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**KUPERBERG AND TURAEV-VIRO INVARIANTS
IN UNIMODULAR CATEGORIES**

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We give a categorical setting in which Penrose graphical calculus naturally extends to graphs drawn on the boundary of a handlebody. We use it to introduce invariants of 3-manifolds presented by Heegaard splittings. We recover Kuperberg invariants when the category comes from an involutory Hopf algebra and Turaev–Viro invariants when the category is semisimple and spherical.

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1. Introduction

A remarkable achievement of low-dimensional topology in the last 30 years was a discovery of deep relations between topology and the theory of monoidal (tensor) categories. This development was initiated by V. Jones' introduction of his famous knot polynomial; by now it encompasses many aspects of low-dimensional topology including 3-manifold invariants, representations of mapping class groups of surfaces, topological quantum field theories in dimensions 2 and 3, etc. In particular, it was shown that monoidal categories satisfying certain conditions and carrying appropriate additional structures give rise to topological invariants of 3-dimensional manifolds, see [Turaev 1994; Turaev and Virelizier 2017]. This has instigated extensive research in the theory of monoidal categories aiming at construction (and eventually classification) of monoidal categories with required properties. At the

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same time, this development has provoked an appearance of parallel approaches not involving monoidal categories but using related algebraic objects. One such approach is due to G. Kuperberg [1991] who derived invariants of 3-manifolds from involutory Hopf algebras. The initial aim of this paper was to recover Kuperberg's invariants in terms of monoidal categories. To this end we introduce here a new construction of 3-manifold invariants from monoidal categories. We show that our method produces both the Kuperberg invariants and the standard Turaev–Viro invariants. Other generalizations of Kuperberg invariants of 3-manifolds were considered by Kashaev and Virelizier [2019].

The first main result of our paper is that in a general categorical setting there exists an invariant of graphs on the boundary of a handlebody.

With some additional categorical structure we show that this invariant extends to an invariant of bichrome handlebody graphs which are graphs on the boundary of a handlebody which are the union of two subgraphs: blue and red. Here the red curves have the nice property: any edge of the graph can be slid over a red curve. This is used when we define a 3-manifold invariant. In particular, we use Heegaard decompositions of 3-manifolds which are unions of two handlebodies glued along their common boundary. Such a decomposition is encoded by a set of disjoint simple closed curves on the boundary of a handlebody. These curves form a complete set of meridians for the second handlebody and are drawn in red.

On the algebraic side, we use two main tools — the so-called modified traces (m-traces) and the chromatic morphisms. The m-traces generalize the usual trace of endomorphisms of objects of a monoidal category to situations where the standard trace is not defined. The m-traces first appeared in [Geer et al. 2011a; 2013a; 2013b] and have been successfully used to produce 3-manifold invariants, see for example [Beliakova et al. 2018b; Costantino et al. 2014; Geer et al. 2011b]. The chromatic morphisms are introduced here.

2. Statements of main results and open problems

2A. The invariant F' . We first introduce the notation used in the main statements, for more details see Section 3. Let \mathcal{C} be a pivotal \mathbb{k} -category, where \mathbb{k} is a field. Let F be the Penrose functor (defined using the Penrose graphical calculus) from the category of planar \mathcal{C} -colored ribbon graphs to \mathcal{C} , see for example [Geer et al. 2013b]. Finally, let t be a modified trace (or m-trace for short) on an ideal \mathcal{I} in \mathcal{C} . Then t induces an invariant of \mathcal{I} -colored spherical graphs denoted by F' , see [Geer et al. 2013b]. This invariant can be computed by composing t and F on a cutting presentation of the graph (see Equation (8)).

We assume the m -trace is *nondegenerate*, i.e., for any object $P \in \mathcal{I}$, the pairing:

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P) \times \mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1}) \rightarrow \mathbb{k} \quad \text{given by} \quad (x, y) \mapsto t_P(xy)$$

is nondegenerate. Given a basis $\{x_i\}$ of $\text{Hom}_{\mathcal{C}}(\mathbb{1}, P)$ let $\{y_i\}$ be the dual basis of $\text{Hom}_{\mathcal{C}}(P, \mathbb{1})$ for this pairing. Let

$$(1) \quad \Omega_P \in \text{Hom}_{\mathcal{C}}(\mathbb{1}, P) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(P, \mathbb{1}) \quad \text{be given by} \quad \Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i,$$

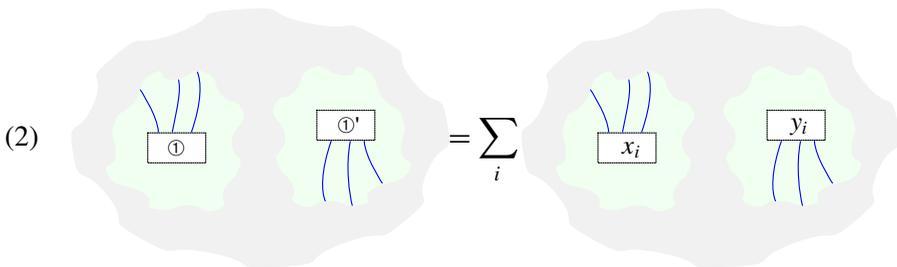
where $\otimes_{\mathbb{k}}$ is the standard tensor product of vector spaces over \mathbb{k} . The following proposition is the first item of [Proposition 4.2](#).

Proposition 2.1. *The element Ω_P is independent of the choice of the basis of $\text{Hom}_{\mathcal{C}}(\mathbb{1}, P)$.*

By a *multihandlebody* we will mean a disjoint union of a finite number of oriented handlebodies. A \mathcal{C} -*colored ribbon graph* on a multihandlebody is a graph on the boundary of the multihandlebody such that each edge is colored with an object of \mathcal{C} and each vertex is thickened to a coupon colored with a morphism of \mathcal{C} . All coupons have a top and a bottom sides which in our pictures will be the horizontal sides of the coupons. Since our graphs are drawn on a surface they have a natural framing and therefore can be considered as ribbon graphs in the usual sense. When all the colors of such a \mathcal{C} -colored ribbon graph are in the ideal \mathcal{I} we say that the graph is \mathcal{I} -*colored*. Let

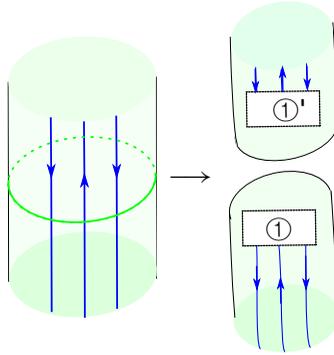
$$\mathcal{H}_{\mathcal{I}} = \{(H, \Gamma) : \Gamma \text{ is a nonempty } \mathcal{I}\text{-colored graph on a multihandlebody } H\}.$$

In what follows we extend the colorings of coupons multilinearly. In particular, we can color an ordered matching pair of coupons with $\Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i$ and we represent such a coupon with the adjacent edges by the figure



To state our first theorem we define a cutting operation on colored graphs. Let $(H, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ and D be a disk in H bounded by a simple oriented curve $\partial D \subset \partial H$ which intersects the edges of Γ nontrivially and transversely without meeting the coupons of Γ . Cutting (H, Γ) along D we obtain a new multihandlebody graph $(\text{cut}_D(H), \text{cut}_D(\Gamma)) \in \mathcal{H}_{\mathcal{I}}$, where $\text{cut}_D(\Gamma)$ is obtained by cutting the edges of Γ intersecting ∂D and then joining the cut points into two new coupons in $\partial(\text{cut}_D(H))$ (one on each side of the cut). The coupons are colored as in Equation (2), see the

following figure:



Note that H and $\text{cut}_D(H)$ can have different numbers of connected components and the orientation of $\text{cut}_D(H)$ is induced by the one of H .

Remark, if $(B^3, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ then Γ is a \mathcal{I} -colored spherical graph in the domain of the invariant F' defined in Equation (8).

The following theorem (proved in Section 4) extends F' to the full set $\mathcal{H}_{\mathcal{I}}$.

Theorem 2.2. *Let \mathcal{C} be a pivotal \mathbb{k} -category equipped with an ideal \mathcal{I} in \mathcal{C} and a nondegenerate m -trace on \mathcal{I} . Then there exists a unique mapping*

$$F' : \mathcal{H}_{\mathcal{I}} \rightarrow \mathbb{k}$$

satisfying the following four properties.

(1) Invariance. *The element $F'(H, \Gamma)$ of \mathbb{k} depends only on the orientation preserving diffeomorphism class of the pair $(H, \Gamma) \in \mathcal{H}_{\mathcal{I}}$.*

(2) Extension of F' . *For any \mathcal{I} -colored ribbon graph (B^3, Γ) in the 3-ball B^3 , we have*

$$F'(B^3, \Gamma) = F'(\Gamma).$$

(3) Disjoint union of multihandlebodies. *For any $(H_1, \Gamma_1), (H_2, \Gamma_2) \in \mathcal{H}_{\mathcal{I}}$ we have*

$$F'(H_1 \sqcup H_2, \Gamma_1 \sqcup \Gamma_2) = F'(H_1, \Gamma_1)F'(H_2, \Gamma_2).$$

(4) Cutting along a disk. *Cutting any $(H, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ along a disk D as described before the statement of the theorem, we always have $F'(\text{cut}_D(H), \text{cut}_D(\Gamma)) = F'(H, \Gamma)$.*

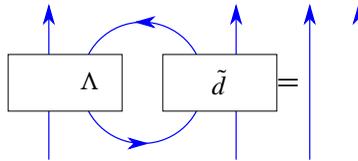
The hypotheses of the above theorem are quite mild. Indeed, by Theorem 5.5 of [Geer et al. 2018] they are satisfied in all the following examples (see [Geer et al. 2013a]): representations of factorizable ribbon Hopf algebras, finite groups and their quantum doubles, Lie (super)algebras, the $(1, p)$ minimal model in conformal field theory, and quantum groups at a root of unity.

Following the ideas of [De Renzi et al. 2018], we extend the invariant of multi-handlebody graphs F' to so-called bichrome multihandlebody graphs as follows. A *bichrome handlebody graph* is a graph on the boundary of a multihandlebody which is split as a disjoint union of two subgraphs: blue and red. The blue subgraph is a \mathcal{C} -colored graph on the multihandlebody. The red subgraph is a collection of disjoint simple closed unoriented curves (which are not required to be \mathcal{C} -colored). We refer to these curves as *red circles*. We say a bichrome handlebody graph is *admissible* if its blue subgraph is \mathcal{I} -colored and meets each connected component of the multihandlebody.

