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Given a closed Riemannian manifold and a pair of multicurves in it, we give a formula relating the linking number of the latter to the spectral theory of the Laplace operator acting on differential 1-forms. As an application, we compute the linking number of any two multigeodesics of the flat torus of dimension 3, generalising a result of P. Dehornoy.

1. Introduction

Let (M, g) be a closed Riemannian manifold of dimension 3. We call *a curve* an embedding of the oriented circle. A *multicurve* is a finite collection of disjoint curves. We say that a multicurve is *homologically trivial* if its homology class vanishes, as a cycle of M.

Given two homologically trivial multicurves Γ, Υ, one defines their *linking number* by taking any surface S_Γ whose boundary is Γ and algebraically intersecting it with Υ;

$$\mathrm{lk}(\Gamma, \Upsilon) := i(S_\Gamma, \Upsilon).$$

For example, see Figure 1.

It is not immediate that this number is well-defined, because of the choice involved about a surface S_Γ. As a general reference to the notion of linking number, one can recommend [Arnold and Khesin 1998, Section 4 of Chapter III; Bott and Tu 1982, Section 28]. Our main theorem relates the linking number with spectral theory.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold and Γ, Υ two disjoint homologically trivial multicurves, they link according to the following formula*

$$\mathrm{lk}(\Gamma, \Upsilon) = \lim_{t \to 0} \sum_{k \geq 0} e^{-\lambda_k t} \int_\Gamma \eta_k \int_\Upsilon * \left(\frac{d\eta_k}{\lambda_k} \right), \tag{1.2}$$

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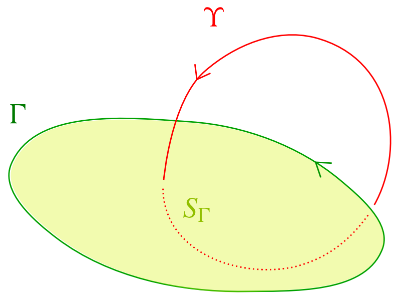

Figure 1. Here, both collections Γ and Υ consist of a single curve. Their linking number is ± 1, depending on the global orientation.

where $(\eta_k)_{k\in\mathbb{N}}$ *denotes an eigenvector basis with corresponding eigenvalues* $(\lambda_k)_{k\in\mathbb{N}}$ *of the Laplace operator* Δ *acting on the Hilbert space of square integrable differential* 1*-forms in* $\ker(\Delta)^\perp$.

Note that this theorem relates a topological number with metric quantities. In particular the right member of (1.2) does not depend on the underlying metric g.

Theorem 1.1 can be used if one has enough knowledge of the spectral theory of (M, g), as in the case of the canonical flat torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$. We prove a general formula for the linking number of multicurves consisting of geodesics of $\mathbb{T}^3$. However, not to burden this introduction, we postpone the statement to Section 3. Specialising our formula to the case of closed orbits of the geodesic flow on the 2-torus $\mathbb{T}^2$ gives the following corollary.

Corollary 1.3. *Let* $\Gamma = (\gamma^i)_{i\in I}$ *and* $\Upsilon = (\upsilon^j)_{j\in J}$ *be two homologically trivial multicurves in* $\mathbb{T}^3$ *consisting of periodic orbits of the* $\mathbb{T}^2$ *geodesic flow. They link according to the formula*

$$\mathrm{lk}(\Gamma, \Upsilon) = \sum_{i\in I,\, j\in J} \langle \gamma^i, \upsilon^j\rangle \frac{1-\theta_{i,j}/\pi}{2},$$

where $\theta_{i,j}$ *denotes the unique determination in* $[0, 2\pi[$ *of the oriented angle* θ *made at each intersection point* (see Figure 2) *and* $\langle\gamma^i, \upsilon^j\rangle$ *denotes the algebraic intersections between the projections on* $\mathbb{T}^2$ *of the curves* γ^i *and* υ^j.

Another formula was found by P. Dehornoy using different methods in his Ph.D. thesis [2011]. Our formula shows, in a clear way, that the linking number entertains some interactions with the intersection number on the curves projected on the basis.

We now briefly survey old and more recent results about the linking number. The first occurrence of the notion of linking number goes back to Gauss's studies on electromagnetism (see [Ricca and Nipoti 2011]). Gauss noticed that integrating the magnetic field generated by an electric power flowing in a close wire γ — for us a differential form ω_γ — along any closed curve υ gives a number which does

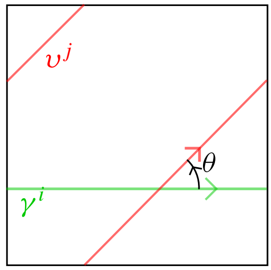

Figure 2. Here, the intersection number is 1. The angle θ defined in Corollary 1.3 is represented in black.

not depend on the homology class of υ in the complement of γ. That is to say, the differential form ω_γ is closed.

In fact, Gauss went further in his study: he gives in $\mathbb{R}^3$ an explicit formula expressing the differential form ω_γ. Let $x \notin \gamma$ and $X(x) \in T_x(\mathbb{R}^3)$, then:

$$(\omega_\gamma)_x(X) = \frac{1}{4\pi} \int_{[0,2\pi]} \det\left(\gamma'(s), X(x), \frac{\gamma(s) - x}{\|\gamma(s) - x\|^3}\right) ds, \tag{1.4}$$

where $\gamma(s)$ denotes any parametrisation compatible with the curve γ orientation, and $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^3$. Back in these days, there was no topological definition of the linking number so that, following Gauss, one could have defined it setting

$$\begin{aligned} \operatorname{lk}(\gamma, \upsilon) &:= \int_\upsilon \omega_\gamma \\ &= \frac{1}{4\pi} \int_{[0,2\pi]} \int_{[0,2\pi]} \det\left(\gamma'(s), \upsilon'(t), \frac{\gamma(s) - \upsilon(t)}{\|\gamma(s) - \upsilon(t)\|^3}\right) ds\, dt. \end{aligned} \tag{1.5}$$

Gauss's formula was related later with the definition of the linking number introducing this article, see for example [Arnold and Khesin 1998, Section 4 of Chapter 3]. It is still an active research field to try to get Gauss-like formulas for the linking number and its natural generalisations [DeTurck et al. 2008; DeTurck and Gluck 2008].

Formula (1.5) also suggests the existence of a universal object which, integrated over a pair of homologically trivial multicurves, gives back their linking number. A *linking form* Ω is an integrable $(1, 1)$-differential form, satisfying for any two homologically trivial disjoint multicurves Γ and Υ,

$$\operatorname{lk}(\Gamma, \Upsilon) = \int_\Gamma \int_\Upsilon \Omega.$$

One can think of a $(1, 1)$-differential form as a 2-differential form; we will get back on the $(1, 1)$-form precise definition in Section 2B.

The definition of linking form was introduced by Arnold (see [Arnold and Khesin 1998, Section 4 of Chapter III]) to generalise Moffatt's interpretation [1969] of the helicity. Let us recall briefly how the latter is defined.

Let X be a vector field preserving a probability measure μ in the Lebesgue class whose asymptotic cycle vanishes. This assumption implies that the 2-differential form $i_X\mu$ is exact, meaning that there is a differential 1-form α such that $d\alpha = i_X\mu$. One can show that

$$\mathcal{H}(X) := \int_M \alpha \wedge d\alpha$$

does not depend on the choice involving the primitive α. We call this number the helicity of the vector field X. This notion was introduced by [Moreau 1961; Woltjer 1958] to study certain energies associated to vector field solutions of some partial differential equations. Note that the asymptotic cycle assumption is automatically satisfied in some natural situations, for example when the ambient manifold is a homology sphere or if X is the Reeb flow associated to a contact structure.

