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Congruences of Fourier coefficients of modular forms have long been an object of central study. By comparison, the arithmetic of other expansions of modular forms, in particular Taylor expansions around points in the upper half-plane, has been much less studied. Recently, Romik made a conjecture about the periodicity of coefficients around $\tau_0 = i$ of the classical Jacobi theta function θ_3. Here, we generalize the phenomenon observed by Romik to a broader class of modular forms of half-integral weight and, in particular, prove the conjecture.

1. Introduction

Fourier coefficients of modular forms are well-known to encode many interesting quantities, such as the number of points on elliptic curves over finite fields, partition numbers, divisor sums, and many more. Thanks to these connections, the arithmetic of modular form Fourier coefficients has long enjoyed a broad study, and remains a very active field today. However, Fourier expansions are just one sort of canonical expansion of modular forms. Petersson [1941] also defined the so-called *hyperbolic* and *elliptic* expansions, which instead of being associated to a cusp of the modular curve, are associated to a pair of real quadratic numbers or a point in the upper half-plane, respectively. A beautiful exposition on these different expansions and some of their more recent connections can be found in [Imamoḡlu and O'Sullivan 2009]. In particular, Imamoḡlu and O'Sullivan point out that Poincaré series with respect to hyperbolic expansions include the important examples of Katok [1985] and Zagier [1975], which are the functions which Kohnen [1985] later used to construct the holomorphic kernel for the Shimura–Shintani lift.

Here, we focus on elliptic expansions, which are essentially Taylor expansions. While a Fourier expansion of a given modular form $f \in M_k(\Gamma)$ for some weight k and congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is an expansion at a *cusp* of Γ, i.e., at the boundary of the completed upper half-plane $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, one might also

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consider expansions around an interior point $\tau_0 \in \mathbb{H}$. The classical Taylor expansion in the sense of complex analysis,

$$f(\tau) = \sum_{n=0}^{\infty} \left(\frac{d^n f}{d\tau^n} \right)(\tau_0) \cdot \frac{(\tau - \tau_0)^n}{n!},$$

only converges on an open disc of radius $y_0 := \mathrm{Im}(\tau_0)$ around τ_0, which is not optimal because the natural domain of holomorphy of f is the full upper half-plane $\mathbb{H}$. Instead of this naive construction, one uses a Cayley-type transformation

$$\tau \mapsto w = \frac{\tau - \tau_0}{\tau - \overline{\tau_0}}$$

to map the upper half-plane to the open unit disc, sending the point $\tau_0 = x_0 + i y_0$ to the origin, and considers f as a function in w instead. Taking the usual Taylor expansion with respect to w around $w = 0$ yields the relation

$$(1-w)^{-k} f\left(\frac{\tau_0 - \overline{\tau_0} w}{1 - w} \right) = \sum_{n=0}^{\infty} \partial^n f(\tau_0) \frac{(4\pi y_0 w)^n}{n!}, \quad |w| < 1, \tag{1-1}$$

where

$$\partial := \partial_k := D - \frac{k}{4\pi \,\mathrm{Im}(\tau)} \quad \text{with } D := \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}, \; q := e^{2\pi i \tau} \tag{1-2}$$

denotes the renormalized Maaß raising operator with the abbreviations $\partial^0 = \mathrm{id}$ and $\partial^n := \partial_k^n := \partial_{k+2(n-1)} \circ \cdots \circ \partial_{k+2} \circ \partial_k$ for $n > 0$; see for instance [Zagier 2008, Proposition 17]. Note that for any smooth function $f : \mathbb{H} \to \mathbb{C}$ and $g \in \mathrm{SL}_2(\mathbb{R})$ we have

$$(\partial_k f)|_{k+2}\, g = \partial_k (f|_k g),$$

where $|_k$ denotes the weight k slash operator (see Section 2 for the definition), so in particular the operator ∂_k preserves modularity, but not holomorphy (except in weight 0).

Remark. We note that for $k \notin \mathbb{Z}$, there is an ambiguity on the left-hand side of (1-1), while the right-hand side is well-defined for any k. Since we have $(1-w) \neq 0$ for $|w| < 1$ and the unit disc is simply connected, we can fix the branch of the holomorphic square-root that is positive for positive real arguments to make (1-1) consistent for any half-integer $k \in \frac{1}{2}\mathbb{Z}$, as can be seen by restricting w to the open interval $(-1, 1)$ in the proof of [Zagier 2008, Proposition 17].

It follows from the theory of complex multiplication that the coefficients in the Taylor expansion of a modular form with algebraic Fourier coefficients — a condition we assume throughout the paper if not specified otherwise — around a CM point (suitably normalized) are again algebraic numbers. In special cases, these are also known to have deep arithmetic meaning. For example, it was shown by Rodríguez Villegas and Zagier [1993] that Taylor coefficients of Eisenstein series are essentially special values of Hecke L-functions, a fact which later allowed them to give an explicitly computable criterion to decide whether or not a prime $p \equiv 1 \pmod 9$ is the sum of two rational cubes [Rodríguez Villegas and Zagier 1995]; see also [Zagier 2008, pp. 89–90 and 97–99].

Remark. Loosely speaking, this relation between special values of L-functions and Taylor coefficients may already suggest their periodicity modulo primes in special cases since for instance in the simplest case of an L-function, the Riemann ζ-function, the special values are essentially Bernoulli numbers, whose periodicity properties modulo primes are well-known.

Given these applications, it is natural to ask for arithmetic properties, for instance congruences, of Taylor expansions of modular forms. The articles [Datskovsky and Guerzhoy 2008; Larson and Smith 2014] have previously given conditions under which Taylor expansions of integral weight modular forms are periodic. Recently, Romik [2020] studied the Taylor coefficients of the classical Jacobi theta function

$$\theta_3(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$$

around the point $\tau_0 = i$. He gives explicit recursions for these coefficients and, based on numerical examples, he conjectures a certain behavior of these coefficients modulo primes. To be more precise, let $\Phi = \Gamma(1/4)^4/(8\pi^2\sqrt{2})$ and define the numbers $d(n)$ by

$$(1-w)^{-1/2}\theta_3\left(\frac{i+wi}{1-w}\right) =: \theta_3(i)\sum_{n=0}^{\infty}\frac{d(n)}{(2n)!}(\Phi w)^{2n}, \quad |w|<1. \tag{1-3}$$

For instance, the first few values of $d(n)$ are $1, 1, -1, 51, 849, -26199$. Comparing this to (1-1), we point out that the derivatives $\partial^n\theta_3(i)$ vanish for odd n since i is a fixed point of the transformation $\tau \mapsto -1/\tau$, under which θ_3 and all its nonholomorphic derivatives are equivariant. Romik [2020, Theorem 1] showed that the numbers $d(n)$ are all integers and posed the following conjecture.

Conjecture 1.1 [Romik 2020, Conjecture 13]. *Let p be an odd prime.*

(1) *If $p \equiv 3 \pmod 4$, then $d(n) \equiv 0 \pmod p$ for sufficiently large n.*

(2) *If $p \equiv 1 \pmod 4$, the sequence $\{d(n) \pmod p\}_{n=1}^{\infty}$ is periodic.*

In particular, regardless of the case, the sequence modulo p is always eventually periodic. Romik [2020, Section 8] also asks the question if a similar pattern persists modulo higher powers of primes. Recently, part of Conjecture 1.1 has been proven by Scherer [2020].

