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# ISOTYPIC MULTIHARMONIC POLYNOMIALS AND GELBART–HELGASON RECIPROCITY

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**A result of Gelbart and Helgason that describes the harmonic analysis of the space of functions on certain Stiefel manifolds and a result of Folland that describes the harmonic analysis of the de Rham complex on a sphere are generalized to describe the harmonic analysis of the de Rham complex on certain Stiefel manifolds. Equivalently, the cobranching law from a fundamental representation of an orthogonal group to a larger orthogonal group is determined.**

## 1. Introduction

Let $n = n_1 + n_2$. The group $\mathrm{SO}(n)$ contains $\mathrm{SO}(n_1)$ and $\mathrm{SO}(n_2)$ as subgroups embedded in the upper-right and lower-left corners, respectively. These subgroups commute with one another and so if $(\rho, E_\rho)$ is a representation of $\mathrm{SO}(n)$ then the space $E_\rho^{\mathrm{SO}(n_2)}$ of $\mathrm{SO}(n_2)$-fixed vectors in $E_\rho$ affords a representation $\rho^{\mathrm{SO}(n_2)}$ of $\mathrm{SO}(n_1)$. Gelbart [1974] investigated the nature of $\rho^{\mathrm{SO}(n_2)}$ when $\rho$ is irreducible and $n_1 < n_2$. He identified those $\rho$ for which $\rho^{\mathrm{SO}(n_2)} \neq \{0\}$ and showed [Gelbart 1974, Proposition 2.2] that for these $\rho$ the dimension of $\rho^{\mathrm{SO}(n_2)}$ is equal to the dimension of a certain irreducible representation of $\mathrm{GL}(n_1)$ that is determined by $\rho$.

Gelbart's remarkable observation was one starting point for a substantial amount of subsequent research. A high point in this literature is the work of Gross and Kunze [1984]. In that paper, the authors introduce the concept of a bitriangular structure in a Lie group, develop a general theory of such structures, and then apply this theory to reprove Gelbart's result, improve the coincidence of dimensions to what they call a spatial isomorphism between $\rho^{\mathrm{SO}(n_2)}$ and the restriction of the associated $\mathrm{GL}(n_1)$ representation to $\mathrm{SO}(n_1)$, and also to generalize to many other situations with similar features (such as that of $\mathrm{U}(n_1)$ and $\mathrm{U}(n_2)$ as subgroups of $\mathrm{U}(n)$). In an appendix to Gross' and Kunze's paper it is explained, based on comments made by the referee, how their results may be understood in the framework of Kostant's work [1963] on group representations on polynomial rings. Conceptually, one may think of Gross' and Kunze's results as giving a kind of algebraic Poisson

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transform from the space of the $\mathrm{GL}(n_1)$ representation to the space of $\rho^{\mathrm{SO}(n_2)}$. This is clearer in the subsequent work of Johnson [1987], who generalizes the result still further and gives a shorter proof of the generalization not using the machinery of bitriangular systems. In the same work, Johnson also explains that his main result is a generalization of a famous result of Helgason's [1970]. Knapp [2001] reviews the history of this circle of ideas, identifying Gelbart's observation and Helgason's results as the twin roots from which they grow.

Now suppose that $(\nu, E_\nu)$ is an irreducible representation of $\mathrm{SO}(n_2)$. Then one may seek to generalize the above-described results by considering the isotype $\rho^{\mathrm{SO}(n_2),\nu}$ of $\nu$ in $\rho$. Since $\mathrm{SO}(n_1)$ and $\mathrm{SO}(n_2)$ commute, the isotype is a representation of $\mathrm{SO}(n_1)$. A successful generalization would take the form

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}\big(\nu, \mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho)\big) \cong \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{SO}(n_1)}(\Pi_{\rho,\nu}) \tag{1-1}$$

as representations of $\mathrm{SO}(n_1)$, where $\Pi_{\rho,\nu}$ is some representation of $\mathrm{GL}(n_1)$ depending on $\rho$ and $\nu$. In light of the history sketched above, the author proposes to use the name Gelbart–Helgason reciprocity for results of this form and also for similar results associated with other pairs of subgroups such as those mentioned above.

The main results of the current work are Theorems 5.2 and 5.4. The former is a Gelbart–Helgason reciprocity result of the form

$$\mathrm{Hom}_{\mathrm{O}(n_2)}\big(\nu_a, \mathrm{res}^{\mathrm{O}(n)}_{\mathrm{O}(n_2)}(\rho)\big) \cong \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{O}(n_1)}(\Pi_{\rho,\nu_a}),$$

where $\nu_a$ is the $a$-th exterior power of the standard representation of $\mathrm{O}(n_2)$ with $a \le n_2/2$, $\rho$ is any irreducible representation of $\mathrm{O}(n)$, and $\Pi_{\rho,\nu_a}$ is described explicitly in terms of $\rho$ and $a$. The latter is a similar result with special orthogonal groups in place of orthogonal groups. The original example of Gelbart–Helgason reciprocity that was described above follows from the $a = 0$ case of the second of these results. Just as with the original result, both of these results require restrictions on $n_1$ and $n_2$: roughly, $n_2$ must be large enough relative to $n_1$.

Gelbart's original observation arose in the context of analysis on Stiefel manifolds, where it was used to decompose the space of functions on the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n_2)$ under the action of $\mathrm{SO}(n)$. Theorem 5.4 may be interpreted in a similar way as giving the decomposition of the space of sections of the bundle of differential $a$-forms on the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(n_2)$. This interpretation is obtained by a standard argument using Frobenius reciprocity. In Gelbart's original approach to his result another theorem of Helgason's [1963] played an important role. This theorem ensures that a surface harmonic on $\mathrm{SO}(n)/\mathrm{SO}(n_2)$ (meaning a function belonging to the isotypic subspace of some irreducible representation of $\mathrm{SO}(n)$) extends to a solid harmonic (meaning a multiharmonic polynomial on a suitable space of matrices). This allowed Gelbart to obtain his result by analyzing the space of solid harmonics. The corresponding fact does not appear explicitly in our approach because we can take advantage of subsequent progress in the theory of harmonic tensors due to Howe and to Kashiwara and Vergne.

The case where $n_1 = 1$, so that the Stiefel manifold referred to in the previous paragraph is a sphere, was previously studied by Folland [1989], who gave the complete harmonic analysis of the de Rham complex on a sphere. In a similar way, Theorem 5.4 gives the complete harmonic analysis of the de Rham complex on the associated Stiefel manifold provided that $2n_1 \leq n_2$. In fact, Folland did more than just determine the abstract harmonic analysis of the de Rham complex on a sphere. Namely, he gave explicit embeddings of the abstract irreducible constituents into the space of differential forms. The motivation for the work reported here was to obtain similarly explicit models in the more general case in order to facilitate computations involving the $(\mathrm{SO}(n_2), \nu_a)$-isotypes in representations of $\mathrm{SO}(n)$. Note that Gross and Kunze [1984] did give a model of the isotypic subspace in the case where $a = 0$, but we found their model unsuitable for the type of computation that is the present goal, whereas the models that follow from the method of proof used here are suitable. We chose not to emphasize this aspect of the problem in this work, as it might prove distracting to do so and the explicit models will be explained at length elsewhere.

It remains to describe the approach that we use to prove Theorems 5.2 and 5.4. We make extensive use of the classic results of Howe [1989] and Kashiwara and Vergne [1978], for which Howe's inspiring essay [1995] is a concise source. Specifically, from this essay, we use Theorem 2.1.2 (which Howe calls $(\mathrm{GL}(n), \mathrm{GL}(m))$-duality), and Theorem 3.5.2 and Proposition 3.6.3 (which together constitute $(\mathrm{O}(n), \mathrm{GL}(m))$-duality and which these days often go under the name of Howe duality). The necessary results are recalled in a suitable form in Section 3 after a brief discussion of multiharmonic polynomials in Section 2. Using these results, one could prove that it is always possible to find an explicit $\Pi_{\rho,\nu}$ such that the left-hand side of (1-1) is a quotient of the right-hand side. This statement by itself could be of some use, but the central issue is to show that the kernel of the surjection is zero under suitable hypotheses. The heart of the proof is Theorem 4.1, which expresses the $(\mathrm{O}(n_2), \nu_a)$-isotype in a space of multiharmonic polynomials as a quotient of a simpler space and gives a numerical criterion for the kernel of the surjection to be zero. The method of proof is elementary, although it raises an interesting concrete problem in commutative algebra that we do not need to solve completely for the present application; a complete solution would be welcome. From this result, we deduce Theorem 5.2, and then Theorem 5.4 follows by purely representation-theoretic methods. Finally, we remark that this approach extends to all the other classical cases of Gelbart–Helgason reciprocity and results in similar generalizations of them. In fact, the proofs in other cases are somewhat simpler because we do not have to contend with a disconnected group such as $\mathrm{O}(n)$. A combined exposition proved awkward and so these results will appear elsewhere.

Finally, the author would like to thank the referee for helping him to improve the clarity of the introduction and for pointing out a helpful reference.

