Pacific Journal of Mathematics

ON THE IRREDUCIBLE COMPONENTS OF A GELFAND–GRAEV REPRESENTATION OF A FINITE CHEVALLEY GROUP

CHARLES W. CURTIS

Volume 307 No. 1 July 2020

PACIFIC JOURNAL OF MATHEMATICS
Vol. 307, No. 1, 2020
https://doi.org/10.2140/pjm.2020.307.109

ON THE IRREDUCIBLE COMPONENTS OF A GELFAND–GRAEV REPRESENTATION OF A FINITE CHEVALLEY GROUP

CHARLES W. CURTIS

This paper contains a construction of the irreducible representations in the field of complex numbers of the Hecke algebra of a Gelfand–Graev representation of a finite Chevalley group, based on formulas for the structure constants of the Hecke algebra. Using this information, formulas for the corresponding irreducible characters of the finite Chevalley group are obtained.

1. Introduction

This note is a supplement to two earlier papers [Curtis 1988; 2015], and contains applications of the main result of [Curtis 2015] on the structure constants of the Hecke algebra H of a Gelfand–Graev representation. In it, we first prove, in Section 2, that the irreducible representations of the Hecke algebra H are given by the eigenvalues of certain matrices whose entries are structure constants of H. The main problem is to calculate the representations of the Chevalley group G itself. From the theory of Hecke algebras [Curtis and Reiner 1981, §11], the irreducible representations of the Hecke algebra H are in a bijective correspondence with uniquely determined irreducible representations of G. In Section 3, we review this correspondence, and obtain formulas for the characters of G, using a theorem of Rimhak Ree.

We begin with some notation and terminology. Let G be a Chevalley group over a finite field $k = \mathbb{F}_q$ of characteristic p (as in [Chevalley 1955] or [Steinberg 1968]). Let B be a Borel subgroup of G with $U = O_p(B)$ (the unipotent radical of B), and let T be a maximal torus such that $B = UT$. Let W be the Weyl group of G. Then W is a finite Coxeter group with distinguished generators $S = \{s_1, \dots, s_n\}$.

Let Φ be the root system associated with W, with $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots corresponding to the generators $s_i \in S$, and $\Phi_\pm$ the set of positive roots (respectively, negative roots) associated with them. For each root α, let U_α be the root subgroup of G corresponding to it. The subgroup U is generated by the root subgroups U_α, $\alpha > 0$.

MSC2010: primary 20C33; secondary 20C08.
Keywords: representation theory, finite Chevalley groups, Hecke algebras.

From [Steinberg 1968, §3], the Chevalley group G has a (B, N)-pair, with Borel subgroup B, N the subgroup generated by all elements $w_\alpha(t)$, and $B \cap N = T$, with T the subgroup generated by all elements $h_\alpha(t)$ (see the definitions of $w_\alpha(t)$ and $h_\alpha(t)$ in Section 2). Then $N/T \cong W$. (If the field k contains more than three elements, then N is the normalizer $N = N_G(T)$ [Steinberg 1968, p. 36].)

By the Bruhat decomposition, the (U, U)-double cosets are parametrized by the elements of N, while the (B, B)-double cosets are parametrized by the elements of W.

We consider induced representations γ of the form ψ^G, for a linear representation ψ of U in the field of complex numbers. Let

$$e = |U|^{-1} \sum_{u \in U} \psi(u^{-1})u$$

be the primitive idempotent affording ψ in the group algebra $\mathbb{C}U$ of U over the field of complex numbers. Then $\gamma = \psi^G$ is afforded by the left $\mathbb{C}G$-module $\mathbb{C}Ge$. The *Hecke algebra* of γ is the subalgebra $H = e\mathbb{C}Ge$ of $\mathbb{C}G$, and is isomorphic to $(\operatorname{End}_{\mathbb{C}G} \mathbb{C}Ge)^\circ$. These representations and their Hecke algebras were first investigated by Gelfand and Graev [1962a; 1962b]. In particular they introduced the important class of *Gelfand–Graev representations* of G, which are the induced representations ψ^G, for a linear representation ψ of U in *general position*, that is, $\psi|U_{\alpha_i} \neq 1$ for each simple root subgroup U_{α_i}, $1 \le i \le n$, and $\psi|U_\alpha = 1$ for each positive and not simple root α.

