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SYMMETRY BREAKING DIFFERENTIAL OPERATORS, THE SOURCE OPERATOR AND RODRIGUES FORMULAE

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A Rodrigues type formula is obtained for the symbols of the covariant bidifferential operators on a simple real Jordan algebra.

Introduction

Symmetry breaking differential operators (SBDO for short) have been used for a long time in physics, and they have interested many authors in mathematics during recent years. T. Kobayashi has designed a program to study the existence, the uniqueness and the construction of such operators. We will use a restricted version of the notion of SBDO, adapted to the present article, and we will be concerned mostly with the constructive part of the program, in an effort to give explicit expressions for these operators. Here is a (nonexhaustive) list of recent papers on the subject: [Beckmann and Clerc 2012; Ben Saïd et al. 2020a; 2020b; Clerc 2016; 2017a; 2017b; 2017c; Fischmann et al. 2019; Juhl 2009; Kobayashi 2014; Kobayashi and Pevzner 2016; Kobayashi and Speh 2015; Ovsienko and Redou 2003; Somberg 2013]. Let M be a manifold, G (called the big group) a Lie group acting on M, N a submanifold of M, and H (called the little group) a closed Lie subgroup of G which preserves N. Let π be a smooth representation of G on $C^\infty(M)$ and ρ a smooth representation of H on $C^\infty(N)$. Let D be a differential operator from $C^\infty(M)$ into $C^\infty(N)$. Then D is said to be a *symmetry breaking differential operator* if D intertwines $\pi_{|H}$ and ρ, i.e.,

$$D \circ \pi(h) = \rho(h) \circ D \quad \text{for all } h \in H.$$

A specific case, the most important in this article, is the tensor product situation. Let G_1 be a Lie group acting on a manifold M_1. Let $G = G_1 \times G_1$ act on $M = M_1 \times M_1$, and let H be the diagonal subgroup $\operatorname{diag}(G_1 \times G_1) \simeq G_1$. Let π_1, π_2, π_3 be three smooth representations of G_1 on $C^\infty(M_1)$. Let π be the (exterior) tensor product representation $\pi_1 \otimes \pi_2$ viewed as a representation of $G_1 \times G_1$ on $C^\infty(M_1 \times M_1)$. Let D be a differential operator from $C^\infty(M_1 \times M_1)$

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into $C^\infty(M_1)$. Then D is said to be a *covariant bidifferential* operator with respect to $(\pi_1 \otimes \pi_2, \pi_3)$ if

$$D \circ (\pi_1(g) \otimes \pi_2(g)) = \pi_3(g) \circ D \quad \text{for all } g \in G_1.$$

The present paper is a continuation of work with S. Ben Saïd and K. Koufany [2020b], where covariant bidifferential were constructed in the realm of *real simple Jordan algebras*.

Departing from the convention for notation, let V be a simple real Jordan algebra and let G the conformal group of V.[1] The group G is simple, and the subgroup P of affine conformal transformations is a parabolic subgroup with abelian unipotent radical, and moreover its opposite parabolic $\overline{P}$ is conjugate to P.

To any $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}$ is associated a smooth representation $\widetilde{\pi}_{\lambda,\epsilon}$ induced by a character $\chi_{\lambda,\epsilon}$ of $\overline{P}$ (a degenerate nonunitary scalar principal series) realized on the smooth sections of a line bundle over $X = G/\overline{P}$.

The action of G on X can be transferred as a rational action of G on V, and the representation can be realized as a (not everywhere defined) action $\pi_{\lambda,\epsilon}$ of G on smooth functions on V (the *noncompact picture* realization of $\widetilde{\pi}_{\lambda,\epsilon}$).

Given two such couples (λ_1, ϵ_1), (λ_2, ϵ_2), the tensor product $\pi_{\lambda_1,\epsilon_1} \otimes \pi_{\lambda_2,\epsilon_2}$ can be realized on $C^\infty(V \times V)$. Let $k \in \mathbb{N}$. In [Ben Saïd et al. 2020b], we construct bidifferential operators $B^{(k)}_{(\lambda_1,\lambda_2)}$ acting from $C^\infty(V \times V)$ into $C^\infty(V)$ which are covariant with respect to $(\pi_{\lambda_1,\epsilon_1} \otimes \pi_{\lambda_2,\epsilon_2}, \pi_{\lambda_1+\lambda_2+2k,\epsilon_1\epsilon_2})$.[2]

When $V = \mathbb{R}$, these covariant bidifferential operators coincide (up to normalization) with the classical *Rankin–Cohen brackets*. More details on this example can be found in [Clerc 2017b, Section 5].

However, the operators obtained in [Ben Saïd et al. 2020b] were not given by closed formulae. The purpose of the present paper is to give more explicit formulae for these operators.

Let n be the dimension of V, det be the determinant of the Jordan algebra and let r be the rank of V. Consider the product $V \times V$ and let $\operatorname{diag}(V) \simeq V$ be the diagonal of $V \times V$. Let

$$\operatorname{res} : C^\infty(V \times V) \to C^\infty(V)$$

be the restriction map to the diagonal

$$\{(x, x), x \in V\} \simeq V.$$

The main result in this article is the following theorem, which is a new result even for the classical case of the Rankin–Cohen brackets.

[1] Strictly speaking, G is a twofold covering of the proper conformal group of V.

[2] The expression of the covariant bidifferential operator does not depend on the signs ϵ_1, ϵ_2; hence the signs have been eliminated in the notation.

Theorem 0.1. *Let* $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$. *Then for any* $s, t \in \mathbb{C}$ *and for any* $k \in \mathbb{N}$, *there exists a polynomial* $c^{(k)}_{\lambda,\mu}$ *on* $V \times V$ *such that*

$$\det\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^k (\det x)^{s+k} (\det y)^{t+k} = c^{(k)}_{s,t}(x, y)(\det x)^s (\det y)^t. \tag{1}$$

Let $B^{(k)}_{\lambda,\mu} : C^\infty(V \times V) \to C^\infty(V)$ *be the bidifferential operator given by*

$$B^{(k)}_{\lambda,\mu} = \operatorname{res} \circ c^{(k)}_{\lambda-\frac{n}{r},\mu-\frac{n}{r}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

Then $B^{(k)}_{\lambda,\mu}$ *satisfies the following covariance relation for any* $g \in G$:

$$B^{(k)}_{\lambda,\mu} \circ (\pi_{\lambda,\epsilon}(g) \otimes \pi_{\mu,\eta}(g)) = \pi_{\lambda+\mu+2k,\epsilon\eta}(g) \circ B^{(k)}_{\lambda,\mu}. \tag{2}$$

The proof goes through four steps.

• Construction of the *source operator*.

The *source operator* is a differential operator $E_{\lambda,\mu}$ with polynomial coefficients on $V \times V$ which is, under the diagonal action of G, covariant with respect to $((\pi_{\lambda,\epsilon} \otimes \pi_{\mu,\eta}), (\pi_{\lambda+1,-\epsilon} \otimes \pi_{\mu+1,-\eta}))$. The construction of the source operator is based on the *Knapp–Stein intertwining operators* and requires delicate computations of Fourier transforms of singular distributions on V (see [Ben Saïd et al. 2020b]).

• Construction of SBDO from the source operator.

The construction is by induction on the parameter k, namely,

$$B^{(k)}_{\lambda,\mu} = \operatorname{res} \circ E_{\lambda+k-1,\mu+k-1} \circ \cdots \circ E_{\lambda,\mu}, \tag{3}$$

and the fact that the resulting bidifferential operators satisfy a covariance relation is an elementary consequence of the covariance property satisfied by the source operator (see, again, [Ben Saïd et al. 2020b]).

• Deduction of the recursion relation for the symbols of the SBDO.

This is new and requires a proper definition of the symbols of some singular operators on V and the development of an ad hoc calculus, inspired by the classical pseudodifferential calculus. The recursion relation, which relates the symbol of $B^{(k)}_{\lambda,\mu}$ and the symbol of the operator $B^{(k-1)}_{\lambda+1,\mu+1}$ is deduced from (3).

• Obtaining the Rodrigues formula for the symbols of the SBDO.

The recurrence relation obtained in the third step is reminiscent of the recurrence relation that can be deduced from the classical *Rodrigues formulae* for families of orthogonal polynomials. This observation allows us to find an explicit solution for the recurrence relation.

The method presented for the bidifferential operators (i.e., the tensor product situation) is valid for more general SBDO (i.e., more general geometric situations). The rôle of the Rodrigues formula was discovered in three "elementary" cases by explicit calculations: the Rankin–Cohen brackets, the Juhl operators and the conformally covariant bidifferential operators on $\mathbb{R}^n$. These results are presented at the end of the paper. For the two first cases, the relation of the symbols of the operators to families of orthogonal polynomials (the Jacobi polynomials for the Rankin–Cohen brackets and the Gegenbauer polynomials for the Juhl operators) had been observed before (see [Kobayashi and Speh 2015; Kobayashi and Pevzner 2016; Juhl 2009]). In both cases, the recurrence relation obtained for the symbols of the differential operators could be compared to the classical Rodrigues formula for the families of orthogonal polynomials involved. The third case is new; no connection to families of orthogonal polynomials or special functions is known.

Here is a finer description of the different sections of this article.

Section 1 contains, in a quite general context, a development of an ad hoc calculus, designed to gain flexibility in some computations done in [Ben Saïd et al. 2020b], typically for rewriting a differential operator with polynomial coefficients in its *normal form*, that is to say, differentiation before multiplication. The calculus is inspired by the symbolic calculus in the Weyl algebra and/or the pseudodifferential symbolic calculus. However, it is more an algebraic calculus as no asymptotic analysis is needed.

In Section 2, the construction of the source operator is recalled, mostly relying on [Ben Saïd et al. 2020b] and rephrased in terms of the symbolic calculus developed in Section 1.

Section 3 introduces the covariant bidifferential operators and the recurrence relation satisfied by their symbols.

Section 4 is devoted to the Rodrigues formula for the symbols of the covariant bidifferential operators.

Section 5 presents a more direct proof of the Rodrigues formula for the symbols of the SBDO in the three "elementary" examples.

1. Dual symbolic calculi on E and E^*

The symbolic calculus on E. Let E be a real vector space of dimension N. Choosing a basis of E allows us to identify E with $\mathbb{R}^N$. The corresponding coordinates of an element $x \in E$ will be denoted by $(x_1, x_2, \dots, x_N)$. For $1 \leq j \leq N$, let $\partial_j = \partial/\partial x_j$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ an N-multiindex, we let

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_N, \qquad \alpha! = \alpha_1!\alpha_2! \cdots \alpha_N!,$$
$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_1} \cdots \partial_N^{\alpha_N}.$$

Let E^* be the dual of E, also identified with $\mathbb{R}^N$ via the basis of E^* dual to the chosen basis for E. Elements of E^* will be denoted by Greek letters and their coordinates by $\xi = (\xi_1, \xi_2, \dots, \xi_N)$. The duality between E and E^* is denoted by $x.\xi,\ x \in E,\ \xi \in E^*$. The dual $(E^*)^*$ is identified with E.