## 2. Background on multiharmonic polynomials

Let $V = \mathrm{Mat}(n, m)$ denote the space of $n$-by-$m$ matrices over a field $F$ of characteristic zero. The group $G = \mathrm{O}(n) \times \mathrm{GL}(m)$ acts on this space by $(g, h)M = gMh^{-1}$. The induced action on the algebra $F[V]$ of polynomials on $V$ is given by $((g, h) f)(M) = f((g, h)^{-1}M)$. For $l \in \mathbb{N}$, let $[l] = \{1, \ldots, l\}$. The space $V$ has a standard basis consisting of the matrix units $E_{ir}$ with $i \in [n]$ and $r \in [m]$. Let $x_{ir} \in F[V]$ be the coordinate dual to $E_{ir}$ and $x = [x_{ir}]$ be the matrix of these coordinates. The action of $G$ on the space of linear polynomials on $V$ may be summarized by the equation $[(g, h)x_{ir}] = g^\top xh$. Similarly, let $\partial_{ir}$ be the partial derivative operator with respect to $x_{ir}$ and $\partial = [\partial_{ir}]$. The algebra of constant-coefficient differential operators on $V$ is naturally identified with the symmetric algebra $\mathrm{sym}(V)$ and so has an induced action of $G$. This action satisfies $[(g, h)\partial_{ir}] = g\partial h^{-1}$.

We shall identify $\mathrm{O}(n)$ and $\mathrm{GL}(m)$ as subgroups of $G$ in the standard way. Define polynomials $\varphi_{rs}$ for $r, s \in [m]$ by $[\varphi_{rs}] = x^\top x$ and differential operators $\Delta_{rs}$ by $[\Delta_{rs}] = \partial^\top \partial$. Then $[(g, h)\varphi_{rs}] = h^\top [\varphi_{rs}]h$ and $[(g, h)\Delta_{rs}] = h^{-\top}[\Delta_{rs}]h^{-1}$. The polynomials $\varphi_{rs}$ generate the algebra $F[V]^{\mathrm{O}(n)}$ of $\mathrm{O}(n)$-invariant polynomials on $V$ and the differential operators $\Delta_{rs}$ generate the algebra of $\mathrm{O}(n)$-invariant constant-coefficient differential operators on $V$. We call a polynomial $h \in F[V]$ *multiharmonic* if $\Delta_{rs}h = 0$ for all $r, s \in [m]$ and denote by $\mathcal{H} < F[V]$ the space of all multiharmonic polynomials. In addition, we denote by $\mathcal{I}$ the ideal of $F[V]$ generated by $\varphi_{rs}$ for $r, s \in [m]$. It follows from the identities noted above for the transformation of $\varphi_{rs}$ and $\Delta_{rs}$ under the action of $G$ that both $\mathcal{H}$ and $\mathcal{I}$ are $G$-invariant.

If $p \in F[V]$ then we denote by $p(\partial)$ the operator that results from substituting $\partial_{ir}$ for $x_{ir}$ in $p$. We then introduce a bilinear form on $F[V]$ by

$$\langle p, q\rangle = \mathrm{CT}(p(\partial)q(x)),$$

where CT denotes the constant term. If $(g, h) \in G$ then we have

$$\langle (g, h)p, q\rangle = \langle p, (g^\top, h^\top)q\rangle$$

and, in particular, the bilinear form is $\mathrm{O}(n)$-invariant. If $m = x^a$ (using multi-index notation) is a monomial then we have $\langle m, q\rangle = a!c_m(q)$, where $c_m(q)$ denotes the coefficient of the monomial $m$ in $q$. It follows from this that the restriction of the bilinear form to the space of polynomials of given total degree is nondegenerate. Moreover, we have $\langle pr, q\rangle = \langle p, r(\partial)q\rangle$ for all $p, q, r \in F[V]$.

**Lemma 2.1.** *We have* $\mathcal{H} = \mathcal{I}^\perp$ *and* $F[V] = \mathcal{H} \oplus^\perp \mathcal{I}$.

*Proof.* The nondegeneracy of the bilinear form implies that $h \in \mathcal{H}$ if and only if $\langle p, \Delta_{ir} h\rangle = 0$ for all $p \in F[V]$. This is equivalent to $\langle p\varphi_{ir}, h\rangle = 0$ for all $p$, which in turn is equivalent to $\langle \varphi, h\rangle = 0$ for all $\varphi \in \mathcal{I}$. This establishes the first claim. For the second, we initially assume that $F = \mathbb{Q}$. Then the bilinear form is positive definite. Both $\varphi_{ir}$ and $\Delta_{ir}$ are homogeneous with respect to total degree and it follows that both $\mathcal{I}$ and $\mathcal{H}$ are homogeneous. Using a subscript to denote total degree, we have $F[V]_k = \mathcal{I}_k^{\perp} \oplus^{\perp} \mathcal{I}_k = \mathcal{H}_k \oplus^{\perp} \mathcal{I}_k$, from which the second claim follows in this case. In general, $\varphi_{ir} \in \mathbb{Q}[V]$ and $\Delta_{ir}$ preserves $\mathbb{Q}[V]$ and it follows that $\mathcal{I} = F \otimes_{\mathbb{Q}} \mathcal{I}_{\mathbb{Q}}$ and $\mathcal{H} = F \otimes_{\mathbb{Q}} \mathcal{H}_{\mathbb{Q}}$ within $F[V] = F \otimes_{\mathbb{Q}} \mathbb{Q}[V]$. Thus the second claim follows in general by extension of scalars. □

We denote by $\boldsymbol{P} : F[V] \to \mathcal{H}$ the orthogonal projection map associated with the decomposition $F[V] = \mathcal{H} \oplus^{\perp} \mathcal{I}$. Note that the $G$-invariance of $\mathcal{H}$ and $\mathcal{I}$ implies that $\boldsymbol{P}$ is $G$-equivariant.

## 3. Background on representations

The purpose of this section is to recall some results about representations of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ in a form suitable for our use and to summarize some results of Howe and Kashiwara and Vergne that we shall require. As mentioned in the introduction, Howe's essay [1995] is a suitable source for much of this material. In particular, Theorem 2.1.2 of that work is recalled as (3-2), Proposition 3.6.3 is recalled as (3-1), and the description of the singular vectors given below is drawn from the discussion preceding the latter result. The other facts recalled here, such as the relationship between the representations of $\mathrm{SO}(n)$ and those of $\mathrm{O}(n)$, are common knowledge in representation theory. The terminology and notation introduced in this section will be used subsequently.

In this section we assume that $F$ is algebraically closed. Let $l = \lfloor n/2 \rfloor$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{so}(n)$ and $\mathfrak{t} \oplus \mathfrak{u}$ be a Borel subalgebra of $\mathfrak{so}(n)$ with nilradical $\mathfrak{u}$. Let $T < \mathrm{O}(n)$ be the intersection of the normalizers of $\mathfrak{t}$ and $\mathfrak{u}$. There is an element $\tau \in \mathrm{O}(n)$ of order two such that $T = \{e, \tau\} \ltimes T^{\circ}$. By a representation we shall mean a finite-dimensional rational representation over $F$. If $(\rho, E_\rho)$ is an irreducible representation of $\mathrm{SO}(n)$ then the space $E_\rho^{\mathfrak{u}}$ of *singular vectors* in $E_\rho$ has dimension one and the isomorphism class of $(\rho, E_\rho)$ is determined by the action of $T^{\circ}$ on $E_\rho^{\mathfrak{u}}$. We may equally well consider the action of $\mathfrak{t}$ on $E_\rho^{\mathfrak{u}}$ and the functional corresponding to this action is called the *highest weight* of $\rho$. If $(\rho, E_\rho)$ is an irreducible representation of $\mathrm{O}(n)$ then the space $E_\rho^{\mathfrak{u}}$ has dimension either one or two and the isomorphism class of $(\rho, E_\rho)$ is determined by the action of $T$ on $E_\rho^{\mathfrak{u}}$. Moreover, $E_\rho^{\mathfrak{u}}$ is an irreducible representation of $T$ with this action.

Let $(\rho, E_\rho)$ be an irreducible representation of $\mathrm{SO}(n)$ whose highest weight is stable under $\tau$. Then $(\rho, E_\rho)$ extends to a representation of $\mathrm{O}(n)$ on the same space in two different ways. We denote these extensions by $(\rho_+, E_{\rho_+})$ and $(\rho_-, E_{\rho_-})$, where $E_{\rho_+} = E_{\rho_-} = E_\rho$, $\rho_+(\tau)|_{E_\rho^{\mathfrak{u}}} = 1$ and $\rho_-(\tau)|_{E_\rho^{\mathfrak{u}}} = -1$. Now let $(\rho, E_\rho)$ be an irreducible representation of $\mathrm{SO}(n)$ whose highest weight is not stable under $\tau$. Then there is a unique irreducible representation $(\hat\rho, E_{\hat\rho})$ of $\mathrm{O}(n)$ such that the restriction of $(\hat\rho, E_{\hat\rho})$ to $\mathrm{SO}(n)$ is isomorphic to the direct sum of $(\rho, E_\rho)$ and $(\rho^\tau, E_{\rho^\tau})$, where $\rho^\tau$ denotes the composition of $\rho$ with conjugation by $\tau$ and $E_{\rho^\tau} = E_\rho$. A model of this representation is obtained by inducing $(\rho, E_\rho)$ from $\mathrm{SO}(n)$ to $\mathrm{O}(n)$. Every irreducible representation of $\mathrm{O}(n)$ arises by one of these two constructions from an irreducible representation of $\mathrm{SO}(n)$ or, more precisely, from the orbit under $\{e, \tau\}$ of an irreducible representation of $\mathrm{SO}(n)$.