Definition 2.3. Let G be an object of \mathcal{I} . Set $\Lambda = \sum_i x_i y_i \in \text{End}_{\mathcal{C}}(G \otimes G^*)$, where $\Omega_{G \otimes G^*} = \sum x_i \otimes_{\mathbb{k}} y_i$. A *chromatic morphism* for G is a morphism $\tilde{d} : G \otimes G \rightarrow G \otimes G$ such that

$$(3) \quad (\text{Id}_G \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_G) \circ (\Lambda \otimes \tilde{d}) \circ (\text{Id}_G \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_G) = \text{Id}_{G \otimes G} .$$

This equation is represented pictorially as



where all blue strands are colored by G .

We use the word chromatic here because the morphism \tilde{d} is used to change a red circle into a blue graph, as explained in Theorem 2.4. In the case of Hopf algebras this corresponds to the evaluation of the integral on the red circles (see Section 6).

A *generator* of an ideal \mathcal{I} is an object $G \in \mathcal{I}$ such that for any $P \in \mathcal{I}$ there exists morphisms $f_j : G \rightarrow P$ and $g_j : P \rightarrow G$ indexed by a finite set J such that

$$(4) \quad \text{Id}_P = \sum_{j \in J} f_j g_j .$$

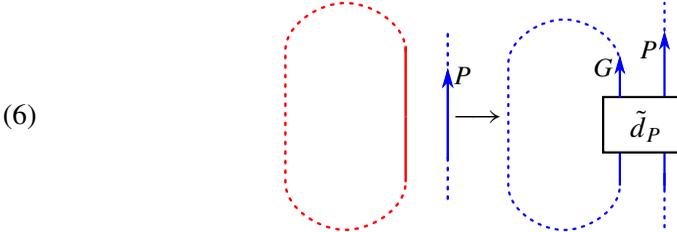
Remark there are many different notions of a generator of an ideal. Here we use the word generator because in the case of Hopf algebras G will be the projective generator.

Let \tilde{d} be a chromatic morphism for a generator G of an ideal \mathcal{I} . If $P \in \mathcal{I}$ and $\text{Id}_P = \sum_j f_j g_j$ as in Equation (4) we define

$$(5) \quad \tilde{d}_P = \sum_{j \in J} (\text{Id}_G \otimes f_j) \tilde{d} (\text{Id}_G \otimes g_j) : G \otimes P \rightarrow G \otimes P .$$

We prove the following theorem in Section 5C.

Theorem 2.4. *Let \mathcal{C} be a pivotal \mathbb{k} -category equipped with a nondegenerate m-trace on an ideal \mathcal{I} and a chromatic morphism \tilde{d} on a generator G . Then there exist a unique extension of F' to admissible bichrome handlebody graphs which is preserved under the following transformation making a red circle blue in the presence of a nearby blue edge colored with an object P as shown in the figure*



Moreover, if (H, Γ) is a bichrome handlebody graph then $F'(H, \Gamma)$ only depends on the orientation preserving diffeomorphism class of (H, Γ) .

Later we will see that F' is also invariant under sliding an edge of Γ over a red circle, see Proposition 5.4.

2B. The invariant \mathcal{K} . Let \mathcal{C} be a pivotal \mathbb{k} -category equipped with a nondegenerate m-trace t on an ideal \mathcal{I} and a chromatic morphism on a generator G . Since t is nondegenerate, there exists a morphism $h : G \rightarrow G$ such that $t_G(h) \neq 0$. By renormalizing the m-trace we can assume $t_G(h) = 1$. Let \mathbf{O}_G be the ribbon graph in \mathbb{R}^2 formed by the braid closure of the coupon filled with h . Consider the bichrome handlebody graph (B^3, \mathbf{O}_G) where \mathbf{O}_G is blue and viewed as a graph on the boundary of B^3 . By definition, one has $F'(B^3, \mathbf{O}_G) = F'(\mathbf{O}_G) = t_G(h) = 1$.

Let M be a closed connected orientable 3-manifold. Next we use some standard topological definitions, see Section 5 for more details. A Heegaard diagram of M is a prescription for a Heegaard splitting $M = H_\alpha \cup_\Sigma H_\beta$ determined by upper and lower reducing sets of bounding circles $\{\alpha_i\}$ and $\{\beta_i\}$ in $\Sigma = \partial H_\alpha = \partial H_\beta$. A Heegaard diagram determines an admissible bichrome handlebody graph on H_α where the red subgraph is the set of circles $\{\beta_i\}$ (on the boundary of H_α) and the blue subgraph is \mathbf{O}_G embedded in a small disk in ∂H_α . We call such a handlebody graph a bichrome diagram for M . See Figure 1 for an example.

Next we state the main theorem of this paper which will be proved in Section 5C.

Theorem 2.5. *If (H, Γ) is a bichrome diagram for a closed connected orientable 3-manifold M , then $F'(H, \Gamma)$ only depends of the diffeomorphism class of M .*

We denote the invariant of Theorem 2.5 by $\mathcal{K}_{\mathcal{C}}(M)$. Let us now discuss some examples.

2C. Hopf algebras and Kuperberg’s invariant. We use standard terminology of Hopf algebras, for details see Section 6. Let A be a finite dimensional unibalanced

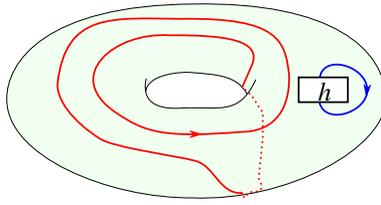


Figure 1. Bichrome diagram for the lens space $L(2, 1)$.

unimodular pivotal Hopf algebra. Let $A\text{-mod}$ be the category of finite dimensional modules over A . Let Proj be the ideal of projective objects in $A\text{-mod}$. The Hopf algebra A itself with its left regular representation is a generator of Proj .

In Section 6 we prove the following theorem:

Theorem 2.6. *There exists a nondegenerate m -trace on Proj in $A\text{-mod}$ and a chromatic morphism \tilde{d} on the generator A .*

Let us discuss the hypothesis on A briefly. The existence of the chromatic morphism comes from the theory of (co)integrals in Hopf algebras. The requirement that A is pivotal implies that the category $A\text{-mod}$ is pivotal. The unimodularity of A ensures that the ideal Proj has a nondegenerate right m -trace. That A is unibalanced implies that this m -trace is also a left m -trace, see [Beliakova et al. 2018a].

Theorems 2.5 and 2.6 yield an invariant $\mathcal{K}_{A\text{-mod}}$. From Lemmas 6.2 and 6.3, the chromatic morphism is essentially determined by the integral $\lambda : A \rightarrow \mathbb{k}$ and cutting along a bounding circle is determined by the cointegral $\Lambda \in A$, where $\lambda(\Lambda) = 1$.

In Section 6C we will prove the following:

Theorem 2.7. *If A is involutive (the square of the antipode is the identity map) then*

$$\mathcal{K}_{A\text{-mod}}(M) = \text{Ku}_A(M),$$

where $\text{Ku}_A(M)$ is the Kuperberg invariant associated to A (see [Kuperberg 1991]).

2D. Turaev–Viro invariant. Let \mathcal{C} be a finite semisimple spherical \mathbb{k} -category (see [Barrett and Westbury 1999]). Here we claim that such a category satisfies the hypothesis of Theorem 2.5. If the dimension of \mathcal{C} (see Equation (7)) is not zero in \mathbb{k} , the resulting 3-manifold invariant $\mathcal{K}_{\mathcal{C}}$ is the Turaev–Viro invariant $\text{TV}_{\mathcal{C}}$ [Turaev and Viro 1992; Barrett and Westbury 1996] associated to \mathcal{C} .

Let $\{S_i\}_{i \in I}$ be a set of representatives of the isomorphism classes of simple objects of \mathcal{C} . Then $G = \bigoplus_{i \in I} S_i$ is a generator. By definition of a semisimple spherical \mathbb{k} -category, the quantum trace $\text{t} = \text{qTr}_{\mathcal{C}}$ is a nondegenerate m -trace on $\mathcal{I} = \mathcal{C}$. Then it follows that

$$\left\{ x_i = \frac{1}{\text{qdim}(S_i)} \overleftarrow{\text{coev}}_{S_i} \right\}_{i \in I} \quad \text{and} \quad \{ y_i = \overrightarrow{\text{ev}}_{S_i} \}_{i \in I}$$

are dual bases of $\text{Hom}_{\mathcal{C}}(\mathbb{1}, G \otimes G^*)$ and $\text{Hom}_{\mathcal{C}}(G \otimes G^*, \mathbb{1})$, respectively. Using the expansion $\Omega_{G \otimes G^*} = \sum_{i \in I} x_i \otimes_{\mathbb{k}} y_i$, it is straightforward to check that

$$\tilde{d} = \sum_{i \in I} \text{qdim}(S_i) \text{Id}_{S_i} \otimes \text{Id}_G$$

is a chromatic morphism for G .

In [Barrett and Westbury 1996], the Turaev–Viro invariant $\text{TV}_{\mathcal{C}}$ is generalized to the context of a finite semisimple spherical \mathbb{k} -category \mathcal{C} when the following scalar D , called the dimension of \mathcal{C} , does not vanish in \mathbb{k} :

$$(7) \quad D = \sum_{i \in I} \text{qdim}(S_i)^2 \in \mathbb{k}.$$

In Section 7 we prove the following theorem.

Theorem 2.8. *If \mathcal{C} is a finite semisimple nondegenerate spherical \mathbb{k} -category with the chromatic morphism \tilde{d} and generator G then the invariant $\mathcal{K}_{\mathcal{C}}$ is proportional to the Turaev–Viro invariant of closed 3-manifolds associated to \mathcal{C} ,*

$$\text{TV}_{\mathcal{C}} = D^{-1} \mathcal{K}_{\mathcal{C}}.$$

2E. Open Problems. A strong point of our approach is its great generality. Besides the categories studied here, it certainly applies in other settings. Here we list (from least to most general) three further categories where our constructions should work:

- (1) the categories of finite dimensional modules over nice (quantum) Lie superalgebras, see [Scheunert and Zhang 2001; 2005],
- (2) the categories of finite dimensional modules over nice quasi-Hopf algebras, see [Bulacu and Caenepeel 2003; 2012; Hausser and Nill 1999; Panaite and Van Oystaeyen 2000],
- (3) general unimodular finite tensor categories, see [Shimizu 2019].

Here the adjective “nice” means that the category satisfies the hypothesis of Theorem 2.4, in particular, admits an m -trace and a chromatic morphism. The theory of [Geer et al. 2018] should imply the existence of an m -trace in the above contexts. Here, we may need to choose an appropriate pivotal structure to make the m -trace two sided, and it may be useful to work with the ideal of projective modules. It seems likely that the references listed above can help to construct chromatic morphisms in the categories in question. It also looks plausible that the results of [Geer et al. 2013a] may help to generalize our approach to nonunimodular categories.

