], hence T_d is retract rational. Since $BT[d]$ is stably birational to T_d , it follows that $BT[d]$ is also retract rational.

(b) If G is not cyclic, by Lemma 4.3(a) it contains a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Since $p \mid d$, the conclusion follows from Lemmas 5.3 and 5.1. $\square$

6. A flasque resolution of $\hat{T}_d$

In order to complete the proof of Theorem 1.1, it remains to consider the case $p = 2$. In this section we construct an explicit flasque resolution of $\hat{T}_d$, when G is assumed to be a p -group. The construction works for any prime p , but we will use it only in the case when $p = 2$.

Assume that G is a p -group, and let d be a divisor of $|G|$. Let M_d be the character lattice of the torus dual to T_d . Consider the sequence

$$(6-1) \quad 0 \rightarrow M_d \rightarrow \mathbb{Z}[G] \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \rightarrow 0,$$

obtained by dualizing (4-1). The map α is defined by $\alpha(e_g, 0) = 1$ for every $g \in G$ and $\alpha(0, 1) = -d$, so

$$M_d = \left\{ ((a_g)_{g \in G}, a) \in \mathbb{Z}[G] \oplus \mathbb{Z} : \sum_{g \in G} a_g - da = 0 \right\}.$$

We need to construct an explicit coflasque resolution of M_d . The standard coflasque resolution (see, e.g., [Colliot-Thélène and Sansuc 1977, Lemme 3]) applied to M_d is too unwieldy for the computations that we want to perform. Thus we produce an ad hoc coflasque resolution of M_d . Consider the short exact sequence

$$(6-2) \quad 0 \rightarrow N_d \rightarrow P_d \xrightarrow{\beta} M_d \rightarrow 0.$$

Here $P_d := \mathbb{Z}[G \times G] \oplus R_d$, where $R_d := \bigoplus_{G' < G} \mathbb{Z}[G/G']$, the sum being over all subgroups G' of G ; in particular P_d is a permutation lattice. We denote by (g, g') , $g, g' \in G$, the elements of the standard basis of $\mathbb{Z}[G \times G]$. For every subgroup G' of G , denote by $e_{G'}$ the element of P_d supported on the summand $\mathbb{Z}[G/G']$ and having a 1 in the $\mathbb{Z}$ -component corresponding to the coset G' in G/G' , and 0 everywhere else. Recall that $\text{Hom}_G(\mathbb{Z}[G/G'], M_d) \cong \text{Hom}_{G'}(\mathbb{Z}, M_d) \cong M_d^{G'}$, the isomorphism being given by $\varphi \mapsto \varphi(e_{G'}) \in M_d^{G'}$. It follows that a map $R_d \rightarrow M_d$ is specified by the images of all the $e_{G'}$, and for every subgroup G' the image of $e_{G'}$ must be G' -invariant. Then β , as a map $P_d \rightarrow \mathbb{Z}[G] \oplus \mathbb{Z}$, is defined by $\beta((g, g'), 0) := (e_g - e_{g'}, 0)$ and

$$\beta(e_{G'}) := \begin{cases} (\sigma_{G'}, |G'|/d) & \text{if } d \mid |G'|, \\ ((d/|G'|)\sigma_{G'}, 1) & \text{if } d \nmid |G'|. \end{cases}$$

Note that since G is a p -group and $d \mid |G|$, either $d \mid |G'|$ or $|G'| \nmid d$. To see that β maps P_d to M_d , observe that clearly $\alpha(\beta((g_1, g_2), 0)) = 0$ and that

$$\alpha(\beta(e_{G'})) = \begin{cases} |G'| - d \cdot |G'|/d = 0 & \text{if } d \mid |G'|, \\ (d/|G'|) \cdot |G'| - d = 0 & \text{if } d \nmid |G'|. \end{cases}$$

Moreover, $\beta(e_{G'}) \in M_d^{G'}$, so β is G -equivariant. If we fix a subgroup $G_0 < G$ of order d , then M_d is generated as a G -module by elements of the form $(e_g - e_{g'}, 0)$, $g, g' \in G$, and $(\sigma_{G_0}, 1)$. Now, $e_g - e_{g'} = \beta((g, g'), 0)$, and $(\sigma_{G_0}, 1) = \beta(e_{G_0})$, so β is surjective. We define $N_d := \text{Ker } \beta$.

Lemma 6.1. *The G -lattice N_d is coflasque, therefore (6-2) is a coflasque resolution of M_d .*

Proof. Let G' be a subgroup of G . Since P_d is a permutation lattice, we have $H^1(G', P_d) = 0$. Recall that we denote degree 0 Tate cohomology by $\hat{H}^0$. The cohomology long exact sequence associated to (6-2) then yields

$$\hat{H}^0(G', P_d) \xrightarrow{\hat{H}^0(\beta)} \hat{H}^0(G', M_d) \rightarrow H^1(G', N_d) \rightarrow 0,$$

so to prove that $H^1(G', N_d) = 0$ it suffices to show that $\hat{H}^0(\beta)$ is surjective. Since $H^{-1}(G', \mathbb{Z}) = 0$, passing to cohomology in (6-1) gives

$$0 \rightarrow \hat{H}^0(G', M_d) \rightarrow \hat{H}^0(G', \mathbb{Z}[G] \oplus \mathbb{Z}) \rightarrow \hat{H}^0(G', \mathbb{Z}),$$

which can be rewritten as

$$(6-3) \quad 0 \rightarrow \hat{H}^0(G', M_d) \rightarrow \mathbb{Z}/|G'|\mathbb{Z} \xrightarrow{-d} \mathbb{Z}/|G'|\mathbb{Z}.$$

We have $\hat{H}^0(G', P_d) = \hat{H}^0(G', R_d) = \bigoplus_{G'' < G} \hat{H}^0(G', \mathbb{Z}[G/G''])$. To prove that $\hat{H}^0(\beta)$ is surjective it suffices to show that the map

$$\gamma : \hat{H}^0(G', \mathbb{Z}[G/G']) \rightarrow \hat{H}^0(G', M_d)$$

given by the summand relative to $G'' = G'$ is surjective. If we view $\hat{H}^0(G', M_d)$ as the d -torsion of $\hat{H}^0(G', \mathbb{Z}[G] \oplus \mathbb{Z})$ via (6-3), then $\gamma(e_{G'})$ coincides with the $\mathbb{Z}$ -component of $\beta(e_{G'}) \in \mathbb{Z}[G] \oplus \mathbb{Z}$ modulo $|G'|$, that is:

$$\gamma(e_{G'}) = \begin{cases} |G'|/d & \text{if } d \mid |G'|, \\ 1 & \text{if } d \nmid |G'|. \end{cases}$$

In both cases $\gamma(e_{G'})$ generates $\hat{H}^0(G', M_d)$. It follows that γ is surjective, as desired. $\square$

Proposition 6.2. *Let L/k be a Galois extension whose Galois group G is a p -group, and set $T := R_{L/k}^{(1)}(\mathbb{G}_m)$. If $BT[d]$ is retract rational for some $d \mid |G|$, then any subgroup of G of order d is cyclic.*

Proof. Assume that $BT[d]$ is retract rational. Since $BT[d]$ is stably birational to T_d , T_d is also retract rational. By Lemma 6.1, the dual of (6-2) is a flasque resolution of $\hat{T}_d$. By [Saltman 1984, Theorem 3.14(a)], the dual of N_d is invertible, hence so is N_d . This means that there exists a G -lattice U such that $N_d \oplus U = P$ for some permutation G -lattice P . In particular, for every subgroup G' of G , we have an embedding $H^2(G', N_d) \hookrightarrow H^2(G', P)$. Since $H^{-1}(G', P_d) = 0$, the long exact sequence associated to (6-2) shows that $H^1(G', M_d)$ embeds in $H^2(G', N_d)$, and hence in $H^2(G', P)$. The long exact sequence associated to (6-1) gives

$$H^0(G', \mathbb{Z}[G] \oplus \mathbb{Z}) \rightarrow H^0(G', \mathbb{Z}) \rightarrow H^1(G', M_d) \rightarrow 0.$$

This shows that if $d \mid |G'|$, then $H^1(G', M_d) \cong \mathbb{Z}/d\mathbb{Z}$. If $|G'| = d$, it follows that $H^2(G', P)$ contains an element of order $|G'|$. Now, P is a permutation G' -module, that is, a direct sum of G' -modules of the form $\mathbb{Z}[G'/G'']$, where G'' is a subgroup of G' . We have

$$H^2(G', \mathbb{Z}[G'/G'']) = H^2(G'', \mathbb{Z}) = H^1(G'', \mathbb{Q}/\mathbb{Z}) \cong (G'')_{ab}$$

for every subgroup G'' of G' , therefore the existence of an element of order $|G'|$ in $H^2(G', P)$ implies that G' must be cyclic. $\square$

Note that, when p is odd, Proposition 6.2 is weaker than Theorem 1.1(b), which we have already proved. We will use Proposition 6.2 to prove Theorem 1.1 when $p = 2$.

7. Proof of Theorem 1.1 for $p = 2$

For every even integer $r \geq 4$, we denote by D_r the dihedral group of r elements. In particular, $D_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Lemma 7.1. *Let L/k be a finite Galois extension with Galois group G , and let $T := R_{L/k}^{(1)}(\mathbb{G}_m)$.*

- (a) *Assume that $G = D_{2^n}$ for some $n \geq 1$. Then $BT[2]$ is retract rational.*
- (b) *Let $G = Q_8$ be the quaternion group. Then $BT[2]$ is not retract rational.*
- (c) *Let $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then $BT[2]$ is not retract rational.*

Proof. (a) We start by showing that $\text{III}^2(D_{2^n}, \mathbb{Z}/2\mathbb{Z}) = 0$. Recall that the nontrivial element α of $H^2(D_{2^n}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is given by the central extension

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow Q_{2^{n+1}} \rightarrow D_{2^n} \rightarrow 1.$$

Let $g \in Q_{2^{n+1}}$ be an element of order 2^n , and let $\bar{g} \in D_{2^n}$ be the image of g . Then $\bar{g}$ has order 2^{n-1} and the following sequence is exact:

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \langle g \rangle \rightarrow \langle \bar{g} \rangle \rightarrow 1.$$

This shows that the restriction map $H^2(D_{2^n}, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(\langle \bar{g} \rangle, \mathbb{Z}/2\mathbb{Z})$ sends α to a nonzero class. This means that $\text{III}^2(D_{2^n}, \mathbb{Z}/2\mathbb{Z}) = 0$, as claimed.

The conclusion now follows from known results. If G' is a dihedral subgroup of D_{2^n} , then we have just shown that $\text{III}^2(G', \mathbb{Z}/2\mathbb{Z}) = 0$. By (4-2), we obtain $H^1(G', [\hat{T}_2]^\text{fl}) = 0$. If G' is a cyclic subgroup of G , then $H^1(G', [\hat{T}_2]^\text{fl}) = 0$ by [Colliot-Thélène and Sansuc 1977, Proposition 2]. Since every subgroup of G is either cyclic or dihedral, it follows that $[\hat{T}_2]^\text{fl}$ is coflasque. By [Voskresenski 1998, §4, Theorem 5], $[\hat{T}_2]^\text{fl}$ is invertible, so by [Saltman 1984, Proposition 3.14(a)], the torus T_2 is retract rational. Since $BT[2]$ is stably birational to T_2 , we conclude that $BT[2]$ is retract rational.

(b) By Lemma 4.2 it suffices to show that $\text{III}^2(Q_8, \mathbb{Z}/2\mathbb{Z}) \neq 0$. Recall that $H^2(Q_8, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the three nontrivial classes correspond to central extensions of the form

$$(7-1) \quad 1 \rightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \Gamma \xrightarrow{\pi} Q_8 \rightarrow 1,$$

where

$$\Gamma := \langle a, b \mid a^4 = b^4 = e, bab^{-1} = a^{-1} \rangle \cong \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$$

is the unique nontrivial semidirect product of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$. Note that $Z(\Gamma) = \{e, a^2, b^2, a^2b^2\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $\iota(1) = a^2b^2$. Let $\alpha \in H^2(Q_8, \mathbb{Z}/2\mathbb{Z})$ be the class of (7-1), in the case when $\pi(a) = i$ and $\pi(b) = j$. Since

$$\pi^{-1}(\langle -1 \rangle) = \langle a^2, b^2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

the sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi^{-1}(\langle -1 \rangle) \rightarrow \langle -1 \rangle \rightarrow 1$$

splits, so α restricts to 0 in $H^1(\langle -1 \rangle, \mathbb{Z}/2\mathbb{Z})$. Moreover, the subgroups

$$\pi^{-1}(\langle i \rangle) = \langle a, b^2 \rangle, \quad \pi^{-1}(\langle j \rangle) = \langle a, b \rangle, \quad \pi^{-1}(\langle i \rangle) = \langle ab, a^2b^2 \rangle$$

all admit a set of two commuting generators of orders 4 and 2, hence they are isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. It follows that α restricts to 0 in $H^2(G', \mathbb{Z}/2\mathbb{Z})$ for every proper subgroup G' of Q_8 , so $0 \neq \alpha \in \text{III}^2(Q_8, \mathbb{Z}/2\mathbb{Z})$.

(c) Let $a := (1, 0)$ and $b := (0, 1)$ be two generators of $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $\Gamma := G \rtimes \mathbb{Z}/2\mathbb{Z}$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on G by sending $a \mapsto ab$ and $b \mapsto b$. The center of Γ is $\langle a^2, b \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and we have a short exact sequence

$$(7-2) \quad 1 \rightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \rightarrow 1,$$

where $\iota(1) := b$. Let $\gamma \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and consider the exact sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \Gamma' \rightarrow \langle \gamma \rangle \rightarrow 1,$$

where $\Gamma' := \pi^{-1}(\langle \gamma \rangle)$. It is easy to check that $(b, 0)$ is not the square of any element

of Γ , hence $\pi^{-1}(\langle \gamma \rangle)$ is a noncyclic extension of $\langle \gamma \rangle$ by $\mathbb{Z}/2\mathbb{Z}$. If $\langle \gamma \rangle \cong \mathbb{Z}/2\mathbb{Z}$, then necessarily $\Gamma' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and if $\langle \gamma \rangle \cong \mathbb{Z}/4\mathbb{Z}$, then $\Gamma' \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. This shows that the class of (7-2) in $H^2(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is a nontrivial element of $\text{III}^2(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$. $\square$

In the next proposition we collect the two pieces of information needed to complete the proof of Theorem 1.1 that we obtained via a computer calculation. We use Algorithm F2 of [Hoshi and Yamasaki 2017, §5.2], implemented in the computer algebra system GAP. I thank Thomas Rüd for helping me understand the code in [Hoshi and Yamasaki 2017].

Proposition 7.2. *If $G = Q_8$, then $BT[4]$ is not retract rational. If $G = (\mathbb{Z}/2\mathbb{Z})^3$, then $BT[2]$ is not retract rational.*

Proof. If (G, d) equals $(Q_8, 4)$ or $((\mathbb{Z}/2\mathbb{Z})^3, 2)$, it is enough to check that T_d is not retract rational, that is, that $[T_d]^{\text{fl}}$ is not invertible. Using the presentation (6-1), this can be done using Algorithm F2 of [Hoshi and Yamasaki 2017, §5.2]. $\square$

Remark 7.3. Much as in the proof of Lemma 7.1(a) (or indeed by another GAP computation), one may show that $\text{III}^2(Q_8, \mathbb{Z}/4\mathbb{Z}) = 0$ and $\text{III}^2((\mathbb{Z}/2\mathbb{Z})^3, \mathbb{Z}/2\mathbb{Z}) = 0$. In other words, in the two cases considered in Proposition 7.2, T_d is coflasque but not retract rational; cf. [Voskresenski 1998, §4 Example 2, Theorem 5]. We will not make use of this in the sequel.

Proof of Theorem 1.1 for $p = 2$. If d is odd, the conclusion of the theorem follows from Proposition 3.3(d). From now on, we assume that d is even.

(a') The proof is the same as that of (a). When G is cyclic, every torus split by G is retract rational by [Colliot-Thélène and Sansuc 1977, Proposition 2], hence T_d is retract rational, and so $BT[d]$ is also retract rational.

(b') When $4 \nmid d$, $BT[d]$ is retract rational by Lemma 7.1(a) and Proposition 3.3(d), so we may assume that $4 \mid d$. If $G = D_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $e(T) = 4$; see the first paragraph of Section 4. Since BT is rational, by Proposition 3.3(b) we deduce that $BT[d]$ is stably birational to T . The torus T is not retract rational by [Colliot-Thélène and Sansuc 1977, Proposition 2], hence $BT[d]$ is not retract rational in this case. If $G = D_{2^n}$ for some $n \geq 3$, then G contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the claim follows from Lemma 5.1.

(c') Let $|G| = 2^n$. Assume first that $BT[d]$ is retract rational for some d divisible by 4. We prove by induction on n that G is cyclic. If $n = 1$, there is nothing to prove. If $n = 2$, the claim follows from (b'). When $n = 3$, by the case $n = 2$ and Lemma 5.1, G does not contain a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, hence $G = \mathbb{Z}/8\mathbb{Z}$ or Q_8 . When $G = Q_8$, $BT[d]$ is not retract rational by Proposition 7.2, hence $G = \mathbb{Z}/8\mathbb{Z}$. If $n \geq 4$, by Lemma 5.1 and the inductive assumption, every subgroup of G is either cyclic or dihedral, hence by Lemma 4.3(c), G is also cyclic or dihedral.

Assume now that $BT[d]$ is retract rational for some $d \equiv 2 \pmod{4}$. We prove that G is either cyclic or dihedral by induction on n . If $n = 1, 2$, there is nothing to prove. If $n = 3$, then by Lemma 7.1, the group G is not isomorphic to Q_8 or $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and by Proposition 7.2, it is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, hence it is either $\mathbb{Z}/8\mathbb{Z}$ or D_8 , as desired. If $n \geq 4$, by inductive assumption, every proper subgroup of G is either cyclic or dihedral. By Lemma 4.3(c), G is either cyclic or dihedral. $\square$

Example 7.4. Let L/k be a Galois extension with Galois group a symmetric group S_n , $d \geq 1$, be an integer, and p be an odd prime. Recall that if $n/2 < p \leq n$, so that $p \mid n!$ but $p^2 \nmid n!$, then any p -Sylow subgroup of S_n is cyclic of order p , and that if $p \leq n/2$ then any p -Sylow subgroup of S_n contains a subgroup of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. The 2-Sylow subgroups of S_n are cyclic for $n = 2, 3$, dihedral for $n = 4, 5$, and neither cyclic nor dihedral for $n \geq 6$ (they contain subgroups of the form $D_8 \times \mathbb{Z}/2\mathbb{Z}$). Applying Theorem 1.1 and Lemma 5.2, we obtain:

- If there exists an odd prime p such that $2p \leq n$ and $p \mid d$, then $BT[d]$ is not retract rational.
- If $n = 4, 5$, and $4 \mid d$, then $BT[d]$ is not retract rational.
- If $n \geq 6$, and d is even, then $BT[d]$ is not retract rational.

In all other cases, $BT[d]$ is retract rational.

Assume further that $d \mid |S_n| = n!$; by Proposition 3.3(d) one can reduce to this case by multiplying by some invertible element in $\mathbb{Z}/e(T)\mathbb{Z}$. Define $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\sigma(1) = 0, \quad \sigma(2) = 2, \quad \sigma(3) = 6, \quad \sigma(4) = 6, \quad \sigma(5) = 30$$

and

$$\sigma(n) = \prod_{n/2 < p \leq n} p, \quad n \geq 6.$$

Then we have just shown that $BT[d]$ is p -retract rational if and only if $d \mid \sigma(n)$.

8. An application to the Grothendieck ring of stacks

The ring $K_0(\text{Stacks}_k)$ was introduced by Ekedahl [2009b]. By definition, it is the quotient of the free abelian group on the equivalence classes $\{\mathcal{X}\}$ of all algebraic stacks $\mathcal{X}$ of finite type over k with affine stabilizers, by the scissor relations $\{\mathcal{X}\} = \{\mathcal{Y}\} + \{\mathcal{X} \setminus \mathcal{Y}\}$ for every closed substack $\mathcal{Y} \subseteq \mathcal{X}$ and the relations $\{\mathcal{E}\} = \{\mathcal{X} \times_k \mathbb{A}_k^n\}$ for every vector bundle $\mathcal{E}$ of rank n over $\mathcal{X}$. The product is defined by $\{\mathcal{X}\} \cdot \{\mathcal{Y}\} := \{\mathcal{X} \times_k \mathcal{Y}\}$ on the generators and extended by linearity; the multiplicative identity is $1 = \{\text{Spec } k\}$.

If G is a finite or connected group, there appears to be a connection between properties of $\{BG\}$ and the Noether Problem for G ; as of now, the link between

the two is largely unexplained in either direction. Consider the following equations in $K_0(\text{Stacks}_k)$, called expected class formulas:

$$\begin{aligned} \{BG\} &= 1, & G \text{ a finite group;} \\ \{BG\} &= \{G\}^{-1}, & G \text{ a connected linear group.} \end{aligned}$$

As shown in [Ekedahl 2009a], it frequently happens that $\{BG\} = 1$ for a finite group G . This is true, for example, if G is a symmetric group; see [Ekedahl 2009a, Theorem 4.3]. It is striking that all the known counterexamples G to $\{BG\} = 1$ are also counterexamples to the Noether Problem.