Suppose that $n$ is odd. Then $\tau$ lies in the center of $\mathrm{O}(n)$ and so $T$ is abelian. Thus every irreducible representation of $\mathrm{O}(n)$ has the form $(\rho_\pm, E_{\rho_\pm})$ where $(\rho, E_\rho)$ is an irreducible representation of $\mathrm{SO}(n)$. In the standard model of the root system of type $\mathrm{B}_l$, the highest weight of an irreducible representation of $\mathrm{SO}(n)$ has the form $\lambda_1\varepsilon_1 + \cdots + \lambda_l\varepsilon_l$ with $\lambda_1, \ldots, \lambda_l \in \mathbb{N}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. We associate this weight with the partition $[\lambda_1, \ldots, \lambda_l]$. In this way, the highest weight of an irreducible representation of $\mathrm{SO}(n)$ is associated to a partition with at most $l$ parts or, equivalently, to a Young diagram of depth at most $l$. If $D$ is a Young diagram then we denote its depth by $\mathrm{dp}(D)$. Suppose that $D$ is a Young diagram with $\mathrm{dp}(D) \leq l$. We define the diagram $D'$ to be the result of changing the length of the first column in $D$ from $\mathrm{dp}(D)$ to $n - \mathrm{dp}(D)$. Note that $\mathrm{dp}(D') = n - \mathrm{dp}(D)$ is at least $l + 1$ and at most $n$. The diagram $D$ may be recovered from $D'$ by repeating the operation that produced $D'$ from $D$. We call a diagram $D$ *special* if $\mathrm{dp}(D) \leq n$ and if $\mathrm{dp}(D) > l$ then the $k$-th row in $D$ has length 1 for all $k > n - \mathrm{dp}(D)$. Let $(\rho, E_\rho)$ be the irreducible representation of $\mathrm{SO}(n)$ associated to the diagram $D$. Define the *sign* of the diagram $D$ to be

$$\mathrm{sgn}(D) = (-1)^{\lambda_1 + \cdots + \lambda_l}.$$

We associate the diagram $D$ to the irreducible representation $(\rho_{\mathrm{sgn}(D)}, E_{\rho_{\mathrm{sgn}(D)}})$ of $\mathrm{O}(n)$ and the diagram $D'$ to the irreducible representation $(\rho_{-\,\mathrm{sgn}(D)}, E_{\rho_{-\,\mathrm{sgn}(D)}})$ of $\mathrm{O}(n)$. In this way, we obtain a one-to-one correspondence between isomorphism classes of irreducible representations of $\mathrm{O}(n)$ and special diagrams.

Now suppose that $n$ is even. Then $T$ is not abelian and two-dimensional spaces of singular vectors are possible. In the standard model of the root system of type $\mathrm{D}_l$, the highest weight of an irreducible representation of $\mathrm{SO}(n)$ has the form $\lambda_1\varepsilon_1 + \cdots + \lambda_l\varepsilon_l$ with $\lambda_1, \ldots, \lambda_{l-1} \in \mathbb{N}$, $\lambda_l \in \mathbb{Z}$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l-1} \geq |\lambda_l|$. We associate this weight to the partition $[\lambda_1, \lambda_2, \ldots, \lambda_{l-1}, |\lambda_l|]$. The action of $\tau$ changes the sign of $\lambda_l$ and so in this way we associate the $\{e, \tau\}$-orbit of the weight to a partition with at most $l$ parts or, equivalently, to a Young diagram of depth at most $l$. The operation $D \mapsto D'$ on special Young diagrams has fixed points in this case. We have $D = D'$ precisely when $\mathrm{dp}(D) = l$. These diagrams are those corresponding to weights that are not fixed by $\tau$ and we associate such a diagram to $(\hat{\rho}, E_{\hat{\rho}})$ where $(\rho, E_\rho)$ is an irreducible representation of $\mathrm{SO}(n)$ whose highest weight is associated with the diagram. Those diagrams $D$ with $\mathrm{dp}(D) < l$ correspond to weights that are fixed by $\tau$. If we have such a diagram $D$ and $(\rho, E_\rho)$ is an irreducible representation of $\mathrm{SO}(n)$ whose highest weight corresponds to $D$ then we associate the representation $(\rho_+, E_{\rho_+})$ of $\mathrm{O}(n)$ to the diagram $D$ and the representation $(\rho_-, E_{\rho_-})$ of $\mathrm{O}(n)$ to the diagram $D'$. In this way, we obtain a one-to-one correspondence between isomorphism classes of irreducible representations of $\mathrm{O}(n)$ and special diagrams, just as in the case where $n$ is odd.

There is a similar, but simpler, correspondence between certain irreducible representations of $\mathrm{GL}(m)$ and Young diagrams of depth at most $m$. We briefly recall this because we shall need the notation below. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{gl}(m)$ and $\mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{gl}(m)$ with nilradical $\mathfrak{n}$. If $(\pi, E_\pi)$ is an irreducible representation of $\mathrm{GL}(m)$ then the space $E_\pi^{\mathfrak{n}}$ of singular vectors is one-dimensional and the isomorphism class of $(\pi, E_\pi)$ is determined by the action of $\mathfrak{h}$ on $E_\pi^{\mathfrak{n}}$. The functional associated with this action is the highest weight of $(\pi, E_\pi)$. In the standard model, this weight has the form $\lambda_1\varepsilon_1 + \cdots + \lambda_m\varepsilon_m$ with $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$ and $\lambda_1 \geq \cdots \geq \lambda_m$. The representation $(\pi, E_\pi)$ is said to be *polynomial* if $\lambda_m \geq 0$. If $(\pi, E_\pi)$ is an irreducible polynomial representation of $\mathrm{GL}(m)$ then we associate $(\pi, E_\pi)$ to the partition $[\lambda_1, \ldots, \lambda_m]$ or equivalently to the associated Young diagram.

If $D$ is a special Young diagram then we write $\sigma_D$ for the irreducible representation of $\mathrm{O}(n)$ that is associated to it. If $D$ is a Young diagram of depth at most $m$ then we write $\pi_{m,D}$ for the irreducible representation of $\mathrm{GL}(m)$ that is associated to it. The representation of $G = \mathrm{O}(n) \times \mathrm{GL}(m)$ that is afforded by $\mathcal{H}$ decomposes as

$$\mathcal{H} \cong \bigoplus_D \sigma_D \boxtimes \pi_{m,D}, \tag{3-1}$$

where the sum is over special diagrams of depth at most $m$. The action of $G$ on $F[V]$ that we have been considering is the restriction to $G$ of the action of $\mathrm{GL}(n) \times \mathrm{GL}(m)$ on $F[V]$ that is induced from the action $(h_1, h_2)M = h_1^{-\top} M h_2^{-1}$ of $\mathrm{GL}(n) \times \mathrm{GL}(m)$ on $V$. The representation of $\mathrm{GL}(n) \times \mathrm{GL}(m)$ that is afforded by $F[V]$ decomposes as

$$F[V] \cong \bigoplus_D \pi_{n,D} \boxtimes \pi_{m,D}, \tag{3-2}$$

where the sum is over diagrams of depth at most $\min(n, m)$.

We shall need some explicit information about the summand corresponding to each diagram in (3-1) and (3-2). To this end, fix $i \in F$ such that $i^2 = -1$ and introduce new coordinates $y_{jr},\ j \in [n],\ r \in [m]$, on $V$. If $n$ is even then these coordinates are defined by

$$y_{jr} = \begin{cases} x_{2j-1,r} + i x_{2j,r} & \text{if } 1 \le j \le n/2, \\ x_{2n+1-2j,r} - i x_{2n+2-2j,r} & \text{if } n/2+1 \le j \le n. \end{cases}$$

If $n$ is odd then they are instead defined by

$$y_{jr} = \begin{cases} x_{2j-1,r} + i x_{2j,r} & \text{if } 1 \le j \le (n-1)/2, \\ \sqrt{2} x_{nr} & \text{if } j = (n+1)/2, \\ x_{2n+1-2j,r} - i x_{2n+2-2j,r} & \text{if } (n+1)/2+1 \le j \le n. \end{cases}$$

Let $\eth_{jr}$ be the partial derivative operator with respect to $y_{jr}$. In its action on $F[V]$, the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{O}(n)$ is spanned by the operators

$$Z_{jk} = \sum_{r \in [m]} (y_{jr} \eth_{kr} - y_{n+1-k,r} \eth_{n+1-j,r})$$

for $j, k \in [n]$. Note that $Z_{n+1-k,n+1-j} = -Z_{jk}$. The subalgebra $\mathfrak{t}$ of $\mathfrak{so}(n)$ spanned by $Z_{11}, \dots, Z_{ll}$ is a Cartan subalgebra. If $\mathfrak{u}$ is the subalgebra spanned by $Z_{jk}$ with $j \le l$ and $j < k$ then $\mathfrak{t} \oplus \mathfrak{u}$ is a Borel subalgebra of $\mathfrak{so}(n)$ with nilradical $\mathfrak{u}$. With this choice of Borel subalgebra, the element $\tau \in \mathrm{O}(n)$ may be taken to act by $\tau y_{jr} = -y_{jr}$ for $j \in [n]$ if $n$ is odd and by

$$\tau y_{jr} = \begin{cases} y_{jr} & \text{if } j < n/2 \text{ or } j > n/2+1, \\ y_{n/2+1,r} & \text{if } j = n/2, \\ y_{n/2,r} & \text{if } j = n/2+1, \end{cases}$$

if $n$ is even.