Theorem [Scherer 2020, Theorem 1]. (1) *Part* (1) *of Conjecture 1.1 is true.*

(2) *We have that $d(n) \equiv (-1)^{n+1} \pmod 5$.*

Apart from the congruences modulo 5, part (2) of Romik's conjecture has not been settled by Scherer's work. In this paper, we prove and considerably generalize this half of the conjecture.

In order to state our main result we need to introduce an additional notation. As usual, define the weight k Eisenstein series for even integers $k > 2$ by

$$E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n, \tag{1-4}$$

where B_k denotes the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. Letting $\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$, the modular function $\phi_k := E_k/\Theta^{2k}$ is modular on $\Gamma_0(4)$, and as such takes algebraic values at CM points.

Theorem 1.2. *Suppose that $k, N \in \mathbb{N}$ and let $f \in M_{k-1/2}(\Gamma_1(4N))$ be a modular form with algebraic integer Fourier coefficients. Further suppose that $p > 3$ is a split prime in $\mathbb{Q}(\tau_0)$ for a CM point τ_0.*

Assume furthermore that the absolute norm of the algebraic number $\phi_{p-1}(\tau_0)$ is p-integral and is not divisible by p. Then there exists $\Omega \in \mathbb{C}^\times$, which can be chosen to depend only on τ_0 and p, such that for $n_1, n_2 > A$ satisfying

$$n_1 \equiv n_2 \pmod{(p-1)p^A}$$

we have the congruence

$$\partial^{n_1} f(\tau_0)/\Omega^{2k+4n_1-1} \equiv \partial^{n_2} f(\tau_0)/\Omega^{2k+4n_2-1} \pmod{p^{A+1}}.$$

Remark. The condition that $\phi_{p-1}(\tau_0)$ be a p-adic unit is entirely technical, and the theorem probably holds true without it. However, this condition simplifies our proof considerably, and so we have chosen to state the theorem in this way.

Remark. The condition $n_1, n_2 > A$ in the theorem originates from the application of the Euler–Fermat totient theorem in the proof. Therefore, our theorem predicts in complete generality when the sequence of Taylor coefficients of any half-integer weight modular form becomes periodic and what its maximal period length is.

Remark. Work by Mori [1995] states a kind of q-expansion principle which says roughly that if an integer weight modular form has Fourier coefficients in some suitable extension of $\mathbb{Z}_p$ then its Taylor coefficients at a CM point (suitably normalized) lie again in the same extension and satisfy so-called Kummer–Serre congruences and vice versa. For a precise statement, see [Mori 1995, Theorem 3]. This work, however, only immediately applies to integer weight forms and also doesn't obviously give the congruences we need in our result.

Remark. It is worth noting that the inert prime case was studied in detail for integral weight forms by Larson and Smith [2014]. There, they found similar eventual vanishing results modulo p as in part (1) of Conjecture 1.1. Although it appears numerically that more general versions of their work hold, it appears that new techniques are required to prove a general phenomenon since their proofs use the structure of the algebra of integer weight modular forms on $\mathrm{SL}_2(\mathbb{Z})$ in an essential way.

Note that in Theorem 1.2 the number $\Omega \in \mathbb{C}^*$ depends on the prime p. In order to prove and generalize Romik's conjecture, we need to remove this dependence on the prime, which we do in the following theorem.

Theorem 1.3. *Let $\tau_0 \in \mathbb{H}$ be a CM point such that the class number of $K = \mathbb{Q}(\tau_0)$ is 1. Assume further that the CM elliptic curve E defined by $\mathbb{C}/\langle \omega, \omega\tau_0\rangle_{\mathbb{Z}}$ for a real period ω is defined over $\mathbb{Q}$ and the conditions and notation in Theorem 1.2. Then there exists a number $\widetilde{\Omega} \in \mathbb{C}^\times$ which only depends on τ_0 such that for every prime $p > 3$ that splits in K and at which E has good reduction we have the congruence*

$$\partial^{n_1} f(\tau_0)/\widetilde{\Omega}^{2k+4n_1-1} \equiv \partial^{n_2} f(\tau_0)/\widetilde{\Omega}^{2k+4n_2-1} \pmod{p^{A+1}}$$

for any $n_1, n_2 > A$ with

$$n_1 \equiv n_2 \pmod{(p-1)p^A}.$$

Part (2) of Conjecture 1.1 follows by taking $f(\tau) = \Theta(\tau) \in M_{1/2}(\Gamma_0(4))$ as defined above and $\tau_0 = i/2$ in Theorem 1.3. By combining with Scherer's result, this proves Conjecture 1.1.

Corollary 1.4. *Conjecture 1.1 is true.*

Remark. The assumptions on the class number and elliptic curve E in Theorem 1.3 are again of a technical nature to simplify the proof and the statement of the result. A suitably modified version of the result should still hold without these assumptions.

The rest of this paper is organized as follows. In Section 2 we collect some necessary background about quasimodular and almost holomorphic modular forms. Section 3 contains the proof of Theorems 1.2 and 1.3, which makes use of an important result following from the theory of Katz (see Proposition 3.2). We conclude the paper by discussing examples of Taylor expansions of modular forms for $\Gamma_0(4)$ around various CM points in Section 4.

2. Quasimodular and almost holomorphic modular forms of half-integer weight

In this section, we review the basic theory of quasimodular and almost holomorphic forms, which we require in our proofs of the main results. Quasimodular forms and almost holomorphic modular forms generalize classical modular forms. The first example of a quasimodular form is the Eisenstein series of weight 2,

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty} n\frac{q^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n.$$

While E_2 is not modular, it very nearly is. In general, quasimodular forms have a slightly deformed modularity transformation, and every quasimodular form has an associated almost holomorphic modular form. An almost holomorphic modular form is simply a modular form which, instead of being holomorphic, is a polynomial in $Y := 1/(-4\pi y)$, where $y := \mathrm{Im}(\tau)$, with holomorphic functions as coefficients. In the case of E_2, the associated almost holomorphic modular form is the function

$$E_2^*(\tau) := E_2(\tau) + 12Y,$$

which transforms as a modular form of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$. More precise definitions follow below.

The systematic study of these functions originates[1] from work of Kaneko and Zagier [1995] on a theorem of Dijkgraaf [1995]. In the last few years, these functions (in integral weight) have received a lot of attention in the context of the celebrated Bloch–Okounkov theorem [Bloch and Okounkov 2000; Zagier 2016].

In this section, we record special cases of Lemma 1.1 and Proposition 1.2 of [Zemel 2015], where Zemel generalizes the concepts of quasimodular and almost holomorphic modular forms to the setting of real-analytic modular forms, possibly with singularities, of arbitrary (real or complex) weights, arbitrary (vector-valued) multiplier systems for arbitrary Fuchsian groups.