Arnold interpreted the helicity of a vector field X as some average of the asymptotic linking number of two trajectories of the flow. Given $x, y \in M$, we consider the trajectories starting off x and y of the flow X at times t and s. We close them by gluing their extremities using a small path, we compute the linking number, we divide by the product ts and one would like to let $s, t \to \infty$. To do it, one needs to show this limit to be almost everywhere well-defined; this is one of the reasons why Arnold introduced the notion of linking form. Actually, he showed that the linking form is *integrable*, see Remark 4.14, which allows one to define the previous limit using Birkhoff's ergodic theorem. See [Arnold and Khesin 1998, Section 4 of Chapter III] for more details. This perspective on the helicity was developed in [Vogel 2003; Kotschick and Vogel 2003; DeTurck et al. 2013].

Arnold also noticed that linking forms always exists on compact manifolds. This was made more precise by T. Vogel [2003], relying on G. de Rham's work [1984, Section 28] on Hodge theory. We denote by $g^1(x, y)$ the kernel of the *Green operator*, the inverse operator of the Laplace one. We have

Theorem 1.6 [Vogel 2003, Theorem 3]. *Let (M, g) a compact Riemannian manifold. The $(1, 1)$-differential form*

$$\Omega(x, y) = *_y d_y g^1(x, y) \tag{1.7}$$

is an integrable linking form. We call this linking form the ***de Rham–Vogel linking form****.*

Vogel's proof relies on Arnold's remark that any inverse operator of d gives rise to a linking form (up to some duality). This theorem shows the existence of linking forms on closed manifolds, but does not come with a simple formula like

Gauss's one (1.5). There is, to the author's knowledge, only two others known formulas of this type, found in [DeTurck and Gluck 2013]. The first one holds for the hyperbolic 3-space and the second one for the round 3-sphere. The authors find such a formula in exhibiting a "fundamental solution of Maxwell's equations," meaning in exhibiting the de Rham–Vogel linking form defined above.

Outline of the article. In Section 2 we recall some basics on Laplace operators acting on differential forms and on their inverses, Green operators. We shall also recall that (1, 1)-forms are kernels of operators acting on differential 1-forms as well as how to integrate them along a pair of curves.

With this operator perspective in mind, Theorem 1.6 implies that the operator coming from the de Rham–Vogel linking form (the linking operator) commutes with the Laplace operator. The latter is well known to be diagonalisable on the space of square integrable differential 1-forms. Expanding the linking operator with respect to such a basis of 1-eigenforms will allow us to find a formula relating the spectral theory of differential 1-forms to the linking number.

However, the integration current over a closed curve is not square integrable, which prevents readily obtaining such a spectral-linking formula. To circumvent this difficulty, in order to reach some more regularity, we will use the heat operator to smooth the integration currents. This smoothing is responsible for the limit $t \to 0$ appearing in Theorem 1.1. This is the heart of Section 3 which finishes with the proof of Theorem 1.1.

In Section 4, using Theorem 1.1, we compute the linking numbers of collections of geodesics in the flat 3-torus $\mathbb{T}^3$ for which the spectral theory of differential 1-forms is well known. As an application, we prove Corollary 1.3.

2. Kernels of Green operators and linking forms

This section is devoted to introduce all the objects we will use later on.

2A. ***The Laplace and the Green operators.*** Let (M, g) be a closed manifold of dimension p. We denote by

- μ_g the volume form associated to the metric g;
- $\Omega^*(M) = \bigoplus_{0\le k\le p} \Omega^k(M)$ the space of all differential forms, graded with respect to the degree k;
- $*$ the Hodge operator, or Hodge star, which satisfies the following identity

$$** = (-1)^{k(p-k)}; \tag{2.1}$$

Note that we abuse the notation by omitting the degree k of the underlying differential form.

- d the exterior differential operator on $\Omega^*(M)$.

The Hodge star is defined in order to endow the vectorial space $\Omega^k(M)$ with a scalar product

$$\langle \alpha \cdot \beta \rangle = \int_M \alpha \wedge *\beta.$$

With respect to it, the operator d has a unique adjoint operator, denoted by δ, satisfying by definition

$$\langle d\alpha \cdot \beta \rangle = \langle \alpha \cdot \delta\beta \rangle.$$

A straightforward computation involving the Hodge star definition and the Stokes formula gives

$$\delta = (-1)^{p(k+1)+1} * d * . \tag{2.2}$$

We now have all the material required to define the Laplace operator.

Definition 2.3. The *Laplace operator* acting on $\Omega^*(M)$, denoted by Δ, is defined by

$$\Delta := d\delta + \delta d.$$

Note that the Laplace operator stabilises all differential forms spaces of fixed degree, and that it is self-adjoint with respect to the scalar product $\langle \cdot , \cdot \rangle$. We denote by Δ^k its restriction to $\Omega^k(M)$.

A differential form is said to be harmonic if it lies in the kernel of the Laplace operator, denoted by $\ker \Delta$. The space $\mathcal{H}_k$ of all harmonic k-forms being identified, by a famous theorem of Hodge, to the k-th homology group of M (see [Rosenberg 1997, page 46] for example), the Laplace operator Δ^k cannot be invertible in general. However, nothing prevents it from being invertible when restricted to the space orthogonal to its kernel. We denote by $\pi_{\mathcal{H}^k}$ the orthogonal projection on $\mathcal{H}_k = \ker(\Delta^k)$.

Definition 2.4. A *Green operator*, denoted by G^k, is any operator satisfying the following equation on the space of smooth differential forms of degree k:

$$G^k \circ \Delta^k = \Delta^k \circ G^k = \mathrm{Id} - \pi_{\mathcal{H}^k} \tag{2.5}$$

Such an operator always exists, provided that M is closed. There is a slight ambiguity about G^k, which is fully determined up to its restriction on the space $\ker(\Delta^k)$. From now on, we will suppose that $G(\ker(\Delta^k)) = \{0\}$, allowing one to speak of *the* Green operator. One can recommend [de Rham 1984, Chapter 3] for a general introduction to Green operators (and their kernels).

Green operators are *kernel operators*, meaning that there is a smooth family of endomorphisms — what we call a $(1, 1)$-form — $g^k(x, y) : \Lambda^k(T_x M) \to \Lambda^k(T_y M)$, indexed by $M \times M \setminus \mathrm{Diag}$ such that for all smooth differential forms α of degree k

$$G^k(\alpha)_y = \int_{x \in M} g^k(x, y)(\alpha_x)\, d\mu_g(x).$$

2B. ***Differential* (1, 1)*-forms.*** We give in this subsection the precise definition of a (1, 1)-form. We also explain how to integrate them over a pair of multicurves. Given an Euclidean space E, we denote by $\sharp$ the musical endomorphism which maps some vector $X \in E$ on its dual linear form, so in E^*, according to the Euclidean structure on E.

Definition 2.6. Let M be a manifold, We call (1, 1)-*form* a family of morphisms $T_x^*(M) \to T_y^*(M)$ indexed by $M \times M$.

Let γ and υ two curves parametrised by s and t. We define the integral over the pair of curves (γ, ν) of a (1, 1)-form Ω as

$$\int_\gamma \int_\upsilon \Omega := \int_\gamma \left(\int_0^1 \Omega(\upsilon(s), y)(\upsilon'(s)^\sharp)\, ds \right).$$

Moreover, the following integral — an element of $T_y^*(M)$ —

$$\int_0^1 \Omega(\upsilon(s), y)(\upsilon'(s)^\sharp)\, ds$$

does not depend on a choice of parametrisation, since $\Omega(x, y)$ is linear. So that we will prefer to denote it for short as

$$\int_\upsilon \Omega((\cdot)^\sharp, y),$$

omitting the underlying parametrisation.

