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For a coefficient-free cluster algebra $\mathcal{A}$, we study the cluster automorphism group $\mathrm{Aut}(\mathcal{A})$ and the automorphism group $\mathrm{Aut}(E_{\mathcal{A}})$ of its exchange graph $E_{\mathcal{A}}$. We show that these two groups are isomorphic with each other, if $\mathcal{A}$ is of finite type excepting types of rank 2 and type F_4, or if $\mathcal{A}$ is of skew-symmetric finite mutation type.

1. Introduction

Cluster algebras were introduced by Sergey Fomin and Andrei Zelevinsky [2002]. In this paper we consider cluster algebras with trivial coefficients, which can be defined through a skew-symmetrizable square matrix. Such a cluster algebra is a $\mathbb{Z}$-subalgebra of a rational function field with n indeterminates. More precisely, a *seed* is a pair consisting of a set (*cluster*) of n indeterminates (*cluster variables*) in the field and a skew-symmetrizable square matrix (*exchange matrix*) of size n. Starting from an initial seed, we get a new seed by an operation called mutation. Then the cluster algebra is algebraic-generated by all the cluster variables obtained by iterated mutations. The cluster algebra has nice combinatorial structures which are (in some sense) given by mutations, and these structures are captured by its exchange graph, which is a graph with seeds as vertices and with mutations as edges.

We focus in this paper on two special types of cluster algebras: the *finite type* and the *finite mutation type*. Cluster algebras of finite type are those algebras with a finite number of clusters. They are classified in [Fomin and Zelevinsky 2003a], which corresponds to the Killing–Cartan classification of complex semisimple Lie algebras, or, equivalently, corresponds to the classification of root systems in Euclidean space. If there are finitely many matrix classes in the seeds of a cluster algebra, then we say it is of finite mutation type, where two matrices are in the same class if one of them can be obtained from the other by simultaneous relabeling of the rows and columns. The cluster algebras of finite mutation type with skew-symmetric

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exchange matrices are classified in [Felikson et al. 2012b]; a large class of them arises from marked Riemann surfaces (possibly with boundary) [Fomin et al. 2008], and there are 11 exceptional ones. The classification of skew-symmetrizable cluster algebras of finite mutation type is given in [Felikson et al. 2012a] via operations called unfoldings upon the skew-symmetric cluster algebras of finite mutation type.

We consider the relations in this paper between two groups associated to the cluster algebras. One is the cluster automorphism group consisting of *cluster automorphisms*, which are permutations of the clusters that commute with mutations. This group is introduced in [Assem et al. 2012] for a coefficient-free cluster algebra, and in [Chang and Zhu 2016b] for a cluster algebra with coefficients, it reveals the combinatorial and algebraic symmetries of the cluster algebra. Another is the automorphism group of the exchange graph, which consists of graph automorphism of the exchange graph. This group describes the symmetries of the exchange graph; in other words, it describes combinatorial symmetries of the cluster algebra. The problem that considers the relations between these two groups is stated in [Saleh 2014].

The exchange graph is a fairly coarse invariant of a cluster algebra, e.g., all infinite type cluster algebras of rank 2 have the same exchange graph. This article suggests that, nonetheless, the exchange graph is already rich enough to capture most of the symmetries of the cluster algebra.

For a coefficient-free cluster algebra $\mathcal{A}$ with exchange graph $E_{\mathcal{A}}$, we write the cluster automorphism group of $\mathcal{A}$ and the automorphism group of $E_{\mathcal{A}}$ as $\mathrm{Aut}(\mathcal{A})$ and $\mathrm{Aut}(E_{\mathcal{A}})$, respectively. In general, $\mathrm{Aut}(\mathcal{A})$ is a subgroup of $\mathrm{Aut}(E_{\mathcal{A}})$, and may be a proper subgroup; see Examples 3.3 and 3.5. The main result of this paper is that these two groups are isomorphic with each other if $\mathcal{A}$ is of finite type, excepting types of rank two and type F_4 (Theorem 3.16), or $\mathcal{A}$ is of skew-symmetric finite mutation type (Theorem 3.18). Therefore in some degree, for these cluster algebras, the algebraic symmetries are also captured by the exchange graphs. In particular, we compute the automorphism group of the exchange graph of a finite type cluster algebra in Table 1; see Remark 3.17.

To prove these results, we describe $E_{\mathcal{A}}$ more precisely. In Section 3A, we define layers of geodesic loops of $E_{\mathcal{A}}$ by using the distance of a vertex to a fixed vertex on $E_{\mathcal{A}}$. An easy observation is that an isomorphism of exchange graphs should maintain the combinatorial numbers of the layers of geodesic loops based on the corresponding vertices; see Remark 3.2(4). By this observation, we directly show in Examples 3.6, 3.7, 3.10 and 3.11 that for a cluster algebra of type A_3, B_3, C_3, $\tilde{A}_2$ or T_3 (the cluster algebra from a once-punctured torus), we have $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Aut}(E_{\mathcal{A}})$. For the general cases we reduce them to above five cases (Theorems 3.16 and 3.18).

The paper is organized as follows: we recall preliminaries on cluster algebras, cluster algebras of finite mutation type and cluster automorphisms in Section 2, then we prove the main theorems in Section 3.

Dynkin type	automorphism group $\mathrm{Aut}(E_{\mathcal{A}})$
$A_n (n \geqslant 2)$	$\mathbb{D}_{n+3}$
B_2	$\mathbb{D}_6$
$B_n (n \geqslant 3)$	$\mathbb{D}n+1$
C_2	$\mathbb{D}_6$
$C_n (n \geqslant 3)$	$\mathbb{D}_{n+1}$
D_4	$\mathbb{D}_4 \times S_3$
$D_n (n \geqslant 5)$	$\mathbb{Z}_2$
E_6	$\mathbb{D}_{14}$
E_7	$\mathbb{D}_{10}$
E_8	$\mathbb{D}_{16}$
F_4	$\mathbb{D}_7 \rtimes \mathbb{Z}_2$
G_2	$\mathbb{D}_8$

Table 1. Automorphism groups of exchange graphs of cluster algebras of finite type.

2. Preliminaries

2A. *Cluster algebras.*

Definition 2.1. [Fomin and Zelevinsky 2002] (labeled seeds). A *labeled seed* is a pair $\Sigma = (\boldsymbol{x}, B)$, where

- $\boldsymbol{x} = \{x_1, x_2, \dots, x_n\}$ is an ordered set of n indeterminates;
- $B = (b_{x_j x_i})_{n\times n} \in M_{n\times n}(\mathbb{Z})$ is a skew-symmetrizable matrix labeled by $\boldsymbol{x} \times \boldsymbol{x}$; that is, there exists a diagonal matrix D with positive integer entries such that DB is skew-symmetric.

The set $\boldsymbol{x}$ is called the *cluster* with elements the *cluster variables*, and B is called the *exchange matrix*. An element $b_{x_j x_i}$ in B is also written as b_{ji} for brevity. We assume throughout the paper that B is indecomposable; that is, for any $1 \leqslant i, j \leqslant n$, there is a sequence $i_0 = i, i_1, \dots, i_m, i_{m+1} = j$, such that $b_{i_k, i_{k+1}} \neq 0$ for any $0 \leqslant k \leqslant m$. We also assume that $n > 1$ for convenience. One may produce a new labeled seed by a mutation at direction k for any cluster variable x_k.