We begin by recalling the slash operator. For a function $f : \mathbb{H} \to \mathbb{C}$, a weight $k \in \frac{1}{2}\mathbb{Z}$, and a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, let

$$(f|_k\gamma)(\tau) := \begin{cases} (c\tau+d)^{-k} f\left(\dfrac{a\tau+b}{c\tau+d}\right) & \text{if } k \in \mathbb{Z}, \\ \left(\dfrac{c}{d}\right)\varepsilon_d^{-2k}\left(\sqrt{c\tau+d}\right)^{-2k} f\left(\dfrac{a\tau+b}{c\tau+d}\right) & \text{if } k \in \frac{1}{2}+\mathbb{Z}, \end{cases}$$

where for $k \in \frac{1}{2}+\mathbb{Z}$ we assume additionally that $\gamma \in \Gamma_0(4)$ (i.e., $4 \mid c$), $\left(\frac{c}{d}\right)$ denotes the extended Jacobi symbol in the sense of Shimura [1973], we choose the branch of the square root so that $-\frac{\pi}{2} < \arg\sqrt{z} \le \frac{\pi}{2}$, which is consistent with the choice made in the remark following (1-1), and

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

With this notation in mind, we can make the following definition.

Definition 2.1. Suppose that $k \in \frac{1}{2}\mathbb{Z}$ and $d \in \mathbb{N}_0$. Let $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ if $k \in \mathbb{Z}$ and $\Gamma \le \Gamma_0(4)$ if $k \in \frac{1}{2}+\mathbb{Z}$. Then a *quasimodular form* of weight k and depth d is a holomorphic function f on $\mathbb{H}$ with moderate growth when τ approaches any cusp

[1] Essentially the same concepts under slightly different names have been introduced independently by Shimura [1986].

in $\mathbb{Q} \cup \{\infty\}$ satisfying

$$(f|_k\gamma)(\tau) = \sum_{j=0}^{\mathsf{d}} \Big(\frac{c}{c\tau+d}\Big)^j f_j(\tau) \tag{2-1}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathbb{H}$, where $f_0\,(= f), \dots, f_{\mathsf{d}}$ are certain holomorphic functions, depending only on f but not on γ, which satisfy the same growth conditions.

We also say that the *depth* of a quasimodular form f is the largest integer d in (2-1), such that f_{d} does not vanish identically. The space of quasimodular forms of weight k and depth $\le \mathsf{d}$ is denoted by $\widetilde{M}_k^{\le \mathsf{d}}(\Gamma)$. If we allow arbitrarily large depth (which is actually at most $k/2$; see Proposition 2.2), we omit the superscript.

A closely related notion is that of almost holomorphic modular forms of weight $k \in \frac{1}{2}\mathbb{Z}$, which are defined — as mentioned at the beginning of this section — as polynomials in $Y = 1/(-4\pi y)$ with holomorphic coefficients, transforming like modular forms. The space of such functions is denoted by $\widehat{M}_k^{\le \mathsf{d}}(\Gamma)$, where d denotes the maximal degree of the polynomial. Again, an omitted superscript indicates that the degree can be unbounded. The following proposition makes the aforementioned close connection between quasimodular forms and almost holomorphic modular forms explicit.

Proposition 2.2. *Let $f \in \widetilde{M}_k^{\le \mathsf{d}}(\Gamma)$ be a quasimodular form of weight k and depth $\le \mathsf{d}$ with corresponding functions $f_0, \dots, f_{\mathsf{d}}$ as in* (2-1).

(1) *For $j = 0, \dots, \mathsf{d}$ we have $f_j \in \widetilde{M}_{k-2j}^{\le \mathsf{d}-j}(\Gamma)$, the corresponding functions being given by $\binom{j}{r} f_r$, $j \le r \le \mathsf{d}$. In particular, the function f_{d} is a modular form of weight $k - 2\mathsf{d}$.*

(2) *The function*

$$f^*(\tau) = \sum_{j=0}^{\mathsf{d}} f_j(\tau)\Big(\frac{1}{2\pi i} Y\Big)^j$$

transforms like a modular form of weight k. Conversely, if we have

$$G(\tau) = \sum_{j=0}^{\mathsf{d}} g_j(\tau)\Big(\frac{1}{2\pi i} Y\Big)^j \in \widehat{M}_k^{\le \mathsf{d}}(\Gamma),$$

then $g_0 \in \widetilde{M}_k^{\le \mathsf{d}}(\Gamma)$ with corresponding functions $g_0, \dots, g_{\mathsf{d}}$.

In particular, the graded rings $\widetilde{M}_*(\Gamma) = \bigoplus_k \widetilde{M}_k(\Gamma)$ and $\widehat{M}_*(\Gamma) = \bigoplus_k \widehat{M}_k(\Gamma)$ are canonically isomorphic.

In the case of integer weight, this proposition goes back to [Kaneko and Zagier 1995]; for half-integer weight it is, as mentioned earlier, a special case of [Zemel

2015, Lemma 1.1 and Proposition 1.2]. We record the following version of [Zagier 2008, Proposition 20]. The proof of this result carries over almost literally, making occasional use of Proposition 2.2; thus, we omit it here.

Proposition 2.3. (1) *The differential operator D maps quasimodular forms to quasimodular forms, i.e., for $f \in \widetilde{M}_k^{\leq \mathsf{d}}(\Gamma)$, we have $Df \in \widetilde{M}_{k+2}^{\leq \mathsf{d}+1}(\Gamma)$.*

(2) *Every quasimodular form is a polynomial in E_2 whose coefficients are modular forms, i.e., we have a decomposition $\widetilde{M}_k^{\leq \mathsf{d}}(\Gamma) = \bigoplus_{j=0}^{\mathsf{d}} M_{k-2j}(\Gamma) \cdot E_2^j$.*

(3) *Every quasimodular form is a linear combination of derivatives of modular forms and derivatives of E_2. More precisely, we have*

$$\widetilde{M}_k^{\leq \mathsf{d}}(\Gamma) = \begin{cases} \bigoplus_{j=0}^{\mathsf{d}} D^j(M_{k-2j}(\Gamma)) & \text{if } \mathsf{d} < k/2, \\ \bigoplus_{j=0}^{k/2-1} D^j(M_{k-2j}(\Gamma)) \oplus \mathbb{C} \cdot D^{k/2-1} E_2 & \text{if } \mathsf{d} = k/2. \end{cases}$$

In the proof of Theorem 1.2, we require the following easy consequence of the above.

Corollary 2.4. *Let $H \in M_k(\Gamma)$ and $G \in M_\ell(\Gamma)$ for $k, \ell \in \frac{1}{2}\mathbb{Z}$. Then we have that $G \cdot (D^n H) \in \widetilde{M}_{k+\ell+2n}(\Gamma)$ and the associated almost holomorphic modular form is given by $G \cdot (\partial^n H)$.*

Proof. It is clear that it suffices to show that the almost holomorphic modular form associated to $D^n H$ is given by $\partial^n H$. As remarked above, we may apply Proposition 2.3 in this setting, wherefore $D^n H \in \widetilde{M}_{k+2n}(\Gamma)$. Furthermore, $\partial^n H$ is an almost holomorphic modular form of the same weight whose constant term with respect to Y is precisely $D^n H$, as one sees immediately from the formula

$$\partial_k^n = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \frac{(k+n-1)!}{(k+m-1)!} Y^{n-m} D^m$$

for the iterated raising operator, which is easily shown by induction; see for instance [Zagier 2008, equation (56)]. □

3. Proofs of Theorems 1.2 and 1.3

In this section, we prove the main results.