A basis for the Hecke algebra H of a Gelfand–Graev representation ψ^G is given by the nonzero elements of the form ene, $n \in N$, because N is a set of representatives of the (U, U)-double cosets. The *standard basis elements* are the nonzero elements of the form $c_n = \operatorname{ind}(n)ene$, where $\operatorname{ind}(n) = |U : nUn^{-1} \cap U|$. The structure constants $[c_\ell c_m : c_n]$ for the standard basis elements, $c_\ell, c_m, \dots$ are complex numbers defined by the formulas

$$c_\ell c_m = \sum_n [c_\ell c_m : c_n] c_n,$$

with $\ell, m, n \in N^*$, and are algebraic integers (here N^* is the set of elements $n \in N$ such that $ene \neq 0$).

The *structure constants* of H are given by the formula

$$[c_\ell c_m : c_n] = \sum_{u\ell u_1 = nvm^{-1} \in U\ell U \cap nU_{m^{-1}}m^{-1}} \psi((uu_1)^{-1}v),$$

by [Curtis and Reiner 1981, Proposition 11.30] and the fact that $U\ell U \cap nU_{m^{-1}}m^{-1}$ is a set of representatives of the left U-cosets in $U\ell U \cap nUm^{-1}U$. As in [Curtis 1988] and [Curtis 2009], $U_n = U \cap nU_- n^{-1}$ for $n \in N$. The main purpose of this paper is to show how the irreducible representations of the Hecke algebra H of a

Gelfand–Graev representation, and the characters of corresponding representations of the Chevalley group G, are obtained from the structure constants of the Hecke algebra H. The structure constants are calculated more precisely in [Curtis 2015].

Other decompositions of Gelfand–Graev representations have been obtained by Chang [1976], for the groups $\mathrm{GL}(3, q)$, and for a general finite Chevalley group by Deligne and Lusztig [1976, §10] and by myself [Curtis 1993], in both cases using the Deligne–Lusztig theory of representations of algebraic groups on the ℓ-adic cohomology of varieties on which G acts.

2. The irreducible representations of *H*

As the endomorphism algebra of an induced representation ψ^G, H is a semisimple algebra over $\mathbb{C}$. A fundamental property of H was proved by Gelfand and Graev [1962b], for the Chevalley groups $\mathrm{SL}_n(k)$, by Yokonuma [1967] for a general Chevalley group, and with a simplified proof and extended to the case of twisted Chevalley groups by Steinberg [1968, Theorem 49].

Theorem 2.1 (Yokonuma, Steinberg). *The Hecke algebra H of a Gelfand–Graev representation of a finite Chevalley group is a commutative algebra. As a consequence, a Gelfand–Graev representation is multiplicity free: each irreducible component occurs with multiplicity one.*

The irreducible representations of H were obtained for the Chevalley groups $G = \mathrm{SL}_2(k)$ by Gelfand and Graev [1962a], using formulas which they called *Bessel functions over finite fields*. The formulas were obtained using the structure constants of the Hecke algebra of a Gelfand–Graev representation. The approach to the irreducible representations of H to follow is based on formulas for the structure constants of H given in [Curtis 2015, Corollary 4.2]. These are stated later, after a brief review of the necessary background, and are proved in [Curtis 2009; 2015]. One of the main points is to describe how the equations $u\ell u_1 = nvm^{-1}$, for certain elements $u, u_1, v \in U$ and $\ell, m, n \in N$, are solved in a general Chevalley group using ideas about refinements of the Bruhat decomposition due to Kawanaka [1975] and Deodhar [1985].