Definition 2.2. [Fomin and Zelevinsky 2002] (seed mutations). The labeled seed $\mu_k(\Sigma) = (\mu_k(\boldsymbol{x}), \mu_k(B))$ obtained by the *mutation* of Σ in the direction k is given by:

- $\mu_k(\boldsymbol{x}) = (\boldsymbol{x} \setminus \{x_k\}) \sqcup \{\mu_{x_k,\boldsymbol{x}}(x_k)\}$ where

$$x_k \mu_{x_k,\boldsymbol{x}}(x_k) = \prod_{\substack{1\leqslant j\leqslant n;\\ b_{jk}>0}} x_j^{\,b_{jk}} + \prod_{\substack{1\leqslant j\leqslant n;\\ b_{jk}<0}} x_j^{\,-b_{jk}}.$$

- $\mu_k(B) = (b'_{ji})_{n\times n} \in M_{n\times n}(\mathbb{Z})$ is given by

$$b'_{ji} = \begin{cases} -b_{ji} & \text{if } i = k \text{ or } j = k; \\ b_{ji} + \frac{1}{2}(|b_{ji}|b_{ik} + b_{ji}|b_{ik}|) & \text{otherwise.} \end{cases}$$

It is easy to check that a mutation is an involution; that is, $\mu_k\mu_k(\Sigma) = \Sigma$.

Definition 2.3. [Fomin and Zelevinsky 2007] (n-cluster patterns). An n-regular tree $\mathbb{T}_n$ is a diagram, whose edges are labeled by $1, 2, \dots, n$, such that the n edges emanating from each vertex receive different labels. A *n-cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\boldsymbol{x}_t, B_t)$ to every vertex $t \in \mathbb{T}_n$, so that the labeled seeds assigned to the endpoints of any edge labeled by k are obtained from each other by the seed mutation in direction k. The elements of Σ_t are written as follows:

$$\boldsymbol{x}_t = (x_{1;t}, \dots, x_{n;t}), \quad B_t = (b^t_{ij}). \tag{1}$$

Note that $\mathbb{T}_n$ is in fact determined by any fixed labeled seed on it. Now we are ready to define cluster algebras.

Definition 2.4. [Fomin and Zelevinsky 2007] (cluster algebras). Given a seed Σ and a cluster pattern $\mathbb{T}_n$ associated to it, we denote

$$\mathscr{X} = \bigcup_{t\in\mathbb{T}_n} \boldsymbol{x}_t = \{x_{i,t} : t \in \mathbb{T}_n,\ 1 \le i \le n\}, \tag{2}$$

the union of clusters of all the seeds in the pattern. We call the elements $x_{i,t} \in \mathscr{X}$ the *cluster variables*. The *cluster algebra $\mathcal{A}$* associated with Σ is the $\mathbb{Z}$-subalgebra of the rational function field $\mathcal{F} = \mathbb{Q}(x_1, x_2, \dots, x_n)$, generated by all cluster variables, $\mathcal{A} = \mathbb{Z}[\mathscr{X}]$.

To a skew-symmetrizable matrix $B = (b_{ji})_{n\times n}$, one can associate a *valued quiver* (quiver for brevity) $Q = (Q_0, Q_1, \upsilon)$ as follows: $Q_0 = \{1, 2, \dots, n\}$ is a set of vertices. For any two vertices j and i, if $b_{ji} > 0$, then there is an arrow α from j to i to which we assign a pair of values $(\upsilon_1(\alpha), \upsilon_2(\alpha)) = (b_{ji}, -b_{ij})$. These arrows form the set Q_1. Since B is an indecomposable skew-symmetrizable matrix, the defined valued quiver Q is connected and there are no loops nor 2-cycles in Q. Then we can define a mutation of the valued quiver by the mutation of the matrix; we refer to [Fomin and Zelevinsky 2002; Keller 2012] for details. We say two quivers Q and Q' are *mutation equivalent* if the corresponding matrices are mutation equivalent; that is, one of them can be obtained from the other one by a finite sequence of mutations. We also write $(\boldsymbol{x}, Q)$ for the labeled seed $(\boldsymbol{x}, B)$, and write $\mathcal{A}_Q$ for the cluster algebra defined by Σ. The quiver and the defined cluster algebra are called skew-symmetric if the corresponding matrix is skew-symmetric.

If the cluster algebra is of finite type [Fomin and Zelevinsky 2003a] or of skew-symmetric type, then the cluster determines the quiver [Gekhtman et al. 2008], and we denote the quiver of a cluster $\boldsymbol{x}$ by $Q(\boldsymbol{x})$.

Example 2.5. Let B be the following skew-symmetrizable matrix with skew-symmetrizer $D = \operatorname{diag}\{2, 2, 1, 1\}$:

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

The quiver corresponding to B is Q, where we always delete the trivial pairs of values (1, 1), and replace a arrow assigning pair (m, m) by m arrows:

$$Q : 1 \longrightarrow 2 \xleftarrow{(2,1)} 3 \rightrightarrows 4.$$

Definition 2.6. [Fomin and Zelevinsky 2007] (seeds). Given two labeled seeds $\Sigma = (\boldsymbol{x}, B)$ and $\Sigma' = (\boldsymbol{x}', B')$, we say that they define the same *seed* if Σ' is obtained from Σ by simultaneous relabeling of the sets $\boldsymbol{x}$ and the corresponding relabeling of the rows and columns of B.

We denote by $[\Sigma]$ the seed represented by a labeled seed Σ. The cluster $\boldsymbol{x}$ of a seed $[\Sigma]$ is an unordered n-element set. For any $x \in \boldsymbol{x}$, there is a well-defined mutation $\mu_x([\Sigma]) = [\mu_k(\Sigma)]$ of $[\Sigma]$ at direction x, where $x = x_k$. For two same rank skew-symmetrizable matrices B and B', we say $B \cong B'$ if B' is obtained from B by simultaneous relabeling of the rows and columns of B. Then the exchange matrices in any two labeled seeds representing a same seed are isomorphic. The isomorphism of two exchange matrices induces an isomorphism of corresponding quivers. For convenience, in the rest of the paper, we also denote by Σ the seed $[\Sigma]$ represented by Σ.

Definition 2.7. [Fomin and Zelevinsky 2007] (exchange graphs). The *exchange graph* of a cluster algebra is the n-regular graph whose vertices are the seeds of the cluster algebra and whose edges connect the seeds related by a single mutation. We denote by $E_{\mathcal{A}}$ the exchange graph of a cluster algebra $\mathcal{A}$.

Clearly, the exchange graph of a cluster algebra is a quotient graph of the n-regular tree; its vertices are equivalent classes of labeled seeds. The exchange graph need not be a finite graph; if it is finite, then we say the corresponding cluster algebra (and its cluster pattern) are of *finite type*.

Definition 2.8. [Fomin and Zelevinsky 2003a, page 70] (cluster complexes). A cluster complex Δ of $\mathcal{A}$ is a simplicial complex on the ground set $\mathscr{X}$ with the clusters as the maximal simplices.

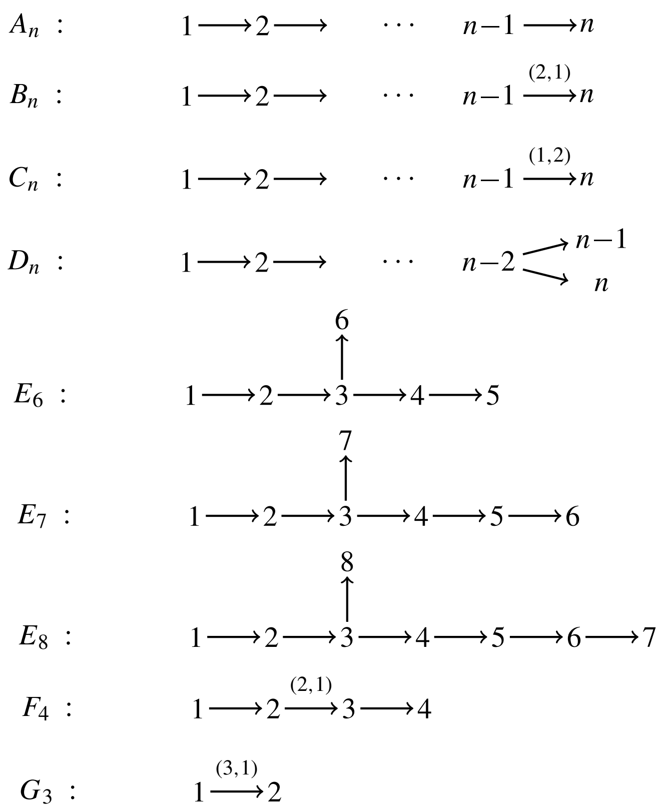

Figure 1. Quivers of finite type.

