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A systematic geometric theory for the ultradifferentiable (nonquasianalytic and quasianalytic) wavefront set similar to the well-known theory in the classic smooth and analytic setting is developed. In particular an analogue of Bony's theorem and the invariance of the ultradifferentiable wavefront set under diffeomorphisms of the same regularity is proven using a theorem of Dynkin about the almost-analytic extension of ultradifferentiable functions. Furthermore, we prove a microlocal elliptic regularity theorem for operators defined on ultradifferentiable vector bundles. As an application, we show that Holmgren's theorem and several generalizations hold for operators with quasianalytic coefficients.

1. Introduction

The aim of this work is to establish a geometric theory for the wavefront set in ultradifferentiable classes introduced by Hörmander [1971a] analogous to the one for the classical wavefront set. There are a number of recent works dealing with this question; see, e.g., [Adwan and Hoepfner 2015; Berhanu and Hailu 2017; Hoepfner and Medrado 2018]. In this paper we present a unified approach to the problem, which also allows us to treat quasianalytic classes, which the methods introduced up to now were not able to cover. We note that the geometric theory of the ultradifferentiable wavefront set developed here has numerous possible applications, including for example to problems studied by Baouendi and Métivier [1982], Berhanu, Cordaro and Hounie [Berhanu et al. 2008] or Castellanos, Cordaro and Petronilho [Castellanos et al. 2013].

Regarding questions of the regularity of solutions of PDEs, the wavefront set is a crucial notion introduced by Sato [1970] in the analytic category and by

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Hörmander [1971b] in the smooth case. Their refinement of the singular support simplifies, for example, the proof of the classical elliptic regularity theorem considerably.

One of the basic features of both the smooth and analytic wavefront sets is that they are invariant under smooth and real-analytic changes of coordinates, respectively. Hence it is possible to define the smooth (or analytic) wavefront set of a distribution given on a smooth (or analytic) manifold. This is mainly due to the fact the smooth (resp. analytic) wavefront set can either be described by the Fourier transform (Hörmander's approach), boundary values of almost analytic (resp. holomorphic) functions (Sato's definition) or by the FBI transform (due to Bros and Iagolnitzer [1975]). The proof of the equivalence of these descriptions in the analytic category is due to Bony [1977].

Various other notions of wavefront sets associated to microlocalizable structures have since then been introduced; e.g., for Sobolev spaces, see, e.g., Lerner [2010]. In this paper we are interested in ultradifferentiable classes, that is, spaces of smooth functions which include strictly all real analytic functions. The most well known example of such classes are the Gevrey classes; see, e.g., [Rodino 1993].

Generally, spaces of ultradifferentiable functions are defined by putting growth conditions either on the derivatives or the Fourier transform of its elements. One family of ultradifferentiable classes, which includes the Gevrey classes, is the category of Denjoy–Carleman classes. The elements of a Denjoy–Carleman class satisfy generalized Cauchy estimates of the form

$$|\partial^\alpha f(x)| \leq Ch^{|\alpha|}M_{|\alpha|}$$

on compact sets, where C and h are constants independent of α and $\mathcal{M} = (M_j)_j$ is a sequence of positive real numbers, the weight sequence associated to the Denjoy–Carleman class. Such classes of smooth functions were first investigated by Borel and Hadamard, but were named after Denjoy and Carleman who characterized independently the quasianalyticity of such a class using its weight sequence; see the survey [Thilliez 2008].

There is a rich literature concerning the Denjoy–Carleman classes and their properties. It turns out that conditions on the weight sequence translate to stability conditions of the associated class. For example, if $\mathcal{M}$ is a regular weight sequence in the sense of Dynkin [1976], then it is known that the Denjoy–Carleman class is closed under composition and solving ordinary differential equations and that the implicit function theorem holds in the class; see, e.g., [Bierstone and Milman 2004]. Hence it makes sense in this situation to consider manifolds of Denjoy–Carleman type. In Section 2 we give a brief introduction in the modern theory of Denjoy–Carleman classes and include a survey of the statements from the literature that are needed later on for the convenience of the reader. We note also that using the theory it is straightforward to generalize Nagano's theorem [1966] to orbits of quasianalytic vector fields.

There have been several attempts to define wavefront sets with respect to Denjoy–Carleman classes; see, e.g., [Komatsu 1991] and [Chung and Kim 1997]. The definition in the latter uses the FBI transform and also allows us to define $\mathrm{WF}_{\mathcal{M}} u$ for ultradistributions u in the nonquasianalytic case; see, e.g., [Adwan and Hoepfner 2010]. But if we restrict ourselves to distributions, the most wide-reaching definition of an ultradifferentiable wavefront set both with respect to the conditions imposed on the weight sequence and scope of achieved results was given by Hörmander [1971a] utilizing the Fourier transform. Due to the relatively weak conditions that he imposed on the weight sequence, Hörmander was only able to define the ultradifferentiable wavefront set $\mathrm{WF}_{\mathcal{M}} u$ of distributions u on real-analytic manifolds but not distributions defined on general ultradifferentiable manifolds. Hörmander's results are reviewed in Section 3.

The main result we need in order to proceed is a theorem of Dynkin [1976]. He showed that for regular weight sequences each function in a regular Denjoy–Carleman class has an almost-analytic extension, whose $\bar{\partial}$-derivative satisfies near $\operatorname{Im} z = 0$ a certain exponential decrease in terms of the weight sequence. We apply this result and several statements of Hörmander [1983] in Section 4 to prove that the Denjoy–Carleman wavefront set can be characterized by such $\mathcal{M}$-almost-analytic extensions. Using this characterization it is possible to modify Hörmander's proof of the invariance of the wavefront set in the real-analytic case to show that in our situation the ultradifferentiable wavefront set for distributions on Denjoy–Carleman manifolds can be well defined.

In Section 5 we show that $\mathrm{WF}_{\mathcal{M}} u$ can be characterized by the generalized FBI transform introduced by Berhanu and Hounie [2012]. This shows, in the case of distributions, the equivalence of the wavefront set introduced by Kim and Chung with the definition of Hörmander and generalizes, in that situation, results of Berhanu and Hailu [2017] and Hoepfner and Medrado [2018], especially to quasianalytic classes.

We may note that if we combine our methods with the arguments in [Hoepfner and Medrado 2018] then it is possible to generalize the above results to ultradistributions, in particular the invariance of the wavefront set on ultradifferentiable manifolds. In fact, it should be possible to give variants for ultradistributions of most of the statements that are proven in this paper. However, in order to give a unified presentation, especially regarding the assumptions on the weight sequence $\mathcal{M}$, we consider here only distributions.

As mentioned in the beginning, one of the fundamental results regarding the classical wavefront set is the elliptic regularity theorem which states in its microlocal form that we have for all partial differential operators P with smooth coefficients that $\mathrm{WF}\, u \subseteq \mathrm{WF}\, Pu \cup \mathrm{Char}\, P$, where $\mathrm{Char}\, P$ is the set of characteristic points of P, for all distributions u. Similarly Hörmander proved that $\mathrm{WF}_{\mathcal{M}}\, u \subseteq \mathrm{WF}_{\mathcal{M}}\, u \cup \mathrm{Char}\, P$ holds for operators with real-analytic coefficients. However, recently several authors, e.g., Albanese, Jornet and Oliaro[Albanese et al. 2010] and Pilipović, Teofanov and Tomić [Pilipović et al. 2018], have used the pattern of Hörmander's proof to show this inclusion for ultradifferentiable wavefront sets and operators with ultradifferentiable coefficients for variously defined ultradifferentiable classes.

Arguing similarly, we prove in Section 6 that, if $\mathcal{M}$ is a regular weight sequence that satisfies an additional condition, which is usually referred to in the literature as *moderate growth*, (see, e.g., [Thilliez 2008]), then $\mathrm{WF}_{\mathcal{M}}\, u \subseteq \mathrm{WF}_{\mathcal{M}}\, Pu \cup \mathrm{Char}\, P$ for operators P with coefficients in the Denjoy–Carleman class associated to $\mathcal{M}$. In fact, we show this inclusion for operators with ultradifferentiable coefficients acting on distributional sections of ultradifferentiable vector bundles.

Following the approach given separately by Kawai [Sato et al. 1973] and Hörmander [1971a] in the analytic case, we use the elliptic regularity theorem in Section 7 to prove a generalization of Holmgren's uniqueness theorem to operators with coefficients in quasianalytic Denjoy–Carleman classes. Finally we give quasianalytic versions of the generalizations of the analytic Holmgren's theorem due to Bony [1976], Hörmander [1993], Sjöstrand [1982] and Zachmanoglou [1972].

2. Denjoy–Carleman classes

Throughout this article, Ω denotes an open subset of $\mathbb{R}^n$. A *weight sequence* is a sequence of positive real numbers $(M_j)_{j\in\mathbb{N}_0}$ such that

$$M_0 = 1,$$
$$M_j^2 \le M_{j-1}M_{j+1}, \quad j \in \mathbb{N}.$$

Definition 2.1. Let $\mathcal{M} = (M_j)_j$ be a weight sequence. We say that a smooth function $f \in \mathcal{E}(\Omega)$ is *ultradifferentiable of class* $\{\mathcal{M}\}$ if and only if for every compact set $K \Subset \Omega$ there exist constants C and h such that for all multi-indices $\alpha \in \mathbb{N}_0^n$,

$$|D^\alpha f(x)| \le C h^{|\alpha|} M_{|\alpha|}, \qquad x \in K. \tag{2-1}$$

We denote the space of ultradifferentiable functions of class $\{M\}$ on Ω as $\mathcal{E}_{\mathcal{M}}(\Omega)$. Note that $\mathcal{E}_{\mathcal{M}}(\Omega)$ is always a subalgebra of $\mathcal{E}(\Omega)$ [Komatsu 1973].

Example 2.2. For any $s \ge 0$ consider the sequence $\mathcal{M}^s = ((k!)^{s+1})_k$. The space of ultradifferentiable functions associated to $\mathcal{M}^s$ is the well-known space of Gevrey functions $\mathcal{G}^{s+1} = \mathcal{E}_{\mathcal{M}^s}$ of order $s+1$; see, e.g., [Rodino 1993]. If $s=0$ then $\mathcal{G}^1 = \mathcal{E}_{\mathcal{M}^0} = \mathcal{O}$ is the space of real-analytic functions.

Remark 2.3. It is easy to see that $\mathcal{E}_{\mathcal{M}}(\Omega)$ is an infinite-dimensional vector space, since it contains all polynomials. In fact $\mathcal{E}_{\mathcal{M}}(\Omega)$ is a complete locally convex space; see, e.g., [Komatsu 1973]. The topology on $\mathcal{E}_{\mathcal{M}}(\Omega)$ is defined as follows. If $K \Subset \Omega$ is a compact set such that $K = \overline{K^\circ}$ then we define for $f \in \mathcal{E}(K)$,

$$\|f\|_K^h := \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \left| \frac{D^\alpha f(x)}{h^{|\alpha|} M_{|\alpha|}} \right|$$

and set

$$\mathcal{E}_{\mathcal{M}}^h(K) := \{f \in \mathcal{E}(K) \mid \|f\|_K^h < \infty\}.$$

It is easy to see that $\mathcal{E}_{\mathcal{M}}^h(K)$ is a Banach space. Moreover, $\mathcal{E}_{\mathcal{M}}^h(K) \subsetneq \mathcal{E}_{\mathcal{M}}^k(K)$ for $h < k$ and the inclusion mapping $\iota_h^k : \mathcal{E}_{\mathcal{M}}^h(K) \to \mathcal{E}_{\mathcal{M}}^k(K)$ is compact. Hence the space

$$\mathcal{E}_{\mathcal{M}}(K) := \{f \in \mathcal{E}(K) \mid \text{there exists } h > 0 \text{ such that } \|f\|_K^h < \infty\} = \varinjlim_h \mathcal{E}_{\mathcal{M}}^h(K)$$

is an (LB)-space. We can now write

$$\mathcal{E}_{\mathcal{M}}(\Omega) = \varprojlim_K \mathcal{E}_{\mathcal{M}}(K)$$

as a projective limit. For more details on the topological structure of $\mathcal{E}_{\mathcal{M}}(\Omega)$, see [Komatsu 1973].

We also call $\mathcal{E}_{\mathcal{M}}(\Omega)$ the *Denjoy–Carleman class on* Ω *associated to the weight sequence* $\mathcal{M}$.

If $\mathcal{M}$ and $\mathcal{N}$ are two weight sequences then

$$\mathcal{M} \preccurlyeq \mathcal{N} :\Longleftrightarrow \sup_{k\in\mathbb{N}_0} \left(\frac{M_k}{N_k}\right)^{1/k} < \infty$$

defines a reflexive and transitive relation on the space of weight sequences. Furthermore it induces an equivalence relation by setting

$$\mathcal{M} \approx \mathcal{N} :\Longleftrightarrow \mathcal{M} \preccurlyeq \mathcal{N} \text{ and } \mathcal{N} \preccurlyeq \mathcal{M}.$$

It holds that $\mathcal{E}_{\mathcal{M}} \subseteq \mathcal{E}_{\mathcal{N}}$ if and only if $\mathcal{M} \preccurlyeq \mathcal{N}$ and $\mathcal{E}_{\mathcal{M}} = \mathcal{E}_{\mathcal{N}}$ if and only if $\mathcal{M} \approx \mathcal{N}$; see [Mandelbrojt 1952] and also [Rainer and Schindl 2014; Thilliez 2008]. For example, if $r < s$ then $\mathcal{G}^{r+1} \subsetneq \mathcal{G}^{s+1}$.

The weight function $\omega_{\mathcal{M}}$ (see [Mandelbrojt 1952; Komatsu 1973]) associated to the weight sequence $\mathcal{M}$ is defined by

$$\omega_{\mathcal{M}}(t) := \sup_{j\in\mathbb{N}_0} \log \frac{t^j}{M_j}, \quad t > 0,$$
$$\omega_{\mathcal{M}}(0) := 0.$$

We note that $\omega_{\mathcal{M}}$ is a continuous increasing function on $[0, \infty)$ and vanishes on the interval [0, 1], and $\omega_{\mathcal{M}} \circ \exp$ is convex. In particular $\omega_{\mathcal{M}}(t)$ increases faster than $\log t^p$ for any $p > 0$ as t tends to infinity. It is possible to extract the weight sequence from the weight function, i.e.,

$$M_k = \sup_t \frac{t^k}{e^{\omega_{\mathcal{M}}(t)}};$$

see [Mandelbrojt 1952; Komatsu 1973].

If f and g are two continuous functions defined on $[0, \infty)$ then we write $f \sim g$ if and only if $f(t) = O(g(t))$ and $g(t) = O(f(t))$ for $t \to \infty$. It can be shown that the weight function ω_s for the Gevrey space $\mathcal{G}^{s+1}$ satisfies

$$\omega_s(t) \sim t^{1/(s+1)}.$$

Sometimes the classes $\mathcal{E}_{\mathcal{M}}$ are defined using the sequence $m_k = M_k/(k!)$ instead of $(M_k)_k$ and (2-1) is replaced by

$$|D^\alpha f(x)| \leq C h^{|\alpha|} |\alpha|! m_{|\alpha|}.$$

Infrequently the sequences $\mu_k = M_{k+1}/M_k$ or $L_k = M_k^{1/k}$ are also used, with an accordingly modified version of (2-1); see also Remark 3.3. The main reason for the different ways of defining the Denjoy–Carleman classes is the following. In order to show that these classes satisfy certain properties, like the inverse function theorem, one has to put certain conditions on the defining data of the spaces, i.e., the weight sequence; see, e.g., [Rainer and Schindl 2016]. Often these conditions are easier to write down in terms of these other sequences instead of using $(M_j)_j$. In the following our point of view is that the sequences $(M_k)_k$, $(m_k)_k$, $(\mu_k)_k$ and $(L_k)_k$ are all associated to the weight sequence $\mathcal{M}$. We are going to use especially the two sequences $(m_j)_j$ and $(M_j)_j$ indiscriminately.

We may note that sometimes ultradifferentiable functions associated to the weight sequence $\mathcal{M}$ are defined as smooth functions satisfying (2-1) for all $h > 0$ on each compact K; see, e.g., [Ehrenpreis 1970]. One says then that f is ultradifferentiable of class $(\mathcal{M})$ and the corresponding space is the Beurling class associated to $\mathcal{M}$. On the other hand $\mathcal{E}_{\mathcal{M}}$ is then usually called the Roumieu class associated to $\mathcal{M}$; see, e.g., [Komatsu 1973; Rainer and Schindl 2016].

From now on we shall put certain conditions on the weight sequences under consideration.

Definition 2.4. We say that a weight sequence $\mathcal{M}$ is *regular* if and only if it satisfies the following conditions, with $k \in \mathbb{N}$:

(M1) $$m_0 = m_1 = 1.$$

(M2) $$\sup_k \sqrt[k]{\frac{m_{k+1}}{m_k}} < \infty.$$

(M3) $$m_k^2 \le m_{k-1} m_{k+1}.$$

(M4) $$\lim_{k\to\infty} \sqrt[k]{m_k} = \infty.$$

The last condition just means that the space $\mathcal{O}$ of all real-analytic functions is strictly contained in $\mathcal{E}_{\mathcal{M}}$ whereas the first is a useful normalization condition that will help simplify certain computations. It is obvious that if we replace in (M1) the number 1 with some other positive real number we would not change the resulting space $\mathcal{E}_{\mathcal{M}}$.

If $\mathcal{M}$ is a regular weight sequence, then it is well known that the associated Denjoy–Carleman class satisfies certain stability properties; see, e.g., [Bierstone and Milman 2004; Rainer and Schindl 2016]. For example $\mathcal{E}_{\mathcal{M}}$ is *closed under differentiation*, i.e., if $f \in \mathcal{E}_{\mathcal{M}}(\Omega)$ then $D^\alpha f \in \mathcal{E}_{\mathcal{M}}(\Omega)$ for all $\alpha \in \mathbb{N}_0^n$.