For $1 \le j \le N$, let $\partial_{*j} = \partial/\partial\xi_j$, and as before, for an N-multiindex α, let

$$\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_N^{\alpha_N}, \qquad \partial_*^\alpha = \partial_{*1}^{\alpha_1}\partial_{*2}^{\alpha_2}\cdots\partial_{*N}^{\alpha_N}.$$

The Fourier transform is viewed, loosely speaking, as the map from functions on E to functions on E^* given by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_E e^{-ix.\xi} f(x)\,dx.$$

On E^*, the inverse Fourier transform is defined by

$$(\mathcal{F}^* g)(x) = \check{g}(x) = (2\pi)^{-N}\int_{E^*} e^{ix.\xi} g(\xi)\,d\xi.$$

The classical formulae of Fourier analysis are

$$\begin{aligned}
&\mathcal{F}\circ x^\alpha = (i\partial_*)^\alpha\circ\mathcal{F}, && \mathcal{F}\circ\Big(\frac{1}{i}\partial\Big)^\alpha = \xi^\alpha\circ\mathcal{F}\\
&\mathcal{F}^*\circ\xi^\alpha = \Big(\frac{1}{i}\partial\Big)^\alpha\circ\mathcal{F}^*, && \mathcal{F}^*\circ(i\partial_*)^\alpha = x^\alpha\circ\mathcal{F}^*.
\end{aligned}\tag{4}$$

More precisely, let $\mathcal{S}(E)$ be the Schwartz space of rapidly decreasing smooth functions on E and let $\mathcal{S}'(E)$ be the space of tempered distributions on E, both equipped with their usual topology. The Fourier transform maps $\mathcal{S}(E)$ (resp. $\mathcal{S}'(E)$) into $\mathcal{S}(E^*)$) (resp. $\mathcal{S}'(E^*)$). Define $\mathrm{Op}(E)$ to be the space of operators from $\mathcal{S}(E)$ into $\mathcal{S}'(E)$ generated by a finite combination of products of a convolution operator by a tempered distribution on E followed by a multiplication by a polynomial function on E. For k a tempered distribution, denote by K the corresponding convolution operator, viewed as an operator from $\mathcal{S}(E)$ into $\mathcal{S}'(E)$ given by

$$K\varphi(x) = (k\star\varphi)(x).$$

For p a polynomial function on E, the multiplication operator

$$f \mapsto (pf)(x) = p(x)f(x),$$

seen as an operator from $\mathcal{S}(E)$ into $\mathcal{S}(E)$ or from $\mathcal{S}'(E)$ into $\mathcal{S}'(E)$, is denoted by $p(x)$ or p depending on the context.

Hence an element of $\mathrm{Op}(E)$ is a finite linear combination of operators of the form $p(x)\circ K$, respecting the convention of normal order (convolution first, then multiplication). Any element of $\mathrm{Op}(E)$ can be written in a unique way as

$$\sum_\alpha x^\alpha L_\alpha,$$

where for each N-multiindex α, L_α is a convolution operator with a tempered distribution E, and such that L_α is 0 except for a finite number of α's.

The space $\mathrm{Op}(E)$ contains the Weyl algebra $\mathcal{W}(E)$ (algebra of differential operators with polynomial coefficients). In fact, if D is a constant coefficients differential operator, then for a test function $f \in \mathcal{S}(E)$,

$$Df = D(\delta_0 \star f) = (D\delta_0) \star f,$$

where δ_0 is the Dirac distribution at 0. As $D\delta_0$ is a tempered distribution, D belongs to $\mathrm{Op}(E)$. Moreover, if p is a polynomial function on E, the operator $p(x)D$ also belongs to $\mathrm{Op}(E)$ and hence the Weyl algebra $\mathcal{W}(E)$ is included in $\mathrm{Op}(E)$.

The *symbol* is the Fourier side version of these operators. For K the convolution operator with the tempered distribution $k(x)$, the symbol is defined as the distribution on E^* equal to the Fourier transform $\hat{k}(\xi)$ of k, whereas for p a polynomial function on E, its symbol is just $p(x)$. For the product in normal order $p(x)K$, the symbol is defined to be $p(x)\hat{k}(\xi)$. This corresponds to the following formula (to be understood as an equality of tempered distributions) valid for any $f \in \mathcal{S}(E)$:

$$p(x)Kf(x) = (2\pi)^{-N} \int_{E^*} e^{ix.\xi} p(x)\hat{k}(\xi)\hat{f}(\xi)d\xi.$$

Definition 1.1. Let $L = \sum_\alpha x^\alpha K_\alpha$ be an element of $\mathrm{Op}(E)$. Its symbol $\mathrm{symb}(L)$ is the element of $\mathcal{P}ol(E, \mathcal{S}'(E^*))$ defined by

$$\mathrm{symb}(L)(x, \xi) = \sum_\alpha x^\alpha \hat{k}_\alpha(\xi).$$

Conversely, given $s(x,\xi) = \sum_\alpha x^\alpha l_\alpha(\xi) \in \mathcal{P}ol(E, \mathcal{S}'(E^*))$, the operator

$$L = \sum_\alpha x^\alpha K_\alpha,$$

where K_α is the convolution operator by the tempered distribution $\mathcal{F}^* l_\alpha(x)$, belongs to $\mathrm{Op}(E)$ and its symbol is equal to $s(x, \xi)$.

Let $D \in \mathcal{W}(E)$. Recall that its symbol (in the usual sense) $\sigma(D)$ is the polynomial function on $E \times E^*$ defined by

$$D(e^{ix.\xi}) = \sigma(D)(x,\xi)e^{ix.\xi}.$$

Proposition 1.2. *Let $D \in \mathcal{W}(E)$. Then*

$$\mathrm{symb}(D) = \sigma(D).$$

Proof. Let first $D = \partial^\alpha$ for some N-multiindex α. Then $\sigma(D)(x,\xi) = (i\xi)^\alpha$. On the other hand, D is the convolution operator with the distribution $\partial^\alpha\delta_0$ and $\mathcal{F}(\partial^\alpha\delta_0)(\xi) = (i\xi)^\alpha\mathcal{F}\delta_0(\xi) = (i\xi)^\alpha$, so $\mathrm{symb}(\partial^\alpha)(\xi) = (i\xi)^\alpha = \sigma(\partial^\alpha)(\xi)$. The general case follows easily. □

The space $\mathrm{Op}(E)$ is not an algebra, but some compositions are possible, and then the symbol of the composed operator is given by a formula similar to the composition formula in the pseudodifferential calculus or the composition formula for the symbols in the Weyl algebra.

Let $L \in \mathrm{Op}(E)$ and $D \in \mathcal{W}(E)$. Then $L \circ D$ is well defined as an operator from $\mathcal{S}(E)$ into $\mathcal{S}'(E)$.

Proposition 1.3. *Let L be an element of* $\mathrm{Op}(E)$, *and let $D \in \mathcal{W}(E)$. Then $L \circ D$ belongs to* $\mathrm{Op}(E)$ *and its symbol is given by*

$$\mathrm{symb}(L)\sharp\,\mathrm{symb}(D) := \mathrm{symb}(L\circ D) = \sum_\alpha \frac{1}{\alpha!}\left(\frac{1}{i}\partial_\xi\right)^\alpha \mathrm{symb}(L)\partial_x^\alpha\,\mathrm{symb}(D). \tag{5}$$

Proof. First assume that $L = K$ is a convolution operator by a tempered distribution k, and D is the multiplication operator by $p(x)$, where p is a polynomial function on E. Then

$$\begin{aligned}(K\circ p)\varphi(x) &= \int_E k(x-y)p(y)\varphi(y)dy\\ &= (2\pi)^{-N}\int_E\int_{E^*} e^{i(x-y).\xi}\hat{k}(\xi)p(y)\varphi(y)\,dy\,d\xi,\end{aligned}$$

which by the change of variable $y \mapsto z = y - x$ becomes

$$(2\pi)^{-N}\int_E\int_{E^*} e^{-iz.\xi}\hat{k}(\xi)p(x+z)\varphi(x+z)\,dz\,d\xi.$$

Use the (exact) Taylor expansion for the polynomial p at x to get

$$(2\pi)^{-N}\sum_\alpha\frac{1}{\alpha!}\partial_x^\alpha p(x)\int_E\int_{E^*} e^{-iz.\xi}\hat{k}(\xi)z^\alpha\varphi(x+z)\,dz\,d\xi.$$

Substituting

$$z^\alpha\int_{E^*} e^{-i\xi.z}\hat{k}(\xi)d\xi = \int_{E^*} e^{-iz.\xi}\left(\frac{1}{i}\partial_\xi\right)^\alpha\hat{k}(\xi)\,d\xi$$

gives

$$(2\pi)^{-N}\sum_\alpha\frac{1}{\alpha!}\partial_x^\alpha p(x)\int_E\int_{E^*} e^{-iz.\xi}\left(\frac{1}{i}\partial_\xi\right)^\alpha\hat{k}(\xi)\varphi(x+z)\,dz\,d\xi,$$

and then substituting

$$\int_E e^{-iz.\xi}\varphi(x+z)\,dz = e^{ix.\xi}\hat{\varphi}(\xi)$$

gives

$$(2\pi)^{-N}\sum_\alpha \frac{1}{\alpha!}\partial_x^\alpha p(x)\int_{E^*} e^{ix.\xi}\left(\frac{1}{i}\partial^*\right)^\alpha \hat{k}(\xi)\hat{\varphi}(\xi)d\xi,$$

and the conclusion follows. The general case is an easy consequence of this special case, as multiplication on the left by a polynomial function p (resp. composition on the right with a constant coefficients differential operator) corresponds for the symbols to a multiplication by $p(x)$ (resp. to a multiplication by $\sigma(D)(\xi)$). □

In particular, the composition law for the symbols of two such differential operators coincides with the classical formula for the symbol of a product in the Weyl algebra

$$\text{(6)}\qquad \operatorname{symb}(D\circ F) = d(x,\xi)\sharp f(x,\xi) = \sum_\alpha \frac{1}{\alpha!}\left(\frac{1}{i}\partial_\xi\right)^\alpha d(x,\xi)\,\partial_x^\alpha f(x,\xi).$$

The dual symbolic calculus on E*. The symbolic calculus has its dual (via the Fourier transform) version on E^*. The family $\mathrm{Op}(E^*)$ is the family of finite combinations of multiplications by tempered distributions composed with constant coefficients differential operators on E^*, viewed as operators from $\mathcal{S}(E^*)$ into $\mathcal{S}'(E^*)$. The family $\mathrm{Op}(E^*)$ is obtained from $\mathrm{Op}(E)$ by transmutation by the Fourier transform. Any element of $\mathrm{Op}(E^*)$ can be written in a unique way as

$$L = \sum_\alpha l_\alpha(\xi)\partial_\xi{}^\alpha,$$

where for each N-multiindex α, l_α is a tempered distribution and $l_\alpha = 0$ but for a finite number of α's.

By definition, the symbol of the multiplication operator by the tempered distribution k on E^* is $k(\xi)$. On the other hand, if D is a differential operator on E^* with polynomial coefficients, its (usual) symbol is defined by the polynomial function $\sigma^*(D)$ on $E^*\times E$ defined by

$$D(e^{i\xi.x}) = \sigma^*(D)(\xi,x)e^{i\xi.x}.$$

Notice that this definition implies that

$$\sigma^*\left(\frac{1}{i}\partial_{*j}\right)(\xi,x) = x_j.$$

Definition 1.4. Let $L = \sum_\alpha l_\alpha \partial_\xi^\alpha$. Its symbol is defined by

$$\operatorname{symb}^*(L)(\xi, x) = l_\alpha(\xi)(ix)^\alpha.$$

The space $\operatorname{Op}(E^*)$ contains the Weyl algebra $\mathcal{W}(E^*)$ of differential operators on E^* with polynomial coefficients and for $D \in \mathcal{W}(E^*)$, $\operatorname{symb}^*(D)$ coincides with its usual symbol $\sigma^*(D)$.

