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ELIMINATING TAME RAMIFICATION GENERALIZATIONS OF ABHYANKAR'S LEMMA

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A basic version of Abhyankar's lemma states that for two finite extensions L and F of a local field K, if $L|K$ is tamely ramified and if the ramification index of $L|K$ divides the ramification index of $F|K$, then the compositum $L.F$ is an unramified extension of F. In this paper, we generalize the result to valued fields with value groups of rational rank 1, and show that the latter condition is necessary. Replacing the condition on the ramification indices by the condition that the value group of L be contained in that of F, we generalize the result further in order to give a necessary and sufficient condition for the elimination of tame ramification of an arbitrary extension $F|K$ by a suitable algebraic extension of the base field K. In addition, we derive more precise ramification theoretical statements and give several examples.

1. Introduction

In this paper we consider valued fields (K, v), i.e., fields K with a Krull valuation v. The valuation ring of v on K will be denoted by $\mathcal{O}_K$. The value group of (K, v) will be denoted by vK, and its residue field by Kv. The value of an element a will be denoted by va, and its residue by av. By $(L|K, v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. For background on valuation theory, see [Endler 1972; Engler and Prestel 2005; Kuhlmann $\geq$ 2020; Zariski and Samuel 1960]. Basic facts that we will need, in particular from ramification theory, will be presented in Section 2.

Throughout, we will consider the following general situation. We let (M, v) be an arbitrary algebraically closed extension of some valued field (K, v). Every subfield E of M will be endowed with the restriction of v, which we will again denote by v; note that (M, v) contains a unique henselization of (E, v), which we denote by (E^h, v). Further, we take an arbitrary subextension $F|K$ and an algebraic subextension $L|K$ of $M|K$. The *compositum of the fields* F and L within M is the smallest subfield of M that contains both F and L, and we denote it by $L.F$. The restriction of v from M to $L.F$ is then a simultaneous extension of the restrictions

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to L and F. Similarly, the *compositum of the value groups* vF and vL within vM is the smallest subgroup of vM that contains both vF and vL, and we denote it by $vL+vF$.

An algebraic extension $(L|K, v)$ of henselian fields is called *tame* if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) The ramification index $(vE : vK)$ is not divisible by $\operatorname{char} Kv$.

(TE2) The residue field extension $Ev|Kv$ is separable.

(TE3) The extension $(E|K, v)$ is *defectless*, i.e.,

$$[E:K] = (vE:vK)[Ev:Kv].$$

Note that the extension $(L|K, v)$ is called *tamely ramified* if (TE1) and (TE2) hold for all finite subextensions $E|K$, so a finite tame extension is the same as a finite defectless tamely ramified extension. The extension $(L|K, v)$ is called *unramified* if the canonical embedding of vK in vL is onto and the residue field extension $Lv|Kv$ is separable; this does not necessarily imply that the extension is defectless.

In the case of a henselian discretely valued field (K, v), condition (TE3) is known to hold as soon as $L|K$ is separable. Therefore, if in addition $\operatorname{char} K = 0$, then a finite extension of (K, v) is tame once it is tamely ramified. If in addition (K, v) is complete, then condition (TE3) always holds.

For henselian discretely valued fields, Abhyankar's lemma provides a sufficient condition to eliminate tame ramification of a finite extension $(F|K, v)$ by lifting through a finite extension. In this case we can choose M to be the algebraic closure of K, and the extension of v from K to L, F and $L.F$ is uniquely determined.

Theorem 1 (Abhyankar's lemma). *Let (K, v) be a henselian discretely valued field, $(L|K, v)$ be a finite tame extension and $(F|K, v)$ a finite extension. If the ramification index of $(L|K, v)$ divides the ramification index of $(F|K, v)$, then the extension $(L.F/F, v)$ is unramified.*

In [Chabert and Halberstadt 2018] the following version of Abhyankar's lemma is shown: the ramification index of the compositum of two finite extensions of local fields is equal to the least common multiple of the ramification indices corresponding to the finite extensions, provided at least one of the extensions is tame. This version is a special case of a more general theorem that we will present next.

The condition on the ramification indices in Theorem 1 is also necessary. Indeed, $(L.F|F, v)$ being unramified implies that $v(L.F) = vF$. Thus,

$$(vF : vK) = (v(L.F) : vK) = (v(L.F) : vL)(vL : vK),$$

hence $(vL : vK)$ divides $(vF : vK)$.

The question naturally arises how far the above formulation of Abhyankar's lemma can be generalized. The next theorem, which implies Theorem 1, shows that the result remains true whenever vK has rational rank 1; the rational rank of an abelian group is the $\mathbb{Q}$-dimension of the divisible hull $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ of Γ.

From now on we will assume the general situation as introduced in the beginning, i.e., $F|K$ is an arbitrary extension, and $L|K$ is a (not necessarily finite) algebraic extension.

Theorem 2. *Assume that the value group of* (K, v) *is of rational rank* 1, *that the extension* $(L.K^h|K^h, v)$ *is tame and that the ramification indices* $(vL : vK)$ *and* $(vF : vK)$ *are finite. Then* $(v(L.F) : vK)$ *is the least common multiple of* $(vL : vK)$ *and* $(vF : vK)$. *In particular*, $(L.F|F, v)$ *is unramified if and only if the ramification index of* $(L|K, v)$ *divides the ramification index of* $(F|K, v)$.