In a different direction, recall that Kuperberg [1996] used framings of 3-manifolds to generalize the invariant in [Kuperberg 1991] to arbitrary finite dimensional Hopf algebras. As explained above, a finite dimensional unibalanced unimodular pivotal

Hopf algebra A gives rise to a framing-independent 3-manifold invariant $\mathcal{K}_{A\text{-mod}}$ which is computed in a way similar to the invariants in [Kuperberg 1996] using an integral and a cointegral (see Section 6). With Theorem 2.7 in mind, we ask if $\mathcal{K}_{A\text{-mod}}(M) = \text{Ku}_A(M, f)$ for some framing f of a 3-manifold M ? Is the Kuperberg invariant associated to a *unibalanced unimodular pivotal Hopf algebra* framing-independent? (This is known not to be true for all finite dimensional Hopf algebras.)

3. Algebraic setup

3A. Pivotal and ribbon categories. In this paper, we consider strict tensor categories with tensor product \otimes and unit object $\mathbb{1}$. Let \mathcal{C} be such a category. The notation $V \in \mathcal{C}$ means that V is an object of \mathcal{C} .

The category \mathcal{C} is a *pivotal category* if it has duality morphisms

$$\begin{aligned} \overleftarrow{\text{coev}}_V: \mathbb{1} &\rightarrow V \otimes V^*, & \overleftarrow{\text{ev}}_V: V^* \otimes V &\rightarrow \mathbb{1}, \\ \overrightarrow{\text{coev}}_V: \mathbb{1} &\rightarrow V^* \otimes V, & \overrightarrow{\text{ev}}_V: V \otimes V^* &\rightarrow \mathbb{1}, \end{aligned}$$

which satisfy compatibility conditions (see for example [Barrett and Westbury 1999; Geer et al. 2013a]).

3B. \mathbb{k} -categories. Let \mathbb{k} be a field. A \mathbb{k} -category is a category \mathcal{C} such that its hom-sets are left \mathbb{k} -modules, the composition of morphisms is \mathbb{k} -bilinear, and the canonical \mathbb{k} -algebra map $\mathbb{k} \rightarrow \text{End}_{\mathcal{C}}(\mathbb{1})$, $k \mapsto k \text{Id}_{\mathbb{1}}$ is an isomorphism. A *tensor \mathbb{k} -category* is a tensor category \mathcal{C} such that \mathcal{C} is a \mathbb{k} -category and the tensor product of morphisms is \mathbb{k} -bilinear.

3C. m -traces on ideals in pivotal categories. Let \mathcal{C} be a pivotal \mathbb{k} -category. Here we recall the definition of an m -trace on an ideal in \mathcal{C} , for more details see [Geer et al. 2013a; Geer et al. 2013b]. By a *ideal* of \mathcal{C} we mean a full subcategory, \mathcal{I} , of \mathcal{C} such that:

Closed under tensor products. If V is an object of \mathcal{I} and W is any object of \mathcal{C} , then $V \otimes W$ and $W \otimes V$ are objects of \mathcal{I} .

Closed under retracts. If V is an object of \mathcal{I} , W is any object of \mathcal{C} , and there exists morphisms $f: W \rightarrow V$, $g: V \rightarrow W$ such that $gf = \text{Id}_W$, then W is an object of \mathcal{I} .

An m -trace on an ideal \mathcal{I} is a family of linear functions

$$\{t_V: \text{End}_{\mathcal{C}}(V) \rightarrow \mathbb{k}\}_{V \in \mathcal{I}}$$

such that following two conditions hold:

Cyclicity. If $U, V \in \mathcal{I}$ then for any morphisms $f: V \rightarrow U$ and $g: U \rightarrow V$ in \mathcal{C} we have

$$t_V(gf) = t_U(fg).$$

Partial trace properties. If $U \in \mathcal{I}$ and $W \in \mathcal{C}$ then for any $f \in \text{End}_{\mathcal{C}}(U \otimes W)$ and $g \in \text{End}_{\mathcal{C}}(W \otimes U)$ we have

$$\begin{aligned} \text{t}_{U \otimes W}(f) &= \text{t}_U \left((\text{Id}_U \otimes \overrightarrow{\text{ev}}_W)(f \otimes \text{Id}_{W^*})(\text{Id}_U \otimes \overleftarrow{\text{coev}}_W) \right), \\ \text{t}_{W \otimes U}(g) &= \text{t}_U \left((\overleftarrow{\text{ev}}_W \otimes \text{Id}_U)(\text{Id}_{W^*} \otimes g)(\overrightarrow{\text{coev}}_W \otimes \text{Id}_U) \right). \end{aligned}$$

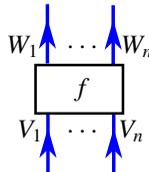
As above, an m -trace is *nondegenerate* if for any object $P \in \mathcal{I}$, the following pairing is nondegenerate:

$$\text{Hom}_{\mathcal{C}}(\mathbb{1}, P) \times \text{Hom}_{\mathcal{C}}(P, \mathbb{1}) \rightarrow \mathbb{k} \text{ given by } (x, y) \mapsto \text{t}_P(xy).$$

Remark that using the pivotal structure and the partial trace property one can show the nondegeneracy condition is equivalent to the following: for all $P \in \mathcal{I}$ and $V \in \text{Obj}(\mathcal{C})$, the pairing $\text{Hom}_{\mathcal{C}}(V, P) \times \text{Hom}_{\mathcal{C}}(P, V) \rightarrow \mathbb{k}$ given by $(x, y) \mapsto \text{t}_P(xy)$ is nondegenerate.

3D. Projective objects. Many of our examples use an m -trace on the ideal of projective objects. An object P of \mathcal{C} is *projective* if for any epimorphism $p: X \rightarrow Y$ and any morphism $f: P \rightarrow Y$ in \mathcal{C} , there exists a morphism $g: P \rightarrow X$ in \mathcal{C} such that $f = pg$. An object Q of \mathcal{C} is *injective* if for any monomorphism $i: X \rightarrow Y$ and any morphism $f: X \rightarrow Q$ in \mathcal{C} , there exists a morphism $g: Y \rightarrow Q$ in \mathcal{C} such that $f = gi$. Denote by Proj the full subcategory of projective objects. In a pivotal category projective and injective objects coincide (see [Geer et al. 2013b]). Also, Proj is an ideal.

3E. Invariants of colored ribbon graphs. Let \mathcal{C} be a pivotal \mathbb{k} -category. A morphism $f: V_1 \otimes \cdots \otimes V_n \rightarrow W_1 \otimes \cdots \otimes W_m$ in \mathcal{C} can be represented by a box and arrows:



which are called *coupons*. All coupons have a top and a bottom sides which in our pictures will be the horizontal sides of the coupons. By a ribbon graph in an oriented manifold Σ , we mean an oriented compact surface embedded in Σ which is decomposed into elementary pieces: bands, annuli, and coupons (see [Turaev 1994]) and is the thickening of an oriented graph. In particular, the vertices of the graph lying in $\text{Int } \Sigma = \Sigma \setminus \partial \Sigma$ are thickened to coupons. A \mathcal{C} -colored ribbon graph is a ribbon graph whose (thickened) edges are colored by objects of \mathcal{C} and whose coupons are colored by morphisms of \mathcal{C} . The intersection of a \mathcal{C} -colored ribbon graph in Σ with $\partial \Sigma$ is required to be empty or to consist only of vertices of

valency 1. When Σ is a surface the ribbon graph is just a tubular neighborhood of the graph.

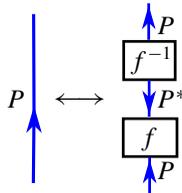
A \mathcal{C} -colored ribbon graph in \mathbb{R}^2 (resp. $S^2 = \mathbb{R}^2 \cup \{\infty\}$) is called *planar* (resp. *spherical*). Let Rib be the category of planar \mathcal{C} -colored ribbon graphs and let $F : \text{Rib} \rightarrow \mathcal{C}$ be the pivotal functor¹ (defined using the Penrose graphical calculus) corresponding to \mathcal{C} , see for example [Geer et al. 2013b]. Let \mathcal{L}_{adm} be all spherical \mathcal{C} -colored ribbon graphs L such that L is the braid closure of a $(1,1)$ -ribbon graph T_V whose open edge is colored with an object $V \in \mathcal{I}$.

Given an m -trace t on \mathcal{I} we can renormalize F to an invariant

$$(8) \quad F' : \mathcal{L}_{\text{adm}} \rightarrow \mathbb{k} \quad \text{given by} \quad F'(L) = t_V(F(T_V)),$$

where T_V is any $(1,1)$ -ribbon graph as above. The properties of the m -trace imply F' is a isotopy invariant of L , see [Geer et al. 2013b].

Remark 3.1. If \mathcal{I} is an ideal and $P \in \mathcal{I}$ then Lemma 2 of [Geer et al. 2013b] implies that $P^* \in \mathcal{I}$. Moreover, the pivotal structure gives an isomorphism $f : P \rightarrow P^{**}$ for all P . This morphism can be used to change the orientation of an edge of a graph as shown in the following diagram:



4. An invariant of \mathcal{I} -colored graphs on a multihandlebody

In this section we prove Proposition 2.1 and Theorem 2.2.

4A. Algebraic preliminaries. The following lemma contains standard facts from linear algebra; we leave the proof to the reader.

Lemma 4.1. *Let X_j and Y_j be finite dimensional \mathbb{k} -modules, for $j = 1, 2$. Let $\langle \cdot, \cdot \rangle_{X_j, Y_j} : X_j \otimes_{\mathbb{k}} Y_j \rightarrow \mathbb{k}$ be a pairing whose right and left kernels are zero. Given a basis $\{x_i^j\}$ of X_j let $\{y_i^j\}$ be the dual basis of Y_j determined by $\langle x_i^j, y_j^j \rangle_{X_j, Y_j} = \delta_{i,j}$. Then*

- (1) *the element $\Omega_j = \sum_i x_i^j \otimes_{\mathbb{k}} y_i^j \in X_j \otimes Y_j$ is independent of the choice of the basis $\{x_i^j\}$,*

¹We call this functor the pivotal functor because such a functor is associated to each pivotal category. However, the functor does not preserve the duality.

(2) if $h : X_1 \rightarrow X_2$ and $k : Y_2 \rightarrow Y_1$ are \mathbb{k} -linear maps such that $\langle h(x), y \rangle_{X_2, Y_2} = \langle x, k(y) \rangle_{X_1, Y_1}$ for all $x \in X_1$ and $y \in Y_2$ then

$$(9) \quad (h \otimes \text{Id}_{Y_1})(\Omega_1) = (\text{Id}_{X_2} \otimes k)(\Omega_2).$$

Proposition 4.2. Recall the element $\Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i$ defined in Equation (1). Let $\Lambda_P = \sum_i x_i y_i \in \text{End}_{\mathcal{C}}(P)$. We have:

(1) The element Ω_P is independent of the choice of the basis of $\text{Hom}_{\mathcal{C}}(\mathbb{1}, P)$.