Then Δ is an n-dimensional complex. In particular, if $\mathcal{A}$ is of finite type or skew-symmetric, then the vertices of $E_{\mathcal{A}}$ are clusters, so the dual graph of Δ is $E_{\mathcal{A}}$.

2B. ***Finite types and finite mutation types.*** By the classification of cluster algebras of finite type [Fomin and Zelevinsky 2003a], a cluster algebra is of finite type if and only if there is a seed whose quiver is one of the quivers depicted in Figure 1. Note that the underlying graphs of quivers in Figure 1 are trees, thus any two quivers with the same underlying graph are mutation-equivalent.

Definition 2.9. [Fomin et al. 2008; Felikson et al. 2012b] A *block* is a quiver isomorphic to one of the quivers with black or white colored vertices shown in Figure 2. Vertices marked in white are called *outlets*. A connected quiver Q is called *block-decomposable* (*decomposable* for brevity) if it can be obtained from a collection of blocks by identifying outlets of different blocks along some partial matching (matching of outlets of the same block is not allowed), where two arrows with the same endpoints and opposite directions cancel out. If Q is not block-decomposable then we call Q *nondecomposable*.

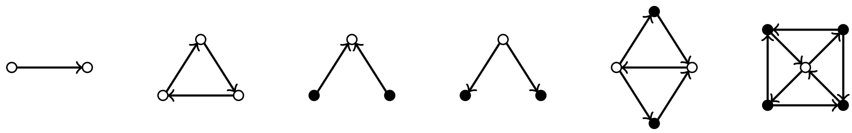

Figure 2. Blocks. Outlets are colored white, dead ends are black.

It is proved in [Fomin et al. 2008, Theorem 13.3] that a quiver is decomposable if and only if it is a quiver of a triangulation of an oriented marked Riemann surface, and thus a quiver mutation equivalent to a decomposable quiver is also decomposable. Note that all arrow multiplicities of a decomposable quiver are 1 or 2. Therefore decomposable quivers are mutation finite. It is clear that a quiver of rank 2, that is, a quiver with two vertices, is mutation finite. Besides these two kinds of quivers, there are exactly 11 exceptional skew-symmetric quivers of finite mutation type; see Theorem 6.1 in [Felikson et al. 2012b]. We list the exceptional quivers in Figure 3.

2C. ***Automorphism groups.*** In this section, we recall the cluster automorphism group [Assem et al. 2012] of a cluster algebra, and the automorphism group of the corresponding exchange graph [Chang and Zhu 2016b].

Definition 2.10. [Assem et al. 2012] *(*cluster automorphisms). For a cluster algebra $\mathcal{A}$ and a $\mathbb{Z}$-algebra automorphism $f:\mathcal{A}\to\mathcal{A}$, we call f a *cluster automorphism* if there exists a labeled seed $(\boldsymbol{x}, B)$ of $\mathcal{A}$ such that the following conditions are satisfied:

(1) $f(\boldsymbol{x})$ is a cluster.

(2) f is compatible with mutations; that is, for every $x\in\boldsymbol{x}$ and $y\in\boldsymbol{x}$, we have

$$f(\mu_{x,\boldsymbol{x}}(y)) = \mu_{f(x),f(\boldsymbol{x})}(f(y)).$$

Then a cluster automorphism maps a labeled seed $\Sigma = (\boldsymbol{x}, B)$ to a labeled seed $\Sigma' = (\boldsymbol{x}', B')$. Under our assumption that B is indecomposable, we have the following:

Lemma 2.11 [Assem et al. 2012]. *A $\mathbb{Z}$-algebra automorphism $f:\mathcal{A}\to\mathcal{A}$ is a cluster automorphism if and only if there exists a labeled seed $\Sigma=(\boldsymbol{x}, B)$ of $\mathcal{A}$, such that $f(\boldsymbol{x})$ is the cluster in a labeled seed $\Sigma'=(\boldsymbol{x}', B')$ of $\mathcal{A}$ with $B'=B$ or $B'=-B$.*

We call the cluster automorphism such that $B=B'$ (resp. $B=-B'$) a *direct cluster automorphism* (resp. an *inverse cluster automorphism*). Clearly, all the cluster automorphisms of a cluster algebra $\mathcal{A}$ form a group with homomorphism

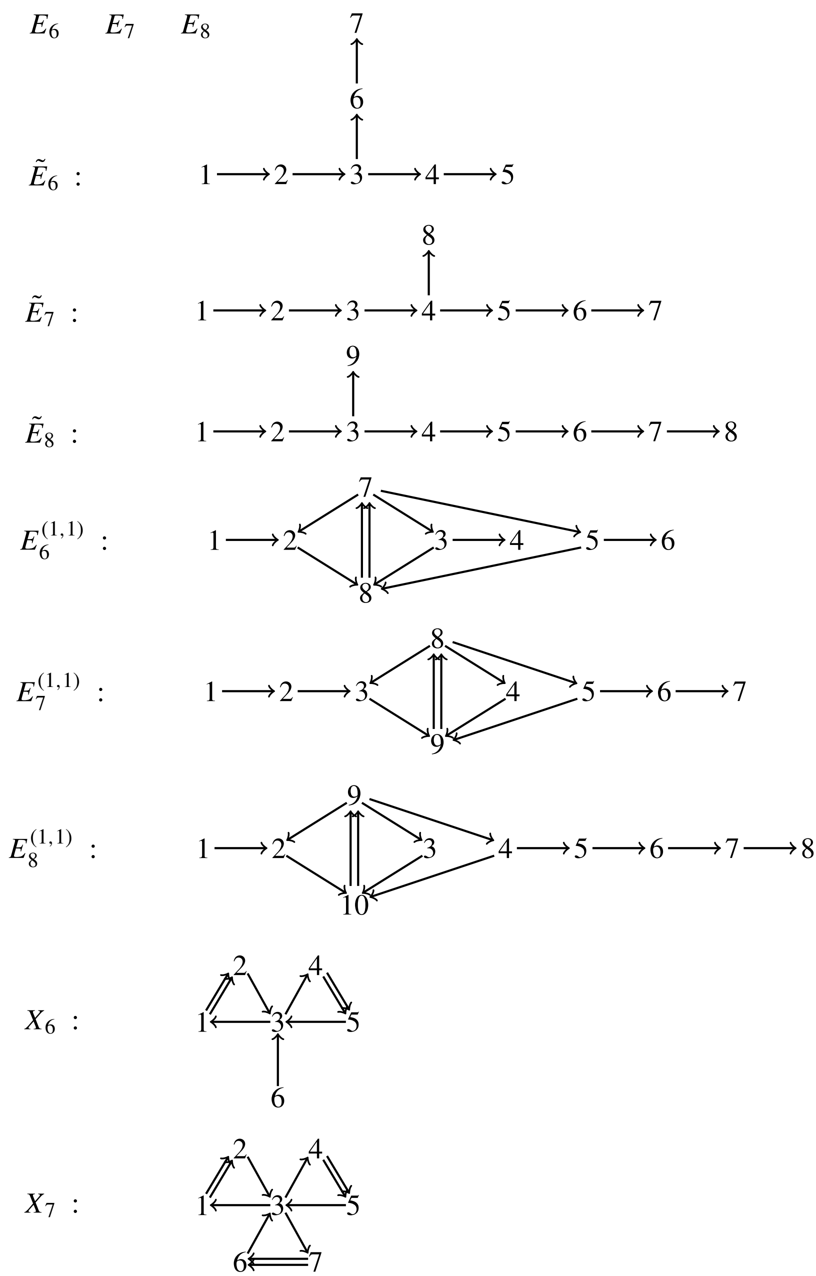

Figure 3. Representatives of nondecomposable quivers of finite mutation type.

composition as multiplication. We call this group the *cluster automorphism group* of $\mathcal{A}$, and denote it by $\mathrm{Aut}(\mathcal{A})$. We call the group $\mathrm{Aut}^+(\mathcal{A})$ consisting of the direct cluster automorphisms of $\mathcal{A}$ the *direct cluster automorphism group* of $\mathcal{A}$, which is a subgroup of Aut($\mathcal{A}$) with index at most two; see [Assem et al. 2012].