Preliminary results and work of Katz. The periodicity phenomenon in Theorem 1.2 is ultimately a consequence of the very general theory of Katz [1976]. However, Katz's work does not contain a statement which is exactly sufficient for our purposes here. The key result we need is Proposition 3.2 below which is an extension of Lemma 1 from [Datskovsky and Guerzhoy 2008]. This statement, as well as the theory developed in [Katz 1976], is formulated for the case of integral weight modular forms, and all weights are assumed to be integral throughout this subsection. Also, p is always assumed to be a prime larger than 3.

The mantra we need here, which requires some work for its precise specialization which we will employ later, is that *p-adically close modular forms have p-adically close values*. That is nothing but a specialization of the *q-expansion principle* from [Katz 1976, Section 5.2].

To make this precise, we first of all need a version of Damerell's theorem which allows for making all quantities under consideration algebraic. The idea is simple: while *quasimodular forms* have q-expansions, *almost holomorphic modular forms* take essentially algebraic values at τ_0 (see Proposition 3.1 below), and as described in Proposition 2.2, the two rings are canonically isomorphic. Thus, we can assign algebraic values to algebraic q-expansions in order to study congruences between them. Proposition 2.3 allows us to assign a q-expansion to every quasimodular form $f \in \widetilde{M}_*(\Gamma)$: we simply plug in the q-expansions of modular forms and E_2 into the expression. Namely, for $g \in \widetilde{M}_k(\Gamma)$, Proposition 2.3(2) implies

$$g = \sum_{r=0}^{\lfloor k/2 \rfloor} F_{k-2r} E_2^r \in \mathbb{C}[\![q]\!] \quad \text{with } F_{k-2r} \in M_{k-2r}(\Gamma).$$

We will identify $g \in \widetilde{M}_k(\Gamma)$ with the associated *almost holomorphic* form $g^* \in \widehat{M}_k(\Gamma)$ via the isomorphism between $\widetilde{M}_*(\Gamma)$ and $\widehat{M}_*(\Gamma)$ which preserves the gradation, and set the value

$$g^*(\tau_0) = \sum_{r=0}^{\lfloor k/2 \rfloor} F_{k-2r}(\tau_0)(E_2^*(\tau_0))^r \in \mathbb{C} \quad \text{with } F_{k-2r} \in M_{k-2r}(\Gamma).$$

From now on, we fix an algebraic number field K which is large enough to contain the relevant quantities below, and we denote its ring of integers by $\mathcal{O}$.

With these notations, we have the following algebraicity statement.

Proposition 3.1 (Katz's version of Damerell's theorem, [Katz 1976, Theorem 4.0.4]). *If $\tau_0 \in K$, then there exists an $\omega \in \mathbb{C}^*$ such that*

$$\textit{if } g \in \widetilde{M}_k(\Gamma) \cap K[\![q]\!], \textit{ then } g^*(\tau_0)/\omega^k \in K.$$

We now pass to the question about congruences. Given two formal power series $g_1 = \sum_{n=0}^{\infty} b_1(n)q^n$, $g_2 = \sum_{n=0}^{\infty} b_2(n)q^n \in K[\![q]\!]$, we say that $g_1 \equiv g_2 \pmod{p^A}$ if their coefficients are congruent modulo p^A, i.e., if $b_1(n) - b_2(n) \in p^A\mathcal{O}$ for all n. This notation applies, in particular, to the situation when g_1 and g_2 are (the q-expansions of) quasimodular forms in $\widetilde{M}_*(\Gamma) \cap K[\![q]\!]$.

Clearly, if $\omega \in \mathbb{C}^\times$ works in Proposition 3.1 then so does any $K^\times$-multiple. Note furthermore that if $\omega \in \mathbb{C}^\times$ satisfies Proposition 3.1 for any single $g \in \widetilde{M}_k(\Gamma) \cap K[\![q]\!]$ then so it also does for all $g \in \widetilde{M}_k(\Gamma) \cap K[\![q]\!]$, and we make a specific choice now.

The discussion above puts no restrictions on the prime p under consideration. From now on, we assume that p splits in $\mathbb{Q}(\tau_0)$. The first consequence of this choice is that $E_{p-1}(\tau_0) \neq 0$ (see Section 2.1 of [Katz 1973]), which allows us to pick $\omega \neq 0$ in the following proposition.

This proposition is nothing but a specialization to our notations of the (more general) q-expansion principle from [Katz 1976, Section 5.2] combined with a p-adic version of Damerell's theorem [Katz 1976, Comparison Theorem 8.0.9]. It was formulated and proved as [Datskovsky and Guerzhoy 2008, Lemma 1] in the special case when $N = 1$. For the general case which we need here, the proof follows mutatis mutandis; we have omitted this simple translation of the proof given in [Datskovsky and Guerzhoy 2008] for notational simplicity.

Proposition 3.2 [Datskovsky and Guerzhoy 2008, Lemma 1]. *Assume that p splits in $\mathbb{Q}(\tau_0)$. Pick a complex number ω_p so that $\omega_p^{p-1} = E_{p-1}(\tau_0)$. For $i = 1, 2$ let*

$$g_i = \sum_{n=0}^{\infty} b_i(n) q^n \in \widetilde{M}_{k_i}(\Gamma) \cap \mathcal{O}[\![q]\!].$$

If

$$g_1 \equiv g_2 \pmod{p^A}$$

for a positive integer A, then

$$g_1^*(\tau_0)/\omega_p^{k_1} \equiv g_2^*(\tau_0)/\omega_p^{k_2} \pmod{p^A}.$$

Remark. A naive explanation for the choice of ω_p is as follows. Since $E_{p-1} \equiv 1 \pmod{p}$ by the von Staudt–Clausen theorem, we also ought to have $E_{p-1}(\tau_0) \equiv 1 \pmod{p}$. In other words, if Proposition 3.2 is true for some choice of ω_p, then this should be a correct choice. Note however that the proposition is simply false as stated for inert primes, although it may still happen that $E_{p-1}(\tau_0) \neq 0$. For example, the prime 13 is inert in $\mathbb{Q}(\sqrt{7})$, but for $\tau_0 = \frac{1}{2}(1 + i\sqrt{7})$ we have $E_{12}(\tau_0) \approx 0.98818418 \neq 0$. Now the two weight 12 modular forms E_{12} and $E_{12} + 13\Delta$ with $\Delta := (E_4^3 - E_6^2)/1728$ denoting the usual Δ function (which has integer Fourier coefficients) are obviously congruent modulo 13, but choosing ω_p as specified in Proposition 3.2, we find that

$$E_{12}(\tau_0)/\omega_p^{12} = 1 \quad \text{and} \quad (E_{12}(\tau_0) + 13\Delta(\tau_0))/\omega_p^{12} = \frac{211934}{212625} \equiv 6 \pmod{13}.$$

Multiplication by the Θ-function and passage to half-integral weight. Here we sketch how to generalize the results of the preceding subsection to half-integral weight. A generalization of Katz's theory to half-integral weight has also been developed by Ramsey [2006]. Although it is based on similar ideas, Ramsey's generalization is less explicit than our approach here, and is not intended in the specific direction which we require, and so is less convenient for our purposes. To move from integral weight to half-integral weight, we use the simple (and common) technique of multiplying by Jacobi's $\Theta(\tau)$. Thanks to Jacobi's identity

$$\Theta(\tau)=\frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)}, \tag{3-1}$$

where $\eta(\tau):=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ denotes the Dedekind eta function, Θ does not vanish in the interior of the upper half-plane. We then need the following result on the action of this multiplication by Θ operation on quasimodular forms.