Similarly, in its action on $F[V]$, the Lie algebra $\mathfrak{gl}(m)$ of $\mathrm{GL}(m)$ is spanned by the operators

$$W_{rs} = \sum_{j \in [n]} y_{jr} \eth_{js} = \sum_{j \in [n]} x_{jr} \partial_{js}$$

for $r, s \in [m]$. The subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(m)$ spanned by $W_{11}, \dots, W_{mm}$ is a Cartan subalgebra. If $\mathfrak{n}$ is the subalgebra spanned by $W_{rs}$ with $r < s$ then $\mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{gl}(m)$ with nilradical $\mathfrak{n}$.

Let $y = [y_{jr}]$ and $\delta_k(y)$ be the $k$-by-$k$ leading minor of the matrix $y$, with the convention that a minor that is larger than the matrix from which it is drawn is to be interpreted as 0. Thus $\delta_k(y) = 0$ if and only if $k > \min(n, m)$. With this convention in mind, we eschew saying "unless it is zero" repeatedly in the following discussion. If $a_1, \dots, a_l \in \mathbb{N}$ then the monomial $\delta^a(y) = \delta_1^{a_1}(y) \cdots \delta_l^{a_l}(y)$ is a singular vector in $\mathcal{H}$. If $j$ is the largest index such that $a_j \ne 0$ then we define

$$\eta(\delta^a(y)) = \delta_1^{a_1}(y) \cdots \delta_{j-1}^{a_{j-1}}(y) \delta_j^{a_j - 1}(y) \delta_{n-j}(y),$$

with the special case $\eta(1) = \delta_n$ when all the $a_j$ are zero. The monomial $\eta(\delta^a(y))$ is a singular vector in $\mathcal{H}$. The monomial $\tau(\delta^a(y))$ is also a singular vector in $\mathcal{H}$. Let $D$ be the diagram associated with the partition $[a_1+\cdots+a_l, a_2+\cdots+a_l, \ldots, a_{l-1}+a_l, a_l]$. If $n$ is odd then $\delta^a(y)$ spans the space of singular vectors in the summand $\sigma_D \boxtimes \pi_{m,D}$ in (3-1) corresponding to $D$ and $\eta(\delta^a(y))$ spans the space of singular vectors in the summand corresponding to $D'$. If $n$ is even then the same is true if $a_l = 0$, but if $a_l \neq 0$ then $\delta^a(y)$ and $\tau(\delta^a(y))$ span the space of singular vectors in the summand corresponding to $D = D'$.

The description of the singular vectors in the summands in (3-2) is similar but simpler. Let $\delta_k(x)$ be the $k$-by-$k$ leading minor of the matrix $x = [x_{jr}]$, with the same convention as above. Let $c = \min(m, n)$ and $a_1, \ldots, a_c \in \mathbb{N}$. The monomial $\delta^a(x) = \delta_1^{a_1}(x) \cdots \delta_c^{a_c}(x)$ is a singular vector in $F[V]$ and spans the space of singular vectors in the summand $\pi_{n,D} \boxtimes \pi_{m,D}$ in (3-2), where $D$ is the diagram associated to the partition $[a_1+\cdots+a_c, a_2+\cdots+a_c, \ldots, a_{c-1}+a_c, a_c]$.

## 4. Isotypic multiharmonic polynomials

In this section we initially assume that $F$ is algebraically closed. Let $n = n_1 + n_2$. Then $V = V_1 \oplus V_2$ where $V_1$ is the subspace of matrices of the form $\binom{M^{(1)}}{0}$ with $M^{(1)} \in \mathrm{Mat}(n_1, m)$ and $V_2$ is the subspace of matrices of the form $\binom{0}{M^{(2)}}$ with $M^{(2)} \in \mathrm{Mat}(n_2, m)$. We identify $V_i$ with $\mathrm{Mat}(n_i, m)$ and embed $\mathrm{O}(n_1)$ and $\mathrm{O}(n_2)$ into $\mathrm{O}(n)$ in the upper-left and lower-right corners, respectively. We systematically use the superscripts (1) and (2) to denote the upper and lower block matrices in this way. When this notation is being used, we number the rows of the lower block matrix by the elements of $[n_2]$, so that $M^{(2)}_{jr} = M_{n_1+j,r}$. We require the alternate coordinates $y^{(2)}_{jr}$ that are related to the coordinates $x^{(2)}_{jr}$ as described in the previous section. Let $y^{(2)} = [y^{(2)}_{jr}]$ be the matrix of these coordinates. Note that we only make this change of coordinates in the lower block, leaving the upper block unchanged.

For $0 \leq a \leq n_2/2$, we denote by $\nu_a$ the $a$-th exterior power of the standard representation of $\mathrm{O}(n_2)$. In particular, $\nu_0$ is the trivial representation of $\mathrm{O}(n_2)$. We denote by $\mathcal{H}^{\mathrm{O}(n_2),\nu_a}$ the $\nu_a$-isotypic subspace of $\mathcal{H}$ as a representation of $\mathrm{O}(n_2)$.

**Theorem 4.1.** *Let*

$$\mathcal{M}_a = \bigoplus_{I\subset[n_2],\ J\subset[m],\ |I|=|J|=a} F[V_1]\det(y^{(2)}_{IJ}),$$

*where $y^{(2)}_{IJ}$ denotes the minor matrix of $y^{(2)}$ drawn from the rows in $I$ and the columns in $J$. Then we have $\boldsymbol{P}(\mathcal{M}_a) = \mathcal{H}^{\mathrm{O}(n_2),\nu_a}$. If $\min(m, n_1) + a \leq n_2$ then we have $\mathcal{M}_a \cap \mathcal{I} = \{0\}$.*

*Proof.* Let $\mathcal{H}^{(2)}$ be the space of multiharmonic polynomials in $F[V_2]$. The diagram $D$ associated with the representation $\nu_a$ is a column of $a \leq n_2/2$ boxes and it follows from the results recalled in Section 3 that $(\mathcal{H}^{(2)})^{\mathrm{O}(n_2),\nu_a}$ contains the singular vector $\delta_a(y)$ and that $(\mathcal{H}^{(2)})^{\mathrm{O}(n_2),\nu_a}$ is generated by this vector under the actions of $\mathfrak{so}(n_2) \oplus \mathfrak{gl}(m)$ and $\tau$. The expressions for the action of the Lie algebra and of $\tau$ given in Section 3 make it clear that by applying them repeatedly to $\delta_a(y)$ we obtain linear combinations of $a$-by-$a$ minors from $y^{(2)}$ and so

$$(\mathcal{H}^{(2)})^{\mathrm{O}(n_2),\nu_a} \subset \sum_{I\subset[n_2],\ J\subset[m],\ |I|=|J|=a} F \det(y^{(2)}_{IJ}).$$

On the other hand, (3-1) implies that

$$\dim(\mathcal{H}^{(2)})^{\mathrm{O}(n_2),\nu_a} = \dim(\sigma_D)\dim(\pi_{m,D}) = \binom{n_2}{a}\binom{m}{a},$$

which is equal to the number of elements in the spanning set just found. Thus

$$(\mathcal{H}^{(2)})^{\mathrm{O}(n_2),\nu_a} = \bigoplus_{I\subset[n_2],\ J\subset[m],\ |I|=|J|=a} F \det(y^{(2)}_{IJ}).$$

It is a general fact that

$$F[V_2]^{\mathrm{O}(n_2),\nu_a} = F[V_2]^{\mathrm{O}(n_2)}(\mathcal{H}^{(2)})^{\mathrm{O}(n_2),\nu_a}.$$

The map $F[V_1]\otimes F[V_2] \to F[V]$ that satisfies $f_1 \otimes f_2 \mapsto f_1 f_2$ is an isomorphism and this and the fact that $\mathrm{O}(n_2)$ acts trivially on $F[V_1]$ now imply that

$$F[V]^{\mathrm{O}(n_2),\nu_a} = \sum_{I\subset[n_2],\ J\subset[m],\ |I|=|J|=a} F[V_1]F[V_2]^{\mathrm{O}(n_2)} \det(y^{(2)}_{IJ}). \tag{4-1}$$