Remark 2.5. The fact that $\mathcal{E}_{\mathcal{M}}(\Omega)$ is closed under differentiation implies immediately another stability condition, namely *closedness under division by a coordinate* (see [Bierstone and Milman 2004]):

Suppose that $f \in \mathcal{E}_{\mathcal{M}}(\Omega)$ and $f(x_1, \dots, x_{j-1}, a, x_{j+1}, \dots, x_n) = 0$ for some fixed $a \in \mathbb{R}$ and all $x_k,\ k \neq j$, with the property $(x_1, \dots, x_{j-1}, a, x_{j+1}, \dots, x_n) \in \Omega$. Then we apply the fundamental theorem of calculus to the function

$$f_j : t \longmapsto f(x_1, \dots, x_{j-1}, t(x_j - a) + a, x_{j+1}, \dots, x_n)$$

and obtain

$$\begin{aligned} f(x) &= \int_0^1 \frac{\partial f_j}{\partial t}(t)\, dt \\ &= (x_j - a) \int_0^1 \frac{\partial f}{\partial x_j}(x_1, \dots, x_{j-1}, t(x_j - a) + a, x_{j+1}, \dots, x_n)\, dt \\ &= (x_j - a) g(x). \end{aligned}$$

It is easy to see that $g \in \mathcal{E}_{\mathcal{M}}(\Omega)$ using $\partial f / \partial x_j \in \mathcal{E}_{\mathcal{M}}(\Omega)$.

For the proof of the properties above, only (M2) was used. If we apply also (M3) then it is possible to show that $\mathcal{E}_{\mathcal{M}}(\Omega)$ is *inverse closed*, i.e., if $f \in \mathcal{E}_{\mathcal{M}}(\Omega)$ does not vanish at any point of Ω then

$$\frac{1}{f} \in \mathcal{E}_{\mathcal{M}}(\Omega);$$

see [Rainer and Schindl 2016].

In fact, if $\mathcal{M}$ is a regular weight sequence then the associated Denjoy–Carleman class satisfies also the following stability properties.

Theorem 2.6. *Let $\mathcal{M}$ be a regular weight sequence and $\Omega_1 \subseteq \mathbb{R}^m$ and $\Omega_2 \subseteq \mathbb{R}^n$ be open sets. Then the following holds*:

(1) *The class $\mathcal{E}_\mathcal{M}$ is **closed under composition** (see [Roumieu 1962] and also [Bierstone and Milman 2004]), i.e., let $F\colon \Omega_1 \to \Omega_2$ be an $\mathcal{E}_\mathcal{M}$-mapping, that is, each component F_j of F is ultradifferentiable of class $\{\mathcal{M}\}$ in Ω_1, and $g \in \mathcal{E}_\mathcal{M}(\Omega_2)$. Then also $g \circ F \in \mathcal{E}_\mathcal{M}(\Omega_1)$.*

(2) *The **inverse function theorem** holds in the Denjoy–Carleman class $\mathcal{E}_\mathcal{M}$ (see [Komatsu 1979]): Let $F : \Omega_1 \to \Omega_2$ be an $\mathcal{E}_\mathcal{M}$-mapping and $p_0 \in \Omega_1$ such that the Jacobian $F'(p_0)$ is invertible. Then there exist neighborhoods U of p_0 in Ω_1 and V of $q_0 = F(x_0)$ in Ω_2 and an $\mathcal{E}_\mathcal{M}$-mapping $G : V \to U$ such that $G(q_0) = p_0$ and $F \circ G = \mathrm{id}_V$.*

(3) *The **implicit function theorem** is valid in $\mathcal{E}_\mathcal{M}$ (see [Komatsu 1979]): Let $F : \mathbb{R}^{n+d} \supseteq \Omega \to \mathbb{R}^d$ be an $\mathcal{E}_\mathcal{M}$-mapping and $(x_0, y_0) \in \Omega$ such that $F(x_0, y_0) = 0$ and $\partial F/\partial y(x_0, y_0)$ is invertible. Then there exist open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^d$ with $(x_0, y_0) \in U \times V \subseteq \Omega$ and an $\mathcal{E}_\mathcal{M}$-mapping $G\colon U \to V$ such that $G(x_0) = y_0$ and $F(x, G(x)) = 0$ for all $x \in V$.*

Furthermore it is true that $\mathcal{E}_\mathcal{M}(\Omega)$ is *closed under solving ODEs*; to be more specific, the following theorem holds.

Theorem 2.7 (Yamanaka [1991]; see also Komatsu [1980]). *Let $\mathcal{M}$ be a regular weight sequence, $0 \in I \subseteq \mathbb{R}$ an open interval, $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^d$ be open and $F \in \mathcal{E}_\mathcal{M}(I \times U \times V)$.*

Then the initial value problem

$$\begin{aligned} x'(t) &= F(t, x(t), \lambda), \quad && t \in I,\ \lambda \in V, \\ x(0) &= x_0, && x_0 \in U, \end{aligned}$$

has locally a unique solution x that is ultradifferentiable near 0.

More precisely, there is an open set $\Omega \subseteq I \times U \times V$ that contains the point $(0, x_0, \lambda)$ and an $\mathcal{E}_\mathcal{M}$-mapping $x = x(t, y, \lambda) : \Omega \to U$ such that the function $t \mapsto x(t, y_0, \lambda_0)$ is the solution of the initial value problem

$$\begin{aligned} x'(t) &= F(t, x(t), \lambda_0), \\ x(0) &= y_0. \end{aligned}$$

For any regular weight sequence $\mathcal{M}$ we can define the associated weight by

$$h_\mathcal{M}(t) = \inf_k t^k m_k \quad \text{if } t > 0 \text{ and } h_\mathcal{M}(0) = 0. \tag{2-2}$$

As above, we have that

$$m_k = \sup_t \frac{h_\mathcal{M}(t)}{t^k}.$$

In order to describe the connection between the weight and the weight function associated to a regular weight sequence we set

$$\tilde{\omega}_{\mathcal{M}}(t) := \sup_{j\in\mathbb{N}_0} \log \frac{t^j}{m_j},$$
$$\tilde{h}_{\mathcal{M}}(t) = \inf_k t^k M_k,$$

for $t > 0$ and $\tilde{\omega}_{\mathcal{M}}(0) = \tilde{h}_{\mathcal{M}}(0) = 0$.

Lemma 2.8. *If $\mathcal{M}$ is a regular weight sequence then*

$$\begin{aligned} h_{\mathcal{M}}(t) &= e^{-\tilde{\omega}_{\mathcal{M}}(1/t)}, \\ \tilde{h}_{\mathcal{M}}(t) &= e^{-\omega_{\mathcal{M}}(1/t)}. \end{aligned} \tag{2-3}$$

Proof. We prove only the equality for $h_{\mathcal{M}}$. Of course, the verification of the other equation is completely analogous. If $t > 0$ is chosen arbitrarily we have by the monotonicity of the exponential function that

$$\exp\left(\tilde{\omega}_{\mathcal{M}}\left(\frac{1}{t}\right)\right) = \exp\left(\sup_k \log \frac{1}{m_k t^k}\right) = \sup_k \frac{1}{m_k t^k} = \frac{1}{\inf_k m_k t^k} = \frac{1}{h_{\mathcal{M}}(t)}. \qquad \square$$

We obtain that $h_{\mathcal{M}}$ is continuous with values in $[0, 1]$, equals 1 on $[1, \infty)$ and goes more rapidly to 0 than t^p for any $p > 0$ for $t \to 0$. Although the weight function is the prevalent concept, the weight has been used, e.g., by Dynkin [1976] and Thilliez [2003].

Example 2.9. If $\mathcal{M} = \mathcal{M}^s$ is the Gevrey sequence of order s then we know already that the associated weight function satisfies $\omega_s(t) \sim t^{1/(1+s)}$. Hence (2-3) shows for $s > 0$ that if we set

$$f_s(t) = e^{-1/t^s}$$

then there are constants C_1, C_2, Q_1 and $Q_2 > 0$ such that

$$C_1 f_s(Q_1 t) \le h_s(t) \le C_2 f_s(Q_2 t)$$

for $t > 0$.

It is well known (see, e.g., [Mather 1971] or [Melin and Sjöstrand 1975]) that a function f is smooth on Ω if and only if there is an almost-analytic extension F of f, i.e., there exists a smooth function F on some open set $\tilde{\Omega} \subseteq \mathbb{C}^n$ with $\tilde{\Omega} \cap \mathbb{R}^n = \Omega$ such that

$$\bar{\partial}_j F = \frac{\partial}{\partial \bar{z}_j} F = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right) F$$

is flat on Ω and $F|_\Omega = f$. The idea is now that if f is ultradifferentiable then one should find an extension F of f such that the regularity of f is translated in a certain uniform decrease of $\bar{\partial}_j F$ near Ω (see [Dynkin 1993]). Such extensions were constructed, e.g., by Petzsche and Vogt [1984] and Adwan and Hoepfner [2010] under relative restrictive conditions on the weight sequence. The most general result in this regard though was given by Dynkin [1976].

Theorem 2.10. *Let $\mathcal{M}$ be a regular weight sequence and $K \Subset \mathbb{R}^n$ be a compact convex set with $K = \overline{K^\circ}$. Then $f \in \mathcal{E}_\mathcal{M}(K)$ if and only if there exists a test function $F \in \mathcal{D}(\mathbb{C}^n)$ with $F|_K = f$ and if there are constants $C, Q > 0$ such that*

$$|\bar{\partial}_j F(z)| \le C h_\mathcal{M}(Q d_K(z)), \tag{2-4}$$

where $1 \le j \le n$ and d_K is the distance function with respect to K on $\mathbb{C}^n \setminus K$.

We shall note that Dynkin used the function $h_1(t) = \inf_{k\in\mathbb{N}} m_k t^{k-1}$ instead of the weight $h_\mathcal{M}$.[1] But we observe that

$$h_\mathcal{M}(t) = \inf_{k\in\mathbb{N}_0} m_k t^k \le t \inf_{k\in\mathbb{N}} m_k t^{k-1} = t h_1(t) \le Ct \inf_{k\in\mathbb{N}} m_{k-1} t^{k-1} = Ct h_\mathcal{M}(t),$$

where we used (M2). Since $h_\mathcal{M}$ is rapidly decreasing for $t \to 0$ we can interchange these two functions in the formulation of Theorem 2.10. In fact, Dynkin's proof gives immediately the following result.

Corollary 2.11. *Let $\mathcal{M}$ be a regular weight sequence, $p \in \Omega$ and $f \in \mathcal{D}'(\Omega)$. If f is ultradifferentiable of class $\{\mathcal{M}\}$ near p, i.e., there exists a compact neighborhood K of p such that $f|_K \in \mathcal{E}_\mathcal{M}(K)$, then there are an open neighborhood $W \subseteq \Omega$ of p, a constant $\rho > 0$ and a function $F \in \mathcal{E}(W + iB(0, \rho))$ such that $F|_W = f|_W$ and*

$$|\bar{\partial}_j F(x + iy)| \le C h_\mathcal{M}(Q|y|) \tag{2-5}$$

for some positive constants C, Q and all $1 \le j \le n$ and $x + iy \in W + iB(0, \rho)$.

One of the main questions in the study of ultradifferentiable functions is if the class under consideration behaves more like the ring of real-analytic functions or the ring of smooth functions. E.g., does the class contain flat functions, that means nonzero elements whose Taylor series at some point vanishes? That leads to following definition.

Definition 2.12. Let $E \subseteq \mathcal{E}(\Omega)$ be a subalgebra. We say that E is quasianalytic if and only if for $f \in E$ the fact that $D^\alpha f(p) = 0$ for some $p \in \Omega$ and all $\alpha \in \mathbb{N}_0^n$ implies that $f \equiv 0$ in the connected component of Ω that contains p.

In the case of Denjoy–Carleman classes quasianalyticity is characterized by the following theorem.

[1] h_1 is in fact the weight associated to the shifted sequence $(m_{k+1})_k$.

Theorem 2.13 [Denjoy 1921; Carleman 1923a; 1923b]. *The space* $\mathcal{E}_{\mathcal{M}}(\Omega)$ *is quasianalytic if and only if*

$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} = \infty. \tag{2-6}$$

We say that a weight sequence is quasianalytic if and only if it satisfies (2-6) and nonquasianalytic otherwise.

Example 2.14. Let $\sigma > 0$ be a parameter. We define a family $\mathcal{N}^\sigma$ of regular weight sequences by $N_0 = N_1 = 1$ and

$$N_k^\sigma = k!(\log(k+e))^{\sigma k}$$

for $k \geq 2$. The weight sequence $\mathcal{N}^\sigma$ is quasianalytic if and only if $0 < \sigma \leq 1$; see [Thilliez 2008].

Remark 2.15. Obviously $\mathcal{D}_{\mathcal{M}}(\Omega) = \mathcal{D}(\Omega) \cap \mathcal{E}_{\mathcal{M}}(\Omega)$ is nontrivial if and only if $\mathcal{E}_{\mathcal{M}}(\Omega)$ is nonquasianalytic; see, e.g., [Rudin 1966]. It is well known that the sequences $\mathcal{M}^s$ are nonquasianalytic if and only if $s > 0$. In fact there is a nonquasianalytic regular weight sequence $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{M}} \preccurlyeq \mathcal{M}^s$ for all $s > 0$; see [Rainer and Schindl 2014, page 125]. Hence

$$\mathcal{O} \subsetneq \mathcal{E}_{\tilde{\mathcal{M}}} \subsetneq \bigcap_{s>0} \mathcal{G}^{s+1}.$$

Using Theorem 2.6 we produce the following definition:

Definition 2.16. Let M be a smooth manifold and $\mathcal{M}$ a regular weight sequence. We say that M is an ultradifferentiable manifold of class $\{\mathcal{M}\}$ if and only if there is an atlas $\mathcal{A}$ of M that consists of charts such that

$$\varphi' \circ \varphi^{-1} \in \mathcal{E}_{\mathcal{M}}$$

for all $\varphi, \varphi' \in \mathcal{A}$.

A mapping $F\colon M \to N$ between two manifolds of class $\{\mathcal{M}\}$ is ultradifferentiable of class $\{\mathcal{M}\}$ if and only if $\psi \circ F \circ \varphi^{-1} \in \mathcal{E}_{\mathcal{M}}$ for any charts φ and ψ of M and N, respectively. We can now consider the category of ultradifferentiable manifolds of class $\{\mathcal{M}\}$. We denote by

$$\mathfrak{X}_{\mathcal{M}}(M) = \mathcal{E}_{\mathcal{M}}(M, TM)$$

the Lie algebra of ultradifferentiable vector fields on M. Note that, if $\mathcal{M}$ is a regular weight sequence, an integral curve of an ultradifferentiable vector field of class $\{\mathcal{M}\}$ is an $\mathcal{E}_{\mathcal{M}}$-curve by Theorem 2.7.

These considerations allow us to state a quasianalytic version of Nagano's theorem [1966].

Theorem 2.17. *Let U be an open neighborhood of $p_0 \in \mathbb{R}^n$ and $\mathcal{M}$ a quasianalytic regular weight sequence. Furthermore let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{X}_{\mathcal{M}}(U)$ that is also an $\mathcal{E}_{\mathcal{M}}$-module, i.e., if $X \in \mathfrak{g}$ and $f \in \mathcal{E}_{\mathcal{M}}(U)$ then $fX \in \mathfrak{g}$.*

There exists an ultradifferentiable submanifold W of class $\{\mathcal{M}\}$ in U, such that

$$T_pW = \mathfrak{g}(p) \quad \text{for all } p \in W. \tag{2-7}$$

Moreover, the germ of W at p_0 is uniquely defined by this property.

The proof of Theorem 2.17 is the same as in the analytic version; see, e.g., Baouendi, Ebenfelt and Rothschild [Baouendi et al. 1999]. We call the uniquely defined germ $\gamma_{p_0}(\mathfrak{g})$ of the manifold constructed in Theorem 2.17 the local Nagano leaf of $\mathfrak{g}$ at p_0. From now on all Lie algebras of ultradifferentiable vector fields that are considered are assumed to be also $\mathcal{E}_{\mathcal{M}}$-modules.

Following Nagano [1966] (see also [Baouendi et al. 1999]), we can also give a global version of Theorem 2.17.

Theorem 2.18. *Let $\mathcal{M}$ be a quasianalytic regular weight sequence. If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{X}_{\mathcal{M}}(\Omega)$ then $\mathfrak{g}$ admits a foliation of Ω, that is a partition of Ω by maximal integral manifolds.*

Before we close this section we need to introduce another condition for weight sequences. Let $\mathcal{M}$ be a weight sequence. We say that $\mathcal{M}$ is of *moderate growth* if and only if there are constants C and ρ such that

$$M_{j+k} \le C\rho^{j+k} M_j M_k \tag{M2$'$}$$

for all $(j,k) \in \mathbb{N}_0^2$. Both the Gevrey sequences $\mathcal{M}^s$ and the sequences $\mathcal{N}^\sigma$ from Example 2.14 satisfy (M2$'$) for all s and σ, respectively.

For a discussion of this condition, see, e.g., [Komatsu 1973]. Here we only mention two facts. First, for any weight sequence $\mathcal{M}$, if (M2$'$) holds then (M2) is also satisfied. Furthermore, if $\mathcal{M}$ satisfies (M2$'$) then there is some $s > 0$ such that $\mathcal{E}_{\mathcal{M}} \subseteq \mathcal{G}^{1+s}$; see, e.g., [Thilliez 2003]. On the other hand consider the regular weight sequence $\mathcal{L}$ given by $L_0 = L_1 = 1$ and $L_k = k!2^{k^2}$ for $k \ge 2$. Then $\mathcal{G}^{1+s} \subsetneq \mathcal{E}_{\mathcal{L}}$ for all $s \ge 0$ and therefore $\mathcal{L}$ cannot satisfy (M2$'$).