This formula clearly shows that the linking form enjoys some bilinearity. In fact, if we denote by $\Upsilon = \bigcup_{i \in I} \upsilon^i$ and $\Gamma = \bigcup_{j \in J} \gamma^j$ we have

$$\mathrm{lk}(\Gamma, \Upsilon) = \int_{\Gamma \times \Upsilon} \Omega = \int_{\bigcup \gamma_i \times \bigcup \gamma_j} \Omega = \sum_{i \in I,\, j \in J} \int_{\gamma_i \times \gamma_j} \Omega.$$

Note that we did not require that either of the curves γ_i or υ_j be homologically trivial.

2C. ***The de Rham–Vogel linking form.*** Recall that the de Rham–Vogel linking form is defined as

$$*_y d_y g^1(x, y),$$

which may be slightly confusing at first. What does it mean to consider the image by $*_y d_y$ of a family of morphism from $T_x^* M$ to $T_y^* M$?

Given $\alpha \in T_x^*(M)$, the Green kernel defines a differential form by

$$\alpha_y := y \mapsto g(x, y)(\alpha(x)) \in T_y^*(M).$$

This differential form is smooth on $M \setminus \{x\}$, which allows one to take its image by the operator $*d$ wherever it makes sense. This gives rise to another linear morphism

$$T_x^*(M) \to T_y^*(M), \quad \alpha \mapsto \alpha_y,$$

which turns out to correspond to the kernel of the operator $\alpha \mapsto (*dG^1)(\alpha)$. Then, in the end, de Rham's notation $*_y d_y g^1(x, y)$ is to be understood as the kernel of the operator $\alpha \mapsto (*dG^1)(\alpha)$ that we call the *linking operator*.

Remark 2.7. (1) As pointed out by Arnold, any kernel associated to the inverse operator of the exterior differential d is a linking form. The operator $\alpha \mapsto (*dG)(\alpha)$ is actually one of them, up to Hodge duality. See [Vogel 2003, Lemma 2].

(2) The singularity of the $(1, 1)$-form $g^1(x, y)$ along the diagonal is roughly equivalent to r^{-1}. Thus, after one differentiation, this singularity turns to be in r^{-2}, which is still integrable in dimension three, see [de Rham 1984, Theorem 23 page 134]. So that what we meant by integrable is that for every x the function

$$y \mapsto \| *_y d_y g(x, y)\|$$

is integrable on M with respect to μ_g. The notation $\| \cdot \|$ stands for the linear morphism norm induced by the metric g.

2D. ***Behaviour of the linking form under isometries.*** The de Rham–Vogel linking form being constructed from a metric, it is natural to wonder how it behaves under an isometry Φ. The isometry Φ commutes with the Hodge star as well as with the exterior differential d. In particular, it commutes with every operators made out of this two ones, as the Laplace operator and its inverse, the Green operator. Looking at the kernel of the latter, this commutation relation can be read as

$$(\Phi_1)_* g^1(x, y) = (\Phi_2)^* g^1(x, y),$$

where Φ_1 and Φ_2 denote the Φ-action on the first and second factor, respectively, of the product $M \times M$. In particular, the diagonal action of Φ on the product $M \times M \setminus \mathrm{Diag}$ preserves the Green kernel, and thus the de Rham–Vogel linking form. Since we will use this remark to simplify a bit the computations performed in Section 4, we present it as:

Proposition 2.8. *Let γ and υ two curves* (not necessary homologically trivial) *and Φ an isometry of (M, g), then*

$$\int_\gamma \int_\upsilon *_y d_y g^1 = \int_{\Phi^{-1}(\gamma)} \int_{\Phi^{-1}(\upsilon)} *_y d_y g^1.$$

3. The spectral-linking formula

Let us recall that Δ is self-adjoint with respect to $\langle\,\cdot\,,\cdot\,\rangle$. It is well known that self-adjoint operators are diagonalisable in finite dimension; it is actually still the case for the Laplace operator, provided that the underlying manifold is closed.

Theorem 3.1 [Rosenberg 1997, Theorem 1.30]. *Let (M, g) be a closed Riemannian manifold. There is a orthonormal basis $(\eta_n)_{n\in\mathbb{N}}$ of differential 1-forms, meaning that $\langle\eta_i, \eta_j\rangle = \delta_i(j)$, and a sequence of nonnegative numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that*

$$\Delta\eta_n = \lambda_n\eta_n.$$

In particular, if $\alpha \in \ker(\Delta)^\perp$ we have

$$\alpha = \sum_{n\in\mathbb{N}} \langle\eta_n \cdot \alpha\rangle\eta_n.$$

Formally, one would like to write the Green operator as

$$g^1(x, y) := \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n}\eta_n(x) \otimes \eta_n(y),$$

which gives the following expression for the de Rham–Vogel linking form:

$$*_y d_y g^1(x, y) := \sum_{n\in\mathbb{N}} \eta_n(x) \otimes *d\left(\frac{\eta_n(y)}{\lambda_n}\right).$$

Keeping it formal, one would like then to integrate each factor along γ and υ to get

$$\operatorname{lk}(\gamma, \upsilon) = \sum_{n\in\mathbb{N}} \int_\gamma \eta_n \int_\upsilon *\left(\frac{d\eta_n}{\lambda_n}\right).$$

However, the previous series does not converge a priori. In fact, an integration current over a curve is not square integrable and therefore cannot be decomposed with respect to the orthonormal basis (η_n). To circumvent this difficulty, we will regularise them thanks to the use of the heat kernel, from which the term $e^{-\lambda_n t}$ of formula (1.2) comes from. As a corollary of this approach, we are able to prove the following stronger version of Theorem 1.1.

Theorem 3.2. *Let (M, g) be a closed Riemannian manifold and Ω the de Rham–Vogel linking form, then for all pairs of curves γ and υ* (not necessary homologically trivial) *we have*

$$\int_\gamma \int_\upsilon \Omega = \lim_{t\to 0} \sum_{k>0} e^{-\lambda_k t} \int_\gamma \eta_k \int_\upsilon *\left(\frac{d\eta_k}{\lambda_k}\right), \tag{3.3}$$

where $(\eta_k)_{k\in\mathbb{N}}$ *denotes an eigenvector basis of the Laplace operator* Δ *acting on the Hilbert space* $\ker(\Delta)^{\perp}$ *— viewed as a subspace of square integrable differential forms — and* (λ_k) *the associated eigenvalues.*

All the rest of this section is dedicated to the proof of the above theorem.

3A. ***The heat operator on 1-differential forms.*** The following definition is the key to regularise the integration currents. More details about generalised heat kernels can be found in [Berline et al. 1992, Section 2.3].

Definition 3.4. Let (M, g) be a closed Riemannian manifold and η be a continuous, bounded differential 1-form. The following Cauchy problem of unknown $(\eta_t)_{t\in\mathbb{R}_+}$:

$$\begin{cases} \Delta\eta_t + \partial_t\eta_t = 0, \\ \eta_0 = \eta, \end{cases}$$

has a unique solution. We denote by $e^{-t\Delta^1}$ the *heat operator* which maps η to the time t solution of the above Cauchy problem.We denote by p_t^1 the *heat kernel*, which satisfies by definition

$$(\eta_t)_y = \int_M p_t^1(x, y)(\eta_x)\, d\mu_g(x).$$

Moreover, one has

$$e^{-t\Delta^1}(\eta) \xrightarrow{t\to 0} \eta$$

for the uniform convergence topology.

In particular, if U and V are two closed disjoin subsets of M, one has

$$p_t^1(x, y) \xrightarrow{t\to 0} 0$$

uniformly on $U \times V$.

The heat kernel has the interesting property of being smooth for all $t > 0$, as opposed to the Green operator. In particular, it can be decomposed according to an orthonormal basis of eigenforms.

3B. ***The diffused curves.*** Let γ be a curve of M. We denote by $L^1(\Omega^1(M))$ the space of integrable differential 1-form, meaning forms whose punctual norm is integrable over M with respect to the Riemannian measure.