Now suppose that $h \in \mathcal{H}^{\mathrm{O}(n_2),\nu_a}$. For $r, s \in [m]$ we may write $\varphi_{rs} = \varphi^{(1)}_{rs} + \varphi^{(2)}_{rs}$ with $\varphi^{(1)}_{rs} \in F[V_1]$ and $\varphi^{(2)}_{rs} \in F[V_2]$. We have $F[V_2]^{\mathrm{O}(n_2)} = F[\varphi^{(2)}_{rs}]$ and so (4-1) implies that we may write

$$h = \sum_{I\subset[n_2],\ J\subset[m],\ |I|=|J|=a} \sum_b p_{I,J,b}(\varphi^{(2)})^b \det(y^{(2)}_{IJ}),$$

where $b$ is a multi-index and $p_{I,J,b} \in F[V_1]$. By replacing $\varphi^{(2)}$ by $\varphi - \varphi^{(1)}$ in this expression, we obtain

$$h \equiv \sum_{I\subset[n_2],\ J\subset[m],\ |I|=|J|=a} \sum_b p_{I,J,b}(-\varphi^{(1)})^b \det(y^{(2)}_{IJ}) \pmod{\mathcal{I}}$$

and so

$$h = \boldsymbol{P}\Bigg( \sum_{I\subset[n_2],\ J\subset[m],\ |I|=|J|=a} \sum_b p_{I,J,b}(-\varphi^{(1)})^b \det(y^{(2)}_{IJ}) \Bigg) \in \boldsymbol{P}(\mathcal{M}_a).$$

Note that the sum defining $\mathcal{M}_a$ is direct because if we choose $I_0 \subset [n_2]$ and $J_0 \subset [m]$ with $|I_0| = |J_0| = a$ and then specialize $y^{(2)}$ in such a way that the minor matrix $y^{(2)}_{I_0 J_0}$ is the $a$-by-$a$ identity matrix and all entries not forced by this condition are 0 then we obtain $\det(y^{(2)}_{IJ}) = 1$ if $I = I_0$ and $J = J_0$ and $\det(y^{(2)}_{IJ}) = 0$ otherwise.

This observation implies that the minors $\det(y^{(2)}_{IJ})$ are independent over $F[V_1]$, as required. The argument to this point shows that $\boldsymbol{P}(\mathcal{M}_a) \supset \mathcal{H}^{\mathrm{O}(n_2),\nu_a}$. The reverse inclusion is clear and so the first claim is proved.

The minors $\det(y^{(2)}_{IJ})$ lie in the $F$-span of the minors $\det(x^{(2)}_{IJ})$ and conversely. It follows that these two collections of minors are $F$-bases for the same space and so we have

$$\mathcal{M}_a = \bigoplus_{I\subset[n_2],\ J\subset[m],\ |I|=|J|=a} F[V_1]\det(x^{(2)}_{IJ}).$$

Now suppose that $\min(m, n_1) + a \le n_2$. We wish to show that $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ and to do so, we shall make use of the commutative algebra

$$A = F[\varepsilon_1, \dots, \varepsilon_a]/(\varepsilon_1^2, \dots, \varepsilon_a^2).$$

Suppose first that $n_1 + a \le n_2$. Fix $I_0 \subset [n_2]$ and $J_0 \subset [m]$ of cardinality $a$. Let $M^{(1)} \in V_1$. We shall construct a matrix $M \in V(A)$, the $A$-points of $V$, from this data. As implied by the notation, the upper block of $M$ will be the given matrix $M^{(1)}$. Next we choose an injection $\iota : [n_1] \to [n_2] \setminus I_0$, which is possible since $n_1 \le n_2 - a$, and partially define the matrix $M^{(2)}$ by

$$M^{(2)}_{jr} = \begin{cases} iM^{(1)}_{\iota^{-1}(j),r} & \text{if } j \in \operatorname{im}(\iota), \\ 0 & \text{if } j \in [n_2] \setminus I_0 \text{ but } j \notin \operatorname{im}(\iota). \end{cases}$$

This has the effect of inserting the matrix $iM^{(1)}$ into certain rows in the matrix $M^{(2)}$. To complete the definition of $M^{(2)}$ suppose that $I_0 = \{j_1 < \cdots < j_a\}$ and $J_0 = \{r_1 < \cdots < r_a\}$ and define

$$M^{(2)}_{jr} = \begin{cases} \varepsilon_k & \text{if } j = j_k \text{ and } r = r_k, \\ 0 & \text{otherwise.} \end{cases}$$

This has the effect of inserting the matrix $\operatorname{diag}(\varepsilon_1, \dots, \varepsilon_k)$ into the rows and columns of $M^{(2)}$ indexed by $I_0$ and $J_0$. Since no two $\varepsilon_k$ lie in the same row as one another and $\varepsilon_k^2 = 0$ for all $k$, we have

$$M^\top M = (M^{(1)})^\top M^{(1)} + (iM^{(1)})^\top (iM^{(1)}) = 0.$$

That is, $M$ is an $A$-point of the scheme defined by the ideal $\mathcal{I}$. Now suppose that

$$f = \sum_{I,J} f_{IJ}(x^{(1)}) \det(x^{(2)}_{IJ}) \in \mathcal{M}_a \cap \mathcal{I}.$$

Then $f(M) = 0$ and so

$$\sum_{I,J} f_{IJ}(M^{(1)}) \det(M^{(2)}_{IJ}) = 0. \tag{4-2}$$

Among the minors $\det(M^{(2)}_{IJ})$, only $\det(M^{(2)}_{I_0J_0}) = \varepsilon_1 \cdots \varepsilon_a$ involves all the quantities $\varepsilon_k$. The minors $\det(M^{(2)}_{IJ})$ other than this one are linear combinations of products of fewer than $a$ of the $\varepsilon_k$ with coefficients drawn from $F$. Since the products $\varepsilon_S = \prod_{k\in S} \varepsilon_k$ with $S \subset [a]$ are an $F$-basis for $A$, it follows from (4-2) that $f_{I_0J_0}(M^{(1)}) = 0$. Since $M^{(1)} \in V_1$ was arbitrary, this implies that $f_{I_0J_0} = 0$, and since $I_0$ and $J_0$ were arbitrary, we conclude that $f = 0$, as required. This proves the second assertion when $n_1 + a \le n_2$.

Now assume that $m + a \le n_2$. We shall show that $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ by the same method that we just used in the case $n_1 + a \le n_2$, albeit that the construction of the matrix $M$ is more elaborate in the present case. Fix $I_0 \subset [n_2]$ and $J_0 \subset [m]$ of cardinality $a$ and let $M^{(1)} \in V_1$. Let $E < F^{n_1}$ be the column space of $M^{(1)}$. Equip $F^{n_1}$ with the bilinear form $(v, w) = v^\top w$ and write $E = E_0 \oplus^\perp E_1$ where the form is zero on $E_0$ and nondegenerate on $E_1$. Let $w_1, \dots, w_d$ be an ordered basis for $E_1$ and define $Y = [w_1|\cdots|w_d] \in \mathrm{Mat}(n_1, d)$. The symmetric matrix $-Y^\top Y \in \mathrm{Mat}(d)$ is invertible and hence the associated bilinear form on $F^d$ is equivalent to the standard bilinear form. That is, there is an invertible matrix $Z \in \mathrm{Mat}(d)$ such that $Z^\top Z = -Y^\top Y$. Let $Z = [z_1|\cdots|z_d]$. Note that

$$d = \dim(E_1) \le \dim(E) \le m \le n_2 - a,$$

so that we may choose an injection $\iota : [d] \to [n_2] \setminus I_0$. Let $T : E \to F^n$ be the linear map that satisfies

$$T(v) = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

for $v \in E_0$ and

$$T(w_j) = \begin{pmatrix} w_j \\ u_j \end{pmatrix}$$

where $u_j \in F^{n_2}$ is defined by

$$(u_j)_k = \begin{cases} (z_j)_{\iota^{-1}(k)} & \text{if } k \in \mathrm{im}(\iota), \\ 0 & \text{if } k \notin \mathrm{im}(\iota). \end{cases}$$

Note that if $j_1, j_2 \in [d]$ then

$$\begin{aligned} (T(w_{j_1}))^\top T(w_{j_2}) &= w_{j_1}^\top w_{j_2} + u_{j_1}^\top u_{j_2} = w_{j_1}^\top w_{j_2} + z_{j_1}^\top z_{j_2} \\ &= (Y^\top Y)_{j_1 j_2} + (Z^\top Z)_{j_1 j_2} \\ &= 0 \end{aligned}$$

and that by the choice of $E_0$, we also have $(T(v))^\top T(w) = 0$ for any $v, w \in E_0$. Suppose that $M^{(1)} = [v_1|\cdots|v_m]$ and define $M' = [T(v_1)|\cdots|T(v_m)] \in \mathrm{Mat}(n, m)$, which is possible since $v_j \in E$ for all $j \in [m]$. The facts that we just noted imply that $(M')^\top M' = 0$. Suppose further that $I_0 = \{j_1 < \cdots < j_a\}$ and $J_0 = \{r_1 < \cdots < r_a\}$, and define $M'' \in \mathrm{Mat}(n, m)$ by

$$(M'')_{jr} = \begin{cases} \varepsilon_k & \text{if } j = n_1 + j_k \text{ and } r = r_k, \\ 0 & \text{otherwise.} \end{cases}$$