We begin the review by recalling some notation. For each root α, there is a homomorphism (see [Steinberg 1968, p. 46]) $\varphi = \varphi_\alpha : \mathrm{SL}_2(k) \to G$ such that φ takes

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \to x_\alpha(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \to x_{-\alpha}(t), \quad \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \to w_\alpha(t) \in N,$$
$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to h_\alpha(t) \in T$$

for all $t \in k$. The elements $w_\alpha(t)$ and $h_\alpha(t)$ are given by

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \quad h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1},$$

by [Steinberg 1968, p. 30]. If $w = s_k \cdots s_1$ is a reduced expression of an element $w \in W$ then $\dot{w} = \dot{s}_k \cdots \dot{s}_1$, with $\dot{s}_i = w_{\alpha_i}(t_i)$ for some fixed choice of $t_i \in k^* = k - \{0\}$, is a representative in N of w which is independent of the choice of the reduced expression chosen, by [Steinberg 1968, Lemma 83, p. 242]. In what follows we assume that representatives $\dot{x} \in N$ of all elements $x \in W$ have been chosen in this way, for a fixed choice of representatives $\dot{s}_i$ of the generators $s_i \in S$.

As in [Deodhar 1985], a *subexpression* τ of a fixed reduced expression $w = s_k \cdots s_1$ is a sequence $\tau = (\tau_k, \dots, \tau_1, \tau_0)$ of elements of W such that $\tau_i \tau_{i-1}^{-1} \in \{1, s_i\}$ for $i = 1, \dots, k$ and $\tau_0 = 1$. Then the set of terminal elements τ_k of subexpressions of $w = s_k \cdots s_1$ coincides with the set of elements $x \in W$ such that $x \le w$ in the Chevalley–Bruhat order. In what follows, the *length* of an element $w \in W$ in terms of the generators $s_i \in S$ is denoted by $\ell(w)$.

For each element $w \in W$, let $U_w = U \cap^w U_-$, where $U_- =^{w_0} U$ and w_0 is the element of maximal length in W. Then $U = U_w U_{ww_0}$ and $BwB = U_w \dot{w} B$, in both cases with uniqueness of expression. Let $w = s_k \cdots s_1$ be a reduced expression of $w \in W$. Then $U_w = U_{\alpha_k} \dot{s}_k U_{s_{k-1} \cdots s_1} \dot{s}_k^{-1}$ with uniqueness of expression.

The calculation of the structure constants of the Hecke algebra H of a Gelfand–Graev representation, and in particular the solutions of the equations in G described earlier, is based on an examination of the following subset of G. Let w, x, y be elements of W, and $\dot{w}, \dot{x}, \dot{y}$ corresponding elements of N. Let

$$U(w, x, y) = \{u \in U_w : u\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1} \ne \varnothing\}.$$

Then the set $U(w, x, y)$ is independent of the choice of representatives $\dot{w}, \dot{x}, \dot{y}$ of w, x, y in N. Moreover, $U_w \dot{w} B \cap \dot{y} U_{x^{-1}} \dot{x}^{-1}$ is a set of representatives of the left B-cosets in $BwB \cap y(BxB)^{-1}$, and its cardinality is the structure constant $[e_w e_x : e_y]$ of the standard basis elements e_w, e_x, e_y, for $w, x, y \in W$, in the Iwahori–Hecke algebra. For a fixed reduced expression $w = s_k \cdots s_1$ of an element $w \in W$, and elements x, y in W, Kawanaka defined a family of subexpressions of the expression for w, called K-sequences in [Curtis 2015], and proved in [Kawanaka 1975, Lemma 2.14b] that the cardinality of the set $U(w, x, y)$ and the nonzero structure constants of the Iwahori–Hecke algebra are given by the formula

$$|U(w, x, y)| = [e_w e_x : e_y] = |B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}| = \sum_\tau q^{a(\tau)}(q-1)^{b(\tau)},$$

where e_w, e_x, e_y are standard basis elements of the Iwahori–Hecke algebra $H(G, B)$, and the sum is taken over all K-sequences τ for w, x, y, and $a(\tau)$ and $b(\tau)$ are the nonnegative integers associated with a K-sequence, as in [Curtis 2015]. As a consequence, it follows that $U(w, x, y) \neq \varnothing$ if and only if there exist K-sequences for w, x, y; see also [Borel and Tits 1972, (3.19)], where the conditions are stated in a different way.