3. The ultradifferentiable wavefront set

In this and the following two sections we always assume that $\mathcal{M}$ is a regular weight sequence.

Hörmander [1971a] proved the following local characterization of $\mathcal{E}_{\mathcal{M}}$ via the Fourier transform:

Proposition 3.1. *Let $u \in \mathcal{D}'(\Omega)$ and $p_0 \in \Omega$. Then u is ultradifferentiable of class $\{\mathcal{M}\}$ near p_0 if and only if there are an open neighborhood V of p_0, a* *bounded sequence* $(u_N)_N \subseteq \mathcal{E}'(U)$ *such that* $u|_V = (u_N)|_V$ *and some constant* $Q > 0$ *so that*

$$\sup_{\substack{\xi\in\mathbb{R}^n\\ N\in\mathbb{N}_0}} \frac{|\xi|^N |\hat{u}_N(\xi)|}{Q^N M_N} < \infty.$$

Hörmander then used this characterization to define the ultradifferentiable wavefront set:

Definition 3.2. Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. We say that u is *microlocally ultradifferentiable of class* $\{\mathcal{M}\}$ at (x_0, ξ_0) if and only if there are a neighborhood V of x_0, a bounded sequence $(u_N)_N \subseteq \mathcal{E}'(\Omega)$ with $u_{N|V} \equiv u_{|V}$ and a conic neighborhood Γ of ξ_0 such that $u_N|_V \equiv u|_V$, where $V \in \mathcal{U}(x_0)$, and a conic neighborhood Γ of ξ_0 such that for some constant $Q > 0$

$$\sup_{\substack{\xi\in\Gamma\\ N\in\mathbb{N}_0}} \frac{|\xi|^N |\hat{u}_N|}{Q^N M_N} < \infty. \tag{3-1}$$

The ultradifferentiable wavefront set $\mathrm{WF}_{\mathcal{M}}\, u$ is then defined as

$$\mathrm{WF}_{\mathcal{M}}\, u := \big\{(x, \xi) \in T^*\Omega \setminus \{0\} \mid \\ u \text{ is not microlocally ultradifferentiable of class } \{\mathcal{M}\} \text{ at } (x, \xi)\big\}.$$

Remark 3.3. Hörmander [1971a] defined $\mathrm{WF}_{\mathcal{M}}$ for weight sequences that satisfy weaker conditions than those we imposed in Definition 2.4. He required, as we have done, (M2) and that $\mathcal{O} \subseteq \mathcal{E}_{\mathcal{M}}$, but (M3) is replaced by the monotonic growth of the sequence

$$L_N = (M_N)^{1/N}. \tag{3-2}$$

This condition still implies that $\mathcal{E}_{\mathcal{M}}$ is an algebra but gives only that $\mathcal{E}_{\mathcal{M}}$ is closed under composition with analytic mappings.

More precisely, in terms of the sequence $(L_N)_N$, the conditions that Hörmander imposed take the following form. First, $N \le L_N$ and $L_{N+1} \le C L_N$ for all N and a constant $C > 0$ independent of N. Furthermore, as mentioned before, the sequence $(L_N)_N$ is also assumed to be increasing.

Note that Hörmander's classes might not even be defined by weight sequences in the sense of Section 2. Hence Hörmander [1983] was able to define $\mathrm{WF}_{\mathcal{M}}\, u$ for distributions u on real analytic manifolds but not on arbitrary ultradifferentiable manifolds of class $\{\mathcal{M}\}$; note that the implicit function theorem may not hold in an arbitrary ultradifferentiable class defined by weight sequences obeying his conditions. Similarly he proved that

$$\mathrm{WF}_{\mathcal{M}}\, u \subseteq \mathrm{WF}_{\mathcal{M}}\, Pu \cup \mathrm{Char}\, P$$

for linear partial differential operators P with analytic coefficients but not for operators whose coefficients might be only of class $\{\mathcal{M}\}$.

As mentioned before it is possible to modify the arguments of Hörmander in the case of regular weight sequences to show that the above inclusion holds for partial differential operators with ultradifferentiable coefficients as long as $\mathcal{M}$ is regular and of moderate growth. Similarly we are able to define $\mathrm{WF}_{\mathcal{M}}\, u$ for distributions defined on manifolds of class $\{\mathcal{M}\}$ (for regular $\mathcal{M}$), in this instance using Dynkin's almost-analytic extension of ultradifferentiable functions (i.e., Corollary 2.11).

However, since regular weight sequences also fulfill the conditions of Hörmander we can use all of his results on $\mathrm{WF}_{\mathcal{M}}$. Indeed, in terms of L_N, we have that (M4) implies that $k \leq \gamma L_k$ for all $k \in \mathbb{N}_0$ and a constant $\gamma > 0$ independent of k by Sterling's formula whereas (M2) is equivalent to the existence of a constant $A > 0$ such that $L_k \leq A L_{k-1}$. We note that the last estimate implies $L_N \leq A^N$ for $N \in \mathbb{N}_0$ since $L_1 = 1$. On the other hand, it is well known that if $(M_N)_N$ satisfies (M3) then $(L_N)_N$ is an increasing sequence; see, e.g., [Mandelbrojt 1952].

The following result shows we may choose the distributions u_N in Definition 3.2 in a special manner.

Proposition 3.4 [Hörmander 1983, Lemma 8.4.4]. *Let $u \in \mathcal{D}'(\Omega)$ and let $K \subset \Omega$ be compact, let $F \subseteq \mathbb{R}^n$ be a closed cone such that $\mathrm{WF}_{\mathcal{M}}\, u \cap (K \times F) = \varnothing$. If $\chi_N \in \mathcal{D}(K)$ and for all α*

$$|D^{\alpha+\beta}\chi_N| \leq C_\alpha h_\alpha^{|\beta|} M_N^{|\beta|/N}, \quad |\beta| \leq N,$$

for some constants $C_\alpha, h_\alpha > 0$ then it follows that $\chi_N u$ is bounded in $\mathcal{E}'^S$ if u is of order S in a neighborhood of K, and further

$$|\widehat{\chi_N u}(\xi)| \leq C \frac{Q^N M_N}{|\xi|^N}, \quad N \in \mathbb{N},\ \xi \in F,$$

for some constants $C, Q > 0$.

We summarize the basic properties of $\mathrm{WF}_{\mathcal{M}}$ according to Hörmander [1983].

Here $\operatorname{sing\,supp} u \subseteq \Omega$ is defined to be the complement of the largest open subset $V \subseteq \Omega$ with $u|_V \in \mathcal{E}_{\mathcal{M}}(V)$.

Theorem 3.5 [Hörmander 1983, Theorems 8.4.5–8.4.7]. *Let $u \in \mathcal{D}'(\Omega)$ and $\mathcal{M}, \mathcal{N}$ be two weight sequences. Then:*

(1) *$\mathrm{WF}_{\mathcal{M}}\, u$ is a closed conic subset of $\Omega \times \mathbb{R}^n \backslash \{0\}$.*

(2) *The projection of $\mathrm{WF}_{\mathcal{M}}\, u$ in Ω is*

$$\pi_1(\mathrm{WF}_{\mathcal{M}}\, u) = \operatorname{sing\,supp}_{\mathcal{M}} u.$$

(3) *$\mathrm{WF}\, u \subseteq \mathrm{WF}_{\mathcal{N}}\, u \subseteq \mathrm{WF}_{\mathcal{M}}\, u$ if $\mathcal{M} \preccurlyeq \mathcal{N}$.*

(4) *If* $P = \sum p_\alpha D^\alpha$ *is a partial differential operator with ultradifferentiable coefficients of class* $\{\mathcal{M}\}$ *then* $\mathrm{WF}_{\mathcal{M}}\, Pu \subseteq \mathrm{WF}_{\mathcal{M}}\, u$.

Additionally we note that $\mathrm{WF}_{\mathcal{M}}\, u$ satisfies the following *microlocal reflection property*:

$$(x, \xi) \notin \mathrm{WF}_{\mathcal{M}}\, u \Longleftrightarrow (x, -\xi) \notin \mathrm{WF}_{\mathcal{M}}\, \bar{u}. \tag{3-3}$$

In particular, if u is a real-valued distribution, that is, $\bar{u} = u$, then $\mathrm{WF}_{\mathcal{M}}\, u|_x := \{\xi \in \mathbb{R}^n \mid (x, \xi) \in \mathrm{WF}_{\mathcal{M}}\, u\}$ is symmetric at the origin.

Example 3.6. It is easy to see that $\mathrm{WF}_{\mathcal{M}}\, \delta_p = \{p\} \times \mathbb{R}^n \backslash \{0\}$ for any regular weight sequence $\mathcal{M}$.

Remark 3.7. The complicated form of Definition 3.2 compared with the definition of the smooth wavefront set stems from the fact that quasianalytic weight sequences are allowed. Thus in general there may not be any nontrivial test functions of class $\{\mathcal{M}\}$. However if $\mathcal{D}_{\mathcal{M}} \neq \{0\}$ then we can choose in Definition 3.2 the constant sequence $u_N = \varphi u$ for some $\varphi \in \mathcal{D}_{\mathcal{M}}(\Omega)$ with $\varphi(x_0) = 1$, and (3-1) is equivalent to the existence of constants $C, Q > 0$ such that

$$|\widehat{\varphi u}(\xi)| \le C \inf_N Q^N M_N |\xi|^{-N} \quad \text{for all } \xi \in \Gamma;$$

thus (2-3) implies

$$|\widehat{\varphi u}(\xi)| \le C \tilde{h}_{\mathcal{M}}\left(\frac{Q}{|\xi|}\right) \le C \exp\left(-\omega_{\mathcal{M}}\left(\frac{|\xi|}{Q}\right)\right).$$

We conclude (see, e.g., [Rodino 1993] in the case of Gevrey-classes) that for nonquasianalytic weight sequences $\mathcal{M}$, (3-1) is equivalent to

$$\sup_{\xi \in \Gamma} e^{\omega_{\mathcal{M}}(Q|\xi|)} |\widehat{\varphi u}(\xi)| < \infty \quad \text{for some } Q > 0.$$

Proposition 3.1 is then only a restatement of the well-known fact that for nonquasianalytic weight sequences we have that $\varphi \in \mathcal{D}_{\mathcal{M}}$ if and only if $\hat{\varphi} \le C e^{-\omega_{\mathcal{M}}(Q|\xi|)}$ for some constants C, Q. Therefore it is possible to define ultradifferentiable classes using appropriately defined weight functions instead of weight sequences; see, e.g., in a somehow generalized setting, [Björck 1966]. However, this approach leads only to nonquasianalytic spaces. This restriction was removed by Braun, Meise and Taylor [Braun et al. 1990], who reformulated the defining estimates of these classes to allow also quasianalytic classes. A wavefront set relative to these classes was introduced in [Albanese et al. 2010]; see Section 6. The complicated connection between the classes defined by weight sequences and those given by weight functions was investigated in Bonet, Meise and Melikhov [Bonet et al. 2007]. Recently a new approach to define spaces of ultradifferentiable functions was introduced in [Rainer and Schindl 2014], which encompasses the classes given by weight sequences and weight functions; see also [Rainer and Schindl 2016].

4. Invariance of the wavefront set under ultradifferentiable mappings

Our aim in this section is to develop, using the almost-analytic extension of functions in $\mathcal{E}_{\mathcal{M}}$ given by Dynkin, a geometric description of $\mathrm{WF}_{\mathcal{M}}$ similar to the one that was presented, e.g., by Liess [1999, Section 4], for the smooth wavefront set.

We need to fix some notation: If $\Gamma \subseteq \mathbb{R}^d$ is a cone and $r > 0$ then

$$\Gamma_r := \{y \in \Gamma \mid |y| < r\}.$$

If $\Gamma' \subseteq \Gamma$ is also a cone we write $\Gamma' \Subset \Gamma$ if and only if $(\Gamma' \cap S^{d-1}) \Subset (\Gamma \cap S^{d-1})$.

Analogous to Liess [Liess 1999, Section 2.1] in the smooth category we say that, if $\mathcal{M}$ is a weight sequence, a function $F \in \mathcal{E}(\Omega \times U \times \Gamma_r)$, $U \subseteq \mathbb{R}^d$ open, is $\mathcal{M}$-*almost analytic* in the variables $(x, y) \in U \times \Gamma_r$ with parameter $x' \in \Omega$ if and only if for all $K \Subset \Omega$, $L \Subset U$ and cones $\Gamma' \Subset \Gamma$ there are constants $C, Q > 0$ such that for some r' we have

$$\left|\frac{\partial F}{\partial \bar{z}_j}(x', x, y)\right| \le C h_{\mathcal{M}}(Q|y|), \quad (x', x, y) \in K \times L \times \Gamma'_{r'},\ j = 1, \dots, d, \tag{4-1}$$

where $\partial/\partial \bar{z}_j = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and $h_{\mathcal{M}}$ is the weight associated to the regular weight sequence $\mathcal{M}$ as defined by (2-2).

We may also say generally that a function $g \in \mathcal{C}(\Omega \times U \times \Gamma_r)$ is of *slow growth* in $y \in \Gamma_r$ if for all $K \Subset \Omega$, $L \Subset U$ and $\Gamma' \Subset \Gamma$ there are constants $c, k > 0$ such that

$$|g(x', x, y)| \le c|y|^{-k}, \quad (x', x, y) \in K \times L \times \Gamma'_r. \tag{4-2}$$

The next theorem is a generalization of [Hörmander 1983, Theorem 4.4.8]; see [Adwan and Hoepfner 2015].

Theorem 4.1. *Let $F \in \mathcal{E}(\Omega \times U \times \Gamma_r)$ be $\mathcal{M}$-almost analytic in the variables $(x, y) \in U \times \Gamma_r$ and of slow growth in the variable $y \in \Gamma_r$. Then the distributional limit u of the sequence $u_\varepsilon = F(\,.\,,\,.\,,\varepsilon) \in \mathcal{E}(\Omega \times U)$ exists. We say that $u = b_\Gamma(F) \in \mathcal{D}'(\Omega \times U)$ is the boundary value of F. Furthermore, we have*

$$\mathrm{WF}_{\mathcal{M}}\, u \subseteq (\Omega \times U) \times (\mathbb{R}^n \times \Gamma^\circ),$$

where $\Gamma^\circ = \{\eta \in \mathbb{R}^d \mid \langle y, \eta\rangle \ge 0$ for all $y \in \Gamma\}$ is the dual cone of Γ in $\mathbb{R}^d$.

Proof. Let $\varphi \in \mathcal{D}(\Omega \times U)$ and $Y_0 \in \Gamma_\delta$. Then there are $K \Subset \Omega$ and $L \Subset U$ such that $\operatorname{supp}\varphi \subseteq K \times L$, and constants $c, k > 0$ exist such that (4-2) holds. We set

$$\Phi_\kappa(x', x, y) = \sum_{|\alpha| \le \kappa} \partial_x^\alpha \varphi(x', x) \frac{(iy)^\alpha}{\alpha!}$$

for $\kappa \ge k$. Obviously $F \cdot \Phi_\kappa$ can be extended to a smooth function on $\mathbb{R}^n \times \mathbb{R}^d \times \Gamma_\delta$

that vanishes outside $K \times L \times \Gamma_\delta$. We consider the function

$$u_\varepsilon : \mathbb{R}^2 \ni (\sigma, \tau) \longmapsto F(x', \tilde{x} + \sigma Y_0, \varepsilon + \tau Y_0)\Phi_\kappa(x', \sigma Y_0, \tau Y_0),$$

where $x' \in \mathbb{R}^n$, $\tilde{x} \in Y_0^\perp = \{z \in \mathbb{R}^d \mid \langle z, Y_0\rangle = 0\}$. If $a < b$ are chosen such that $\varphi(x', \tilde{x} + \sigma Y_0) = 0$ for all $x' \in \mathbb{R}^n$, $\tilde{x} \in Y_0^\perp$ and $\sigma \le a$ or $\sigma \ge b$ then $u_\varepsilon(\sigma, \tau) = 0$ for all $\tau \in [0, 1]$. If $R = [a, b] \times [0, 1]$ then Stokes' theorem states that

$$\int_{\partial R} u_\varepsilon \, d\zeta = \int_R \frac{\partial u_\varepsilon}{\partial \bar{\zeta}} \, d\bar{\zeta} \wedge d\zeta, \tag{4-3}$$

where we have set $\zeta = \sigma + i\tau$.