If a certain row in $M'$ is nonzero then the corresponding row in $M''$ is zero and it follows that $(M')^\top M'' = 0$ and $(M'')^\top M' = 0$. Also, $(M'')^\top M'' = 0$ since $\varepsilon_k^2 = 0$ for all $k$. Let $M = M' + M'' \in \mathrm{Mat}(n, m)$. Observe that the upper block in $M$ is indeed $M^{(1)}$, so this notation is legitimate. We have $M^\top M = 0$; that is, $M$ is an $A$-point of the scheme defined by the ideal $\mathcal{I}$. From this point, the same argument that we used in the previous case verifies that $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ as required. This completes the proof of the second assertion. □

When $F$ is not algebraically closed, Theorem 4.1 cannot be formulated in quite the same way because the coordinates $y_{jr}$ are not necessarily available. However, as we saw in the course of the proof, the theorem can be reformulated with $\det(x^{(2)}_{IJ})$ in place of $\det(y^{(2)}_{IJ})$ and in this form it is intelligible over a general field $F$ of characteristic 0. The reformulation is true in the same generality. The essential observation for this conclusion is that all the objects in the statement are defined over $F$, so that we may deduce it by extending scalars to an algebraic closure.

Note that $\mathcal{M}_0 = F[V_1]$. When $a = 0$ the numerical criterion given in Theorem 4.1 for it to be the case that $\mathcal{M}_0 \cap \mathcal{I} = \{0\}$ is sharp. Indeed, suppose that $n_2 < \min(m, n_1)$. We wish to show that $F[V_1] \cap \mathcal{I} \neq \{0\}$. Write

$$x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}.$$

The entries in the matrix $x^\top x = (x^{(1)})^\top x^{(1)} + (x^{(2)})^\top x^{(2)}$ lie in $\mathcal{I}$ and so $(x^{(1)})^\top x^{(1)} \equiv -(x^{(2)})^\top x^{(2)} \pmod{\mathcal{I}}$. Let $k = \min(m, n_1)$. Since $n_2 < k$, the $k$-by-$k$ minors in the matrix $(x^{(2)})^\top x^{(2)}$ are identically zero. Thus the $k$-by-$k$ minors in the matrix $(x^{(1)})^\top x^{(1)}$ lie in $\mathcal{I}$. They also lie in $F[V_1]$ and so it suffices to see that at least one of these minors is not identically zero. This may be done by specializing $x^{(1)}$ to either $\binom{I_k}{0}$ or $(I_k\ 0)$, depending on whether $m \le n_1$ or $m > n_1$. On specialization, the leading $k$-by-$k$ minor of $(x^{(1)})^\top x^{(1)}$ specializes to 1 and hence this polynomial is not identically zero.

If $\mathcal{M}_0 \cap \mathcal{I} \neq \{0\}$ then $\mathcal{M}_a \cap \mathcal{I} \neq \{0\}$ for all $a$ and so this is always the case when $n_2 < \min(m, n_1)$. By Theorem 4.1, $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ if $n_2 \ge \min(m, n_1) + a$. It remains to decide whether or not $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ in the range $\min(m, n_1) \le n_2 < \min(m, n_1) + a$. Note that this question makes sense even without imposing the condition $a \le n_2/2$, which played no role in the proof of the third claim in Theorem 4.1. In this range, the answer to the question of whether or not $\mathcal{M}_a \cap \mathcal{I}$ is equal to $\{0\}$ varies depending on the values of $n_1$ and $m$. The following exemplifies this and its verification illustrates some useful techniques that might contribute to a complete resolution of the problem.

**Example.** When $n_1 = n_2 = 2$ we have $\mathcal{M}_1 \cap \mathcal{I} = \{0\}$ if $m = 2$ and $\mathcal{M}_1 \cap \mathcal{I} \neq \{0\}$ if $m = 3$.

To make the arguments easier to follow, we give the variables more memorable names. First consider $m = 2$ and let

$$x = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \\ u_4 & v_4 \end{pmatrix}.$$

The ideal $\mathcal{I}$ is generated by $\varphi_{11} = u_1^2 + \cdots + u_4^2$, $\varphi_{12} = u_1 v_1 + \cdots + u_4 v_4$, and $\varphi_{22} = v_1^2 + \cdots + v_4^2$. We first show that $\mathcal{I}$ is a radical ideal. It follows from Lemma 2.1 that $\sqrt{\mathcal{I}} = \mathcal{I} \oplus (\mathcal{H} \cap \sqrt{\mathcal{I}})$ and so it suffices to show that $\mathcal{H} \cap \sqrt{\mathcal{I}} = \{0\}$. Now $\mathcal{H} \cap \sqrt{\mathcal{I}}$ is $G$-invariant and so if it were nonzero then it would contain some singular vector. The description of the singular vectors given in Section 3 then implies that we would have $\delta_1^{a_1} \delta_2^{a_2} \in \sqrt{\mathcal{I}}$ for some $a_1, a_2 \in \mathbb{N}$, where $\delta_1 = u_1 + i u_2$ and

$$\delta_2 = \begin{vmatrix} u_1 + i u_2 & v_1 + i v_2 \\ u_3 + i u_4 & v_3 + i v_4 \end{vmatrix}.$$

The matrix

$$N = \begin{pmatrix} 1 & -1 \\ -i & -i \\ 1 & 1 \\ i & -i \end{pmatrix}$$

lies in the variety defined by $\sqrt{\mathcal{I}}$ and we have $\delta_1(N) = 2$, $\delta_2(N) = 4$. It follows that $\mathcal{H} \cap \sqrt{\mathcal{I}} = \{0\}$ and so $\mathcal{I}$ is a radical ideal. Let $\mathbb{X} = Z(\mathcal{I})$. It is a consequence of Witt's theorem that the set of matrices of rank two in $\mathbb{X}$ is a single $G$-orbit. It is not, however, a single orbit under $\mathrm{SO}(4) \times \mathrm{GL}(2)$, since a totally isotropic plane in $F^4$ gives rise to a volume form on $F^4$ that is preserved by $\mathrm{SO}(4) \times \mathrm{GL}(2)$ but not by $G$. Thus $\mathbb{X}$ decomposes into the union of two irreducible components that are interchanged by $\tau$, say $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$. Correspondingly, we have $\mathcal{I} = \mathcal{P}_1 \cap \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct prime ideals that are interchanged by $\tau$. Suppose now that

$$g = f_1 u_3 + f_2 v_3 + f_3 u_4 + f_4 v_4 \in \mathcal{M}_1 \cap \mathcal{I}$$

is not 0, with $f_1, \ldots, f_4 \in F[u_1, u_2, v_1, v_2]$. Since $\mathcal{I}$ is a homogeneous ideal and $u_3, v_3, u_4, v_4$ are homogeneous, we may assume that $g$ and $f_1, \ldots, f_4$ are homogeneous. We may further suppose that $g$ has been chosen to have the minimal possible degree for a nonzero homogeneous element of $\mathcal{M}_1 \cap \mathcal{I}$. Suppose that $M^{(1)} \in \mathrm{Mat}(2)$ has rank at most one. Then we have $M^{(1)} = \left(\begin{smallmatrix} tp & sp \\ tq & sq \end{smallmatrix}\right)$ for some $p, q, t, s \in F$. Working over the algebra $A = F[\varepsilon]/(\varepsilon^2)$ we set

$$M = \begin{pmatrix} tp & sp \\ tq & sq \\ \varepsilon & 0 \\ it\sqrt{p^2+q^2} & is\sqrt{p^2+q^2} \end{pmatrix}$$

and note that $M$ is an $A$-point of the scheme defined by $\mathcal{I}$ so that $g(M)=0$. It follows that $f_1(M^{(1)})=0$. We similarly conclude that $f_2(M^{(1)}) = f_3(M^{(1)}) = f_4(M^{(1)})=0$. That is, $f_1, \dots, f_4$ lie in the vanishing ideal of the variety of matrices of rank at most one. This ideal is the principal ideal generated by the polynomial $\psi = u_1v_2 - u_2v_1$ and so $f_1, \dots, f_4$ are divisible by $\psi$. We have $\psi(N) = -2i$ and $\psi(\tau(N)) = 2$ and so $\psi$ does not vanish on either component of $\mathbb{X}$. It follows that $g/\psi \in \mathcal{M}_1 \cap \mathcal{I}$. But the homogeneous degree of $g/\psi$ is less than that of $g$, contrary to assumption. Thus $\mathcal{M}_1 \cap \mathcal{I} = \{0\}$, as claimed.