(2) If $P' \in \mathcal{I}$ and $\phi : P \rightarrow P'$ is a morphism then

$$(\phi \otimes_{\mathbb{k}} \text{Id}_{\mathbb{1}})\Omega_P = \Omega_{P'}(\text{Id}_{\mathbb{1}} \otimes_{\mathbb{k}} \phi) \quad \text{and} \quad \phi \circ \Lambda_P = \Lambda_{P'} \circ \phi.$$

(3) If $f : \mathbb{1} \rightarrow P$ and $g : P \rightarrow \mathbb{1}$ are morphisms then

$$\text{t}_P(fg) = \sum_i \text{t}_P(fy_i)\text{t}_P(x_i g).$$

Proof. We use the above lemma to prove this proposition. Let

$$\begin{aligned} X_1 &= \text{Hom}_{\mathcal{C}}(\mathbb{1}, P), & Y_1 &= \text{Hom}_{\mathcal{C}}(P, \mathbb{1}), \\ X_2 &= \text{Hom}_{\mathcal{C}}(\mathbb{1}, P'), & Y_2 &= \text{Hom}_{\mathcal{C}}(P', \mathbb{1}). \end{aligned}$$

For $j = 1, 2$, define the bilinear pairing $\langle \cdot, \cdot \rangle_{X_j, Y_j} : X_j \otimes Y_j \rightarrow \mathbb{k}$ by

$$\langle x, y \rangle_{X_1, Y_1} = \text{t}_P(x \circ y) \quad \text{and} \quad \langle x, y \rangle_{X_2, Y_2} = \text{t}_{P'}(x \circ y), \quad \text{respectively,}$$

for $x \in X_j$ and $y \in Y_j$. Now the first statement of the proposition is a direct consequence of first statement of [Lemma 4.1](#). Similarly, the second statement of the proposition follows from the second statement of the lemma: let

$$\begin{aligned} h : \text{Hom}(\mathbb{1}, P) &\rightarrow \text{Hom}(\mathbb{1}, P') & \text{be given by} & \quad f \mapsto \phi \circ f \\ k : \text{Hom}(P', \mathbb{1}) &\rightarrow \text{Hom}(P, \mathbb{1}) & \text{be given by} & \quad g \mapsto g \circ \phi \end{aligned}$$

then Equation (9) becomes the first equality in the second statement of the proposition. If we write this equality explicitly we get

$$\sum_i (\phi \circ x_i) \otimes y_i = \sum_i x'_i \otimes (y'_i \circ \phi),$$

where $\Omega_{P'} = \sum_i x'_i \otimes_{\mathbb{k}} y'_i$ and $\Lambda_{P'} = \sum_i x'_i y'_i \in \text{End}_{\mathcal{C}}(P')$ and the second equality follows.

Concerning the last statement of the proposition, let $\{x_i\}$ be a basis of $X_1 = \text{Hom}_{\mathcal{C}}(\mathbb{1}, P)$ and let $\{y_i\}$ be the dual basis of Y_1 determined by $\langle \cdot, \cdot \rangle_{X_1, Y_1}$. Using these bases we can find coefficients a_i and b_i such that $f = \sum a_i x_i$ and $g = \sum b_i y_i$. Then

$$\text{t}_P(fg) = \sum_{i,j} a_i b_j \text{t}_P(x_i y_j) = \sum_i a_i b_i.$$

But we also have that $\mathfrak{t}_P(fy_i) = \sum_j a_j \mathfrak{t}_P(x_j y_i) = a_i$ and similarly $\mathfrak{t}_P(x_i g) = b_i$ so the third statement of the proposition follows. \square

4B. Proof of Theorem 2.2. By the genus of a multihandlebody we mean the sum of the genera of all its components. We will show that the last three properties of Theorem 2.2 determine a well defined invariant by inducting on the genus of the multihandlebody. We first do this for a fixed multihandlebody then at the end of the proof we show it is invariant under orientation preserving diffeomorphisms.

The base induction case is graphs on the disjoint union of 3-balls. The extension of F' Property (2) defines F' for graphs up to isotopy on the boundary of the 3-ball. The disjoint union of handlebodies Property (3) extends the invariant F' uniquely to the disjoint union of 3-balls. To conclude the base case we need to show that F' satisfies the cutting-along-a-disk Property (4) of Theorem 2.2. Let $(B^3, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ be a \mathcal{I} -colored graph on the boundary of the 3-ball. Let D be a disk properly embedded in the 3-ball such that the boundary of D is a simple curve γ on S^2 (which intersects Γ transversely).

Orient γ arbitrarily and up to isotopy assume it is the equator of S^2 . The curve γ intersects Γ in several points which correspond to the tensor product of objects of \mathcal{I} . We denote by P this tensor product. Let Γ_l and Γ_u be the graphs in the lower and upper hemispheres, respectively (here we use the orientation of γ to distinguish them). Let $(\text{cut}_{\gamma}(B^3), \text{cut}_{\gamma}(\Gamma)) = (B^3, \Gamma'_l) \sqcup (B^3, \Gamma'_u)$ be the \mathcal{I} -colored graph on the disjoint union of two 3-balls obtained by cutting along the disk where the new graphs are obtained by closing Γ_l and Γ_u with coupons determined by $\Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i$ as described in the statement above the theorem.

Then $F(\Gamma_l) : \mathbb{1} \rightarrow P$ and $F(\Gamma_u) : P \rightarrow \mathbb{1}$ are morphisms and by definition

$$F'(B^3, \Gamma) = \mathfrak{t}_P(F(\Gamma_l)F(\Gamma_u)).$$

But by Part (3) of Proposition 4.2 we have

$$\mathfrak{t}_P(F(\Gamma_l)F(\Gamma_u)) = \sum_i \mathfrak{t}_P(F(\Gamma_l)y_i)\mathfrak{t}_P(x_i F(\Gamma_u))$$

where by definition $F'(\Gamma'_l) = \mathfrak{t}_P(F(\Gamma_l)y_i)$ and $F'(\Gamma'_u) = \mathfrak{t}_P(x_i F(\Gamma_u))$. This shows that we can cut along D^2 and also that the choice of the orientation on $\gamma = \partial D^2$ does not change the result of F' as the left hand side does not depend on γ . This concludes the base induction step.

Let us assume F' is well defined and satisfies Properties (2), (3) and (4) of Theorem 2.2 for all multihandlebodies with genus strictly less than $g \geq 1$. Let $(H, \Gamma) \in \mathcal{H}_{\mathcal{I}}$ where H is a multihandlebody of genus g . Choose a disk D which is properly embedded in H such that the boundary of D is an essential simple circle γ on ∂H intersecting Γ transversely. Cutting along D produces a genus $g - 1$ multihandlebody $(\text{cut}_{\gamma}(H), \text{cut}_{\gamma}(\Gamma))$ which for simplicity we also denote by $\text{cut}_{\gamma}(\Gamma)$. Then

Property (4) and induction says we can assign the value $F'(H, \Gamma) = F'(\text{cut}_\gamma(\Gamma))$. We need to show that this value is well defined, i.e., independent of the disk which is cut along and of the orientation of its boundary chosen to draw the picture used in the description of the property. To do this, let D and D' be two discs which bound essential simple oriented circles γ and γ' , respectively. We want to show that by cutting along D or D' we obtain the same value for F' . We prove this in two steps.

First, let us assume γ and γ' are disjoint. Then one can choose disjoint discs D and D' which both cut the multihandlebody to produce genus $g - 1$ multihandlebodies. We have

$$F'(\text{cut}_{\gamma'}(\Gamma)) = F'(\text{cut}_\gamma(\text{cut}_{\gamma'}(\Gamma))) = F'(\text{cut}_{\gamma'}(\text{cut}_\gamma(\Gamma))) = F'(\text{cut}_\gamma(\Gamma)),$$

where the first and the last equalities hold because Property (3) holds in genus less than g . This in particular proves that the orientation of γ is irrelevant: choose γ' to be parallel to γ but oriented the opposite way.

Second, it follows from [Masur and Minsky 2004] that for any two essential bounding curves γ and γ' , there exists a sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \gamma'$ of circles that bound discs D_i with $D_i \cap D_{i+1} = \emptyset$. Hence the first step implies the value of F' is constant on this sequence:

$$F'(\text{cut}_{\gamma_0}(\Gamma)) = F'(\text{cut}_{\gamma_1}(\Gamma)) = \dots = F'(\text{cut}_{\gamma_n}(\Gamma)).$$

This completes the induction step.

We are now left to prove that F' is invariant under the action of the group of orientation preserving diffeomorphisms of H up to isotopy. Again, we argue by induction on the genus. Since this group is trivial for B^3 (i.e., genus 0), there is nothing to prove in the base case. Assume F' is invariant for all multihandlebodies of genus less than g . Let (H, Γ) be a \mathcal{L} -colored graph on a multihandlebody of genus g . If γ is a simple essential curve bounding in H then applying the cut-along-a-disc property of F' we get

$$F'(H, \Gamma) = F'(\text{cut}_\gamma(H), \text{cut}_\gamma(\Gamma)).$$

If $f : H \rightarrow H$ is a positive self-diffeomorphism, then we have:

$$F'(f(H), f(\Gamma)) = F'(H, f(\Gamma)) = F'(\text{cut}_{f(\gamma)}(H), \text{cut}_{f(\gamma)}(f(\Gamma))).$$

But f induces a diffeomorphism between the genus $g - 1$ multihandlebody graphs $(\text{cut}_\gamma(H), \text{cut}_\gamma(\Gamma))$ and $(\text{cut}_{f(\gamma)}(H), \text{cut}_{f(\gamma)}(f(\Gamma)))$ so by induction we have

$$\begin{aligned} F'(H, \Gamma) &= F'(\text{cut}_\gamma(H), \text{cut}_\gamma(\Gamma)) = F'(\text{cut}_{f(\gamma)}(H), \text{cut}_{f(\gamma)}(f(\Gamma))) \\ &= F'(f(H), f(\Gamma)). \end{aligned}$$

Thus, we have proved [Theorem 2.2](#).

5. Invariant of bichrome graphs and of 3-manifolds

5A. Bichrome graphs on handlebodies. Recall by a *multihandlebody* we will mean a disjoint union of a finite number of oriented connected handlebodies. By the *genus* of a multihandlebody we mean the sum of the genera of all its components. Recall a *bichrome handlebody graph* is a pair (H, Γ) where H is a multihandlebody and Γ is a graph on ∂H (seen up to isotopy) composed of two disjoint subgraphs $\Gamma = \Gamma_{\text{blue}} \sqcup \Gamma_{\text{red}}$ where Γ_{blue} is an \mathcal{I} -colored graph and $\Gamma_{\text{red}} \subset \partial H$ a disjoint union of simple closed unoriented circles. When convenient, we denote a bichrome handlebody graph by Γ with the understanding that it is on a multihandlebody. Recall, we say a bichrome handlebody graph is *admissible* if its blue subgraph is \mathcal{I} -colored and meets each connected component of the multihandlebody. Let \mathcal{H}_b be the set of orientation preserving diffeomorphism classes of admissible bichrome handlebody graphs.