If G is a connected group and k is algebraically closed, no counterexample to $\{BG\}\{G\} = 1$ is known; this is again in line with the Noether Problem, for which no negative answer is known among connected groups. In [Scavia 2018, Theorem 1.5], we exhibited the first connected counterexample T to the expected class formula, in the case when k admits a biquadratic field extension and $\text{char } k = 0$. More precisely, $T := R_{E_1 \times E_2/k}^{(1)}(\mathbb{G}_m)$, where E_1 and E_2 are distinct quadratic extensions of k ; see [Scavia 2018, §3]. Using this result, we showed that $\{BT[2]\} = \{BT\}\{T\} \neq 1$ in [Scavia 2018, Theorem 1.6]. The same argument also shows that $\{BT[2d]\} = \{BT\}\{T\} \neq 1$ for every $d \geq 1$. On the other hand, one has $\{BT[2d-1]\} = 1$ for every $d \geq 1$. Proposition 3.3 provides a conceptual explanation for this periodicity. Moreover, it allows us to compute $\{BT[n]\}$ for every $n \geq 1$.

Proposition 8.1. *Let T be a k -torus, and $m, n \geq 1$ be integers.*

- (a) *If $n \equiv \pm m \pmod{e(T)}$, then $\{BT[n]\} = \{BT[m]\}$.*
- (b) *If $n \equiv \pm 1 \pmod{e(T)}$, then $\{BT[n]\} = 1$.*
- (c) *If $e(T) \mid n$, then $\{BT[n]\} = \{BT\}\{T\}$.*

Proof. Fix an embedding of T into a quasisplit torus S , and consider the corresponding diagram (3-3) for some $d \geq 1$. Since the quasisplit torus S is special, by [Bergh 2016, Corollary 2.4], we have $\{BS\}\{S\} = 1$ in $K_0(\text{Stacks}_k)$, hence $\{S\}$ is invertible. By [Bergh 2016, Proposition 2.9], we have

$$(8-1) \quad \{BT\} = \{Q\}/\{S\}$$

and

$$(8-2) \quad \{BT[d]\} = \{T_d\}/\{S\}$$

in $K_0(\text{Stacks}_k)$.

- (a) If $n \equiv m \pmod{e(T)}$, by Proposition 3.1 we have $T_n \cong T_m$, hence the claim follows from (8-2).
- (b) If $n \equiv \pm 1 \pmod{e(T)}$ then by (a) we have $\{BT[n]\} = \{BT[1]\} = 1$.

(c) If $e(T) \mid n$, by Proposition 3.1 we have $T_n \cong T \times Q$, so $\{T_n\} = \{T\}\{Q\}$. Combining this with (8-1) and (8-2) (for $n = d$), we obtain

$$\{BT[n]\} = \{T_n\}/\{S\} = \{T\}\{Q\}/\{S\} = \{T\}\{BT\}. \quad \square$$

Proposition 8.2. *Let $T := R_{E/k}^{(1)}(\mathbb{G}_m)$, where E is a product of two distinct separable quadratic extensions of k . For every $d \geq 1$, we have*

$$\{BT[2d-1]\} = 1, \quad \{BT[2d]\} = \{BT\}\{T\}$$

in $K_0(\text{Stacks}_k)$.

Proof. Using the norm exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[C_2] \oplus \mathbb{Z}[C_2] \rightarrow \hat{T} \rightarrow 0,$$

the argument of [Merkurjev 2010, Example 5.1] adapts verbatim to a proof that $e(T) = 2$. The conclusion now follows from Proposition 8.1. $\square$

Since $\{BT\}\{T\} \neq 1$ by [Scavia 2018, Theorem 1.5], the previous proposition gives a more conceptual proof of [Scavia 2018, Theorem 1.6].

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FEDERICO SCAVIA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER BC
CANADA
scavia@math.ubc.ca

EXPLICIT POLYNOMIAL BOUNDS ON PRIME IDEALS IN POLYNOMIAL RINGS OVER FIELDS

WILLIAM SIMMONS AND HENRY TOWNSNER

Consider an ideal $I \subseteq k[x_1, \dots, x_n]$ of a polynomial ring over a field with the property that for some b , if $fg \in I$ for f, g of degree $\leq b$, then $f \in I$ or $g \in I$. It is known that if b is sufficiently large, then I is prime. We construct an explicit bound on b , polynomial in the degree of the generators of I (the existence of such a bound was established by Schmidt-Götsch in 1989). We also give a similar bound for detecting maximal ideals in $k[x_1, \dots, x_n]$.

1. Introduction

Suppose I is an ideal of a polynomial ring over a field, $I \subseteq k[x_1, \dots, x_n]$, and whenever $fg \in I$ for f, g of degree $\leq b$, then either $f \in I$ or $g \in I$. When b is sufficiently large, it turns out that I is prime. Schmidt-Götsch [1989] proved that “sufficiently large” can be taken to be a polynomial in the degree of generators of I (with the degree of this polynomial depending on n). However Schmidt-Götsch used model-theoretic methods to show this, and did not give any indication of how large the degree of this polynomial is. In this paper we obtain an explicit bound on b , polynomial in the degree of the generators of I . We also give a similar bound for detecting maximal ideals in $k[x_1, \dots, x_n]$.

Our problem belongs to a thread of algorithmic algebra going back over a century. Drawing on the work of Kronecker, Noether, and others, Hermann published in 1926 a seminal paper entitled “Die Frage der endlich vielen Schritte in der Theorie der Polynomideale” [Hermann 1926] (see [Hermann 1998], “The question of finitely many steps in polynomial ideal theory”, for an English translation with commentary). Hermann obtained bounds on calculations in polynomial rings over fields such as witnessing ideal membership and finding bases of syzygies. Other tasks she showed to be algorithmically possible include primary decomposition, intersection, and quotients of ideals.

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Seidenberg [1974] analyzed, extended, and in some cases corrected Hermann's work. He noted (see [1974, p. 311]) that one can decide primality of polynomial ideals over "explicitly given fields" by determining if the ideal is primary and, if so, whether the ideal is equal to its associated prime. Seidenberg gave bounds on some steps of the process but did not analyze the complexity of deciding primality.

Buchberger's method of Gröbner bases (introduced in his thesis [1965]; see Section 3 for basic results) provided strong impetus to the development of computational algebra. While many problems are in principle solvable by the methods of Hermann and Seidenberg, the theory of Gröbner bases has grown substantially and is often more convenient. In particular, much work has been done on the complexity of Gröbner bases. Mayr and Meyer [1982] established doubly exponential worst-case bounds on the complexity of Gröbner bases (as a function of the number of indeterminates). Building on subsequent efforts, Dubé [1990] gave a sharper (but still doubly exponential) bound that we take as a starting point for our analysis in Theorem 3.18.

Gröbner basis methods can be used as building blocks for algorithms that test primality in polynomial rings. Gianni, Trager, and Zacharias [Gianni et al. 1988, pp. 157–158] described an algorithm for deciding primality by, among other things, inductively reducing the number of variables to the univariate case. Eisenbud, Huneke, and Vasconcelos [Eisenbud et al. 1992] gave a method for carrying out primary decomposition of ideals in polynomial rings over perfect fields; one may then follow Seidenberg's observation to decide primality. To our knowledge no one has explicitly analyzed the complexity of these algorithms. However, many of our bounds in Section 3 concern steps of the algorithm from [Gianni et al. 1988] as presented in [Adams and Lousaunau 1994].

Bounds and algorithms are closely related, but existence of one does not necessarily imply existence of the other. If a field k is finite, then our results yield a trivial, albeit inefficient, algorithm for checking whether the ideal generated by a given finite set of polynomials over k is prime: see if there is a counterexample among the finite number of polynomials having degree no greater than our bound. (This process uses the fact that ideal membership is decidable.) Nevertheless, primality cannot be effectively decided for arbitrary fields. To see this, recall that if F is a field, a proper ideal generated by a nonzero polynomial g in $F[x]$ is prime if and only if g is irreducible. However, Fröhlich and Shepherdson [1955] gave explicit examples of fields for which irreducibility is algorithmically undecidable.

Another result related to our problem is Lemma 10 of [Chistov 2008]. There Chistov shows that there is a constant $c \in \mathbb{N}$ such that, given an algebraic variety defined by polynomials in n variables of total degree at most d , irreducible components of the variety are defined by polynomials of total degree at most d^{2^c} .

Our main result is as follows:

Theorem 1.1. *There exists a function $b(n, d)$ that is polynomially bounded in d and such that for any field k and proper ideal $I \subseteq k[x_1, \dots, x_n]$ with generators of total degree at most d , if $fg \in I$ implies that either $f \in I$ or $g \in I$ for all f, g of degree less than or equal to $b(n, d)$, then I is prime. Moreover,*

$$b(n, d) \leq \max\{(2d)^{2^{3n^2+2n}}, (2nd)^{n(n+1)2^{4n+6}}\}$$

for all $n, d \geq 1$. If either $d > 1 = n$ or $d = 1$ then the values $b(1, d) = d$ or $b(n, 1) = 0$, respectively, suffice to ensure primality.

See Theorems 4.26 and 4.35 for the final steps of the proof. The function $b(n, d)$ is defined in the course of the argument and a more compact bound easily follows (see Corollary 4.36):

Corollary 1.2. *We have $b(n, d) \leq (2nd)^{n^3 2^{6n^2}}$ for all $n, d \geq 1$.*

Schmidt-Göttsch [1989] showed that $b(n, d)$ is polynomially bounded in d using ultraproduct methods (as in [van den Dries and Schmidt 1984; Schoutens 2010; Göral 2018]), but did not give any explicit calculation of the bound. Abstract results from proof theory [Kohlenbach 2008; Towsner 2018] suggest that it is possible to extract explicit bounds from proofs that use ultraproducts. The authors demonstrated [Simmons and Towsner 2019] that such methods can be used to extract explicit bounds from ultraproduct proofs in commutative and differential algebra [Harrison-Trainor et al. 2012; van den Dries and Schmidt 1984]. (For instance, in Theorem 2.8 and Lemma 7.7 of [Simmons and Towsner 2019] we gave another bound on $b(n, d)$, but that bound was not polynomial in the degree of the generators.) However, the arguments here, while inspired by Schmidt-Göttsch's approach, were not purely extracted by proof theory; instead we bring in known results using Gröbner bases, etc., where possible and add new arguments to simplify (or clarify) the most complicated parts of Schmidt-Göttsch's argument.

2. Outline of the argument

We briefly describe our strategy. We are interested in *counterexamples to primality*, that is, polynomials $f, g \notin I$ such that $fg \in I$. Our goal is to show that if there are no counterexamples to primality of I having degree $b(n, d)$ or less (we say I is *prime up to $b(n, d)$*), then there are no counterexamples of any degree (i.e., I is prime). The main ingredients are localization of polynomial rings, bounds on Gröbner bases for various auxiliary ideals, and bounds on solutions to systems of linear equations over polynomial rings.

- Relabeling if necessary, choose a maximal set of indeterminates $x_1, \dots, x_r$ that is algebraically independent modulo I . By independence mod I we have $k[x_1, \dots, x_r] \cap I = \{0\}$; by maximality $k[x_1, \dots, x_r, x_j] \cap I \neq \{0\}$ for any $j > r$.

Without loss of generality, we may assume that no $x_i \in I$ (otherwise we solve the problem for smaller n).

- Localize with respect to the variables $x_1, \dots, x_r$; that is, consider the ideal

$$J = Ik(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$$

generated by I in the ring $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$. The first main task is to show that $k[x_1, \dots, x_n] \cap J = I$. This step requires induction to prove that there are no counterexamples to primality of I in which one factor belongs to $k[x_1, \dots, x_r]$. Here we use bounds on Gröbner bases of saturation and quotient ideals.

- The second major task is to show that J is maximal. We introduce a maximal ideal M extending J and prove that $J = M$. This step uses various tools such as the primitive element theorem, flat extensions, and bounds on Gröbner bases for elimination ideals. We also require some complicated inductive arguments on the number of variables $x_{r+1}, \dots, x_n$. Once we know that J is maximal and hence prime, we conclude that I is prime.

We are principally concerned with the inductive definition of $b(n, d)$ and obtaining a relatively clean estimate rather than precise values of constants and lower-order terms. For this reason (and for readability) our estimate of the bound is somewhat relaxed compared to the actual content of the proofs. By following closely, the interested reader may keep track of such details.

3. Preliminary definitions and results

Throughout the section, k denotes an arbitrary field (except for Theorems 3.7 and 3.8, which require that k have positive characteristic).

We use the following algebraic result to control degrees when going from rational functions to polynomials:

Theorem 3.1 (Gauss' lemma). *Let R be a unique factorization domain (UFD) and let F be the fraction field of R . If $p(x) \in R[x]$ factors as $q_1(x)q_2(x)$ in $F[x]$ such that q_1, q_2 have degrees d_1, d_2 in x , then $p(x)$ factors as $p_1(x)p_2(x)$ in $R[x]$ where the degrees of p_1, p_2 are also d_1, d_2 , respectively.*

Proof. See, for example, Corollary 2.2 in [Lang 1965]. □

The following version of a well-known theorem in field theory plays a prominent role in Section 4 when we consider the case of zero-dimensional ideals.

Theorem 3.2 (primitive element theorem). *Let $F \subseteq K$ be a field extension, with F being an infinite field. Let $a, b \in K$ be algebraic of degrees r, s , respectively, over F . Further assume that b is separable over F . Then for all but at most $r(s-1)$ choices of $c \in F$, the linear combination $a + cb$ generates the field $F(a, b)$ over F .*

Proof. See [van der Waerden 1970, p. 139]. Though the bound is clear from the proof, we summarize the argument here because it reappears in the proof of Theorem 4.21. Let $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ be the roots (counting multiplicity) of the minimal polynomials f of a and g of b , respectively, over F . (We work in an extension of K containing the roots.) We may assume that $\alpha_1 = a$ and $\beta_1 = b$. By separability, $\beta_i \neq \beta_j$ for $i \neq j$; the list $\alpha_1, \dots, \alpha_r$ may contain repetition. Consider the $r(s-1)$ -many equations $\alpha_i + x\beta_j = a + xb$, where $1 \leq i \leq r$ and $j \neq 1$. Since $\beta_j \neq b$, each equation has at most one solution in F and, except for at most $r(s-1)$ elements, none of the infinitely many remaining elements $c \in F$ satisfies any of the equations.

Let $c \in F$ satisfy none of the equations; we claim that $F(a, b) = F(a + cb)$. Consider the polynomials $g(x)$ and $f((a + cb) - cx)$, both of which belong to $F(a + cb)[x]$ and have a common root b . They have no other root in common: another root β_j of g , with $j \neq 1$, yields an argument $(a + cb) - c\beta_j$ of f that cannot equal α_i for any i lest c satisfy a forbidden equation. It follows that g and $f((a + cb) - cx)$ have a GCD of degree 1 in $F(a + cb)[x]$ (if they were coprime, they could have no common root; if the GCD had degree greater than 1, they would have another common root or g would have a repeated root, contradicting separability). This GCD must be $x - b$, whence $b \in F(a + cb)$ and also $a \in F(a + cb)$. $\square$

3A. Bounds on solutions to linear equations in polynomial rings. Much of our work is done with Gröbner bases, but it is sometimes convenient to use an older tool from [Hermann 1926].

Theorem 3.3. *Let k be a field and let $N_1, N_2 \in \mathbb{N}$. Consider the system of homogeneous equations $\{\sum_{j \leq N_2} f_{ij} X_j = 0\}_{i \leq N_1}$, where the coefficients f_{ij} are polynomials in $k[x_1, \dots, x_n]$ of total degree at most d . Then every solution in $(k[x_1, \dots, x_n])^{N_2}$ of the system may be written as a linear combination (over $k[x_1, \dots, x_n]$) of solutions that each have total degree at most $(2N_1 d)^{2^n}$.*

Proof. See Theorem 3.2 of [Aschenbrenner 2004] for this version of the bound; other proofs of the basic result are Theorem 1 of [Seidenberg 1974], and Theorem 2 of [Hermann 1926] for the original. $\square$

Remark 3.4. In [Simmons and Towsner 2019, Theorem 2.5], we call this fact “internal flatness” because it is equivalent to flatness of extensions of polynomial rings by internal polynomial rings in the sense of nonstandard analysis; [Aschenbrenner 2004] refers to such results as “effective flatness”.

The preceding theorem generalizes to inhomogeneous equations; here the relationship is with faithfully flat extensions.

Theorem 3.5. *Let k be a field and let $N_1, N_2 \in \mathbb{N}$. Consider the system of equations $\{\sum_{j \leq N_2} f_{ij} Y_j = h_i\}_{i \leq N_1}$, where the $f_{ij}, h_i \in k[x_1, \dots, x_n]$ have total degree at*

most d . If the system has a solution in $(k[x_1, \dots, x_n])^{N_2}$, then there exists a solution $y = (y_1, \dots, y_{N_2})$ such that each y_j has total degree at most $(2N_1d)^{2^n}$.

Proof. See Theorem 3.4 of [Aschenbrenner 2004] or p. 92 of [Renschuch 1980] for this bound. Seidenberg [Seidenberg 1974] and Hermann [Hermann 1998] gave earlier proofs but contain some errors (see p. 1, [Aschenbrenner 2004]). $\square$

We also need a version of faithful flatness over the function field $k(x_1, \dots, x_n)$.

Theorem 3.6. *Let k be a field and let $N_1, N_2 \in \mathbb{N}$. Consider the system of equations $\{\sum_{j \leq N_2} f_{ij} Z_j = h_i\}_{i \leq N_1}$, where the $f_{ij}, h_i \in k[x_1, \dots, x_n]$ have total degree at most d . If the system has a solution in $(k(x_1, \dots, x_n))^{N_2}$, then there exists a solution $z = (z_1, \dots, z_{N_2}) \in (k(x_1, \dots, x_n))^{N_2}$ such that each z_j may be written as a ratio of polynomials of total degree at most $(2N_1d)^{2^n}$.*

Proof. Let $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_{N_2}) \in (k(x_1, \dots, x_n))^{N_2}$ be a solution to the system. Multiply every equation in the system by the product of all denominators of $\tilde{z}_1, \dots, \tilde{z}_{N_2}$. It follows that there is a solution in $(k[x_1, \dots, x_n])^{N_2+1}$ to the homogeneous system $\{\sum_{j \leq N_2} f_{ij} \tilde{Z}_j = h_i \tilde{Z}\}_{i \leq N_1}$ such that $\tilde{Z} \neq 0$. Hence Theorem 3.3 implies that there is a solution in $(k[x_1, \dots, x_n])^{N_2+1}$ with each entry having total degree at most $(2N_1d)^{2^n}$ and $\tilde{Z} \neq 0$. Divide each entry by the nonzero value of $\tilde{Z}$ to obtain the desired solution z to the original system. $\square$

The remaining flatness results require the field k to have characteristic $p > 0$.

Theorem 3.7 (cf. [Schmidt-Götsch 1989, Lemma 2.11]). *Let $p, m, N_1, N_2 \in \mathbb{N}$, with p prime, and let k be a field of characteristic p . Consider the system of homogeneous equations $\{\sum_{j \leq N_2} f_{ij} Y_j^{p^m} = 0\}_{i \leq N_1}$, where the f_{ij} are polynomials in $k[x_1, \dots, x_n]$ of total degree at most B . Then every solution in $(k[x_1, \dots, x_n])^{N_2}$ of the system may be written as a linear combination (over $k[x_1, \dots, x_n]$) of solutions that each have total degree at most $(2N_1^2 N_2 \binom{B+n}{n} p^{m(n-1)} B)^{2^n}$.*

Proof. Let V_0 be a basis in k of the set of k^{p^m} -linear combinations of the coefficients of all the f_{ij} . Define V_1 to be the set of all power products $\{x_1^{l_1} \cdots x_n^{l_n} \mid 0 \leq l_j < p^m\}$. Finally, let $V = \{v_0 v_1 \mid v_0 \in V_0, v_1 \in V_1\}$. Since V_0 is a basis and each variable in an element of V_1 has power less than p^m , it follows that V is linearly independent over $k^{p^m}[x_1^{p^m}, \dots, x_n^{p^m}]$. Each f_{ij} may thus be written uniquely as a sum $\sum_{v \in V} f_{ij,v} v$, where $f_{ij,v} \in k^{p^m}[x_1^{p^m}, \dots, x_n^{p^m}]$.