In contrast, in Section 7 we will show that the result fails for higher rational rank (see Lemma 18). In particular, the result fails for generalized discretely valued fields, i.e., those valued fields whose value group is a lexicographically ordered product of finitely many copies of $\mathbb{Z}$.

By reformulating the condition on the ramification indices in a different way, using the value groups themselves instead, one can prove a far-reaching generalization of Abhyankar's lemma. The *absolute ramification field* (K^r, v) *of* (K, v) is the ramification field of the normal extension $(K^{\mathrm{sep}}|K, v)$, where K^{sep} denotes the separable-algebraic closure of K. Likewise, the *absolute inertia field* (K^i, v) *of* (K, v) is the inertia field of the extension $(K^{\mathrm{sep}}|K, v)$. Since M is assumed to be algebraically closed, just as for henselizations, it contains a unique ramification field and a unique inertia field for every subfield (E, v). We have that $E^h \subseteq E^i \subseteq E^r$ and hence, (E^i, v) and (E^r, v) are henselian.

An extension $(L|K, v)$ of valued fields is called *immediate* if the canonical embeddings of vK in vL and of Kv in Lv are onto. Recall that the henselization is an immediate extension.

In Section 3, we will prove the following:

Theorem 3. (1) *Assume that* (L, v) *is contained in the absolute ramification field of* (K, v). *Then* $(L.F, v)$ *is contained in the absolute ramification field of* (F, v) *and* $v(L.F) = vL + vF$. *Further*, $(L.F, v)$ *is contained in the absolute inertia field of* (F, v) (*which implies that the extension* $(L.F|F, v)$ *is unramified*) *if and only if* vL *is a subgroup of* vF.

(2) *Assume that* (L, v) *is contained in the absolute inertia field of* (K, v). *Then* $(L.F, v)$ *is contained in the absolute inertia field of* (F, v) *and* $(L.F)v = Lv.Fv$. *Further*, $(L.F, v)$ *is contained in the henselization of* (F, v) (which implies that the extension $(L.F|F, v)$ is immediate) *if and only if* Lv *is a subfield of* Fv.

In Section 7 we will show that this theorem implies Theorem 2 and hence also Theorem 1.

Note that if char $Kv = 0$, then the absolute ramification field is algebraically closed, so (L, v) is contained in it as soon as $L|K$ is algebraic. If char $Kv > 0$ and $L|K$ is algebraic, then for (L, v) to lie in the absolute ramification field (K^r, v) of (K, v), the following three conditions are necessary and sufficient (the letters "PT" stand for "pre-tame"):

(PT1) char Kv does not divide the order of any nonzero element in vL/vK,

(PT2) the residue field extension $Lv|Kv$ is separable,

(PT3) for every finite subextension $E|K$ of $L|K$, the extension $(E^h|K^h, v)$ of their respective henselizations (in (M, v)) is defectless.

This means that if (K, v) is henselian, then (L, v) lies in its absolute ramification field if and only if $(L|K, v)$ is a tame extension; in other words, (K^r, v) is the unique maximal tame extension of (K, v).

Similarly, (L, v) lies in the absolute inertia field of (K, v) if and only if $L|K$ is algebraic, $vL = vK$, and conditions (PT2) and (PT3) hold.

Assume now that char $Kv = p > 0$. Does elimination of tame ramification also hold if the extension $(L^h|K^h, v)$ is not tame? The answer is yes if we restrict the scope to normal extensions. We denote by $(vL)_{p'}$ the maximal subgroup of vL containing vK and such that p does not divide the order of any of its nonzero element modulo vK. Further, we denote by $(Lv)_s$ the maximal subfield of Lv separable over Kv. A p-extension is a (not necessarily finite) Galois extension with Galois group a p-group.

Theorem 4. *Assume that $L|K$ is normal, $F|K$ is an arbitrary extension, and* char $Kv = p > 0$. *Then the following assertions hold*:

(1) *The quotient group $v(L.F)/((vL)_{p'} + vF)$ is a p-group, and in particular, $v(L.F)/vF$ is a p-group if and only if $(vL)_{p'} \subseteq vF$.*

(2) *If $(vL)_{p'} = vK$, then the maximal separable subextension of $(L.F)v|(Lv)_s.Fv$ is a p-extension.*

Trivial examples of ramification that can easily be eliminated appear when the base field K is smaller than the constant field of the function field F. More sophisticated examples will therefore present situations where the base field K is equal to the constant field, i.e., is relatively algebraically closed in F. But this does not imply that K is equal to the relative algebraic closure of K in a fixed henselization of (F, v). In [Kuhlmann 2004], for valued rational function fields $(K(x)|K, v)$ the *implicit constant field* $\mathrm{IC}(K(x)|K, v)$ is defined to be the relative algebraic closure of K in a fixed henselization of $(K(x), v)$. While it depends on the chosen henselization, it is unique up to valuation preserving isomorphism over K.

Theorem 5 [Kuhlmann 2004, Theorem 1.3]. *Let $(L|K, v)$ be a countably generated separable-algebraic extension of nontrivially valued fields. Then there is an extension of v from L to the algebraic closure $L(x)^{\mathrm{ac}} = K(x)^{\mathrm{ac}}$ of the rational function field $K(x)$ such that, upon taking henselizations in $(K(x)^{\mathrm{ac}}, v)$,*

$$L^h = \mathrm{IC}(K(x)|K, v).$$

This means that $L \subset K(x)^h$, so that $L(x) = L.K(x)$ lies in the henselization of $K(x)$ and all ramification, whether tame or wild, is eliminated. We will construct specific examples in Section 6.