A simple computation gives

$$2i \frac{\partial}{\partial \bar{\zeta}} \big(\Phi_\kappa(x', \tilde{x} + \sigma Y_0, \tau Y_0)\big) = (\kappa + 1)\tau^\kappa \sum_{|\alpha| = \kappa+1} \partial_x^\alpha \varphi(x', \tilde{x} + \sigma Y_0) \frac{(iY_0)^\alpha}{\alpha!}.$$

Hence formula (4-3) means in detail that

$$\begin{aligned}\int_a^b & F(x', \sigma Y_0, \varepsilon)\varphi(x', \sigma Y_0)\, d\sigma \\ &= \int_a^b F(x', \sigma Y_0, \varepsilon + Y_0)\Phi_\kappa(x', \sigma Y_0, Y_0)\, d\sigma \\ &\quad + 2i \int_a^b \int_0^1 \langle \bar{\partial} F(x', \sigma Y_0, \varepsilon + \tau Y_0), Y_0\rangle \Phi_\kappa(x', \sigma Y_0, \tau Y_0)\, d\tau d\sigma \\ &\quad + (\kappa + 1) \int_a^b \int_0^1 F(x', \sigma Y_0, \varepsilon + \tau Y_0)\tau^\kappa \sum_{|\alpha| = \kappa+1} \frac{\partial_x^\alpha \varphi}{\beta!}\, d\tau d\sigma\end{aligned}$$

and thus integrating over $\Omega \times Y_0^\perp$ yields

$$\begin{aligned}\int_{\Omega\times U} & F(x', x, \varepsilon)\varphi(x', x)\, d\lambda(x', x) \\ &= \int_{\Omega\times U} F(x', x, \varepsilon{+}Y_0)\Phi_\kappa(x', x, Y_0)\, d\lambda(x', x) \\ &\quad +2i \int_{\Omega\times U}\int_0^1 \langle \bar{\partial} F(x', x, \varepsilon{+}\tau Y_0), Y_0\rangle \Phi_\kappa(x', x, \tau Y_0)\, d\tau d\lambda(x', x) \\ &\quad +(\kappa{+}1)\int_{\Omega\times U}\int_0^1 F(x', x, \varepsilon{+}\tau Y_0)\tau^\kappa \sum_{|\alpha| = \kappa+1} \partial_x^\alpha \varphi(x', x) \frac{(iY_0)^\alpha}{\alpha!} d\lambda(x', x).\end{aligned} \tag{4-4}$$

Since by assumption $|\tau^\kappa F(x', x, \varepsilon + \tau Y_0)| \le c$ for some constant c and $\bar{\partial}_j F$ decreases rapidly for $\Gamma_r \ni y \to 0$ (see the remarks after Lemma 2.8) the bounded convergence theorem implies that the right-hand side converges for $\varepsilon \to 0$. Hence we define

$$\begin{aligned}(4\text{-}5)\quad \langle u,\varphi\rangle := &\int_{\Omega\times U} F(x',x,Y_0)\Phi_\kappa(x',x,Y_0)\,d\lambda(x',x)\\ &+2i\int_{\Omega\times U}\int_0^1 \langle \bar\partial F(x',x,\tau Y_0),Y_0\rangle \Phi_\kappa(x',x,\tau Y_0)\,d\tau d\lambda(x',x)\\ &+(\kappa+1)\int_{\Omega\times U}\int_0^1 F(x',x,\tau Y_0)\tau^\kappa \sum_{|\alpha|=\kappa+1}\partial_x^\alpha\varphi(x',x)\frac{(iY_0)^\alpha}{\alpha!}\,d\tau d\lambda(x',x).\end{aligned}$$

Since there is a constant $\tilde{C}$ only depending on F and $K\times L$ such that

$$|\langle u,\varphi\rangle| \le \tilde{C}\sup_{(x',x)\in K\times L}\Bigl(\sum_{|\beta|\le\kappa+1}|\partial_x^\beta\varphi(x',x)|\Bigr),$$

we deduce that the linear form u on $\mathcal{D}(\Omega\times U)$ given by (4-5) is a distribution.

Now, let $p_0\in\Omega\times U$ and $\omega_2\times V_2\Subset\omega_1\times V_1\Subset\Omega\times U$ be two open neighborhoods of p_0. Using [Hörmander 1983, Theorem 1.4.2] we can choose a sequence $(\varphi_\kappa)_\kappa\subset\mathcal{D}(\omega_1\times V_1)$ such that $\varphi_\kappa|_{\omega_2\times V_2}\equiv 1$ and for all $\gamma\in\mathbb{N}_0^{n+d}$ we have that

$$|D^{\gamma+\beta}\varphi_\kappa|\le (C_\gamma(\kappa+1))^{|\beta|},\quad |\beta|\le\kappa+1, \tag{4-6}$$

for a constant $C_\gamma\ge 1$ independent of κ. As before we set for each κ

$$\Phi_\kappa(x',x,y)=\sum_{|\alpha|\le\kappa}\partial_x^\alpha\varphi_\kappa(x',x)\frac{(iy)^\alpha}{\alpha!}.$$

We aim to estimate $\widehat{\varphi_\kappa u}$. In order to do so, let $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^d$ and notice that (4-5) implies for $\kappa\ge k$,

$$\begin{aligned}\widehat{\varphi_\kappa u}(\xi,\eta) &= \langle u, e^{-i\langle\,.\,,(\xi,\eta)\rangle}\varphi_\kappa\rangle\\ &=\int_{\Omega\times U}F(x',x,Y_0)e^{-i(x'\xi+(x+iY_0)\eta)}\Phi_\kappa(x',x,Y_0)\,d\lambda(x',x)\\ &\quad+2i\int_{\Omega\times U}\int_0^1\langle\bar\partial F(x',x,\tau Y_0),Y_0\rangle e^{-i(x'\xi+(x+i\tau Y_0)\eta)}\\ &\qquad\qquad\times\Phi_\kappa(x',x,\tau Y_0)\,d\tau d\lambda(x',x)\\ &\quad+(\kappa+1)\int_{\Omega\times U}\int_0^1 F(x',x,\tau Y_0)e^{-i(x'\xi+(x+i\tau Y_0)\eta)}\tau^\kappa\\ &\qquad\qquad\times\sum_{|\alpha|=\kappa+1}\partial_x^\alpha\varphi(x',x)\frac{(iY_0)^\alpha}{\alpha!}\,d\tau d\lambda(x',x)\end{aligned}$$

for some fixed, but arbitrary $Y_0\in\Gamma_r$ (note that k depends on u, $\omega_1\times V_1$ and Y_0).

Condition (4-6) gives the following estimate for $0 \le \mu \le \kappa + 1$:

$$\left| \sum_{|\alpha|=\mu} \partial_x^\alpha \varphi_\kappa(x', x) \frac{(iY)^\alpha}{\alpha!} \right| \le C_0^\mu (\kappa+1)^\mu \sum_{|\alpha|=\mu} \frac{|Y^\alpha|}{\alpha!} = C_0^\mu (\kappa+1)^\mu \frac{|Y|_1^\mu}{\mu!},$$

where $|Y|_1 = \sum_j |Y_j|$ for $Y = (Y_1, \dots, Y_d) \in \mathbb{R}^d$. Hence we have

$$|\Phi_\kappa(x', x, \tau Y_0)| \le C_1^{\kappa+1},$$
$$\left| (\kappa+1) \sum_{|\alpha|=\kappa+1} \partial_x^\alpha \varphi_\kappa(x', x) \frac{(iY_0)^\alpha}{\alpha!} \right| \le C_1^{\kappa+1},$$

for $C_1 = 2e^{C_0|Y_0|_1}$ and $\tau \in [0, 1]$. We obtain

$$\begin{aligned} &|\widehat{\varphi_\kappa u}(\xi, \eta)| \\ &\quad \le C_1^{\kappa+1} e^{\eta Y_0} + 2C_1^{\kappa+1} C \int_0^1 h_{\mathcal{M}}(Q\tau|Y_0|) e^{\tau\eta Y_0}\, d\tau + C_1^{\kappa+1} \int_0^1 \tau^{\kappa-k} e^{\tau\eta Y_0}\, d\tau \\ &\quad \le C_2 Q_1^\kappa \Big(e^{\eta Y_0} + m_{\kappa-k} \int_0^1 \tau^{\kappa-k} e^{\eta Y_0} \Big) \\ &\quad = C_2 Q_1^\kappa \big(e^{\eta Y_0} + m_\kappa (\kappa-k)! (-Y_0\eta)^{k-\kappa-1} \big) \end{aligned}$$

for some constants C_2, Q_1 and $Y_0\eta < 0$. If we set $\tilde{Y}_0 = (0, Y_0) \in \mathbb{R}^n \times \mathbb{R}^d$ then obviously

$$\langle \tilde{Y}_0, (\xi, \eta) \rangle = \langle Y_0, \eta \rangle.$$

Therefore we have for $\kappa \ge k$ and $\zeta = (\xi, \eta)$ that

$$|\widehat{\varphi_\kappa u}(\zeta)| = C_3 Q_1^\kappa \big(e^{\tilde{Y}_0\zeta} + m_{\kappa-k} (\kappa-k)! (-\tilde{Y}_0\zeta)^{k-\kappa-1} \big)$$

and $\tilde{Y}_0\zeta < 0$.

Now for any $\zeta_0 \in \mathbb{R}^{n+d}$ with $\langle \tilde{Y}_0, \zeta_0 \rangle < 0$ we can choose an open cone $V \subseteq \mathbb{R}^{n+d}$ such that $\zeta_0 \in V$ and for some constant $c > 0$ we have $\langle \tilde{Y}_0, \zeta \rangle < -c|\zeta|$ if $\zeta \in V$. Furthermore we set $u_\kappa = \varphi_{k+\kappa-1} u$. Clearly the sequence $(u_\kappa)_\kappa$ is bounded in $\mathcal{E}'(\Omega \times U)$ and $u_\kappa|_{\omega_2 \times V_2} \equiv u|_{\omega_2 \times V_2}$. Also using the inequality $e^{-c|\zeta|} \le \kappa!(c|\zeta|)^{-\kappa}$ we conclude

$$|\hat{u}_\kappa(\zeta)| = C_3 Q_1^\kappa \big(\kappa!(c|\zeta|)^{-\kappa} + m_{\kappa-1}(\kappa-1)!(c|\zeta|)^{-\kappa} \big) \le C_3 Q_2^\kappa m_\kappa \kappa! |\zeta|^{-\kappa}, \quad \zeta \in V.$$

Hence $(p_0, \zeta_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ and therefore

$$\mathrm{WF}_{\mathcal{M}}\, u \subseteq (\Omega \times U) \times (\mathbb{R}^n \times \Gamma^\circ) \setminus \{(0, 0)\}. \qquad \square$$

It is clear that the proof would only require $F \in \mathcal{C}^1$. From now the constants used in the proofs will be generic, i.e., they may change from line to line.

Remark 4.2. If $F \in \mathcal{E}(\Omega \times U \times V)$ is $\mathcal{M}$-almost analytic with respect to the variables $(x, y) \in U \times V$ we will often write $F(x', x+iy)$ or $F(x', z, \bar{z})$ and consider F as a smooth function on $\Omega \times (U + iV)$. If $\Omega = \varnothing$ then we just say that F is $\mathcal{M}$-almost analytic.

Even though in the remainder of this paper we shall only use the assertion of Lemma 4.5 in the special case $\Omega = \varnothing$ (i.e., without parameters), we have decided to include the general statement because we think it is of independent interest. We also have an application for the parameter version of the theorem in our paper [Fürdös 2019].

Example 4.3. Consider the holomorphic function $F(z) = \frac{1}{z}$ on $\mathbb{C}\backslash\{0\}$. It is well known that the boundary values of F onto the real line from above and beneath, commonly denoted by

$$\frac{1}{x+i0} = b_+ F = \lim_{y \to 0+} \frac{1}{x+iy},$$

$$\frac{1}{x-i0} = b_+ F = \lim_{y \to 0+} \frac{1}{x-iy},$$

satisfy the jump relations (see, e.g., [Duistermaat and Kolk 2010]); in particular,

$$2i\delta = \frac{1}{x-i0} - \frac{1}{x+i0}.$$

We have that both $1/(x+i0)$ and $1/(x-i0)$ are real-analytic outside the origin. Hence the application of Theorem 4.1 together with the jump relations imply that

$$\mathrm{WF}_{\mathcal{M}}\left(\frac{1}{x \pm i0}\right) = \{0\} \times \mathbb{R}_{\pm}.$$

There is a partial converse to the last theorem.

Theorem 4.4. *Let $\Gamma \subseteq \mathbb{R}^n$ be an open convex cone and $u \in \mathcal{D}'(\Omega)$ with $\mathrm{WF}_{\mathcal{M}}\, u \subseteq \Omega \times \Gamma^\circ$. If $V \Subset \Omega$ and Γ' is an open convex cone with $\overline{\Gamma}' \subseteq \Gamma \cup \{0\}$ then there is an $\mathcal{M}$-almost analytic function F on $V + i\Gamma'_r$ of slow growth for some $r > 0$ such that $u|_V = b_{\Gamma'}(F)$*

Proof. By [Hörmander 1983, Theorem 8.4.15] we have that u can be written on a bounded neighborhood U of V as a sum of a function $f \in \mathcal{E}_{\mathcal{M}}(U)$ and the boundary value of a holomorphic function of slow growth on $U + i\Gamma'_r$ for some r. To obtain the assertion use Corollary 2.11 to extend f almost-analytically on V. □

In order to proceed we need a further refinement of a result of Hörmander.

Lemma 4.5. *Let $\Gamma_j \subseteq \mathbb{R}^n\backslash\{0\}$, $j = 1, \ldots, N$, be closed cones such that $\bigcup_j \Gamma_j = \mathbb{R}^n\backslash\{0\}$ and $V \Subset \Omega$ is convex. Any $u \in \mathcal{D}'(\Omega)$ can be written on V as a linear* *combination* $u|_V = \sum_j u_j$ *of distributions* $u_j \in \mathcal{D}'(V)$ *that satisfy*

$$\mathrm{WF}_{\mathcal{M}}\, u_j \subseteq \mathrm{WF}_{\mathcal{M}}\, u \cap (V \times \Gamma_j)$$

Proof. Set $v = \varphi u$ where $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \equiv 1$ on V. The existence of $v_j \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\mathrm{WF}_{\mathcal{M}}\, v_j \subseteq \mathrm{WF}_{\mathcal{M}}\, v \cap (\mathbb{R}^n \times \Gamma_j)$$

is given in [Hörmander 1983, Corollary 8.4.13]. Set $u_j = (v_j)|_V$. □

Combining Theorem 4.4 with Lemma 4.5 we obtain:

Corollary 4.6. *Let* $u \in \mathcal{D}'(\Omega)$ *and* $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \backslash \{0\}$. *Then* $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ *if and only if there are a neighborhood* U *of* x_0, *open convex cones* $\Gamma_1, \dots, \Gamma_N$ *with the properties* $\xi_0 \Gamma_j < 0,\ \ j = 1, \dots N$ *and* $\Gamma_j \cap \Gamma_k = \varnothing$ *for* $j \neq k$, *and* $\mathcal{M}$*-almost analytic functions* h_j *on* $U + i\Gamma_{r_j},\ \ r_j > 0$, *of slow growth such that*

$$u|_U = \sum_{j=1}^{N} b_{\Gamma_j}(h_j).$$

Proof. Without loss of generality, assume that $\mathrm{WF}_{\mathcal{M}}\, u \neq \varnothing$. If $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ one can find closed cones $V_1, \dots, V_N$ with nonempty interior and $V_j \cap V_k$ has measure zero for $j \neq k$ such that ξ_0 is contained in the interior of V_1 and $V_1 \cap \mathrm{WF}_{\mathcal{M}}\, u = \varnothing$ whereas $\xi_0 \notin V_j$ are acute cones and $\mathrm{WF}_{\mathcal{M}}\, u \cap V_j \neq \varnothing$ for $j = 2, \dots, N$. By Lemma 4.5 we can write u on an open neighborhood U of x_0 as a sum $u = u_1 + \sum_{j=2}^{N} u_j$ with u_1 being an ultradifferentiable function defined on U and $u_j \in \mathcal{D}'(U)$ such that $\mathrm{WF}_{\mathcal{M}}\, u_j \subseteq \mathrm{WF}_{\mathcal{M}}\, u \cap V_j,\ \ j = 2, \dots, N$. The cones $V_2, \dots, V_N$ are the dual cones of open convex cones $\Gamma_2, \dots, \Gamma_N$, i.e., $\Gamma_j^\circ = V_j$. We can choose cones $\Gamma'_j \Subset \Gamma_j$ and using Theorem 4.4 we find $\mathcal{M}$-almost analytic functions h_j on $U + i\Gamma'_{j,r}$ of slow growth such that $u_j = b_{\Gamma'_j}(h_j)$. It remains to note that $\xi_0 y < 0$ for all $y \in \Gamma'_j,\ \ j = 2, \dots, N$. □

Let $\Omega_1 \subseteq \mathbb{R}^m$ and $\Omega_2 \subseteq \mathbb{R}^n$ be open. If $F : \Omega_1 \to \Omega_2$ is a $\mathcal{E}_{\mathcal{M}}$-mapping then we denote as in [Hörmander 1983, page 263] the set of normals by

$$N_F = \{(F(x), \eta) \in \Omega_2 \times \mathbb{R}^n :\ DF(x)\eta = 0\},$$

where DF denotes the transpose of the Jacobian of F. The following is a generalization of [Hörmander 1983, Theorem 8.5.1].

Theorem 4.7. *For any* $u \in \mathcal{D}'(\Omega_2)$ *with* $N_F \cap \mathrm{WF}_{\mathcal{M}}\, u = \varnothing$ *we obtain that the pull-back* $F^*u \in \mathcal{D}'(\Omega_1)$ *is well defined and*

$$\mathrm{WF}_{\mathcal{M}}(F^*u) \subseteq F^*(\mathrm{WF}_{\mathcal{M}}\, u). \tag{4-7}$$

Proof. The first part of the statement is [Hörmander 1983, Theorem 8.2.4]. For the proof of the second part of the theorem assume first that there is an open convex cone Γ such that u is the boundary value of an $\mathcal{M}$-almost analytic function Φ on $\Omega_2 + i\Gamma_r$ of slow growth. Hence $\operatorname{WF}_{\mathcal{M}} u \subseteq \Omega_2 \times \Gamma^\circ$. If $x_0 \in \Omega_1$ and $DF(x_0)\eta \neq 0$ for $\eta \in \Gamma^\circ \setminus \{0\}$ then $DF(x_0)\Gamma^\circ$ is a closed convex cone. We claim that

$$\operatorname{WF}_{\mathcal{M}}(F^*u)|_{x_0} \subseteq \{(x_0, DF(x_0)\eta) : \eta \in \Gamma^\circ \setminus \{0\}\}.$$

We adapt as usual the argument of Hörmander [1983]. We can write (see [Hörmander 1983, page 296])

$$DF(x_0)\Gamma^\circ = \{\xi \in \mathbb{R}^n \mid \langle h, \xi \rangle \geq 0,\ F'(x_0)h \in \Gamma\}.$$

If $\tilde{F}$ denotes an $\mathcal{M}$-almost analytic extension of F onto $X_0 + i\mathbb{R}^n$, where the neighborhood X_0 of x_0 is convex and relatively compact in Ω_1, which exists due to Theorem 2.10, then Taylor's formula implies that

$$\operatorname{Im} \tilde{F}(x + i\varepsilon h) \in \Gamma, \quad x \in X_0,$$

for $F'(x_0)h \in \Gamma$ if X_0 and $\varepsilon > 0$ are small.