Substituting for f_{ij} and changing the order of addition, the system

$$\{\sum_{j \leq N_2} f_{ij} Y_j^{p^m} = 0\}_{i \leq N_1}$$

becomes

$$\{\sum_{v \in V} (\sum_{j \leq N_2} f_{ij,v} Y_j^{p^m}) v = 0\}_{i \leq N_1}.$$

By linear independence, we may replace this with an equivalent system

$$\{\sum_{j \leq N_2} f_{ij,v} Y_j^{p^m} = 0\}_{v \in V, i \leq N_1}$$

whose coefficients belong to $k^{p^m}[x_1^{p^m}, \dots, x_n^{p^m}]$. This is a polynomial ring over the field k^{p^m} , so by Theorem 3.3 every solution of $\{\sum_{j \leq N_2} f_{ij,v} Z_j = 0\}_{v \in V, i \leq N_1}$ in $(k^{p^m}[x_1^{p^m}, \dots, x_n^{p^m}])^{N_2}$ is a $k^{p^m}[x_1^{p^m}, \dots, x_n^{p^m}]$ -linear combination of solutions having bounded degree. That bound depends on the number of equations in the system and on the total degrees of $f_{ij,v}$ as polynomials in $x_1^{p^m}, \dots, x_n^{p^m}$. The former is $N_1|V|$, where $|V|$ is the cardinality of V . In turn, $|V|$ is the product of the cardinalities $|V_0|$ and $|V_1|$. Lemma 3.10 implies that

$$|V_0| \leq N_1 N_2 (\text{the number of monomials in any } f_{ij}) \leq N_1 N_2 \binom{B+n}{n}.$$

Note that $|V_1| \leq p^{mn}$, so $|V| = |V_0||V_1| \leq N_1 N_2 \binom{B+n}{n} p^{mn}$. Since the $f_{ij,v}$ are obtained by factoring out powers of $x_1^{p^m}, \dots, x_n^{p^m}$ from monomials of the f_{ij} , the degree of the $f_{ij,v}$ as polynomials in $k^{p^m}[x_1^{p^m}, \dots, x_n^{p^m}]$ is at most $\frac{B}{p^m}$. Thus, by Theorem 3.3, the desired bound is

$$(2N_1|V|\frac{B}{p^m})^{2^n} \leq (2N_1^2 N_2 \binom{B+n}{n} p^{m(n-1)} B)^{2^n}. \quad \square$$

Theorem 3.8. *Let $p, m, N_1, N_2 \in \mathbb{N}$, with p prime, and let k be a field of characteristic p . Consider the system of equations $\{\sum_{j \leq N_2} f_{ij} Y_j^{p^m} = h_i\}_{i \leq N_1}$, where f_{ij}, h_i are polynomials in $k[x_1, \dots, x_n]$ of total degree at most B . If the system has a solution in $(k(x_1, \dots, x_n))^{N_2}$, then there is a solution $y = (y_1, \dots, y_{N_2})$ in $(k(x_1, \dots, x_n))^{N_2}$ such that each y_j may be written as a ratio of polynomials of total degree at most*

$$(2N_1^2(N_2 + 1) \binom{B+n}{n} p^{m(n-1)} B)^{2^n}.$$

Proof. We use the same approach as in Theorem 3.6. Consider the homogeneous system $\{\sum_{j \leq N_2} f_{ij} Y_j^{p^m} - h_i Y^{p^m} = 0\}_{i \leq N_1}$, where Y is a new indeterminate. By Theorem 3.7, every solution in $(k[x_1, \dots, x_n])^{N_2+1}$ of the homogeneous system is a $k[x_1, \dots, x_n]$ -linear combination of solutions that each have total degree at most $(2N_1^2(N_2 + 1) \binom{B+n}{n} p^{m(n-1)} B)^{2^n}$. There is a solution in $(k(x_1, \dots, x_n))^{N_2}$ to the original system, so by clearing denominators we obtain a solution in $(k[x_1, \dots, x_n])^{N_2+1}$ to the homogeneous system for which $Y \neq 0$. Therefore there is a bounded solution $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_{N_2}, \tilde{y}_{N_2+1}) \in (k[x_1, \dots, x_n])^{N_2+1}$ such that $\tilde{y}_{N_2+1} \neq 0$. Divide $\tilde{y}$ by $\tilde{y}_{N_2+1}$ to obtain a bounded solution $(\tilde{y}_1/\tilde{y}_{N_2+1}, \dots, \tilde{y}_{N_2}/\tilde{y}_{N_2+1}, 1)$ in $(k(x_1, \dots, x_n))^{N_2+1}$. Then $y := (\tilde{y}_1/\tilde{y}_{N_2+1}, \dots, \tilde{y}_{N_2}/\tilde{y}_{N_2+1})$ is the desired solution to the original equation. $\square$

3B. Gröbner Bases. The basic properties of Gröbner bases are laid out in many places; see, for instance, [Adams and Loustaunau 1994; Buchberger and Winkler 1998; Becker and Weispfenning 1993; Cox et al. 1992; Kreuzer and Robbiano 2000]. We need the following notions and facts.

Definition 3.9. In the polynomial ring $k[x_1, \dots, x_n]$, a *power product* of variables $x_1, \dots, x_n$ is a product $x_1^{r_1} \cdots x_n^{r_n}$ for some nonnegative integers r_i (if all r_i are zero, we write 1 for the product). A *monomial ordering* $<$ on $k[x_1, \dots, x_n]$ is a well-ordering of the set of power products such that

- (1) the product 1 is the least element with respect to $<$, and
- (2) $<$ respects multiplication by power products: if $x_1^{r_1} \cdots x_n^{r_n} < x_1^{s_1} \cdots x_n^{s_n}$, then

$$(x_1^{t_1} \cdots x_n^{t_n})(x_1^{r_1} \cdots x_n^{r_n}) = x_1^{r_1+t_1} \cdots x_n^{r_n+t_n} \\ < x_1^{s_1+t_1} \cdots x_n^{s_n+t_n} = (x_1^{t_1} \cdots x_n^{t_n})(x_1^{s_1} \cdots x_n^{s_n}).$$

Lemma 3.10. For positive integers n, d , there are $\binom{d+n-1}{n-1}$ power products of total degree d in the indeterminates $x_1, \dots, x_n$. Counting 1 as a power product with each variable having degree 0, there are $\binom{d+n}{n}$ power products of total degree at most d .

Proof. This follows from an elementary counting argument; see, e.g., the “stars and bars” method in [Stanley 1986]. $\square$

Definition 3.11. Let $<$ be a monomial ordering on $k[x_1, \dots, x_n]$. The *leading term* $\text{LT}(f)$ of a nonzero polynomial f is the monomial $ax^{r_1} \cdots x^{r_n}$ of f such that the power product $x^{r_1} \cdots x^{r_n}$ is maximal with respect to $<$. The coefficient a of the leading term is the *leading coefficient* $\text{LC}(f)$ of f . (Note that a monomial is simply a power product multiplied by a field element.)

Definition 3.12. Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal and let $<$ be a monomial ordering on $k[x_1, \dots, x_n]$. A set of nonzero polynomials $\{g_1, \dots, g_t\}$ belonging to I is a *Gröbner basis* (with respect to $<$) for I if for every $f \in I$ there exists some i such that the leading term $\text{LT}(g_i)$ of g_i divides the leading term $\text{LT}(f)$ of f .

Definition 3.13. Let $F = \{f_1, \dots, f_s\} \subseteq k[x_1, \dots, x_n]$ be a finite set of nonzero polynomials. A polynomial $f \in k[x_1, \dots, x_n]$ is *reducible* with respect to F if f contains a nonzero monomial that is divisible by $\text{LT}(f_i)$ for some $1 \leq i \leq s$. Otherwise we say that f is *reduced modulo F* . A *reduced Gröbner basis* is one such that each generator is reduced with respect to the set containing all the others.

Theorem 3.14 (division algorithm for $k[x_1, \dots, x_n]$). Let $F = \{f_1, \dots, f_s\}$ be a finite set of nonzero polynomials in $k[x_1, \dots, x_n]$ and let $<$ be a monomial ordering on $k[x_1, \dots, x_n]$. For every polynomial f , there exist $u_1, \dots, u_s, r \in k[x_1, \dots, x_n]$ such that

- (1) $f = u_1 f_1 + \cdots + u_s f_s + r$,
- (2) r is reduced modulo F ,
- (3) $\text{LT}(f) \geq \text{LT}(u_i f_i)$ for all $1 \leq i \leq s$, and

- (4) either r is zero, or f is reduced with respect to F (in which case $u_i = 0$ and $f = r$), or $f \neq r \neq 0$ and $\text{LT}(r) \leq \text{LT}(f)$.

Proof. See Theorem 1.5.9 of [Adams and Lousaunau 1994]. $\square$

When $f = u_1 f_1 + \cdots + u_s f_s + r$ as in the theorem, we say f reduces to r modulo F .

Theorem 3.15. *Let G be a Gröbner basis for a nonzero ideal I . A polynomial $f \in k[x_1, \dots, x_n]$ belongs to I if and only if f reduces to 0 modulo G .*

Proof. See Theorem 1.6.2 of [Adams and Lousaunau 1994]. $\square$

Definition 3.16. Let $f, g \in k[x_1, \dots, x_n]$. Let $\text{LCM}(t_1, t_2)$ denote the least common multiple of two power products t_1, t_2 . We define the S -polynomial $S(f, g)$ of f, g to be

$$\left(\frac{\text{LCM}(\text{LT}(f), \text{LT}(g))}{\text{LT}(f)} \right) \cdot f - \left(\frac{\text{LCM}(\text{LT}(f), \text{LT}(g))}{\text{LT}(g)} \right) \cdot g.$$

Theorem 3.17 (Buchberger's criterion). *Given a monomial ordering, let G be a finite set of nonzero polynomials in $k[x_1, \dots, x_n]$. G is a Gröbner basis of the ideal generated by G if and only if $S(f, g)$ reduces to 0 modulo G for all $f, g \in G$.*

Proof. See Theorem 1.7.4 of [Adams and Lousaunau 1994]. $\square$

Theorem 3.18. *Let I be a nonzero ideal of $k[x_1, \dots, x_n]$ generated by polynomials of total degree at most d . Then for any monomial ordering there is a reduced Gröbner basis of I whose elements have total degree at most*

$$b_1(n, d) := \min \left\{ 2d^{2^n}, 2 \left(\frac{1}{2}d^2 + d \right)^{2^{n-1}} \right\}.$$

Proof. The bound $2 \left(\frac{1}{2}d^2 + d \right)^{2^{n-1}}$ was given by Dubé [1990, Corollary 8.3]. Note also that $2d^{2^n}$ is greater than or equal to Dubé's bound whenever $d > 1$. For $d = 1$, the constant bound 1 suffices; it follows from Buchberger's criterion (Theorem 3.17) that after Gaussian elimination we are left with a Gröbner basis; see Theorem 5.68 in [Becker and Weispfenning 1993]. $\square$

Remark 3.19. We consider the more generous bound $2d^{2^n}$ because later we compose bounds with each other and we desire a relatively clean final answer.

Theorem 3.20. *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal generated by polynomials of total degree at most d . If $G = \{g_1, \dots, g_t\}$ is a reduced Gröbner basis of I whose elements have degree at most $b_1(n, d)$, then $t \leq (b_1(n, d) + 1)^{n-1}$.*

Proof. Since G is reduced, no leading term of g_i divides a leading term of g_j for $i \neq j$. It suffices to find an upper bound on sets of monomials of total degree at most $b_1(n, d)$ such that no monomial in the set divides another. We induct on the number of variables. The constant value 1 clearly works for $n = 1$. Assume the claim holds for $n - 1$. For any $0 \leq \alpha \leq b_1(n, d)$, consider the set $G_{x_1, \alpha}$ of leading

terms of G such that x_1 appears with degree α . Since no element of $G_{x_1, \alpha}$ divides another, the same is true of the set $G_{x_1, \alpha}/(x_1^\alpha)$ of monomials formed by dividing the elements of $G_{x_1, \alpha}$ by x_1^α . By the inductive hypothesis, $G_{x_1, \alpha}/(x_1^\alpha)$ (and hence $G_{x_1, \alpha}$) has at most $(b_1(n, d) + 1)^{n-2}$ elements. G is partitioned into $b_1(n, d) + 1$ sets of size at most $(b_1(n, d) + 1)^{n-2}$, so G has at most $(b_1(n, d) + 1)^{n-1}$ elements, as desired. $\square$

Remark 3.21. As noted earlier, we aim for simplicity rather than sharpness. For a more detailed analysis of the number of generators in a Gröbner basis, see [Robbiano 1991]. The issue is closely related to Dickson’s lemma; see [Figueira et al. 2011; León Sánchez and Ovchinnikov 2016] for bounds in that setting.

We now convert the basic bound $b_1(n, d)$ into bounds on the auxiliary ideals that appear in the next section.

Definition 3.22. A monomial ordering $<$ of $k[y_1, \dots, y_m, x_1, \dots, x_n]$ is an *elimination ordering* if every power product in $k[x_1, \dots, x_n]$ (other than 1) is greater than every power product in $k[y_1, \dots, y_m]$. If $I \subseteq k[y_1, \dots, y_m, x_1, \dots, x_n]$ is an ideal, then $I \cap k[y_1, \dots, y_m]$ is an *elimination ideal* that eliminates the x -variables.

Theorem 3.23. Let $<$ be an elimination ordering of $k[y_1, \dots, y_m, x_1, \dots, x_n]$ such that power products in the x -variables are greater than products in the y -variables. Let G be a Gröbner basis with respect to $<$ of an ideal $I \subseteq k[y_1, \dots, y_m, x_1, \dots, x_n]$ such that the elimination ideal $I \cap k[y_1, \dots, y_m]$ is nonzero. Then $G \cap k[y_1, \dots, y_m]$ is a Gröbner basis of $I \cap k[y_1, \dots, y_m]$.

Proof. See Theorem 2.3.4 of [Adams and Loustaunau 1994]. $\square$

Theorem 3.24. Let $<$ be an elimination ordering of $k[y_1, \dots, y_m, x_1, \dots, x_n]$ such that power products in the x -variables are greater than products in the y -variables. Let $I \subseteq k[y_1, \dots, y_m, x_1, \dots, x_n]$ be generated by polynomials of total degree at most d . Also suppose that the elimination ideal $I \cap k[y_1, \dots, y_m]$ is nonzero. Then there is a Gröbner basis (with respect to $<$) of $I \cap k[y_1, \dots, y_m]$ whose elements have total degree at most $b_1(m + n, d)$.

Proof. By Theorem 3.23, the elimination ideal $I \cap k[y_1, \dots, y_m]$ has a Gröbner basis that is a subset of the Gröbner basis of I given by Theorem 3.18. $\square$

Theorem 3.25. Let $I \subseteq k[x_1, \dots, x_n]$ be generated by polynomials of total degree at most d and let $u \in k[x_1, \dots, x_n]$ also have total degree at most d . If there exists a nonzero polynomial $p \in k[x_1, \dots, x_r][Y]$ (where Y is a new indeterminate and $r \leq n$) such that $p(u) \in I$, then there exists a nonzero polynomial $q \in k[x_1, \dots, x_r][Y]$ of total degree at most $b_1(n + 1, d)$ such that $q(u) \in I$.

Proof. Consider the ideal $\tilde{I} = (I, Y - u) \subseteq k[x_1, \dots, x_n, Y]$. Since $p(u) \in I$, it follows that $p(Y) \in \tilde{I}$. Use an elimination ordering in which power products of

$x_{r+1}, \dots, x_n$ are greater than power products in $x_1, \dots, x_r, Y$. Since the elimination ideal $\tilde{I} \cap k[x_1, \dots, x_r, Y]$ is nonzero, by Theorem 3.24, there is a Gröbner basis of $\tilde{I} \cap k[x_1, \dots, x_r, Y]$ whose elements have total degree at most $b_1(n+1, d)$. Let q be an element of this Gröbner basis; note that $q(Y) \in \tilde{I}$ implies that $q(u) \in I$. $\square$

3C. Quotients and Saturations.

Definition 3.26. Let I and J be ideals of a commutative ring R . The *quotient ideal* of I by J (denoted $I : J$) is defined to be

$$\{f \in R \mid Jf \subseteq I\} = \{f \in R \mid gf \in I \text{ for all } g \in J\}.$$

Theorem 3.27. (1) Let I and J be ideals of $k[x_1, \dots, x_n]$ and let w be a new variable. Then $I \cap J = (wI, (1-w)J) \cap k[x_1, \dots, x_n]$.

(2) Let I be an ideal of $k[x_1, \dots, x_n]$ and let f be a nonzero polynomial. Then the quotient ideal $I : (f)$ is $\frac{1}{f}(I \cap (f))$.

Proof. See [Adams and Loustau 1994, Proposition 2.3.5 and Lemma 2.3.11]. $\square$

Theorem 3.28. If $I \subseteq k[x_1, \dots, x_n]$ is a nonzero ideal generated by polynomials of total degree at most d and f is a nonzero polynomial of total degree at most d , then the quotient ideal $I : (f)$ has a Gröbner basis whose elements have total degree at most $b_1(n+1, d+1)$.

Proof. By properties of a monomial ordering, the leading term of a product is the product of the leading terms. It follows from this and Theorem 3.27(2) that by dividing the elements of a Gröbner basis for $I \cap (f)$ by f , we obtain a Gröbner basis for $I : (f)$. Hence it suffices to give a bound on Gröbner bases of $I \cap (f)$. By Theorem 3.27(1), $I \cap (f) = (wI, (1-w)f) \cap k[x_1, \dots, x_n]$, where w is a new variable. Use an elimination ordering with w the greatest variable to eliminate w from an ideal generated by polynomials of total degree at most $d+1$. Then Theorems 3.18 and 3.24 imply that $I \cap (f)$ has a Gröbner basis of total degree at most $b_1(n+1, d+1)$. $\square$

Definition 3.29. Let A be a commutative ring with 1. A subset S of A is a *multiplicative set* if $1 \in S$, $0 \notin S$, and $xy \in S$ for all $x, y \in S$. If I is an ideal of A , the *saturation* of I by S is the set of all elements $a \in A$ such that for some $s \in S$ we have $sa \in I$. We write $I : S^\infty$ to denote the saturation of I by S . If b is a nonnilpotent element of A , we denote the multiplicative set $\{1, b, b^2, \dots\}$ by b^∞ and write the saturation as $I : b^\infty$.

Remark 3.30. It is easy to see that $I : S^\infty$ is an ideal containing I . An equivalent characterization is that the saturation is the intersection $(S^{-1}I) \cap A$, where $S^{-1}I$ is the ideal generated by I in the localization $S^{-1}A$.

In Theorem 4.4 we saturate an ideal $I \subseteq k[x_1, \dots, x_n]$ by the multiplicative set $k[x_i] \setminus \{0\}$. To compute bounds on generators of this saturation we must consider Gröbner bases of polynomials with coefficients from more general rings. The definition of a Gröbner basis is slightly different from the field case to account for the fact that coefficients might be zero-divisors or fail to be units. (Monomial orderings on the set of power products are the same, however. Also, in this paper we only consider the case when the coefficient ring is a Noetherian integral domain.) For the precise definition, see Definition 4.1.13 of [Adams and Loustau 1994]. The details are not necessary for our purposes because of the following theorem:

Theorem 3.31. *Let G be a Gröbner basis (in the sense of fields) of an ideal I of $k[y, x_1, \dots, x_n]$ with respect to an elimination order with the x -variables greater than y . Then G is a Gröbner basis (in the sense of Noetherian integral domains) of I viewed as an ideal of $(k[y])[x_1, \dots, x_n]$.*

Proof. See Theorem 4.1.18 in [Adams and Loustau 1994]. $\square$

Theorem 3.32. *Let R be a Noetherian integral domain, let $g \in R[y_1, \dots, y_m]$ be a nonzero polynomial, and let I be an ideal of $R[y_1, \dots, y_m]$. Letting w be a new variable, we have $I : g^\infty = (I, wg - 1) \cap R[y_1, \dots, y_m]$.*

Proof. See Proposition 4.4.1 in [Adams and Loustau 1994]. $\square$

Theorem 3.33. *Let R be a Noetherian integral domain and let $I \subseteq R[y_1, \dots, y_m]$ be an ideal having a Gröbner basis $G = \{g_1, \dots, g_t\}$ (in the sense of Noetherian integral domains). Let $g = \text{LC}(g_1) \text{LC}(g_2) \cdots \text{LC}(g_t)$ be the product of the leading coefficients of the elements of G . Then $I : (R \setminus \{0\})^\infty = I : g^\infty$.*

Proof. See Proposition 4.4.4 in [Adams and Loustau 1994]. $\square$

Theorem 3.34. *If $I \subseteq k[x_1, \dots, x_n]$ is a nonzero ideal generated by polynomials of total degree at most d , then the saturation $I : (k[x_1] \setminus \{0\})^\infty$ has a Gröbner basis whose elements have total degree at most*

$$b_2(n, d) := b_1(n + 1, (b_1(n, d) + 1)^{n-1} b_1(n, d) + 1).$$

Proof. Choose an elimination ordering having x_1 as least variable. By Theorems 3.18 and 3.20, I has a reduced Gröbner basis $G = \{g_1, \dots, g_t\}$ with respect to this ordering such that the elements of G have total degree at most $b_1(n, d)$ and $t \leq (b_1(n, d) + 1)^{n-1}$. Now view I as an ideal of $(k[x_1])[x_2, \dots, x_n]$, the polynomial ring in $n - 1$ variables over $k[x_1]$. By Theorem 3.31 we know that G is still a Gröbner basis in the sense of coefficient rings that are Noetherian integral domains. By Theorem 3.33, $I : (k[x_1] \setminus \{0\})^\infty = I : g^\infty$, where $g = \text{LC}(g_1) \text{LC}(g_2) \cdots \text{LC}(g_t)$ is the product of the leading coefficients of the elements of G . Thus the degree of g in x_1 is bounded by $tb_1(n, d)$.

Theorem 3.32 shows that $I : g^\infty = (I, wg - 1) \cap (k[x_1])[x_2, \dots, x_n]$, where w is a new variable. Hence $I : (k[x_1] \setminus \{0\})^\infty = (I, wg - 1) \cap k[x_1, x_2, \dots, x_n]$ and we may view $(I, wg - 1)$ as an ideal of $k[x_1, x_2, \dots, x_n, w]$ generated by polynomials of total degree at most $tb_1(n, d) + 1$.