The space $\operatorname{Op}(E^*)$ is not an algebra, but some compositions are possible, and then the symbol of the composed operator is given by a formula similar to the composition formula already seen for $\operatorname{Op}(E)$. If $L \in \operatorname{Op}(E^*)$ and $D \in \mathcal{W}(E^*)$, then $D \circ L$ is well defined as an operator from $\mathcal{S}(E^*)$ into $\mathcal{S}'(E^*)$.

Proposition 1.5. *Let $L \in \operatorname{Op}(E^*)$ and let $D \in \mathcal{W}(E^*)$. Then $D \circ L$ belongs to $\operatorname{Op}(E^*)$ ad its symbol is given by*

$$\text{(7)}\quad \operatorname{symb}^*(D) \flat \operatorname{symb}^*(L) := \operatorname{symb}^*(D \circ L) = \sum_\alpha \frac{1}{\alpha!} \partial_x^\alpha \operatorname{symb}^*(D) \left(\frac{1}{i}\partial_\xi\right)^\alpha \operatorname{symb}^*(L).$$

Proof. First assume that L is a multiplication operator by a tempered distribution $k(\xi)$ and D is a constant coefficient differential operator, with symbol equal to a polynomial $p(x)$. As both sides of (7) are linear in p, we may even assume that p is a monomial, say $p(x) = x^\beta$ for some multiindex α. As $(p(x)q(x)) \flat k(\xi) = p(x) \flat (q(x) \flat k(\xi))$ for $p, q \in \mathcal{P}(E)$, it suffices to prove the formula for $p(x) = x_j^{\beta_j}$, or equivalently for $D = \left(\frac{1}{i}\partial_j^*\right)^{\beta_j}$. In this case, let $\varphi \in \mathcal{S}(E^*)$. Then

$$(i\partial_{*j})^{\beta_j}(k(\xi)\varphi(\xi)) = \sum_{\alpha_j=0}^{\beta_j} \binom{\beta_j}{\alpha_j} \left(\frac{1}{i}\partial_{*j}\right)^{\alpha_j} k(\xi) \left(\frac{1}{i}\partial_{*j}\right)^{\beta_j-\alpha_j} \varphi(\xi).$$

Now use

$$\operatorname{symb}^*\left(\left(\frac{1}{i}\partial_{*j}\right)^{\beta_j-\alpha_j}\right) = x_j^{\beta_j-\alpha_j} = \frac{(\beta_j-\alpha_j)!}{\beta_j!}$$

to get

$$\operatorname{symb}^*(x^{\beta_j} \circ k(\xi)) = x^{\beta_j} \flat k(\xi) = \sum_{\alpha_j} \frac{1}{\alpha_j!} \left(\frac{1}{i}\partial_j^*\right)^{\alpha_j} k(\xi) \partial_j^{\alpha_j} x_j^{\beta_j}.$$

This proves (7) for the case where L is a multiplication operator by a tempered distribution and D is a constant coefficient differential operator. The general case follows easily. □

Again, the space $\operatorname{Op}(E^*)$ contains the Weyl algebra $\mathcal{W}(E^*)$, the space of differential operators with polynomial coefficients on E^*, and the symbol of such an operator coincides with its usual definition. Also the composition formula for the symbol of two differential operators with polynomials coefficients is given by

$$c(\xi,x)\flat d(\xi,x)=\sum_\alpha \frac{1}{\alpha!}\,\partial_x^\alpha c(\xi,x)\left(\frac{1}{i}\partial_\xi\right)^\alpha d(\xi,x).$$

Proposition 1.6. *Let D be a differential operator on E^* with polynomial coefficients, let $d^*(\xi,x)$ be its symbol and let p be a polynomial function on E^*. Let $c^*(\xi,x)=d^*(\xi,x)\flat p(\xi)$. Then*

$$c(\xi,0)=Dp(\xi).$$

Proof. By linearity, it is sufficient to prove the result when $D=\xi^\gamma\left(\frac{1}{i}\partial_\xi\right)^\delta$ for γ,δ arbitrary N-multiindices. Then $d^*(\xi,x)=\xi^\gamma x^\delta$ so

$$c(\xi,0)=(\xi^\gamma x^\delta\flat p(\xi))(\xi,0)=\sum_{\alpha!}\frac{1}{\alpha}\xi^\gamma\partial_x^\alpha(x^\delta)(0)\left(\frac{1}{i}\partial_\xi\right)^\alpha p(\xi).$$

But $\partial_x^\alpha(x^\delta)(0)=0$ unless $\alpha=\delta$, in which case

$$\partial_x^\delta(x^\delta)(0)=\delta!$$

so

$$c(\xi,0)=\xi^\gamma\left(\frac{1}{i}\partial_\xi\right)^\delta p(\xi)=Dp(\xi).$$ □

Proposition 1.7. *Let $e(x,\xi)$ be a symbol on $E\times E^*$ and $d(x,\xi)$ be a polynomial function on $E\times E^*$. Let*

$$e^*(\xi,x)=e(x,\xi),\qquad d^*(\xi,x)=d(x,\xi).$$

Then

$$e(x,\xi)\sharp d(x,\xi)=d^*(\xi,x)\flat e^*(\xi,x). \tag{8}$$

The same result is valid when $e(x,\xi)$ is a polynomial function and $d(x,\xi)$ is a symbol.

Proof. The identity (8) is a consequence of the two composition formulae for the symbols (5) and (7). □

2. The source operator on a real simple Jordan algebra

Let V be a simple real Jordan algebra,[3] of dimension n and rank r. Recall that the *conformal group* of V is the group generated by the translations, the structure group

[3] Reference [Ben Saïd et al. 2020b] contains a long introduction to real simple Jordan algebras including their classification and further references.

of V and the inversion. In [Ben Saïd et al. 2020b], we introduced a group G which is locally isomorphic to the conformal group (a twofold covering of the proper conformal group; see details in [Ben Saïd et al. 2020b, Section 7]). Let P be the parabolic subgroup of G corresponding to the conformal affine maps (generated by the translations and the structure group). The unipotent radical N of P is abelian and can be identified with V. Let $\overline{P}$ be the parabolic subgroup opposite to P. Then $G/\overline{P}$ is the *conformal completion* of V and the map $V \simeq N \to G/\overline{P}$ gives the embedding of V in its completion.

The characters of $\overline{P}$ are described by $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}$, and for each (λ, ϵ) there is an induced representation (belonging to the degenerate nonunitary scalar principal series) $\widetilde{\pi}_{\lambda,\epsilon}$ of G acting on the space $\mathcal{H}_{(\lambda,\epsilon)}$ of smooth sections of a line bundle over X. The *Knapp–Stein operators* are a meromorphic (with respect to λ) family of operators

$$\widetilde{I}_{\lambda,\epsilon} : \mathcal{H}_{\lambda,\epsilon} \to \mathcal{H}_{\frac{2n}{r}-\lambda,\epsilon}$$

of intertwining operators, i.e., for any $g \in G$, they satisfy

$$\widetilde{I}_{\lambda,\epsilon} \circ \widetilde{\pi}_{\lambda,\epsilon}(g) = \widetilde{\pi}_{\frac{2n}{r}-\lambda,\epsilon}(g) \circ \widetilde{I}_{\lambda,\epsilon}. \tag{9}$$

There is another realization of the representations $\pi_{\lambda,\epsilon}$ called the *noncompact picture*, obtained by using the embedding $V \to X$. The action of G on V is a rational (not everywhere defined) action, the space considered for the representation is the space of smooth functions $C^\infty(V)$ and the action can be explicitly written in this realization.

Recall the following notation: for $s \in \mathbb{C}$, $\epsilon \in \{\pm\}$, and for $t \in \mathbb{R}^*$,

$$t^{s,\epsilon} = \begin{cases} |t|^s & \text{if } \epsilon = +, \\ \operatorname{sgn}(t)|t|^s & \text{if } \epsilon = -. \end{cases}$$

Keeping same notation as above, the representation $\pi_{\lambda,\epsilon}$ is given by

$$\pi_{\lambda,\epsilon}(g) f(x) = a(g^{-1}, x)^{-\lambda,\epsilon} f(g^{-1}(x)), \tag{10}$$

where a is a smooth cocycle on $G \times V$, and $a(g^{-1}, x) = 0$ precisely when g^{-1} is not defined at x.

The Knapp–Stein intertwining operators can also be transferred and they are given (up to a scalar) by

$$I_{\lambda,\epsilon} f(x) = \int_V \det(x-y)^{-\frac{2n}{r}+\lambda,\epsilon} f(y)\,dy, \tag{11}$$

where det is the *determinant* polynomial of V.

The Knapp–Stein operators are convolution operators, and the family $\det(x)^{s,\epsilon}$ which is well defined for $\Re(s) \gg 0$ is extended by analytic continuation to a meromorphic family of tempered distributions (see more details in [Ben Saïd et al. 2020b]). The following result plays an important rôle in the sequel.

Proposition 2.1. *The Fourier transform of the kernel of the intertwining operator is given by*

$$\ell_{\lambda,\epsilon}(\xi) := \mathcal{F}(\det x^{-\frac{2n}{r}+\lambda,\epsilon})(\xi) = \kappa_+(\lambda,\epsilon)\det\xi^{\frac{n}{r}-\lambda,+} + \kappa_-(\lambda,\epsilon)\det\zeta^{\frac{n}{r}-\lambda,-}, \tag{12}$$

where $\kappa_\pm(\lambda)$ are meromorphic functions on $\mathbb{C}$, not both identically 0.

This result was obtained in [Ben Saïd et al. 2020b, Section 5], where the explicit values of $\kappa_\pm(\lambda,\epsilon)$ can be found.

Consider now (λ,ϵ) and (μ,η) in $\mathbb{C}\times\{\pm\}$ and form the tensor product $\widetilde{\pi}_{\lambda,\epsilon}\otimes\widetilde{\pi}_{\mu,\eta}$. In the compact realization, the space of the representation (after completion) $\mathcal{H}_{(\lambda,\epsilon),(\mu,\eta)}$ is the space of smooth sections of a line bundle over $X\times X$. Let $\pi_{\lambda,\epsilon}\otimes\pi_{\mu,\eta}$ be the corresponding tensor product representation in the noncompact picture, acting on $C^\infty(V\times V)$.