Definition 2.12. [Saleh 2014; Chang and Zhu 2016b] (automorphism of exchange graphs). An automorphism of the exchange graph $E_{\mathcal{A}}$ of a cluster algebra $\mathcal{A}$ is an automorphism of $E_{\mathcal{A}}$ as a graph, that is, a permutation σ of the vertex set, such that the pair of vertices (u, v) forms an edge if and only if the pair $(\sigma(u), \sigma(v))$ also forms an edge.

Clearly, the natural composition of two automorphisms of $E_{\mathcal{A}}$ is again an automorphism. We define an *automorphism group* Aut($E_{\mathcal{A}}$) of $E_{\mathcal{A}}$ as a group consisting of automorphisms of $E_{\mathcal{A}}$. It is clear that a cluster automorphism induces a unique automorphism of the exchange graph. Thus Aut($\mathcal{A}$) is a subgroup of Aut($E_{\mathcal{A}}$); see [Chang and Zhu 2016b]. By the definition, an automorphism σ of an exchange graph maps clusters to clusters, and induces an automorphism of its dual graph, the cluster complex Δ; we denote this automorphism by σ_{Δ}. Then σ_{Δ} is a permutation of cluster variables in $\mathscr{X}$, which maps a maximal simplex to a maximal simplex, but the map may not be compatible with the algebra relations among cluster variables in $\mathcal{A}$, thus it is not necessarily a cluster automorphism. In fact, Aut($\mathcal{A}$) may be a proper subgroup of Aut($E_{\mathcal{A}}$); see Examples 3.3 and 3.5. The following lemma can be viewed as a description of Aut($\mathcal{A}$) as a subgroup of Aut($E_{\mathcal{A}}$), as those exchange graph automorphisms which happen to preserve B-matrices (perhaps up to global reversal of sign) up to simultaneously relabeling of the rows and columns. In this point of view, the main thrust of this paper is to show that, typically for the cluster algebras we consider, any graph automorphism has the property of preserving B-matrices.

Lemma 2.13. *Let $\Phi : E_{\mathcal{A}} \to E_{\mathcal{A}}$ be an automorphism which maps a seed $\Sigma = (\boldsymbol{x}, B)$ to a seed $\Sigma' = (\boldsymbol{x}', B')$. If $B \cong B'$ or $B \cong -B'$ under the correspondence $\boldsymbol{x} \to \boldsymbol{x}'$, then the map $\boldsymbol{x} \to \boldsymbol{x}'$ induces a cluster automorphism Ψ of $\mathcal{A}$ and the induced automorphism $\Psi_E : E_{\mathcal{A}} \to E_{\mathcal{A}}$ coincides with Φ.*

Proof. Since $B \cong B'$ or $B \cong -B'$, the map $\boldsymbol{x} \to \boldsymbol{x}'$ induces a cluster automorphism Ψ of $\mathcal{A}$ by Lemma 2.11. Notice that $\Phi(\boldsymbol{x}) = \Psi(\boldsymbol{x})$; then by inductions on the mutations, we have $\Phi = \Psi$ on each cluster of $E_{\mathcal{A}}$, so $\Phi = \Psi_E$ as automorphisms of the exchange graph $E_{\mathcal{A}}$. □

3. Automorphism groups of exchange graphs

In this section we consider relations between the groups Aut($\mathcal{A}$) and Aut($E_{\mathcal{A}}$) for a cluster algebra $\mathcal{A}$ of finite type or of skew-symmetric finite mutation type. For this, we need to describe $E_{\mathcal{A}}$ more precisely. In the following we will recall the basic structures of $E_{\mathcal{A}}$ from [Fomin and Zelevinsky 2002; 2003a], and then introduce layers of geodesic loops on $E_{\mathcal{A}}$.

3A. ***Layers of geodesic loops.*** Let $\Sigma = (\boldsymbol{x}, B)$ be a labeled seed on the cluster pattern of $\mathcal{A}$. Let $\boldsymbol{x}'$ be a proper subset of $\boldsymbol{x}$, then $\boldsymbol{x}'$ is a nonmaximal simplex in the cluster complex Δ. We denote by $\Delta_{\boldsymbol{x}'}$ the *link* of $\boldsymbol{x} \setminus \boldsymbol{x}'$, which is the simplicial complex on the ground set

$$\mathscr{X}_{\boldsymbol{x}'} = \{\alpha \in \mathscr{X} - (\boldsymbol{x} \setminus \boldsymbol{x}') : (\boldsymbol{x} \setminus \boldsymbol{x}') \cup \{\alpha\} \in \Delta\},$$

such that $\boldsymbol{x}''$ is a simplex in $\Delta_{\boldsymbol{x}'}$ if and only if $\boldsymbol{x} \setminus \boldsymbol{x}' \cup \boldsymbol{x}''$ is a simplex in Δ. Let $\Gamma_{\boldsymbol{x}'}$ be the dual graph of $\Delta_{\boldsymbol{x}'}$. We view $\Gamma_{\boldsymbol{x}'}$ as a subgraph of $E_{\mathcal{A}}$ whose vertices are the maximal simplices in Δ that contain $\boldsymbol{x} \setminus \boldsymbol{x}'$. In fact, as we explain now, $\Gamma_{\boldsymbol{x}'}$ is the exchange graph of a cluster algebra $\mathcal{A}_f$ defined by a frozen seed

$$\Sigma_f = (\boldsymbol{x}', \boldsymbol{x} \setminus \boldsymbol{x}', B_f),$$

which is the freezing of Σ at $\boldsymbol{x} \setminus \boldsymbol{x}'$ (see [Chang and Zhu 2016c, Definition 2.25]), where B_f is obtained from B by deleting the columns labeled by variables in $\boldsymbol{x} \setminus \boldsymbol{x}'$. Then elements in $\boldsymbol{x} \setminus \boldsymbol{x}'$ are coefficients of $\mathcal{A}_f$ (we refer to [Fomin and Zelevinsky 2002; 2007] for a cluster algebra with coefficients). Let $\mathcal{A}'$ be a cluster algebra defined by a seed $\Sigma' = (\boldsymbol{x}', B')$, where B' is obtained from B by deleting rows and columns labeled by variables in $\boldsymbol{x} \setminus \boldsymbol{x}'$. In our setting, that is, where cluster algebras are of finite type or of skew-symmetric finite type, the exchange graph of a cluster algebra (with coefficients) only depends on the principal part of the exchange matrix (see [Fomin and Zelevinsky 2003a; Cerulli Irelli et al. 2013]) which is the submatrix labeled by $\boldsymbol{x} \setminus \boldsymbol{x}' \times \boldsymbol{x} \setminus \boldsymbol{x}'$; thus the graph $\Gamma_{\boldsymbol{x}'}$ coincides with the exchange graph $E_{\mathcal{A}'}$.