Recalling (4-4) we see that the map

$$\mathbb{R}_{\geq 0} \times (\Gamma \cup \{0\}) \ni (\varepsilon, y) \longmapsto \tilde{\Phi}(\varepsilon, y) := \Phi(\tilde{F}(\,.\, + i\varepsilon h) + iy) \in \mathcal{D}'(X_0)$$

is continuous. If $\varepsilon \to 0$ then $\tilde{\Phi}(\varepsilon, y) \to \tilde{\Phi}(0, y) = \Phi(\tilde{F}(\,.\, + 0i) + iy)$ in $\mathcal{D}'$ and if now $y \to 0$ we have by definition $\tilde{\Phi}(0, y) \to F^*u$. On the other hand if first $y \to 0$ then

$$\tilde{\Phi}(\varepsilon, y) \to \tilde{\Phi}(\varepsilon, 0) = \Phi(\tilde{F}(\,.\, + i\varepsilon h)).$$

Hence by continuity

$$F^*u = \lim_{\varepsilon \to 0} \Phi(\tilde{F}(\,.\, + i\varepsilon h))$$

in $\mathcal{D}'(X_0)$ and by the proof of Theorem 4.1,

$$\operatorname{WF}_{\mathcal{M}} F^*u|_{x_0} \subseteq \{(x_0, \xi) \mid \langle h, \xi \rangle \geq 0\}.$$

The claim follows.

Now suppose that $(F(x_0), \eta_0) \notin \operatorname{WF}_{\mathcal{M}} u$. By Corollary 4.6 we can write a general distribution u on some neighborhood U_0 of $F(x_0)$ as $\sum_{j=1}^N u_j$ where the distributions u_j, $j = 1, \dots, N$, are the boundary values of some $\mathcal{M}$-almost analytic functions Φ_j on $U_0 + i\Gamma_j$, where the Γ_j are some open convex cones such that $\eta_0\Gamma_j < 0$ for all $j = 1, \dots, N$. By assumption $DF(x)\eta \neq 0$ when $(F(x), \eta) \in \operatorname{WF}_{\mathcal{M}} u$ for $x \in F^{-1}(U_0)$. Hence we can assume that $DF(x)\eta \neq 0$ for $\eta \in \Gamma_j^\circ$ for all $j = 1, \dots, N$ and $x \in F^{-1}(U_0)$ since in the proof of Corollary 4.6 the cones Γ_j, $j = 1, \dots, N$, can be chosen such that $\Gamma^\circ \cap S^{n-1}$ and $\Gamma_j^\circ \cap \operatorname{WF}_{\mathcal{M}} u|_x \neq \varnothing$

for $x \in U_0$. By the arguments above we have for a small neighborhood V of x_0 that

$$F^*u|_V = \sum_{j=1}^{N} F^*u_j|_V$$

and $\mathrm{WF}_{\mathcal{M}}(F^*u_j)|_{x_0} \subseteq \{(x_0, DF(x_0)\eta) \mid \eta \in \Gamma_j^\circ \setminus \{0\}\}$ for all j. However, since $\eta_0\Gamma_j < 0$ it follows that $(x_0, DF(x_0)\eta_0) \notin \mathrm{WF}_{\mathcal{M}}(F^*u_j)$ and so $(x_0, DF(x_0)\eta_0) \notin \mathrm{WF}_{\mathcal{M}}(F^*u)$. □

Remark 4.8. If F is an $\mathcal{E}_{\mathcal{M}}$-diffeomorphism we obtain from Theorem 4.7 that

$$\mathrm{WF}_{\mathcal{M}}(F^*u) = F^*(\mathrm{WF}_{\mathcal{M}}\, u).$$

Hence if M is an $\mathcal{E}_{\mathcal{M}}$-manifold and $u \in \mathcal{D}'(M)$ we can define $\mathrm{WF}_{\mathcal{M}}\, u$ invariantly as a subset of $T^*M\backslash\{0\}$. More precisely, there is a subset K_u of T^*M such that the diagram

$$\begin{array}{ccccc} & & K_u & & \\ & \swarrow & & \searrow & \\ T^*\varphi(U\cap V) \supseteq \mathrm{WF}_{\mathcal{M}}\, v_1 & & \xrightarrow{\ \rho^*\ } & & \mathrm{WF}_{\mathcal{M}}\, v_2 \subseteq T^*\psi(U\cap V) \end{array}$$

commutes for any two charts φ and ψ of M on $U \subseteq M$ and $V \subseteq M$, respectively. We have set $\rho = \psi \circ \varphi^{-1}$, $v_1 = \varphi^* u \in \mathcal{D}'(\varphi(U \cap V))$ and $v_2 = \psi^* u \in \mathcal{D}'(\psi(U \cap V))$. It follows that $K_u \subseteq T^*M \setminus \{0\}$ has to be closed and fiberwise conic. We set $\mathrm{WF}_{\mathcal{M}}\, u := K_u$.

Analogously we define the wavefront set of a distribution $u \in \mathcal{D}'(M, E)$ with values in an ultradifferentiable vector bundle locally by setting

$$\mathrm{WF}_{\mathcal{M}}\, u|_V = \bigcup_{j=1}^{\nu} u_j,$$

where $V \subseteq M$ is an open subset such that there is a local basis $\omega^1, \ldots, \omega^\nu$ of $\mathcal{E}_{\mathcal{M}}(V, E)$ and $u_j \in \mathcal{D}'(V)$ are distributions on V such that

$$u|_V = \sum_{j=1}^{\nu} u_j \omega^j.$$

We close this section by observing that Theorem 4.7 allows us to strengthen a uniqueness result of Boman [1995]:

Theorem 4.9. *Let $\mathcal{M}$ be a quasianalytic weight sequence and $S \subseteq \mathbb{R}^n$ an $\mathcal{E}_{\mathcal{M}}$-submanifold. If u is a distribution defined on a neighborhood of S such that*

$$\mathrm{WF}_{\mathcal{M}}\, u \cap N^*S = \varnothing$$

and

$$\partial^\alpha u|_S = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^n,$$

then u vanishes on some neighborhood of S.

Indeed, locally S is diffeomorphic to

$$S' = \{(x', x'') \in \mathbb{R}^{m+d} \mid x'' = 0\} \subseteq \mathbb{R}^n$$

and the assumptions of the theorem translate to the corresponding conditions for the pullback $w = F^*u$ where $F : \mathbb{R}^n \to \mathbb{R}^n$ is the local $\mathcal{E}_{\mathcal{M}}$-diffeomorphism that maps S' to S. Then the proof of Theorem 1 in [Boman 1995] gives $w = 0$ in a neighborhood of S'.

5. A generalized version of Bony's Theorem

We have seen that for a distribution u the wavefront set $\mathrm{WF}_{\mathcal{M}}\, u$ can be described either using the Fourier transform or by its $\mathcal{M}$-almost analytic extensions. A similar fact is true for the analytic wavefront set using holomorphic extensions. The latter was the original approach of Sato [1970]. However, Bros and Iagolnitzer [1975] used the classical FBI transform to describe the set of microlocal analytic singularities. It was Bony [1977] who proved that all three methods actually describe the same set. In the ultradifferentiable setting Chung and Kim [1997] (see also Kim, Chung and Kim [Kim et al. 2001]) used the FBI transform to define an ultradifferentiable singular spectrum for Fourier hyperfunctions. However, they did not mention how this singular spectrum in the case of distributions may be related to $\mathrm{WF}_{\mathcal{M}}$ as defined by Hörmander. Our next aim is to show an ultradifferentiable version of Bony's theorem. We will work in the generalized setting of [Berhanu and Hounie 2012]. We shall note that recently Berhanu and Hailu [2017] showed that the Gevrey classes can be characterized by this generalized FBI transform and Hoepfner and Medrado [2018] also proved a characterization of the ultradifferentiable wavefront set of ultradistributions for a certain class of nonquasianalytic weight sequences.

Let p be a real, homogeneous, positive, elliptic polynomial of degree $2k$, $k \in \mathbb{N}$, on $\mathbb{R}^n$, i.e.,

$$p(x) = \sum_{\alpha = 2k} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{R},$$

and let there be constants $c, C > 0$ such that

$$c|x|^{2k} \leq p(x) \leq C|x|^{2k}, \quad x \in \mathbb{R}^n.$$

Let

$$c_p^{-1} = \int e^{-p(x)} dx.$$

As in [Berhanu and Hounie 2012, Section 4] we consider the generalized FBI transform with generating function e^{-p} of a distribution of compact support $u \in \mathcal{E}'(\mathbb{R}^n)$, i.e.,

$$\mathfrak{F}u(t, \xi) = c_p \langle u(x), e^{i\xi(t-x)} e^{-|\xi| p(t-x)} \rangle.$$

The inversion formula is

$$u = \lim_{\varepsilon\to\infty} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F}u(t,\xi)|\xi|^{n/(2k)}\, dt d\xi, \tag{5-1}$$

where of course the distributional limit is meant.

Theorem 5.1. *Let $u \in \mathcal{D}'(\Omega)$ and*

$$(x_0, \xi_0) \in T^*\Omega \setminus \{0\}.$$

Then $(x_0,\xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ if and only if there is a test function $\psi \in \mathcal{D}(\Omega)$ with $\psi|_U \equiv 1$ for some neighborhood U of x_0 such that

$$\sup_{(t,\xi)\in V\times\Gamma} e^{\omega_{\mathcal{M}}(\gamma|\xi|)} |\mathfrak{F}(\psi u)(t,\xi)| < \infty \tag{5-2}$$

for some conic neighborhood $V\times\Gamma$ of (x_0,ξ_0) and some constant $\gamma > 0$.

Proof. We modify the proof of [Berhanu and Hounie 2012, Theorem 4.2]; see also the proof of [Hoepfner and Medrado 2018, Theorem 5.2].

First, assume that $(x_0,\xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$. By Corollary 4.6 we know that for some neighborhood U of x_0,

$$u|_U = \sum_{j=1}^{N} b_{\Gamma^j}(F_j),$$

where F_j are $\mathcal{M}$-almost analytic on $U\times\Gamma^j_{r_j}$ for cones Γ^j with $\xi_0\Gamma^j < 0$. Hence it suffices to prove the necessity of (5-2) for $u = b_\Gamma(F)$ being the boundary value of an $\mathcal{M}$-almost analytic function F on $U\times\Gamma_d$ where Γ is a cone with the property that $\xi_0\Gamma < 0$. Without loss of generality, $x_0 = 0$, and let $r > 0$ be such that $B_{2r}(0) \Subset U$ and $\psi \in \mathcal{D}(B_{2r}(0))$ be such that $\psi|_{B_r(0)} \equiv 1$. Furthermore we choose $v \in \Gamma_d$ and set

$$Q(t,\xi,x) = i\xi(t-x) - |\xi|p(t-x).$$

Then

$$\mathfrak{F}(\psi u)(t,\xi) = \lim_{\tau\to 0+} \int_{B_{2r}(0)} e^{Q(t,\xi,x+i\tau v)} \psi(x) F(x+i\tau v)\, dx.$$

As in the proof of Theorem 4.2 in [Berhanu and Hounie 2012] we put $z = x+iy$, $\psi(z) = \psi(x)$ and

$$D_\tau := \{x + i\sigma v \in \mathbb{C}^n \mid x \in B_{2r} = B_{2r}(0),\ \tau \le \sigma \le \lambda\}$$

for some $\lambda > 0$ to be determined later and consider the n-form

$$e^{Q(t,\xi,z)} \psi(z) F(z)\, dz_1 \wedge \cdots \wedge dz_n.$$

Since $\psi \in \mathcal{D}(B_{2r}(0))$, Stokes' theorem implies

$$\begin{aligned}
\text{(5-3)}\quad \int_{B_{2r}} & e^{Q(t,\xi,x+i\tau v)}\psi(x)F(x+i\tau v)\,dx \\
&= \int_{B_{2r}} e^{Q(t,\xi,x+i\lambda v)}\psi(x)F(x+i\lambda v)\,dx \\
&\quad + \sum_{j=1}^{n}\int_{D_\tau} e^{Q(t,\xi,z)}\frac{\partial}{\partial \bar{z}_j}(\psi(z)F(z))\,d\bar{z}_j\wedge dz_1\wedge\cdots\wedge dz_n \\
&= \int_{B_{2r}} e^{Q(t,\xi,x+i\lambda v)}\psi(x)F(x+i\lambda v)\,dx \\
&\quad + \sum_{j=1}^{n}\int_{B_{2r}}\int_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)}\frac{\partial\psi}{\partial\bar{z}_j}(x+i\sigma v)F(x+i\sigma v)\,d\sigma dx \\
&\quad + \sum_{j=1}^{n}\int_{B_{2r}}\int_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)}\psi(x+i\sigma v)\frac{\partial F}{\partial\bar{z}_j}(x+i\sigma v)\,d\sigma dx.
\end{aligned}$$

We need to estimate the integrals on the right-hand side of (5-3). Using the arguments of [Berhanu and Hounie 2012] we see that there is an open cone Γ containing ξ_0 and a bounded neighborhood V of 0 such that the first two integrals can be estimated by $Ce^{-\gamma|\xi|}$ where $C,\gamma>0$ are constants, as long as $\xi\in\Gamma$ and $t\in V$. Since (M4) implies that $\omega_{\mathcal{M}}(t)=O(t)$ for $t\to\infty$; see, e.g., [Komatsu 1973] or [Bonet et al. 2007], we obtain that both integrals can in fact be bounded by $Ce^{-\omega_{\mathcal{M}}(\gamma|\xi|)}$ if $(t,\xi)\in V\times\Gamma$.

In order to estimate the third integral in (5-3) we recall from [Berhanu and Hounie 2012] that for λ small enough there is a constant $c_0>0$ such that

$$\operatorname{Re} Q(t,\xi,x+i\lambda v)\le -\frac{c_0}{2}\lambda|v||\xi|$$

if $\xi\in\Gamma,\ x\in B_{2r}$ and $t\in V$. Hence we have for a generic constant $C_3>0$ and all $k\in\mathbb{N}_0$ that

$$\begin{aligned}
&\left|\sum_{j=1}^{n}\int_{B_{2r}}\int_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)}\psi(x)\frac{\partial F}{\partial\bar{z}_j}(x+i\sigma v)\,d\sigma dx\right| \\
&\qquad\le C_3\int_0^\infty e^{-c_0\sigma|v||\xi|}h_{\mathcal{M}}(\rho\sigma|v|)\,d\sigma \le C_3\int_0^\infty e^{-c_0\sigma|v||\xi|}\rho^k\sigma^k|v|^k m_k\,d\sigma \\
&\qquad= C_3\rho_1^k M_k|\xi|^{-k}.
\end{aligned}$$

Lemma 2.8 gives

$$\left|\sum_{j=1}^{n}\int_{B_{2r}}\int_{\tau}^{\lambda} e^{Q(t,\xi,x+i\sigma v)}\psi(x)\frac{\partial F}{\partial\bar{z}_j}(x+i\sigma v)\,d\sigma dx\right|\le C_3\tilde{h}_{\mathcal{M}}(\rho_1|\xi|^{-1})\le C_3e^{-\omega_{\mathcal{M}}(\rho_1|\xi|)}.$$

In view of (5-3) we have shown that for $\xi \in \Gamma$ and t in a small enough neighborhood of 0 there are constants $C, \gamma > 0$ such that

$$\left| \int_{B_{2r}} e^{Q(t,\xi,x+i\tau v)} \psi(x) F(x+i\tau v)\, dx \right| \leq C e^{-\omega_{\mathcal{M}}(\gamma|\xi|)}.$$

Note that in the estimate the constants C and γ depend on λ but not on $\tau < \lambda$. Thus (5-2) is proven.

On the other hand, assume that (5-2) holds for a point (x_0, ξ_0), i.e., that there are a neighborhood V of x_0, an open cone $\Gamma \subseteq \mathbb{R}^n$ containing ξ_0 and constants $C, \gamma > 0$ such that

$$|\mathfrak{F}(\psi u)(x,\xi)| \leq C e^{-\omega_{\mathcal{M}}(\gamma|\xi|)}, \quad x \in V,\ \xi \in \Gamma, \tag{5-4}$$

for some test function $\psi \in \mathcal{D}(\Omega)$ that is 1 near x_0. We may assume that $x_0 = 0$. We have to prove that $(0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$ or, equivalently, $(0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, v$ where $v = \psi u$. We invoke the inversion formula (5-1) for the FBI transform

$$v = \lim_{\varepsilon \to \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F} v(t,\xi) |\xi|^{n/(2k)}\, dt d\xi$$

and split the occurring integral into two parts

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F} v(t,\xi) |\xi|^{n/(2k)}\, dt d\xi = I_1^\varepsilon(x) + I_2^\varepsilon(x) \tag{5-5}$$

where

$$I_1^\varepsilon(x) = \int_{\mathbb{R}^n} \int_{|t| \leq a} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F} v(t,\xi) |\xi|^{n/(2k)}\, dt d\xi$$
$$I_2^\varepsilon(x) = \int_{\mathbb{R}^n} \int_{a \leq |t|} e^{i\xi(x-t)} e^{-\varepsilon|\xi|^2} \mathfrak{F} v(t,\xi) |\xi|^{n/(2k)}\, dt d\xi$$

for a constant a to be determined. Following Berhanu and Hounie [2012] we see that for any choice of a the second integral converges to a holomorphic function in a neighborhood of the origin for $\varepsilon \to 0$.