Definition 5.1. Let H be a multihandlebody.

- (1) We say that a set of simple closed curves (referred to as circles) on ∂H *bounds* in H if the circles bound a disjoint union of disks embedded in H .
- (2) A bounding set of circles is a *reducing set* if the complement of the circles is a disjoint union of spheres with holes.
- (3) The *complexity* of a circle γ on $\partial H \subset H$ is the minimal number of intersections of γ with any reducing set of circles.

Definition 5.2 (red capping and digging moves). Let (H, Γ) and (H', Γ') be bichrome handlebody graphs. We say that (H, Γ) is obtained from (H', Γ') by a *red capping move* along a red circle $c \subset \Gamma'_{\text{red}}$ if there is a properly embedded disc D in H' such that

- (1) $D \cap c = \{\text{pt}\}$,
- (2) H is obtained from H' by gluing a 2-handle along c , and
- (3) $\Gamma = \Gamma' \setminus c$, where we identify $\partial H'$ set minus a neighborhood of c with a subset of ∂H .

Conversely, we say (H', Γ') is obtained from (H, Γ) by a *red digging move* (resulting in the red circle c). See [Figure 2](#) for a visual representation of these moves.

Proposition 5.3. *A red circle in a bichrome handlebody graph is the result of a red digging move if and only if it has complexity one.*

Proof. Let c be a red circle in a bichrome handlebody graph (H', Γ') . If c has complexity one then there exists a reducing set S of circles such that $c \cap S$ is exactly one point. Now gluing a 2-handle along c one gets a multihandlebody H , and the

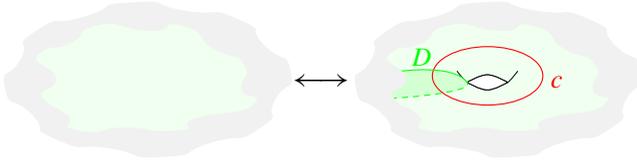


Figure 2. Red digging / capping moves.

graph $\Gamma := \Gamma' \setminus c$ is obtained by a red capping move which is the inverse of the sought digging move.

Reciprocally, if c is obtained by a red digging move then it intersects exactly once the disc D in [Definition 5.2](#). □

Proposition 5.4. *Let $(H, \Gamma), (H, \Gamma') \in \mathcal{H}_b$ be bichrome graphs such that Γ' is obtained from Γ by sliding a blue or red edge of Γ over one of its simple red circles. Then Γ and Γ' are related by a sequence of red digging and capping moves.*

Proof. Let $\Gamma = \Gamma_{\text{blue}} \cup \Gamma_{\text{red}}$ and let $c \subset \Gamma_{\text{red}}$ be the simple circle on which we want to slide an edge e (where e is contained in either Γ_{blue} or Γ_{red}). We claim that up to applying one red digging move which transforms (H, Γ) into a new bichrome graph (H_1, Γ_1) and creates a new red circle $c' \subset \Gamma_1$, we can reduce to the case where c is a red circle created by a red digging move. Proving this claim would imply the proposition because after applying a red capping move on c , one gets a bichrome graph (H_2, Γ_2) in which e can be slid, by an isotopy, over the disc added by the red capping. Then redigging along the same disc and recapping along c' produces exactly the bichrome graph (H, Γ') obtained by sliding e over c .

We now prove the claim by describing a suitable red digging move. Let $I \subset \partial H$ be a parametrized segment, i.e., the image of an embedding $i : [-1, 1] \hookrightarrow \partial H$ such that $\partial I \cap \Gamma = i(\{0\})$ is formed by a single point belonging to c . Let I' be a properly embedded arc in H obtained by slightly pushing I inside H (keeping fixed ∂I). Also, let

$$H_1 = H \setminus \text{Tub}(I'), \quad c' = \partial \text{Tub}(i(1)),$$

$$(\Gamma_1)_{\text{blue}} = \Gamma_{\text{blue}}, \quad (\Gamma_1)_{\text{red}} = \Gamma_{\text{red}} \sqcup c',$$

where Tub denotes a tubular neighborhood and we identify $\partial H \setminus \text{Tub}(\{i(\pm 1)\})$ with a subset of ∂H_1 .

Then (H_1, Γ_1) is obtained from (H, Γ) by a red digging move along I which creates the red circle c' . But now, with the notation of [Definition 5.2](#), c intersects the disc D bound by $I \cup I'$ exactly once so it has complexity 1 and thus, by [Proposition 5.3](#) it is the result of a red digging. □

5B. Heegaard splittings. A Heegaard splitting for a closed oriented connected 3-manifold M is an ordered triple $(H_\alpha, H_\beta, \Sigma)$ such that the following conditions hold:

- (1) Σ is a closed surface embedded in M ,
- (2) H_α and H_β are oriented handlebodies positively embedded in M ,
- (3) $\partial H_\beta = \Sigma = \partial H_\alpha = H_\alpha \cap H_\beta$,
- (4) $M = H_\alpha \cup H_\beta$ and
- (5) Σ is oriented as the boundary of H_α (with the outgoing vector first convention).

A Heegaard diagram for M compatible with this Heegaard splitting is a triple $(\Sigma, \{\alpha_i\}, \{\beta_i\})$, where $\{\alpha_i\}$ and $\{\beta_i\}$ are minimal reducing sets of circles bounding in H_α and H_β , respectively. Then one can recover H_α, H_β and $M \simeq H_\alpha \cup_\Sigma H_\beta$ up to diffeomorphism from the Heegaard diagram.

Recall in Section 2B given a Heegaard diagram $(\Sigma, \{\alpha_i\}, \{\beta_i\})$ of a closed connected orientable 3-manifold M we defined a bichrome diagram for M : a bichrome handlebody graph $(H, \Gamma_{\text{blue}} \cup \Gamma_{\text{red}})$, where $H = H_\alpha$, $\Gamma_{\text{red}} = \{\beta_i\}$ and Γ_{blue} is the planar ribbon graph \mathcal{O}_G which is the braid closure of the coupon filled with h such that $t_G(h) = 1$. (Observe that since $H \setminus \Gamma_{\text{red}}$ is connected, the position of \mathcal{O}_G is unique up to isotopy.)

Theorem 5.5. *If (H, Γ) and (H', Γ') are two bichrome diagrams for M , then they are related by a finite sequence of red digging and capping moves.*

Proof. It is well known that (H, Γ_{red}) and $(H', \Gamma'_{\text{red}})$ can be connected by a finite sequence of stabilization moves (and their inverses) as well as handle slide moves (see [Singer 1933]). Clearly a stabilization move is a special case of a red digging move (and its inverse is a red capping move). Furthermore by Proposition 5.4 we can achieve handle slides via isotopy and red digging and capping moves. Finally, since $\Gamma_{\text{blue}} = \mathcal{O}_G$ is contained in a disc in ∂H we can always operate the above moves. □

5C. Proof of Theorems 2.4 and 2.5. We need to prove a few lemmas.

Lemma 5.6. *Let $P, P' \in \mathcal{I}$ and set $\Lambda_{P' \otimes G^*} = \sum_i y_i x_i \in \text{End}_{\mathcal{C}}(P' \otimes G^*)$, where $\Omega_{P' \otimes G^*} = \sum x_i \otimes_{\mathbb{k}} y_i$. Then*

$$(10) \quad (\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ (\Lambda_{P' \otimes G^*} \otimes \tilde{d}_P) \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P) = \text{Id}_{P' \otimes P}.$$

Proof. Since G is a generator of \mathcal{I} there exist $g_i : P \rightarrow G$, $f_i : G \rightarrow P$ and $g'_i : P' \rightarrow G$, $f'_i : G \rightarrow P'$ such that $\sum_i f_i \circ g_i = \text{Id}_P$ and $\sum_i f'_i \circ g'_i = \text{Id}_{P'}$. Then we have

$$\begin{aligned} \text{Id}_{P' \otimes P} &= \sum_{i,j} (f'_i \otimes f_j) \circ \text{Id}_{G \otimes G} \circ (g'_i \otimes g_j) \\ &= \sum_{i,j} (f'_i \otimes \overleftarrow{\text{ev}}_G \otimes f_j) \circ (\Lambda \otimes \tilde{d}) \circ (g'_i \otimes \overrightarrow{\text{coev}}_G \otimes g_j) \\ &= \sum_i (f'_i \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ (\Lambda \otimes \tilde{d}_P) \circ (g'_i \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P) \end{aligned}$$

$$\begin{aligned}
 &= \sum_i (\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ \left(((f'_i \otimes \text{Id}_{G^*}) \Lambda (g'_i \otimes \text{Id}_{G^*})) \otimes \tilde{d}_P \right) \\
 &\hspace{20em} \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P) \\
 &= \sum_i (\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ \left((\Lambda_{P' \otimes G^*} \circ (f'_i g'_i \otimes \text{Id}_{G^*})) \otimes \tilde{d}_P \right) \\
 &\hspace{20em} \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P) \\
 &= \sum_i (\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ (\Lambda_{P' \otimes G^*} \otimes \tilde{d}_P) \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P),
 \end{aligned}$$

where the second equality uses the definition of the chromatic morphism (Equation (3)), the third equality uses the definition of \tilde{d}_P (Equation (5)) and the second to last equality comes from Part (2) of Proposition 4.2 where $\phi = f'_i \otimes \text{Id}_{G^*}$. \square

Given a bichrome handlebody graph (H, Γ) we can produce a new bichrome handlebody graph (H', Γ') by doing a red digging move on (H, Γ) then changing the newly created red circle into a blue graph using the chromatic morphism, as in Equation (6). We say (H', Γ') is obtained from (H, Γ) by a *blue digging move*. Conversely, we say (H, Γ) is obtained from (H', Γ') by a *blue capping move*. See Figure 3 for a pictorial representation of these moves.

Lemma 5.7. *The invariant F' of \mathcal{I} -colored ribbon graphs on multihandlebodies defined in Theorem 2.2 is invariant under blue digging and capping moves.*

Proof. Let (H, Γ) and (H', Γ') be bichrome handlebody graphs such that (H', Γ') is obtained from (H, Γ) by a blue digging move. Up to cutting along a reducing set of curves for H , we can assume that the component of H to which we are applying the blue digging has genus 0. To prove the statement we will compute the morphisms associated to the subsurfaces of ∂H and $\partial H'$ drawn in Figure 3.