For a 2-dimensional subcomplex $\boldsymbol{x}'$ of Δ, we call the dual graph $\Gamma_{\boldsymbol{x}'}$ a *geodesic loop* of $E_{\mathcal{A}}$. We mention that the definition of geodesic loop is slightly different from the definition used in [Fomin and Zelevinsky 2003a], where a line is not a geodesic loop. If $\mathcal{A}$ is of finite type, then $E_{\mathcal{A}}$ is a finite graph, and $\Gamma_{\boldsymbol{x}'}$ is a polygon. Notice that in the seed $\Sigma' = (\boldsymbol{x}', B')$ constructed above, B' is of Dynkin type, that is, one of types A_2, B_2, C_2 or G_2. Therefore $\Gamma_{\boldsymbol{x}'}$ is a $(h{+}2)$-polygon, where h is the Coxeter number of the corresponding Dynkin type; see [Fomin and Zelevinsky 2003a]. If $\mathcal{A}$ is of finite mutation type, then $\Gamma_{\boldsymbol{x}'}$ may be a line. We fix a basepoint $\Sigma = (\boldsymbol{x}, B)$ and introduce the following concept.

Definition 3.1. (1) Letting Σ' be a point of $E_{\mathcal{A}}$, the *distance* $\ell(\Sigma, \Sigma')$ between Σ and Σ' is the minimal length of paths between Σ and Σ'.

(2) Letting L be a geodesic loop of $E_{\mathcal{A}}$, the *distance* $\ell_\Sigma(L)$ between Σ and L is the minimal length $\min\{\ell(\Sigma, \Sigma'), \Sigma' \in L\}$.

(3) Letting $m \in \mathbb{Z}_{\geqslant 0}$ be a nonnegative integer, denote by ℓ_Σ^m the set of geodesic loops whose distance to Σ is m. We call it the *m-layer* of geodesic loops of $E_{\mathcal{A}}$ based on Σ.

(4) For any $m \in \mathbb{Z}_{\geqslant 0}$, denote by $N(\ell_\Sigma^m)$ the set of amounts of edges belonging to geodesic loops in the *m-layer* ℓ_Σ^m.

Remark 3.2. The following observations are directly derived from the definitions:

(1) The elements in ℓ^0_Σ are those geodesic loops $\Gamma_{\boldsymbol{x}'}$ for the 2-dimensional subcomplex $\boldsymbol{x}'$ of Δ, where $\boldsymbol{x}'$ is a subset of the cluster $\boldsymbol{x}$ in Σ.

(2) For $m_1 \neq m_2$, $\ell^{m_1}_\Sigma \cap \ell^{m_2}_\Sigma = \varnothing$.

(3) The disjoint union $\bigsqcup_{m\geqslant 0} \ell^m_\Sigma$ is the set of all the geodesic loops of $E_{\mathcal{A}}$.

(4) If $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ is an isomorphism of graphs such that the image of Σ is Σ', then for every $m \in \mathbb{Z}_{\geqslant 0}$, $N(\ell^m_\Sigma) = N(\ell^m_{\Sigma'})$ as sets.

3B. *Cases of rank 2 and rank 3.* In this subsection, we consider the relations between $\mathrm{Aut}(\mathcal{A})$ and $\mathrm{Aut}(E_{\mathcal{A}})$ for a cluster algebra $\mathcal{A}$ of rank 2 or rank 3.

Example 3.3. For a finite type cluster algebra $\mathcal{A}$ of rank 2, that is, one of types A_2, B_2, C_2 or G_2, its exchange graph $E_{\mathcal{A}}$ is a $(h+2)$-polygon, thus $\mathrm{Aut}(E_{\mathcal{A}})$ is isomorphic to the dihedral group $\mathbb{D}_{h+2}$, where h is the Coxeter number. If $\mathcal{A}$ is of type A_2, then $\mathrm{Aut}(\mathcal{A}) \cong \mathbb{D}_5$ [Assem et al. 2012], thus $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Aut}(E_{\mathcal{A}})$. If $\mathcal{A}$ is of type B_2, C_2 or G_2, [Chang and Zhu 2016a, Theorem 3.5] shows that $\mathrm{Aut}(\mathcal{A}) \cong \mathbb{D}_{(h+2)/2}$, thus $\mathrm{Aut}(\mathcal{A}) \subsetneqq \mathrm{Aut}(E_{\mathcal{A}})$.

Example 3.4. For an infinite type skew-symmetric cluster algebra $\mathcal{A}$ of rank 2, its exchange graph $E_{\mathcal{A}}$ is a line, thus $\mathrm{Aut}(E_{\mathcal{A}}) = \langle s\rangle \times \langle r\rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2 = \mathbb{D}_\infty$, where s is a left shift of $E_{\mathcal{A}}$ which maps a cluster to the left adjacent cluster and r is a reflection with respect to a fixed cluster. Then s corresponds to a direct cluster automorphism of $\mathcal{A}$ and r corresponds to an inverse cluster automorphism of $\mathcal{A}$; thus by Lemma 2.13, $\mathrm{Aut}(E_{\mathcal{A}}) \subseteq \mathrm{Aut}(\mathcal{A})$. Therefore $\mathrm{Aut}(E_{\mathcal{A}}) \cong \mathrm{Aut}(\mathcal{A}) \cong \mathbb{D}_\infty$.

Example 3.5. For an infinite type non-skew-symmetric cluster algebra $\mathcal{A}$ of rank 2, its exchange graph $E_{\mathcal{A}}$ is also a line, thus as shown in Example 3.4, $\mathrm{Aut}(E_{\mathcal{A}}) = \langle s\rangle \times \langle r\rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, where s corresponds to a direct cluster automorphism of $\mathcal{A}$, while r dose not correspond to any cluster automorphism of $\mathcal{A}$, since there is no nontrivial symmetry of the quiver in any seed of $\mathcal{A}$. Thus $\mathrm{Aut}(\mathcal{A}) \cong \mathbb{Z} \subsetneqq \mathrm{Aut}(E_{\mathcal{A}})$.

Example 3.6. We consider the cluster algebra $\mathcal{A}$ of type A_3 with an initial labeled seed $\Sigma_0 = (\{x_1, x_2, x_3\}, Q)$, where Q is $1 \to 2 \leftarrow 3$. Its exchange graph $E_{\mathcal{A}}$ is depicted in Figure 4. Note that there are three quadrilaterals and six pentagons in $E_{\mathcal{A}}$. Then as shown in [Chang and Zhu 2016a, Example 3], $\mathrm{Aut}(\mathcal{A}) = \langle f_-, f_+\rangle \cong \mathbb{D}_6$, where f_- is defined by

$$f_- : \begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto \mu_2(x_2), \\ x_3 \mapsto x_3. \end{cases} \tag{3}$$

It maps Σ_0 to Σ_1, and induces a reflection with respect to the horizontal central

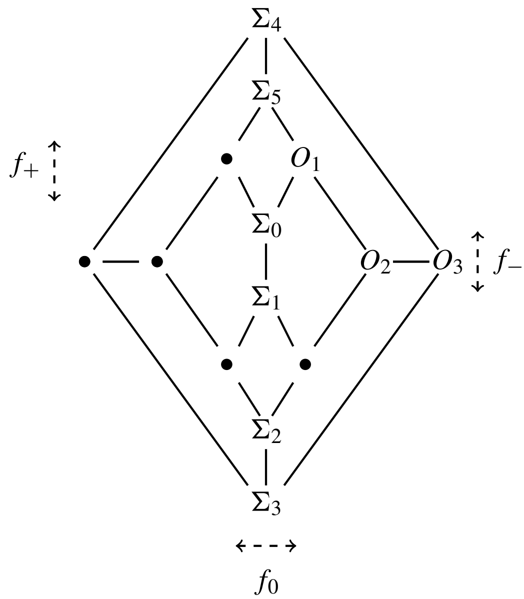

Figure 4. The exchange graph of a cluster algebra of type A_3.

axis of $E_{\mathcal{A}}$. The cluster automorphism f_+ is defined by

$$f_+ : \begin{cases} x_1 \mapsto \mu_1(x_1), \\ x_2 \mapsto x_2, \\ x_3 \mapsto \mu_3(x_3). \end{cases} \tag{4}$$

It gives a reflection on $E_{\mathcal{A}}$, which maps Σ_0 to Σ_5. In fact, as shown in [Chang and Zhu 2016a], a direct cluster automorphism of $\mathcal{A}$ is of the form $(f_+f_-)^m$, $0 \leq m \leq 5$, which induces a rotation of seeds in $\{\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$, thus $\mathrm{Aut}^+(\mathcal{A})$ can be viewed as the symmetry group of the *bipartite belt* consisting of seeds in $\{\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$, where the quivers in these seeds are the bipartite quivers isomorphic to Q.