Now consider $m = 3$ and let

$$x = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{pmatrix}.$$

The ideal $\mathcal{I}$ is generated by the polynomials $\varphi_{11}$, $\varphi_{12}$, $\varphi_{13}$, $\varphi_{22}$, $\varphi_{23}$, and $\varphi_{33}$, where $\varphi_{rs}$ is the dot product of the $r$-th column of $x$ with the $s$-th column. Let

$$\delta = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

and note that $\delta$ is also the determinant of the transpose matrix. Thus

$$\delta^2 = \begin{vmatrix} \varphi_{11} - u_4^2 & \varphi_{12} - u_4v_4 & \varphi_{13} - u_4w_4 \\ \varphi_{12} - u_4v_4 & \varphi_{22} - v_4^2 & \varphi_{23} - v_4w_4 \\ \varphi_{13} - u_4w_4 & \varphi_{23} - v_4w_4 & \varphi_{33} - w_4^2 \end{vmatrix}. \tag{4-3}$$

Now the matrix

$$\begin{pmatrix} u_4^2 & u_4v_4 & u_4w_4 \\ u_4v_4 & v_4^2 & v_4w_4 \\ u_4w_4 & v_4w_4 & w_4^2 \end{pmatrix}$$

has rank one and so when we expand (4-3) we obtain

$$\delta^2 = \begin{vmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{12} & \varphi_{22} & \varphi_{23} \\ \varphi_{13} & \varphi_{23} & \varphi_{33} \end{vmatrix} - \begin{vmatrix} u_4^2 & \varphi_{12} & \varphi_{13} \\ u_4v_4 & \varphi_{22} & \varphi_{23} \\ u_4w_4 & \varphi_{23} & \varphi_{33} \end{vmatrix} - \begin{vmatrix} \varphi_{11} & u_4v_4 & \varphi_{13} \\ \varphi_{12} & v_4^2 & \varphi_{23} \\ \varphi_{13} & v_4w_4 & \varphi_{33} \end{vmatrix} - \begin{vmatrix} \varphi_{11} & \varphi_{12} & u_4w_4 \\ \varphi_{12} & \varphi_{22} & v_4w_4 \\ \varphi_{13} & \varphi_{23} & w_4^2 \end{vmatrix}, \tag{4-4}$$

a combination of determinants each of which has at least two columns from $\mathcal{I}$. We now differentiate both sides of (4-4) with respect to $w_3$ and divide by 2. On the left-hand side we obtain

$$g = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \cdot \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

and on the right-hand side we obtain a linear combination of nine determinants each of which has at least one column from $\mathcal{I}$. It follows that $g \in \mathcal{I}$. On the other hand, expanding $\delta$ along the third row shows that $g \in F[V_1]u_3 + F[V_1]v_3 + F[V_1]w_3 \subset \mathcal{M}_1$. Thus $g \in \mathcal{M}_1 \cap \mathcal{I}$ and so $\mathcal{M}_1 \cap \mathcal{I} \neq \{0\}$, as claimed. This completes the verification of the example. We remark that $\delta$ itself does not lie in $\mathcal{I}$ in this case, whereas $\delta^2$ does, and so $\mathcal{I}$ is not a radical ideal.

## 5. Gelbart–Helgason reciprocity

In this section, we assume that $F$ is algebraically closed. Given $\lambda_1, \dots, \lambda_m \in \mathbb{N}$, we will write $F[V](\lambda_1, \dots, \lambda_m)$ for the subspace of $F[V]$ consisting of polynomials that are column-multihomogeneous of multidegree $\lambda_1, \dots, \lambda_m$; that is, those polynomials $\varphi$ that satisfy

$$\varphi(t_1 M_1 | \cdots | t_m M_m) = t_1^{\lambda_1} \cdots t_m^{\lambda_m} \varphi(M)$$

for all $M = [M_1 | \cdots | M_m] \in V$. Recall that in its action on $F[V]$, $\mathfrak{gl}(m)$ is spanned by the operators

$$W_{rs} = \sum_{j \in [n]} x_{jr} \partial_{js},$$

that $\mathfrak{h}$ denotes the span of $W_{11}, \dots, W_{mm}$ and that $\mathfrak{n}$ denotes the span of $W_{rs}$ with $r < s$. The operator $W_{rr}$ is the Euler operator associated with the $r$-th column and so $\varphi \in F[V](\lambda_1, \dots, \lambda_m)$ if and only if $\varphi$ is an $\mathfrak{h}$-weight vector with weight $\lambda_1\varepsilon_1 + \cdots + \lambda_m\varepsilon_m$. We write $\mathcal{H}(\lambda_1, \dots, \lambda_m) = F[V](\lambda_1, \dots, \lambda_m) \cap \mathcal{H}$ and extend the notation similarly to other subspaces of $F[V]$.

**Lemma 5.1.** *Let $D$ be a special diagram of depth $m$ with associated partition $[\lambda_1, \dots, \lambda_m]$. Then the representation of* $\mathrm{O}(n)$ *afforded by $\mathcal{H}^{\mathfrak{n}}(\lambda_1, \dots, \lambda_m)$ is isomorphic to $\sigma_D$.*

*Proof.* The elements of $\mathcal{H}^{\mathfrak{n}}(\lambda_1, \dots, \lambda_m)$ are singular vectors which have weight $\lambda_1\varepsilon_1 + \cdots + \lambda_m\varepsilon_m$. The only irreducible representation $\mathrm{GL}(m)$ that contains such vectors is $\pi_{m,D}$ and $\pi_{m,D}^{\mathfrak{n}}(\lambda_1, \dots, \lambda_m)$ is one-dimensional. Given this observation, the assertion follows from (3-1). □

**Theorem 5.2.** *Suppose that $0 \leq a \leq n_2/2$ and that $D$ is a special diagram. Let $\mathbb{S}_a(D)$ denote the set of all diagrams of depth at most $n_1$ that may be obtained* *from $D$ by deleting a total of $a$ boxes, no two in the same row. Then, as an* $\mathrm{O}(n_1)$*-representation,*

$$\mathrm{Hom}_{\mathrm{O}(n_2)}\big(\nu_a, \sigma_D^{\mathrm{O}(n_2),\nu_a}\big)$$

*is isomorphic to a quotient of*

$$\bigoplus_{E\in\mathbb{S}_a(D)} \mathrm{res}_{\mathrm{O}(n_1)}^{\mathrm{GL}(n_1)}(\pi_{n_1,E}). \tag{5-1}$$

*If* $\mathcal{M}_a \cap \mathcal{I} = \{0\}$ *then* $\mathrm{Hom}_{\mathrm{O}(n_2)}(\nu_a, \sigma_D^{\mathrm{O}(n_2),\nu_a})$ *is isomorphic to* (5-1)*. In particular, this holds if* $\min(\mathrm{dp}(D), n_1) + a \le n_2$.

*Proof.* Let $m = \mathrm{dp}(D)$ and $[\lambda_1, \ldots, \lambda_m]$ be the associated partition. We use the model of $\sigma_D$ provided by Lemma 5.1 to compute the $\nu_a$-isotype in $\sigma_D$ under $\mathrm{O}(n_2)$. The result is that

$$\sigma_D^{\mathrm{O}(n_2),\nu_a} = (\mathcal{H}^{\mathfrak{n}}(\lambda_1, \ldots, \lambda_m))^{\mathrm{O}(n_2),\nu_a} = (\mathcal{H}^{\mathrm{O}(n_2),\nu_a})^{\mathfrak{n}}(\lambda_1, \ldots, \lambda_m), \tag{5-2}$$

because the actions of $\mathrm{O}(n_2)$ and $\mathfrak{gl}(m)$ commute. By Theorem 4.1, this space is the image under the $G$-equivariant map $\boldsymbol{P}$ of the space $\mathcal{M}_a^{\mathfrak{n}}(\lambda_1, \ldots, \lambda_m)$. The kernel of this map is the space $(\mathcal{M}_a \cap \mathcal{I})^{\mathfrak{n}}(\lambda_1, \ldots, \lambda_m)$. As a representation of $\mathrm{GL}(n_1) \times \mathrm{GL}(m) \times \mathrm{O}(n_2)$ we have

$$\begin{aligned}\mathcal{M}_a &\cong (F[V_1] \otimes \eta_a) \boxtimes \nu_a \\ &\cong \bigoplus_{E_1} \pi_{n_1,E_1} \boxtimes (\pi_{m,E_1} \otimes \eta_a) \boxtimes \nu_a \\ &\cong \bigoplus_{E_1} \bigoplus_{E_2} \pi_{n_1,E_1} \boxtimes \pi_{m,E_2} \boxtimes \nu_a,\end{aligned}$$

where $\eta_a$ denotes the $a$-th exterior power of the standard representation of $\mathrm{GL}(n_2)$, $E_1$ runs over diagrams of depth at most $\min(m, n_1)$ and, for each $E_1$, $E_2$ runs over diagrams that may be obtained from $E_1$ by adding a total of $a$ boxes to $E_1$, with no two in the same row. The first isomorphism relies on the observation that as a representation of $\mathrm{GL}(m) \times \mathrm{O}(n_2)$ the space spanned by the minors $\det(y_{IJ}^{(2)})$ is isomorphic to $\eta_a \boxtimes \nu_a$. The second isomorphism follows from (3-2) applied to $V_1$. The third isomorphism is a consequence of Pieri's rule. We now reverse the order of the sum in the above isomorphism to obtain