It remains to look at I_1^ε. Suppose that a is small enough such that $B_a(0) \subseteq V$. Let $\mathcal{C}_j,\ 1 \leq j \leq N$, be open, acute cones such that

$$\mathbb{R}^n = \bigcup_{j=1}^N \overline{\mathcal{C}}_j$$

and the intersection $\overline{\mathcal{C}}_j \cap \overline{\mathcal{C}}_k$ has measure zero for $j \neq k$. Furthermore, let $\xi_0 \in \mathcal{C}_1$, $\mathcal{C}_1 \subseteq \Gamma$ and $\xi_0 \notin \overline{\mathcal{C}}_j$ for $j \neq 1$. In particular that means that (5-4) holds on $B_a(0) \times \mathcal{C}_1$, i.e.,

$$|\mathfrak{F}(\psi u)(x,\xi)| \leq C e^{-\omega_{\mathcal{M}}(\gamma|\xi|)}, \quad x \in B_a(0),\ \xi \in \mathcal{C}_1. \tag{5-6}$$

Furthermore, arguing as in [Berhanu and Hounie 2012], we can choose open cones Γ_j with the property that $\xi_0\Gamma_j < 0$ for $j = 2, \dots, N$ such that the functions

$$f_j(x+iy) = \int_{\mathcal{C}_j} \int_{B_a(0)} e^{i\xi(x+iy-t)} \mathfrak{F}v(t,\xi)|\xi|^{n/(2k)}\, dt d\xi$$

are holomorphic on $\mathbb{R}^m \times i\Gamma_j$.

In the remaining case we have to modify the arguments in [Berhanu and Hounie 2012] a little bit. We set

$$f_1^\varepsilon(x) = \int_{\mathcal{C}_1} \int_{B_a(0)} e^{i\xi(x-t)-\varepsilon|\xi|^2} \mathfrak{F}v(t,\xi)|\xi|^{n/(2k)}\, dt d\xi$$

and

$$f_1(x) = \int_{\mathcal{C}_1} \int_{B_a(0)} e^{i\xi(x-t)} \mathfrak{F}v(t,\xi)|\xi|^{n/(2k)}\, dt d\xi.$$

The functions f_1^ε, $\varepsilon > 0$, extend to entire functions whereas f_1 is smooth due to (5-6) since $e^{-\omega_{\mathcal{M}}}$ is rapidly decreasing (see the remark after the proof of (2-3)). This decrease also shows that f_1^ε converges uniformly to f_1 in a neighborhood of 0 since

$$\begin{aligned} |f_1(x) - f_1^\varepsilon(x)| &\le \int_{\mathcal{C}_1} \int_{B_a(0)} |\mathfrak{F}v(t,\xi)||\xi|^{n/(2k)}|1 - e^{-\varepsilon|\xi|^2}|\, dt d\xi \\ &\le C \int_{\mathcal{C}_1} |\xi|^{n/(2k)} e^{-\omega_{\mathcal{M}}(\gamma|\xi|)} |1 - e^{-\varepsilon|\xi|^2}|\, d\xi \end{aligned}$$

and the last integral converges to 0 by the monotone convergence theorem.

In fact $f_1 \in \mathcal{E}_{\mathcal{M}}$ because

$$\begin{aligned} |D^\alpha f_1(x)| &\le \int_{\mathcal{C}_1} |\xi|^{n/(2k)} |\xi^\alpha \mathfrak{F}v(t,\xi)|\, dt d\xi \\ &\le C \int_{\mathcal{C}_1} |\xi|^{n/(2k)+|\alpha|} e^{-\omega_{\mathcal{M}}(\gamma|\xi|)}\, d\xi \\ &\le C\gamma^{|\alpha|} M_{|\alpha|}, \end{aligned}$$

where in the last step (2-3) and (M2) are used.

So we have showed that on an open neighborhood U of the origin and some open cones Γ_j, $j = 2, \dots, N$, which satisfy $\xi_0\Gamma_j < 0$ we can write

$$v|_U = v_0 + \sum_{j=2}^N b_{\Gamma_j} f_j$$

with $v_0 \in \mathcal{E}_{\mathcal{M}}(U)$ and f_j holomorphic on $U + i\Gamma_j$ for $j = 2, \dots, N$. Hence $(0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, v$. □

We summarize our results regarding the description of $\mathrm{WF}_{\mathcal{M}}\, u$ in order to obtain the generalized Bony's theorem alluded to in the beginning of this section (see [Hoepfner and Medrado 2018]).

Theorem 5.2. *Let $u \in \mathcal{D}'(\Omega)$. For $(x_0, \xi_0) \in T^*\Omega\backslash\{0\}$ the following statements are equivalent*:

(1) $(x_0, \xi_0) \notin \mathrm{WF}_{\mathcal{M}}\, u$.

(2) *There are $U \in \mathcal{U}(x_0)$, open convex cones $\Gamma^j \subseteq \mathbb{R}^n$ with $\xi_0 \Gamma^j < 0$ and $\mathcal{M}$-almost analytic functions F_j of slow growth in $U \times \Gamma^j_{\rho_j}$, $\rho_j > 0$ and $j = 1, \dots, N$ for some $N \in \mathbb{N}$ such that*

$$u|_U = \sum_{j=1}^{N} b_{\Gamma^j} F_j.$$

(3) *There are $\varphi \in \mathcal{D}(\Omega)$ with $\varphi \equiv 1$ near x_0, with a neighborhood V of x_0, and an open cone Γ containing ξ_0 such that* (5-2) *holds.*

We can also give a local version of Theorem 5.2.

Corollary 5.3. *Let $u \in \mathcal{D}'(\Omega)$ and $p \in \Omega$. Then the following are equivalent*:

(1) *The distribution u is of class $\{\mathcal{M}\}$ near p.*

(2) *There are a bounded sequence $(u_N)_N \subseteq \mathcal{E}'(\Omega)$ and an open neighborhood $V \subseteq \Omega$ of p such that $u_N|_V = u|_V$ for all $N \in \mathbb{N}_0$ and* (3-1) *holds for $\Gamma = \mathbb{R}^n$ and some constant $Q > 0$.*

(3) *There exist an open neighborhood $W \subseteq \Omega$ of p, $r > 0$ and a smooth function F on $W + iB(0, r)$ such that $F|_W = u|_W$ and* (2-5) *holds for some constants $C, Q > 0$.*

(4) *There are a test function $\psi \in D(\Omega)$ with $\psi \equiv 1$ near p, a neighborhood V of p and constants $C, \gamma > 0$ such that*

$$\sup_{(t,\xi)\in V\times\mathbb{R}^n} e^{\omega_{\mathcal{M}}(\gamma|\xi|)} |\mathfrak{F}(\psi u)(t, \xi)| < \infty.$$

Proof. The equivalence of (1) and (2) is just Proposition 3.1, whereas Corollary 2.11 shows that (1) implies (3). For the fact that (4) implies (1) we note that by Theorem 5.1 we have $(p, \xi) \notin \mathrm{WF}_{\mathcal{M}}\, u$ for all $\xi \in \mathbb{R}^n\backslash\{0\}$. Therefore u has to be ultradifferentiable of class $\{\mathcal{M}\}$ near p. Now we show that (4) follows from (3): Suppose that $u \in \mathcal{E}_{\mathcal{M}}(V)$ on a neighborhood of p and let $F \in \mathcal{E}(W + i\mathbb{R}^n)$ be an $\mathcal{M}$-almost analytic extension of u on a relatively compact neighborhood $W \Subset V$ of p. We choose $\varphi \in \mathcal{D}(W)$, $0 \le \varphi \le 1$, and $\varphi \equiv 1$ near p. We consider the map

$$\theta : y \longmapsto \theta(y) = y - is\varphi(y)\frac{\xi}{|\xi|}$$

for some $1 > s > 0$ to be determined. Finally let $\psi \in \mathcal{D}(V)$ such that $\psi \equiv 1$ on W. As in the proof of Theorem 5.1 we set $\psi(z) = \psi(x)$ for $z = x + iy \in \mathbb{C}^n$. We put $v = \psi F$ and consider the n-form

$$e^{Q(t,\xi,z)} v(z)\, dz_1 \wedge \cdots \wedge dz_n$$

on

$$D_s = \left\{ x + i\sigma\varphi(x)\frac{\xi}{|\xi|} \in \mathbb{C}^n \;\middle|\; 0 < \sigma < s,\ x \in \operatorname{supp} v \right\}.$$

Stokes' theorem gives us

$$\mathfrak{F}v(t,\xi) = c_p \int_{\theta(\mathbb{R}^n)} e^{Q(t,\xi,z)} v(z,\bar{z})\, dz_1 \wedge \cdots \wedge dz_n \\ + c_p \sum_{j=1}^{n} \int_{D_s} e^{Q(t,\xi,z)} \frac{\partial v}{\partial \bar{z}_j}(z,\bar{z})\, d\bar{z}_j \wedge dz_1 \wedge \cdots \wedge dz_n.$$

The second integral above is estimated in the same way as the last integral in (5-3). On the other hand the first integral on the right-hand side equals

$$G(t,\xi) = c_p \int_{\mathbb{R}^n} e^{Q(t,\xi,\theta(y))} v(\theta(y)) \det \theta'(y)\, dy.$$

We note that

$$\operatorname{Re} Q(t,\xi,\theta(y)) \leq -s\varphi(y)|\xi|(1 + O(s\varphi(y)\big) - c_0|t-y|^{2k}$$

and hence

$$|G(t,\xi)| \leq C \int_{B_\delta(p)} e^{\operatorname{Re} Q(t,\xi,\theta(y))}\, dy + C \int_{\substack{\mathbb{R}^n \setminus B_\delta(p) \\ y \in \operatorname{supp}(v\circ\theta)}} e^{\operatorname{Re} Q(t,\xi,\theta(y))}\, dy \\ = I_1(t,\xi) + I_2(t,\xi),$$

where $B_\delta(p) \subseteq \{x \in \mathbb{R}^n \mid \varphi(x) = 1\}$, can be estimated as follows; see [Berhanu et al. 2008]. Set $s = \delta/4$. We obtain

$$I_1(t,\xi) \leq Ce^{-c|\xi|}$$

for all $\xi \in \mathbb{R}^n$ if t is in some bounded neighborhood of p. Furthermore

$$I_2(t,x) \leq C \int_{\substack{\mathbb{R}^n \setminus B_r(p) \\ y \in \operatorname{supp}(u\circ\theta)}} e^{-|\xi||t-y|^{2k}}\, dy \leq Ce^{-(\delta/2)^{2k}|\xi|}$$

for all ξ and $|t-p| \leq \frac{\delta}{2}$.

Hence we have showed that there are constants $c, C > 0$ such that

$$|\mathfrak{F}u(t,\xi)| \leq Ce^{-\omega_{\mathcal{M}}(c|\xi|)}$$

for all $\xi \in \mathbb{R}^n$ and t in a bounded neighborhood of p. □

6. Elliptic regularity

As mentioned in the introduction, Albanese, Jornet and Oliaro [2010] used the pattern of the proof of [Hörmander 1983, Theorem 8.6.1] (see Remark 3.3) to prove elliptic regularity for operators with coefficients that are all in the same ultradifferentiable class defined by a weight function; see Remark 3.7. Similarly Hörmander's methods were applied by Pilipović, Teofanov and Tomić [2016; 2018] for certain classes that are defined by more degenerate sequences.

It should be noted that the assumptions Albanese, Jornet and Oliaro put on the weight functions guarantee that the associated class is closed under composition and the inverse function theorem holds. So it would be a reasonable conjecture that the regularity of the defining weight sequence is necessary for elliptic regularity to hold in the category of Denjoy–Carleman classes. But there are weight functions obeying these conditions such that the associated function class cannot be described by regular weight sequences and on the other hand there are regular Denjoy–Carleman classes that cannot be defined by such weight functions; see [Bonet et al. 2007]. It turns out, however, that the regularity of the weight sequence is not enough for the proof of the elliptic regularity theorem, we also have to assume that (M2′) holds. In that case the main result of [Bonet et al. 2007] implies that the Denjoy–Carleman class can be described by a weight function that satisfies the conditions of [Albanese et al. 2010]. Hence, we could use their elliptic regularity theorem, but we would have to show that their definition of the ultradifferentiable wavefront set coincides with the definition of Hörmander. Instead we give here a proof in full detail partially in preparation for the forthcoming paper by Fürdös, Nenning, Rainer and Schindl [Fürdös et al. 2020], where we deal with the problem in the far more general setting of the ultradifferentiable classes introduced in [Rainer and Schindl 2014]; see Remark 3.7.

Furthermore, we show here that Hörmander's proof can be modified in a way to investigate the regularity of solutions of a determined system of linear partial differential equations

$$\begin{aligned} P_{11}u_1 + \cdots + P_{1\nu}u_\nu &= f_1 \\ \vdots \qquad\quad & \quad\ \vdots \\ P_{\nu 1}u_1 + \cdots + P_{\nu\nu}u_\nu &= f_\nu \end{aligned}$$

where the $P_{jk},\ 1 \le j, k \le \nu$, are partial differential operators with $\mathcal{E}_{\mathcal{M}}$-coefficients.

More precisely, using the geometric theory for the ultradifferentiable wavefront set developed in Section 4, we can work in the following setting (see [Hörmander 1983, Chapter 6] or [Chazarain and Piriou 1982]).

Let M be an ultradifferentiable manifold of class $\{\mathcal{M}\}$ and E and F two vector bundles of class $\{\mathcal{M}\}$ on M with the same fiber dimension ν. An ultradifferentiable partial differential operator $P : \mathcal{E}_{\mathcal{M}}(M, E) \to \mathcal{E}_{\mathcal{M}}(M, F)$ of class $\{\mathcal{M}\}$ is given locally by

$$Pu = \begin{pmatrix} P_{11} & \cdots & P_{1\nu} \\ \vdots & \ddots & \vdots \\ P_{\nu 1} & \cdots & P_{\nu\nu} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_\nu \end{pmatrix}, \tag{6-1}$$

where the P_{jk} are linear partial differential operators with ultradifferentiable coefficients defined in suitable chart neighborhoods. If

$$Q(x, D) = \sum_{|\alpha| \le m} q_\alpha(x) D^\alpha$$

is a differential operator of order $\le m$ on some open set $\Omega \subseteq \mathbb{R}^n$ then the principal symbol q is defined to be

$$q(x, \xi) = \sum_{|\alpha| = m} q_\alpha(x) \xi^\alpha.$$

Hence the order of P is of order $\le m$ if and only if no operator P_{jk} on any chart neighborhood is of order higher than m and P is of order m if the operator is not of order $\le m - 1$. The principal symbol p of P is an ultradifferentiable mapping defined on T^*M with values in the space of fiber-linear maps from E to F that is homogenous of degree m in the fibers of T^*M. It is given locally by

$$p(x, \xi) = \begin{pmatrix} p_{11}(x, \xi) & \cdots & p_{1\nu}(x, \xi) \\ \vdots & \ddots & \vdots \\ p_{\nu 1}(x, \xi) & \cdots & p_{\nu\nu}(x, \xi) \end{pmatrix}, \tag{6-2}$$

where p_{jk} is the principal symbol of the operator P_{jk}. See [Chazarain and Piriou 1982] for more details. We say that P is not characteristic (or noncharacteristic) at a point $(x, \xi) \in T^*M \setminus \{0\}$ if $p(x, \xi)$ is an invertible linear mapping. We define the set of all characteristic points

$$\operatorname{Char} P = \{(x, \xi) \in T^*M \setminus \{0\} : P \text{ is characteristic at } (x, \xi)\}.$$

Theorem 6.1. *Suppose that $\mathcal{M}$ is a regular weight sequence that satisfies also* (M2′). *Let M be an $\mathcal{E}_{\mathcal{M}}$-manifold and E, F two ultradifferentiable vector bundles on M of the same fiber dimension. If $P(x, D)$ is a differential operator between E and F with $\mathcal{E}_{\mathcal{M}}$-coefficients and p its principal symbol, then*

$$\operatorname{WF}_{\mathcal{M}} u \subseteq \operatorname{WF}_{\mathcal{M}}(Pu) \cup \operatorname{Char} P, \qquad u \in \mathcal{D}'(M, E). \tag{6-3}$$

Proof. We write $f = Pu$. Since the problem is local we work on some chart neighborhood Ω such that in suitable trivializations of E and F we may write $u = (u_1, \dots, u_\nu) \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$, $f = (f_1, \dots, f_\nu) \in \mathcal{D}'(\Omega, \mathbb{C}^\nu)$, and P and its principal symbol p are of the forms (6-1) and (6-2), respectively. In particular, P is of order m on Ω.