Lemma 3.3. *Let $H\in\mathbb{C}[\![q]\!]$ be such that the product*

$$\Theta H\in\widetilde{M}_k(\Gamma)$$

is (a q-expansion of) a quasimodular form of weight $k\in\mathbb{Z}$ on $\Gamma=\Gamma_1(N)$, where $4\mid N$. Then

$$\Theta DH\in\widetilde{M}_{k+2}(\Gamma).$$

Proof. Since $\Theta H\in\widetilde{M}_k(\Gamma)$, we have that

$$D(\Theta H)=\Theta DH+HD\Theta\in\widetilde{M}_{k+2}(\Gamma),$$

and it suffices to show that $HD\Theta\in\widetilde{M}_{k+2}(\Gamma)$. It follows from (3-1) that

$$\begin{aligned}24\frac{D\Theta}{\Theta}(\tau)&=10E_2(2\tau)-2E_2(z)-8E_2(4\tau)\\&=10\Big(E_2(2\tau)-\frac{1}{2}E_2(\tau)\Big)-8\Big(E_2(4\tau)-\frac{1}{4}E_2(\tau)\Big)+E_2(\tau)\in\widetilde{M}_{k+2}(\Gamma_0(4)),\end{aligned}$$

and therefore

$$HD\Theta=H\frac{D\Theta}{\Theta}\Theta\in\widetilde{M}_{k+2}(\Gamma)$$

as required. □

Periodicity of Taylor coefficients. We now have all the pieces in place to prove our main results.

Proof of Theorem 1.2. Let p be a splitting prime in $\Omega_p\in\mathbb{C}^\times$ such that $\Omega_p^2=\omega_p$, with ω_p as in Proposition 3.2. Suppose further that

$$f\in M_{k-1/2}(\Gamma)\cap\mathcal{O}[\![q]\!]$$

is a half-integral weight modular form with algebraic integer Fourier coefficients, and assume that both $f(\tau_0)/\Omega_p^{2k-1}$ and $\Theta(\tau_0)/\Omega_p$ lie in K. By Euler's totient theorem, if both $n_1,n_2>A$, we have

$$n_1\equiv n_2 \pmod{(p-1)p^A}\quad\text{implies}\quad D^{n_1}(f)\equiv D^{n_2}(f)\pmod{p^{A+1}}.$$

Multiplication by Θ preserves these congruences:

$$\Theta D^{n_1}f\equiv\Theta D^{n_2}f\pmod{p^{A+1}}.$$

Lemma 3.3 (applied repeatedly) implies that both products

$$\Theta D^{n_1} f \in \widetilde{M}_{k+2n_1}(\Gamma) \quad \text{and} \quad \Theta D^{n_1} f \in \widetilde{M}_{k+2n_2}(\Gamma)$$

are quasimodular forms and we can apply Proposition 3.2 to derive the congruence

$$(\Theta D^{n_1} f)^*(\tau_0)/\omega_p^{k+2n_1} \equiv (\Theta D^{n_2} f)^*(\tau_0)/\omega_p^{k+2n_2} \pmod{p^{A+1}}$$

for the (normalized) values at τ_0. We now apply Corollary 2.4 to evaluate the quasimodular forms $\Theta D^{n_1}(f)$ and $\Theta D^{n_2}(f)$ of integral weight at τ_0

$$(\Theta D^{n_i} f)^*(\tau_0) = \Theta(\tau_0)\partial^{n_i} f(\tau_0) \quad \text{for } i = 1, 2,$$

and factor out $\Theta(\tau_0)$, which by (3-1) is not 0. The extra assumption on p-integrality of the value $\phi_{p-1}(\tau_0)$ allows us to guarantee that

$$v_p(\mathrm{Nm}_{\mathbb{Q}}^{K}(\Theta^2(\tau_0)/\omega)) = 0,$$

where Nm is the norm map, and v_p is the p-adic valuation. We then can cancel this quantity, and obtain the desired periodicity modulo powers of the splitting prime:

$$\partial^{n_1} f(\tau_0)/\Omega^{2k+4n_1-1} \equiv \partial^{n_2} f(\tau_0)/\Omega^{2k+4n_2-1} \pmod{p^{A+1}}. \qquad \square$$

Deligne's congruence and proof of Theorem 1.3. So far, Theorem 1.2 claims the existence of $\Omega_p \in \mathbb{C}^\times$, which depends on the splitting prime p. However, the conjecture of Romik is stated for a global choice, common for all primes. In this subsection, we show how to make a global choice, and compare that with the choice made by Romik [2020]. The fact that these two choices differ by a p-adic unit for every splitting prime $p > 2$ will allow us to derive Conjecture 1.1 from our Theorem 1.2.

Proof of Theorem 1.3. Let $K = \mathbb{Q}(\tau_0)$ be an imaginary quadratic field. Define $\omega = \omega_{\tau_0}$ to be the real period of the CM elliptic curve $E = \mathbb{C}/\langle \omega, \omega\tau_0 \rangle_{\mathbb{Z}}$, and let

$$\wp(z) := \frac{1}{z^2} + \sum_{n \geq 2} c_n z^{2n-2}$$

be the associated Weierstrass $\wp$-function.

By assumption, the elliptic curve E is defined over $\mathbb{Q}$ and the class number of K is 1, which implies that $c_n \in \mathbb{Q}$, where

$$c_n = (2n-1)\omega^{-2n} \sum_{(0,0) \neq (m,n) \in \mathbb{Z} \times \mathbb{Z}} \frac{1}{(m\tau_0 + n)^{2n}}.$$

These quantities are nothing but the values of Eisenstein series at τ_0, properly normalized. Namely, we have

$$c_n = 2\Big(\frac{2\pi i}{\omega}\Big)^{2n} \frac{1}{(2n-2)!} \mathbb{G}_{2n}(\tau_0).$$

See [Zagier 2008, Section 2.2] for the notation and the normalizations of Eisenstein series $\mathbb{G}_k$ and $E_k := -(2k/B_k)\mathbb{G}_k$.

We now define Bernoulli–Hurwitz numbers $\mathrm{BH}(2n)$ for integers $n \geq 1$, following [Katz 1975], as

$$c_n =: \frac{\mathrm{BH}(2n)}{2n} \frac{1}{(2n-2)!},$$

and with the above notation we have that

$$\mathrm{BH}(2n) = -\Big(\frac{2\pi i}{\omega}\Big)^{2n} B_{2n} E_{2n}(\tau_0).$$

By assumption, E has good reduction at p. Denote now by $A(p) \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the Hasse invariant of its modulo p reduction. All we need to know here is that $A(p) \neq 0$ if and only if the elliptic curve has good ordinary reduction at p, i.e., the prime p splits in $\mathbb{Q}(\tau_0)$.