Use an elimination ordering with w the greatest variable to eliminate w from $(I, wg - 1)$. By 3.18, 3.24, and the bound on t , the resulting elimination ideal has a Gröbner basis of total degree at most

$$b_1(n + 1, tb_1(n, d) + 1) \leq b_1(n + 1, (b_1(n, d) + 1)^{n-1}b_1(n, d) + 1). \quad \square$$

Theorem 3.35. $b_2(n, d) \leq 2^{n2^{n+3}} d^{n2^{2n+1}}.$

Proof. We get a bound of the desired form through the following chain of inequalities:

$$\begin{aligned} b_2(n, d) &= b_1(n + 1, (b_1(n, d) + 1)^{n-1}b_1(n, d) + 1) \\ &\leq 2((b_1(n, d) + 1)^{n-1}b_1(n, d) + 1)^{2^{n+1}} \leq 2((2d^{2^n} + 1)^{n-1} \cdot 2d^{2^n} + 1)^{2^{n+1}} \\ &\leq 2((2^2 d^{2^n})^{n-1} \cdot 2^2 d^{2^n})^{2^{n+1}} = 2(2^{2n} d^{n2^n})^{2^{n+1}} \leq 2^{n2^{n+3}} d^{n2^{2n+1}}. \quad \square \end{aligned}$$

4. Proof of the main theorem

We continue to use k to denote an arbitrary field. I denotes a proper ideal generated by polynomials $f_1, \dots, f_s \in k[x_1, \dots, x_n]$ of total degree at most d . As explained in the outline, we choose a maximal set of indeterminates that are algebraically independent modulo I . Relabeling if necessary, we may assume they are $x_1, \dots, x_r$. Since 0 is the only polynomial in $k[x_1, \dots, x_r] \cap I$, whenever $fg \in I$ for $f \in k[x_1, \dots, x_r] \setminus \{0\}$, either $g \in I$ or f, g are counterexamples to primality of I .

Definition 4.1. If $I \subseteq k[x_1, \dots, x_n]$ is a proper ideal and b is a natural number, we say that I is *prime up to b* if for all $f, g \in k[x_1, \dots, x_n]$ of total degree at most b such that $fg \in I$, either $f \in I$ or $g \in I$. By *maximal up to b* , we mean that for every $f \notin I$ with degree $\leq b$, f has an inverse mod I : there is some g with $fg - 1 \in I$.

Before proceeding, we take care of the simple cases in the main theorem:

Theorem 4.2. *Let $I \subseteq k[x_1, \dots, x_n]$ be a proper ideal generated by polynomials of degree at most d . Let $\{x_1, \dots, x_r\}$ be a maximal subset of $\{x_1, \dots, x_r, \dots, x_n\}$ that is algebraically independent modulo I .*

- If $n = 1$, then I is prime if I is prime up to d .
- If $d = 1$ (with n arbitrary), then I is prime.
- If $r = 0$, then I is prime if I is prime up to $nb_1(n, d)$.
- If $r = n$, then $I = \{0\}$ and is prime.

Proof. First let $n = 1$. The ring $k[x_1]$ is a principal ideal domain and I is of the form (f) , where f is the greatest common divisor of the given generators of I . Then f has degree at most d and I is prime if and only if f is irreducible (if f is reducible, no proper factor belongs to I). If I is prime up to d , then f is irreducible.

When $d = 1$, then I is generated by linear polynomials, and I is necessarily prime; this follows geometrically from the irreducibility of affine linear varieties. Therefore primality up to any natural number (e.g., 0) vacuously guarantees primality if $d = 1$.

If $r = 0$, by Theorem 3.24 there is a nonzero polynomial $\alpha_i(x_i) \in I \cap k[x_i]$ of degree at most $b_1(n, d)$ for each $x_1, \dots, x_n$. Using α_i to reduce higher powers of $x_i \bmod I$, we see that any f is equivalent mod I to some $\tilde{f}$ of total degree at most $nb_1(n, d)$. Hence $fg \in I$ if and only if $\tilde{f}\tilde{g} \in I$ and then primality up to $nb_1(n, d)$ ensures primality.

The final claim is clear. $\square$

Remark 4.3. If any $x_i \in I$, then modulo I we may assume that x_i does not appear in any polynomials we consider. Our claimed bound on detecting prime ideals increases with n , so having fewer variables only makes the bound easier to prove.

Due to the preceding theorem and remark, we may safely assume in the following work that $n, d > 1$, $0 < r < n$ and that $x_i \notin I$ for any $1 \leq i \leq n$.

Denote by J the ideal generated by I in $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$. The key properties of J are that if I is prime up to a sufficient level, then $I = J \cap k[x_1, \dots, x_n]$ and J is maximal. Together these prove that I is prime.

4A. Proof of $I = J \cap k[x_1, \dots, x_n]$. To obtain this result we rely heavily on the bounds from Section 3 on Gröbner bases for various kinds of ideals.

Theorem 4.4. *Let $1 \leq i \leq n$. Suppose that $f \in k[x_i]$ is irreducible and that there exists $g \in k[x_1, \dots, x_n]$ such that $fg \in I$ but $g \notin I$. Then the degree of f is at most*

$$b_3(n, d) := b_1(n + 1, b_2(n, d) + 1).$$

Proof. Let $q_1, \dots, q_N$ be generators of the saturation ideal

$$I : (k[x_i] \setminus \{0\})^\infty \subseteq k[x_1, \dots, x_n].$$

By Theorem 3.34, the total degree of each generator is at most $b_2(n, d)$. For each $1 \leq l \leq N$, consider the ideal $(I : (q_l)) \cap k[x_i]$. Since $k[x_i]$ is a PID, each ideal $(I : (q_l)) \cap k[x_i]$ is generated by some polynomial $u_l \in k[x_i]$; in particular, $u_l q_l \in I$. By Theorems 3.28 and 3.24, the degree of u_l is at most $b_3(n, d) := b_1(n + 1, b_2(n, d) + 1)$. Let $f \in k[x_i]$ be irreducible and let g be an element of $k[x_1, \dots, x_n] \setminus I$ with the property that $fg \in I$. We claim that

$$\deg(f) \leq \max_{1 \leq l \leq N} \{\deg(u_l)\} \leq b_3(n, d).$$

Note that $g \in I : (k[x_i] \setminus \{0\})^\infty$ and so there are $\alpha_l \in k[x_1, \dots, x_n]$ such that $g = \sum_{l=1}^N \alpha_l q_l$. Multiplying both sides by $\prod_{l=1}^N u_l$, we obtain

$$(\prod_{l=1}^N u_l)g = (\prod_{l=1}^N u_l)(\sum_{l=1}^N \alpha_l q_l) = \sum_{l=1}^N (\prod_{l=1}^N u_l) \alpha_l q_l \in I,$$

so $\prod_{l=1}^N u_l \in (I : (g)) \cap k[x_i]$. Also note that $f \in (I : (g)) \cap k[x_i]$ and that $(I : (g)) \cap k[x_i]$ is proper since $g \notin I$. Given that f belongs to the same proper ideal of $k[x_i]$ as $\prod_{l=1}^N u_l$ and is irreducible, f must generate the ideal and hence divide u_l for some l . In particular, the degree of f is at most the maximum of the degrees of the u_l and is less than or equal to $b_3(n, d)$ as claimed. $\square$

Remark 4.5. Note that the result holds for any variable x_i , not necessarily one of the $x_1, \dots, x_r$. Also, the proof goes through unchanged for any field L and any ideal of $L[x_1, \dots, x_n]$ generated by elements of degree at most d . This is important because in Theorem 4.8 we apply Theorem 4.4 after changing the field from k to $k(x_{i_1}, \dots, x_{i_l})$ for some subset $\{x_{i_1}, \dots, x_{i_l}\}$ of the variables.

Theorem 4.6. $b_3(n, d) \leq 2^{n2^{2n+6}} d^{n2^{3n+2}}.$

Proof. We compute as follows:

$$\begin{aligned} b_3(n, d) &= b_1(n+1, b_2(n, d) + 1) \leq 2(2^{n2^{n+3}} d^{n2^{2n+1}} + 1)^{2^{n+1}} \\ &\leq 2(2^{n2^{n+4}} d^{n2^{2n+1}})^{2^{n+1}} = 2(2^{n2^{2n+5}} d^{n2^{3n+2}}) \leq 2^{n2^{2n+6}} d^{n2^{3n+2}}. \end{aligned} \quad \square$$

We aim to show that $J \cap k[x_1, \dots, x_n] = I$. We consider arbitrary subsets $\{x_{i_1}, \dots, x_{i_l}\} \subseteq \{x_1, \dots, x_r\}$ of cardinality l having certain properties and show inductively that $\{x_{i_1}, \dots, x_{i_l}, x_{i_{l+1}}\}$ retains the properties. Let

$$J_l := Ik(x_{i_1}, \dots, x_{i_l})[\text{remaining variables from } x_1, \dots, x_n],$$

$$J_{l+1} := Ik(x_{i_1}, \dots, x_{i_l}, x_{i_{l+1}})[\text{remaining variables from } x_1, \dots, x_n].$$

With this notation we have $J = J_r$.

Theorem 4.7. *Let $1 \leq l \leq r$ and suppose that $I : (f) = I$ for all irreducible $f \in k[x_{i_1}, \dots, x_{i_l}]$ and every subset $\{x_{i_1}, \dots, x_{i_l}\} \subseteq \{x_1, \dots, x_r\}$ of cardinality l . Then*

- (1) $I : (f) = I$ for all $f \in k[x_{i_1}, \dots, x_{i_l}] \setminus \{0\}$ (irreducible or not) and
- (2) $J_l \cap k[x_1, \dots, x_n] = I$.

Proof. The reverse containment is trivial for both parts. For the forward direction, let f factor as $\prod_{l=1}^N p_l$ for some irreducible polynomials $p_l \in k[x_{i_1}, \dots, x_{i_l}]$ and suppose $fg \in I$. It follows that $(\prod_{l=2}^N p_l)g \in I : (p_1)$, which is equal to I by the hypothesis since p_1 is irreducible. Continuing this way, we conclude that $g \in I$. This proves (1).

For the second claim, let $g \in J_l \cap k[x_1, \dots, x_n]$. Clearing denominators, we see that $fg \in I$ for some $f \in k[x_{i_1}, \dots, x_{i_l}]$. Part (1) implies that $g \in I$. $\square$

Theorem 4.8. *Let $1 \leq l < r$ and $n > 1$. Suppose $J_l \cap k[x_1, \dots, x_n] = I$ for all $\{x_{i_1}, \dots, x_{i_l}\} \subseteq \{x_1, \dots, x_r\}$ and $I : (f) = I$ for all irreducible $f \in k[x_{i_1}, \dots, x_{i_l}, x_{i_{l+1}}]$ of total degree at most $(n-1) \cdot b_3(n, d)$. Then $J_{l+1} \cap k[x_1, \dots, x_n] = I$.*

Proof. By Theorem 4.7, it suffices to show that $I : (f) = I$ for all irreducible $f \in k[x_{i_1}, \dots, x_{i_l}, x_{i_{l+1}}]$. Pick any such f . By hypothesis we already have $I : (f) = I$ if the total degree of f is less than or equal to $(n-1) \cdot b_3(n, d)$, so suppose that the degree of f exceeds this. At least one of the $l+1 \leq r \leq n-1$ variables must appear with degree greater than $b_3(n, d)$. Without loss of generality we may assume that variable is $x_{i_{l+1}}$ (because we assume the properties in the statement for arbitrary subsets of $\{x_1, \dots, x_r\}$ of size l). By Gauss' lemma, f remains irreducible over $k(x_{i_1}, \dots, x_{i_l})$. Since Theorem 4.4 applies to irreducible univariate polynomials over any field and the $x_{i_{l+1}}$ -degree of f is greater than $b_3(n, d)$, we deduce that $J_l : (f) = J_l$. Let $g \in k[x_1, \dots, x_n]$ be such that $fg \in I$. We conclude from $J_l : (f) = J_l$ that $g \in J_l$. It follows that $g \in I$ because by hypothesis $J_l \cap k[x_1, \dots, x_n] = I$. This proves that $I : (f) = I$ as required. $\square$

Theorem 4.9. *Let $n, d > 1$. If I is prime up to*

$$b_4(n, d) := b_1(n+1, (n-1) \cdot b_3(n, d) + 1),$$

then for all $1 \leq l \leq r$, subsets $\{x_{i_1}, \dots, x_{i_l}\} \subseteq \{x_1, \dots, x_r\}$, and irreducible polynomials $f \in k[x_{i_1}, \dots, x_{i_l}]$ of total degree at most $(n-1) \cdot b_3(n, d)$, we have $I : (f) = I$.

Proof. It suffices for the generators of $I : (f)$ to belong to I . By Theorem 3.28, such a generator g has total degree at most $b_4(n, d) := b_1(n+1, (n-1) \cdot b_3(n, d) + 1)$. Since $fg \in I$ but $f \notin I$, by primality up to $b_4(n, d)$ we conclude that $g \in I$. $\square$

Theorem 4.10. *Let $n, d > 1$. If I is prime up to $b_4(n, d)$, then $J \cap k[x_1, \dots, x_n] = I$.*

Proof. We induct on the size l of subsets $\{x_{i_1}, \dots, x_{i_l}\}$ of $\{x_1, \dots, x_r\}$ to prove that $J_l \cap k[x_1, \dots, x_n] = I$ for each $1 \leq l \leq r$. By Theorem 4.9, $I : (f) = I$ for all irreducible $f \in k[x_{i_1}, \dots, x_{i_l}]$ of total degree at most $(n-1) \cdot b_3(n, d)$. (This satisfies one of the two hypotheses of Theorem 4.8.) Letting $l = 1$, Theorem 4.4 now implies that $I : (f) = I$ for irreducible $f \in k[x_{i_l}]$ of any degree. Theorem 4.7 then yields the base case $J_1 \cap k[x_1, \dots, x_n] = I$. We can now apply Theorem 4.8 repeatedly to obtain $J \cap k[x_1, \dots, x_n] = I$. $\square$

4B. J is maximal. We start with an important lemma that allows us to control the degrees of $x_{r+1}, \dots, x_n$ modulo I .

Lemma 4.11. *Let I be defined as before and suppose I is prime up to $b_1(n, d)$.*

- (1) For each $r + 1 \leq j \leq n$, there is a polynomial $w_j \in I \cap k[x_1, \dots, x_r][x_j]$ such that w_j has total degree at most $b_1(n, d)$, is irreducible, and remains irreducible over $k(x_1, \dots, x_r)[x_j]$.
- (2) The $k(x_1, \dots, x_r)$ -vector space $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]/J$ is finite dimensional of dimension at most $\prod_{j=r+1}^n d_j \leq (b_1(n, d))^{n-r}$, where d_j is the x_j -degree of w_j in the first part of the lemma.

Proof. By definition of $\{x_1, \dots, x_r\}$ as a maximal algebraically independent set mod I , we know that $I \cap k[x_1, \dots, x_r] = \{0\}$ and for all $j > r$ there exist elements of positive x_j -degree in $I \cap k[x_1, \dots, x_r][x_j]$. By Theorem 3.24, $I \cap k[x_1, \dots, x_r][x_j]$ has a Gröbner basis whose elements have total degree at most $b_1(n, d)$; the x_j -degree of each generator is positive. Consider the set

$$W = \{w \in I \cap k[x_1, \dots, x_r][x_j] \mid \text{the total degree of } w \text{ is at most } b_1(n, d)\}$$

and let N be the minimal x_j -degree of any element of W . Let $W_N \subseteq W$ be the subset of W whose elements have x_j -degree N . Choose $w_j = a_N x_j^N + \dots + a_0 \in W_N$ such that $a_N \in k[x_1, \dots, x_r]$ has minimal total degree out of all coefficients of x_j^N in W_N .

We claim that w_j is irreducible in $k[x_1, \dots, x_r][x_j]$. To see this, suppose toward contradiction that w_j properly factors as $fg = (b_l x_j^l + \dots + b_0)(c_m x_j^m + \dots + c_0)$. Then either $b_l \in k[x_1, \dots, x_r]$ would have lower total degree than a_N or $l < N$, and likewise for c_m and m . Since f, g have total degree at most $b_1(n, d)$, by primality up to $b_1(n, d)$ we have $f \in I$ or $g \in I$. But this contradicts either minimality of N or that of a_N , so w_j does not factor in $k[x_1, \dots, x_r][x_j]$. By Gauss' lemma (see Theorem 3.1), w_j remains irreducible over $k(x_1, \dots, x_r)[x_j]$.

The final claim follows from the existence of the w_j and the definition of d_j . $\square$

Lemma 4.12. J is contained in a maximal ideal M of $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$.

Proof. We need only show that J is proper. Suppose on the contrary that 1 is a $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ -linear combination of $f_1, \dots, f_s$. Then by clearing denominators in an equation witnessing that $1 \in J$, we obtain a nonzero element $k[x_1, \dots, x_r] \cap I$. This contradicts algebraic independence of $x_1, \dots, x_r$ mod I . $\square$

We prove that J is maximal by showing that $J = M$. A crucial step is the following:

Proof that elements of M having small degree belong to J . When working in the ring $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$, we must keep track of the variables $x_1, \dots, x_r$ as well as $x_{r+1}, \dots, x_n$. For convenience we use the following terminology:

Definition 4.13. Let $\alpha/\beta \in k(x_1, \dots, x_r) \setminus \{0\}$ with α, β coprime as polynomials in $k[x_1, \dots, x_r]$. We say the $(x_1, \dots, x_r)$ -degree (or just degree) of α/β is the maximum of the total degrees of α and β . A monomial $(\alpha/\beta)T$, where T is a power product in $x_{r+1}, \dots, x_n$, is defined to have $(x_1, \dots, x_r)$ -degree equal

to the sum of the $(x_1, \dots, x_r)$ -degree of α/β and the total degree of T . Given $f \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n] \setminus \{0\}$, we define the $(x_1, \dots, x_r)$ -degree of f to be the maximum of the $(x_1, \dots, x_r)$ -degrees of the monomials appearing in f .

Let $\mathcal{J} \subseteq k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ be a proper ideal and B a natural number. We say $\mathcal{J}$ is $(x_1, \dots, x_r)$ -prime up to B if for all $f, g \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ of $(x_1, \dots, x_r)$ -degree at most B such that $fg \in \mathcal{J}$, either $f \in \mathcal{J}$ or $g \in \mathcal{J}$. Likewise, if every $f \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n] \setminus \mathcal{J}$ of $(x_1, \dots, x_r)$ -degree at most B is invertible mod $\mathcal{J}$, we say that $\mathcal{J}$ is $(x_1, \dots, x_r)$ -maximal up to B .

Lemma 4.14. *Let $f \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n] \setminus \{0\}$ have $(x_1, \dots, x_r)$ -degree at most B . Then there exists a polynomial $g \in k[x_1, \dots, x_r]$ of total degree at most $B \cdot \binom{B+n}{n}$ such that the product $gf \in k[x_1, \dots, x_r, x_{r+1}, \dots, x_n]$ has total degree at most $B(1 + \binom{B+n}{n})$.*

Proof. Write f as $\sum_i (\alpha_i/\beta_i)T_i$, where α_i, β_i are coprime in $k[x_1, \dots, x_r]$ and T_i is a power product in $x_{r+1}, \dots, x_n$. Since f has $(x_1, \dots, x_r)$ -degree bounded by B , each α_i, β_i , and T_i has total degree at most B . Clear denominators and keep track of the degrees. By Lemma 3.10, there are $\binom{B+n}{n}$ power products of degree at most B in n variables, so f contains at most that many monomials. Let g be $\prod_i \beta_i$, which has total degree at most $B \cdot \binom{B+n}{n}$. It follows that

$$gf = \left(\prod_i \beta_i\right) \cdot \sum_i (\alpha_i/\beta_i)T_i \in k[x_1, \dots, x_r, x_{r+1}, \dots, x_n]$$

and has total degree at most $B(1 + \binom{B+n}{n})$. $\square$

Lemma 4.15. *For any d_0 , if I is prime up to $\max\{b_4(n, d), d_0(1 + \binom{d_0+n}{n})\}$, then $J = Ik(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ is $(x_1, \dots, x_r)$ -prime up to d_0 .*

Proof. Let $g, h \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ each have $(x_1, \dots, x_r)$ -degree $\leq d_0$ and suppose $gh \in J$. By Lemma 4.14, there are $g_0, h_0 \in k[x_1, \dots, x_r]$ such that $g_0g, h_0h \in k[x_1, \dots, x_n]$ and the total degrees of g_0g, h_0h are at most $d_0(1 + \binom{d_0+n}{n})$. By Theorem 4.10, $(g_0g)(h_0h) \in I$ since $(g_0g)(h_0h) \in J \cap k[x_1, \dots, x_n]$ and I is prime up to at least $b_4(n, d)$. Since I is also prime up to at least $d_0(1 + \binom{d_0+n}{n})$, either $g_0g \in I$ or $h_0h \in I$, showing that either g or h belongs to J . $\square$

Lemma 4.16 (cf. [Schmidt-Götttsch 1989, Lemma 2.3]). *Suppose the $(x_1, \dots, x_r)$ -degree of $f \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ is at most d_0 , where $d \leq d_0$. For each $j = r+1, \dots, n$, denote by d_j the x_j -degree of the polynomial w_j obtained in Lemma 4.11. There exists $\tilde{f} \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ such that the following hold:*

- (1) $\deg_{x_j} \tilde{f} < d_j$,
- (2) $f - \tilde{f} \in J = Ik(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$, and
- (3) the $(x_1, \dots, x_r)$ -degree of $\tilde{f}$ is bounded by $d_0(1 + b_1(n, d))^{n-r}$.