Finally, let us mention that there are various other versions and generalizations of Abhyankar's lemma. Here we list only a few. When the valued field (K, v) is a formally $\wp$-adic field, then Theorem 1 is Corollary 4 in [Narkiewicz 2004, Chapter 5]. Elimination of ramification by so-called strongly solvable extensions of the base field has been presented in [Ponomarëv 1998; 1999]. Generalizations are also discussed in the Stacks Project [Stacks 2005–], some of which we will cite in Section 7. Finally, a "perfectoid Abhyankar lemma" has recently been presented in [André 2018].

2. Preliminaries

We recall some aspects of ramification theory and of general valuation theory see, e.g., [Abhyankar 1959; Endler 1972; Engler and Prestel 2005; Kuhlmann ≥ 2020; Neukirch 1992; Zariski and Samuel 1960]. We take a normal algebraic extension $(L|K, v)$ of valued fields and set $G = \mathrm{Aut}\, L|K$. The *decomposition group* of the extension is defined as

$$G^d(L|K, v) := \{\sigma \in G \mid v \circ \sigma = v \text{ on } L\},$$

the *inertia group* as

$$G^i(L|K, v) := \{\sigma \in G \mid \forall x \in \mathcal{O}_L : v(\sigma x - x) > 0\},$$

and the *ramification group* as

$$G^r(L|K, v) := \{\sigma \in G \mid \forall x \in L^\times : v(\sigma x - x) > vx\}.$$

The corresponding fixed fields in K^{sep} will be denoted as $(L|K, v)^d$, $(L|K, v)^i$ and $(L|K, v)^r$ and are called the *decomposition field*, *inertia field* and *ramification field* of $(L|K, v)$, respectively. We have:

$$G^r(L|K, v) \trianglelefteq G^i(L|K, v) \trianglelefteq G^d(L|K, v) \leq G$$

and

$$G^r(L|K,v) \trianglelefteq G^d(L|K,v),$$

so $(L|K,v)^d \subseteq (L|K,v)^i \subseteq (L|K,v)^r$ with both extensions as well as $(L|K,v)^d \subseteq (L|K,v)^r$ Galois.

In the above notation, the absolute decomposition field, absolute inertia field and absolute ramification field of (K,v) that we mentioned in the introduction are $K^d = (K^{\mathrm{ac}}|K,v)^d = (K^{\mathrm{sep}}|K,v)^d$, $K^i = (K^{\mathrm{ac}}|K,v)^i = (K^{\mathrm{sep}}|K,v)^i$ and $K^r = (K^{\mathrm{ac}}|K,v)^r = (K^{\mathrm{sep}}|K,v)^r$, respectively.

We collect the main facts of ramification theory that we will need in this paper in the next theorem. To simplify notation, we set $L_d = (L|K,v)^d$, $L_i = (L|K,v)^i$, $L_r = (L|K,v)^r$, and denote by L_s the maximal separable extension of K inside of L.

Theorem 6. (1) *The extension $(L_d|K,v)$ is immediate and v has a unique extension from L_d to L.*

(2) *The extension $L_iv|L_dv$ is separable, and $L_rv = L_iv$.*

(3) *We have that $vL_i = vL_d$, and the order of no element in $vL_r/vL_i = vL_r/vK$ is divisible by* char Kv.

(4) *If* char $Kv = p > 0$, *then $G^r(L|K,v)$ is a p-group, so $L_s|L_r$ is a p-extension. If* char $Kv = 0$, *then $G^r(L|K,v)$ is trivial and $L_r = L$. The extension $Lv|L_rv$ is purely inseparable, and vL/vL_r is a p-group.*

(5) *If $K \subseteq K_1 \subseteq K_2 \subseteq L_r$, $K_2|K_1$ is finite and (K_1,v)* (and thus also (K_2,v)) *is henselian, then the extension $(K_2|K_1,v)$ is defectless.*

(6) *We have that $(L|L_d,v)^i = L_i$ and $(L|L_d,v)^r = (L|L_i,v)^r = L_r$.*

(7) *If $K \subseteq L' \subseteq L$, then $(L|L',v)^d = L'.L_d$, $(L|L',v)^i = L'.L_i$ and $(L|L',v)^r = L'.L_r$.*

(8) *Whenever $F|K$ is an arbitrary extension and the valuation v is fixed on some field containing the algebraic closure of F, then $K^d \subseteq F^d$, $K^i \subseteq F^i$ and $K^r \subseteq F^r$.*

(9) *If $K \subseteq K_1 \subseteq K^d$, then $K_1^d = K^d$. If $K \subseteq K_1 \subseteq K^i$, then $K_1^i = K^i$. If $K \subseteq K_1 \subseteq K^r$, then $K_1^r = K^r$.*

Corollary 7. *If $K \subseteq K_1 \subseteq K_1' \subseteq L_r$, $(K_1'|K_1,v)$ is immediate and (K_1,v)* (and thus also (K_1',v)) *is henselian, then $K_1 = K_1'$.*

Proof. Take $K_2|K_1$ to be any finite subextension of $K_1'|K_1$. Since $(K_1'|K_1,v)$ is immediate by assumption, the same holds for $(K_2|K_1,v)$. As this extension is also defectless by part 5) of Theorem 6, we have that

$$[K_2:K_1] = (vK_2:vK_1)[K_2v:K_1v] = 1,$$

whence $K_1 = K_2$. It follows that $K_1 = K_1'$. □

Here is a crucial lemma for the proof of Theorems 3 and 4:

Lemma 8. *Take any extension (L, v) of (K, v), elements $\beta \in vL$, $c \in K$ and a positive integer n such that $n\beta = vc$. Suppose that p does not divide n. Then the polynomial $X^n - c$ splits in the absolute inertia field L^i of (L, v) and $\beta \in vL^i$.*

Proof. Take some $b \in L$ such that $vb = \beta$. Then $vcb^{-n} = 0$ and therefore, $cb^{-n}v \neq 0$. Since p does not divide n, the polynomial $X^n - cb^{-n}v$ has n distinct roots in $(Lv)^{\text{sep}} = L^i v$. By Hensel's lemma, it follows that the polynomial $X^n - cb^{-n}$ splits completely in the henselian field (L^i, v). Hence, so does $X^n - c$. □

Further, we will need the *fundamental inequality*, of which we state only a simple form here: for every finite extension $(L|K, v)$,

$$[L : K] \geq (vL : vK)[Lv : Kv]. \tag{1}$$

Finally, we will need:

Proposition 9. *Take any prime p and an arbitrary extension $F|K$ and a normal algebraic extension $L|K$. If the maximal separable subextension of $L|K$ is a p-extension, then the same holds for $L.F|F$.*

Proof. Let $L_s|K$ be the maximal separable subextension of $L|K$ and set $E := L_s \cap F$. Then both $L_s|K$ and $L_s|E$ are normal and separable, and $\operatorname{Aut} L_s|E$ is a subgroup of $\operatorname{Aut} L_s|K$. Since the latter is a p-group by assumption, so is the former.

Since $L_s \cap F = E$ and $L_s|E$ is normal and separable, F and L_s are linearly disjoint over E and it follows that $\operatorname{Aut} L_s.F|F = \operatorname{Aut} L_s|E$, which shows that $L_s.F|F$ is a p-extension. Since $L|L_s$ is purely inseparable, also $L.(L_s.F) = L.F$ is a purely inseparable extension of $L_s.F$, so $L_s.F|F$ is the maximal separable subextension of $L.F|F$. □

3. Proof of Theorem 3

In this and the next two sections, we will freely use the facts collected in Theorem 6 as well as the fundamental inequality (1) without citing them.

We assume the extensions $(F|K, v)$ and $(L|K, v)$ to be as in the introduction. Since $L|K$ is algebraic, vL/vK is a torsion group.

Let us first assume that $vL \subseteq vF$ and that (L, v) is contained in the absolute ramification field K^r of (K, v), so $vL \subseteq vK^r$. Take any set $\{\beta_j \mid j \in J\}$ of generators of vL over vK, and let n_j be positive integers such that $n_j\beta_j \in vK$ for each $j \in J$. Since $\operatorname{char} Kv$ does not divide the order of any element in vK^r/vK, the same holds for vL/vK. Therefore, we can assume that $\operatorname{char} Kv$ does not divide any of the n_j. Applying Lemma 8, we can find elements $b_j \in L^i$ such that $vb_j = \beta_j$ and $c_j := b_j^{n_j} \in K$. Since $K^i \subseteq L^i$, we obtain that

$$vL \subseteq vK^i(b_j \mid j \in J) \subseteq vL^i = vL,$$

showing that equality must hold everywhere. Since $Lv|Kv$ is separable by condition (TE2), we have that $K^i v = (Kv)^{\text{sep}} = (Lv)^{\text{sep}} = L^i v$ and thus,

$$K^i v \subseteq K^i(b_j \mid j \in J)v \subseteq L^i v = K^i v,$$

showing again that equality must hold everywhere. We have proved that

$$(L^i|K^i(b_j \mid j \in J), v)$$

is an immediate extension.

By assumption, (L, v) is an extension of (K, v) within the absolute ramification field (K^r, v) of (K, v). Hence also (L^i, v) is contained in (K^r, v). Therefore, we can apply Corollary 7 to find that

$$L^i = K^i(b_j \mid j \in J).$$

Since $K \subseteq F$, it follows that $K^i \subseteq F^i$. Since $\beta_j \in vL \subseteq vF$, we know from Lemma 8 that the polynomials $X^{n_j} - c_j$ split completely over F^i. Consequently, we also have $b_j \in F^i$ for each $j \in J$. This yields that

$$L \subseteq L^i = K^i(b_j \mid j \in J) \subseteq F^i.$$

We conclude that

$$L.F \subseteq F^i,$$

so the extension $(L.F|F, v)$ is unramified.

Now we prove the assertion in the general case, where vL is not necessarily a subgroup of vF. We construct an extension (F_1, v) of (F, v) within its absolute ramification field (F^r, v) such that $vF_1 = vL + vF$. Take (F_1, v) to be a maximal extension of (F, v) within (F^r, v) such that $vF_1 \subseteq vL + vF$; this exists by Zorn's lemma. We have to show that $vF_1 = vL + vF$. Suppose otherwise and take an element $\beta \in vL \setminus vF_1$. Let n be the order of β over vF_1; as it must be a divisor of the order of β over vK and (L, v) lies in the absolute ramification field of (K, v), it is not divisible by char Kv. It follows that $\beta \in vF_1^r$. Take an element $c \in F_1$ such that $vc = n\beta$. Then by Lemma 8 there is some $b \in (F_1^r)^i = F_1^r = F^r$ such that $b^n = c$ and therefore, $vb = \beta$. We compute:

$$n = (vF_1 + \mathbb{Z}\beta : vF_1) \leq (vF_1(b) : vF_1) \leq [F_1(b) : F_1] \leq n,$$

so equality holds everywhere and we find that $vF_1(b) = vF_1 + \mathbb{Z}\beta \subseteq vL + vF$. Since $b \notin F_1$, this contradicts the maximality of F_1, showing that $vF_1 = vL + vF$.