In [Curtis 1988] a geometric version of Kawanaka's formula was proved. It states that $U(w, x.y)$, viewed as a subset of the algebraic group $G(\bar{k})$ over the algebraic closure $\bar{k}$ of the finite field k, is a disjoint union of subsets U_τ, which are called *cells* in [Curtis 1988]. The cells U_τ are subsets of $G(\bar{k})$ parametrized by K-sequences τ for w, x, y relative to a fixed reduced expression of the element w, with corresponding subsets U_τ, also called cells (defined in [Curtis 1988]), in the finite Chevalley group $G = G(k)$. A review of the definition and properties of K-sequences τ, and cells U_τ, is given in [Curtis 2015, §2–3]. The main result in [Curtis 1988] extends Deodhar's decomposition [1985] (see also [Curtis 2009, §4]) of the intersection $ByB \cap B_- xB$, viewed as subsets of the flag variety G/B in the algebraic group $G(\bar{k})$, with B_- the Borel subgroup opposite to B. Each cell U_τ is isomorphic (in bijective correspondence as a set, or isomorphic as a variety in $G(\bar{k})$) to a product

$$U_\tau \cong \prod_\alpha U_\alpha \times \prod_\beta U_\beta^*$$

for certain subsets $\{\alpha\}$ and $\{\beta\}$ of cardinalities $a(\tau)$ and $b(\tau)$ of the positive root subgroups determined by τ and where U_β^* is the set of nonidentity elements in U_β. From the decomposition of $U(w, x, y)$ as a union of cells U_τ, it follows that $U_w \dot{w} B \cap \dot{y} U_{x^{-1}} \dot{x}^{-1}$ can be identified with the set of triples (u, b, v) with $u \in U_\tau$ for some τ, $b \in B$ and $v \in U_{x^{-1}}$, satisfying the equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$, with b and v uniquely determined by u by [Curtis 2009, Lemma 2.4]. This completes the review of how the equations $u\ell u_1 = nvm^{-1}$, in a Chevalley group, are solved, in connection with more exact formulas for the structure constants.

We now state the version of the structure constant formula [Curtis 2015, Corollary 4.2] on which the calculation of the irreducible representations is based.

Theorem 2.2. *The structure constants of H are complex numbers, given by the formula*

$$[c_\ell c_m : c_n] = \sum_\tau \sum_{u \in U_\tau} \psi((uu_1)^{-1} v)$$

for all standard basis elements c_ℓ, c_m, c_n, and satisfy the following conditions. For each K-sequence τ for w, x, y, and corresponding cell U_τ, the sum is taken over solutions of the equation $u\dot{w}\hat{u}_1\hat{s} = \dot{y}\hat{v}\dot{x}^{-1}$ (obtained by [Curtis 2015, Theorem 4.1]), *with $u \in U_\tau$ and $\hat{u}_1, \hat{v}, \hat{s}$ satisfying the conditions $\hat{u}_1 = su_1s^{-1} \in U$, $\hat{v} = s''v(s'')^{-1} \in U_{x^{-1}}$, and $\hat{s} = s(\dot{x}s''(s')^{-1}\dot{x}^{-1})^{-1} \in T$. If there are no solutions satisfying these conditions, then the structure constant is zero.*

We proceed to the calculation of the irreducible representations of H. We first make some changes in notation. Let d be the dimension of the Hecke algebra H, and let the standard basis elements of H be $c_1, \dots, c_d$. The structure constants become c_{ijk}, and the multiplication in H is now given by the equations

$$c_i c_j = \sum_{k=1}^{d} c_{ijk} c_k, \quad 1 \le i, j, k \le d.$$

Let $Z^1, \dots, Z^d$ be a basic set of irreducible representations for the commutative semisimple algebra H. The representations are all one dimensional, and are completely described by their values $Z^m(c_j)$ on the standard basis elements $c_1, \dots, c_d$ of H.