We will prove that any automorphism of $E_{\mathcal{A}}$ is induced from an element in $\mathrm{Aut}(\mathcal{A})$, and thus $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Aut}(E_{\mathcal{A}})$. For this purpose, we show the following claims:

(1) There exists no automorphism of $E_{\mathcal{A}}$ which maps Σ_0 to a vertex except for Σ_i, $0 \leq i \leq 5$.

(2) If an automorphism of $E_{\mathcal{A}}$ maps Σ_0 to Σ_i, $0 \leq i \leq 5$, then it is induced from a cluster automorphism of $\mathcal{A}$.

Let σ be an automorphism of $E_{\mathcal{A}}$. Due to symmetries of $E_{\mathcal{A}}$, we only show that $\sigma(\Sigma) \neq O_i$, $i = 1, 2, 3$. By a direct computation, the sets of numbers for the layers of geodesic loops based on these vertices are

$$N(\ell^0_\Sigma)=\{4,5,5\},\quad N(\ell^1_\Sigma)=\{4,5,5\},\quad N(\ell^2_\Sigma)=\{5,5\},\quad N(\ell^3_\Sigma)=\{4\};$$
$$N(\ell^0_{O_1})=\{4,5,5\},\quad N(\ell^1_{O_1})=\{5,5,5\},\quad N(\ell^2_{O_1})=\{4,4\},\quad N(\ell^3_{O_1})=\{5\};$$
$$N(\ell^0_{O_2})=\{5,5,5\},\quad N(\ell^1_{O_1})=\{4,4,4\},\quad N(\ell^2_{O_1})=\{5,5,5\};$$
$$N(\ell^0_{O_3})=\{4,5,5\},\quad N(\ell^1_{O_3})=\{5,5,5\},\quad N(\ell^2_{O_3})=\{4,4\},\quad N(\ell^3_{O_3})=\{5\}.$$

Then by Remark 3.2(4), $\sigma(\Sigma_0)\neq O_i,\ i=1,2,3$. So the first claim is affirmed.

Now we consider the second claim. Still due to the symmetries of the graph, we may assume that $\sigma(\Sigma_0)=\Sigma_0$. Since σ is a graph automorphism, it can be seen that there are two possibilities for σ; one is the identity, the other is the reflection f_0 with respect to the vertical central axis of $E_{\mathcal{A}}$, as depicted in Figure 4. Note that the identity graph automorphism is induced from the identity automorphism of the cluster algebra, while the graph automorphism f_0 is induced from the cluster automorphism $(f_+f_-)^3$ by a direct computation. Therefore the second claim is true and we have $\mathrm{Aut}(E_{\mathcal{A}})\cong\mathrm{Aut}(\mathcal{A})\cong\mathbb{D}_6$.

Example 3.7. It is known from a result in [Fomin and Zelevinsky 2003b] that the cluster algebras of type B_n and type C_n have the same exchange graph. Based on a seed Σ_0, the exchange graph of a cluster algebra $\mathcal{A}$ of type B_3 or type C_3 is depicted in Figure 5. For the cluster algebra of type B_3, the quiver of the initial seed Σ_0 is

$$1\longrightarrow 2\xleftarrow{(2,1)}3.$$

For the cluster algebra of type C_3, the quiver of the initial seed Σ_0 is

$$frm[o]--\longrightarrow 2\xleftarrow{(1,2)}3.$$

Let σ be an automorphism of $E_{\mathcal{A}}$. As shown by Example 4 in [Chang and Zhu 2016a], we have $\mathrm{Aut}(\mathcal{A})\cong\mathbb{D}_4$ and $\{\Sigma_0,\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4,\Sigma_5,\Sigma_6,\Sigma_7\}$ are all the seeds whose quivers are isomorphic to Q. Similarly, to get $\mathrm{Aut}(E_{\mathcal{A}})\cong\mathrm{Aut}(\mathcal{A})$, we prove the two claims stated in Example 3.6. For the first claim, we only need to prove that $\sigma(\Sigma_0)\neq O_i(i=1,2,3,4)$ in Figure 5, and this can be obtained by the fact that these seeds have different combinatorial numbers of layers of geodesic loops:

$$N(\ell^0_{\Sigma_0})=\{4,5,6\},\quad N(\ell^1_{\Sigma_0})=\{4,5,6\};$$
$$N(\ell^0_{O_1})=\{5,6,6\};$$
$$N(\ell^0_{O_2})=\{4,5,6\},\quad N(\ell^1_{O_2})=\{5,6,6\};$$
$$N(\ell^0_{O_3})=\{4,5,6\},\quad N(\ell^1_{O_3})=\{5,6,6\};$$
$$N(\ell^0_{O_4})=\{5,6,6\}.$$

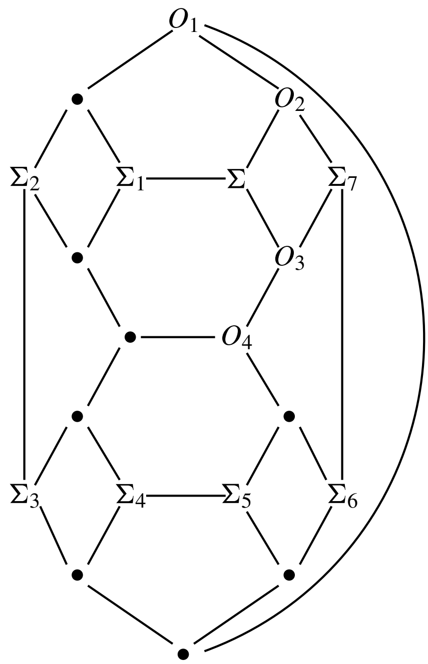

Figure 5. The exchange graph of a cluster algebra of type B_3 or type C_3.

For the second claim, we may also assume $\sigma(\Sigma_0) = \Sigma_0$. Since $N(\ell^0_{\Sigma_0}) = \{4, 5, 6\}$, there are neither rotation symmetries nor reflection symmetries of $E_{\mathcal{A}}$ at Σ_0. So σ must be the identity automorphism of $E_{\mathcal{A}}$, which is induced from the identity automorphism of the cluster algebra. Noticing that there are eight elements in $\mathrm{Aut}(\mathcal{A}) \cong \mathbb{D}_4$, where each one corresponds a graph automorphism which maps Σ_0 to $\Sigma_i,\ 0 \le i \le 7$.

Example 3.8. For cluster algebras of type F_4, let the quiver Q of a seed Σ be

$$1 \longrightarrow 2 \xleftarrow{(2,1)} 3 \longrightarrow 4.$$

Then $\mathrm{Aut}(\mathcal{A}) \cong \mathbb{D}_7$ [Chang and Zhu 2016a]. The variables x_1, x_2, x_3 and the corresponding full subquiver of Q form a seed Σ_1 of type B_3, while x_2, x_3, x_4 and the corresponding full subquiver of Q form a seed Σ_2 of type C_3. By pinning down Σ, rotating the graph $E_{\mathcal{A}}$ induces an automorphism σ of $E_{\mathcal{A}}$, which exchanges the graph $E_{\mathcal{A}_{\Sigma_1}}$ and the graph $E_{\mathcal{A}_{\Sigma_2}}$. However σ does not induce a cluster automorphism of $\mathcal{A}$, and $\mathrm{Aut}(\mathcal{A}) \cong \mathbb{D}_7 \subsetneqq \mathbb{D}_7 \rtimes \mathbb{Z}_2 \cong \mathrm{Aut}(E_{\mathcal{A}})$.