We now quote a special case of part (1) of the theorem proved in [Katz 1975]:

$$p \cdot \mathrm{BH}(p-1) \equiv A(p) \pmod{p}.$$

We translate this congruence using the above notation into

$$-pB_{p-1}\Big(\frac{2\pi i}{\omega}\Big)^{p-1} E_{p-1}(\tau_0) \equiv A(p) \pmod{p},$$

which simplifies using the von Staudt–Clausen congruence $pB_{p-1} \equiv -1 \pmod{p}$ into

$$\Big(\frac{2\pi i}{\omega}\Big)^{p-1} E_{p-1}(\tau_0) \equiv A(p) \pmod{p}, \tag{3-2}$$

that is, the left-hand side is an algebraic integer which is nonzero modulo p if and only if p splits in $\mathbb{Q}(\tau_0)$.

We now compare the local choice of ω_p from Proposition 3.2, which was $\omega_p^{p-1} = E_{p-1}(\tau_0)$, with the global (i.e., independent of p) ω in (3-2), and conclude that the ratio of two omegas is a p-adic unit as we wanted. This completes the proof of Theorem 1.3. □

Remark. In the case $\tau_0 = i$, congruence (3-2) was proved in [Hurwitz 1898]. The above exposition follows closely [Katz 1975], where a short proof based on the q-expansion principle of more general congruences is presented. An independent and elementary proof is presented in [Kaneko and Zagier 1998, Section 3], where a slightly different normalization for Eisenstein series is chosen. In this paper (and in many others, in fact), the congruence is attributed to Deligne.

Remark. In the case when $\tau_0 = i/2$, which is the objective of Conjecture 1.1(2), it is classical [Hurwitz 1898][2] (or see [Husemöller 1987, Section 9.6]) that

$$\omega = \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma(1/4)}{\Gamma(3/4)} = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.62205755429211\ldots.$$

Note that compared to Romik's choice of normalization, we find that $\omega^4 = 2\pi^2\Phi^2$. The factor of 2 is of no importance as it is a p-adic unit for any odd prime and the additional power of π originates from the different normalizations of the series defined in (1-1) and (1-3).

The elliptic curve $\mathbb{C}/\langle\omega, \omega\tau_0\rangle_{\mathbb{Z}}$ in this special case has Weierstrass equation

$$y^2 = 4x^3 - 44x + 56 \quad \text{with the nonvanishing differential } \frac{dx}{y}.$$

4. Examples

In this section, we present several examples for the periodicity of Taylor coefficients at two different CM points, $\tau_0 = i$ and $\tau_0 = \mathfrak{z}_7 = \frac{1}{2}(1 + i\sqrt{7})$. For the sake of being completely explicit, we focus on modular forms for the group $\Gamma_0(4)$. It is a well-known fact, which is easily verified using the dimension formula for spaces of modular forms for this group, that the algebra of modular forms for this group is a free polynomial algebra on two generators. More precisely, we have

$$M_*(\Gamma_0(4)) = \mathbb{C}[\Theta, F_2],$$

where the usual Jacobi theta function $\Theta(\tau)$ was defined in (3-1) and

$$F_2(\tau) := \eta(4\tau)^8\eta(2\tau)^{-4} = \sum_{n \text{ odd}} \sigma_1(n)q^n$$

is a weight 2 modular form; see for instance [Cohen 1975]. Note that in odd integer weight k, we include the spaces $M_k(\Gamma_0(4), \chi_{-4})$ transforming with the nontrivial Nebentypus modulo 4 rather than those with trivial Nebentypus, which would be empty anyway.

It follows from the general theory of complex multiplication (see [Zagier 2008, Proposition 26 and p. 84]) that the values of any weight k modular form for $\Gamma_0(4)$ (with algebraic Fourier coefficients) at a CM point τ_0 of (fundamental) discriminant D are algebraic multiples of Ω_D^k, where

$$\Omega_D = \frac{1}{\sqrt{2\pi|D|}}\left(\prod_{j=1}^{|D|-1}\Gamma(j/|D|)^{\chi_D(j)}\right)^{1/(2h'(D))},$$

[2] Hurwitz does the computation for the CM point $\tau_0 = i$, but one can use the same method to get the result for $i/2$.

$\chi_D = \left(\frac{D}{\cdot}\right)$ denotes the Kronecker character and $h'(D)$ denotes the modified class number of discriminant D, i.e., the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral positive definite binary quadratic forms of discriminant D multiplied by $\frac{1}{3}$ if $D=-3$ or $\frac{1}{2}$ if $D=-4$. Indeed, we find that

$$\Theta(i) = \left(\frac{3+2\sqrt{2}}{2}\right)^{1/4}\Omega_{-4}^{1/2} \quad \text{and} \quad F_2(i) = \frac{3-2\sqrt{2}}{32}\Omega_{-4}^2, \tag{4-1}$$

$$\Theta(\mathfrak{z}_7) = \left(\frac{8+3\sqrt{7}}{4}\right)^{1/4}\Omega_{-7}^{1/2} \quad \text{and} \quad F_2(\mathfrak{z}_7) = -\frac{8-3\sqrt{7}}{2^6}\Omega_{-7}^2. \tag{4-2}$$

Closely following the proof of [Zagier 2008, Proposition 28] we offer the next two propositions, which allow us to compute the Taylor coefficients of any modular form for $\Gamma_0(4)$ at one of the points i and $\mathfrak{z}_7$ recursively. This method can be used completely analogously for Taylor coefficients at any other CM point, which is also why we only give a detailed proof of Proposition 4.1. Generalizing the method to groups other than $\Gamma_0(4)$ is also possible, but some care must be taken if the algebra of modular forms in question is not a free polynomial algebra, which it usually is not.

Before formulating the propositions, we introduce a modification of the Serre derivative (see [Zagier 2008, equation (67)][3]). Let ϕ be any quasimodular form of weight 2 for $\Gamma_0(4)$ such that the associated almost holomorphic modular form is given by $\phi^*(\tau) = \phi(\tau) - 1/(4\pi y)$, and hence transforms like a modular form of weight 2. The Eisenstein series $\frac{1}{12}E_2$ for instance would be a valid choice, but not always the most convenient one, as illustrated for instance in Proposition 4.3. Then we define the modified Serre derivative by

$$\vartheta_\phi f := Df - k\phi f \tag{4-3}$$

for $f \in M_k(\Gamma_0(4))$. This function maps $M_k(\Gamma_0(4))$ to $M_{k+2}(\Gamma_0(4))$, like the usual Serre derivative. The iterated version $\vartheta_\phi^{[n]} : M_k(\Gamma_0(4)) \to M_{k+2n}(\Gamma_0(4))$ of this operator is then defined recursively via

$$\begin{aligned} &\vartheta_\phi^{[0]} f = f, \quad \vartheta_\phi^{[1]} f = \vartheta_\phi f, \\ &\vartheta_\phi^{[n+1]} = \vartheta_\phi(\vartheta_\phi^{[n]} f) + n(k+n-1)\psi\vartheta_\phi^{[n-1]} f \quad (n \geq 1), \end{aligned} \tag{4-4}$$

where $\psi \in M_4(\Gamma_0(4))$ is given by $\psi = D\phi - \phi^2$. In the special case for instance where $\phi = \frac{1}{12}E_2$, we have $\psi = -\frac{1}{144}E_4$. In this particular case, we omit the subscript of the operator, so $\vartheta^{[n]} := \vartheta_{E_2/12}^{[n]}$.