Definition 3.5. We call the γ*-diffused curve*, denoted by $e^{-t\Delta^1}(\gamma)$, the following family of linear forms indexed by $t > 0$:

$$e^{-t\Delta^1}(\gamma) : L^1(\Omega^1(M)) \to \mathbb{R}, \quad \beta \mapsto \int_\gamma e^{-t\Delta^1}(\beta).$$

This diffusing process associates to each $t > 0$ a differential form approximating the integration current over the curve γ: the smaller t, the better the approximation.

Lemma 3.6. *For all $\beta \in L^1(\Omega^1(M))$ continuous on a neighbourhood of U of the curve γ, we have*

$$e^{-t\Delta^1}(\gamma)(\beta) \xrightarrow{t\to 0} \int_\gamma \beta.$$

Proof. We have been careful to consider a differential form β integrable. So, since the heat kernel converges uniformly to 0 away from the diagonal, we have

$$\left| e^{-t\Delta^1}(\gamma)(\beta) - \int_\gamma \int_U p_t^1(x, y)(\beta_x)\, d\mu_g(x) \right| \xrightarrow{t\to 0} 0.$$

The differential form β being continuous on U, from the very definition on the heat kernel we have

$$\int_U p_t^1(x, y)(\beta_x)\, d\mu_g(x) \xrightarrow{t\to 0} \beta_y$$

uniformly. Therefore, one is allowed to permute limit and integral to get

$$\int_\gamma \int_U p_t^1(x, y)(\beta_x)\, d\mu_g(x) \xrightarrow{t\to 0} \int_\gamma \beta,$$

which is the expected result. □

If now υ is a curve disjoint to γ, recall that the differential 1-form

$$(\omega_\upsilon)_y := \int_\upsilon \Omega((\,\cdot\,)^\sharp, y)$$

is integrable, where $\Omega = *_1 d_1 g_1$ is the de Rham–Vogel linking form. Applying Lemma 3.6 readily gives:

Corollary 3.7. *For any two curves γ and υ we have*

$$e^{-t\Delta^1}(\gamma)(\omega_\upsilon) \xrightarrow{t\to 0} \int_\gamma \int_\upsilon \Omega.$$

The goal is now to identify, $t > 0$ being fixed, the left member of the above equation to the series appearing in (3.3). We will conclude by using the above corollary to recover Theorem 3.2 by letting $t \to 0$.

3C. ***The approximating series.*** The benefits of having diffused the integration current is to allow one to write the left member of (3.3) as a scalar product of two smooth differential 1-forms. We will conclude by using Plancherel's formula, allowing one to write down this scalar product with respect to an orthonormal basis.

Lemma 3.8. *For all differential forms $\beta \in L^1(\Omega^1(M))$ and all $t > 0$ we have*

$$e^{-t\Delta^1}(\gamma)(\beta) = \left\langle \beta \cdot \int_\gamma p_t^1((\,\cdot\,)^\sharp, y) \right\rangle.$$

Note the scalar product is well-defined, since the differential form $\int_\gamma p_t^1((\,\cdot\,)^\sharp, y)$ is smooth.

Proof. The operator $e^{-t\Delta}$ being self-adjoint and since $i_X(\alpha)(x) = g_x(X^\sharp \cdot \alpha)$, we have the following identity for any differential 1-form β and any vector field X:

$$i_X(y)(p_t(x, y)\beta_x) = g\big(\beta_y \cdot (p_t(x, y)(X_x^\sharp))\big).$$

Therefore, setting $X_x = \gamma'(s)$ and integrating along γ, one gets

$$\int_\gamma p_t^1(x, \,\cdot\,)(\beta_x) = g_y\bigg(\beta_y, \int_\gamma p_t^1((\,\cdot\,)^\sharp, y)\bigg),$$

which gives, after integration over M with respect to μ_g,

$$\int_M \int_\gamma p_t^1(x, y)(\beta_x)\, d\mu_g(y) = \bigg\langle \beta \cdot \int_\gamma p_t^1((\,\cdot\,)^\sharp, y)\bigg\rangle.$$

We conclude recalling that the form β is integrable, which allows one to switch both integrals of the above equation left member, recovering our definition of a diffused curve. □

We conclude the proof of Theorem 3.2 as announced by identifying the right member of (3.3) with some series.

Lemma 3.9. *For all $t > 0$ we have*

$$e^{-t\Delta^1}(\gamma)(\omega_\upsilon) = \sum_{k>0} e^{-\lambda_k t} \int_\gamma \eta_k \int_\upsilon *\bigg(\frac{d\eta_k}{\lambda_k}\bigg).$$

Proof. We start by using the semigroup property of the heat operator $e^{-t\Delta}$,

$$e^{-t\Delta^1}(\gamma)(\omega_\upsilon) = e^{-t\Delta^1/2}(\gamma)(e^{-t\Delta^1/2}(\omega_\upsilon)),$$

for which we apply Lemma 3.8 to get

$$e^{-t\Delta^1/2}(\gamma)(e^{-t\Delta^1/2}(\omega_\upsilon)) = \bigg\langle \int_\gamma p_{t/2}^1((\,\cdot\,)^\sharp, y) \cdot (e^{-t\Delta^1/2}(\omega_\upsilon))\bigg\rangle.$$

Both differential 1-forms appearing in the above equation being smooth, one is able to write down this scalar product with respect to an orthonormal basis consisting of the Laplace operator eigenforms:

$$e^{-t\Delta^1}(\gamma)(\omega_\upsilon) = \sum_{k\in\mathbb{N}} \bigg\langle \bigg[\int_\gamma p_{t/2}^1((\,\cdot\,)^\sharp, y)\bigg] \cdot \eta_k \bigg\rangle \big\langle e^{-t\Delta^1/2}(\omega_\upsilon) \cdot \eta_k \big\rangle.$$

It remains then to prove both the two following identities:

$$e^{-\lambda_k t/2}\int_\gamma \eta_k = \left\langle \left[\int_\gamma p^1_{t/2}((\,\cdot\,)^\sharp, y)\right]\cdot \eta_k\right\rangle, \tag{3.10}$$

$$\frac{e^{-\lambda_k t/2}}{\lambda_k}\int_\upsilon *d\eta_k = \left\langle e^{-t\Delta^1/2}(\omega_\upsilon)\cdot\eta_k\right\rangle \tag{3.11}$$

We start off with the right member of (3.10). Recalling Lemma 3.8 the other way around, one gets

$$\left\langle \left[\int_\gamma p^1_{t/2}((\,\cdot\,)^\sharp, y)\right]\cdot \eta_k\right\rangle = \int_\gamma e^{-t\Delta^1/2}(\eta_k).$$

Then, η_k being an eigenform of eigenvalue λ_k, we have

$$e^{-t\Delta^1/2}(\eta_k) = e^{-t\lambda_k/2}\int_\gamma \eta_k,$$

which proves that (3.10) holds.