Proposition 3.9. *Let Q be a connected quiver with three vertices which is of finite type. Let $\Sigma = (\boldsymbol{x}, Q)$ and $\Sigma' = (\boldsymbol{x}', Q')$ be two seeds (not necessarily mutation* *equivalent to each other). If there is an isomorphism* $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ *such that* $\sigma(\Sigma) = \Sigma'$, *then* $\Sigma' = (\boldsymbol{x}', Q')$ *is a finite type seed with* Q' *connected, and*

(1) *if* Σ *is of type* A_3, *then* $Q' \cong Q$ *(or* Q^{op}*);*

(2) *if* Σ *is of type* B_3 *and* Σ' *is not of type* C_3, *then* $Q' \cong Q$ *(or* Q^{op}*);*

(3) *if* Σ *is of type* C_3 *and* Σ' *is not of type* B_3, *then* $Q' \cong Q$ *(or* Q^{op}*).*

Proof. Clearly, since $E'_{\mathcal{A}} \cong E_{\mathcal{A}}$ is of finite type, Q' is a Dynkin type quiver with three vertices. If Q is of type A_3, then by Example 3.6,

$$N(\ell^0_\Sigma) = \{4, 5, 5\} \text{ or } \{5, 5, 5\}.$$

If Q is of type B_3 (or C_3), then from Example 3.7,

$$N(\ell^0_\Sigma) = \{4, 5, 6\} \text{ or } \{5, 6, 6\}.$$

If Q' is a union of a quiver of type A_2 and a point, then from Example 3.3,

$$N(\ell^0_\Sigma) = \{4, 4, 5\}.$$

If Q' is a union of a quiver of type B_2 (or C_2) and a point, then from Example 3.3,

$$N(\ell^0_\Sigma) = \{4, 4, 6\}.$$

If Q' is a union of a quiver of type G_2 and a point, then from Example 3.3,

$$N(\ell^0_\Sigma) = \{4, 4, 8\}.$$

Thus we get the proof by Remark 3.2. □

Example 3.10. Let Q be the quiver in Figure 6; we say it is of type $\tilde{A}_2$. Then it is not hard to see that if a quiver in the mutation class of Q is not isomorphic to Q, then it must be isomorphic to the quiver Q' in Figure 6. Let $\mathcal{A}$ be a cluster algebra with an initial seed

$$\Sigma = (\{x_1, x_2, x_3\}, Q),$$

as in the above examples. To show that $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Aut}(E_{\mathcal{A}})$, we only need to notice that

$$N(\ell^0_\Sigma) = \{5, 5, \infty\},$$
$$N(\ell^0_{\Sigma'}) = \{5, 5, 5\},$$

where Σ' is a seed of $\mathcal{A}$ with quiver isomorphic to Q'. In fact, from [Assem et al. 2012, Section 3.3],

$$\mathrm{Aut}(\mathcal{A}) = \langle r_1, r_2 \mid r_1 r_2 = r_2 r_1, {r_1}^2 = r_2 \rangle \rtimes \langle \sigma \mid \sigma^2 = 1 \rangle \cong \mathbb{H}_{2,1} \rtimes \mathbb{Z}_2,$$

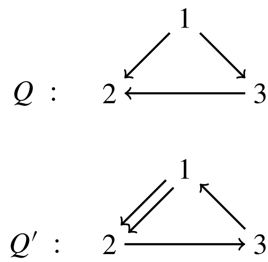

Figure 6. Quivers of type $\tilde{A}_2$.

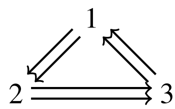

Figure 7. Quiver of type T_3.

where

$$r_1 : \begin{cases} x_1 \mapsto x_3, \\ x_2 \mapsto \mu_1(x_1), \\ x_3 \mapsto x_2, \end{cases} \tag{5}$$

$$r_2 : \begin{cases} x_1 \mapsto x_2, \\ x_2 \mapsto \mu_3\mu_1(x_3), \\ x_3 \mapsto \mu_1(x_1), \end{cases} \tag{6}$$

$$\sigma : \begin{cases} x_1 \mapsto x_2, \\ x_2 \mapsto x_1, \\ x_3 \mapsto x_3. \end{cases} \tag{7}$$

Thus $\mathrm{Aut}(E_{\mathcal{A}}) \cong \mathbb{H}_{2,1} \rtimes \mathbb{Z}_2$.

Example 3.11. Let $\mathcal{A}$ be a cluster algebra from a once punctured torus, which we call a cluster algebra of type T_3; then it is of finite mutation type with quiver always isomorphic to the quiver in Figure 7. By Lemma 2.13, we have $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Aut}(E_{\mathcal{A}})$.

Corollary 3.12. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two cluster algebras of finite type, or of skew-symmetric finite mutation type, with rank equal to* 2 *or* 3. *Let $\Sigma = (\boldsymbol{x}, B)$ and $\Sigma' = (\boldsymbol{x}', B')$ be two seeds of $\mathcal{A}$ and $\mathcal{A}'$, respectively. If $N(\ell^k_\Sigma) = N(\ell^k_{\Sigma'})$ for any $k \in \mathbb{Z}_{\geq 0}$, then there exists an isomorphism $\Phi : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ such that $\Phi(\boldsymbol{x}) = \boldsymbol{x}'$.*

Proof. This follows from Examples 3.6, 3.7, 3.10 and 3.11. □

We expect the result in the corollary to be true for any finite type cluster algebras and finite mutation type cluster algebras. This means that for any seed Σ, the set $N(\ell^k_\Sigma)$ characterizes the exchange graph.

Lemma 3.13. *Let Q be a connected skew-symmetric quiver of finite mutation type.*

(1) *If there are* 3 *vertices in Q, then Q is one of the following types*:

  (a) *A_3 type.*
  (b) *$\tilde{A}_2$ type.*
  (c) *T_3 type.*

(2) *If there are at least four vertices in Q, then any full subquiver of Q with three vertices is of type A_3 or of type $\tilde{A}_2$.*

Proof. (1) From the classification of cluster algebras of finite mutation type, Q must be block-decomposable, so the proof is a straightforward check by gluing the blocks in Figure 2.

(2) We only need to notice that a quiver of type T_3 is obtained by gluing two blocks of type II in Figure 2, and thus one cannot further glue it with a block to obtain a connected quiver of finite mutation type. □

It is clear that if for any quiver in the mutation equivalent class of Q the number of arrows between any two vertices is at most 2, then Q is of finite mutation type. The above lemma shows that the inverse statement is also true for the cases when there are at least three vertices; that is, we have the following corollary, which has been stated in [Derksen and Owen 2008, Corollary 8].

Corollary 3.14. *A connected quiver Q with at least three vertices is of finite mutation type if and only if for any quiver in its mutation class the number of arrows between any two vertices is at most* 2.

Proposition 3.15. *Let Q be a connected skew-symmetric quiver with three vertices which is of finite mutation type. Let $\Sigma = (\boldsymbol{x}, Q)$ and $\Sigma' = (\boldsymbol{x}', Q')$ be two seeds. If there is an isomorphism $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ such that $\sigma(\Sigma) = \Sigma'$, then $Q' \cong Q$ or $Q' \cong Q^{\mathrm{op}}$.*

Proof. Similar to Proposition 3.9, this follows from Lemma 3.13 and Examples 3.11, 3.6 and 3.10. □

3C. *General cases.*

Theorem 3.16. *Let $\mathcal{A}$ be a cluster algebra of finite type. Assuming that it is not of type F_4, let $\Sigma = (\boldsymbol{x}, Q)$ be a labeled seed of $\mathcal{A}$, where Q is a connected quiver with at least three vertices. Then we have* $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Aut}(E_{\mathcal{A}})$.