Theorem 2.3. *The values $Z^m(c_j)$ of the irreducible representations are eigenvalues of the $d \times d$ matrices $A_i = (c_{ijk})$, $1 \le j, k \le d$ and $1 \le i \le d$.*

In the proof, we use the notation ω_j^m for $Z^m(c_j)$, $1 \le j \le d$, for a fixed irreducible representation Z^m. Because the representation Z^m is a homomorphism of algebras, it preserves the structure equations, so

$$\omega_i^m \omega_j^m = \sum_{k=1}^{d} c_{ijk} \omega_k^m, \quad 1 \le i, j, k \le d.$$

Let w_m be the column vector with entries $\omega_1^m, \dots, \omega_d^m$, corresponding to an irreducible representation Z^m of H. By a straightforward computation it follows that

$$A_i w_m = \omega_i^m w_m,$$

so w_m is an eigenvector of the matrix A_i in the statement of the theorem, with eigenvalue ω_i^m, for $1 \le i \le d$.

We now prove that w_m is the unique (up to scalar multiples) common eigenvector of the matrices A_i with eigenvalues $\omega_1^m, \dots, \omega_d^m$. This follows because it is easily proved, using the structure equations again and the fact that H is a commutative algebra, that the map $c_i \to A_i$ affords the left regular representation of H. It follows that the vector space V of d-rowed column vectors on which the matrices A_i act affords a faithful representation of H, and the subspaces of V generated by the vectors w_m are a basic set of simple modules appearing with multiplicity one, for the commutative semisimple algebra H. This completes the proof of the uniqueness result stated above.

To complete the picture, we give another proof, with historical background, that the values of the irreducible representations $Z^m(c_j)$ are eigenvalues of matrices whose entries are structure constants.

Let us fix the index i. Then the elements $\omega_1^m, \dots, \omega_d^m$ are a nontrivial solution to the system of homogeneous equations (involving a Kronecker delta)

$$\sum_{k=1}^{d} (\delta_{jk}\omega_i^m \omega_j^m - c_{ijk}\omega_k^m) = 0, \quad 1 \le j, k \le d,$$

with coefficient matrix

$$(\delta_{jk}\omega_j^m - c_{ijk}), \quad 1 \le j, k \le d.$$

The solution is nontrivial because the identity element e of H is one of the standard basis elements c_j, and as $Z^m(e) \ne 0$, some one of the elements ω_j^m is nonzero.

From what has been proved, it follows that the determinant of the coefficient matrix of the system is equal to zero, so that the element ω_i^m is an eigenvalue of the coefficient matrix of the system. Therefore, the elements $\omega_1^m, \dots, \omega_d^m$ are all eigenvalues of matrices whose entries are structure constants.

Remark. The idea that the irreducible representations of a commutative semisimple algebra can be obtained from a knowledge of the structure constants of the algebra, as far as I know, is due to Frobenius. In his first paper on characters of a finite group [Frobenius 1896], he found the structure constants of the centers of the group algebras of the finite groups PSL(2, p), and used them along with other information to calculate the character tables of those groups. This result, many of us believe, was the starting point of the representation theory (in the field of complex numbers) of finite Chevalley groups.

3. On the irreducible characters of a finite Chevalley group G appearing as constituents of a Gelfand–Graev representation of G

In Section 2, the irreducible representations of the Hecke algebra H of a Gelfand–Graev representation of a finite Chevalley group G were calculated. They are all one dimensional, and are given by eigenvalues of matrices whose entries are structure constants of the Hecke algebra H. We now wish to investigate the irreducible characters ζ of G such that $(\zeta, \psi^G) \ne 0$. As usual, we extend characters of G to functions on the group algebra $\mathbb{C}G$, and view the Hecke algebra H as a subalgebra of the group algebra $\mathbb{C}G$. We first have, by [Curtis and Reiner 1981, Theorem 11.25], the following theorem:

Theorem 3.1. *There is a bijection from the set of irreducible characters ζ of G such that $(\zeta, \psi^G) \ne 0$ to the set of all irreducible characters φ of H, given by $\zeta \to \zeta|H = \varphi$.*

We include a few remarks about the proof in [Curtis and Reiner 1981, §11]. Let ζ be an irreducible character of G as in the statement of the theorem, and let M be a simple $\mathbb{C}G$-module affording ζ. Recall that $H = e\mathbb{C}Ge$, where the idempotent $e = |U|^{-1} \sum_{u \in U} \psi(u^{-1})u$ in H affords the character ψ of U in general position. Then the induced character ψ^G is afforded by the left ideal $\mathbb{C}Ge$, and $(\zeta, \psi^G) \neq 0$ implies that eM is a left H-module affording the character $\zeta|H$ of H. Moreover, eM is a simple H-module, and $\varphi = \zeta|H$ is an irreducible character of H afforded by eM.

The main result of this section is a character formula, due to Rimhak Ree, for the irreducible character ζ of G as above. The character formula gives the value $\zeta(t)$ of the character of G at an element t of G in terms of the values of the character φ of H, where $\varphi = \zeta|H$. We require first some remarks about dual bases in the Hecke algebra H.

We recall the standard basis elements of H, $c_n = \text{ind}(n)ene$, with $n \in N^*$, where $N^* = n \in N, ene \neq 0$. A second basis of H is given by the elements $\hat{c}_n = c_{n^{-1}}$ with $n \in N^*$. A linear function λ on H is defined by the formula $\lambda\left(\sum \xi_n c_n\right) = \xi_1$, for an element $\sum \xi_n c_n$ in H with coefficients ξ_n and $c_1 = e$. A bilinear form B on H is then defined by setting $B(h, k) = \lambda(hk)$, for $h, k \in H$. One checks that the bilinear form B is symmetric, associative, and nondegenerate. It is then not difficult to prove that the bases are dual in the sense that $B(\hat{c}_n, c_m) = 0$ if $n \neq m$ and $B(\hat{c}_n, c_n) = \text{ind}(n)$, for all $n \in N^*$.

Theorem 3.2 (R. Ree). *Let ζ be an irreducible character of G such that $(\zeta, \psi^G) \neq 0$, where ψ^G is the character of a Gelfand–Graev representation of G. Then $\zeta|H \neq 0$, and $\varphi = \zeta|H$ is an irreducible character of H. Let $t \in G$, let $\mathfrak{C}$ be the conjugacy class of t, and let C be the conjugacy class sum. Then*

$$\zeta(t) = |C_G(t)|\varphi(eCe)\left(|U| \sum_{n \in N^*} (\text{ind}(n))^{-1} \varphi(\hat{c}_n)\varphi(c_n)\right)^{-1}.$$

Proof. The proof is similar to the proof of [Curtis and Reiner 1981, Theorem 11.28] with some changes. Let $\boldsymbol{M}$ be a matrix representation of G affording ζ. Then $eCe = Ce$ because C is in the center of the group algebra, and

$$\boldsymbol{M}(C) = \omega \boldsymbol{I},$$

where $\omega = |\mathfrak{C}|\zeta(t)\zeta(1)^{-1}$, as one sees by taking traces on both sides.Then

$$\boldsymbol{M}(eCe) = \boldsymbol{M}(C)\boldsymbol{M}(e).$$

Taking traces again, we have

$$\zeta(eCe) = \omega\zeta(e) = |\mathfrak{C}|\,\zeta(t)\zeta(e)\zeta(1)^{-1}.$$

Now let

$$\varepsilon = \zeta(1)|G|^{-1} \sum_{x \in G} \zeta(x^{-1})x$$

be the central primitive idempotent in $\mathbb{C}G$ corresponding to ζ. Then

$$\varepsilon e = \zeta(1)|G|^{-1} \sum_{x \in G} \zeta(x^{-1}) exe.$$

Let $e = \sum_{u \in U} \alpha_u u$, for complex coefficients α_u. Because $e^2 = e$, we have

$$\varepsilon e = \zeta(1)|G|^{-1} \sum_{x \in G} \sum_{u,v \in U} \zeta(x^{-1}) \alpha_u \alpha_v euxve.$$