Proof. The claim follows from successively dividing f by $w_{r+1}, \dots, w_n$. By Lemma 4.11, each such $w_j \in I \cap k[x_1, \dots, x_r][x_j] \subseteq J$ has total degree at most $b_1(n, d)$, so $d_j \leq b_1(n, d)$. The first division step involving w_{r+1} at most adds the $(x_1, \dots, x_r)$ -degrees of f and w_{r+1} while the x_{r+1} -degree decreases. (If $\deg_{x_{r+1}} f$ is already less than d_{r+1} , we may skip to w_{r+2} .) At each step we increase the original $(x_1, \dots, x_r)$ -degree of f (which was at most d_0) by at most $b_1(n, d)$, and we can continue at most d_0 -many times. Therefore, the $(x_1, \dots, x_r)$ -degree of the remainder after completing division by w_{r+1} is bounded by $d_0 + d_0 \cdot b_1(n, d) = d_0(1 + b_1(n, d))$.

After dividing by w_j , the degree of the remainder in x_j is less than d_j and can never increase thereafter because x_j does not appear in w_l for $r < l \neq j$. The remainder $\tilde{f}$ after considering all $n - r$ polynomials w_j thus satisfies the first two claims. The pattern repeats, giving us a bound of

$$d_0 + d_0 \cdot \binom{n-r}{1} \cdot b_1(n, d) + \dots + d_0 \cdot \binom{n-r}{n-r-1} \cdot (b_1(n, d))^{n-r-1} + d_0 \cdot (b_1(n, d))^{n-r} \\ = d_0(1 + b_1(n, d))^{n-r}$$

when the process terminates. $\square$

Lemma 4.17 (cf. [Schmidt-Göttsch 1989, Lemma 2.4]). *Suppose the $(x_1, \dots, x_r)$ -degree of $f \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ is at most d_0 , where $d \leq d_0$. For each $j = r + 1, \dots, n$, denote by d_j the x_j -degree of the polynomial w_j obtained in Lemma 4.11. There is a monic polynomial $\theta_f \in k(x_1, \dots, x_r)[Y]$ such that*

- (1) *the Y -degree of θ_f is at most $\prod_{j=r+1}^n d_j$,*
- (2) *$\theta_f(f) \in J = Ik(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$, and*
- (3) *the $(x_1, \dots, x_r)$ -degree of θ_f is at most $(\prod_{j=r+1}^n d_j)(d_0(1 + b_1(n, d))^{n-r})$.*

Proof. Let $\{v_1, \dots, v_t\}$ be power products in $x_{r+1}, \dots, x_n$ that form a basis of $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n] \bmod J$. By Lemma 4.11, t is at most $\prod_{j=r+1}^n d_j$. By Lemma 4.16, for each v_i and product $f v_i$ there is some $\tilde{f} v_i$ such that

- for $r < j \leq n$, the x_j -degree of $\tilde{f} v_i$ is less than d_j ,
- $f v_i - \tilde{f} v_i \in J$, and
- $\tilde{f} v_i$ has $(x_1, \dots, x_r)$ -degree at most $d_0(1 + b_1(n, d))^{n-r}$.

So we may write

$$\tilde{f} v_i = \sum_{l=1}^t a_{il} v_l,$$

where each $a_{il} \in k(x_1, \dots, x_r)$ has $(x_1, \dots, x_r)$ -degree at most $d_0(1 + b_1(n, d))^{n-r}$.

We use the determinant trick. Consider the $t \times t$ -matrix $A(Y) = [\delta_{il} Y - a_{il}]$ and note that the determinant of $A(Y)$ is monic and $\det(A(f))$ must belong to J : if v is the column vector with entries v_l , then $[a_{il}]v = f v \bmod J$. Therefore $[\delta_{il} f - a_{il}]v = 0 \bmod J$. Multiply both sides by the adjugate of $[\delta_{il} f - a_{il}]$

to obtain $[\delta_{il} \det(A(f))]v \in J$. The power product 1 appears among the v_i , so $\det(A(f)) \in J$.

Let $\theta_f = \det(A(Y))$. Then θ_f has Y -degree at most $\prod_{j=r+1}^n d_j$ and $(x_1, \dots, x_r)$ -degree at most $t \cdot (d_0(1 + b_1(n, d))^{n-r}) \leq (\prod_{j=r+1}^n d_j)(d_0(1 + b_1(n, d))^{n-r})$. $\square$

Theorem 4.18 (cf. [Schmidt-Götsch 1989, Lemma 2.7]). *Let $d_0 \in \mathbb{N}$ and let*

$$\tilde{d} = (\prod_{j=r+1}^n d_j)(d_0)(1 + (1 + b_1(n, d))^{n-r}),$$

with d_j defined in Lemma 4.11. If I is prime up to $\max\{b_4(n, d), \tilde{d}(1 + \binom{\tilde{d}+n}{n})\}$, then $J = Ik(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ is $(x_1, \dots, x_r)$ -maximal up to d_0 .

Proof. Suppose we have $f \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n] \setminus J$ with $(x_1, \dots, x_r)$ -degree $\leq d_0$. By Lemma 4.17, there is a monic $\theta_f \in k(x_1, \dots, x_n)[Y]$ with Y -degree $\leq \prod_{j=r+1}^n d_j$ and $(x_1, \dots, x_r)$ -degree $\leq (\prod_{j=r+1}^n d_j)(d_0(1 + b_1(n, d))^{n-r})$ so that $\theta_f(f) \in J$. Note that $\theta_f(f)$ has $(x_1, \dots, x_r)$ -degree at most

$$\begin{aligned} (d_0)(\prod_{j=r+1}^n d_j) + (\prod_{j=r+1}^n d_j)(d_0(1 + b_1(n, d))^{n-r}) \\ = (\prod_{j=r+1}^n d_j)(d_0)(1 + (1 + b_1(n, d))^{n-r}) = \tilde{d}. \end{aligned}$$

Let us write $\theta_f = Y^m \theta'$ with m maximal, so $f^m \theta'(f) \in J$. By Lemma 4.15, primality of I up to $\max\{b_4(n, d), \tilde{d}(1 + \binom{\tilde{d}+n}{n})\}$ implies $(x_1, \dots, x_r)$ -primality of J up to $\tilde{d}$. Since $f^m \theta'(f)$ has $(x_1, \dots, x_r)$ -degree $\leq \tilde{d}$ and $f \notin J$, we must have $\theta'(f) \in J$. This also implies that the constant term of $\theta'(Y)$ cannot be 0.

Since $\theta'(Y) = Y^s + \sum_{i=1}^{s-1} a_i Y^i + a_0$ with $a_0 \neq 0$ and each $a_i \in k(x_1, \dots, x_r)$, we have

$$\left(a_0^{-1} f^{s-1} + \sum_{i=1}^{s-1} a_0^{-1} a_i f^{i-1}\right) f + 1 \in J,$$

and therefore f is invertible mod J . $\square$

Corollary 4.19. *Let M be the maximal ideal in Lemma 4.12 and let $d_0, \tilde{d}, I, J$ be defined as in Theorem 4.18, with I being prime up to*

$$\max\{b_4(n, d), \tilde{d}(1 + \binom{\tilde{d}+n}{n})\}.$$

If $f \in M \subseteq k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ has total $(x_1, \dots, x_r)$ -degree less than or equal to d_0 , then $f \in J$.

Proof. If $f \notin J$, then f is invertible mod J because J is $(x_1, \dots, x_r)$ -maximal up to d_0 by Theorem 4.18. But since $f \in M$, it follows that $1 \in M$, contradicting that M is a proper ideal. $\square$

Proof that $J = M$. We argue that $J = M$ using induction on the number of variables $x_{r+1}, \dots, x_n$. It is convenient to first treat the case that $x_{r+2} + M, \dots, x_n + M$ are separable over $k(x_1, \dots, x_r)$ (technically, the isomorphic copy $k(x_1, \dots, x_r) + M$ contained in the field $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]/M$). Theorem 4.21, which is based on Theorem 2.8 in [Schmidt-Götsch 1989], contains the heart of the matter.

Lemma 4.20. *Let M be as above and let $M_j := M \cap k(x_1, \dots, x_r)[x_{r+1}, \dots, x_j]$ for $r+1 \leq j \leq n$. Then M_j is a maximal ideal of $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_j]$ and the field $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_j]/M_j$ embeds in $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]/M$.*

Proof. Consider the generators $x_{r+1} + M, \dots, x_n + M$ of the field extension $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]/M$ over $k(x_1, \dots, x_r)$. Because each generator is algebraic, the $k(x_1, \dots, x_r)$ -algebra $k(x_1, \dots, x_r)[x_{r+1} + M, \dots, x_j + M]$ is a subfield of $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]/M$. The $k(x_1, \dots, x_r)$ -algebra is isomorphic to $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_j]/M_j$ via the obvious map sending $x_{r+1} + M_j$ to $x_{r+1} + M$, etc. $\square$

Theorem 4.21 (cf. [Schmidt-Götsch 1989, Theorem 2.8]). *Take I, J, M as above. Let I be prime up to $b_5(n, d) := (2d)^{2^{3n+2n}}$ and $x_{r+2} + M, \dots, x_n + M$ separable over $k(x_1, \dots, x_r)$. For each $r+1 \leq j \leq n$ there are $c_{r+1,j}, \dots, c_{j,j} \in k[x_1, \dots, x_r]$, $h_j \in k[x_1, \dots, x_r][Y]$, and $\phi_{r+1,j}, \dots, \phi_{j,j} \in k(x_1, \dots, x_r)[Y]$ such that, if we define $U_j := \sum_{i=r+1}^j c_{i,j}x_i$ and $M_j := M \cap k(x_1, \dots, x_r)[x_{r+1}, \dots, x_j]$, then*

- (1) $U_j + M_j$ generates $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_j]/M_j$ over $k(x_1, \dots, x_r)$,
- (2) h_j is the minimal polynomial of $U_j + M_j$ over $k(x_1, \dots, x_r)$,
- (3) $h_j(U_j) \in I$ and $x_{r+1} - \phi_{r+1,j}(U_j), \dots, x_j - \phi_{j,j}(U_j)$ all belong to J , and
- (4) the total degrees of $c_{r+1,j}, \dots, c_{j,j} \in k[x_1, \dots, x_r]$ and $h_j \in k[x_1, \dots, x_r][Y]$ are at most $b_5(n, d)$.

Proof. We induct on $r+1 \leq j \leq n$. Let $w_j(x_j)$ be as defined in Lemma 4.11. For the base case, define $c_{r+1,r+1} = 1$, $h_{r+1} = w_{r+1}(Y)$, and $\phi_{r+1,r+1} = x_{r+1} - w_{r+1}(Y)$. Clearly $U_{r+1} + M_{r+1} = x_{r+1} + M_{r+1}$ generates $k(x_1, \dots, x_r)[x_{r+1}]/M_{r+1}$ over $k(x_1, \dots, x_r)$. Lemma 4.11 implies that (2) and (4) hold for $c_{r+1,r+1}$ and h_{r+1} . The bounds suffice since the total degree of w_{r+1} is at most $b_1(n, d) \leq 2d^{2^n} \leq b_5(n, d)$. Claim (3) holds because $x_{r+1} - \phi_{r+1,r+1}(U_{r+1}) = w_{r+1}(x_{r+1}) \in I \subseteq J$.

Suppose (1)–(4) hold for $U_j, c_{r+1,j}, \dots, c_{j,j}, h_j$ and $\phi_{r+1,j}, \dots, \phi_{j,j}$. We may also assume that the total degrees of U_j and h_j are at most $(2d)^{2^{3jn+2j}}$ (this holds for the base case $j = r+1$). Set $c_{r+1,j+1}, \dots, c_{j,j+1}$ equal to $c_{r+1,j}, \dots, c_{j,j}$, respectively. By Lemma 4.20, $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_{j+1}]/M_{j+1}$ is a field extending (an isomorphic copy of) $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_j]/M_j$. From the inductive hypothesis, h_j is the minimal polynomial of $U_j + M_{j+1}$ over $k(x_1, \dots, x_r)$. By assumption $x_{j+1} + M_{j+1}$ is separable and by Lemma 4.11 w_{j+1} is the minimal polynomial of $x_{j+1} + M_{j+1}$ over $k(x_1, \dots, x_r)$. We also know that $k(x_1, \dots, x_r)$ is an infinite field

and $U_j + M_j$ generates $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_j]/M_j$. So the primitive element theorem (Theorem 3.2) implies that there exists $c_{j+1, j+1} \in k[x_1, \dots, x_r]$ of total degree at most $(\deg h_j)(b_1(n, d) - 1) \leq (2d)^{2^{3jn+2j}} \cdot (2d^{2^n} - 1)$ such that $U_{j+1} + M_{j+1} = U_j + c_{j+1, j+1}x_{j+1} + M_{j+1}$ generates $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_{j+1}]/M_{j+1}$ over $k(x_1, \dots, x_r)$. Note that the total degree of U_{j+1} is at most

$$(2d)^{2^{3jn+2j}} \cdot (2d^{2^n} - 1) + 1 \leq (2d)^{2^{3jn+2j}} \cdot 2d^{2^n} \leq (2d)^{2^{3jn+2j+2^n}}.$$

Because $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]/J$ is a finite-dimensional $k(x_1, \dots, x_r)$ -vector space, U_{j+1} is algebraic mod J (and hence mod I). By Theorem 3.25, the set

$$H = \{h \in k[x_1, \dots, x_r][Y] \setminus \{0\} \mid \text{the total degree of } h \text{ is at most } b_1(n+1, \max\{d, \deg U_{j+1}\}) \text{ and } h(U_{j+1}) \in I\}$$

is nonempty. The total degrees of elements of H are bounded by

$$\begin{aligned} b_1(n+1, \max\{d, \deg U_{j+1}\}) &\leq b_1(n+1, (2d)^{2^{3jn+2j+2^n}}) \leq 2((2d)^{2^{3jn+2j+2^n}})^{2^{n+1}} \\ &\leq (2d)^{(2^{3jn+2j+2^n})2^{n+1}+1} \leq (2d)^{2^{3jn+2j+n+1+2^{2n+1}+1}} \leq (2d)^{2^{3(j+1)n+2(j+1)}}. \end{aligned}$$

By the same argument we used in Lemma 4.11, there exists some irreducible $h_{j+1} \in H$; by definition of H , the total degree of h_{j+1} is at most

$$(2d)^{2^{3(j+1)n+2(j+1)}} \leq b_5(n, d).$$

(Since $j \leq n$, primality up to $b_5(n, d) := (2d)^{2^{3n^2+2n}}$ suffices for each j . The bound $(2d)^{2^{3(j+1)n+2(j+1)}}$ has the correct form to continue the induction.)

Thus h_{j+1} is the minimal polynomial of $U_{j+1} + M_{j+1}$ over $k(x_1, \dots, x_r)$. Note that $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_{j+1}]/M_{j+1}$ is isomorphic to $k(x_1, \dots, x_r)[Y]/(h_{j+1})$ under the map sending $U_{j+1} + M_{j+1}$ to $Y + (h_{j+1})$.

It remains to define $\phi_{r+1, j+1}, \dots, \phi_{j+1, j+1}$. We start with $\phi_{j+1, j+1}$ and use it to determine the others. Consider the polynomial ring $k(x_1, \dots, x_r)[Y, Z]$. The proof of Theorem 3.2 shows that there exist $A, B \in k(x_1, \dots, x_r)[Y, Z]$ such that

$$\begin{aligned} Z - (x_{j+1} + M_{j+1}) &= h_j((U_{j+1} + M_{j+1}) - c_{j+1, j+1}Z)A(U_{j+1} + M_{j+1}, Z) \\ &\quad + w_{j+1}(Z)B(U_{j+1} + M_{j+1}, Z). \end{aligned}$$

(This uses the fact that h_j and w_{j+1} are the minimal polynomials of $U_j + M_{j+1} = U_{j+1} - c_{j+1, j+1}x_{j+1} + M_{j+1}$ and $x_{j+1} + M_{j+1}$, respectively.)

Since U_{j+1} generates $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_{j+1}] \bmod M_{j+1}$, there is some polynomial $C \in k(x_1, \dots, x_r)[Y]$ such that $x_{j+1} - C(U_{j+1}) \in M_{j+1}$. It follows that

$$\begin{aligned} Z - (C(U_{j+1}) + M_{j+1}) &= h_j((U_{j+1} + M_{j+1}) - c_{j+1, j+1}Z)A(U_{j+1} + M_{j+1}, Z) \\ &\quad + w_{j+1}(Z)B(U_{j+1} + M_{j+1}, Z). \end{aligned}$$

By the isomorphism sending $U_{j+1} + M_{j+1}$ to $Y + (h_{j+1})$, we have $Z - C(Y) = h_j(Y - c_{j+1,j+1}Z)A(Y, Z) + w_{j+1}(Z)B(Y, Z) + h_{j+1}(Y)D(Y, Z)$ for some D in $k(x_1, \dots, x_r)[Y, Z]$. Define $\phi_{j+1,j+1}(Y)$ to be $C(Y)$ and observe that upon substituting x_{j+1} for Z and U_{j+1} for Y we get $x_{j+1} - \phi_{j+1,j+1}(U_{j+1}) \in J$.

By the induction hypothesis on (3), $x_{r+1} - \phi_{r+1,j}(U_j), \dots, x_j - \phi_{j,j}(U_j)$ belong to J . Since $U_j = U_{j+1} - c_{j+1,j+1}x_{j+1}$ and $x_{j+1} - \phi_{j+1,j+1}(U_{j+1}) \in J$, we define $\phi_{l,j+1}(Y)$ to be $\phi_{l,j}(Y - c_{j+1,j+1}\phi_{j+1,j+1}(Y))$ for $r+1 \leq l \leq j$. This implies that $x_l - \phi_{l,j+1}(U_{j+1}) \in J$, completing the proof. $\square$

Theorem 4.22. *Let I, J, M be defined as above. Suppose I is prime up to $b_5(n, d)$ and $x_{r+2} + M, \dots, x_n + M$ are separable over $k(x_1, \dots, x_r)$. Then J is a maximal ideal of $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$.*

Proof. We prove that $J = M$ by showing for any $p \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ that if $p \notin J$, then $p \notin M$ (noting that $J \subseteq M$ by definition). We use the polynomials $U_n, x_{r+1} - \phi_{r+1,n}(U_n), \dots, x_n - \phi_{n,n}(U_n)$ and $h_n(U_n)$ from Theorem 4.21. By the theorem, $U_n + M_n = U_n + M$ generates the field $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]/M$, $x_{r+1} - \phi_{r+1,n}(U_n), \dots, x_n - \phi_{n,n}(U_n)$ all belong to J , and $h_n(U_n) \in I$ with h_n being the minimal polynomial of $U_n + M$. Hence we may replace, modulo J , each $x_{r+1}, \dots, x_n$ in p with a polynomial in U_n ; i.e., $p = p_1(U_n) + q$ for some $q \in J$ and some $p_1 \in k(x_1, \dots, x_r)[Y]$. Since $p \notin J$ but $h_n(U_n) \in I$, we know that $p_1(U_n) \notin J$ and the remainder $p_2(U_n)$ from dividing $p_1(U_n)$ by $h_n(U_n)$ is not zero.

Because $p_2(U_n)$ is a nonzero polynomial in U_n of lower degree than $h_n(U_n)$, minimality of h_n implies that $p_2(U_n) \notin M$. Since $p_1(U_n)$ is equal to $p_2(U_n) \bmod J$, it follows that $p \notin M$. $\square$

We now reduce the inseparable case to the separable. Theorem 4.25 is based on Theorems 2.8 and 2.12 in [Schmidt-Götsch 1989]. We use induction, with the base step depending on Theorem 4.22. The inductive step requires a specialization of the faithful flatness results from Section 3A. In Lemma 4.23 we continue to use the same field k and ideals $I \subseteq k[x_1, \dots, x_r, x_{r+1}, \dots, x_n]$ and $J \subseteq k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ as before.

Lemma 4.23 (cf. [Schmidt-Götsch 1989, Lemma 2.10]). *Let $g \in k[x_1, \dots, x_n]$ have total degree at most $B \geq d$ and let $D = \{t_1, \dots, t_N\}$ be a set of power products $x_{r+1}^{i_{r+1}} \cdots x_n^{i_n}$, with each element of D having total degree at most B . Suppose there exist $\beta_t \in k(x_1, \dots, x_r)$ for each $t \in D$ such that $g + \sum_{t \in D} \beta_t t \in J$. Then there exist $\gamma_t \in k(x_1, \dots, x_r)$ such that $g + \sum_{t \in D} \gamma_t t \in J$ and each γ_t may be written as a ratio of polynomials of total degree at most $(2^{(2B)^{2^n+d+n-r}} B)^{2^r}$.*

Proof. Considering g as well as the generators $f_1, \dots, f_s$ of I as polynomials in $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$, we may apply Theorem 3.5 to conclude that there are $g_i \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ having total degree (in $x_{r+1}, \dots, x_n$) at most

$(2B)^{2^n}$ such that $g + \sum_{t \in D} \beta_t t = \sum_{i=1}^s g_i f_i$. The monomials in $x_{r+1}, \dots, x_n$ that appear in the equation have total degree at most $(2B)^{2^n} + d$. By Lemma 3.10, there are

$$N_1 := \binom{(2B)^{2^n} + d + n - r}{n - r}$$

power products in $x_{r+1}, \dots, x_n$ of total degree at most $(2B)^{2^n} + d$.