Proof. We need to show that $\mathrm{Aut}(E_{\mathcal{A}}) \subseteq \mathrm{Aut}(\mathcal{A})$. Let Φ be any automorphism of $E_{\mathcal{A}}$. Then it induces an automorphism ϕ of the complex Δ, in particular, which gives a permutation on the cluster variable set $\mathscr{X}$. Let $\boldsymbol{x}' \subseteq \boldsymbol{x}$ be a 3-dimensional complex, and let $Q(\boldsymbol{x}')$ be the full subquiver of $Q(\boldsymbol{x})$ with vertices indexed by the variables in $\boldsymbol{x}'$. Let $\mathcal{A}'$ be the cluster algebra defined by the seed $\Sigma' = (\boldsymbol{x}', Q(\boldsymbol{x}'))$.

Notice that since ϕ is an automorphism of a complex, it maps a simplex to a simplex, and thus induces a bijection from $\mathscr{X}_{\boldsymbol{x}'} = \{\alpha \in \mathscr{X} - (\boldsymbol{x} \setminus \boldsymbol{x}') : \boldsymbol{x} \setminus \boldsymbol{x}' \cup \{\alpha\} \in \Delta\}$ to $\mathscr{X}_{\phi(\boldsymbol{x}')} = \{\alpha \in \mathscr{X} - \phi(\boldsymbol{x} \setminus \boldsymbol{x}') : \phi(\boldsymbol{x} \setminus \boldsymbol{x}') \cup \{\alpha\} \in \Delta\}$, and also induces an isomorphism $\phi_{\boldsymbol{x}'}$ from the link $\Delta_{\boldsymbol{x}'}$ to the link $\Delta_{\phi(\boldsymbol{x}')}$. Moreover, the duality of the isomorphism $\phi_{\boldsymbol{x}'}$ gives an isomorphism between the dual graphs of the complexes; that is, we have an isomorphism

$$\Phi_{\boldsymbol{x}'} : \Gamma_{\boldsymbol{x}'} \to \Gamma_{\phi(\boldsymbol{x}')}. \tag{8}$$

Let $\overline{\Sigma}' = (\phi(\boldsymbol{x}'), Q(\phi(\boldsymbol{x}')))$ be a seed, where $Q(\phi(\boldsymbol{x}'))$ is the full subquiver of $Q(\Phi(\boldsymbol{x}))$ whose vertices are those labeled by elements in $\phi(\boldsymbol{x}')$. Let $\overline{\mathcal{A}}'$ be the cluster algebra defined by $\overline{\Sigma}'$. As shown in the beginning of Section 3A, there are isomorphisms $\Gamma_{\boldsymbol{x}'} \cong E_{\mathcal{A}'}$ and $\Gamma_{\phi(\boldsymbol{x}')} \cong E_{\overline{\mathcal{A}}'}$. Combining these with the isomorphism (8), we have $E_{\mathcal{A}'} \cong E_{\overline{\mathcal{A}}'}$. Since $\mathcal{A}$ is not of type F_4, if $Q(\boldsymbol{x}')$ is of type B_3 (resp. type C_3), then $Q(\phi(\boldsymbol{x}'))$ is not of type C_3 (resp. type B_3). Thus by Proposition 3.9, $Q(\boldsymbol{x}')) \cong Q(\phi(\boldsymbol{x}'))$ or $Q(\boldsymbol{x}') \cong Q(\phi(\boldsymbol{x}'))^{\mathrm{op}}$.

Let $\boldsymbol{x}' = \{x_1, x_2, x_3\} \subseteq \boldsymbol{x}$ and $\boldsymbol{x}'' = \{x_2, x_3, x_4\} \subseteq \boldsymbol{x}$ be two 3-dimensional complexes with exactly two common elements. By the above discussion, we have $Q(\boldsymbol{x}')) \cong Q(\phi(\boldsymbol{x}'))$ or $Q(\boldsymbol{x}') \cong Q(\phi(\boldsymbol{x}'))^{\mathrm{op}}$, and $Q(\boldsymbol{x}'')) \cong Q(\phi(\boldsymbol{x}''))$ or $Q(\boldsymbol{x}'') \cong Q(\phi(\boldsymbol{x}''))^{\mathrm{op}}$. Now assume $b_{x_2x_3} \neq 0$, that is, there exists at least one arrow in $Q(\boldsymbol{x})$ between the vertices labeled by x_2 and x_3, then simultaneously we have $Q(\boldsymbol{x}')) \cong Q(\phi(\boldsymbol{x}'))$ and $Q(\boldsymbol{x}'')) \cong Q(\phi(\boldsymbol{x}''))$, or $Q(\boldsymbol{x}') \cong Q(\phi(\boldsymbol{x}'))^{\mathrm{op}}$ and $Q(\boldsymbol{x}'') \cong Q(\phi(\boldsymbol{x}''))^{\mathrm{op}}$. Finally, due to the arbitrariness of the choice of $\boldsymbol{x}'$ and the connectedness of the quiver, one may show that $Q(\Phi(\boldsymbol{x})) \cong Q$ or $Q(\Phi(\boldsymbol{x})) \cong Q^{\mathrm{op}}$. See the inductive process in the following picture:

$$\cdots\ x_0 - - - \underbrace{x_1 - - - \overbrace{x_2 - - - x_3}^{\boldsymbol{x}''}}_{\boldsymbol{x}'} \underbrace{- - - x_4}_{\boldsymbol{x}'''} - - - x_5 - - - x_6 \cdots$$

Therefore $\Phi : E_{\mathcal{A}} \to E_{\mathcal{A}}$ induces a cluster automorphism of $\mathcal{A}$ by Lemma 2.13. Thus $\mathrm{Aut}(E_{\mathcal{A}}) \subseteq \mathrm{Aut}(\mathcal{A})$ and we have $\mathrm{Aut}(E_{\mathcal{A}}) \cong \mathrm{Aut}(\mathcal{A})$. □

Remark 3.17. By combining the above theorem, Table 1 in [Assem et al. 2012] and Theorem 3.5 in [Chang and Zhu 2016a], we may compute the automorphism groups of the exchange graphs of cluster algebras of finite type; see Table 1. The cases of rank 2 and type F_4 are computed in Examples 3.3 and 3.8, respectively.

Theorem 3.18. *Let $\mathcal{A}$ be a connected skew-symmetric cluster algebra of finite mutation type*; *then* $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{Aut}(E_{\mathcal{A}})$.

Proof. If $\mathcal{A}$ is of finite type of rank 2, that is, of type A_2, then the result follows from Example 3.3. If $\mathcal{A}$ is of infinite type of rank 2, the result follows from Example 3.4. When the rank of $\mathcal{A}$ is at least 3, the proof is similar to the proof of Theorem 3.16 by using the connectedness of the cluster algebra and Proposition 3.15. □

Corollary 3.19. *Let $\mathcal{A}$ be a connected cluster algebra of finite type or of skew-symmetric finite mutation type*, *then an automorphism of $E_{\mathcal{A}}$ is determined by the image of any fixed seed Σ and the images of the seeds adjacent to Σ. More precisely*, *let $\Sigma = (\boldsymbol{x}, B)$ be a seed on $E_{\mathcal{A}}$, then an automorphism $\Phi : E_{\mathcal{A}} \to E_{\mathcal{A}}$ is determined by a pair (Σ', ϕ), where $\Sigma' = (\boldsymbol{x}', B')$ is a seed on $E_{\mathcal{A}}$ and $\phi : \boldsymbol{x} \to \boldsymbol{x}'$ is a bijection such that $\Phi(\Sigma) = \Sigma'$ and $\Phi(\mu_x(\boldsymbol{x})) = \mu_{\phi(x)}(\boldsymbol{x}')$ for any $x \in \boldsymbol{x}$.*

Proof. If $\mathcal{A}$ is of finite type of rank 2 and of type F_4, then the conclusion is clear. Otherwise, note that a cluster automorphism is determined by such a pair (Σ', ϕ); thus the proof follows from Theorems 3.16 and 3.18. □

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