The right hand side is just $\text{Id}_{P'} \otimes \text{Id}_P$. In order to compute the left hand side, let γ be the curve bounding the disc in the far left part of Figure 3. Cutting along γ , as discussed before Theorem 2.2, we get that the morphism associated to the left hand side is:

$$(\text{Id}_{P'} \otimes \overleftarrow{\text{ev}}_G \otimes \text{Id}_P) \circ (\Lambda_{P' \otimes G^*} \otimes \tilde{d}_P) \circ (\text{Id}_{P'} \otimes \overrightarrow{\text{coev}}_G \otimes \text{Id}_P).$$

But by Lemma 5.6 this morphism is equal to $\text{Id}_{P'} \otimes \text{Id}_P$. \square

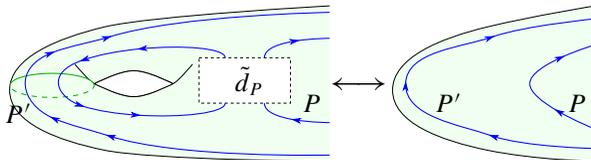


Figure 3. *Blue digging and capping moves.* Here one can assume the orientation of the left most strand is as shown in the figure because of Remark 3.1.

Proof of Theorem 2.4. To prove the theorem we will prove a slightly stronger statement:

Statement. There exists a unique extension of F' to an invariant of admissible bichrome handlebody graphs satisfying Equation (6) and invariant under the blue capping and digging moves.

We prove the statement by induction on the number of red circles. When there are no red circles the statement is just [Lemma 5.7](#).

Now suppose the statement is true for all admissible bichrome handlebody graphs with $n - 1$ red circles. Let (H, Γ) be an admissible bichrome handlebody graph with n red circles. There exists a not self-intersecting path γ_0 in $\partial H \setminus \Gamma$ going from a point on a blue edge to a point on one red circle β_0 . Using this path we can pull a small segment of the blue edge to β_0 then use the chromatic morphism to change β_0 into a blue graph. We obtain an admissible bichrome handlebody graph (H, Γ_{γ_0}) with $n - 1$ red circles. Then by induction $F'(H, \Gamma_{\gamma_0})$ is well defined, so if F' exists it is unique. To prove F' exists we need to show it is independent of the choice of the path γ_0 . Let γ_1 be another such path going from a point on a blue edge to a point on one of the red circles β_1 and let (H, Γ_{γ_1}) be the admissible bichrome handlebody graph obtained by using γ_1 to make β_1 blue. We consider two cases.

Case 1. $\beta_0 = \beta_1$. Here we have two subcases. First, suppose the red circle $\beta_0 = \beta_1$ has complexity one. By [Proposition 5.3](#) we have β_0 is the result of a digging move. Therefore, when we use either γ_0 or γ_1 with the chromatic morphism to change β_0 to a blue graph and we arrive at a diagram of the form given in the left side of [Figure 3](#). In both cases we can do a blue capping move to arrive at the same \mathcal{I} -colored ribbon graph on a multihandlebody with $n - 1$ red circles. Thus, induction implies

$$F'(H, \Gamma_{\gamma_0}) = F'(H, \Gamma_{\gamma_1})$$

so in this subcase the extension of F' does not depend on the choice of γ_0 .

Second, assume the red curve $\beta_0 = \beta_1$ has any complexity. Apply a blue digging move to Γ along a small interval I intersecting β_0 in a single point. The result is a new bichrome handlebody graph (H', Γ') with n red circles in which the image β'_0 of β_0 is a red curve with complexity 1. Since this digging move only modified (H, Γ) in a neighborhood of I , we can identify γ_0 and γ_1 as paths in $\partial H' \setminus \Gamma'$ and we can apply the chromatic morphism to β'_0 either through γ_0 and γ_1 getting, respectively, bichrome handlebody graphs Γ'_{γ_0} and Γ'_{γ_1} with $n - 1$ red components. By the preceding case $F'(H', \Gamma'_{\gamma_0}) = F'(H', \Gamma'_{\gamma_1})$. But now observing that Γ'_{γ_0} (resp. Γ'_{γ_1}) is obtained from Γ_{γ_0} (resp. Γ_{γ_1}) by a blue digging move, we have

$$F'(H, \Gamma_{\gamma_0}) = F'(H', \Gamma'_{\gamma_0}) = F'(H', \Gamma'_{\gamma_1}) = F'(H, \Gamma'_{\gamma_1}).$$

Case 2. $\beta_0 \neq \beta_1$. In this situation we have two subcases. First, suppose γ_0 and γ_1 are nonintersecting. In (H, Γ_{γ_0}) (resp. (H, Γ_{γ_1})) we can use γ_1 (resp. γ_0) to change β_1 (resp. β_0) into a blue graph and obtain an admissible bichrome handlebody graph (H, Γ_1) with $n - 2$ red circles. Then by induction we have:

$$F'(H, \Gamma_{\gamma_0}) = F'(H, \Gamma_1) = F'(H, \Gamma_{\gamma_1}).$$

Second, suppose $\gamma_0 \cap \gamma_1 \neq \emptyset$; we claim that there exists another path γ'_1 connecting Γ_{blue} to β_1 such that $\gamma_0 \cap \gamma'_1 = \emptyset$ so that by the preceding subcase we have $F'(H, \Gamma_{\gamma_0}) = F'(H, \Gamma_{\gamma'_1})$ then Case 1 implies $F'(H, \Gamma_{\gamma'_1}) = F'(H, \Gamma_{\gamma_1})$ and the proof follows. To prove our claim observe that since $\gamma_0 \cap \gamma_1 \neq \emptyset$ then γ_0 and γ_1 are contained in the same connected component R of $\partial H \setminus \Gamma$. Moreover, R is an open orientable surface which is the interior of a compact surface with at least 3 distinct boundary components: $\partial_{\text{blue}} \subset \Gamma_{\text{blue}}$, β_0 and β_1 . Then γ_0 and γ_1 are embedded arcs in R connecting ∂_{blue} to β_0 and β_1 , respectively. But $R \setminus \gamma_0$ is connected as γ_0 intersects the closed curve β_0 once. Thus, there exists another path in $R \setminus \gamma_0$ connecting ∂_{blue} and β_1 . \square

Proof of Theorem 2.5. By Theorem 5.5 it is sufficient to prove that F' is invariant under red digging and capping moves. Let (H, Γ) and (H', Γ') be bichrome handlebody graphs such that (H', Γ') is obtained from (H, Γ) by a red digging move. Suppose c is the red circle created in this move. Let (H', Γ'') be the bichrome handlebody graph obtained from using an edge of the blue graph and the chromatic morphism to change the red circle c into a blue graph. Then by definition the composition of these two moves is a blue digging move. Thus, by Lemma 5.7 we have

$$F'(H, \Gamma) = F'(H', \Gamma'').$$

Moreover, since (H', Γ') and (H', Γ'') differ by an isotopy and a move represented in Equation (6) then Theorem 2.4 implies

$$F'(H', \Gamma') = F'(H', \Gamma'').$$

Combining the last two equalities we have $F'(H, \Gamma) = F'(H', \Gamma')$ which concludes the lemma. \square

6. Details on Hopf algebras and Kuperberg invariants

In this section we prove Theorems 2.6 and 2.7. First, we briefly recall some well known facts about Hopf algebras, see for example [Radford 2012].

6A. Hopf algebra preliminaries. Let A be a finite-dimensional Hopf algebra over a field \mathbb{k} with a multiplication $m : A \otimes A \rightarrow A$, a unit $\eta : \mathbb{k} \rightarrow A$, a coproduct $\Delta : A \rightarrow A \otimes A$, a counit $\epsilon : A \rightarrow \mathbb{k}$, an antipode $S : A \rightarrow A$. A *right integral* of

A is a linear form $\lambda \in A^*$ satisfying $\lambda f = f(1_A) \cdot \lambda$ for every $f \in A^*$. This means that $(\lambda \otimes \text{Id}_A)(\Delta(x)) = \lambda(x) \cdot 1_A$ for every $x \in A$. A *left* (resp. *right*) *cointegral* of A is a vector $\Lambda \in A$ satisfying $x\Lambda = \epsilon(x)\Lambda$ (resp. $\Lambda x = \epsilon(x)\Lambda$) for every $x \in A$. Since A is finite-dimensional, right integrals form a 1-dimensional ideal in A^* and left cointegrals form a 1-dimensional ideal in A . Moreover, every nonzero right integral $\lambda \in A^*$ and every nonzero left cointegral $\Lambda \in A$ satisfy $\lambda(\Lambda) \neq 0$. We fix a choice of a right integral $\lambda \in A^*$ and of a left cointegral $\Lambda \in A$ satisfying $\lambda(\Lambda) = 1$. In this section we use sumless Sweedler notation to describe the application of the coproduct, for example $\Delta^3(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$.

The Hopf algebra A being *unimodular* means that $S(\Lambda) = \Lambda$, or equivalently, Λ is both a right and left cointegral. We say A is *pivotal* if there exists $g \in A$ such that $S^2(x) = gxg^{-1}$ for all $x \in A$. Let Proj be the ideal of projective modules over A (for a definition of projective object see [Section 3D](#)). By Theorem 1 of [\[Beliakova et al. 2018a\]](#) every finite dimensional unimodular pivotal Hopf algebra has a left m-trace on Proj . Such a Hopf algebra is *unibalanced* if this left m-trace is also a right m-trace (see [\[Beliakova et al. 2018a\]](#)).

6B. Proof of Theorem 2.6. Let A be a finite dimensional unibalanced unimodular pivotal Hopf algebra over a field \mathbb{k} and $A\text{-mod}$ be the category of its finite dimensional left modules. Denote A as the *regular representation* of A , which is the left A -module structure on A itself determined by the action $L : A \rightarrow \text{End}_{\mathbb{k}}(A)$ given by $L_h(x) = hx$ for all $h, x \in A$. Let t be the nondegenerate m-trace on Proj with normalization: $t_A(f_\Lambda \circ \epsilon) = 1$, where $f_\Lambda : \mathbb{k} \rightarrow A$ denotes the unique morphism determined by $f_\Lambda(1) = \Lambda$.

Lemma 6.1. *The regular representation A is a generator of Proj in $A\text{-mod}$.*

Proof. Let P be an indecomposable projective A -module. Then P being projective implies it is a direct summand of a free module $\bigoplus_n A$. But P is indecomposable so P is a direct summand of A . Since Krull-Schmidt Theorem holds in $A\text{-mod}$ we have that every element of Proj is a direct sum of indecomposable projective modules and the lemma follows. □

For $P \in \text{Proj}$, observe that the map $\Lambda'_P : P \rightarrow P$ given by the left action of Λ on P is a morphism because Λ is both a left and right cointegral. On the other hand, consider the morphism $\Lambda_P : P \rightarrow P$ given by $\Lambda_P = \sum_i x_i y_i$, where $\Omega_P = \sum_i x_i \otimes_{\mathbb{k}} y_i$ is defined in [Proposition 2.1](#).