From the $k(x_1, \dots, x_r)$ -coefficients of the power products of $x_{r+1}, \dots, x_n$ in $g + \sum_{t \in D} \beta_t t = \sum_{i=1}^s g_i f_i$, we obtain a system of N_1 -many linear equations over $k[x_1, \dots, x_r]$ whose coefficients have degree at most B . Theorem 3.6 implies that the system has solutions in $k(x_1, \dots, x_r)$ bounded by $(2 \binom{(2B)^{2^n} + d + n - r}{n - r} B)^{2^r}$. Thus there are $\gamma_t, g'_i \in k(x_1, \dots, x_r)$ that satisfy the claimed bound such that $g + \sum_{t \in D} \gamma_t t = \sum_{i=1}^s g'_i f_i \in J$. $\square$

We are ready to handle the general case. To keep track of the bounds, we use the following names in Theorem 4.25. While $b_1(n, d)$, $b_4(n, d)$, and $b_5(n, d)$ are the same as before, note that the other values are not necessarily identical to those in the statements of earlier theorems.

Notation 4.24. Define

$$\begin{aligned} N_1 &:= \left(2 \binom{(2(n-r)b_1(n,d))^{2^n} + d + n - r}{n - r} (n - r) b_1(n, d) \right)^{2^r}, \\ N_2 &:= \left(2 \binom{(2(n-r)(b_1(n,d))^2)^{2^n} + d + n - r}{n - r} (n - r) (b_1(n, d))^2 \right)^{2^r}, \\ N_3 &:= b_1(n, d)^{n-r}, \\ N_4 &:= N_3 N_2 + N_1, \\ N_5 &:= (2N_3^2(N_3 + 1) \binom{N_4 + r}{r} (b_1(n, d))^{(r-1)} N_4)^{2^r}, \\ N_6 &:= N_5 + (n - r)(b_1(n, d))^2, \\ \tilde{d} &:= N_3 N_6 (1 + (1 + b_1(n, d))^{n-r}), \\ b_6(n, d) &:= \max\{b_4(n, d), b_5(n, d), \tilde{d}(1 + \binom{\tilde{d} + n}{n})\}. \end{aligned}$$

Theorem 4.25 (cf. [Schmidt-Götsch 1989, Theorems 2.8 and 2.12]). *If I is prime up to $b_6(n, d)$, then J is a maximal ideal of $k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$.*

Proof. Since I is prime at least up to $b_5(n, d)$, by Theorem 4.22 we are done unless k has prime characteristic p and at least some of $x_{r+1}, \dots, x_n$ are inseparable mod M . For each $r + 1 \leq j \leq n$, let $h_j(Y) \in k(x_1, \dots, x_r)[Y]$ be the minimal polynomial of x_j mod M . If h_j happens to be separable, define m_j to be 0. If h_j is inseparable, then every exponent of Y that appears in h_j is divisible by p . Let m_j be the greatest power of p such that p^{m_j} divides all exponents of Y that appear in h_j . Recall that the polynomial $w_j(x_j) \in k[x_1, \dots, x_r][x_j]$ from Lemma 4.11 belongs to $I \subseteq M$. Being a minimal polynomial, $h_j(Y)$ divides $w_j(Y)$ in $k(x_1, \dots, x_r)[Y]$. Lemma 4.11 then implies that $p^{m_j} \leq \deg_Y(h_j) \leq \deg_Y(w_j) = d_j \leq b_1(n, d)$.

Let $\tilde{h}_j(Z) \in k(x_1, \dots, x_r)[Z]$ be the polynomial obtained from h_j by replacing $Y^{p^{m_j}}$ with Z . Note that $\tilde{h}_j$ remains irreducible. It follows from our choice of m_j that not every exponent of Z appearing in $\tilde{h}_j$ is divisible by p . Hence $\tilde{h}_j$ is the minimal polynomial of $x_j^{p^{m_j}} \bmod M$ and $x_j^{p^{m_j}} + M$ is separable over $k(x_1, \dots, x_r)$.

Let $m = \max\{m_{r+1}, \dots, m_n\}$. For convenience, let us define the polynomial ring

$$P_{r,l,n} := k(x_1, \dots, x_r)[x_{r+1}, \dots, x_l, x_{l+1}^{p^m}, \dots, x_n^{p^m}]$$

(where $P_{r,r,n}$ is understood to be $k(x_1, \dots, x_r)[x_{r+1}^{p^m}, \dots, x_n^{p^m}]$). Note that $M \cap P_{r,r,n}$ is still a maximal ideal: if $f \in P_{r,r,n} \setminus M$, then by maximality of M , there exists $g \in k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ such that $fg - 1 \in M$. Since k has prime characteristic p , $(fg - 1)^{p^m} = f^{p^m}g^{p^m} - 1 = f(f^{p^m-1}g^{p^m}) - 1 \in M$, with $f^{p^m-1}g^{p^m} \in P_{r,r,n}$. This shows that $M \cap P_{r,r,n}$ is maximal. Since $x_{r+1}^{p^m} + M \cap P_{r,r,n}, \dots, x_n^{p^m} + M \cap P_{r,r,n}$ are separable over $k(x_1, \dots, x_r)$, by Theorem 4.22 the ideal $J \cap P_{r,r,n} = M \cap P_{r,r,n}$. We use induction to show that $J = M$.

For $r \leq l \leq n$, define $S_l = J \cap P_{r,l,n}$ and $M_l = M \cap P_{r,l,n}$. (So in particular $M_n = M \cap k(x_1, \dots, x_r)[x_{r+1}, \dots, x_n] = M$.) As was the case with $l = r$, each M_l is a maximal ideal of $P_{r,l,n}$. Define the quotient rings

$$A_l = P_{r,l,n}/S_l \quad \text{and} \quad B_l = P_{r,l,n}/M_l.$$

The previous paragraph established that $S_r = M_r$, the base case of our induction. Now suppose that $S_j = M_j$ for some $r \leq j < n$. In particular, $A_j = B_j$. We claim that $S_{j+1} = M_{j+1}$.

Note that A_{j+1} surjects as a ring onto the field B_{j+1} via the natural map sending $\alpha + S_{j+1}$ to $\alpha + M_{j+1}$ for $\alpha \in P_{r,j+1,n}$. We prove this map is injective by showing that the vector space dimension of B_{j+1} over B_j is greater than or equal to that of A_{j+1} over the field $A_j = B_j$. This will show that $A_{j+1} = B_{j+1}$ and $S_{j+1} = M_{j+1}$.

Since $x_{j+1}^{p^m} + M_{j+1} \in B_j$ and B_{j+1} is generated as a field over B_j by $x_{j+1} + M_{j+1}$, B_{j+1}/B_j is a purely inseparable extension. Hence the minimal polynomial of $x_{j+1} + M_{j+1}$ over B_j has the form $X^{p^\mu} + b$ for some $\mu \in \mathbb{N}$ and $b \in B_j$. In particular, the degree of the extension B_{j+1}/B_j is p^μ . It remains to show that the vector space dimension of A_{j+1} over $A_j = B_j$ is at most p^μ .

Denote by τ the minimal natural number such that $x_{j+1}^{p^\tau} + S_{j+1} \in A_j$. Note that $\mu \leq \tau \leq m$. Let $D = \{t_1 + S_j, \dots, t_N + S_j\}$ be a $k(x_1, \dots, x_r)$ -basis of $A_j = B_j$, where $t_1, \dots, t_N$ are representative power products in the variables $x_{r+1}, \dots, x_j, x_{j+1}^{p^m}, \dots, x_n^{p^m}$. Thus there are $\alpha_t, \beta_t \in k(x_1, \dots, x_r)$ such that

$$x_{j+1}^{p^\mu} + \sum_{t \in D} \alpha_t t \in M \quad \text{and} \quad x_{j+1}^{p^\tau} + \sum_{t \in D} \beta_t t \in J.$$

(From now on we write $t \in D$ to refer to any of $t_1, \dots, t_N$.) It follows from Lemma 4.11 that we may assume that a power product $t \in D$ has total degree at most $(n - r)b_1(n, d)$ as a polynomial in $x_{r+1}, \dots, x_j, x_{j+1}^{p^m}, \dots, x_n^{p^m}$. We have

$p^\tau \leq p^m \leq b_1(n, d)$, so by Lemma 4.23 we may assume that the β_t are ratios of polynomials of total degree at most

$$N_1 := \left(2^{\binom{(2(n-r)b_1(n,d))^{2^n} + d + n - r}{n-r}} (n-r)b_1(n, d)\right)^{2^r}.$$

If $\tau = \mu$, then $x_{j+1}^{p^\tau} + \sum_{t \in D} \beta_t t \in J$ implies that the dimension of A_{j+1} over A_j is at most p^μ . Hence it suffices to obtain a contradiction from the assumption $\mu < \tau$. Suppose then that $\mu < \tau$ and consider

$$(1) \quad \left(x_{j+1}^{p^\mu} + \sum_{t \in D} \alpha_t t\right)^{p^{\tau-\mu}} - \left(x_{j+1}^{p^\tau} + \sum_{t \in D} \beta_t t\right) = \sum_{t \in D} \alpha_t^{p^{\tau-\mu}} t^{p^{\tau-\mu}} - \sum_{t \in D} \beta_t t.$$

This belongs to $M_j = M \cap P_{r,j,n}$, which by hypothesis equals $S_j = J \cap P_{r,j,n}$. By Lemma 4.23 we see that for each $t \in D$, there exist $\gamma_{t,t'} \in k(x_1, \dots, x_r)$ such that $t^{p^{\tau-\mu}} = \sum_{t' \in D} \gamma_{t,t'} t' \bmod S_j$ and $\gamma_{t,t'}$ is a ratio of polynomials having total degree at most

$$N_2 := \left(2^{\binom{(2(n-r)(b_1(n,d))^2)^{2^n} + d + n - r}{n-r}} (n-r)(b_1(n, d))^2\right)^{2^r}.$$

Substituting $\sum_{t' \in D} \gamma_{t,t'} t'$ for $t^{p^{\tau-\mu}}$, we get the following equalities mod S_j :

$$(2) \quad \begin{aligned} \sum_{t' \in D} \beta_{t'} t' &= \sum_{t \in D} \beta_t t = \sum_{t \in D} \alpha_t^{p^{\tau-\mu}} t^{p^{\tau-\mu}} = \sum_{t \in D} \alpha_t^{p^{\tau-\mu}} \left(\sum_{t' \in D} \gamma_{t,t'} t' \right) \\ &= \sum_{t \in D} \sum_{t' \in D} (\alpha_t^{p^{\tau-\mu}} \gamma_{t,t'}) = \sum_{t' \in D} \left(\sum_{t \in D} \alpha_t^{p^{\tau-\mu}} \gamma_{t,t'} \right) t'. \end{aligned}$$

The $t_i + S_j$ form a $k(x_1, \dots, x_r)$ -basis of $A_j = B_j$, so for each $t' \in D$, we have

$$\sum_{t \in D} \alpha_t^{p^{\tau-\mu}} \gamma_{t,t'} = \beta_{t'}.$$

Note that the cardinality $|D|$ of D is at most $N_3 := b_1(n, d)^{n-r}$ because each variable $x_{r+1}, \dots, x_j, x_{j+1}^{p^m}, \dots, x_n^{p^m}$ in t has power at most $b_1(n, d)$. Clearing denominators in each equation (separately for each t'), we obtain a system of equations

$$\left\{ \sum_{t \in D} \alpha_t^{p^{\tau-\mu}} \tilde{\gamma}_{t,t'} = \tilde{\beta}_{t'} \right\}_{t' \in D}$$

such that $\tilde{\gamma}_{t,t'}, \tilde{\beta}_{t'} \in k[x_1, \dots, x_r]$ and have total degree at most $N_4 := N_3 N_2 + N_1$.

By Theorem 3.8, for each $t \in D$ there exists $\tilde{\alpha}_t \in k(x_1, \dots, x_r)$ that is a ratio of polynomials of total degree at most $N_5 := (2N_3^2(N_3 + 1) \binom{N_4 + r}{r} p^{(\tau-\mu)(r-1)} N_4)^{2^r}$ satisfying $\left\{ \sum_{t \in D} \tilde{\alpha}_t^{p^{\tau-\mu}} \tilde{\gamma}_{t,t'} = \tilde{\beta}_{t'} \right\}_{t' \in D}$. Dividing to “unclear denominators”, we have

$$\left\{ \sum_{t \in D} \tilde{\alpha}_t^{p^{\tau-\mu}} \gamma_{t,t'} = \beta_{t'} \right\}_{t' \in D}.$$

Referring to the equalities in (1) and (2) and substituting for $\beta_{t'}$, we obtain the following equalities mod S_j :

$$\begin{aligned} \sum_{t \in D} \alpha_t^{p^{\tau-\mu}} t^{p^{\tau-\mu}} &= \sum_{t \in D} \beta_t t = \sum_{t' \in D} \beta_{t'} t' \\ &= \sum_{t' \in D} \left(\sum_{t \in D} \tilde{\alpha}_t^{p^{\tau-\mu}} \gamma_{t,t'} \right) t' = \sum_{t' \in D} \sum_{t \in D} (\tilde{\alpha}_t^{p^{\tau-\mu}} \gamma_{t,t'} t') \\ &= \sum_{t \in D} \tilde{\alpha}_t^{p^{\tau-\mu}} \left(\sum_{t' \in D} \gamma_{t,t'} t' \right) = \sum_{t \in D} \tilde{\alpha}_t^{p^{\tau-\mu}} t^{p^{\tau-\mu}} = \sum_{t \in D} (\tilde{\alpha}_t t)^{p^{\tau-\mu}}. \end{aligned}$$

Since $\tau - \mu > 0$ by assumption, $p^{\tau-\mu}$ -th roots are unique and $\sum_{t \in D} \alpha_t t = \sum_{t \in D} \tilde{\alpha}_t t \pmod{S_j}$. It follows that $x_{j+1}^{p^\mu} + \sum_{t \in D} \tilde{\alpha}_t t \in M$. Observe that the $(x_1, \dots, x_r)$ -degree of this polynomial is at most $N_6 := N_5 + (n-r)(b_1(n, d))^2$. Let d_0 in Corollary 4.19 be N_6 and consequently define

$$\tilde{d} := (b_1(n, d)^{n-r})(d_0)(1 + (1 + b_1(n, d))^{n-r}) = N_3 N_6 (1 + (1 + b_1(n, d))^{n-r}).$$

Since I is prime up to $b_6(n, d) := \max\{b_4(n, d), b_5(n, d), \tilde{d}(1 + \binom{\tilde{d}+n}{n})\}$, we have by Corollary 4.19 that $x_{j+1}^{p^\mu} + \sum_{t \in D} \tilde{\alpha}_t t$ is in J , in fact in $S_{j+1} = J \cap P_{r,j+1,n}$. Thus $x_{j+1}^{p^\mu} + S_{j+1} \in A_j$, contradicting the minimality of τ and proving that $S_{j+1} = M_{j+1}$. By induction, $J = S_n = M_n = M$ and J is maximal. $\square$

4C. Polynomial bounds on primality. Let $b(n, d)$ be the function $b_6(n, d)$ from the preceding theorem.

Theorem 4.26. *If I is prime up to $b(n, d)$, then I is prime.*

Proof. The cases $n = 1$ and $d = 1$ are covered by Theorem 4.2. If $n, d > 1$, then $J \cap k[x_1, \dots, x_n] = I$ by Theorem 4.10. The ideal J is maximal by Theorem 4.25 and restrictions of prime ideals to subrings are prime, so I is prime. $\square$

We conclude this section with concrete upper bounds on $b(n, d)$. For each lemma the claimed bound is greater than d , so the cases $n = 1$ and $d = 1$ are covered and we may assume $n, d > 1$ where necessary.

Lemma 4.27. $b_4(n, d) \leq (2nd)^{n2^{4n+8}}$ for all $n, d \geq 1$.

Proof. Using earlier bounds on $b_1(n, d)$ and $b_3(n, d)$ we obtain

$$\begin{aligned} b_4(n, d) &= b_1(n+1, (n-1) \cdot b_3(n, d) + 1) \leq 2((n-1) \cdot b_3(n, d) + 1)^{2^{n+1}} \\ &\leq 2((n-1) \cdot 2^{2^{2n+6}} d^{n2^{3n+2}} + 1)^{2^{n+1}} \leq 2(n2^{2^{2n+6}} d^{n2^{3n+2}})^{2^{n+1}} \\ &\leq (n2^{2^{2n+7}} d^{n2^{3n+2}})^{2^{n+1}} \leq (2nd)^{n2^{4n+8}}. \end{aligned} \quad \square$$

Similarly, we obtain bounds on the values defined in Notation 4.24:

Lemma 4.28. $N_1 \leq N_2 \leq (2nd)^{n2^{3n+2}}$.

Proof. Clearly $N_1 \leq N_2$. For the second inequality, compute

$$\begin{aligned}
 N_2 &= \left(2 \binom{(2(n-r)(b_1(n,d))^2)^{2^n} + d + n - r}{n-r} (n-r)(b_1(n,d))^2 \right)^{2^r} \\
 &\leq \left(2 \binom{(2n(b_1(n,d))^2)^{2^n} + d + n}{n} (n)(b_1(n,d))^2 \right)^{2^n} \\
 &\leq \left(2 \binom{(2n(2d^{2^n})^2)^{2^n} + d + n}{n} (n)(2d^{2^n})^2 \right)^{2^n} \\
 &\leq \left(2 \left((2n(2d^{2^n})^2)^{2^n} + d + n \right)^n (n)(2d^{2^n})^2 \right)^{2^n} \\
 &\leq \left(2 \left(2(2n(2d^{2^n})^2)^{2^n} \right)^n (n)(2d^{2^n})^2 \right)^{2^n} \\
 &\leq 2^{5n2^{2n+1}} n^{2n2^{2n}} d^{2n2^{3n+1}} \leq (2nd)^{n2^{3n+2}}. \quad \square
 \end{aligned}$$

Lemma 4.29. $N_3 \leq 2^n d^{n2^n}.$

Proof. We have

$$N_3 = b_1(n, d)^{n-r} \leq b_1(n, d)^n \leq (2d^{2^n})^n = 2^n d^{n2^n}. \quad \square$$

Lemma 4.30. $N_4 \leq (2nd)^{n2^{3n+3}}.$

Proof. We compute

$$\begin{aligned}
 N_4 &= N_3 N_2 + N_1 \leq N_2(N_3 + 1) \leq (2nd)^{n2^{3n+2}} (2^n d^{n2^n} + 1) \\
 &\leq (2nd)^{n2^{3n+2}} (2^{n+1} d^{n2^n}) \leq (2nd)^{n2^{3n+3}}. \quad \square
 \end{aligned}$$

Lemma 4.31. $N_5 \leq (2nd)^{n(n+1)2^{4n+4}}.$

Proof. We compute

$$\begin{aligned}
 N_5 &= (2N_3^2(N_3 + 1) \binom{N_4+r}{r} (b_1(n, d))^{(r-1)} N_4)^{2^r} \\
 &\leq \left(2^2 N_3^3 \binom{N_4+n}{n} (2d^{2^n})^{(n-1)} N_4 \right)^{2^n} \\
 &\leq \left(2^2 N_3^3 (N_4 + n)^n (2d^{2^n})^{(n-1)} N_4 \right)^{2^n} \\
 &\leq \left(2^2 N_3^3 (2N_4)^n (2d^{2^n})^{(n-1)} N_4 \right)^{2^n} \\
 &\leq \left(2^2 (2^n d^{n2^n})^3 (2((2nd)^{n2^{3n+3}}))^{(n-1)} (2d^{2^n})^{(n-1)} (2nd)^{n2^{3n+3}} \right)^{2^n} \\
 &\leq 2^{(1+2n)2^n} (2^n d^{n2^n})^{3 \cdot 2^n} ((2nd)^{n2^{3n+3}})^{(n+1)2^n} d^{(n-1)2^{2n}} \\
 &\leq 2^{(1+2n)2^n + 3n2^n + n(n+1)2^{4n+3}} n^{n(n+1)2^{4n+3}} d^{3n2^{2n} + n(n+1)2^{4n+3} + (n-1)2^{2n}} \\
 &\leq (2nd)^{(4n-1)2^{2n} + n(n+1)2^{4n+3}} \leq (2nd)^{n(n+1)2^{4n+4}}. \quad \square
 \end{aligned}$$

Lemma 4.32. $N_6 \leq 2(2nd)^{n(n+1)2^{4n+4}}.$

Proof. We have

$$\begin{aligned} N_6 &= N_5 + (n-r)(b_1(n, d))^2 \leq N_5 + n(2d^{2^n})^2 \\ &\leq (2nd)^{n(n+1)2^{4n+4}} + n2^2 d^{2^{n+1}} \leq 2(2nd)^{n(n+1)2^{4n+4}}. \end{aligned} \quad \square$$

Lemma 4.33. $\tilde{d} \leq (2nd)^{n(n+1)2^{4n+5}}.$

Proof. We compute

$$\begin{aligned} \tilde{d} &= N_3 N_6 (1 + (1 + b_1(n, d))^{n-r}) \leq N_3 N_6 (1 + (1 + 2d^{2^n})^n) \leq N_3 N_6 (2^{2n+1} d^{n2^n}) \\ &\leq (2^n d^{n2^n}) (2(2nd)^{n(n+1)2^{4n+4}}) (2^{2n+1} d^{n2^n}) \\ &\leq 2^{3n+2+n(n+1)2^{4n+4}} n^{n(n+1)2^{4n+4}} d^{2n2^n+n(n+1)2^{4n+4}} \\ &\leq (2nd)^{n(n+1)2^{4n+5}}. \end{aligned} \quad \square$$

Lemma 4.34. $\tilde{d}(1 + \binom{\tilde{d}+n}{n}) \leq (2nd)^{n(n+1)2^{4n+6}}.$

Proof. We have

$$\begin{aligned} \tilde{d}(1 + \binom{\tilde{d}+n}{n}) &\leq \tilde{d}(1 + (\tilde{d} + n)^n) \leq \tilde{d}(1 + (2\tilde{d})^n) \leq (2\tilde{d})^{n+1} \\ &\leq (2(2nd)^{n(n+1)2^{4n+5}})^{n+1} \leq (2nd)^{n(n+1)2^{4n+6}}. \end{aligned} \quad \square$$

Theorem 4.35. $b(n, d) \leq \max\{(2d)^{2^{3n^2+2n}}, (2nd)^{n(n+1)2^{4n+6}}\}$ for all $n, d \geq 1$.