Let us show in the same way that (3.11) occurs as well. We start again from the right member:

$$\left\langle e^{-t\Delta^1/2}(\omega_\upsilon)\cdot\eta_k\right\rangle.$$

The differential 1-form ω_υ being integrable and the operator $e^{-t\Delta/2}$ being self-adjoint we have:

$$\left\langle e^{-t\Delta^1/2}(\omega_\upsilon)\cdot\eta_k\right\rangle = \left\langle \omega_\upsilon\cdot e^{-t\Delta^1/2}(\eta_k)\right\rangle.$$

Therefore, η_k being an eigenform of eigenvalue λ_k, we have

$$e^{-t\Delta^1/2}(\eta_k) = e^{-t\lambda_k/2}\eta_k,$$

and thus

$$\left\langle e^{-t\Delta^1/2}(\omega_\upsilon)\cdot\eta_k\right\rangle = e^{-\lambda_k t/2}\langle \omega_\upsilon\cdot\eta_k\rangle.$$

The linking form Ω being integrable, one can use Fubini's theorem again to get

$$\langle \omega_\upsilon\cdot\eta_k\rangle = \int_\upsilon\left[\int_M \Omega((\,\cdot\,)^\sharp, y)(\eta_k)_x\, d\mu_g(x)\right].$$

But, by construction of the linking form as the operator $*dG$ kernel, we have

$$\int_M \Omega(x, y)(\eta_k)_x\, d\mu_g(y) = ((*dG)(\eta_k))_y.$$

The operator G commutes with d, in particular all terms of this series corresponding to a closed differential form vanish. The remaining terms are given by

$$*dG(\eta_k) = \frac{*d\eta_k}{\lambda_k}.$$

Finally we have

$$\langle \omega_\upsilon \cdot \eta_k \rangle = \int_\upsilon * \left(\frac{d\eta_k}{\lambda_k} \right),$$

which gives

$$\left\langle e^{-t\Delta^1/2}(\omega_\upsilon) \cdot \eta_k \right\rangle = e^{-t\lambda_k/2} \int_\upsilon * \left(\frac{d\eta_k}{\lambda_k} \right), \tag{3.12}$$

the expected result. □

4. Application to torus geodesics linking

This section is devoted to the use of Theorem 3.2 to compute the linking number of homologically trivial multigeodesics of the canonical 3-tore torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$. The spectral theory of $\mathbb{T}^3$ is fully understood: we will describe it in Section 4B. One would like to use it to give a more or less explicit expression to the following series, $t > 0$ being fixed:

$$\sum_{k>0} e^{-\lambda_k t} \int_\gamma \eta_k \int_\upsilon * \left(\frac{d\eta_k}{\lambda_k} \right).$$

Right after, we will identify the limit of this series when $t \to 0$ to the Fourier development of some function. In the meantime, we will recall Theorem 3.2, which guarantees that this sequence of series actually converges to the linking number.

4A. ***Statement of the generalised torus linking theorem.*** We call a *multigeodesic* a multicurve consisting of geodesics. This subsection goal is to state a formula giving the linking number of any two collections of multigeodesics of $\mathbb{T}^3$.

Let us fix some notation. Given a closed geodesic γ of $\mathbb{T}^3$, we parametrise it as

$$\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{T}^3, \quad t \mapsto \begin{pmatrix} \gamma_1 t + \nu_1 \\ \gamma_2 t + \nu_2 \\ \gamma_3 t + \nu_3 \end{pmatrix} \quad \bmod \mathbb{Z}^3,$$

where $\gamma' = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3$ is the *parametrised slope* of γ. With the above notation $\gamma(0) = \gamma(1)$. Note that these curves are automatically oriented by the parametrisation. Note also that we did not require that the geodesic γ is primitive.

We call the point $\nu = \gamma(0) = (\nu_1, \nu_2, \nu_3) \in \mathbb{T}^3$ the *origin* of γ. Such a choice is not canonical since any point $\nu \in \mathrm{Im}(\gamma)$ can also define the origin. See Figure 3.

The parametrisation proposed above is not the arc-length one, but has the benefit to be very closely related to the homology class that γ defines. The 3-torus fundamental group being Abelian, one can check that the vector $\gamma' \in \mathbb{Z}^3$ is canonically identified to the homology class of the closed curve γ in $\mathbb{Z}^3$. We denote by $[\gamma]$ the vector γ' to emphasis its topological flavour. From that remark comes the

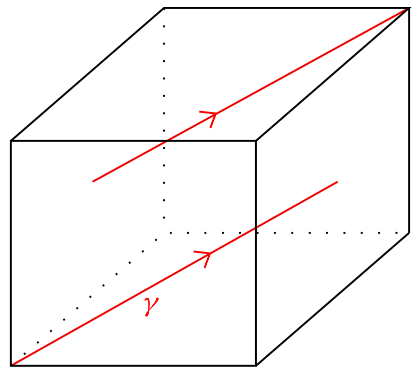

Figure 3. The cube is identified to $\mathbb{T}^3$ by gluing opposed faces with translations. The red curve γ has parametrised slope $\gamma' = (2, 1, 1)$. The origin of γ can be taken anywhere on the image of γ. For example, one can choose $\nu = 0_{\mathbb{T}^3}$ here.

following necessary and sufficient condition for a multigeodesics $\Gamma = (\gamma^i)_{i \in I}$ to be homologically trivial:

$$\sum_{i \in I} [\gamma^i] = 0_{\mathbb{R}^3}.$$

The following construction is needed to state our theorem.

Given two vectors $u, v \in \mathbb{Z}^3$, we define the vector $\beta^{u,v} \in \mathbb{Z}^3$ as the unique one verifying the following conditions:

- $\beta^{u,v} \in \mathrm{vect}(u, v)^{\perp}$;
- $\det(u, v, \beta^{u,v}) > 0$;
- its Euclidean norm $\|\beta^{u,v}\|$ is minimal for the two first properties.

Given two geodesics γ and ν, we still simply denote by $\beta^{\gamma,\nu}$ the vector $\beta^{[\gamma],[\nu]}$. Our torus linking theorem can then be stated as follows.

Theorem 4.1. *Let $\Gamma = (\gamma^i)_{i \in I}$ and $\Upsilon = (\upsilon^j)_{j \in J}$ two homologically trivial multigeodesics of $\mathbb{T}^3$. They link according to the following formula*:

$$\mathrm{lk}(\Gamma, \Upsilon) = \sum_{i \in I, j \in J} \det\left([\gamma^i], [\upsilon^j], \frac{\beta^{i,j}}{\|\beta^{i,j}\|}\right) \frac{1 - 2\lfloor(\nu^{i,j} \cdot \beta^{i,j})\rfloor}{2\|\beta^{i,j}\|}, \tag{4.2}$$

where $\nu^{i,j} = \gamma^i(0) - \upsilon^j(0)$ is the difference between the two origins and $\lfloor \alpha \rfloor$ denotes the unique representative in $[0, 1[$ of the class $(\alpha \mod \mathbb{Z})$.

Remark 4.3. (1) One can define the linking number in every dimension n, provided that we consider two homologically disjoint submanifolds of dimension p and q

satisfying $p+q=n-1$. Our method is likely to be generalised for a flat torus in any dimension.

(2) A priori, (4.2) depends on a choice of parametrisation. We will clarify this point later on with Remark 4.14.

4B. ***Spectral theory of differential 1-forms of*** $\mathbb{T}^3$**.** We start by introducing some notation.

- We denote by a lower index i the i-th coordinate of a vector and by an upper index its belonging to a family of vectors. For example, γ_i^j denotes the i-th coordinate of the j-th vector of a family indexed by $j \in J$.
- Given a vector
$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3,$$
we denote by v^* the differential form $v_1\,dx_1 + v_2\,dx_2 + v_3\,dx_3$. This one being invariant by translations, it defines a harmonic differential form on $\mathbb{T}^3$. *We continue to denote by v^* the induced-on-$\mathbb{T}^3$ differential form.*
- The scalar product of two vectors a and b in $\mathbb{R}^3$ is denoted by $(a \cdot b)$ and the associated Euclidean norm by $\|\cdot\|$.
- The $\mathbb{R}^3$ vectorial product is denoted by $\wedge$.

Let us describe the differential 1-forms spectral theory of $\mathbb{T}^3$ thanks to the following set of datum:

- a vector
$$\boldsymbol{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \in \mathbb{Z}^3,$$
- an orthonormal basis (v^1, v^2, v^3) of $\mathbb{R}^3$.
- a function $f \in \{\cos, \sin\}$

Note that here we have the choice of an orthonormal basis of $\mathbb{R}^3$.

Lemma 4.4. *The differential* 1*-form of* $\mathbb{T}^3$

$$\eta(x) = \sqrt{2} f(2\pi(\boldsymbol{k} \cdot x))(v^i)^* \tag{4.5}$$

is an eigenform of Δ^1 *with associated eigenvalue* $\lambda = (2\pi\|\boldsymbol{k}\|)^2$.