Lemma 6.2. $\Lambda_P = \Lambda'_P.$

Proof. First, consider the case when P is the generator A . Since the space of cointegrals is one dimensional then $\text{Hom}_{A\text{-mod}}(\mathbb{1}, A)$ is one dimensional and generated by the morphism $f_\Lambda : \mathbb{k} \rightarrow A$ determined by $f_\Lambda(1) = \Lambda$. Combining this with the fact

that t is nondegenerate (with the normalization given above) we have $\Omega_A = f_\Lambda \otimes_{\mathbb{k}} \epsilon$. Thus, for all $x \in A$ we have

$$\Lambda_A(x) = f_\Lambda \epsilon(x) = \epsilon(x) \Lambda = x \Lambda = \Lambda'_A(x).$$

So the lemma holds for $P = A$.

Let P be an object in Proj . Since A is a generator there exist $f_j : A \rightarrow P$ and $g_j : P \rightarrow A$ such that $\text{Id}_P = \sum_{j \in J} f_j g_j$. From [Proposition 4.2](#) Part (2) we have

$$\Omega_P = \sum_{j \in J} ((f_j \otimes \text{Id}_1) \Omega_A (\text{Id}_1 \otimes g_j)) = \sum_{j \in J} ((f_j \circ f_\Lambda) \otimes (\epsilon \circ g_j)).$$

So by definition of Λ_P , for $y \in P$, we have

$$\begin{aligned} \Lambda_P(y) &= \sum_{j \in J} (f_j \circ f_\Lambda \circ \epsilon \circ g_j)(y) = \sum_{j \in J} f_j(\epsilon(g_j(y)) \Lambda) \\ &= \sum_{j \in J} f_j(\Lambda g_j(y)) \\ &= \sum_{j \in J} \Lambda f_j(g_j(y)) = \Lambda \cdot y, \end{aligned}$$

and the lemma follows. \square

In [\[De Renzi et al. 2018\]](#) it is shown that $f_{\lambda, 1_A} : A \rightarrow A^* \otimes A$ defined by $1_A \mapsto \lambda \otimes 1_A$ is a morphism.

Lemma 6.3. *The morphism $\tilde{d} : A \otimes A \rightarrow A \otimes A$ defined by*

$$\tilde{d} = (\vec{\text{ev}}_A \otimes \text{Id}_{A \otimes A}) (\text{Id}_A \otimes f_{\lambda, 1_A} \otimes \text{Id}_A) (\text{Id}_A \otimes \Delta)$$

is a chromatic morphism for the generator A .

Proof. First, we set $h = \lambda(S(\Lambda_{(2)})) \Lambda_{(1)}$ and prove

$$(11) \quad h = \lambda(S(\Lambda_{(2)})) \Lambda_{(1)} = 1_A.$$

To prove this we consider

$$\begin{aligned} S(h) &= (\text{Id}_A \otimes \lambda)(S \otimes S)(\Delta(\Lambda)) = (\text{Id}_A \otimes \lambda)(\Delta^{\text{op}}(S(\Lambda))) \\ &= (\lambda \otimes \text{Id}_A)(\Delta(S(\Lambda))) \\ &= \lambda(S(\Lambda)) 1_A = \lambda(\Lambda) 1_A = 1_A. \end{aligned}$$

Then A being finite dimensional implies S is invertible and so Equation (11) follows.

For $x \in X$ let L_x be the left action of x on A or A^* . We also need the following equation

$$(12) \quad \overleftarrow{\text{ev}}_A \circ (\text{Id}_{A^*} \otimes L_x) \circ f_{\lambda \otimes 1_A} = \lambda(x) \cdot \epsilon$$

which is proved, for all $x \in A$, in the proof of Lemma 3.8 of [De Renzi et al. 2018].

Let g be the morphism on the left side of Equation (3). Let $x, y \in A$ then

$$\begin{aligned}
 g(x \otimes y) &= (\text{Id}_A \otimes \overleftarrow{\text{ev}}_A)(\Lambda \otimes \text{Id}_A)(\text{Id}_A \otimes f_{\lambda \otimes 1_A} \otimes \text{Id}_A)(x \otimes \Delta(y)) \\
 &= \Lambda_{(1)}x \otimes (\overleftarrow{\text{ev}}_A(L_{\Lambda_{(2)}} \otimes \text{Id}_A)f_{\lambda \otimes 1_A}(y_{(1)}))y_{(2)} \\
 &= \Lambda_{(1)}x \otimes (\overleftarrow{\text{ev}}_A(\text{Id}_{A^*} \otimes L_{S(\Lambda_{(2)})})f_{\lambda \otimes 1_A}(y_{(1)}))y_{(2)} \\
 &= \Lambda_{(1)}x \otimes \lambda(S(\Lambda_{(2)}))\epsilon(y_{(1)})y_{(2)} \\
 &= \lambda(S(\Lambda_{(2)}))\Lambda_{(1)}x \otimes y \\
 &= x \otimes y,
 \end{aligned}$$

where the fourth equality follows from Equation (12), the fifth from definition of the Hopf algebra, i.e., $(\epsilon \otimes \text{Id}_A)\Delta = \text{Id}_A$ and the last equality follows from Equation (11). \square

Thus, the results of this subsection complete the proof of Theorem 2.6.

6C. Proof of Theorem 2.7. In the last subsection we showed $A\text{-mod}$ satisfies the hypotheses of Theorems 2.4 and 2.5. Thus we have:

Theorem 6.4. *The category of finite dimensional modules over a finite dimensional unibalanced unimodular pivotal Hopf algebra A gives rise to an invariant of bichrome handlebody graphs F' and an invariant of 3-manifolds $\mathcal{K}_{A\text{-mod}}$.*

We will now show how to compute the invariant $\mathcal{K}_{A\text{-mod}}$. Let M be a closed connected orientable 3-manifold M . Here we use the generator $G = A$ and choose \mathbf{O}_G to be the ribbon graph formed by the braid closure of the coupon filled with $h = f_\Lambda \circ \epsilon$, see Section 2B. Let (H_α, Γ) be a bichrome diagram determined by a Heegaard splitting $M = H_\alpha \cup_\Sigma H_\beta$ with lower and upper minimal reducing sets of circles $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_1, \dots, \beta_g\}$, respectively. Using the blue graph \mathbf{O}_G and the chromatic morphism we change all the red circles $\{\beta_i\}$ into a blue graph. Now $\{\alpha_i\}$ is a minimal reducing set of circles on H_α and so using Property (4) of Theorem 2.2 we can cut along the discs in H_α bounded by these circles to obtain a graph Γ' on the boundary of the 3-ball. By definition

$$(13) \quad \mathcal{K}_{A\text{-mod}}(M) = F'(B^3, \Gamma').$$

See Figure 4 for an example.

For the rest of the subsection, we assume A is involutive (the square of the antipode is the identity map). In this case, the invariant $\mathcal{K}_{A\text{-mod}}(M)$ can be computed as follows. To do this we need the following two lemmas. The proof of the first is straightforward and we leave it to the reader.

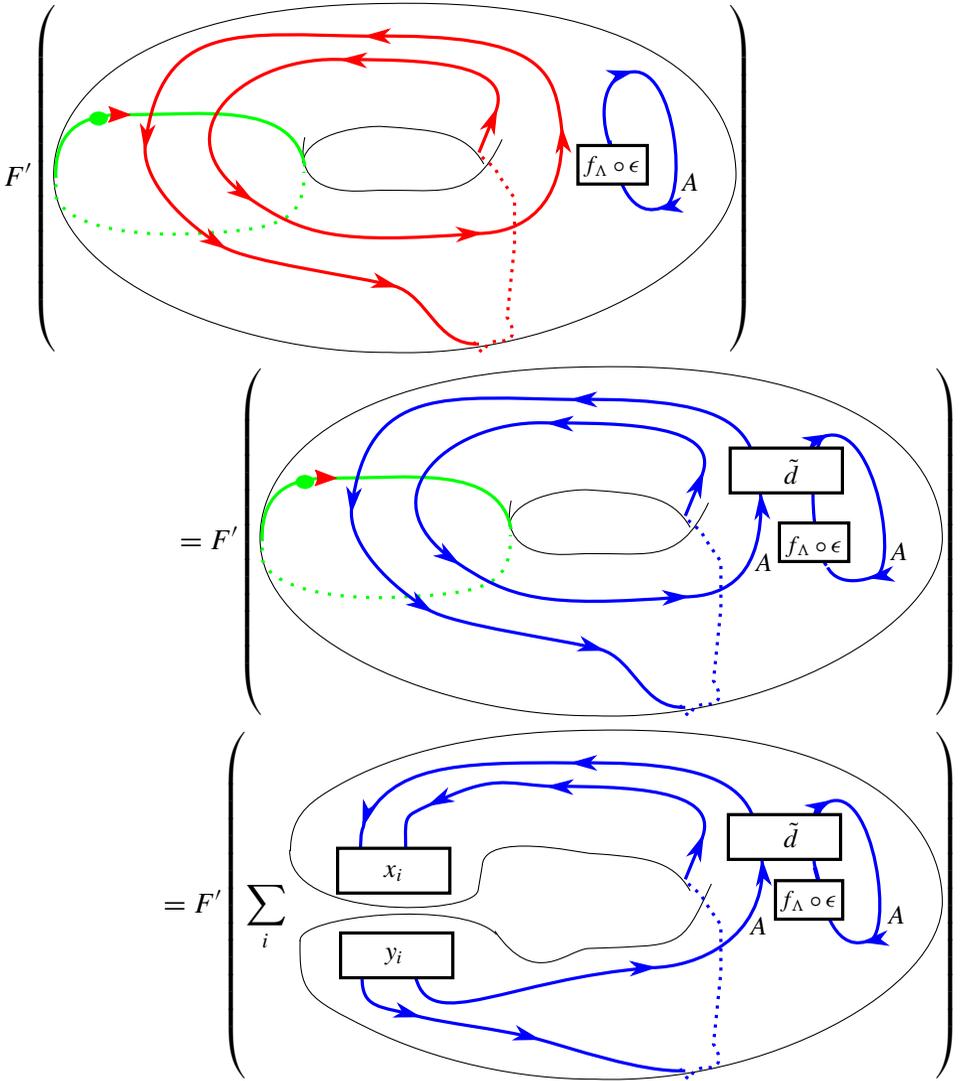


Figure 4. Here we compute $\mathcal{K}_{A\text{-mod}}$ for $L(2, 1)$. The first equality comes from using the chromatic morphism to transform the red circle into a blue graph. Then cutting along the meridian (depicted on the left) one obtains a graph $\Gamma' \subset \partial B^3$. By definition $\mathcal{K}_{A\text{-mod}}(L(2, 1)) = F'(B^3; \Gamma')$.