Proof. We have

$$\begin{aligned} b(n, d) &= b_6(n, d) = \max\{b_4(n, d), b_5(n, d), \tilde{d}(1 + \binom{\tilde{d}+n}{n})\} \\ &\leq \max\{(2nd)^{n2^{4n+8}}, (2d)^{2^{3n^2+2n}}, (2nd)^{n(n+1)2^{4n+6}}\} \\ &\leq \max\{(2d)^{2^{3n^2+2n}}, (2nd)^{n(n+1)2^{4n+6}}\}. \end{aligned} \quad \square$$

If desired, one can continue to simplify and obtain larger, more-readable bounds:

Corollary 4.36. $b(n, d) \leq (2nd)^{n^3 2^{6n^2}}$ for all $n, d \geq 1$.

Proof. We have

$$\begin{aligned} b(n, d) &\leq \max\{(2d)^{2^{3n^2+2n}}, (2nd)^{n(n+1)2^{4n+6}}\} \\ &\leq (2nd)^{\max\{2^{3n^2+2n}, n(n+1)2^{4n+6}\}} \leq (2nd)^{n(n+1)2^{3n^2+2n}} \\ &\leq (2nd)^{n^3 2^{3n^2+2n+2}} \leq (2nd)^{n^3 2^{6n^2}}. \end{aligned} \quad \square$$

5. Maximal Ideals

We note that an analogous result, with a simpler proof, holds for maximal ideals: if an ideal is *maximal up to b* (see Definition 4.1) for large enough b , then the ideal is maximal. Similar remarks apply to the following proof and an ultraproduct argument of Schoutens [2010, 4.1.4] as held for Theorem 1.1 and Schmidt-Göttsch's argument mentioned in the introduction.

The following lemma is a standard step in the proof of the Noether normalization theorem (see, e.g., [Crespo and Hajto 2011, Proposition 1.1.8]):

Lemma 5.1. *Let $f \in k[x_1, \dots, x_n]$ be a nonzero polynomial of total degree d over a field k . There is some $c \in k \setminus \{0\}$ and $a_1, \dots, a_{n-1} \leq (d+1)^{n-1}$ such that the polynomial*

$$cf(y_1 + y_n^{a_1}, y_2 + y_n^{a_2}, \dots, y_{n-1} + y_n^{a_{n-1}}, y_n)$$

is monic in y_n .

To streamline the main proof we first take care of the simple cases and prove a technical inequality.

Lemma 5.2. *For any field k and any proper ideal $I \subseteq k[x_1, \dots, x_n]$ with generators of degree 1, if I is maximal up to 1, then I is maximal. If $n = 1$ and I is proper with generators of degree at most $d \geq 1$, then maximality up to d suffices.*

Proof. Suppose the generators of I have degree 1. By an invertible affine change of variables, we may assume that the generators have the form $x_1, \dots, x_m$ for some $m \leq n$. (Such a change of variables preserves our assumption that linear polynomials not in the ideal are invertible modulo the ideal.) If $m < n$, then x_{m+1} is invertible mod I and $x_{m+1}f - 1 \in I$ for some f . However, $I = (x_1, \dots, x_m)$ cannot have an element with nonzero constant term. Hence $m = n$ and I is maximal.

In the case $n = 1$, $k[x_1]$ is a PID and $I = (f)$ for some f of degree at most d . If $f = 0$, then x is not invertible mod I and I is not maximal up to d . Otherwise, reduction by f gives a nonzero remainder of degree less than d for every element of $k[x_1] \setminus I$. Hence maximality up to d implies that I is maximal. $\square$

Lemma 5.3. *Let k, n, d be natural numbers such that $3 < k+1 < n$ and $d \geq 1$. Define $D := (2d)^{n(n-1)\cdots(n-k)2^{(k-1)n}}$. Then*

$$(2D)^{(n-k-1)2^{n-k+2}} \leq (2d)^{n(n-1)\cdots(n-k)(n-k-1)2^{kn}}.$$

Proof. Applying the definition of D and simplifying, we convert the left side of the claimed inequality into

$$2^{(n-k-1)2^{n-k+2}} (2d)^{n\cdots(n-k-1)2^{kn-k+2}}.$$

Dividing by $(2d)^{n \cdots (n-k-1)2^{kn-k+2}}$ and simplifying, we see the claimed inequality is equivalent to

$$\begin{aligned} 2^{(n-k-1)2^{n-k+2}} &\leq ((2d)^{n(n-1) \cdots (n-k)(n-k-1)})^{(2^{kn}-2^{kn-k+2})} \\ &= ((2d)^{n(n-1) \cdots (n-k)(n-k-1)})^{(2^{kn-k+2})(2^{k-2}-1)}, \end{aligned}$$

which is clear since $k > 2$. $\square$

Theorem 5.4. Define $m(n, d) := (2d)^{(n-1)(n!)2^{(n-2)^n}}$ for all $n > 2, d > 1$. Otherwise let $m(n, 1) := 1$ (for any n) and $m(1, d) := d, m(2, d) := 4d^4$ if $d > 1$. For any field k and any proper ideal $I \subseteq k[x_1, \dots, x_n]$ with generators of total degree at most d , if I is maximal up to $m(n, d)$, then I is maximal.

Proof. By Lemma 5.2, bounds of $m(n, 1) = 1$ and $m(1, d) = d$ suffice if $d = 1$ or $n = 1$ and $d > 1$, respectively. Thus we may assume $n, d > 1$; by Theorem 3.18 $b_1(n, d) \leq 2d^{2^n}$.

The Noether normalization theorem applied to $R = k[x_1, \dots, x_n]/I$ states that there is an injective homomorphism $\pi : k[z_1, \dots, z_r] \rightarrow R$ such that R is a finite extension of a polynomial ring $k[z_1, \dots, z_r]$ for some $0 \leq r \leq n$. The usual proof proceeds by induction on the number of generators n of R as a k -algebra. We reproduce the argument but keep track of bounds along the way. Our first goal is to show that in our scenario, bounded maximality forces r to be 0.

Toward a contradiction, suppose $r > 0$. By maximality up to $m(n, d)$, the ideal I must be nonzero. Starting with a nonzero generator of degree at most d , Lemma 5.1 changes variables by substituting $y_i + y_n^{a_i}$ for x_i if $i < n$ and y_n for x_n using some integers $a_i \leq (d+1)^{n-1}$ such that the transformed generator is monic in y_n . This yields generators of degree at most $(d+1)^{n-1} \cdot d \leq (d+1)^n$ for the ideal $\tilde{I}$ in $k[y_1, \dots, y_n]$ generated by the generators of I (after substitution). The induction continues if there is a nontrivial relation between $y_1, \dots, y_{n-1}$; i.e., if the elimination ideal obtained by eliminating y_n from $\tilde{I}$ is nonzero. Otherwise $\pi(z_i) = (x_i - x_n^{a_i}) + I$ for $i < n$ gives the claimed map; note that $x_n + I$ is integral over $\pi(k[z_1, \dots, x_{n-1}])$. By hypothesis, the process stops after at most $n-1$ steps, leaving us with the desired map for some $r \geq 1$ and $k[z_1, \dots, z_r]$. Theorem 3.24 implies that the generators of the elimination ideal are bounded by $b_1(n, (d+1)^n)$ after one step of substitution and elimination, $b_1(n-1, (b_1(n, (d+1)^n) + 1)^{n-1})$ after two, and so on.

We analyze the alternating substitution and elimination steps to bound the degrees of (representatives of) $\pi(z_1), \dots, \pi(z_r)$ as polynomials in the variables $x_1, \dots, x_n$. Let $E_n = d$. Define D_{n-k} to be $(E_{n-k+1} + 1)^{n-k}$ and E_{n-k} to be $2(D_{n-k} \cdot E_{n-k+1})^{2^{n-k+1}}$ for $0 < k < n$. By induction, D_{n-k} is a bound on the degree of a substitution at the k -th step and E_{n-k} is a bound on the degree of the generators after k -many steps of substitution and elimination. Since we seek the degrees of

the images of $z_1, \dots, z_r$ in the original variables (and not just the variables in the preceding step), our goal is to bound the product $\prod_{i=1}^{n-1} D_{n-i}$. (We eliminate at most $n-1$ variables lest $r=0$.)

Note that $E_{n-k+1} \leq D_{n-k}$ and so

$$E_{n-k} \leq 2(D_{n-k}^2)^{2^{n-k+1}} = 2(D_{n-k})^{2^{n-k+2}}$$

for all $0 < k < n$. It follows that $E_{n-k+1} \leq 2(D_{n-k+1})^{2^{n-k+3}}$ and hence

$$D_{n-k} \leq (2(D_{n-k+1})^{2^{n-k+3}} + 1)^{n-k} \leq (2^2(D_{n-k+1})^{2^{n-k+3}})^{n-k} \leq (2D_{n-k+1})^{(n-k)2^{n-k+3}}$$

for $1 < k < n$.

It is immediate from the definitions that $D_{n-1} = (d+1)^{n-1} \leq (2d)^{n(n-1)}$, and a calculation like that of Lemma 5.3 shows that $D_{n-2} \leq (2d)^{n(n-1)(n-2)2^n}$. Using Lemma 5.3 and the inequality $D_{n-k} \leq (2D_{n-k+1})^{(n-k)2^{n-k+3}}$ for induction on k , we find that

$$D_{n-k} \leq (2d)^{n(n-1)\cdots(n-k)2^{(k-1)n}}$$

for $2 < k < n$; the previous sentence establishes the inequality for $k=1, 2$. It follows that $\prod_{i=1}^k D_{n-i} \leq (2d)^{kn(n-1)\cdots(n-k)2^{(k-1)n}}$ and so $\pi(z_1), \dots, \pi(z_r)$ can be represented by polynomials of degree at most $(2d)^{(n-1)(n!)2^{(n-2)n}}$ in $x_1, \dots, x_n$. One may easily check that $(2d)^{(n-1)(n!)2^{(n-2)n}} \leq m(n, d)$ for all $n, d > 1$.

Then since I is maximal up to $m(n, d)$, $\pi(z_r)$ is invertible in R ($z_r \neq 0$, so by injectivity of π we know $\pi(z_r) \notin I$). Since R is integral over $\pi(k[z_1, \dots, z_r])$, $(\pi(z_r))^{-1}$ is a root of a monic equation with coefficients in $\pi(k[z_1, \dots, z_r])$. Multiplying the monic equation by an appropriate power of $\pi(z_r)$ shows that $(\pi(z_r))^{-1} \in \pi(k[z_1, \dots, z_r])$, whence z_r is invertible in $k[z_1, \dots, z_r]$. But this is absurd, so r is equal to 0 and R is finite over k .

Since R is finite over k , for each $x_1, \dots, x_n$ the restricted ideal $I \cap k[x_i]$ is nonzero. By Theorem 3.24 there is a nonzero polynomial $\alpha_i(x_i) \in I \cap k[x_i]$ of degree at most $b_1(n, d)$ for each $x_1, \dots, x_n$. Using α_i to reduce higher powers of $x_i \bmod I$, we see that any $f \notin I$ is equivalent mod I to some $\tilde{f} \notin I$ of total degree at most $nb_1(n, d) \leq m(n, d)$ for $n, d > 1$. By maximality up to $m(n, d)$, f is invertible mod I and hence I is maximal. $\square$

A single formula for the bound easily follows:

Corollary 5.5. $m(n, d) \leq (2d)^{n(n!)2^{(n-1)n}}$ for all $n, d \geq 1$.

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WILLIAM SIMMONS
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
HOBART AND WILLIAM SMITH COLLEGES
GENEVA, NY
UNITED STATES
wsimmons@hws.edu

HENRY TOWSNER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PENNSYLVANIA
PHILADELPHIA, PA
UNITED STATES
htowsner@math.upenn.edu

A NEW LOCAL GRADIENT ESTIMATE FOR A NONLINEAR EQUATION UNDER INTEGRAL CURVATURE CONDITION ON MANIFOLDS

LIANG ZHAO AND SHOUWEN FANG

In this paper, we consider a nonlinear elliptic equation

$$\Delta u + au \log u + bu = 0$$

on complete Riemannian manifolds under integral curvature condition, where $a \geq 0$, b are two real constants. A new local gradient estimate for positive solutions to this equation under integral curvature condition is derived, and as an application, we give a corresponding Harnack inequality.

1. Introduction

In this paper, we will investigate the nonlinear equation

$$\Delta u + au \log u + bu = 0$$

on Riemannian manifold M^n , where $a \geq 0$, b are two real constants. Replacing u by $e^{-b/a}u$, the equation reduces to

$$(1-1) \quad \Delta u + au \log u = 0.$$

Equation (1-1) originated from Ricci solitons introduced by Hamilton. For Ricci flow, Ricci solitons are its special solutions and can be seen as singular points of the flow. As is well known, if there exist a smooth function f and some constant λ such that

$$(1-2) \quad \text{Ric}(g) + \text{Hess } f = \lambda g$$

on M , we call it a gradient Ricci soliton. A soliton is called shrinking, steady or expanding if the constant λ is, respectively, positive, zero or negative. Assume that (M, g) is a gradient Ricci soliton and set $u = e^f$; we can obtain from (1-2) that

$$\Delta u + 2\lambda u \log u = (C_0 - n\lambda)u$$

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for some constant C_0 . This equation is a special condition of (1-1). For this kind of equation, L. Ma [2006] derived a local gradient estimate for the solutions to a similar equation,

$$\Delta u - au \log u = 0.$$

For a parabolic version of (1-1), we can refer to [Cao et al. 2013; Chen and Chen 2009; Huang and Ma 2010; Huang et al. 2013; Ma 2006; Qian 2010; Wu 2010; Yang 2008]. The first author of this paper also studied the gradient estimates for some nonlinear equations on manifolds [Zhao 2013; 2014b; 2014a]. However, the above results are obtained under Ricci curvature bounded below. Very recently, many mathematicians have considered similar problems on manifolds under integral curvature. In [Dai et al. 2018], the authors obtained a local Sobolev constant estimate for integral Ricci curvature. The Li–Yau gradient bound is derived for collapsing manifolds under integral curvature condition in [Zhang and Zhu 2017].

For each $x \in M$, assume $\rho(x)$ denotes the smallest eigenvalue for the Ricci tensor $\text{Ric} : T_x M \rightarrow T_x M$, and $\text{Ric}_-^K(x) = ((n-1)K - \rho(x))_+ = \max\{0, (n-1)K - \rho(x)\}$, the amount of Ricci curvature lying below $(n-1)K$. Let

$$\|\text{Ric}_-^K\|_q = \sup_{x \in M} \left(\int_{B_o(R)} (\text{Ric}_-^K)^q d\text{vol} \right)^{1/q}.$$

We still need the scale invariant curvature quantity

$$k(x, q, R) = R^2 \left(\int_{B_o(R)} (\text{Ric}_-^K)^q d\text{vol} \right)^{1/q}, \quad k(q, R) = \sup_{x \in R} k(x, q, R),$$

where $\int_{B_o(R)}$ means the average integral. We denote $\|f\|_{q,\Omega}^*$ the normalized q -norm on the domain Ω . Namely,

$$\|f\|_{q,\Omega}^* = \left(\frac{1}{\text{vol}(\Omega)} \int_{\Omega} |f|^q \right)^{1/q}.$$

First, we establish a local gradient estimate for positive solutions to (1-1).

Theorem 1.1. *Let (M, g) be a complete Riemannian manifold. Suppose that u is a positive solution to (1-1) on the ball $B_o(R) \subset M$. For $q > n/2$, $K > 0$ and $R \leq 1$, if there exists a large enough constant b such that $k(q, 1) \leq 1/(b+1)^2$, then*

$$\frac{|\nabla u|}{u} \leq \frac{C}{R} \quad \text{on } B_o\left(\frac{R}{2}\right),$$

where C is a positive constant.

Remark 1.2. From Remark 2.2 in [Dai et al. 2018], we know that

$$k(q, R) \leq 2^{1/q} k(q, 1) \leq 2^{1/q} \frac{1}{(b+1)^2},$$

since $k(q, R) = R^2 \|\text{Ric}_-^K\|_q^*$, therefore, we get

$$(1-3) \quad \|\text{Ric}_-^K\|_q^* \leq \frac{2^{1/q}}{(b+1)^2 R^2}.$$

As an application of Theorem 1.1, we obtain the following corollary.

Corollary 1.3. *Under the same conditions as in Theorem 1.1, for any $x, y \in B_o(R/2)$, and any minimal geodesic $\gamma(s) : [0, 1] \rightarrow B_o(R/2)$ with $\gamma(0) = y$, we have $\gamma(1) = x$. The following Harnack inequality holds:*

$$u(x) \leq u(y)e^C.$$

2. Proof of Theorem 1.1

Assume that u is a positive solution to (1-1). In order to get our main theorem, we need the following computation.

Setting $v = \log u$, we obtain

$$\Delta(|\nabla v|^2) = 2(|\text{Hess } v|^2 + \text{Ric}(\nabla v, \nabla v)) + 2\langle \nabla v, \nabla \Delta v \rangle.$$

Moreover, in the terms of v , (1-1) has the form

$$\Delta u + au \log u = \text{div}(\nabla e^v) + a v e^v = e^v(|\nabla v|^2 + \Delta v + av) = 0.$$

That is to say,

$$(2-1) \quad \Delta v = -av - |\nabla v|^2.$$

Letting $w = |\nabla v|^2$, (2-1) has the equivalent form

$$(2-2) \quad \Delta v = -av - w.$$

Therefore,

$$\Delta(w) = 2[|\text{Hess } v|^2 + \text{Ric}(\nabla v, \nabla v)] - 2\langle \nabla v, \nabla w \rangle - 2aw.$$

Now we will begin to estimate $|\text{Hess } v|^2$. For a local orthonormal frame $\{e_i\}_{i=1}^n$ near any such a given point such that $\nabla v = |\nabla v|e_1$, we know $w = v_1^2$, $w_1 = 2v_{11}v_1$, and $j \geq 2$, $w_j = 2v_{j1}v_1$. Hence $2v_{j1} = w_j/w^{1/2}$.

From (2-2), we derive

$$\sum_{j=2}^n v_{jj} = -av - v_{11} - w = -av - v_{11} - w.$$

We know that

$$|\text{Hess } v|^2 = \sum_{i,j=1}^n v_{ij}^2 = v_{11}^2 + 2 \sum_{j=2}^n v_{1j}^2 + \sum_{i,j=2}^n v_{ij}^2 \geq v_{11}^2 + 2 \sum_{j=2}^n v_{1j}^2 + \frac{1}{n-1} \left(\sum_{j=2}^n v_{jj} \right)^2.$$

Therefore, we get

$$\begin{aligned} |\text{Hess } v|^2 &\geq v_{11}^2 + 2 \sum_{j=2}^n v_{1j}^2 + \frac{1}{n-1}(-av - v_{11} - w)^2 \\ &\geq \alpha \sum_{j=1}^n v_{j1}^2 + \frac{1}{n-1}(av + w)^2 + \frac{2v_{11}}{n-1}(av + w), \end{aligned}$$

where $\alpha = \min\{2, n/(n-1)\}$. Substituting the identities,

$$2wv_{11} = \langle \nabla v, \nabla w \rangle, \quad \sum_{j=1}^n v_{j1}^2 = \frac{1}{4} \frac{|\nabla w|^2}{w},$$

we can derive

$$|\text{Hess } v|^2 \geq \frac{\alpha}{4} \frac{|\nabla w|^2}{w} + \frac{w^2}{n-1}(avw^{-1} + 1)^2 + \frac{1}{n-1}(1 + avw^{-1})\langle \nabla v, \nabla w \rangle.$$

Therefore, we have

$$\begin{aligned} (2-3) \quad \Delta w &\geq \left[\frac{\alpha}{2} \frac{|\nabla w|^2}{w} + \frac{2}{n-1}(w+av)^2 + \frac{2}{n-1}(1+avw^{-1})\langle \nabla v, \nabla w \rangle \right] \\ &\quad + 2\text{Ric}(\nabla v, \nabla v) - 2\langle \nabla v, \nabla w \rangle - 2aw \\ &= \frac{\alpha}{2} w^{-1} |\nabla w|^2 + \frac{2}{n-1} w^2 (1+avw^{-1})^2 + \frac{2}{n-1} (1+avw^{-1})\langle \nabla v, \nabla w \rangle \\ &\quad + 2\text{Ric}(\nabla v, \nabla v) - 2\langle \nabla v, \nabla w \rangle - 2aw \\ &= \frac{\alpha}{2} w^{-1} |\nabla w|^2 + \frac{2}{n-1} w^2 (1+avw^{-1})^2 \\ &\quad + \left[\frac{2}{n-1} (1+avw^{-1}) - 2 \right] \langle \nabla v, \nabla w \rangle + 2\text{Ric}(\nabla v, \nabla v) - 2aw. \end{aligned}$$

We also need the following Sobolev inequality.