Our first proposition now gives the claimed recursion for the Taylor coefficients of a modular form at the point i.

[3]Note that in [loc. cit.], there is a slight typographical error in that the additional application of ϑ_ϕ to $\vartheta_\phi^{[n]} f$ in the definition of $\vartheta_\phi^{[n+1]} f$ is omitted there.

Proposition 4.1. *Let $f \in M_k(\Gamma_0(4))$ with $k \in \frac{1}{2}\mathbb{Z}$ and let $P(X, Y) \in \mathbb{C}[X, Y]$ be a polynomial such that $P(\Theta, F_2) = f$. Then*

$$\partial^n f(i) = \left(\frac{3+2\sqrt{2}}{2}\right)^{n+k/2} p_n\left(\frac{17-12\sqrt{2}}{16}\right)\Omega_{-4}^{2n+k} = p_n\left(\frac{17-12\sqrt{2}}{16}\right)\Theta(i)^{4n+2k},$$

where $p_n(t)$ is the polynomial defined recursively by

$$p_{-1}(t) = 0, \quad p_0(t) = \frac{P(X, tX^4)}{X^{2k}},$$
$$p_{n+1}(t) = \frac{1}{24}(80t-1)(2k+4n)p_n(t) - (16t^2 - t)p_n'(t) - \frac{1}{144}n(n+k-1)(256t^2+224t+1)p_{n-1}(t) \quad (n \ge 0).$$

Proof. Since the completed weight 2 Eisenstein series E_2^* vanishes at i, it follows by comparing the associated Cohen–Kusnetsov series (see [Zagier 2008, equation (68)]) that $\partial^n f(i) = \vartheta^{[n]} f(i)$ for all n. Since $\vartheta^{[n]}$ maps modular forms of weight k to modular forms of weight $k+2n$, we can view $\vartheta^{[n]}$ as an operator on the polynomial ring $\mathbb{C}[\Theta, F_2]$. In particular, there is a polynomial $P_n(X, Y) \in \mathbb{C}[X, Y]$ such that $\vartheta^{[n]} f = P_n(\Theta, F_2)$. Explicitly, we compute

$$\vartheta\Theta = \frac{1}{24}(80F_2\Theta - \Theta^5),$$
$$\vartheta F_2 = \frac{1}{6}(5\Theta^4 F_2 - 16F_2^2),$$
$$E_4 = \Theta^8 + 224\Theta^4 F_2 + 256F_2^2.$$

Hence we can write

$$\vartheta = \frac{1}{24}(80F_2\Theta - \Theta^5)\frac{\partial}{\partial\Theta} + \frac{1}{6}(5\Theta^4 F - 16F_2^2)\frac{\partial}{\partial F_2},$$

which yields the following recursion for P_n:

$$P_{-1}(X, Y) = 0, \quad P_0(X, Y) = P(X, Y),$$
$$P_{n+1}(X, Y) = \frac{1}{24}(-X^5 + 80XY)\frac{\partial P_n(X, Y)}{\partial X} + \frac{1}{6}(5X^4Y - 16Y^2)\frac{\partial P_n(X, Y)}{\partial Y} - \frac{1}{144}n(n+k-1)(X^8 + 224X^4Y + 256Y^2)P_{n-1}(X, Y).$$

The polynomials $P_n(X, Y)$ are weighted homogeneous of weight $k+2n$, where X has weight $\frac{1}{2}$ and Y has weight 2. Thus we can write $P_n(X, Y) = X^{4n+2k} p_n(Y/X^4)$, where $p_n(t) \in \mathbb{C}[t]$ is a single variable polynomial. Since

$$\frac{\partial P_n(X, Y)}{\partial X} = X^{4n+2k-1}[(4n+2k)p_n(Y/X^4) - 4(Y/X^4)p_n'(Y/X^4)],$$
$$\frac{\partial P_n(X, Y)}{\partial Y} = X^{4n+2k-4} p_n'(Y/X^4),$$

we find the following differential recursion for p_n:

$$p_{n+1}(t) = \frac{1}{24}(80t-1)(4n+2k)p_n(t) - (16t^2-t)p_n'(t) - \frac{1}{144}n(n+k-1)(256t^2+224t+1)p_{n-1}(t),$$

i.e., together with the two stated initial values the recursion claimed. Thus we have shown that

$$\vartheta^{[n]}f = \Theta^{4n+2k}p_n(F_2/\Theta^4),$$

and hence

$$\partial^n f(i) = \vartheta^{[n]}f(i) = \Theta(i)^{4n+2k}p_n(F_2(i)/\Theta^4(i)),$$

which yields the claim using the values given in (4-1). □

As an application of Proposition 4.1, we offer the following example.

Example 4.2. The Taylor coefficients of Θ at $\tau_0 = i$ are given as follows:

$$(1-w)^{-1/2}\Theta\Big(i\,\frac{1+w}{1-w}\Big) = \Theta(i)\sum_{n=0}^{\infty}\frac{c(n)}{n!}(\Phi w)^n \quad (|w|<1),$$

where we choose $\Phi = \varepsilon^4\pi\Omega_{-4}^2 = (17+12\sqrt{2})\Gamma(1/4)^4/(16\pi^2)$, and where $\varepsilon = 1+\sqrt{2}$ is the fundamental unit in $\mathbb{Q}(\sqrt{2})$. Concretely, we compute the following table of values from the recursion in Proposition 4.1.

n	0	1	2	3	4	5	6	7	8	9	10	11
$c(n)$	1	ε	1	-3ε	17	9ε	-111	2373ε	12513	86481ε	-146079	-9806643ε

The number Φ here has been chosen in order to make the coefficients $c(n)$ integers in $\mathbb{Q}(\sqrt{2})$, which one may verify by a straightforward induction argument. In view of Proposition 3.2, the period should be chosen depending on the prime modulus p in order to find periodicity, but for the sake of uniformity, we keep this choice of period. By Fermat's little theorem, this still results in a periodic sequence modulo p but with a longer period than with the choice in Proposition 3.2. This motivates the notation $\overline{a_1,\ldots,a_\ell}^{\,b}$ as a shorthand for

$$a_1,\ldots,a_\ell, ba_1,\ldots,ba_\ell, b^2a_1,\ldots,b^2a_\ell,\ldots,$$

i.e., the *quasiperiod* $a_1,\ldots,a_\ell$ is multiplied by b in each repetition. In other words, multiplying the chosen transcendental factor Φ by an ℓ-th root of b yields an actually periodic coefficient sequence.