Now we apply what we have shown already to F_1 in place of F. Since now $vL \subseteq vF_1$, we find that $L.F_1 \subseteq F_1^i \subseteq F_1^r = F^r$ and

$$v(L.F) \subseteq v(L.F_1) \subseteq vF_1^i = vF_1 = vL + vF \subseteq v(L.F),$$

whence $v(L.F) = vL + vF$.

Assume that vL is not a subgroup of vF. Then $vF \subsetneq vL + vF = v(L.F)$, so the extension $(L.F|F, v)$ is not unramified. We have now proved part (1) of Theorem 3.

For the proof of part (2) of Theorem 3, we proceed in a similar way as for part (1), but on a "lower level". By hypothesis, $L \subseteq K^i$. First, we assume that $Lv \subseteq Fv$. We take a set of generators $\{\zeta_j \mid j \in J\}$ of the separable-algebraic field extension $Lv|Kv$. Then we choose monic polynomials $f_j \in K[X]$ such that the reduction $\bar{f}_j$ of f_j modulo v is the minimal polynomial of ζ_j over Kv, for each $j \in J$. Since ζ_j is a simple root of $\bar{f}_j$, we can use Hensel's lemma to find a root $b_j \in L^h$ whose residue is ζ_j. Since $K^h \subseteq L^h$, we have that $K^h(b_j \mid j \in J) \subseteq L^h$ and

$$Lv \subseteq K^h(b_j \mid j \in J)v \subseteq L^h v = Lv,$$

showing that equality must hold. We also have that

$$vL \subseteq vK^i = vK \subseteq vK^h(b_j \mid j \in J) \subseteq vL^h = vL,$$

showing again that equality must hold. Thus, $(L^h|K^h(b_j \mid j \in J), v)$ is an immediate extension of henselian fields inside of the absolute inertia field of (K, v). Hence by Corollary 7 we obtain that

$$L^h = K^h(b_j \mid j \in J).$$

Since $K \subseteq F$, it follows that $K^h \subseteq F^h$. Since $\zeta_j \in Fv$ and ζ_j is a simple root of $\bar{f}_j$, it follows from Hensel's lemma that f_j has a root in F^h with residue ζ_j; this root must be b_j. Consequently,

$$L \subseteq L^h = K^h(b_j \mid j \in J) \subseteq F^h.$$

We conclude that

$$L.F \subseteq F^h,$$

which implies that the extension $(L.F|F, v)$ is immediate.

Next, we prove the assertion in the general case, where Lv is not necessarily a subfield of Fv. We construct an extension (F_1, v) of (F, v) within its absolute inertia field (F^i, v) such that $F_1 v = Lv.Fv$. Take (F_1, v) to be a maximal extension of (F, v) within (F^i, v) such that $F_1 v \subseteq Lv.Fv$; this exists by Zorn's lemma. We have to show that $F_1 v = Lv.Fv$. Suppose otherwise and take an element $\zeta \in Lv \setminus F_1 v$. Since (L, v) lies in the absolute inertia field of (K, v) by hypothesis, ζ is separable-algebraic over Kv and hence also over $F_1 v$. It follows that $\zeta \in F_1^i v$. Take a monic polynomial $f \in F_1[X]$ whose reduction fv modulo v is the minimal polynomial of ζ over F_1v and note that ζ is a simple root of fv. By Hensel's lemma there is a root z of f in the henselian field (F_1^i, v) such that $zv = \zeta$. We compute:

$$\deg f = \deg fv = [F_1v(\zeta) : F_1v] \leq [F_1(z)v : F_1v] \leq [F_1(z) : F_1] \leq \deg f,$$

so equality holds everywhere and we find that $F_1(z)v = F_1v(\zeta) \subseteq Lv.Fv$. Since $z \notin F_1$, this contradicts the maximality of F_1, showing that $F_1v = Lv.Fv$.

Now we apply what we have shown already to F_1 in place of F. Since now $Lv \subseteq F_1v$, we find that $L.F_1 \subseteq F_1^h \subseteq F_1^i = F^i$ and

$$(L.F)v \subseteq (L.F_1)v = F_1^h v = F_1v = Lv.Fv \subseteq (L.F)v,$$

whence $(L.F)v = F_1v = Lv.Fv$.

Finally, assume that Lv is not a subfield of Fv. Then $Fv \subsetneq Lv.Fv = (L.F)v$, so the extension $(L.F|F, v)$ is not immediate. We have now proved part (2) of Theorem 3.

4. Proof of Theorem 4

By assumption, char $Kv = p > 0$. We let L_i, L_r and L_s be as introduced before Theorem 6. Since vL/vL_r is a p-group and no element of vL^r/vK has order divisible by p, we have that $vL_r = (vL)_{p'}$. Further, $L^i = L.K^i$ is a normal extension of K^i and $L_s^i = L_s.K^i$ is a Galois extension of K^i, with ramification field $L_r^i = L_r.K^i$; thus, $L_s^i|L_r^i$ is a p-extension.