$$\mathcal{M}_a \cong \bigoplus_{E_2} \bigoplus_{E_1} \pi_{n_1,E_1} \boxtimes \pi_{m,E_2} \boxtimes \nu_a, \tag{5-3}$$

where $E_2$ runs over diagrams of depth at most $m$ and, for each $E_2$, $E_1$ runs over diagrams of depth at most $n_1$ that may be obtained from $E_2$ by removing a total of $a$ boxes, with no two in the same row. It is important to note that a given diagram $E_1$ in this sum can arise from a given diagram $E_2$ in only one way; we simply decide what set of rows to remove a box from. Similarly, a given diagram $E_2$ in the reversed sum can arise from a given diagram $E_1$ in only one way; we decide what set of rows to add a box to. This means that no multiplicity arises when we interchange the order of the sum. The isomorphism (5-3) implies that as a representation of $\mathrm{GL}(n_1)\times \mathrm{O}(n_2)$,

$$\mathcal{M}_a^{\mathfrak{n}}(\lambda_1,\dots,\lambda_m)\cong \bigoplus_{E\in\mathbb{S}_a(D)} \pi_{n_1,E}\boxtimes \nu_a$$

since

$$(\pi_{m,E_2})^{\mathfrak{n}}(\lambda_1,\dots,\lambda_m)=\begin{cases} F & \text{if } E_2=D,\\ \{0\} & \text{otherwise.}\end{cases}$$

Since $\boldsymbol{P}$ is $(\mathrm{O}(n_1)\times\mathrm{O}(n_2))$-equivariant and the ideal $\mathcal{I}$ is $(\mathrm{O}(n_1)\times\mathrm{O}(n_2))$-invariant, the preceding calculations imply that there is a short exact sequence

$$\{0\}\to(\mathcal{M}_a\cap\mathcal{I})^{\mathfrak{n}}(\lambda_1,\dots,\lambda_m)\to\bigoplus\nolimits_{E\in\mathbb{S}_a(D)}\mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{O}(n_1)}(\pi_{n_1,E})\boxtimes\nu_a\to\sigma_D^{\mathrm{O}(n_2),\nu_a}\to\{0\}$$

of $(\mathrm{O}(n_1)\times\mathrm{O}(n_2))$-representations. By applying the functor $\mathrm{Hom}_{\mathrm{O}(n_2)}(\nu_a,\,\cdot\,)$ to this short exact sequence we obtain the first and second conclusions. Given this, the third conclusion follows directly from Theorem 4.1. This completes the proof. □

**Lemma 5.3.** *If $\rho$ is a representation of* $\mathrm{SO}(n)$ *and $\nu$ is a representation of $O(n_2)$ then we have*

$$\mathrm{Hom}_{\mathrm{O}(n_2)}\big(\nu,\mathrm{res}^{\mathrm{O}(n)}_{\mathrm{O}(n_2)}\big(\mathrm{ind}^{\mathrm{O}(n)}_{\mathrm{SO}(n)}(\rho)\big)\big)\cong\mathrm{Hom}_{\mathrm{SO}(n_2)}\big(\mathrm{res}^{\mathrm{O}(n_2)}_{\mathrm{SO}(n_2)}(\nu),\mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho)\big).$$

*Proof.* By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}\big(\mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho),\mathrm{res}^{\mathrm{O}(n_2)}_{\mathrm{SO}(n_2)}(\nu)\big)\cong\mathrm{Hom}_{\mathrm{O}(n_2)}\big(\mathrm{ind}^{\mathrm{O}(n_2)}_{\mathrm{SO}(n_2)}\big(\mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho)\big),\nu\big).$$

The double coset space $\mathrm{O}(n_2)\backslash\mathrm{O}(n)/\mathrm{SO}(n)$ is a singleton and $\mathrm{O}(n_2)\cap\mathrm{SO}(n)=\mathrm{SO}(n_2)$. Thus Mackey's induction-restriction theorem gives

$$\mathrm{ind}^{\mathrm{O}(n_2)}_{\mathrm{SO}(n_2)}\big(\mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho)\big)\cong\mathrm{res}^{\mathrm{O}(n)}_{\mathrm{O}(n_2)}\big(\mathrm{ind}^{\mathrm{O}(n)}_{\mathrm{SO}(n)}(\rho)\big).$$

From this we obtain

$$\mathrm{Hom}_{\mathrm{O}(n_2)}\big(\mathrm{res}^{\mathrm{O}(n)}_{\mathrm{O}(n_2)}\big(\mathrm{ind}^{\mathrm{O}(n)}_{\mathrm{SO}(n)}(\rho)\big),\nu\big)\cong\mathrm{Hom}_{\mathrm{SO}(n_2)}\big(\mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho),\mathrm{res}^{\mathrm{O}(n_2)}_{\mathrm{SO}(n_2)}(\nu)\big).$$

Since all the representations we are considering are finite-dimensional, we obtain the required isomorphism by applying this to the contragredients of $\rho$ and $\nu$. □

In what follows, by abuse of notation, we shall denote the restriction of $\nu_a$ to $\mathrm{SO}(n_2)$ by the same symbol. Recall that this restriction is irreducible if $a<n_2/2$ but decomposes as $\nu_a^+\oplus\nu_a^-$ if $a=n_2/2$, where $\nu_{n_2/2}^+$ has highest weight $\varepsilon_1+\cdots+\varepsilon_{n_2/2}$ and $\nu_{n_2/2}^-$ has highest weight $\varepsilon_1+\cdots+\varepsilon_{n_2/2-1}-\varepsilon_{n_2/2}$.

**Theorem 5.4.** *Suppose that* $0 \leq a \leq n_2/2$ *and* $n_1 + a \leq n_2$. *Let* $(\rho, E_\rho)$ *be an irreducible representation of* $\mathrm{SO}(n)$ *and* $D$ *the associated diagram. If* $\mathrm{dp}(D) < n/2$ *then*

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}\big(\nu_a, \mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho)\big) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{SO}(n_1)}(\pi_{n_1,E}) \oplus \bigoplus_{E \in \mathbb{S}_a(D')} \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{SO}(n_1)}(\pi_{n_1,E})$$

*as representations of* $\mathrm{SO}(n_1)$. *If* $\mathrm{dp}(D) = n/2$ *then*

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}\big(\nu_a, \mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho)\big) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{SO}(n_1)}(\pi_{n_1,E})$$

*as representations of* $\mathrm{SO}(n_1)$.

*Proof.* First suppose that $\mathrm{dp}(D) < n/2$. Then $\rho^\tau \cong \rho$ and so

$$\mathrm{ind}^{\mathrm{O}(n)}_{\mathrm{SO}(n)}(\rho) \cong \sigma_D \oplus \sigma_{D'}.$$

Lemma 5.3 then implies that

$$\mathrm{Hom}_{\mathrm{SO}(n_2)}\big(\nu_a, \mathrm{res}^{\mathrm{SO}(n)}_{\mathrm{SO}(n_2)}(\rho)\big) \cong \mathrm{Hom}_{\mathrm{O}(n_2)}\big(\nu_a, \mathrm{res}^{\mathrm{O}(n)}_{\mathrm{O}(n_2)}(\sigma_D \oplus \sigma_{D'})\big).$$

Moreover, the naturality of the isomorphisms used to prove Lemma 5.3 implies that this isomorphism respects the $\mathrm{SO}(n_1)$ actions. Since $n_1 + a \leq n_2$, Theorem 5.2 gives

$$\mathrm{Hom}_{\mathrm{O}(n_2)}\big(\nu_a, \mathrm{res}^{\mathrm{O}(n)}_{\mathrm{O}(n_2)}(\sigma_D \oplus \sigma_{D'})\big) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{O}(n_1)}(\pi_{n_1,E}) \oplus \bigoplus_{E \in \mathbb{S}_a(D')} \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{O}(n_1)}(\pi_{n_1,E})$$

as representations of $\mathrm{O}(n_1)$ and hence

$$\mathrm{Hom}_{\mathrm{O}(n_2)}\big(\nu_a, \mathrm{res}^{\mathrm{O}(n)}_{\mathrm{O}(n_2)}(\sigma_D \oplus \sigma_{D'})\big) \cong \bigoplus_{E \in \mathbb{S}_a(D)} \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{SO}(n_1)}(\pi_{n_1,E}) \oplus \bigoplus_{E \in \mathbb{S}_a(D')} \mathrm{res}^{\mathrm{GL}(n_1)}_{\mathrm{SO}(n_1)}(\pi_{n_1,E})$$

as representations of $\mathrm{SO}(n_1)$, as required. Now suppose that $\mathrm{dp}(D) = n/2$, then $\rho^\tau \ncong \rho$ and so

$$\mathrm{ind}^{\mathrm{O}(n)}_{\mathrm{SO}(n)}(\rho) \cong \sigma_D.$$

Given this, the proof is essentially the same as in the previous case. □

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