Putting $y = uxv$ we have $x^{-1} = vy^{-1}u$ and

$$\varepsilon e = \zeta(1)|G|^{-1} \sum_{y \in G} \zeta\Bigg(\sum_{u,v \in U} \alpha_u \alpha_v v y^{-1} u \Bigg) eye.$$

Therefore

$$\varepsilon e = \zeta(1)|G|^{-1} \sum_{x \in G} \zeta(ex^{-1}e) exe.$$

Now apply $\boldsymbol{M}$ and take traces. Noting that $\boldsymbol{M}(\varepsilon) = \boldsymbol{I}$, we obtain

$$\zeta(e) = \zeta(1)|G|^{-1} \sum_{x \in G} \zeta(ex^{-1}e)\zeta(exe).$$

We now bring the dual bases $\{\hat{c}_n\}$ and $\{c_n\}$ of H into the picture. For a (U, U)-double coset UnU, $n \in N^*$, a simple calculation shows that

$$\sum_{x \in UnU} \zeta(ex^{-1}e)\zeta(exe) = |UnU|(\mathrm{ind}(n))^{-2} \zeta(\hat{c}_n)\zeta(c_n).$$

Now apply the facts that $|UnU| = \mathrm{ind}(n)|U|$ and $\zeta|H = \varphi$ to obtain finally

$$\zeta(e) = \zeta(1)|G|^{-1}|U| \sum_{n \in N^*} (\mathrm{ind}(n))^{-1} \varphi(\hat{c}_n)\varphi(c_n).$$

Comparing this formula for $\zeta(e)$ with the one for $\zeta(eCe)$, we obtain the result stated in the theorem. □

Example. We first note that the preceding theorem gives a nice formula for the degree $\zeta(1)$ of an irreducible character of G appearing in the Gelfand–Graev representation. In their first paper, Gelfand and Graev [1962a] summarized their results on the degrees of the irreducible characters of the finite Chevalley groups $\mathrm{SL}(2, q)$ for a finite field of q elements, where q is a power of an odd prime. In this case there are two Gelfand–Graev representations. As they pointed out, all the irreducible characters of a Gelfand–Graev representation have degree $q - 1$, with one exception of degree $\frac{1}{2}(q - 1)$. Let us apply the preceding theorem to this situation. The theorem states that the degree of an irreducible character appearing in a Gelfand–Graev representation is given by the formula

$$\zeta(1) = |G|\varphi(e)\biggl(|U|\sum_n (\operatorname{ind}(n))^{-1}\varphi(\hat{c}_n)\varphi(c_n)\biggr)^{-1}$$
$$= (q+1)(q-1)\biggl(\sum_n (\operatorname{ind}(n))^{-1}\varphi(\hat{c}_n)\varphi(c_n)\biggr)^{-1}.$$

Using the formulas for the irreducible representations of the Hecke algebra of a Gelfand–Graev representation of $\mathrm{SL}(2, q)$ in [Curtis 2015, §5], it can then be shown (in this case) that for all irreducible characters φ of the Hecke algebra H, with one exception, one has $\sum_n (\operatorname{ind}(n))^{-1}\varphi(\hat{c}_n)\varphi(c_n) = q+1$, and that in the exceptional case, the expression becomes $2(q+1)$. It follows that all but one of the irreducible components of a Gelfand–Graev character of $\mathrm{SL}(2, q)$ have degree $q-1$, and that in the exceptional case the degree is $\frac{1}{2}(q-1)$, exactly as Gelfand and Graev stated.

An interesting approach to the rather mysterious characters of degree $\frac{1}{2}(q-1)$ was given by Lusztig [1978, §2.20], using the ℓ-adic cohomology of the Drinfeld curve, on which the finite group $\mathrm{SL}(2, q)$ acts.