Denote by M the operator on $C^\infty(V\times V)$ given by

$$f\in C^\infty(V\times V), \qquad Mf(x,y) = \det(x-y)f(x,y). \tag{13}$$

The operator M satisfies a covariance relation,

$$M\circ(\pi_{\lambda,\epsilon}(g)\otimes\pi_{\mu,\eta}(g)) = (\pi_{\lambda-1,-\epsilon}(g)\otimes\pi_{\mu-1,-\eta}(g))\circ M.$$

Among other things, this relation shows that M can be lifted to the compact picture model, namely as an operator $\widetilde{M}$,

$$\widetilde{M}:\mathcal{H}_{(\lambda,\epsilon),(\mu,\eta)}\to\mathcal{H}_{(\lambda-1,-\epsilon),(\mu-1,-\eta)},$$

and intertwining $\widetilde{\pi}_{\lambda,\epsilon}\otimes\widetilde{\pi}_{\mu,\eta}$ and $\widetilde{\pi}_{\lambda-1,-\epsilon}\otimes\widetilde{\pi}_{\mu-1,-\eta}$. Strictly speaking, the operator $\widetilde{M}$ now depends on the parameters of the representations, but its local expression in the noncompact picture does not.

On the compact realization, now define

$$\widetilde{F}_{(\lambda,\epsilon),(\mu,\eta)} = (\widetilde{I}_{\frac{2n}{r}-1-\lambda,\epsilon}\otimes\widetilde{I}_{\frac{2n}{r}-1-\mu,\eta})\circ\widetilde{M}\circ(\widetilde{I}_{\lambda,\epsilon}\otimes\widetilde{I}_{\mu,\eta}). \tag{14}$$

The next theorem is the main result of [Ben Saïd et al. 2020b].

Theorem 2.2. *The operators $\widetilde{F}_{(\lambda,\epsilon),(\mu,\eta)}$ form a family of differential operators on $X\times X$ depending meromorphically on (λ,μ). They satisfy the following covariance relation, valid for any $g\in G$:*

$$\widetilde{F}_{(\lambda,\epsilon),(\mu,\eta)}\circ(\widetilde{\pi}_{\lambda,\epsilon}(g)\otimes\widetilde{\pi}_{\mu,\eta}(g)) = (\widetilde{\pi}_{\lambda+1,-\epsilon}(g)\otimes\widetilde{\pi}_{\mu+1,-\eta}(g))\circ\widetilde{F}_{(\lambda,\epsilon),(\mu,\eta)}. \tag{15}$$

The covariance property is an obvious consequence of the definition, whereas the proof that it is a *differential* operator requires much work.

The operator $\widetilde{F}_{(\lambda,\epsilon),(\mu,\eta)}$ has a local expression in the noncompact picture, which turns out to be independent of the signs ϵ, η, and is a differential operator on $V \times V$ with polynomial coefficients, henceforth denoted by $F_{\lambda,\mu}$. In the actual proof in [Ben Saïd et al. 2020b], the operator $F_{\lambda,\mu}$ is constructed in the noncompact picture and shown to be covariant, which guarantees its lifting to the compact picture.

Notice however that it is not possible to use the analog of the composition formula (14) directly on $V \times V$ to define $F_{\lambda,\mu}$. In fact, in the compact picture, the space of C^∞ vectors of the representation is the space of C^∞ sections of the line bundle and is stable by the Knapp–Stein operators, which allows us to compose the operators. In the noncompact picture, the space of C^∞ vectors of the representation is difficult to describe and is not adapted to calculations using the Fourier transform. However as M preserves both $\mathcal{S}(V \times V)$ and $\mathcal{S}'(V \times V)$, it is at least possible to compose M and one Knapp–Stein operator (in any order), obtaining an operator mapping $\mathcal{S}(V \times V)$ into $\mathcal{S}'(V \times V)$. It is then possible to apply the symbolic calculus developed in the previous section and to gain some information on the operator $F_{\lambda,\mu}$.

The distinction between V and its dual V^* will be explicitly made, whereas [Ben Saïd et al. 2020b] used the nondegenerate bilinear form $\tau(x, y) = \operatorname{tr}(xy)$ to identify V and V^*.

Proposition 2.3. *The operators* $\widetilde{I}_{\lambda,\epsilon}$ *satisfy*

$$\widetilde{I}_{\lambda,\epsilon} \circ \widetilde{I}_{\frac{2n}{r}-\lambda,\epsilon} = \kappa_2(\lambda, \epsilon)\,\mathrm{Id}, \tag{16}$$

where $\kappa_2(\lambda, \epsilon)$ *is a meromorphic function* $\not\equiv 0$ *on* $\mathbb{C}$.

This result is well known, and can be shown through the noncompact picture. First, when $\Re(\lambda) = \frac{n}{r}$, the representation is unitary on $L^2(V)$ and the Knapp–Stein operator is also unitary (up to a scalar), as can be seen in the noncompact picture using the Fourier transform of the kernel or $I_{\lambda,\epsilon}$ (see (12)). Further, the inverse of $I_{\lambda,\epsilon}$ is equal to $I_{\frac{2n}{r}-\lambda,\epsilon}$ (up to a scalar) and hence the inverse of $\widetilde{I}_{\lambda,\epsilon}$ is equal to $\widetilde{I}_{\frac{2n}{r}-\lambda,\epsilon}$ (up to a scalar). Finally, this relation is extended by analytic continuation to all generic λ (i.e., outside of the poles of the Knapp–Stein operators).

From the definition of $\widetilde{F}_{(\lambda,\epsilon),(\mu,\eta)}$ it is possible to deduce

$$\begin{aligned}\widetilde{F}_{(\lambda,\epsilon),(\mu,\eta)} \circ (\widetilde{I}_{\frac{2n}{r}-\lambda,\epsilon} \otimes \widetilde{I}_{\frac{2n}{r}-\mu,\eta})&\\ = \kappa_3(\lambda, \epsilon; \mu, \eta)(\widetilde{I}_{\frac{2n}{r}-1-\lambda,-\epsilon} &\otimes \widetilde{I}_{\frac{2n}{r}-1-\mu,-\eta}) \circ \widetilde{M},\end{aligned} \tag{17}$$

$$\begin{aligned}(\widetilde{I}_{\lambda+1,-\epsilon} \otimes \widetilde{I}_{\mu+1,-\eta}) \circ \widetilde{F}_{(\lambda,\epsilon),(\mu,\eta)}&\\ = \kappa_4(\lambda, \epsilon; \mu, \eta)\widetilde{M} \circ (\widetilde{I}_{\lambda,\epsilon} &\otimes \widetilde{I}_{\mu,\eta}),\end{aligned} \tag{18}$$

where κ_3 and κ_4 are meromorphic $\not\equiv 0$ functions on $\mathbb{C} \times \mathbb{C}$.[4]

[4] The coefficients κ_3 and κ_4 could be expressed in terms of κ_2, but their exact expressions are not needed in the sequel.

Now transfer these identities to the noncompact picture. The operator $I_{\lambda,\epsilon}$ is a convolution operator and recall that $\ell_{\lambda,\epsilon}$ is the Fourier transform of its kernel (see (12)), so $\ell_{\lambda,\epsilon}$ is the symbol of $I_{\lambda,\epsilon}$. As the operator $F_{\lambda,\mu}$ is a differential operator with polynomial coefficients, let $f_{\lambda,\mu}(x, y, \xi, \zeta)$ be its symbol. The two identities (17) and (18) can be translated using the symbolic calculus on $E = V \times V$.

Proposition 2.4. *The following identities hold true*:

$$\begin{aligned}(19)\quad & f_{\lambda,\mu}(x, y, \xi, \zeta)(\ell_{\frac{2n}{r}-\lambda,\epsilon}(\xi) \otimes \ell_{\frac{2n}{r}-\mu,\eta}(\zeta)) \\ &\qquad = \kappa_3(\lambda, \epsilon; \mu, \eta)(\ell_{\frac{2n}{r}-1-\lambda,-\epsilon}(\xi) \otimes \ell_{\frac{2n}{r}-1-\mu,-\eta}(\zeta)) \sharp \det(x - y), \\ (20)\quad & (\ell_{\lambda+1,-\epsilon}(\xi) \otimes \ell_{\mu+1,-\eta}(\zeta)) \sharp f_{\lambda,\mu}(x, y, \xi, \zeta) \\ &\qquad = \kappa_4(\lambda, \epsilon; \mu, \eta) \det(x - y)(\ell_{\lambda,\epsilon}(\xi) \otimes \ell_{\mu,\eta}(\zeta)).\end{aligned}$$

Theorem 4.8 in [Ben Saïd et al. 2020b] was the key result in the construction of the operator $F_{\lambda,\mu}$. Here it is reinterpreted in order to fit with the symbolic calculus developed in the previous section. Let $\det^*$ be the polynomial on E^* obtained through the identification of V^* with V via the bilinear form τ, and let

$$V^{*\times} = \{\xi \in V^*, \det^*(\xi) \neq 0\}.$$

Proposition 2.5. *For generic* $(s, \epsilon), (t, \eta) \in \mathbb{C} \times \{\pm\}$ *there exists a differential operator* $D_{s,t}$ *with polynomial coefficients on* $V^* \times V^*$ *such that*

$$(21)\qquad \det\Big(\frac{1}{i}(\partial_\xi - \partial_\zeta)\Big) \circ (\det^*\xi^{s,\epsilon} \det^*\zeta^{t,\eta}) = (\det^*\xi^{s-1,-\epsilon} \det^*\zeta^{t-1,-\eta}) \circ D_{s,t}.$$

Proof. The existence of such an operator is proved in [Ben Saïd et al. 2020b], but the equality is proved to be valid only on $V^{*\times} \times V^{*\times}$. We want to prove that (21) is an equality of operators on $V^* \times V^*$. Assume for the moment that $\Re(s) \gg 0$. Then $\det \xi^{s,\epsilon}$, extended by 0 on $\{\xi \in V^*, \det^*\xi = 0\}$ is a smooth enough function on V which vanishes on $V^{*\times}$ up to any given (high) order. Similarly for $\det \zeta^{t,\eta}$, so the stronger statement of the proposition is valid for $\Re(s), \Re(t) \gg 0$. Moreover, the operator $D_{s,t}$ has polynomial coefficients which are also polynomial functions in the parameters (s, t). The statement follows for generic s, t by analytic continuation in (s, t). □

The proposition can be translated in the terminology introduced for the symbolic calculus of the previous section.

Proposition 2.6. *The following identity holds true*:

$$(22)\qquad \det(x - y) \flat \det \xi^{s,\epsilon} \det \zeta^{t,\eta} = \det \xi^{s-1,\epsilon} \det \zeta^{t-1,\eta} d_{s,t}(\xi, \zeta, x, y),$$

where $d(\xi, \zeta, x, y)$ *is the symbol of the operator* $D_{s,t}$.

Proposition 2.7. *Let $f^*_{\lambda,\mu}$ be the polynomial function on $(V^* \times V^*) \times (V \times V)$ defined by*

$$f^*_{\lambda,\mu}(\xi, \zeta, x, y) = f_{\lambda,\mu}(x, y, \xi, \zeta).$$

Then,

$$f^*_{\lambda,\mu}(\xi, \zeta, x, y) = \kappa_5(\lambda, \mu)\, d_{\lambda-\frac{n}{r}+1,\mu-\frac{n}{r}+1}(\xi, \zeta, x, y), \tag{23}$$

where $\kappa_5(\lambda, \mu)$ is a meromorphic $\not\equiv 0$ function on $\mathbb{C} \times \mathbb{C}$.