We have to prove that if $(x_0,\xi_0)\notin \mathrm{WF}_{\mathcal{M}} f\cup \operatorname{Char} P$ then $(x_0,\xi_0)\notin \mathrm{WF}_{\mathcal{M}} u$. Assuming this we find that there has to be a compact neighborhood K of x_0 and a closed conic neighborhood V of ξ_0 in $\mathbb{R}^n\backslash\{0\}$ satisfying

$$\det p(x,\xi)\neq 0,\quad (x,\xi)\in K\times V,\tag{6-4}$$

$$(K\times V)\cap \mathrm{WF}_{\mathcal{M}}(Pu)_j=\varnothing,\quad j=1,\dots,\nu.\tag{6-5}$$

We consider the formal adjoint $Q=P^t$ of P with respect to the pairing

$$\langle f,g\rangle=\sum_{\tau=1}^{\nu}\int f_\tau(x)g_\tau(x)\,dx,\qquad f,g\in\mathcal{D}(\Omega,\mathbb{C}^\nu).$$

If $P=(P_{jk})_{jk}$ then $Q=(Q_{jk})_{jk}=(P^t_{kj})_{jk}$ where P^t_{jk} denotes the formal adjoint of the scalar operator $P_{jk}(x,D)=\sum p^\alpha_{jk}(x)D^\alpha$, i.e., for $v\in\mathcal{E}(\Omega)$,

$$P^t_{jk}(x,D)v=\sum_{|\alpha|\le m}(-D)^\alpha(p^\alpha_{jk}(x)v(x)).$$

Let $(\lambda_N)_N\subseteq\mathcal{D}(K)$ be a sequence of test functions satisfying $\lambda_N|_U\equiv 1$ on a fixed neighborhood U of x_0 for all N and for all $\alpha\in\mathbb{N}_0^n$ there are constants $C_\alpha,h_\alpha>0$ such that

$$|D^{\alpha+\beta}\lambda_N|\le C_\alpha(h_\alpha N)^{|\beta|},\quad |\beta|\le N.\tag{6-6}$$

If $u=(u^1,\dots,u^\nu)\in\mathcal{D}'(\Omega,\mathbb{C}^\nu)$, then the sequence $u^\tau_N=\lambda_{2N}u^\tau$ is bounded in $\mathcal{E}'$ and each of these distributions is equal to u^τ in U for all τ. Hence we have to prove that $(u^\tau_N)_N$ satisfies (3-1), i.e.,

$$\sup_{\substack{\xi\in V\\ N\in\mathbb{N}_0}}\frac{|\xi|^N|\hat{u}^\tau_N|}{Q^NM_N}<\infty$$

for a constant $Q>0$ independent of N.

In order to do so, set $\Lambda^\tau_N=\lambda_N e_\tau\in\mathcal{D}'(\Omega,\mathbb{C}^\nu)$ and observe

$$\hat{u}^\tau_N(\xi)=\langle u^\tau,e^{-i\langle\,.\,,\xi\rangle}\lambda_{2N}\rangle=\langle u,e^{-i\langle\,.\,,\xi\rangle}\Lambda^\tau_{2N}\rangle.$$

Following the argument in the proof of [Hörmander 1983, Theorem 8.6.1] we want to solve the equation $Qg^\tau=e^{-ix\xi}\Lambda^\tau_{2N}$. We make the ansatz

$$g^\tau=e^{-ix\xi}B(x,\xi)w^\tau,$$

where $B(x,\xi)$ is the inverse matrix of the transpose of $p(x,\xi)$, which exists if $(x,\xi)\in K\times V$ and is homogeneous of degree $-m$ in ξ; note that the principal symbol of $Q = P^t$ is $B^{-1}(x, -\xi)$. Using this we conclude that w has to satisfy

$$w^\tau - Rw^\tau = \Lambda_{2N}^\tau. \tag{6-7}$$

Here $R = R_1 + \cdots + R_m$ with $R_j|\xi|^j$ being (matrix) differential operators of order $\leq j$ with coefficients in $\mathcal{E}_{\mathcal{M}}$ that are homogeneous of degree 0 in ξ if $x \in K$ and $\xi \in V$.

A formal solution of (6-7) would be

$$w^\tau = \sum_{k=0}^{\infty} R^k \Lambda_{2N}^\tau.$$

However, this sum may not converge and even if it would converge, in the estimates we want to obtain we are not allowed to consider derivatives of arbitrary high order. Hence we set

$$w_N^\tau := \sum_{j_1+\cdots+j_k \leq N-m} R_{j_1} \cdots R_{j_k} \Lambda_{2N}^\tau$$

and compute

$$w_N^\tau - Rw_N^\tau = \Lambda_{2N}^\tau - \sum_{\sum_{s=1}^k j_s > N-m \geq \sum_{s=2}^k j_s} R_{j_1} \cdots R_{j_k} \Lambda_{2N}^\tau = \Lambda_{2N}^\tau - \rho_N^\tau.$$

Equivalently, we have

$$Q(e^{-ix\xi} B(x, \xi) w_N^\tau) = e^{-ix\xi}(\Lambda_{2N}^\tau(x) - \rho_N^\tau(x, \xi)).$$

We obtain now

$$\begin{aligned} \hat{u}_N^\tau(\xi) &= \langle u, e^{-i\langle\,.\,,\xi\rangle} \Lambda_{2N}^\tau \rangle \\ &= \langle u, Q(e^{-i\langle\,.\,,\xi\rangle} B(\,.\,,\xi) w_N^\tau)\rangle + \langle u, e^{-i\langle\,.\,,\xi\rangle} \rho_N^\tau(\,.\,,\xi)\rangle \\ &= \langle f, e^{-i\langle\,.\,,\xi\rangle} B(\,.\,,\xi) w_N^\tau\rangle + \langle u, e^{-i\langle\,.\,,\xi\rangle} \rho_N^\tau(\,.\,,\xi)\rangle \end{aligned} \tag{6-8}$$

and continue by estimating the right-hand side of (6-8). For this purpose we need the following lemma.

Lemma 6.2. *There exist constants C and h depending only on R and the constants appearing in* (6-6) *such that, if $j = j_1 + \cdots + j_k$ and $j + |\beta| \leq 2N$, we have*

$$|D^\beta(R_{j_1} \cdots R_{j_k} \Lambda_{2N}^\tau)_\sigma| \leq Ch^N M_N^{(j+|\beta|)/N} |\xi|^{-j}, \quad \xi \in V, \sigma = 1, \ldots, \nu. \tag{6-9}$$

Proof. Since both sides of (6-9) are homogeneous of degree $-j$ in $\xi \in V$ it suffices to prove the lemma for $|\xi| = 1$. Moreover we can write

$$(R_{j_1} \cdots R_{j_k} \Lambda_{2N}^\tau)_\sigma = \tilde{R}_\sigma^\tau \lambda_{2N}, \quad \sigma = 1, \ldots, \nu,$$

with $\tilde{R}_\sigma^\tau$ being a certain linear combination of products of components of the operators R_{j_s}. Especially the coefficients of $\tilde{R}_\sigma^\tau$ are all of class $\{\mathcal{M}\}$ on a common neighborhood of K and since there are only finitely many of them we may assume that they all can be considered as elements of $\mathcal{E}^q_{\mathcal{M}}(K)$ for some $q > 0$. We denote the set of the coefficients of the operators $\tilde{R}^\tau_{2N}\sigma$ by $\mathcal{R}$. Recall that (M4) implies that there is a constant $\delta > 0$ such that $N < \delta \sqrt[N]{M_N}$. Hence (6-6) implies that for all $\alpha \in \mathbb{N}_0^n$ we have

$$|D^{\alpha+\beta}\lambda_{2N}| \le C_\alpha h_\alpha^{|\beta|}(2N)^{|\beta|} \le C_\alpha (2h_\alpha\delta)^{|\beta|} M_N^{|\beta|/N} \tag{6-10}$$

for $|\beta| \le 2N$.

Considering all these arguments, the proof of the lemma is a consequence of the following result. □

Lemma 6.3. *Let $K \subseteq \Omega$ be compact, $(\lambda_N)_N \subseteq \mathcal{D}(K)$ be a sequence satisfying* (6-10), *$q \ge 1$ and $a_1, \dots, a_{j-1} \in \mathcal{R} \cup \{1\}$. Then there are constants $C, h > 0$ independent of N such that for $j \le 2N$ we have*

$$|D_{i_1}(a_1 D_{i_2}(a_2 \cdots D_{i_{j-1}}(a_{j-1} D_{i_j}\lambda_{2N})\cdots))| \le C h^j M_N^{j/N}. \tag{6-11}$$

Proof. We begin by noting that (M3) implies that $m_j m_{k-j} \le m_k$ for all $j \le k \in \mathbb{N}$; see [Komatsu 1973]. Furthermore we can assume that there is a constant $C_1 > 1$ such that for all $k \le j-1$,

$$|D^\alpha a_k| \le C_1 q^{|\alpha|} M_{|\alpha|}$$

on K. Obviously the expression $D_{i_1}a_1 D_{i_2}a_2 \cdots D_{i_{j-1}}a_{j-1}D_{i_j}\lambda_{2N}$ can be written as a sum of terms of the form $(D^{\alpha_1}a_1)\cdots(D^{\alpha_{j-1}}a_{j-1})D^{\alpha_j}\lambda_{2N}$ where $|\alpha_1|+\cdots+|\alpha_j| = j$.

We set $h \ge C_1 \max(q, h_0)$. If there are $C_{k_1,\dots,k_j}$ terms with $|\alpha_1| = k_1, \dots, |\alpha_j| = k_j$ then we have the following estimate on K:

$$\begin{aligned}
&|D_{i_1}a_1 D_{i_2}a_2 \cdots D_{i_{j-1}}a_{j-1}D_{i_j}\lambda_{2N}| \\
&\qquad \le C \sum q^{j-k_j} C_1^{j-1} C_{k_1,\dots,k_j} m_{k_1}\cdots m_{k_{j-1}} k_1!\cdots k_{j-1}!\, h_0^{k_j} M_N^{k_j/N} \\
&\qquad \le C h^j \sum m_{j-k_j} C_{k_1,\dots,k_j} k_1!\cdots k_{j-1}!\, M_N^{k_j/N} \\
&\qquad \le C h^j \sum C_{k_1,\dots,k_j} \frac{k_1!\cdots k_{j-1}!}{(j-k_j)!} M_{j-k_j} M_N^{k_j/N}.
\end{aligned}$$

Since $j - k_j \le 2N$, we observe that (M2′) implies that there are two indices $\sigma_1, \sigma_2 \le N$, $\sigma_1 + \sigma_2 = k - k_j$, such that $M_{k-k_j} \le C\rho^{j-k_j} M_{\sigma_1} M_{\sigma_2}$ for some constants C, ρ that are independent of j and N. Now we have

$$M_{j-k_j} M_N^{k_j/N} = C\rho^{j-k_j} M_{\sigma_1} M_{\sigma_2} M_N^{k_j/N} \le C\rho^{j-k_j} M_N^{(\sigma_1+\sigma_2)/N} M_N^{k_j/N} = C\rho^{j-k_j} M_N^{j/N}.$$

since $\sqrt[N]{M_N}$ is increasing. As noted in [Albanese et al. 2010] it is possible to estimate

$$\frac{k_1!\cdots k_{j-1}!}{(j-k_j)!} = \frac{k_1!\cdots k_{j-1}!k_j!j!}{(j-k_j)!k_j!j!} \le 2^j \frac{k_1!\cdots k_j!}{j!},$$

and also (see [Hörmander 1983, page 308])

$$\sum C_{k_1,\dots,k_j} k_1! \cdots k_j! = (2j-1)!!.$$

Since $(2j-1)!!/(j!2^j) \leq 1$ we obtain

$$\begin{aligned}|D_{i_1}a_1 D_{i_2}a_2 \cdots D_{i_{j-1}}a_{j-1}D_{i_j}\lambda_{2N}| &\leq C(4\rho h)^j \frac{(2k-1)!!}{j!2^j} M_N^{j/N} \\ &\leq C(4\rho h)^j M_N^{j/N}. \qquad \square\end{aligned}$$

In order to estimate $\hat{u}_N^\tau$, we note that due to the boundedness of the sequence $(u_N^\tau)_N \subseteq \mathcal{E}'$ the Banach and Steinhaus theorem implies that there are constants μ and c such that

$$|\hat{u}_N^\tau| \leq c(1+|\xi|)^\mu$$

for all N and therefore if $|\xi| \leq \sqrt[N]{M_N}$ then

$$|\xi|^N |\hat{u}_N^\tau| \leq C M_N^{(N+\mu)/N} \leq C\delta^{\mu N} M_N, \tag{6-12}$$

since (M2) implies that there is a constant $\delta > 0$ such that $\sqrt[N]{M_N} \leq \delta \sqrt[N-1]{M_{N-1}}$.

Hence it suffices to estimate the terms on the right-hand side of (6-8) for $\xi \in V$, $|\xi| > \sqrt[N]{M_N}$. We begin with the second term.

As in the scalar case there are constants μ and $C > 0$ that only depend on u and K such that for all $\psi \in \mathcal{D}(\Omega, \mathbb{C}^\nu)$ with $\operatorname{supp}\psi \subseteq K$,

$$|\langle u, \psi\rangle| \leq C \sum_{|\alpha|\leq\mu} \sup_K |D^\alpha \psi|.$$

Note that $\operatorname{supp}_x \rho_N^\tau(\,.\,,\xi) \subseteq K$ for all $\xi \in V$ and $N \in \mathbb{N}$. Thence

$$\begin{aligned}|\langle u, e^{-i\langle\,.\,,\xi\rangle}\rho_N^\tau(\,.\,,\xi)\rangle| &\leq C \sum_{|\alpha|\leq\mu}\sum_{\beta\leq\alpha} |\xi|^{|\alpha|-|\beta|} \sup_{x\in K}|D_x^\beta \rho_N^\tau(x,\xi)| \\ &\leq C \sum_{|\alpha|\leq\mu} |\xi|^{\mu-|\alpha|} \sup_{x\in K}|D_x^\alpha \rho_N^\tau(x,\xi)|\end{aligned}$$

for $\xi \in V$, $|\xi| \geq 1$ and $N \in \mathbb{N}$. There are at most 2^N terms of the form $R_{j_1}\cdots R_{j_k}\Lambda_{2N}^\tau$ in ρ_N^τ and each term can be estimated by (6-9) by setting $N \geq j > N-m$ and hence

$$|D_x^\alpha \rho_N^\tau(x,\xi)| \leq Ch^N 2^N |\xi|^{m-N} M_N^{(N+|\alpha|)/N}$$

for $x \in K$ and $\xi \in V$, $|\xi| > 1$. Applying (M2) therefore gives

$$\begin{aligned}|\langle u, e^{-i\langle\,.\,,\xi\rangle}\rho_N^\tau(\,.\,,\xi)\rangle| &\leq Ch^N 2^N |\xi|^{\mu+m-N} M_N^{(N+\mu)/N} \\ &\leq Ch^N |\xi|^{\mu+m-N} M_N.\end{aligned} \tag{6-13}$$

The first term in (6-8) is more difficult to estimate. To begin with, observe that Lemma 6.2 gives

$$\begin{aligned}|D^\beta w_N^\tau(x,\xi)| &\le Ch^N \sum_{j=0}^{N-m} M_N^{(j+|\beta|)/N}|\xi|^{-j}\\ &\le Ch^N M_N^{|\beta|/N}\sum_{j=0}^{N-m} M_N^{(j-j)/N}\\ &\le Ch^N M_N^{|\beta|/N}(N-m)\\ &\le Ch^N M_N^{|\beta|/N}\end{aligned}$$

for $N>m$, $|\beta|\le N$ and $\xi\in V$, $|\xi|>\sqrt[N]{M_N}$. Recall that for $N\le m$ we have set $w_N^\tau=\Lambda_{2N}^\tau=\lambda_{2N}^\tau e_\tau$. Hence by the above and (6-10) it follows that

$$|D^\beta w_N^\tau(x,\xi)|\le Ch^N M_N^{|\beta|/N} \tag{6-14}$$

for all $N\in\mathbb{N}$, $|\beta|\le N$ and $\xi\in V$, $|\xi|>\sqrt[N]{M_N}$.

On the other hand, since the components of $B(x,\xi)$ are ultradifferentiable of class $\{\mathcal{M}\}$ and homogeneous in $\xi\in V$ of degree $-m$ we note that it is possible to show, as above, using an analogue to Lemma 6.2, the estimate

$$|D_x^\beta(w_N^\tau(x,\xi)|\xi|^m B(x,\xi))|\le Ch^N M_N^{|\beta|/N}\quad |\beta|\le N,\ \xi\in V,\ |\xi|>\sqrt[N]{M_N}. \tag{6-15}$$

In order to finish the proof of Theorem 6.1 we need an additional lemma.

Lemma 6.4. *Let $f\in\mathcal{D}'(\Omega)$, K be a compact subset of Ω and $V\subseteq\mathbb{R}^n\setminus\{0\}$ be a closed cone such that*

$$\mathrm{WF}_{\mathcal{M}} f\cap(K\times V)=\varnothing.$$

Furthermore let $w_N\in(\mathcal{E}(\Omega\times V)$ be such that $\operatorname{supp} w_N(\,.\,,\xi)\subseteq K$ *for all $\xi\in V$ and* (6-14) *holds.*

If μ denotes the order of f in a neighborhood of K then

$$|\widehat{w_N f}(\xi)|=|\langle w_N(\,.\,,\xi)f,e^{-i\langle\,.\,,\xi\rangle}\rangle|\le Ch^N|\xi|^{\mu+n-N}M_{N-\mu-n}, \tag{6-16}$$

for $N>\mu+n$ and $\xi\in\Gamma$, $|\xi|>\sqrt[N]{M_N}$.

Proof. By Proposition 3.4 we can find a sequence $(f_N)_N$ that is bounded in $\mathcal{E}'^\mu$ and equal to f in some neighborhood of K and such that

$$|\hat f_N(\eta)|\le CQ^N M_N/|\eta|^N,\quad \eta\in W, \tag{6-17}$$

where W is a conic neighborhood of Γ. Then $w_N f=w_N f_{N'}$ for $N'=N-\mu-n$.