Proof. We start by showing that these forms are of unit norm:

$$\begin{aligned}\|\eta\|_{L^2} &:= \int_{\mathbb{T}^3} \eta \wedge *\eta \\ &= \int_{\mathbb{T}^3} 2f^2(2\pi(\boldsymbol{k}\cdot x))(v^i)^* \wedge *(v^i)^* \\ &= \int_{\mathbb{T}^3} 2f^2(2\pi(\boldsymbol{k}\cdot x))\, d\mathrm{vol} = 1,\end{aligned}$$

since $f^2 = (1 \pm f(2\,\cdot))/2$.

Recall the Laplace operator definition

$$\Delta\eta = (d\delta + \delta d)\eta.$$

Because $\delta = -*d*$ in dimension 3, one gets

$$d\delta\eta = d(-*d*)\eta = -\sqrt{2}(d*d)\big(f(2\pi(\boldsymbol{k}\cdot x)) * (v^i)^*\big). \tag{4.6}$$

By the Hodge star definition we have

$$*(v^i)^* = (v^j)^* \wedge (v^t)^*,$$

where (i, j, t) is a circular permutation of $(1, 2, 3)$, so that

$$d\delta\eta = -\sqrt{2}d*d\big(f(2\pi(\boldsymbol{k}\cdot x)) \wedge (v^j)^* \wedge (v^t)^*\big). \tag{4.7}$$

And then,

$$\begin{aligned}d\delta\eta &= -\sqrt{2}(d*)\big(2\pi k_i f'(2\pi(\boldsymbol{k}\cdot x))\big((v^i)^* \wedge (v^j)^* \wedge (v^t)^*\big)\big) \\ &= -2\sqrt{2}\pi k_i\, d\big(f'(2\pi(\boldsymbol{k}\cdot x))\big) \\ &= -2\sqrt{2}\pi k_i\, df'(2\pi(\boldsymbol{k}\cdot x)) \\ &= -4\sqrt{2}\pi^2\Big(k_i^2 f''(2\pi(\boldsymbol{k}\cdot x))\, dx_i + k_i k_j f''(2\pi(\boldsymbol{k}\cdot x))\, dx_j \\ &\qquad\qquad + k_i k_t f''(2\pi(\boldsymbol{k}\cdot x))\, dx_t\Big).\end{aligned}$$

We compute $\delta d\eta$ in a similar way to get

$$\begin{aligned}\delta d\eta = -4\sqrt{2}\pi^2&\big(k_j^2 f''(2\pi(\boldsymbol{k}\cdot x))\, dx_i + k_t^2 f''(2\pi(\boldsymbol{k}\cdot x))\big) \\ &+ 4\sqrt{2}\pi^2\big(k_i k_j f''(2\pi(\boldsymbol{k}\cdot x))\, dx_j - k_i k_t f''(2\pi(\boldsymbol{k}\cdot x))\, dx_t\big),\end{aligned}$$

Summing both terms gives

$$\Delta\eta = -4\sqrt{2}\pi^2\big(k_i^2 f''(2\pi(\boldsymbol{k}\cdot x))\, dx_i + k_j^2 f''(2\pi(\boldsymbol{k}\cdot x))\, dx_i + k_t^2 f''(2\pi(\boldsymbol{k}\cdot x))\big)\, dx_i.$$

Since $f'' = -f$ one has

$$\Delta\eta = 4\pi^2(k_1^2 + k_2^2 + k_3^2)\eta,$$

the expected outcome. □

To use Theorem 3.2 we need a basis of eigenforms. Fixing an orthonormal basis of $\mathbb{R}^3$, the family issued from all $\boldsymbol{k} \in \mathbb{Z}^3$ and both the function cos and sin forms a generating family. To see it, one can decomposes in Fourier series the coefficients of a differential 1-form ω written as

$$\omega(x, y, z) = f_1(x, y, z) \cdot (v^1)^* + f_2(x, y, z) \cdot (v^2)^* + f_3(x, y, z) \cdot (v^3)^*.$$

Moreover, this family is free up to the trivial relations $\cos(-\boldsymbol{k} \cdot x) = \cos(\boldsymbol{k} \cdot x)$ and $\sin(-\boldsymbol{k} \sin x) = -\sin(\boldsymbol{k} \cdot x)$.

4C. ***Computation of the approximating series.*** Recall that we parametrised both geodesics γ and υ as

$$\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{T}^3, \quad t \mapsto \begin{pmatrix} \gamma_1 t + \nu_1 \\ \gamma_2 t + \nu_2 \\ \gamma_3 t + \nu_3 \end{pmatrix}, \qquad \upsilon : \mathbb{R}/\mathbb{Z} \to \mathbb{T}^3, \quad t \mapsto \begin{pmatrix} \upsilon_1 t + \mu_1 \\ \upsilon_2 t + \mu_2 \\ \upsilon_3 t + \mu_3 \end{pmatrix},$$

where $\gamma_i, \upsilon_j \in \mathbb{Z}$ and $\mu_j, \nu_j \in [0, 1]$.

First, note that we can assume $\nu = 0$. In fact, since the translation of $\mathbb{R}^3$

$$\tau_\nu := x \to x + \nu,$$

descends to an isometry of $\mathbb{T}^3$, using Proposition 2.8 one has

$$\int_\gamma \int_\upsilon \Omega = \int_{\tau^{-1}(\gamma)} \int_{\tau^{-1}(\upsilon)} \Omega,$$

where now $(0, 0, 0)^\top$ belongs to $\tau^{-1}(\gamma)$. *In order not to burden the notation we will still denote by μ the new origin* (keeping in mind that it actually corresponds to $\mu(\gamma, \upsilon) = \mu - \nu$) of the translated curve υ.

We saw that, given an orthonormal basis of $\mathbb{R}^3$, one can build an orthonormal eigenforms basis of the Laplace operator. To simplify the computation we will perform in (3.3) we make a choice of this orthonormal basis adapted to the curve γ: the first vector is chosen to be $v^1 = [\gamma]/\|[\gamma]\|$, and we arbitrarily complete it to get an orthonormal basis:

$$\left(\left(v^1 = \frac{[\gamma]}{\|[\gamma]\|} \right)^*, v^2, v^3 \right).$$

Recall that we want to compute the following series;

$$\sum_{k>0} e^{-\lambda_k s} \int_\gamma \eta_k \int_\upsilon * \left(\frac{d\eta_k}{\lambda_k} \right). \tag{4.8}$$

We will compute all terms involved in this series separately and we will sum them in the next subsection. These terms are the product of two integrals that we compute independently.

We start with the integral involving the operator $*d$. Let η be an eigenform. One has

$$\int_\gamma \eta = \int_{[0,1]} \sqrt{2} f(2\pi \boldsymbol{k} \cdot \gamma(t))(v^i)^*([\gamma])\, dt, \tag{4.9}$$

where $f \in \{\cos, \sin\}$ and $\boldsymbol{k}$ is a vector of $\mathbb{Z}^3$. The above integral vanishes whenever f is a sinus since the curve γ passes by 0;

$$\begin{aligned}\int_\gamma \eta &= \int_{[0,1]} \sqrt{2}\sin(2\pi \boldsymbol{k} \cdot \gamma(t))(v^i)^*([\gamma])\, dt \\ &= C_1 \int_{[0,1]} \sin(2\pi C_2 t)\, dt = 0\end{aligned}$$

because $C_2 \in \mathbb{Z}$. *We can assume then that f is a cosine.* We keep computing in considering the eigenforms

$$\eta_{\boldsymbol{k},i} = \sqrt{2}\cos(2\pi(\boldsymbol{k} \cdot x))(v^i)^*$$

only, where $\boldsymbol{k} \in \mathbb{Z}^3$ and $i \in \{1, 2, 3\}$. Which, looking backward to (4.9), gives

$$\int_\gamma \eta_{\boldsymbol{k},i} = \int_{t=0}^{1} \cos(2\pi t(\boldsymbol{k} \cdot [\gamma]))(v^i)^*([\gamma])\, dt,$$

where $(v^i)^*([\gamma]) = ([\gamma] \cdot v^i) = \|[\gamma]\|\delta_{i,1}$.