Proof. Use Proposition 1.7 to transform the identity (19) and obtain

$$\kappa_3(\lambda, \epsilon; \mu, \eta) \det(x - y) \flat \left(\ell_{\frac{2n}{r}-1-\lambda,-\epsilon}(\xi)\ell_{\frac{2n}{r}-1-\mu,-\eta}(\zeta)\right) = \left(\ell_{\frac{2n}{r}-\lambda,-\epsilon}(\xi)\ell_{\frac{2n}{r}-\mu,-\eta}(\zeta)\right) f^*_{\lambda,\mu}(\xi, \zeta, x, y). \tag{24}$$

Assume for a moment that $\Re(\lambda), \Re(\mu) \gg 0$. Consider the right-hand side of (24). Recalling formula (12), $\ell_{\frac{2n}{r}-\lambda,-\epsilon}$ is a linear combination of the two distributions $(\det \xi)^{-\frac{n}{r}+\lambda,\pm}$. As $\Re(\lambda) \gg 0$, this distribution is a function, which may be assumed to be smooth up to a large order. Now on the open set $\{\det \xi > 0\} \times \{\det \zeta > 0\}$, $(\det \xi)^{s,\pm} = \det \xi^s$, so that on this open set,

$$\ell_{\frac{2n}{r}-\lambda,-\epsilon}(\xi) = \kappa_6(\lambda, -\epsilon)(\det \xi)^{-\frac{n}{r}+\lambda},$$

where κ_6 is meromorphic in the variable λ.

A similar result holds for $\ell_{\frac{2n}{r}-\mu,-\epsilon}$, so the right-hand side of (24) can be rewritten as

$$\kappa_6(\lambda, -\epsilon)\kappa_6(\mu, -\eta)(\det \xi)^{-\frac{n}{r}+\lambda}(\det \zeta)^{-\frac{n}{r}+\mu} f^*_{\lambda,\mu}(\xi, \zeta, x, y). \tag{25}$$

Consider now the left-hand side of (24). By the previous considerations, on the open set $\{\det \xi > 0\} \times \{\det \zeta > 0\}$,

$$\ell_{\frac{2n}{r}-1-\lambda,-\epsilon}(\xi)\ell_{\frac{2n}{r}-1-\mu,-\eta}(\zeta) = \kappa_6(\lambda+1, \epsilon)\kappa_6(\mu+1, \eta)(\det \xi)^{-\frac{n}{r}+1+\lambda}(\det \zeta)^{-\frac{n}{r}+1+\mu},$$

so that, using (22), the left-hand side of (24) can be rewritten as

$$\kappa_6(\lambda+1, \epsilon)\kappa_6(\mu+1, \eta)(\det \xi)^{-\frac{n}{r}+\lambda}(\det \zeta)^{-\frac{n}{r}+\mu} d_{-\frac{n}{r}+1+\lambda,-\frac{n}{r}+1+\mu}(\xi, \zeta, x, y). \tag{26}$$

Now compare (25) and (26) to conclude that both sides of (23) coincide on the open set $\{\det \xi > 0\} \times \{\det \zeta > 0\}$. As both f^* and d are polynomial functions, the result follows everywhere on $V^* \times V^*$.

By analytic continuation, the conditions $\Re(\lambda) \gg 0$ and $\Re(\mu) \gg 0$ can be removed, thus finishing the proof of the proposition. □

For the last part of this article, renormalize the operator $F_{\lambda,\mu}$ by demanding that its symbol $f_{\lambda,\mu}$ satisfies

$$f_{\lambda,\mu}(x, y, \xi, \zeta) = d_{\lambda-\frac{n}{r}+1,\mu-\frac{n}{r}+1}(\xi, \zeta, x, y). \tag{27}$$

The next proposition gathers the main properties of the operator $F_{\lambda,\mu}$ which have been obtained.

Proposition 2.8. (i) *The operator $F_{\lambda,\mu}$ satisfies the covariance relation, valid for any $g \in G$,*

$$F_{\lambda\mu} \circ (\pi_{\lambda,\epsilon}(g) \otimes \pi_{\mu,\eta}(g)) = (\pi_{\lambda+1,-\epsilon}(g) \otimes \pi_{\mu+1,-\eta}(g)) \circ F_{\lambda,\mu}. \tag{28}$$

(ii) *The coefficients of $F_{\lambda,\mu}$ are polynomial functions on $V \times V$, depending only on $(x - y)$.*

(iii) *The coefficients of $F_{\lambda,\mu}$ depend polynomially on the parameters λ, μ.*

The fact that the coefficients of $F_{\lambda,\mu}$ depend only on $(x - y)$ can be deduced from the covariance property (28) when applied to translations by elements of V, acting diagonally on $V \times V$ by $(x, y) \mapsto (x + v, y + v)$.

The operator $F_{\lambda,\mu}$ is called the *source operator* and, as explained in the introduction, it can be used for constructing covariant bidifferential operators, thus justifying the name of source operator.

3. The covariant bidifferential operators

Let $\operatorname{res} : \mathcal{S}(V \times V) \to \mathcal{S}(V)$ be the restriction operator from $V \times V$ to $\operatorname{diag}(V) = \{(x, x), x \in V\} \simeq V$ given by

$$\operatorname{res}(f)(x) = f(x, x) \quad \text{for } x \in V.$$

Proposition 3.1. *For $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$, for any $g \in G$,*

$$\operatorname{res} \circ (\pi_{\lambda,\epsilon}(g) \otimes \pi_{\mu,\eta}(g) = \pi_{\lambda+\mu,\epsilon\eta}(g) \circ \operatorname{res}.$$

The proof is elementary and left to the reader.

For any positive integer k, let

$$F^{(k)}_{\lambda,\mu} = F_{\lambda+k-1,\mu+k-1} \circ \cdots \circ F_{\lambda,\mu}$$

and

$$B^{(k)}_{\lambda,\mu} = \operatorname{res} \circ F^{(k)}_{\lambda,\mu}.$$

Proposition 3.2. *The operator $B^{(k)}_{\lambda,\mu}$ satisfies the following covariance relation, valid for any $g \in G$:*

$$B^{(k)}_{\lambda,\mu} \circ (\pi_{\lambda,\epsilon}(g) \otimes \pi_{\mu,\eta}(g)) = \pi_{\lambda+\mu+2k,\epsilon\eta}(g) \circ B^{(k)}_{\lambda,\mu}. \tag{29}$$

As the coefficients of $F^{(k)}_{\lambda,\mu}$ depend only on $x - y$, the bidifferential operator $B^{(k)}_{\lambda,\mu}$ has constant coefficients. It is also a consequence of the covariance relation when applied to translations by elements of V.

There is a natural notion of symbol for a bidifferential operator

$$B : C^\infty(V \times V) \to C^\infty(V)$$

(say with polynomial coefficients), extending the classical definition by letting

$$B(e^{i(x.\xi+y.\zeta)})_{x=y} = b(x, \xi, \zeta)e^{i(x.\xi+x.\zeta)}.$$

Proposition 3.3. *The symbol of the operator $B^{(k)}_{\lambda,\mu}$ denoted by $b^{(k)}_{\lambda,\mu}(\xi, \zeta)$ is equal to*

$$b^{(k)}_{\lambda,\mu}(\xi, \eta) = f^{(k)}_{\lambda,\mu}(0, 0, \xi, \eta). \tag{30}$$

Proof. This is a consequence of the fact that the coefficients of $F_{\lambda,\mu}$ and hence of $F^{(k)}_{\lambda,\mu}$ only depend on $(x - y)$. When restricting to the diagonal $\{x = y\}$, all terms of $F_{\lambda,\mu}$ cancel except those with constant coefficients. □

In the next statements and proofs, the signs $\pm$ will be omitted. Identities are proved on $\{\det \xi > 0\} \times \{\det \zeta > 0\}$ and then extended to $V^* \times V^*$.

Proposition 3.4. *The symbols $b^{(k)}_{\lambda,\mu}$ satisfy the recurrence relation*

$$\begin{aligned}(31)\quad & \det\xi^{\lambda-\frac{n}{r}} \det\zeta^{\mu-\frac{n}{r}} b^{(k)}_{\lambda,\mu}(\xi, \zeta) \\ &= \det\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\zeta}\right)\left(\det\xi^{\lambda-\frac{n}{r}+1} \det\zeta^{\mu-\frac{n}{r}+1} b^{(k-1)}_{\lambda+1,\mu+1}(\xi, \zeta)\right).\end{aligned}$$

Proof.

$$\begin{aligned}B^{(k)}_{\lambda,\mu} &= \operatorname{res} \circ (F_{(\lambda+k-1,\mu+k-1)} \circ \cdots \circ F_{\lambda+1,\mu+1}) \circ F_{\lambda,\mu} \\ &= \operatorname{res} \circ F^{(k-1)}_{\lambda+1,\mu+1} \circ F_{\lambda,\mu}\end{aligned}$$

and its symbol satisfies

$$b^{(k)}_{\lambda,\mu}(\xi, \zeta) = (b^{(k-1)}_{\lambda+1,\mu+1} \# f_{\lambda,\mu})(0, 0, \xi, \zeta).$$

Now use Proposition 1.7 and (27) to rewrite this last equation as

$$b_{\lambda,\mu}(\xi, \zeta) = (d_{\lambda-\frac{n}{r}+1,\mu-\frac{n}{r}+1} \flat\, b^{(k)}_{\lambda+1,\mu+1})(\xi, \zeta, 0, 0)$$

and by Proposition 1.6,

$$b_{\lambda,\mu}(\xi, \zeta) = D_{\lambda-\frac{n}{r}+1,\mu-\frac{n}{r}+1}(b^{k-1}_{\lambda+1,\mu+1})(\xi, \zeta).$$

Hence,

$$\det\xi^{\lambda-\frac{n}{r}} \det\zeta^{\mu-\frac{n}{r}} b_{\lambda,\mu}(\xi, \zeta) = (\det\xi^{\lambda-\frac{n}{r}} \det\zeta^{\mu-\frac{n}{r}} D_{\lambda-\frac{n}{r}+1,\mu-\frac{n}{r}+1}) b^{(k-1)}_{\lambda+1,\mu+1}(\xi, \eta)$$

$$= \left(\det\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\zeta}\right) \circ \det\xi^{\lambda-\frac{n}{r}+1} \det\zeta^{\mu-\frac{n}{r}+1}\right) b^{(k-1)}_{\lambda+1,\mu+1}(\xi, \zeta)$$

and by Proposition 2.5 the proposition follows. □

4. The Rodrigues formula

The *Rodrigues formula* is a type of formula which is valid for many orthogonal polynomials or special functions of one variable (see formulae 8.960.1, 8.939.7, 8.949.7, 8.949.8, 8.959.1 of [Gradshteyn and Ryzhik 2007]). These formulae imply recurrence relations which are of the same type as the recurrence relation satisfied by the symbols $d^{(k)}_{\lambda,\mu}$. This remark allows us to solve the recurrence relation and to determine the symbols completely.