We know that $L_s|L_r$ is a p-extension. By Proposition 9, this implies that also $L_s.F|L_r.F$ is a p-extension. Since $L|L_s$ is purely inseparable, it follows that also $L.F|L_s.F$ is purely inseparable. These two facts imply that $v(L.F)/v(L_r.F)$ is a p-group, and that $(L.F)v/(L_r.F)v$ is a normal extension with its maximal separable subextension being a p-extension. Since $v(L_r.F) = (vL)_{p'} + vF$ by part (1) of Theorem 3, the former proves part (1) of Theorem 4.

Now assume that $(vL)_{p'} = vK$. This implies that $L_r = L_i$ and $L_r.F = L_i.F$. Hence from part (2) of Theorem 3 it follows that

$$(L_r.F)v = (L_i.F)v = (Lv)_s.Fv.$$

Together with the facts about $(L.F)v/(L_r.F)v$ that we showed above, this proves part (2) of Theorem 4.

5. A closer analysis of the relevant ramification theory

Throughout this section we will assume that $L|K$ is a (not necessarily finite) Galois extension. Then also $L.F|F$ is a Galois extension, and we denote by res the restriction of automorphisms in Aut $L.F|F$ to L. The following is a consequence of [Neukirch 1992] (see also [Kuhlmann $\geq$ 2020]).

Proposition 10. *In the above situation, we have*:

$$\operatorname{res} G^d(L.F|F, v) \subseteq G^d(L|K, v),$$
$$\operatorname{res} G^i(L.F|F, v) \subseteq G^i(L|K, v),$$
$$\operatorname{res} G^r(L.F|F, v) \subseteq G^r(L|K, v).$$

We set $E := L.F$, let L_d, L_i and L_r be as introduced before Theorem 6, and correspondingly denote by E_d , E_i , E_r the decomposition, inertia and ramification field, respectively, of $(E|F, v)$. As a consequence of Proposition 10, we obtain:

Proposition 11. *With the above assumptions and notation, we have that*

$$L_d \subseteq E_d \cap L, \quad L_i \subseteq E_i \cap L, \quad L_r \subseteq E_r \cap L,$$

and

$$L_d.F \subseteq E_d, \quad L_i.F \subseteq E_i, \quad L_r.F \subseteq E_r.$$

We wish to give examples that show that the inclusion may be strict, even if $F|K$ is finite. In fact, this phenomenon occurs in all instances of elimination of tame or wild ramification.

Example 12. We build on a famous example for an extension with nontrivial defect (see, e.g., [Kuhlmann 2011]). We take (K, v) to be the perfect hull of the Laurent series field $\mathbb{F}_p((t))$ over the field $\mathbb{F}_p$ with p elements. We let ϑ be a root of the Artin–Schreier polynomial $X^p - X - 1/t$. As (K, v) is henselian, there is a unique extension of v to $K(\vartheta)$. Then $(K(\vartheta)|K, v)$ is an immediate Galois extension of degree p, hence has nontrivial defect. The same is true for the extension $(K(\vartheta + a)|K, v)$ where a is a root of $X^p - X - 1$. We set $L = K(\vartheta)$ and $F = K(\vartheta + a)$. We obtain that $L.F = F(a)$. Since $\mathbb{F}_p(a)|\mathbb{F}_p$ is a separable extension of degree p, we see that $L.F = (L.F|F, v)^i$. But as $(K(\vartheta)|K, v)$ has nontrivial defect, $(K(\vartheta), v)$ does not lie in K^r, and consequently, $L_r = K$. With the notation introduced above, we conclude that $K = L_d = L_i = L_r \subsetneq L$, but $F = E_d \subsetneq E_i = E_r = E$ and therefore, $F = L_i.F \subsetneq E_i$ and $F = L_r.F \subsetneq E_r$. ◇

This example shows that the p-extension mentioned in part (2) of Theorem 4 can be nontrivial even if $Lv = (Lv)_s = Kv$ and hence $(Lv)_s.Fv = Fv$. In this example, we have in fact eliminated wild ramification, since $E_r = E$; the wild ramification was turned into a tame unramified extension. It should be noted at this point that eliminating wild ramification cannot increase tame ramification:

Remark 13. If $E_r = E$, then $vE = (vL)_{p'} + vF$. This follows from part 1) of Theorem 4 which states that $vE/((vL)_{p'} + vF)$ is a p-group. But as no element in vE_r/vF has a order divisible by p, the group $vE/((vL)_{p'} + vF)$ must be trivial.

The next example is a basic example of the elimination of tame ramification:

Example 14. We take $K = k(t, x)$ and v to be the t-adic valuation on K. Then $vK = \mathbb{Z}$ and $Kv = k(x)$. We choose an integer $n > 1$ which is not divisible by char k, and n-th roots $t^{1/n}$ and $x^{1/n}$ of t and x, respectively. We assume that k contains a primitive n-th root of unity and set $L = K(t^{1/n})$ and $F = K(t^{1/n}x^{1/n})$, so that $L.F = F(x^{1/n}) = (L.F|F, v)^i$. In this situation, we have that $K = L_d = L_i \subsetneq L_r = L$, but $F = E_d \subsetneq E_i = E_r = E$ and therefore, $F = L_i.F \subsetneq E_i$ and $F \subsetneq L_r.F = E_i$. $\diamond$

Finally, we give an example where a separable extension of the residue field is eliminated. This corresponds to a well known procedure using Hensel's lemma within the henselization of (F, v).