The above integral therefore vanishes whenever

- $(\boldsymbol{k} \cdot [\gamma]) \neq 0$;
- $i \neq 1$.

Moreover, in the case where it does not, the function $t \mapsto \cos(2\pi(\boldsymbol{k} \cdot [\gamma])t)$ is constant, so that

$$\int_\gamma \eta_{\boldsymbol{k},i} = \sqrt{2}\|[\gamma]\|. \tag{4.10}$$

Differential forms giving a nonvanishing term of the series (4.8) are therefore

$$\eta_{\boldsymbol{k},1} = \sqrt{2}\cos(2\pi \boldsymbol{k} \cdot x)\left(\frac{[\gamma]}{\|[\gamma]\|}\right)^*,$$

with $\boldsymbol{k} \in \mathbb{Z}^3$ and $\boldsymbol{k} \cdot [\gamma] = 0$.

From now on, we will denote $\eta_{\boldsymbol{k},1}$ by $\eta_{\boldsymbol{k}}$. We now compute the second term of the series:

$$\int_v *d\eta_{\boldsymbol{k}},$$

starting off computing

$$\begin{aligned}*d\eta_{\boldsymbol{k}} &= *d\big(\sqrt{2}\cos(2\pi(x\cdot\boldsymbol{k}))(v^1)^*\big)\\ &= -2\sqrt{2}\pi\,\sin(2\pi(x\cdot\boldsymbol{k}))*\big(k_1\,dx_1\wedge(v^1)^*+k_2\,dx_2\wedge(v^1)^*+k_3\,dx_3\wedge(v^1)^*\big)\\ &= -2\sqrt{2}\pi\,\sin(2\pi(x\cdot\boldsymbol{k}))(\boldsymbol{k}\wedge v^1)^*.\end{aligned}$$

We then get

$$\begin{aligned}\int_\upsilon *d\eta_{\boldsymbol{k}} &= \int_{t=0}^1 -2\sqrt{2}\pi\,\sin\big(2\pi t([\upsilon]\cdot\boldsymbol{k})+2\pi(\mu\cdot\boldsymbol{k})\big)(\boldsymbol{k}\wedge v^1)^*([\upsilon])\,dt\\ &= -2\sqrt{2}\pi\,\det\left(\frac{[\gamma]}{\|[\gamma]\|},[\upsilon],\boldsymbol{k}\right)\int_0^1\sin\big(2\pi t([\upsilon]\cdot\boldsymbol{k})+2\pi(\mu\cdot\boldsymbol{k})\big)\,dt.\end{aligned}$$

As before, this integral vanishes if one of these conditions holds:

- the vectors $[\gamma]$ and $[\upsilon]$ are collinear;
- $(\boldsymbol{k}\cdot[\upsilon])\neq 0$.

Moreover if $\int_\upsilon *d\eta_{\boldsymbol{k}}\neq 0$, we have

$$\int_\upsilon *d\eta_{\boldsymbol{k}} = -2\sqrt{2}\pi\,\det\left(\frac{[\gamma]}{\|[\gamma]\|},[\upsilon],\boldsymbol{k}\right)\sin(2\pi(\mu\cdot\boldsymbol{k})). \tag{4.11}$$

Multiplying (4.10) and (4.11) one has:

$$\int_\gamma \eta_{\boldsymbol{k}}\int_\upsilon *d\eta_{\boldsymbol{k}} = \begin{cases}4\pi\,\det([\gamma],[\upsilon],\boldsymbol{k})\sin(2\pi(\mu\cdot\boldsymbol{k})) & \text{if } \boldsymbol{k}\in\operatorname{Span}([\gamma],[\upsilon])^\perp,\\ 0 & \text{otherwise.}\end{cases}$$

This leads us to characterise elements of $\operatorname{Span}([\gamma],[\upsilon])^\perp$ with integer coefficients.

Lemma 4.12. *Let b_1 and b_2 two nonzero vectors of $\mathbb{Z}^3$. Then the group*

$$\operatorname{Span}(b_1,b_2)^\perp\cap\mathbb{Z}^3$$

is cyclic. We note by $\pm\beta$ one of these two possible generators.

Proof. As a set it is nonempty; the vector $b_1\wedge b_2$ belongs to $\mathbb{Z}^3$ and is orthogonal to both b_1 and b_2. As the intersection of two subgroups, $\mathbb{Z}^3$ and $\mathbb{R}\cdot b_1\wedge b_2$, it is a subgroup of $\mathbb{R}$. The neutral element of $\operatorname{Span}(b_1,b_2)^\perp\cap\mathbb{Z}^3$ must be isolated since $\mathbb{Z}^3$ is discrete. By characterisation of $\mathbb{R}$ subgroups, this group is cyclic. □

We apply the previous lemma to the pair $([\gamma],[\upsilon])$ to get the following description of elements $\boldsymbol{k}\in\mathbb{Z}^3$ giving a nonvanishing term in the series of (3.3):

$$\operatorname{Span}([\gamma],[\upsilon])^\perp\cap\mathbb{Z}^3 = \{k\beta,\ k\in\mathbb{Z}\}.$$

Among both possible generators, we choose β such that the family $([\gamma],[\upsilon],\beta)$ is positively oriented.

The only nonvanishing terms of the series appearing in (4.8) correspond to the differential forms

$$\eta_{(k\beta)} = \sqrt{2}\cos((k\beta)\cdot x)\left(\frac{[\gamma]}{\|\gamma\|}\right)^{*},$$

and in this case we have

$$\int_\gamma \eta_k \int_\upsilon *d\eta_k = -4\pi k \det([\gamma], [\upsilon], \beta)\sin(2\pi k(\mu\cdot\beta)).$$

From now on we denote by η_k the differential form $\eta_{(k\beta)}$. Recall that the differential forms η_k and η_{-k} are collinear. To get a free family of eigenforms one needs to choose the sign of the integers k: we take them nonnegative. The series of Equation (3.3) then becomes

$$-\sum_{k>0}\frac{e^{-(2\pi\|\beta\|)^2 n^2 s}}{\pi k\|\beta\|}\det\left([\gamma], [\upsilon], \frac{\beta}{\|\beta\|}\right)\sin(2\pi k(\mu\cdot\beta)). \tag{4.13}$$

Remark 4.14. As noticed in Remark 4.3, (4.13) is not independent of the parametrisations involved a priori. In fact, the point $\mu \in \mathbb{T}^3$ appearing in $\sin(2\pi k(\mu\cdot\beta))$ depends of an origin choice for υ. Let us thus check that $k(\mu\cdot\beta)$ actually doesn't, modulo $\mathbb{Z}$. Let $\mu_2 \in \upsilon$ be another origin of υ, by definition there is $t \in \mathbb{R}$ and $\alpha \in \mathbb{Z}^3$ such that

$$\mu_2 - \mu = t[\upsilon] + \alpha,$$

thus

$$(\mu_2\cdot\beta) = (\mu_2 - \mu + \mu\cdot\beta) = (\mu\cdot\beta) + (\alpha\cdot\beta),$$

since $\beta \in [\upsilon]^\perp$. We conclude reducing the above formula modulo $\mathbb{Z}$ to get

$$(\mu_2\cdot\beta) = (\mu\cdot\beta),$$

since $(\alpha\cdot\beta) \in \mathbb{Z}$.