Theorem 4.1. *Let $s,t\in\mathbb{C}$. For any integer k, there exists a (unique) polynomial $c^{(k)}_{s,t}$ on $V\times V$ such that on $\{\det\xi>0\}\times\{\det\zeta>0\}$*

$$\det\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\zeta}\right)^{k}(\det\xi^{s+k}\det\zeta^{t+k})=c^{(k)}_{s,t}(\xi,\zeta)\det\xi^{s}\det\zeta^{t}. \tag{32}$$

Proof. For $k=0$, $c^{(0)}_{s,t}\equiv 1$, so the statement is valid. Now argue by induction on k. Assume that $\det\xi>0$ and $\det\zeta>0$. Then

$$\begin{aligned}\det&\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\zeta}\right)^{k}(\det\xi^{s+k}\det\zeta^{t+k})\\&=\det\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\zeta}\right)\left(\det\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\zeta}\right)^{k-1}\det\xi^{s+1+k-1}\det\zeta^{t+1+k-1}\right)\\&=\det\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\zeta}\right)(\det\xi^{s+1}\det\zeta^{t+1}c^{(k-1)}_{s+1,t+1}(\xi,\zeta)).\end{aligned}$$

As by induction, $c^{(k-1)}_{s+1,t+1}$ is supposed to be a polynomial, the statement to be proved is a consequence of Theorem 4.8 in [Ben Saïd et al. 2020b]. ☐

Homogeneity considerations show that $c^{(k)}_{s,t}$ is a homogeneous polynomial of degree kr on $V\times V$.

A by-product of the proof of Theorem 4.1 is the following proposition.

Proposition 4.2. *The polynomials $c^{(k)}_{s,t}$ satisfy the recurrence relation*

$$\det\xi^{s}\det\zeta^{t}c^{(k)}_{s,t}(\xi,\zeta)=\det\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\zeta}\right)(\det\xi^{s+1}\det\zeta^{t+1}c^{(k-1)}_{s+1,t+1}(\xi,\zeta)). \tag{33}$$

Proposition 4.3. *For $\lambda,\mu\in\mathbb{C}$ and $k\in\mathbb{N}$,*

$$d^{(k)}_{\lambda,\mu}(\xi,\zeta)=c^{(k)}_{\lambda-\frac{n}{r},\mu-\frac{n}{r}}(\xi,\zeta). \tag{34}$$

Proof. The two families of polynomials

$$(d^{(k)}_{\lambda,\mu})_{k\in\mathbb{N}}\quad\text{and}\quad(c^{(k)}_{\lambda-\frac{n}{r},\mu-\frac{n}{r}})_{k\in\mathbb{N}}$$

satisfy the same recurrence relation. Further,

$$c^{(0)}_{\lambda-\frac{n}{r},\mu-\frac{n}{r}}(\xi,\zeta)\equiv 1 \quad\text{and}\quad d^{(0)}_{\lambda,\mu}(\xi,\zeta)\equiv 1,$$

so by induction on k the two families coincide. □

Having computed the symbols of the bidifferential operators, it is possible to express the operators themselves. As the symbols are homogeneous polynomials, we may omit the factors i or $\frac{1}{i}$ and reformulate the previous results as follows, ignoring some powers of i for simplicity.

Theorem 4.4. *Let $c^{(k)}_{s,t}$ be the polynomial defined by the Rodrigues formula* (32). *Let $B^{(k)}_{\lambda,\mu}$ be the bidifferential operator on $V\times V$ defined by*

$$B^{(k)}_{\lambda,\mu}=\operatorname{res}\circ c^{(k)}_{\lambda-\frac{n}{r},\mu-\frac{n}{r}}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right).$$

Then, for any $\epsilon,\eta\in\{\pm\}$,

$$B^{(k)}_{\lambda,\mu}\circ(\pi_{\lambda,\epsilon}(g)\otimes\pi_{\mu,\eta}(g))=\pi_{\lambda+\mu+2k,\epsilon\eta}(g)\circ B^{(k)}_{\lambda,\mu}. \tag{35}$$

It might be of some interest to study the family of polynomials $(c^{(k)}_{\lambda,\mu})_{k\in\mathbb{N}}$ and to explore whether there is any connection with the theory of hypergeometric functions for Jordan algebras as developed in [Faraut and Korányi 1994, Chapter XV].

5. Some examples

There are some situations were the source operator is explicitly known, so the Rodrigues formula can be obtained by a direct calculation.

For the classical *Rankin–Cohen brackets* case, which in the present approach corresponds to the case $V=\mathbb{R}$, the source operator is obtained in [Clerc 2017a, Section 5] and deduced from the *Cayley operator*. The symbols of the Rankin–Cohen operators are known to be related to the Jacobi polynomials (see [Kobayashi and Pevzner 2016]). The Rodrigues formula obtained in this article is shown to correspond to the classical Rodrigues formula for the Jacobi polynomials.

The source operator is also known for the *conformally covariant bidifferential operators on* $\mathbb{R}^n$, $n\geq 2$. The Jordan algebra is $V=\mathbb{R}^n$ with the Jordan multiplication

$$(x_1,x_2,\dots,x_n)(y_1,y_2,\dots,y_n)$$
$$=\big((x_1y_1-x_2y_2-\cdots-x_ny_n),x_1y_2+x_2y_1,\dots,x_1y_n+x_ny_1\big)$$

The conformal group is $SO_0(1,n+1)$ and the determinant is the quadratic form

$$\det x=x_1^2+x_2^2+\cdots+x_n^2.$$

The source operator was computed in [Beckmann and Clerc 2012]; see also [Ben Saïd et al. 2020b, Section 10]. The Rodrigues formula is new.

Our third example is concerned with the *Juhl operators*. The geometric context is different, but still in the realm of symmetry breaking differential operators as presented in the introduction. The Juhl operators are differential operators from S^n to S^{n-1} (from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ in the noncompact picture) and they are covariant with respect to the conformal group of S^{n-1} viewed as the subgroup of the conformal group of S^n which stabilizes $S^{n-1} \subset S^n$. The symbols of these operators were already known to be connected with the Gegenbauer polynomials (see [Juhl 2009; Kobayashi and Pevzner 2016]).

Rodrigues formula for the symbols of the classical Rankin–Cohen brackets. Let $V = \mathbb{R}$ with its usual product. The group G is equal to $\mathrm{SL}(2, \mathbb{R})$ acting by projective transformations on $\mathbb{R}$, and the determinant is given by $\det x = x$. The representations $\pi_{\lambda,\epsilon}$ are given by

$$\pi_{\lambda,\epsilon}(g) f(x) = (cx + d)^{-\lambda,\epsilon} f((ax + b)(cx + d)^{-1}),$$

for $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$.

The source operator $F_{\lambda,\mu}$ is given by

$$F_{\lambda,\mu} = (x - y)\frac{\partial^2}{\partial x \partial y} - \mu\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y} \tag{36}$$

and its symbol is

$$f_{\lambda,\mu}(x, y, \xi, \eta) = -(x - y)\xi\eta + i(-\mu\xi + \lambda\eta). \tag{37}$$

The symbols $b^{(k)}_{\lambda,\mu}$ of the Rankin–Cohen brackets satisfy

$$b^{(k)}_{\lambda,\mu}(\xi, \zeta) = \big(b^{(k-1)}_{\lambda+1,\mu+1}(\xi, \zeta) \,\#\, f_{\lambda,\mu}(x, y, \xi, \zeta)\big)(0, 0, \xi, \zeta).$$

Now use (37) and the composition formula (5) to get

$$b^{(k)}_{\lambda,\mu} = i\left((-\mu\xi + \lambda\zeta)b^{(k-1)}_{\lambda+1,\mu+1} + \xi\zeta\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\zeta}\right)b^{(k-1)}_{\lambda+1,\mu+1}\right). \tag{38}$$

The next proposition introduces a family of polynomials which will be shown to solve the recursion relation.

Proposition 5.1. *Let* $\alpha, \beta \in \mathbb{C}$. *For any* $l \in \mathbb{N}$, *there exists a* (*unique*) *polynomial* $q^{\alpha,\beta}_l(\xi, \eta)$, *homogeneous of degree* l, *such that*

$$\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)^l \xi^{\alpha+l}\eta^{\beta+l} = \xi^\alpha\eta^\beta q^{\alpha,\beta}_l(\xi, \eta). \tag{39}$$

Proposition 5.2. *The polynomials* $q^{\alpha,\beta}_l(\xi, \eta)$ *satisfy the relation*

$$q^{\alpha,\beta}_l = \xi\eta\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)q^{\alpha+1,\beta+1}_{l-1} + \big(-(\beta+1)\xi + (\alpha+1)\eta\big)q^{\alpha+1,\beta+1}_{l-1}. \tag{40}$$

Proof. Observe that

$$\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)^{l}(\xi^{\alpha+l}\eta^{\beta+l}) = \left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\left(\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)^{l-1}\xi^{\alpha+1+l-1}\eta^{\beta+1+l-1}\right)$$
$$= \left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)(\xi^{\alpha+1}\eta^{\beta+1}q_{l-1}^{\alpha+1,\beta+1})(\xi,\eta)$$

and use the Leibniz rule to conclude the claim. □

Proposition 5.3. *For* $\lambda,\mu\in\mathbb{C}$ *and* $k\in\mathbb{N}$,

$$b_{\lambda,\mu}^{(k)}(\xi,\eta) = q_k^{\lambda-1,\mu-1}(i\xi,i\zeta) = i^k q_k^{\lambda-1,\mu-1}(\xi,\zeta). \tag{41}$$

Proof. Set $\alpha=\lambda-1,\ \beta=\mu-1$ and compare (38) and (40) to show that $b_{\lambda,\mu}^{(k)}(\xi,\zeta)$ and $q_k^{\alpha,\beta}(i\xi,i\zeta)$ satisfy the same recurrence relation. As $b_{\lambda,\mu}^{(0)}=q_0^{\lambda-1,\mu-1}\equiv 1$, the conclusion follows by induction on k. □

Theorem 5.4. *Let* $q_k^{\alpha,\beta}$ *be the polynomials defined by the Rodrigues formula* (39). *Let* $(\lambda,\epsilon),(\mu,\eta)\in\mathbb{C}\{\pm\}$ *and let* $k\in\mathbb{N}$. *Let* $B_{\lambda,\mu}^{(k)}$ *be the bidifferential operator on* $V\times V$ *defined by*

$$B_{\lambda,\mu}^{(k)} = q_k^{\lambda-1,\mu-1}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right).$$

Then, for any $g\in G$,

$$B_{\lambda,\mu}^{(k)}\circ(\pi_{\lambda,\epsilon}(g)\otimes\pi_{\mu,\eta}) = \pi_{\lambda+\mu+2k,\epsilon\eta}(g)\circ B_{\lambda,\mu}^{(k)}.$$

This is a consequence of the last results, ignoring the factor i^k in this final formulation.