Lemma 6.5. *Since A is involutive then we can choose the pivotal structure of $A\text{-mod}$ to be trivial and there exist a forgetful pivotal functor from $A\text{-mod}$ to the category of vector spaces $\text{Vect}_{\mathbb{k}}$. Moreover, we have the following commutative*

diagram:

$$\begin{array}{ccc}
 \text{Rib}_{A\text{-mod}} & \xrightarrow{\text{Forgetful}} & \text{Rib}_{\mathcal{V}\text{ect}_{\mathbb{k}}} \\
 \downarrow F & & \downarrow F_{\mathcal{V}\text{ect}_{\mathbb{k}}} \\
 A\text{-mod} & \xrightarrow{\text{Forgetful}} & \mathcal{V}\text{ect}_{\mathbb{k}}
 \end{array}$$

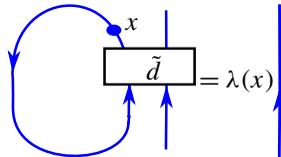
where the horizontal arrows are given by the forgetful functors and the vertical arrows are the pivotal functor from planar colored ribbon graphs Rib to $A\text{-mod}$ or $\mathcal{V}\text{ect}_{\mathbb{k}}$.

The next lemma says that evaluating red circles with the chromatic morphism is essentially the integral.

Lemma 6.6. *For all $x \in A$ the following equality holds in $\mathcal{V}\text{ect}_{\mathbb{k}}$*

$$(\overleftarrow{e\nu}_A \otimes \text{Id}_A)(\text{Id}_{A^*} \otimes L_x \otimes \text{Id}_A)(\text{Id}_{A^*} \otimes \tilde{d})(\overrightarrow{\text{coe}\nu}_A \otimes \text{Id}_A) = \lambda(x) \text{Id}_A$$

this equation is represented pictorially as



where all the strands are colored by A and the dot (or bead) is labeled with x .

Proof. The lemma follows from the definition of the chromatic morphism and Equation (12). □

Now we are ready to compute $\mathcal{K}_{A\text{-mod}}(M)$ when A is involutive and prove [Theorem 2.7](#). To do this we continue the computations given above. In particular, by Equation (13) it suffices to determine the value of $F'(B^3, \Gamma')$ as follows. Let T_A be a $(1,1)$ -ribbon graph whose closure is Γ' . Then Equation (8) implies

$$F'(B^3, \Gamma') = t_A(F(T_A)).$$

Recall the $A\text{-mod}$ morphism $f_\Lambda \circ \epsilon$ defined at the beginning of [Section 6B](#). Since $t_A(f_\Lambda \circ \epsilon) = 1$ the proof of [Theorem 2.7](#) is concluded after proving the following claim:

Claim 6.7. $F(T_A) = \text{Ku}_A(M) \cdot (f_\Lambda \circ \epsilon).$

Proof of Claim 6.7. [Lemma 6.5](#) implies that

$$\text{Forgetful}(F(T_A)) = F_{\mathcal{V}\text{ect}_{\mathbb{k}}}(\text{Forgetful}(T_A)).$$

Thus, it suffices to prove

$$(14) \quad F_{\mathcal{V}\text{ect}_{\mathbb{k}}}(\text{Forgetful}(T_A)) = \text{Ku}_A(M) \cdot \text{Forgetful}(f_\Lambda \circ \epsilon)$$

since Forgetful is a faithful functor. For the rest of the proof, we will work in the category of $\mathcal{Vect}_{\mathbb{k}}$, allowing us to consider left multiplication by an element $x \in A$ as a morphism in $\mathcal{Vect}_{\mathbb{k}}$. Moreover, to simplify notation we identify each morphism in $A\text{-mod}$ with its underlying linear map.

Recall, by definition, when cutting one of the α -circles we replace the blue circles crossing the disk with a ordered matching pair of coupons filled with a sum

$$\sum_i x_i \otimes_{\mathbb{k}} y_i$$

as in Equation (2). By Lemma 6.2 this morphism is equal to the left action by Λ . We now explain how to represent this action with elements called beads.

For each circle in $\{\alpha_i\}$ choose an orientation and a base point (the following procedure is independent of these choices). As above, use the chromatic morphism to change all the red circles $\{\beta_i\}$ into blue graphs (note this induces an orientation on each circle β_i). Then instead of cutting along the bounding circles $\{\alpha_i\}$ we decorate the blue graph with certain sums of elements of A called beads as follows. Each circle α_j intersects the set of upper circles $\{\beta_i\}$ transversely. Let c_1, \dots, c_m be these crossings in the order that they are encountered if we travel from the base point along α_j in the positively oriented direction. For the k -th crossing, set $p_k = 0$ if the tangent vectors of the lower circle α_j and the upper circle, in that order, form a positively oriented basis for the tangent space at c_k , otherwise set $p_k = 1$. We assign to the k -th crossing the bead $S^{p_k}(\Lambda_{(k)})$ where

$$\Delta^m(\Lambda) = \Lambda_{(1)} \otimes \dots \otimes \Lambda_{(m)}.$$

Notice, since $\mathcal{Vect}_{\mathbb{k}}$ has a trivial ribbon structure, then the left hand side of Equation (14) only depends on the abstract graph of T_A or equivalently Γ' . Thus, we can compute $F_{\mathcal{Vect}_{\mathbb{k}}}(\text{Forgetful}(T_A))$ as follows. Each upper circle β_i has an orientation and a base point (determined by where the chromatic morphism is applied). Starting at this point and following the orientation we collect the beads to obtain a word a_i of A written from right to left. Doing this for all upper circles we obtain g beads: a_1, \dots, a_g . For each β_i , apply Lemma 6.6 where x is the bead a_i to obtain

$$F_{\mathcal{Vect}_{\mathbb{k}}}(\text{Forgetful}(T_A)) = \lambda(a_1)\lambda(a_2) \cdots \lambda(a_g) \cdot (f_{\Lambda} \circ \epsilon).$$

For an example of this computation see Figure 5. But when A is involutive, by definition the Kuperberg invariant is

$$\text{Ku}_A(M) = \lambda(a_1)\lambda(a_2) \cdots \lambda(a_g).$$

Thus, we have completed the proof of the claim. □

$$\begin{aligned}
 & \text{Forgetful } F \left(\sum_i \left(\text{Diagram 1} \right) \right) \\
 &= F_{\mathcal{V}\text{ect}_{\mathbb{k}}} \left(\sum_i \left(\text{Diagram 2} \right) \right) \\
 &= F_{\mathcal{V}\text{ect}_{\mathbb{k}}} \left(S(\Lambda_{(1)}) S(\Lambda_{(2)}) \left(\text{Diagram 3} \right) \right) \\
 &= F_{\mathcal{V}\text{ect}_{\mathbb{k}}} \left(S(\Lambda_{(2)}) S(\Lambda_{(1)}) \left(\text{Diagram 4} \right) \right) = \lambda(S(\Lambda_{(2)}) S(\Lambda_{(1)})) \cdot (f_{\Lambda} \circ \epsilon).
 \end{aligned}$$

Figure 5. In Figure 4 we showed $\mathcal{K}_{A\text{-mod}}(L(2, 1)) = F'(B^3, \Gamma')$; here we continue this computations when A is involutive. The first drawing depicts the 1-1 tangle obtained by cutting Γ' along the black disk. In the first equality, we pass to the category $\mathcal{V}\text{ect}_{\mathbb{k}}$ using Lemma 6.5. Since $\mathcal{V}\text{ect}_{\mathbb{k}}$ has a symmetric braiding, the crossings make sense. The third equality re-expresses the cutting through beads (see Lemma 6.2). Finally, we collect the beads and apply Lemma 6.6.

7. The semisimple case and the TV invariant

Here we prove [Theorem 2.8](#). Recall the notation of [Section 2D](#).

Proof of [Theorem 2.8](#). Let \mathcal{T} be a triangulation of a closed connected orientable 3-manifold M . Let t be a maximal tree of edges of \mathcal{T} , then t contains all the v vertices of \mathcal{T} . We define a Heegaard diagram as follows. Let H_β be a regular neighborhood of the 1-skeleton of \mathcal{T} and H_α its complement, then $M = H_\alpha \cup H_\beta$ is a Heegaard splitting. The β -circles are meridians of the edges of \mathcal{T} not in the tree t and the α -circles bound discs formed by the 2-dimensional faces of \mathcal{T} . As in [Section 5B](#) this Heegaard diagram gives a bichrome handlebody graph (H_α, Γ) where the red graph is the β -circles.

By definition of the chromatic morphism we see that the value of a red unknot is the nonzero dimension D of \mathcal{C} (see [Equation \(7\)](#)). Thus, the value of $F'(H_\alpha, \Gamma)$ does not change if we multiply it by $1/D$ and at the same time add a red unknot to the bichrome handlebody graph (H_α, Γ) . With this in mind, we construct a new bichrome handlebody graph as follows. Starting with (H_α, Γ) place a red unknot on the boundary of the neighborhood of each edge of the tree t . Let e be a leaf of t , i.e., an edge of t such that one of its vertices has degree 1. Then we can slide the new red unknot associated to e over all the red meridians of the edges adjacent to this vertex. After these slidings the new unknot becomes a red meridian around e . Continuing this process on the leaves of $t \setminus \{e\}$, we obtain a bichrome handlebody graph (H_α, Γ') where each neighborhood of an edge of \mathcal{T} has a red meridian. Combining the fact that F' satisfies the sliding property for red circles and the discussion above we have

$$(15) \quad \mathcal{K}_{\mathcal{C}}(M) = F'(H_\alpha, \Gamma) = \frac{1}{D^{v-1}} F'(H_\alpha, \Gamma'),$$

where v is the number of vertices of \mathcal{T} .

Now we compute $F'(H_\alpha, \Gamma')$. Use the chromatic morphism to make all the red circles blue. By definition of the chromatic morphism each red circle is changed to a blue circle colored with G or equivalently the weighted sum $\sum_i \text{qdim}(S_i) S_i$ (note this happens for a meridian of each edge of \mathcal{T}). Then cutting along the discs formed by the 2-dimensional faces of \mathcal{T} we obtain a set of spherical tetrahedra (indexed by the set of tetrahedra of \mathcal{T}) whose edges are all colored by G and whose four 3-legged coupons are filled with morphisms coming from the cutting. Using the fact that G splits as an orthogonal direct sum of simple objects one can see that each of these spherical tetrahedra is a sum indexed by colorings of the edges of T by elements of $\{S_i\}$. Moreover, each component of this sum is proportional to the corresponding usual $6j$ -symbol. Therefore, we have

$$\text{TV}_{\mathcal{C}}(\mathcal{T}) = \frac{1}{D^v} F'(H_\alpha, \Gamma'),$$

where $\text{TV}_{\mathcal{C}}(M) = \text{TV}_{\mathcal{C}}(\mathcal{T})$ is the TV invariant associated to the triangulation \mathcal{T} of M , see [Turaev 1994]. Combining this with Equation (15), we have $\frac{1}{D}\mathcal{K}_{\mathcal{C}}(M) = \text{TV}_{\mathcal{C}}(M)$. \square

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