Example 15. We take (K, v) to be as in the previous example, assuming in addition that char $Kv = p > 0$. We let a be a root of the Artin–Schreier polynomial $X^p - X - x$, and b a root of $X^p - X - x - t$. We set $L = K(a)$ and $F = K(b)$. We obtain that $L.F = F(b - a)$. Since $b - a$ is a root of the polynomial $X^p - X - t$ and $vt > 0$, $b - a$ lies in the henselization of (F, v) and it follows that $L.F = E_d$. In this situation, we have that $K = L_d \subsetneq L_i = L_r = L$, but $F \subsetneq E_d = E_i = E_r = E$ and therefore, $F \subsetneq L_i.F = E_d = E$. $\diamond$

6. Examples with rational function fields $F = K(x)$

Example 16. We take a valued field extension $(K(a)|K, v)$ such that $a^n \in K$, the order of va modulo vK is n and n is not divisible by char Kv. It follows that $vK(a) = vK + \mathbb{Z}va$ and $K(a)v = Kv$. We set $L := K(a)$. Further, we consider the Gauß valuation v on the rational function field $L(y)$, that is,

$$v \sum_{i=0}^{k} a_i y^i := \min\{va_i \mid 0 \le i \le k\}.$$

We choose some $d \in K$ such that $vd > va$ and set $x := a + dy$, so $K(x)$ is a rational function field contained in $L(y)$. We consider $K(x)$ equipped with the restriction of the valuation v of $L(y)$.

We wish to prove that $L \subset K(x)^h$. We observe that x/a and x^n/a^n are 1-units and that x/a is a root of the polynomial

$$X^n - \frac{x^n}{a^n} \in K(x)[X] \tag{2}$$

whose reduction modulo v is $X^n - 1$. Since n is not divisible by char Kv, 1 is a simple root of this polynomial and Hensel's lemma shows that $K(x)^h$ contains a unique root z of (2) with residue 1. Consequently, $z = x/a$, whence $a = x/z \in K(x)^h$. This proves that $L \subset K(x)^h$. $\diamond$

Modifications of this example can be obtained by choosing different extensions of v from L to $L(y)$. For example, one can define

$$(3)\qquad v\sum_{i=0}^{k} a_i y^i := \min\{va_i + ivd \mid 0 \le i \le k\},$$

where again $d \in K$ with $vd > va$. In this case we set $x := a + y$ and proceed as in the example. Note that in both constructions, $K(x)v$ is transcendental over Kv; in this case the extensions $(K(x)|K, v)$ are called *residue transcendental*. In the example, we have that $L(y)v = Lv(yv) = Kv(yv)$ is transcendental over Kv and since $L(x)|K(x)$ is algebraic, the same must be true for $K(x)v$. In the modified construction we have that

$$L(y)v = Lv((y/d)v) = Kv((y/d)v).$$

A similar example can be produced with a *value transcendental* extension $(K(x)|K, v)$ where $vK(x)/vK$ has rational rank 1. To achieve this, one replaces vd in definition (3) by some value $\alpha > va$ which is nontorsion over vK. A particular case of this is obtained when one takes v_y to be the y-adic valuation on $L(y)$ and then sets the composition $v_y \circ v$ to be the extension of v from L to $L(y)$.

In all of the above examples the extension $(K(a)|K, v)$ was such that $vK(a) = vK + \mathbb{Z}va$ and $K(a)v = Kv$. However, the examples work in exactly the same way when we assume that $a^n \in K$, $va = 0$, $[Kv(av) : Kv] = n$ and n is not divisible by $\operatorname{char} Kv$. It then follows that $vK(a) = vK$ and $K(a)v = Kv(av)$. In this case it is not tame ramification that is eliminated, but a separable-algebraic extension of the residue field instead.

7. Abhyankar's lemma using ramification indices

Theorem 1 is a consequence of the more general version of Abhyankar's lemma stated in [Stacks 2005–, Tag 0EXT Lemma 15.105.4]. Indeed, in the setup of [Tag 0EXT Lemma 15.105.4] and [Tag 0EXT Remark 15.105.1], we note that the assumptions that $\gcd(e, p) = 1$ and κ_B/κ_A is separable still hold when the valued field extension L/K is tamely ramified. Further, A_1 is a discrete valuation ring of rank 1 by [Tag 0EXT Remark 15.105.1(4)]. Finally, from [Tag 0ASF Definition 15.112.1] and [Tag 09E7 Lemma 15.102.5], it follows that the formally smooth conclusion in [Tag 0EXT Lemma 15.105.4] implies that the extension is unramified.

We will now show how Theorem 2 can be deduced from Theorem 3. We will need the following preparation. If Δ is a torsion free abelian group and $e > 0$ is an integer, then $\frac{1}{e}\Delta$ will denote the abelian group consisting of all α in the divisible hull of Δ such that $e\alpha \in \Delta$.

Lemma 17. *Take an integer $e > 0$, a torsion free abelian group Δ of rational rank 1, and a subgroup Γ of its divisible hull such that $\Delta \subseteq \Gamma$ and $(\Gamma : \Delta) = e$. Then $\Gamma = \frac{1}{e}\Delta$.*

Proof. As Δ of rational rank 1, it can be embedded in $\mathbb{Q}$ by sending any nonzero element in Δ to 1, and the divisible hull of Δ can be identified with $\mathbb{Q}$. As $(\Gamma : \Delta) = e$, we have that $\Gamma \subseteq \frac{1}{e}\Delta$. We wish to show that $(\frac{1}{e}\Delta : \Delta) = e$, which then yields that $\Gamma = \frac{1}{e}\Delta$. It suffices to show that $(\frac{1}{e}\Delta : \Delta) \leq e$.