Finally, the polynomials $q_k^{\alpha,\beta}$ are closely related to the *Jacobi polynomials*. First recall the Rodrigues formula which can be taken as a definition of the Jacobi polynomials (see [Gradshteyn and Ryzhik 2007, 8.960.1, p. 998]):

$$(1-t)^{\alpha}(1+t)^{\beta}P_k^{\alpha,\beta}(t) = \frac{(-1)^k}{2^k l!}\left(\frac{d}{dt}\right)^k((1-t)^{k+\alpha}(1+t)^{k+\beta}). \tag{42}$$

Kobayashi and Pevzner [2016] define a family of homogeneous polynomials $\widetilde{P}_k^{\alpha,\beta}$ of two variables by the formula

$$\widetilde{P}_k^{\alpha,\beta}(\xi,\eta) = (-1)^k(\xi+\eta)^k P_k^{\alpha,\beta}\left(\frac{\eta-\xi}{\xi+\eta}\right). \tag{43}$$

Proposition 5.5. *Let* $\alpha,\beta\in\mathbb{C}$. *Then, for all* $l\in\mathbb{N}$,

$$q_k^{\alpha,\beta} = (-1)^k\, k!\, \widetilde{P}_k^{\alpha,\beta}. \tag{44}$$

Proof. For F a function of two variables (ξ,η), associate the function f of one variable given by
$$f(t)=F(1-t,1+t).$$

Then,
$$\left(\frac{d}{dt}\right)^k f(t)=(-1)^k\left(\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)^k F\right)(1-t,1+t).$$

Apply this result to the function $F(\xi,\eta)=\xi^{k+\alpha}\eta^{k+\beta}$, which corresponds to $f(t)=(1-t)^{k+\alpha}(1+t)^{k+\beta}$. On one hand, by the Rodrigues formula
$$\left(\frac{d}{dt}\right)^k (1-t)^\alpha(1-t)^\beta=(-1)^k2^k\,k!\,(1-t)^\alpha(1+t)^\beta P_k^{\alpha,\beta}(t),$$

whereas on the other hand,
$$\left(\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)^k F\right)(1-t,1+t)=(1-t)^\alpha(1+t)^\beta q_k^{\alpha,\beta}(1-t,1+t).$$

Hence,
$$2^k k!P_k^{\alpha,\beta}(t)=q_k^{\alpha,\beta}(1-t,1+t).$$

Now let $(\xi,\eta)\in\mathbb{R}^2$ such that $\xi+\eta=2$. Write $\xi=1-t$ and $\eta=1+t$, so that $t=(\eta-\xi)/(\xi+\eta)$. Then,
$$k!(\xi+\eta)^k P_k^{\alpha,\beta}\left(\frac{\eta-\xi}{\xi+\eta}\right)=q_k^{\alpha,\beta}(\xi,\eta),$$

or equivalently, using (43),
$$(-1)^k k!\widetilde{P}_k^{\alpha,\beta}(\xi,\eta)=q_k^{\alpha,\beta}(\xi,\eta)$$

whenever $\xi+\eta=2$. As both sides are homogeneous polynomials of degree k, the conclusion follows. □

Conformally covariant bidifferential operators. Let $V=\mathbb{R}^n$ with the Jordan multiplication

$$(x_1,x_2,\dots,x_n)(y_1,y_2,\dots,y_n)$$
$$=\big((x_1y_1-x_2y_2-\cdots-x_ny_n),x_1y_2+x_2y_1,\dots,x_1y_n+x_ny_1\big).$$

The group G is $\mathrm{SO}_0(1,n+1)$ and the determinant is the quadratic form
$$q(x)=|x|^2=x_1^2+x_2^2+\cdots+x_n^2.$$

In this case, as q is positive-definite, there is no use in introducing the signs $\pm$. Hence the representations to be considered belong to the scalar principal series and are given by

$$\pi_\lambda(g)f(x)=\kappa(g^{-1},x)^\lambda f(g^{-1}(x)),$$

where $\lambda\in\mathbb{C}$ and $\kappa(g,x)$ is the conformal factor of g at x.

The source operator was computed in [Beckmann and Clerc 2012]; see also [Ben Saïd et al. 2020b, Section 10].

Denote by Δ_x (resp. Δ_y) the Laplacian on $\mathbb{R}^n$ with respect to the variable x (resp. y) and let

$$\nabla_{xy}=\sum_{j=1}^n\frac{\partial^2}{\partial x_j\partial y_j}.$$

The source operator in this case is the differential operator $F_{\lambda,\mu}$ on $\mathbb{R}^n\times\mathbb{R}^n$ given by

$$\begin{aligned}(45)\quad F_{\lambda,\mu}&=|x-y|^2\Delta_x\Delta_y\\&\quad+2(2\lambda-n+2))\sum_{j=1}^n(x_j-y_j)\frac{\partial}{\partial x_j}\Delta_y+2(2\mu-n+2)\sum_{j=1}^n(y_j-x_j)\frac{\partial}{\partial y_j}\Delta_x\\&\quad+2\mu(2\mu-n+2)\Delta_x-2(2\lambda-n+2)(2\mu-n+2)\nabla_{x,y}+2\lambda(2\lambda-n+2)\Delta_y.\end{aligned}$$

The symbol $f_{\lambda,\mu}$ of the source operator $E_{\lambda,\mu}$ is given by

$$\begin{aligned}(46)\quad &f_{\lambda,\mu}(x,y,\xi,\zeta)\\&=|x-y|^2|\xi|^2|\zeta|^2\\&\quad-i\Big(2(2\lambda-n+2)\sum_{j=1}^n(x_j-y_j)\xi_j|\zeta|^2+2(2\mu-n+2)\sum_{j=1}^n(y_j-x_j)\zeta_j|\xi|^2\Big)\\&\quad-(+2\mu(2\mu-n+2)|\xi|^2-2(2\lambda-n+2)(2\mu-n+2)\langle\xi,\zeta\rangle+2\lambda(2\lambda-n+2)|\zeta|^2).\end{aligned}$$

As before,

$$b^{(k)}_{\lambda,\mu}(\xi,\zeta)=\big(b^{(k-1)}_{\lambda+1,\mu+1}\sharp f_{\lambda,\mu}\big)(0,0,\xi,\zeta),$$

which after computation amounts to the recurrence relation

$$\begin{aligned}(47)\quad &b^{(k)}_{\lambda,\mu}(\xi,\zeta)\\&=i\Big(\big(2\mu(2\mu-n+2)|\xi|^2-2(2\lambda-n+2)(2\mu-n+2)\langle\xi,\zeta\rangle+2\lambda(2\lambda-n+2)|\zeta|^2\big)\\&\qquad\times b^{(k-1)}_{\lambda+1,\mu+1}(\xi,\zeta)\\&\qquad+2(2\lambda-n+2)|\zeta|^2\sum_{j=1}^n\xi_j\Big(\frac{\partial}{\partial\xi_j}-\frac{\partial}{\partial\zeta_j}\Big)b^{(k-1)}_{\lambda+1,\mu+1}(\xi,\zeta)\\&\qquad+2(2\mu-n+2)|\xi|^2\sum_{j=1}^n\eta_j\Big(\frac{\partial}{\partial\zeta_j}-\frac{\partial}{\partial\xi_j}\Big)b^{(k-1)}_{\lambda+1,\mu+1}(\xi,\zeta)\\&\qquad+|\xi|^2|\zeta|^2q\Big(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\zeta}\Big)b^{(k-1)}_{\lambda+1,\mu+1}(\xi,\zeta)\Big).\end{aligned}$$

Together with the condition $b^{(0)}_{\lambda,\mu} \equiv 1$, the recurrence relation (47) determines $b^{(k)}_{\lambda,\mu}$ by induction on k. To solve this recurrence relation, let us introduce for $\alpha, \beta \in \mathbb{C}$ the family of polynomials $p_k^{\alpha,\beta}$ on $(\mathbb{R}^n \times \mathbb{R}^n)^*$ defined by

$$q\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)^k \left(|\xi|^{2(\alpha+k)}|\eta|^{2(\beta+k)}\right) = p_k^{\alpha,\beta}(\xi,\eta)|\xi|^{2\alpha}|\eta|^{2\beta}. \tag{48}$$

It is easily seen that $p_k^{\alpha,\beta}$ thus defined is a homogeneous polynomial of degree $2k$.

Proposition 5.6. *The polynomials* $p_k^{\alpha,\beta}$ *satisfy the recurrence relation*

$$\begin{aligned} p_k^{\alpha,\beta}&(\xi,\eta) \\ &= |\xi|^2|\eta|^2 q\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) p_{k-1}^{\alpha+1,\beta+1} \\ &\quad + 2(2\alpha+2)|\eta|^2 \sum_{j=1}^n \xi_j \left(\frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial \eta_j}\right) p_{k-1}^{\alpha+1,\beta+1} \\ &\quad + 2(2\beta+2)|\xi|^2 \sum_{j=1}^n \eta_j \left(\frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \xi_j}\right) p_{k-1}^{\alpha+1,\beta+1} \\ &\quad + \big((2\alpha+2)(2\alpha+n)|\eta|^2 - 2(2\alpha+2)(2\beta+2)\langle \xi,\eta\rangle + (2\beta+2)(2\beta+n)|\xi|^2\big) \\ &\qquad\qquad \times p_{k-1}^{\alpha+1,\beta+1}. \end{aligned} \tag{49}$$

Proof. Observe that

$$\begin{aligned} q\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)^k &|\xi|^{2(\alpha+k)}|\eta|^{2(\beta+k)} \\ &= q\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\left(q\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)^{k-1} |\xi|^{2(\alpha+1+(k-1))}|\eta|^{2(\beta+1+(k-1))}\right) \\ &= q\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\left(p_{k-1}^{\alpha+1,\beta+1}(\xi,\eta)|\xi|^{2(\alpha+1)}|\eta|^{2(\beta+1)}\right). \end{aligned}$$

A straightforward calculation yields the result. □

Proposition 5.7. *For* $\lambda, \mu \in \mathbb{C}$ *and for* $k \in \mathbb{N}$,

$$\begin{aligned} b^{(k)}_{\lambda,\mu}(\xi,\zeta) &= p_k^{\lambda-\frac{n}{2},\mu-\frac{n}{2}}(i\xi, i\zeta) \\ &= (-1)^k p_k^{\lambda-\frac{n}{2},\mu-\frac{n}{2}}(\xi,\zeta). \end{aligned} \tag{50}$$

Proof. As $d^{(0)}_{\lambda,\mu} = 1$ and $p_0^{\alpha,\beta} = 1$, the conclusion follows by comparing (47) and (49) after setting $\alpha = \lambda - \frac{n}{2}$, $\beta = \mu - \frac{n}{2}$. □

Theorem 5.8. *Let* $(p_k^{\alpha,\beta})_{k\in\mathbb{N}}$ *be the polynomials defined by* (48). *For* $\lambda, \mu \in \mathbb{C}$ *and* $k \in \mathbb{N}$, *let* $B_{\lambda,\mu}^{(k)}$ *be the bidifferential operator defined by*

$$B_{\lambda,\mu}^{(k)} = \operatorname{res} \circ p_k^{\lambda-\frac{n}{2},\mu-\frac{n}{2}}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

Then, *for any* $g \in G$,

$$B_{\lambda,\mu} \circ (\pi_\lambda(g) \otimes \pi_\mu(g)) = \pi_{\lambda+\mu+2k}(g) \circ B_{\lambda,\mu}.$$

This is a consequence of the previous results, ignoring the factor $(-1)^k$.

For generic (λ, μ), the operators $B_{\lambda,\mu}^{(k)}$ exhaust (up to a scalar) the covariant bidifferential operators. For a general discussion of the existence and uniqueness/multiplicity of the covariant bidifferential operators in this case, see [Clerc 2016; 2017c].

The Juhl operators. Some years ago, A. Juhl [2009] introduced a family of differential operators from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ which are covariant for the subgroup of the conformal group of $\mathbb{R}^n$ which preserves the hyperplane $\mathbb{R}^{n-1}$. Recently, I presented a new approach to these operators (see [Clerc 2017a]), based on the source operator method.