4D. ***A uniformly converging family of functions.*** Let us now look into the series (4.13) more in detail. If one is able to let $t \to 0$ within all terms of this series one would get

$$-C\sum_{k>0}\frac{1}{k}\sin(2\pi kx),$$

with $C = (1/(\pi\|\beta\|))\det([\gamma], [\upsilon], \beta/\|\beta\|)$ and $x = (\mu\cdot\beta)$.

One can recognise here the Fourier series development of the defined-on-the-circle-$\mathbb{R}/\mathbb{Z}$ function

$$x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \frac{\pi}{2}(1-2x) & \text{on } (0, 1). \end{cases} \tag{4.15}$$

So that we would have

$$\int_\gamma \int_\upsilon \Omega = \frac{1}{2\|\beta\|} \det\left([\gamma], [\upsilon], \frac{\beta}{\|\beta\|}\right)(1 - 2(\mu \cdot \beta)), \tag{4.16}$$

which is precisely what is expected. To justify the term-by-term convergence of (4.13), we use the following lemma.

Lemma 4.17. *Let $a_k(t)$ and $b_k(t)$ two sequences of functions defined on an interval I containing* 0 *such that*

(1) $\left(\sum_{k\le n} a_k(t)\right)_{n\in\mathbb{N}}$ *is uniformly bounded with respect to t*;

(2) *the sequence of functions $b_k(t)$ is nonincreasing with respect to t and converges uniformly*, *with respect to k*, *to* 0.

Then the series of functions $\sum_{k\in\mathbb{N}} a_k(t)b_k(t)$ converges uniformly on I.

We omit the proof, which consists of a discrete integration by parts of the series.

We set $a_k(t) = \sin(2\pi kx)$, $b_k(t) = e^{-atk^2}/k$ and $I = [0, +\infty]$. One can then check that for all $a \in \mathbb{R}^+$ and $x > 0$, all assumptions of Lemma 4.17 hold. We deduce that the series of functions

$$\sum_{k>0} \frac{e^{-atk^2}}{k} \sin(2\pi kx)$$

converges uniformly on $]0, +\infty]$. One is therefore allowed to switch limits and sum in (4.13) to get

$$\begin{aligned}\lim_{t\to 0} \sum_{k>0} \frac{e^{-atk^2}}{k} \sin(2\pi kx) &= \sum_{k>0} \lim_{t\to 0} \frac{e^{-atk^2}}{k} \sin(2\pi kx) \\ &= \sum_{k>0} \frac{\sin(2\pi kx)}{k},\end{aligned}$$

which concludes the proof.

4E. ***The $\mathbb{T}^2$-geodesic flow special case.*** Particularly interesting collections of multi-geodesics of $\mathbb{T}^3$ arise as periodic orbits of the $\mathbb{T}^2$-geodesic flow. More generally, linking number of collections of periodic orbits have been studied by E. Ghys [2007] and Dehornoy [2017; 2011] for dynamical purposes. The latter showed that, for a large class of examples given by geodesic flows on surfaces, these collections all link positively, up to a choice of global orientation. This implies the existence of Birkhoff sections and, as a corollary, that periodic orbits of this flows display fibred knots. In the setting of $\mathbb{T}^2$, we will see that Theorem 4.1 specifies easily giving a new linking number formula.

We start by noticing that $\mathbb{T}^3$ is identified to the unitary tangent bundle $U\mathbb{T}^2$ of the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. In fact, the unitary tangent bundle of $\mathbb{T}^2$ is trivial, $\mathbb{T}^2$

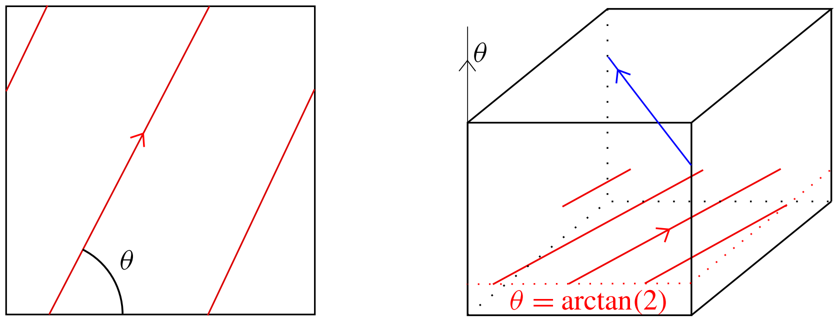

Figure 4. The red curves on the left represents a closed geodesic of $\mathbb{T}^2$. This curve lifts canonically to the red right one on the unitary tangent bundle. This lifted curve remains in the leaf $\theta = \arctan(2)$. The blue curve represents another lifted geodesic.

being a Lie group. One trivialisation consists to choose a direction of $\mathbb{R}^2$, which induces one on $\mathbb{T}^2$, from which one is able to assign an angle to any vector of $U\mathbb{T}^2$. That is to say the map

$$U\mathbb{T}^2 \to \mathbb{T}^3, \quad u \mapsto ((x, y), \theta)$$

is an actual trivialisation. With the unitary tangent bundle of a Riemannian manifold comes always a flow: the geodesic flow. In the case of $\mathbb{T}^2$, one can fully write down the flow in the trivialisation given above

$$\Phi_t : \mathbb{T}^3 \to \mathbb{T}^3, \quad (x, y, \theta) \mapsto (x + t \cos\theta, y + t \sin\theta, \theta).$$

Note that periodic orbits of a flow are naturally parametrised and oriented by the flow itself:

$$\gamma : \mathbb{S}^1 \to \mathbb{T}^3, \quad t \mapsto (x + t \cos\theta, y + t \sin\theta, \theta).$$

See Figure 4.

Remark 4.18. The fact that orbits of the geodesic flow are still geodesics on the unitary tangent bundle is more general, providing that one endows the latter with the right metric; the so called Sasaki metric. In our case, it turns out that the Sasaki metric coincides with the $\mathbb{T}^3$ flat one.

In this setting, one can readily specifies Theorem 3.2 to get:

Corollary 4.19 [Dehornoy 2011, page 11]. *Let $\Gamma = (\gamma^i)_{i\in I}$ and $\Upsilon = (\upsilon^j)_{j\in J}$ be two homologically trivial multigeodesics of $\mathbb{T}^2$. In the $\mathbb{T}^2$-unitary tangent bundle* *they link according to the following formula*

$$\mathrm{lk}(\Gamma, \Upsilon) = \sum_{i \in I,\, j \in J} \langle \gamma^i, \upsilon^j \rangle \frac{1 - \theta_{i,j}/\pi}{2},$$

where $\theta_{i,j}$ *denotes the unique determination in* $[0, 2\pi[$ *of the oriented angle* θ *made at any intersections points* (see Figure 2), *and* $\langle \gamma^i, \upsilon^j \rangle$ *denotes the algebraic intersections between* γ^i *and* υ^j *on* $\mathbb{T}^2$.

Proof. As previously noticed, the orbits of this flow remain in the leaves $\theta = \mathrm{cst}$, so that the vectors $[\gamma^i]$ and $[\upsilon^j]$ belong $\mathbb{R}^2 \subset \mathbb{R}^3$. Our vector $\beta^{i,j}$ defined in Theorem 4.1 becomes

$$\beta^{i,j} = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}$$

for all pairs (i, j), the sign depending whether or not the angle between the curves γ_i and υ_j is greater than π. In particular we have $\|\beta^{i,j}\| = 1$. Moreover, the determinant $\det([\gamma^i], [\upsilon^j], \beta^{i,j})$ becomes $\det_{\mathbb{R}^2}([\gamma_i], [\upsilon_j])$, which corresponds to the algebraic intersection number between γ^i and υ^j seen as curve of $\mathbb{T}^2$. To conclude, the quantity $(\beta^{i,j} \cdot \mu^{i,j})$ turns out to be interpreted as the difference between the angle made by the curve, i.e,

$$(\pi - (\mu^{i,j} \cdot \beta^{i,j})) = (\pi - (\theta_{i,j})). \qquad \square$$

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