Considering the first 200 coefficients, we find that

$$\begin{aligned}\{c(n)\}_{n=0}^{\infty} &\equiv \{1, \overline{\varepsilon, 1}^{\,2}\} \pmod 5\\ &\equiv \{1, \overline{\varepsilon, 1, -3\varepsilon, -8, 9\varepsilon, -11, -2\varepsilon, -12, 6\varepsilon, -4}^{\,7}\} \pmod{5^2},\end{aligned}$$

and that $c(n) \equiv 57c(n+50) \pmod{5^3}$ for $n \geq 11$.

For $p = 13$, we obtain

$$\{c(n)\}_{n=0}^{\infty} \equiv \{1, \overline{\varepsilon, 1, -3\varepsilon, -8, 9\varepsilon, -11, -2\varepsilon, -12, 6\varepsilon, -4}^{7}\} \pmod{13}.$$

With only a small alteration, we obtain the analogous result for the point $\mathfrak{z}_7$.

Proposition 4.3. *Let $f \in M_k(\Gamma_0(4))$ with $k \in \frac{1}{2}\mathbb{Z}$ and $P \in \mathbb{C}[X, Y]$ such that $f = P(\Theta, F_2)$. Then*

$$\begin{aligned}\partial^n f(\mathfrak{z}_7) &= \left(\frac{8+3\sqrt{7}}{4}\right)^{n+k/2} q_n\left(-\frac{127-48\sqrt{7}}{16}\right)\Omega_{-7}^{2n+k} \\ &= q_n\left(-\frac{127-48\sqrt{7}}{16}\right)\Theta(\mathfrak{z}_7)^{4n+2k},\end{aligned}$$

where $q_n(t)$ is defined recursively by

$$q_{-1}(t) = 0, \quad q_0(t) = \frac{P(X, tX^4)}{X^{2k}},$$

$$\begin{aligned}q_{n+1}(t) = \frac{1}{168}&(592t-5)(2k+4n)q_n(t) - (16t^2-t)q_n'(t) \\ &- \frac{1}{7056}n(n+k-1)(6400t^2+15584t+25)q_{n-1}(t) \qquad (n \geq 0).\end{aligned}$$

Proof. Let $\phi = \frac{1}{12}E_2 - \frac{1}{42}(\Theta^4 + 16F_2)$, whence

$$\psi = D\phi - \phi^2 = -\frac{1}{7056}(25\Theta^8 + 15584\Theta^4 F_2 + 6400F_2^2).$$

Then ϕ is a quasimodular form of weight 2 for $\Gamma_0(4)$ and

$$\phi^* = \frac{1}{12}E_2^* - \frac{1}{42}(\Theta^4 + 16F_2) = \phi - \frac{1}{4\pi y}$$

transforms like a modular form. Since $E_2^*(\mathfrak{z}_7) = (3/\sqrt{7})\Omega_{-7}^2$ (see [Zagier 2008, table on p. 87]), one sees easily by comparing to the values given in (4-2) that $\phi^*(\mathfrak{z}_7) = 0$, wherefore it follows, as in the proof of Proposition 4.1, that $\partial^n f(\mathfrak{z}_7) = \vartheta_\phi^{[n]} f(\mathfrak{z}_7)$. The action of $\vartheta_\phi^{[n]}$ on the polynomial algebra $\mathbb{C}[\Theta, F_2]$ is determined by

$$\begin{aligned}\vartheta_\phi \Theta &= -\frac{1}{168}(5\Theta^5 - 592\Theta F_2), \\ \vartheta_\phi F_2 &= \frac{1}{42}(37\Theta^4 F_2 - 80F_2^2),\end{aligned}$$

as one can easily verify. The proof now follows the exact same lines as that of Proposition 4.1, so we leave the rest to the reader. □

Example 4.4. We apply Proposition 4.3 to the Cohen–Eisenstein series

$$\begin{aligned}\mathscr{H}_{5/2}(\tau) &= \frac{1}{120}(\Theta^5(\tau) - 20\Theta(\tau)F_2(\tau)) \\ &= \frac{1}{120}(1-10q-70q^4-48q^5-120q^8-250q^9-240q^{12}-240q^{13}+O(q^{16}))\end{aligned}$$

of weight $\frac{5}{2}$. Choosing $\Phi = (\sqrt{7}/4)\pi\Omega_{-7}^2 = (\Gamma(1/7)\Gamma(2/7)\Gamma(4/7))^2/(64\pi^3)$ and noting that $\varepsilon = 8-3\sqrt{7}$ is the fundamental unit of $\mathbb{Q}(\sqrt{7})$, we find that

$$(1-w)^{-5/2}\mathscr{H}_{5/2}\Big(\frac{\mathfrak{z}_7-\bar{\mathfrak{z}}_7 w}{1-w}\Big) = \frac{\Theta(\mathfrak{z}_7)^5\varepsilon}{480}\sum_{n=0}^{\infty}\frac{b(n)}{n!}(\Phi w)^n,$$

with the first few of the numbers $d(n)$ being given by

$$\begin{aligned} b(n) =& \\ &-3\sqrt{7}+72,\ -60\sqrt{7}-265,\ 1105\sqrt{7}+1160,\ -6300\sqrt{7}-30705,\\ &130485\sqrt{7}+366600,\ -2715900\sqrt{7}-5323465,\ 38437065\sqrt{7}+146660040,\\ &-1220829660\sqrt{7}-2376737265,\ 24402981165\sqrt{7}+78627988680\ldots. \end{aligned}$$

The $b(n)$ are normalized so that they are integers in $\mathbb{Q}(\sqrt{7})$, which one can verify again by an induction analogous to the one employed in Example 4.2. Note that this is not possible if we normalize so that the leading coefficient is 1 and also that they cannot be expressed as integer multiples of powers of the fundamental unit. The factored norms of these numbers are given by

$$\begin{aligned} \mathrm{Nm}(b(n)) =& \\ &3^2\cdot 569,\ 5^2\cdot 1801,\ -3^3\cdot 5^2\cdot 47\cdot 227,\ 3^2\cdot 5^2\cdot 193\cdot 15313,\\ &3^3\cdot 5^2\cdot 22535131,\ -5^2\cdot 7\cdot 401\cdot 331934593,\ 3^4\cdot 5^2\cdot 5514721764001,\\ &-3^2\cdot 5^2\cdot 7\cdot 2797\cdot 1085992448669,\ 3^4\cdot 5^2\cdot 139\cdot 7154532998265547\ldots. \end{aligned}$$

Note that the factor 569 in the norm of $d(0)$ also occurs in the norm of the singular modulus

$$\alpha := \frac{\mathscr{H}_{5/2}(\mathfrak{z}_7)}{\Theta^5(\mathfrak{z}_7)} = \frac{1065-400\sqrt{7}}{800},$$

which equals $\mathrm{Nm}(\alpha) = 2^{-10}\cdot 5^{-2}\cdot 569$. For simplicity, we look at the norms of the numbers $b(n)$ modulo 11 and, computing the first 1000 of them, we find that $\mathrm{Nm}(b(n)) \equiv 3\,\mathrm{Nm}(b(n+110)) \pmod{11}$ for $n \geq 4$. Consistently, we find for the numbers $b(n)$ themselves that $b(n) \equiv 6b(n+110) \pmod{11}$ for $n \geq 4$.

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