Lemma 2.1. *Given $q > n/2$ and $K > 0$, there exists $\epsilon = \epsilon(n, q, K)$ such that if M is a complete manifold with $k(q, 1) \leq \epsilon$, then for any $R \leq 1$, there is a constant C_s such that*

$$\left(\int_{B_o(R)} f^{2q/(q-1)} \right)^{(q-1)/q} \leq C_s V^{-1/q} \left(\int_{B_o(R)} f^2 + R^2 \int_{B_o(R)} |\nabla f|^2 \right),$$

where V means the volume of $B_o(R)$.

Proof. The following Sobolev inequality is obtained from [Dai et al. 2018, 4.6]:

$$\|f\|_{q/(q-1), B_o(R)}^* \leq C_s R \|\nabla f\|_{1, B_o(R)}^*,$$

where C_s denotes the Sobolev constant. In the remainder of this paper, this constant

may change from line to line. We can get

$$\begin{aligned} \left(\int_{B_o(R)} f^{2q/(q-1)} \right)^{(q-1)/q} &\leq C_s V^{-1/q} R \int_{B_o(R)} 2|f| |\nabla f| \\ &\leq C_s V^{-1/q} \left(\int_{B_o(R)} f^2 + R^2 \int_{B_o(R)} |\nabla f|^2 \right). \quad \square \end{aligned}$$

The inequality (2-3) holds wherever w is strictly positive. Assume

$$K = \{x \in \Omega : w(x) = 0\};$$

here $\Omega \subset M$ is an open set. Then for any nonnegative function ψ with compact support in $\Omega \setminus K$, we have

$$\begin{aligned} (2-4) \quad & - \int_{\Omega} \langle \tfrac{1}{2} \nabla w, \nabla \psi \rangle g \\ & \geq \int_{\Omega} \left\{ \frac{\alpha}{4} \frac{|\nabla w|^2}{w} + \frac{w^2}{n-1} (1+avw^{-1})^2 + \left[\frac{1}{n-1} (1+avw^{-1}) - 1 \right] \langle \nabla v, \nabla w \rangle g \right. \\ & \quad \left. + \text{Ric}(\nabla v, \nabla v) - aw \right\} \psi. \end{aligned}$$

In particular, assume $\psi = w^b \eta^2$, where $\eta \in C_0^\infty(\Omega)$ is nonnegative, and $b > 1$ is to be determined later. Then direct computation yields

$$\nabla \psi = bw^{b-1} \eta^2 \nabla w + 2w^b \eta \nabla \eta.$$

In the above inequality (2-4), we can estimate the Ricci term as follows:

$$\begin{aligned} & \int_{\Omega} \text{Ric}(\nabla v, \nabla v) w^b \eta^2 g \\ & \geq (n-1)K \int_{\Omega} w^{1+b} \eta^2 - \int_{\Omega} |\text{Ric}_-^K| w^{1+b} \eta^2 g \\ & \geq (n-1)K \int_{\Omega} w^{1+b} \eta^2 - \|\text{Ric}_-^K\|_q \left(\int_{\Omega} (w^{1+b} \eta^2)^{q/(q-1)} \right)^{(q-1)/q}. \end{aligned}$$

By Lemma 2.1, applying the Sobolev inequality to $w^{1+b} \eta^2$, we have

$$\left(\int_{\Omega} (w^{1+b} \eta^2)^{q/(q-1)} \right)^{(q-1)/q} \leq V^{-1/q} C_s (R^2 \int_{\Omega} |\nabla (w^{(1+b)/2} \eta)|^2 + \int_{\Omega} w^{1+b} \eta^2).$$

Since

$$\begin{aligned} \int_{\Omega} |\nabla (w^{(1+b)/2} \eta)|^2 &= \int_{\Omega} \left| \frac{b+1}{2} w^{(b-1)/2} \eta \nabla w + w^{(b+1)/2} \nabla \eta \right|^2 \\ &= \int_{\Omega} \left[\frac{(b+1)^2}{4} |\nabla w|^2 \eta^2 + w^{1+b} |\nabla \eta|^2 + (b+1) w^b \eta \langle \nabla w, \nabla \eta \rangle \right]. \end{aligned}$$

Letting $\delta = \|\text{Ric}_-^K\|_q$, by the inequality (1-3) and the assumption of Theorem 1.1, we get

$$\delta V^{-1/q} R^2 = \|\text{Ric}_-^K\|_q^* R^2 \leq \frac{c}{(b+1)^2}.$$

Therefore, we obtain

$$\begin{aligned} (2-5) \quad \int_{\Omega} \text{Ric}(\nabla v, \nabla v) w^b \eta^2 &\geq (n-1)K \int_{\Omega} w^{1+b} \eta^2 - C_S \delta V^{-1/q} \int_{\Omega} w^{1+b} \eta^2 \\ &\quad - \frac{C_S \delta V^{-1/q} R^2}{4} \int_{\Omega} [(b+1)^2 w^{1-b} |\nabla w|^2 \eta^2 \\ &\quad + 4w^{b+1} |\nabla \eta|^2 + 4(b+1)w^b \eta \langle \nabla w, \nabla \eta \rangle] \\ &\geq [(n-1)K - C_S \delta V^{-1/q}] \int_{\Omega} w^{1+b} \eta^2 \\ &\quad - \frac{c}{(b+1)^2} \int_{\Omega} [(b+1)^2 w^{1-b} |\nabla w|^2 \eta^2 \\ &\quad + 4w^{b+1} |\nabla \eta|^2 + 4(b+1)w^b \eta \langle \nabla w, \nabla \eta \rangle]. \end{aligned}$$

Inserting (2-5) into (2-4), we can derive

$$\begin{aligned} & - \int_{\Omega} \left\langle \frac{1}{2} \nabla w, b w^{b-1} \eta^2 \nabla w + 2w^b \eta \nabla \eta \right\rangle \\ & \geq \int_{\Omega} \left\{ \left(\frac{\alpha}{4} - c \right) w^{b-1} \eta^2 |\nabla w|^2 + \frac{1}{n-1} w^{2+b} \eta^2 (1 + a v w^{-1}) \right. \\ & \quad \left. + \left[\frac{1}{n-1} (1 + a v w^{-1}) - 1 \right] w^b \eta^2 \langle \nabla v, \nabla w \rangle \right\} \\ & \quad + [(n-1)K - V^{-1/q} C_S \delta - a] \int_{\Omega} w^{1+b} \eta^2 \\ & \quad - c \int_{\Omega} \left[\frac{4}{(b+1)^2} w^{b+1} |\nabla \eta|^2 + \frac{4}{b+1} w^b \eta \langle \nabla w, \nabla \eta \rangle \right]. \end{aligned}$$

Moreover,

$$- \int_{\Omega} w^b \eta \langle \nabla w, \nabla \eta \rangle \leq \int_{\Omega} w^b |\nabla w| |\nabla \eta| \eta,$$

and $1 + a v w^{-1} \geq 1$. Therefore, we obtain

$$\begin{aligned} & \left(1 + \frac{4c}{b+1} \right) \int_{\Omega} w^b |\nabla w| |\nabla \eta| \eta + \int_{\Omega} w^b \eta^2 \langle \nabla v, \nabla w \rangle \\ & \geq \left(\frac{\alpha}{4} + \frac{b}{2} - c \right) \int_{\Omega} w^{b-1} \eta^2 |\nabla w|^2 + [(n-1)K - V^{-1/q} C_S \delta - a] \int_{\Omega} w^{1+b} \eta^2 \\ & \quad + \frac{1}{n-1} \int_{\Omega} w^b \eta^2 \langle \nabla v, \nabla w \rangle + \frac{1}{n-1} \int_{\Omega} w^{2+b} \eta^2 - \frac{4c}{(b+1)^2} \int_{\Omega} w^{b+1} |\nabla \eta|^2. \end{aligned}$$

Since b is a large enough constant, we can make the coefficient $1 + 4c/(b+1)$ less than or equal to 2, and get

$$\begin{aligned} & 2 \int_{\Omega} w^b |\nabla w| |\nabla \eta| \eta + \int_{\Omega} w^b \eta^2 \langle \nabla v, \nabla w \rangle \\ & \geq \left(\frac{\alpha}{4} + \frac{b}{2} - c \right) \int_{\Omega} w^{b-1} \eta^2 |\nabla w|^2 + [(n-1)K - V^{-1/q} C_s \delta - a] \int_{\Omega} w^{1+b} \eta^2 \\ & \quad + \frac{1}{n-1} \int_{\Omega} w^b \eta^2 \langle \nabla v, \nabla w \rangle + \frac{1}{n-1} \int_{\Omega} w^{2+b} \eta^2 - \frac{4c}{(b+1)^2} \int_{\Omega} w^{b+1} |\nabla \eta|^2. \end{aligned}$$

Noting that

$$2 \int_{\Omega} w^b |\nabla w| |\nabla \eta| \eta \leq \frac{b}{12} \int_{\Omega} w^{b-1} |\nabla w|^2 \eta^2 + \int_{\Omega} \frac{a_1}{b} w^{1+b} |\nabla \eta|^2,$$

we also get

$$\int_{\Omega} w^b \eta^2 \langle \nabla w, \nabla v \rangle \leq \int_{\Omega} w^{b+1/2} \eta^2 |\nabla w| \leq \frac{b}{12} \int_{\Omega} w^{b-1} |\nabla w|^2 \eta^2 + \frac{a_2}{b} \int_{\Omega} w^{2+b} \eta^2,$$

and the following inequality holds:

$$\begin{aligned} \frac{1}{n-1} \int_{\Omega} w^b \eta^2 \langle \nabla w, \nabla v \rangle & \geq - \int_{\Omega} \frac{1}{n-1} w^{b+1/2} \eta^2 |\nabla w| \\ & \geq - \frac{b}{12} \int_{\Omega} w^{b-1} |\nabla w|^2 \eta^2 - \frac{a_3}{b} \int_{\Omega} w^{2+b} \eta^2. \end{aligned}$$

Combining these inequalities, we derive

$$\begin{aligned} (2-6) \quad \int_{\Omega} \left[\frac{\alpha}{4} + \frac{b}{4} - c \right] w^{b-1} \eta^2 |\nabla w|^2 & \leq \left(\frac{4c}{(b+1)^2} + \frac{a_1}{b} \right) \int_{\Omega} w^{b+1} |\nabla \eta|^2 \\ & \quad + \left(\frac{a_2}{b} + \frac{a_3}{b} - \frac{1}{n-1} \right) \int_{\Omega} w^{2+b} \eta^2 \\ & \quad + \left[-(n-1)K + V^{\frac{-1}{q}} C_s \delta + a \right] \int_{\Omega} w^{1+b} \eta^2. \end{aligned}$$

We can choose b such that

$$\frac{\alpha}{4} + \frac{b}{4} - c \geq \frac{b}{8}.$$

For the first term on the left-hand side of (2-6), we use

$$|\nabla(w^{(1+b)/2} \eta)|^2 \leq \frac{(b+1)^2}{2} w^{b-1} |\nabla w|^2 \eta^2 + 2w^{b+1} |\nabla \eta|^2.$$

Substituting it into the above inequality, we get

$$\begin{aligned} & \frac{1}{(b+1)^2} \frac{b}{4} \left[\int_{\Omega} |\nabla(w^{(1+b)/2}\eta)|^2 - 2 \int_{\Omega} w^{b+1} |\nabla\eta|^2 \right] \\ & \leq \left(\frac{4c}{(b+1)^2} + \frac{a_1}{b} \right) \int_{\Omega} w^{b+1} |\nabla\eta|^2 + \left(\frac{a_2}{b} + \frac{a_3}{b} - \frac{1}{n-1} \right) \int_{\Omega} w^{2+b} \eta^2 \\ & \quad + [-(n-1)K + V^{-1/q} C_s \delta + a] \int_{\Omega} w^{1+b} \eta^2. \end{aligned}$$

Therefore, we get the following inequality of a simple form:

$$\int_{\Omega} |\nabla(w^{(1+b)/2}\eta)|^2 + bd_1 \int_{\Omega} w^{2+b} \eta^2 \leq a_4 \int_{\Omega} w^{1+b} |\nabla\eta|^2 + bd_2 \int_{\Omega} w^{1+b} \eta^2,$$

where d_1, d_2, a_4 are some constants.

In the following process of proving Theorem 1.1, we will need the following lemma.

Lemma 2.2. *For $b_0 > 0$ large enough and $R > 0$, there exists $d_3 = d_3(m) > 0$ such that*

$$\|w\|_{L^{(b_0+1)m/(m-1)}(B_o(3/4R))} \leq d_3 b_0^{(b_0+3)/(b_0+1)} V^{(m-1)/(m(b_0+1))} R^{-2}.$$

Proof. From Lemma 2.1, we have

$$\begin{aligned} & \left(\int_{\Omega} w^{(1+b)m/(m-1)} \eta^{2m/(m-1)} \right)^{(m-1)/m} \\ & \leq C_s V^{-1/m} \left(R^2 \int_{\Omega} |\nabla(w^{(b+1)/2}\eta)|^2 + \int_{\Omega} w^{b+1} \eta^2 \right) \\ & \leq b_0 V^{-1/m} \left(a_4 R^2 \int_{\Omega} w^{1+b} |\nabla\eta|^2 \right. \\ & \quad \left. + bd_2 R^2 \int_{\Omega} w^{1+b} \eta^2 - bd_1 R^2 \int_{\Omega} w^{2+b} \eta^2 + \int_{\Omega} w^{b+1} \eta^2 \right), \end{aligned}$$

where $b_0 = C_s$, then we have

$$\begin{aligned} (2-7) \quad & \left(\int_{\Omega} w^{(1+b)m/(m-1)} \eta^{2m/(m-1)} \right)^{(m-1)/m} + bd_1 b_0 R^2 V^{-1/m} \int_{\Omega} w^{2+b} \eta^2 \\ & \leq a_4 R^2 b_0 V^{-1/m} \int_{\Omega} w^{1+b} |\nabla\eta|^2 \\ & \quad + bd_2 b_0 R^2 V^{-1/m} \int_{\Omega} w^{1+b} \eta^2 + b_0 V^{-1/m} \int_{\Omega} w^{b+1} \eta^2 \\ & \leq a_4 R^2 b_0 V^{-1/m} \int_{\Omega} w^{1+b} |\nabla\eta|^2 + a_5 b_0^2 b V^{-1/m} \int_{\Omega} w^{b+1} \eta^2. \end{aligned}$$

We note that $a_5 b_0 b w^{1+b} < \frac{1}{2} b d_1 R^2 w^{2+b}$, when $w > a_6 b_0 R^{-2}$. For the second term on the right-hand side of (2-7), we decompose Ω into two parts, one over the region $w > a_6 b_0 R^{-2}$ and the second part is the complement of the first region. With the decomposition we get

$$\begin{aligned} a_5 b_0^2 b V^{-1/m} \int_{\Omega} w^{b+1} \eta^2 &\leq \frac{1}{2} b d_1 b_0 R^2 V^{-1/m} \int_{\Omega} w^{2+b} \eta^2 + a_5 b_0^2 b V^{1-1/m} \left(\frac{a_6 b_0}{R^2} \right)^{b+1} \\ &\leq \frac{1}{2} b d_1 b_0 R^2 V^{-1/m} \int_{\Omega} w^{2+b} \eta^2 + a_7^{b_0} b_0^2 V^{1-1/m} \frac{b_0^{b+1}}{R^{2b+2}}. \end{aligned}$$

Let $\eta_1 \in C_0^\infty(\Omega)$ satisfy $0 \leq \eta_1 \leq 1$, $\eta_1 \equiv 1$ in $B_o(\frac{3}{4}R)$, $|\nabla \eta_1| \leq \tilde{C}/R$, and let $\eta = \eta_1^{p+b}$. The first term on the right-hand side of (2-7) satisfies

$$\begin{aligned} a_4 R^2 b_0 \int_{B_o(R)} w^{1+b} |\nabla \eta|^2 &\leq a_8 b^2 b_0 \int_{B_o(R)} w^{1+b} \eta^{2(b+1)/(b+2)} \\ &\leq a_8 b^2 b_0 \left(\int_{B_o(R)} w^{p+b} \eta^2 \right)^{2(b+1)/(b+2)} V^{1/(b+2)} \\ &\leq \frac{1}{2} b d_1 b_0 R^2 \int_{B_o(R)} w^{2+b} \eta^2 + b_0 b^{b+2} d_1^{1+b} R^{-2(1+b)} V. \end{aligned}$$

Hence

$$\left(\int_{B_o(R)} w^{(1+b)m/(m-1)} \eta^{2m/(m-1)} \right)^{(m-1)/m} \leq d_2 b^{b+3} d_1^{b+1} R^{-2(1+b)} V^{1-1/m}.$$

Assume that $b = b_0$ and $b_1 = (1 + b_0)m/(m-1)$; we get

$$\|w\|_{L^{(b_0+1)m/(m-1)}(B_o(3/(4)R))} \leq d_3 b_0^{(b_0+3)/(b_0+1)} V^{(m-1)/(m(b_0+1))} R^{-2}. \quad \square$$

Now we begin to prove the main theorem. From (2-7), by ignoring the second item on the left-hand side, we derive

$$\left(\int_{B_o(R)} w^{(1+b)m/(m-1)} \eta^{2m/(m-1)} \right)^{(m-1)/m} \leq a_9 \frac{b_0}{V^{1/m}} \int_{B_o(R)} (b_0 b \eta^2 + R^2 |\nabla \eta|^2) w^{1+b}.$$

Set

$$b_{l+1} = b_l \frac{m}{m-1}, \quad \Omega_l = B\left(o, \frac{R}{2} + \frac{R}{4l}\right), \quad l = 1, 2, \dots$$

and choose $\eta_l \equiv 1$ in Ω_{l+1} , $\eta_l \equiv 0$ in $B_o(R) \setminus \Omega_l$, $|\nabla \eta_l| \leq C 4^l/R$. By letting $b+1 = b_l$ and $\eta = \eta_l$, we have

$$\left(\int_{B_o(R)} w^{b_{l+1}} \eta^{2m/(m-1)} \right)^{(m-1)/m} \leq a_9 \frac{b_0}{V^{1/m}} \int_{B_o(R)} w^{b_l} (\eta_l^2 + c^2 4^{2l}),$$

which implies

$$\|w\|_{L^{b_{l+1}}(\Omega_{l+1})} \leq \left[a_{10}(1+16^l) \frac{b_0}{V^{1/m}} \right]^{1/b_l} \|w\|_{L^{b_l}(\Omega_l)}.$$

Hence,

$$\|w\|_{L^{b_{l+1}}(\Omega_{l+1})} \leq \left[a_{10} \frac{b_0}{V^{1/m}} \right]^{\sum_{i=1}^l 1/b_i} 17^{\sum_{i=1}^l i/b_i} \|w\|_{L^{b_1}(\Omega_1)}.$$

Note that $\sum_{l=1}^{\infty} 1/b_l = m/b_1$ and $\sum_{i=1}^{\infty} \frac{i}{b_i} = \frac{m^2}{b_1}$. We obtain

$$\|w\|_{L^\infty(B_o(R/2))} \leq \left(a_{10} \frac{b_0}{V^{1/m}} \right)^{m/b_1} 17^{m^2/b_1} \|w\|_{L^{(b_0+1)m/(m-1)}(B_o(3R/4))}.$$

By Lemma 2.2, we know

$$\|w\|_{L^\infty(B_o(R/2))} \leq a_{11} b_0^{(b_0+3)/(b_0+1)} \frac{1}{R^2}.$$

We can assume b_0 large enough to make $(b_0+3)/(b_0+1) \leq \frac{3}{2}$, which implies

$$\|w\|_{L^\infty(B_o(R/2))} \leq a_{11} C_s^{3/2} \frac{1}{R^2}.$$

We have completed the proof of Theorem 1.1.

Proof of Corollary 1.3. Let minimal geodesic $\gamma(s) : [0, 1] \rightarrow M^n$, so that $\gamma(0) = y$ and $\gamma(1) = x$, then

$$\begin{aligned} \ln \frac{u(x)}{u(y)} &= \int_0^1 \frac{d \ln(u(\gamma(s)))}{ds} = \int_0^1 \frac{\nabla u \cdot \gamma'}{u(\gamma(s))} ds \\ &\leq \int_0^1 \frac{|\nabla u| |\gamma'|}{|u(\gamma(s))|} ds = \rho(x, y) \int_0^1 \frac{|\nabla u|}{|u(\gamma(s))|} ds \\ &\leq \rho(x, y) \int_0^1 \frac{C_s^{3/4}}{R} ds \leq \int_0^1 C, \end{aligned}$$

where $\rho = \rho(x, y)$ denotes the geodesic distance between x and y . □

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LIANG ZHAO
 DEPARTMENT OF MATHEMATICS
 NANJING UNIVERSITY OF AERONAUTICS AND ASTRONAUTICS
 NANJING
 CHINA
 zhaozongliang09@163.com

SHOUWEN FANG
 COLLEGE OF MATHEMATICAL SCIENCE
 YANGZHOU UNIVERSITY
 YANGZHOU
 CHINA
 shwfang@163.com

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