The group $G = \mathrm{SO}_0(1, n+1)$ acts conformally on $\mathbb{R}^n$ by a rational action. For $g \in G$ defined at $x \in \mathbb{R}^n$, let $\kappa(g, x)$ be the conformal factor of g at x, so that for every $v \in \mathbb{R}^n$,

$$|Dg(x)v| = \kappa(g, x)|v| \quad \text{for all } v \in \mathbb{R}^n.$$

For $\lambda \in \mathbb{C}$, the *principal series representation* π_λ of G (in the noncompact picture) is given by

$$\pi_\lambda(g) f(x) = \kappa(g^{-1}, x)^\lambda f(g^{-1}(x)),$$

where $f \in C^\infty(\mathbb{R}^n)$.

Let us identify the hyperplane $\{x \in \mathbb{R}^n, x_n = 0\}$ with $\mathbb{R}^{n-1}$ and write $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$. The subgroup H of G which stabilizes this hyperplane can be identified with $\mathrm{SO}_0(1, n)$. For $\mu \in \mathbb{C}$, the scalar principal series representation π'_μ of H is realized on $C^\infty(\mathbb{R}^{n-1})$ and given by

$$\pi'_\mu(h) f(x') = \kappa(h^{-1}, x')^\mu f(h^{-1}(x')), \quad h \in H,$$

where $f \in C^\infty(\mathbb{R}^{n-1})$.

For $\lambda \in \mathbb{C}$, let E_λ be the differential operator on $\mathbb{R}^n$ given by

$$E_\lambda = x_n \Delta + (2\lambda - n + 2)\frac{\partial}{\partial x_n}, \tag{51}$$

where $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$ is the usual Laplacian on $\mathbb{R}^n$. The operator E_λ has polynomial coefficients and its symbol is given by

$$e_\lambda(x, \xi) = -x_n|\xi|^2 + i(2\lambda - n + 2)\xi_n. \tag{52}$$

Proposition 5.9. *For any $h \in H$,*

$$E_\lambda \circ \pi_\lambda(h) = \pi_{\lambda+1}(h) \circ E_\lambda. \tag{53}$$

See [Clerc 2017a] for the proof. The operator E_λ is called *the source operator* and plays the same rôle for the construction of Juhl operators as the source operator $F_{\lambda,\mu}$ did for the construction of the Rankin–Cohen operators. For $k \geq 1$, define

$$E_\lambda^{(k)} = E_{\lambda+k-1} \circ \cdots \circ E_\lambda. \tag{54}$$

The operator $E_\lambda^{(k)}$ has polynomial coefficients and its symbol is denoted by $e_\lambda^{(k)}$. It satisfies the covariance relation

$$E_\lambda^{(k)} \circ \pi_\lambda(h) = \pi_{\lambda+k}(h) \circ E_\lambda^{(k)} \quad \text{for all } h \in H. \tag{55}$$

Notice that the coefficients of $E_\lambda^{(k)}$ depend only on the variable x_n, a consequence of the covariance relation when applied to the translations by vectors belonging to the hyperplane $\{x_n = 0\}$.

Define the restriction map res by

$$C^\infty(\mathbb{R}^n) \ni f \mapsto \operatorname{res}(f) \in C^\infty(\mathbb{R}^{n-1}), \quad \operatorname{res}(f)(x') = f(x', 0).$$

Notice that for any $\lambda \in \mathbb{C}$,

$$\operatorname{res} \circ \pi_\lambda(h) = \pi'_\lambda(h) \circ \operatorname{res} \quad \text{for all } h \in H.$$

For $k \in \mathbb{N}$, define

$$J_\lambda^{(k)} = \operatorname{res} \circ E_\lambda^{(k)} = \operatorname{res} \circ E_{\lambda+k-1} \circ \cdots \circ E_\lambda. \tag{56}$$

Proposition 5.10. *For any $h \in H$,*

$$J_\lambda^{(k)} \circ \pi_\lambda(h) = \pi'_{\lambda+k}(h) \circ J_\lambda^{(k)}. \tag{57}$$

The operator $J_\lambda^{(k)}$ is a differential operator from $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$ with polynomial coefficients. Moreover, $J_\lambda^{(k)}$ has constant coefficients, as a consequence of the covariance property (57) for h a translation along a vector in $\mathbb{R}^{n-1}$. Hence the symbol $j_\lambda^{(k)}$ of $J_\lambda^{(k)}$ depends only on $\xi \in \mathbb{R}^n$.

The definition of the operators $J_\lambda^{(k)}$ implies a recurrence relation for their symbols.

Proposition 5.11. *The polynomials $j_\lambda^{(k)}$ satisfy the relation*

$$j_\lambda^{(k)} = \frac{1}{i}(2\lambda - n + 2)\xi_n j_{\lambda+1}^{(k-1)} + |\xi|^2 \frac{\partial}{\partial \xi_n} j_{\lambda+1}^{(k-1)}. \tag{58}$$

Proof. As already noted, the coefficients of E_λ depend only on x_n. The same is true (and for the same reason) for the operators $E_\lambda^{(k)}$. As $J_\lambda^{(k)} = \operatorname{res} \circ E_\lambda^{(k)}$,

$$j_\lambda^{(k)}(\xi) = e_\lambda^{(k)}(0, \xi). \tag{59}$$

Next,

$$E_\lambda^{(k)} = (E_{\lambda+k-1} \circ \cdots \circ E_{\lambda+1}) \circ E_\lambda = \circ E_{\lambda+1}^{(k-1)} \circ E_\lambda \tag{60}$$

so that

$$e_\lambda^{(k)} = e_{\lambda+1}^{(k-1)} \# e_\lambda$$

and, hence,

$$j_\lambda^{(k)}(\xi) = \big(j_{\lambda+1}^{(k-1)} \# e_\lambda\big)(0, \xi).$$

Using the composition formula (5) and (52), we get (58). □

Together with the initial condition $j_\lambda^{(0)} \equiv 1$, the recurrence relation (58) determines the polynomials $j_\lambda^{(k)}$ by induction over k. To solve this recurrence relation, define for $\gamma \in \mathbb{C}$ the sequence of polynomials B_k^γ on $\mathbb{R}^n$ by

$$B_0^\gamma = 1, \quad \left(\frac{\partial}{\partial \xi_n}\right)^k |\xi|^{2(\gamma+k)} = B_k^\gamma(\xi)|\xi|^{2\gamma}. \tag{61}$$

Observe that B_k^γ is a homogeneous polynomial of degree k.

Lemma 5.12. *Let $\gamma \in \mathbb{C}$. For any $k \geq 1$,*

$$B_k^\gamma = 2(\gamma+1)\xi_n B_{k-1}^{\gamma+1} + |\xi|^2 \frac{\partial}{\partial \xi_n} B_{k-1}^{\gamma+1}. \tag{62}$$

Proof. We have

$$\begin{aligned} \frac{\partial}{\partial \xi_n}^k (|\xi|)^{2(\gamma+k)} &= \frac{\partial}{\partial \xi_n}\left(\left(\frac{\partial}{\partial \xi_n}\right)^{k-1} |\xi|^{2(\gamma+1+(k-1))}\right) \\ &= \frac{\partial}{\partial \xi_n}(|\xi|^{2(\gamma+1)} B_{k-1}^{\gamma+1}(\xi)) \\ &= |\xi|^{2(\gamma+1)} \frac{\partial}{\partial \xi_n} B_{k-1}^{\gamma+1}(\xi) + 2(\gamma+1)\xi_n |\xi|^{2\gamma} B_{k-1}^{\gamma+1}(\xi), \end{aligned}$$

and the conclusion follows. □

Proposition 5.13. *Let $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$. The following identity is satisfied*:

$$p_k^\lambda(\xi) = B_k^{\lambda-\frac{n}{2}}(i\xi) = i^k B_k^{\lambda-\frac{n}{2}}(\xi). \tag{63}$$

Proof. Notice that $p_0^\lambda = B_0^{\lambda-\frac{n}{2}} = 1$, use the homogeneity of the polynomials and compare (58) and (62) for $\gamma = \lambda - \frac{n}{2}$. □

Theorem 5.14. *Let $(B_k^\gamma)_{k\in\mathbb{N}}$ be the family of polynomials defined by the formula* (61)*. For $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, let $J_\lambda^{(k)}$ be the differential operator defined by*

$$J_\lambda^{(k)} = \operatorname{res} \circ B_k^{\lambda-\frac{n}{2}}.$$

Then, for any $h \in H$,

$$J_\lambda^{(k)} \circ \pi_\lambda(h) = \pi'_{\lambda+k}(h) \circ J_\lambda^{(k)}.$$

This result is a consequence of the previous results, ignoring the factor i^k.

For given λ and k, it is known that the covariant differential operator is unique (up to a scalar) (see [Juhl 2009]).

The polynomials B_k^γ are connected with the classical *Gegenbauer polynomials*. In fact, the latter may be defined through the Rodrigues formula (see [Gradshteyn and Ryzhik 2007, page 993]) as

$$C_k^\lambda(t) = c_k(\lambda)(1-t^2)^{-(\lambda-\frac{1}{2})}\left(\frac{d}{dt}\right)^k (1-t^2)^{k+\lambda-\frac{1}{2}}, \tag{64}$$

where

$$c_k(\lambda) = \frac{(-1)^k\Gamma(\lambda+\frac{1}{2})\Gamma(k+2\lambda)}{2^k k!\Gamma(2\lambda)\Gamma(k+\lambda+\frac{1}{2})}.$$

Observe first that B_k^γ is a homogeneous polynomial of degree k. Next, as $|\xi|^2 = |\xi'| + \xi_n^2$, $B_k^\lambda(\xi',\xi_n)$ can be written as a polynomial in ξ_n and $|\xi'|^2$, so set

$$B_k^\gamma(\xi',\xi_n) = A_k^\gamma(|\xi'|,\xi_n),$$

where A_k^γ is a polynomial of two variables, homogeneous of degree k and even in the first variable. With this notation, (61) implies

$$\left(\frac{\partial}{\partial t}\right)^k (s^2+t^2)^{\gamma+k} = A_k^\gamma(s,t)(s^2+t^2)^\gamma. \tag{65}$$

Proposition 5.15. $A_k^\gamma(s,t) = c_k\big(\gamma+\frac{1}{2}\big)^{-1}(-i)^k s^k C_k^{\gamma+\frac{1}{2}}\left(\frac{t}{is}\right).$

Proof. Let f be a function of one variable, and associate the function of two variables given by $F(s,t) = f\big(\frac{t}{is}\big)$. Then,

$$\left(\frac{\partial}{\partial t}\right)^k F(s,t) = (-i)^k s^{-k} f^{(k)}\left(\frac{t}{is}\right).$$

Apply this relation to

$$f(t) = (1-t^2)^{k+\lambda-\frac{1}{2}}, \quad F(s,t) = s^{-2k-2\lambda+1}(s^2+t^2)^{k+\lambda-\frac{1}{2}}.$$

Now, by (64),

$$f^{(k)}(t) = c(k,\lambda)^{-1}(1-t^2)^{\lambda-\frac{1}{2}}C_k^\lambda(t),$$

whereas by (65) and letting $\gamma = \lambda - \frac{1}{2}$

$$\left(\frac{\partial}{\partial t}\right)^k F(s,t) = s^{-2k-2\lambda+1}(s^2+t^2)^{\lambda-\frac{1}{2}}A_k^{\lambda-\frac{1}{2}}(s,t)\ .$$

The proposition follows. □

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