References

[Borel and Tits 1972] A. Borel and J. Tits, "Compléments à l'article 'Groupes réductifs'", *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 253–276. MR Zbl

[Chang 1976] B. Chang, "Decomposition of Gelfand–Graev characters of $\mathrm{GL}_3(q)$", *Comm. Algebra* **4**:4 (1976), 375–401. MR Zbl

[Chevalley 1955] C. Chevalley, "Sur certains groupes simples", *Tohoku Math. J.* (2) **7** (1955), 14–66. MR Zbl

[Curtis 1988] C. W. Curtis, "A further refinement of the Bruhat decomposition", *Proc. Amer. Math. Soc.* **102**:1 (1988), 37–42. MR Zbl

[Curtis 1993] C. W. Curtis, "On the Gelfand–Graev representations of a reductive group over a finite field", *J. Algebra* **157**:2 (1993), 517–533. MR Zbl

[Curtis 2009] C. W. Curtis, "On problems concerning the Bruhat decomposition and structure constants of Hecke algebras of finite Chevalley groups", *Michigan Math. J.* **58**:1 (2009), 213–230. MR Zbl

[Curtis 2015] C. W. Curtis, "Notes on the structure constants of Hecke algebras of induced representations of finite Chevalley groups", *Pacific J. Math.* **279**:1-2 (2015), 181–202. MR Zbl

[Curtis and Reiner 1981] C. W. Curtis and I. Reiner, *Methods of representation theory, I: With applications to finite groups and orders*, Wiley, New York, 1981. MR Zbl

[Deligne and Lusztig 1976] P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields", *Ann. of Math.* (2) **103**:1 (1976), 103–161. MR Zbl

[Deodhar 1985] V. V. Deodhar, "On some geometric aspects of Bruhat orderings, I: A finer decomposition of Bruhat cells", *Invent. Math.* **79**:3 (1985), 499–511. MR Zbl

[Frobenius 1896] F. G. Frobenius, "Über Gruppencharaktere", *Sitzungsber. Preuss. Akad. Wiss. Berlin* **1896** (1896), 985–1021. Reprinted in *Gesammelte Abhandlungen, III*, Springer (1968), 1–37. Zbl

[Gelfand and Graev 1962a] I. M. Gelfand and M. I. Graev, "Categories of group representations and the classification problem of irreducible representations", *Dokl. Akad. Nauk SSSR* **146**:4 (1962), 757–760. In Russian; translated in *Soviet Math. Dokl.* **3** (1962), 1378–1381. MR

[Gelfand and Graev 1962b] I. M. Gelfand and M. I. Graev, "Construction of irreducible representations of simple algebraic groups over a finite field", *Dokl. Akad. Nauk SSSR* **147**:3 (1962), 529–532. In Russian; translated in *Soviet Math. Dokl.* **3** (1962), 1646–1649.

[Kawanaka 1975] N. Kawanaka, "Unipotent elements and characters of finite Chevalley groups", *Osaka Math. J.* **12**:2 (1975), 523–554. MR Zbl

[Lusztig 1978] G. Lusztig, *Representations of finite Chevalley groups* (Madison, WI, 1977), CBMS Region. Conf. Ser. Math. **39**, Amer. Math. Soc., Providence, RI, 1978. MR Zbl

[Steinberg 1968] R. Steinberg, *Lectures on Chevalley groups*, Yale Univ., 1968. MR Zbl

[Yokonuma 1967] T. Yokonuma, "Sur le commutant d'une représentation d'un groupe de Chevalley fini", *C. R. Acad. Sci. Paris Sér. A-B* **264** (1967), 433–436. MR Zbl

Received August 7, 2019. Revised January 24, 2020.

CHARLES W. CURTIS
DEPARTMENT OF MATHEMATICS AND INSTITUTE OF FUNDAMENTAL SCIENCE
UNIVERSITY OF OREGON
EUGENE, OR
UNITED STATES

cwc@uoregon.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2020 is US $520/year for the electronic version, and $705/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 307 No. 1 July 2020

0030-8730(202007)307:1;1-I