Take any $e + 1$ many elements $\alpha_1, \ldots, \alpha_{e+1} \in \frac{1}{e}\Delta$; we have to show that at least two of them have the same coset modulo Δ. As these elements are rational numbers, we can multiply them by a common denominator s to obtain integers $s\alpha_1, \ldots, s\alpha_{e+1}$. The ideal they generate in $\mathbb{Z}$ is principal, equal to, say, $r\mathbb{Z}$. We know that $(\mathbb{Z} : e\mathbb{Z}) = e$ and hence also $(r\mathbb{Z} : er\mathbb{Z}) = e$. Thus there are distinct $i, j \in \{1, \ldots, e + 1\}$ such that $s\alpha_i - s\alpha_j \in er\mathbb{Z}$. This implies that $\alpha_i - \alpha_j \in e\frac{r}{s}\mathbb{Z}$. Since the elements $s\alpha_1, \ldots, s\alpha_{e+1}$ generate the group $r\mathbb{Z}$, the elements $\alpha_1, \ldots, \alpha_{e+1}$ generate the group $\frac{r}{s}\mathbb{Z}$, which shows that $\frac{r}{s}\mathbb{Z} \subseteq \frac{1}{e}\Delta$, whence $\alpha_i - \alpha_j \in e\frac{r}{s}\mathbb{Z} \subseteq \Delta$. Therefore, α_i and α_j have the same coset modulo Δ. □

As mentioned in the introduction, the assumption that $(L.K^h|K^h, v)$ is tame yields that $(L.K^h, v)$ lies in the absolute ramification field of (K^h, v), which is equal to the absolute ramification field of (K, v). Since vK has rational rank 1, Lemma 17 shows that the value group of (L, v) is $\frac{1}{(vL:vK)}vK$, and likewise, the value group of (F, v) is $\frac{1}{(vF:vK)}vK$. Now we infer from Theorem 3 that

$$v(L.F) = \frac{1}{(vL : vK)}vK + \frac{1}{(vF : vK)}vK.$$

If ℓ is the least common multiple of $(vL : vK)$ and $(vF : vK)$, then the right hand side is equal to $\frac{1}{\ell}vK$. This proves Theorem 2.

We wish to investigate how far Theorem 1 can be generalized while keeping the use of ramification indices. We note that if q is a prime and $a, b \in K^{\mathrm{ac}}$ such that $a^q, b^q \in K$, then $va, vb \in \frac{1}{q}vK$, and that $\frac{1}{q}vK/vK$ is an $\mathbb{F}_q$-vector space.

Lemma 18. *Take a valued field (K, v) and an extension of v to the algebraic closure K^{ac} of K. Assume that there are $a, b \in K^{\mathrm{ac}}$ with $va, vb \notin vK$ and a prime q such that $a^q, b^q \in K$ and $va + vK$ and $vb + vK$ are $\mathbb{F}_q$-linearly independent elements in $\frac{1}{q}vK/vK$. Then we have that $(vK(a) : vK) = q = (vK(b) : K)$ and that*

$$(4) \qquad (vK(a, b) : vK(a)) = q = (vK(a, b) : K(b)).$$

Proof. We compute:

$$(vK(a) : vK) \leq [K(a) : K] \leq q = (vK + \mathbb{Z}va : vK) \leq (vK(a) : vK).$$

Thus, equality holds everywhere, showing that $(vK(a) : vK) = q$. In a similar way, one shows that $(vK(b) : vK) = q$. Further, the equality $(vK + \mathbb{Z}va : vK) = (vK(a) : vK)$ shows that $vK(a) = vK + \mathbb{Z}va$. Similarly, it is shown that $vK(b) = vK + \mathbb{Z}vb$. Obviously, $va, vb \in vK(a, b)$. However, since $va + vK$ and $vb + vK$ are $\mathbb{F}_q$-linearly independent elements in $\frac{1}{q}vK/vK$, we have that $va \notin vK + \mathbb{Z}vb = vK(b)$ and $vb \notin vK + \mathbb{Z}va = vK(a)$. As q is a prime, we conclude that

$$(vK(a, b) : vK(b)) \geq q \quad \text{and} \quad (vK(a, b) : vK(a)) \geq q,$$

and with similar inequalities as above, one proves that (4) holds. □

This lemma shows that Theorem 1 will fail as soon as there exist a prime q different from the residue characteristic and two values $\alpha, \beta \in vK$ such that both are not divisible by q in vK and $\alpha/q + vK$ and $\beta/q + vK$ are $\mathbb{F}_q$-linearly independent elements in $\frac{1}{q}vK/vK$. Then one can pick $a, b \in K^{\mathrm{ac}}$ such that $a^q, b^q \in K$ with $va^q = \alpha$ and $vb^q = \beta$. It follows that $a, b \notin K$, so these elements satisfy the assumptions of Lemma 18.

Quick examples for the above situation are valued fields (K, v) for which vK is isomorphic to $\mathbb{Z}^n$ with $n > 1$, endowed with any ordering. These include all generalized discretely valued fields with $n > 1$.

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