If we denote the partial Fourier transform of $w_N(x,\xi)$ by

$$\hat w_N(\eta,\xi)=\int_\Omega e^{-ix\eta}w_N(x,\xi)\,dx$$

then obviously (6-14) is equivalent to

$$|\eta^\beta \hat{w}_N(\eta,\xi)| \le Ch^N M_N^{|\beta|/N}$$

for $|\beta| \le N$, $\xi \in V$, $|\xi| > \sqrt[N]{M_N}$ and $\eta \in \mathbb{R}^n$. Since $|\eta| \le \sqrt{n}\max|\eta_j|$ we conclude that

$$|\eta|^\ell |\hat{w}_N(\eta,\xi)| \le Ch^N M_N^{\ell/N} \tag{6-18}$$

for $\ell \le N$, $\eta \in \mathbb{R}^n$ and $\xi \in V$, $|\xi| > \sqrt[N]{M_N}$. Hence we obtain

$$\begin{aligned}(|\eta| + M_N^{1/N})^N |\hat{w}_N(\eta,\xi)| &= \sum_{k=0}^N \binom{N}{k} M_N^{k/N} |\eta|^{N-k} |\hat{w}_N(\eta,\xi)| \\ &\le Ch^N \sum_{k=0}^N \binom{N}{k} M_N^{k/N} M_N^{(N-k)/N} \\ &\le Ch^N M_N\end{aligned} \tag{6-19}$$

if $\eta \in \mathbb{R}^n$, $\xi \in V$ and $|\xi| > \sqrt[N]{M_N}$. Like Hörmander [1983] and Albanese, Jornet and Oliaro [2010], we consider

$$\begin{aligned}\widehat{w_N f}(\xi) &= \frac{1}{(2\pi)^n} \int \hat{w}_N(\eta,\xi) \hat{f}_{N'}(\xi-\eta)\,d\eta \\ &= \frac{1}{(2\pi)^n} \int_{|\eta|<c|\xi|} \hat{w}_N(\eta,\xi) \hat{f}_{N'}(\xi-\eta)\,d\eta \\ &\qquad + \frac{1}{(2\pi)^n} \int_{|\eta|>c|\xi|} \hat{w}_N(\eta,\xi) \hat{f}_{N'}(\xi-\eta)\,d\eta\end{aligned}$$

for some $0 < c < 1$. The boundedness of the sequence $(f_N)_N$ in $\mathcal{E}'^\mu$ implies as before that

$$|\hat{f}_N(\xi)| \le C(1+|\xi|)^\mu.$$

Hence we conclude that

$$\begin{aligned}(2\pi)^n |\widehat{w_N f}(\xi)| &\le \|\hat{w}_N(\,.\,,\xi)\|_{L^1} \sup_{|\xi-\eta|<c|\xi|} |\hat{f}_{N'}(\eta)| \\ &\qquad + C\int_{|\eta|>c|\xi|} |\hat{w}_N(\eta,\xi)|(1+c^{-1})^\mu (1+|\eta|)^\mu\,d\eta\end{aligned}$$

since $|\eta| \ge c|\xi|$ gives

$$|\xi - \eta| \le (1+c^{-1})|\eta|.$$

On the other hand, there is a constant $0 < c < 1$ such that $\eta \in W$ when $\xi \in V$ and $|\xi - \eta| \le c|\xi|$. Then $|\eta| \ge (1-c)|\xi|$ and we can replace the supremum above by $\sup_{\eta\in W}|\hat{f}_{N'}(\eta)|$. Furthermore, by (6-19),

$$\begin{aligned}\|\hat{w}_N(\,.\,,\xi)\|_{L_1} &= \int_{\mathbb{R}^n}|\hat{w}_N(\eta,\xi)|\,d\eta \le Ch^N M_N\int_{\mathbb{R}^n}(|\eta|+\sqrt[N]{M_N})^N\,d\eta\\ &\le Ch^N M_N\int_{\sqrt[N]{M_N}}^{\infty}s^{-N'-1}\,ds\\ &\le Ch^N M_N\frac{M_N^{N'/N}}{N'}\le Ch^N M_N^{\mu+n}.\end{aligned}$$

Thence it follows for $\xi\in V,\ |\xi|>\sqrt[N]{M_N}$, that

$$\begin{aligned}|\widehat{w_N f}(\xi)| &\le C_1(1-c)^{-N'}\|\hat{w}_N(\,.\,,\xi)\|_{L^1}|\xi|^{-N'}\sup_{\eta\in W}|\hat{f}_{N'}(\eta)||\eta|^{N'}\\ &\qquad + C_2(1+c^{-1})^{N'+\mu}\int_{|\eta|>c|\xi|}(1+|\eta|)^{\mu}|\hat{w}_N(\eta,\xi)|\,d\eta\\ &\le C_1h^N M_N^{(n+\mu)/N}Q^{N'}M_{N'}|\xi|^{-N'}+C_2\tilde{h}^N M_N\int_{|\eta|>c|\xi|}|\eta|^{-N'-n}\,d\eta\\ &\le Ch^N M_{N'}|\xi|^{-N'},\end{aligned}$$

where we have also used (M2), (6-17) and (6-18). □

Due to (6-15) we can replace w_N in (6-16) with $(w_N^\tau|\xi|^m B)_\sigma,\ \sigma=1,\dots,\nu$, and obtain

$$|\langle f, e^{-i\langle\,.\,,\xi\rangle}B(\,.\,,\xi)w_N^\tau\rangle|\le Ch^N|\xi|^{\mu+n-N}M_{N-\mu-n} \tag{6-20}$$

for $\xi\in V,\ |\xi|>\sqrt[N]{M_N}$.

We consider now the sequence $(v_N^\tau)_N=(u^\tau_{N+m+n+\mu})_N$. If $\xi\in V,\ |\xi|\le(M_{N''})^{1/N''}$, $N''=N+\mu+n+m$, then by (6-12)

$$|\xi|^N|\hat{v}_N^\tau|\le C\delta^N M_{N''}\le C\delta_1^N M_N\quad\text{for some }\delta_1>0.$$

On the other hand (6-8), (6-13) and (6-20) give

$$|\xi|^N|\hat{v}_N^\tau(\xi)|\le C_1h_1^N M_{N+m}|\xi|^{-m}+C_2h_2^N M_{N+\mu+m+n}|\xi|^{-n}\le Ch^N M_N$$

for $\xi\in V,\ |\xi|>(M_{N''})^{1/N''}$.

Thus we have shown for all $\tau=1,\dots,\nu$ that the bounded sequence $(v_N^\tau)_N\subseteq\mathcal{E}'(\Omega)$ satisfies

$$\sup_{\substack{\xi\in V\\ N\in\mathbb{N}}}\frac{|\xi|^N|v_N^\tau(\xi)|}{Q^N M_N}<\infty$$

for some $Q>0$. Clearly $u^\tau|_U\equiv(v_N^\tau)|_U$ and hence

$$(x_0,\xi_0)\notin \mathrm{WF}_{\mathcal{M}}\,u^\tau$$

for all $\tau=1,\dots,\nu$. □

For elliptic operators, i.e., operators P with $\operatorname{Char} P = \varnothing$, the following holds obviously.

Corollary 6.5. *If P is an elliptic operator with ultradifferentiable coefficients of class $\{\mathcal{M}\}$ and $u \in \mathcal{D}'$ then*

$$\mathrm{WF}_{\mathcal{M}}\, Pu = \mathrm{WF}_{\mathcal{M}}\, u.$$

7. Uniqueness theorems

Hörmander [1971a] and Kawai (see [Sato et al. 1973]) independently noticed that results like Theorem 6.1 in the analytic category can be used to prove Holgrem's uniqueness theorem [1901]. We show here that Theorem 6.1 can also be used to give a quasianalytic version of Holmgren's uniqueness theorem. We follow mainly the presentation of [Hörmander 1983].

First recall [Hörmander 1993, Theorem 6.1]:

Proposition 7.1. *Let $I \subseteq \mathbb{R}$ be an interval and $x_0 \in \partial \operatorname{supp} u$, then $(x_0, \pm 1) \in \mathrm{WF}_{\mathcal{M}}\, u$ for any quasianalytic regular weight sequence $\mathcal{M}$.*

As noted in [Hörmander 1993], Proposition 7.1 immediately generalizes to a result for several variables (see [Hörmander 1983, Theorem 8.5.6], and see [Kim et al. 2001] for a similar result):

Theorem 7.2. *Let $\mathcal{M}$ be a quasianalytic regular weight sequence, $u \in \mathcal{D}'(\Omega)$, $x_0 \in \operatorname{supp} u$ and $f : \Omega \to \mathbb{R}$ be a function of class $\{\mathcal{M}\}$ with the properties*

$$df(x_0) \neq 0, \quad f(x) \leq f(x_0) \quad \text{if } x_0 \neq x \in \operatorname{supp} u.$$

Then we have

$$(x_0, \pm df(x_0)) \in \mathrm{WF}_{\mathcal{M}}\, u.$$

Proof. If we replace f by $f(x) - |x - x_0|^2$ we see that we may assume that $f(x) < f(x_0)$ for $x_0 \neq x \in \operatorname{supp} u$. Furthermore, since $df(x_0) \neq 0$ we can assume that $x_0 = 0$ and $f(x) = x_n$. Next we choose a neighborhood U of 0 in $\mathbb{R}^{n-1}$ so that $U \times \{0\} \Subset \Omega$. By assumption $\operatorname{supp} u \cap (\bar{U} \times \{0\}) = \{0\}$. Hence there is an open interval $I \subseteq \mathbb{R}$ with $0 \in I$ such that

$$U \times I \Subset \Omega \quad \text{and} \quad \operatorname{supp} u \cap (\partial U \times I) = \varnothing. \tag{7-1}$$

If A is an entire analytic function in the variables $x' = (x_1, \dots, x_{n-1})$ then we consider the distribution $U_A \in \mathcal{D}'(I)$ given by $\langle U_A, \psi \rangle = \langle u_A \otimes \psi \rangle$. Note U_A is well defined due to (7-1). By [Hörmander 1983, Theorem 8.5.4'] we have that

$$\mathrm{WF}_{\mathcal{M}}(U_A) \subseteq \bigl\{(x_n, \xi_n) \in I \times \mathbb{R} \backslash \{0\} \mid \text{there exists } x' \in U : (x', x_n, 0, \xi_n) \in \mathrm{WF}_{\mathcal{M}}\, u\bigr\}.$$

Note that (x', x_n) above must be close to 0 for x_n small.

Assume, e.g., that $(0, e_n) \notin \mathrm{WF}_{\mathcal{M}}\, u$, $e_n = (0, \dots, 0, 1)$. Then I can be chosen so small that $(x, e_n) \notin \mathrm{WF}_{\mathcal{M}}\, u$ for $x \in U \times I$. We conclude that $(x_n, 1) \notin \mathrm{WF}_{\mathcal{M}}\, U_A$ if $x_n \in I$. Proposition 7.1 implies that $U_A = 0$ on I since $U_A = 0$ on $I \cap \{x_n > 0\}$. That means actually that

$$\langle u|_{U\times I}, A \otimes \varphi\rangle = 0$$

for all $\varphi \in \mathcal{D}(I)$. Since A was chosen arbitrarily from a dense subset of $\mathcal{E}(\mathbb{R}^{n-1})$ it follows that $u = 0$ on $U \times I$. □

For the rest of this section $\mathcal{M}$ is going to be a quasianalytic regular weight sequence that satisfies (M2′).

In order to give Theorem 7.2 a more invariant form we need to recall some facts from [Hörmander 1983].

Definition 7.3. Let F be a closed subset of a $\mathcal{C}^2$ manifold X. The *exterior normal set* $N_e(F) \subseteq T^*X\backslash\{0\}$ is defined as the set of all points (x_0, ξ_0) such that $x_0 \in F$ and there exists a real-valued function $f \in \mathcal{C}^2(X)$ with $df(x_0) = \xi_0 \neq 0$ and $f(x) \le f(x_0)$ when $x \in F$.

In fact, following the remarks in [Hörmander 1983, page 300] we observe that it would be sufficient for f to be defined locally around x_0. Furthermore f could then also be chosen to be real-analytic in a chart neighborhood near x_0. If g is $\mathcal{C}^1$ near a point $\tilde{x} \in F$ and $dg(\tilde{x}) = \tilde{\xi} \neq 0$ then $(\tilde{x}, \tilde{\xi}) \in \overline{N_e(F)} \subseteq T^*X\backslash\{0\}$. It is clear that if $(x_0, \xi_0) \in N_e(F)$ then $x_0 \in \partial F$. In fact, if $\pi : T^*\Omega \to \Omega$ is the canonical projection then $\pi(N_e(F))$ is dense in ∂F; see [Hörmander 1983, Proposition 8.5.8.]. The *interior normal set* $N_i(F) \subseteq T^*X\backslash\{0\}$ consists of all points (x_0, ξ_0) with $(x_0, -\xi_0) \in N_e(F)$. The *normal set* of F is defined as

$$N(F) = N_e(F) \cup N_i(F) \subseteq T^*X\backslash\{0\}.$$

In this notation Theorem 7.2 takes the following form.

Theorem 7.4. *Let $u \in \mathcal{D}'(\Omega)$. Then*

$$\overline{N(\operatorname{supp} u)} \subseteq \mathrm{WF}_{\mathcal{M}}\, u.$$

Theorem 7.4 combined with Theorem 6.1 gives:

Theorem 7.5. *Let P be a partial differential operator with $\mathcal{E}_{\mathcal{M}}$-coefficients and $u \in \mathcal{D}'(\Omega)$ be a solution of $Pu = 0$. Then*

$$\overline{N(\operatorname{supp} u)} \subseteq \operatorname{Char} P,$$

i.e., the principal symbol p_m of P must vanish on $N(\operatorname{supp} u)$.

In fact, we can now derive the *quasianalytic Holmgren uniqueness theorem.* We recall that a $\mathcal{C}^1$-hypersurface M is characteristic at a point x with respect to a partial differential operator P, if and only if for a defining function φ of M near x we have that $(x, d\varphi(x)) \in \operatorname{Char} P$.

Corollary 7.6. *Let P be a partial differential operator with $\mathcal{E}_{\mathcal{M}}$-coefficients. If X is a $\mathcal{C}^1$-hypersurface in Ω that is noncharacteristic at x_0 and $u \in \mathcal{D}'(\Omega)$ is a solution of $Pu = 0$ that vanishes on one side of X near x_0 then $u \equiv 0$ in a full neighborhood of x_0.*

In fact, (see [Zachmanoglou 1969]) it is possible to reformulate Corollary 7.6.

Corollary 7.7. *Let P be a differential operator with coefficients in $\mathcal{E}_{\mathcal{M}}(\Omega)$. Furthermore let $F \in \mathcal{E}_{\mathcal{M}}(\mathbb{R}^n)$ be a real-valued function of the form*

$$F(x) = f(x') - x_n, \quad x' = (x_1, \dots, x_{n-1}),$$

where $f \in \mathcal{E}_{\mathcal{M}}(\mathbb{R}^{n-1})$ and suppose that the level hypersurfaces of F are nowhere characteristic with respect to P in Ω. Set also $\Omega_c = \{x \in \Omega \mid F(x) < c\}$ for $c \in \mathbb{R}$. If $u \in \mathcal{D}'(\Omega)$ is a solution of $P(x, D)u = 0$ and there is $c \in \mathbb{R}$ such that $\Omega_c \cap \operatorname{supp} u$ is relatively compact in Ω, then $u = 0$ in Ω_c.

Proof. We set for $c \in \mathbb{R}$,

$$\omega_c = \{x \in \Omega \mid F(x) = c\}.$$

Note that for each $c \in \mathbb{R}$ the set ω_c is not relatively compact in Ω. Therefore Ω_c is not relatively compact in Ω either for any c since $\partial\Omega_c = \omega_c$.

By assumption there is a $c \in \mathbb{R}$ such that $K = \operatorname{supp} u \cap \overline{\Omega}_c$ is compact in Ω. In particular, K is bounded in Ω. Hence there has to be $\tilde{c} < c$ such that

$$K \subseteq \{x \in \Omega \mid \tilde{c} \leq F(x) \leq c\}.$$

Let $c_1 < c$ be the greatest real number such that the inclusion above holds for $\tilde{c} = c_1$. Since K is compact there is a point $p \in \partial K$ such that $F(p) = c_1$. It follows that $p \in \partial \operatorname{supp} u \cap \omega_{c_1}$. Thus we can apply Corollary 7.6 because ω_{c_1} is nowhere characteristic for P. Hence u vanishes in a full neighborhood of p. This contradicts the choice of c_1. We conclude that u has to vanish on Ω_c. □

Note that Hörmander [1963] used the analytic version of Corollary 7.7 to prove Holmgren's uniqueness theorem.

Remark 7.8. We have formulated our results for scalar operators on open sets of $\mathbb{R}^n$ but they remain of course valid on ultradifferentiable manifolds of class $\{\mathcal{M}\}$. Actually, all the conclusions in this section hold even for determined systems of operators and vector-valued distributions. Indeed, we have only to verify that Theorem 7.2 holds also for distributions with values in $\mathbb{C}^\nu$, but this is trivial: If $f(x) \leq f(x_0)$ for $x \in \operatorname{supp} u$ then $f(x) \leq f(x_0)$ for all $x \in \operatorname{supp} u_j$ and any $1 \leq j \leq n$, since $\operatorname{supp} u = \bigcup_{j=1}^{\nu} \operatorname{supp} u_j$. Hence Theorem 7.2 implies

$$(x_0, \pm df(x_0)) \in \bigcap_{j=1}^{\nu} \mathrm{WF}_{\mathcal{M}}\, u_j \subseteq \mathrm{WF}_{\mathcal{M}}\, u.$$

Following an idea of Bony [1969; 1976], it is possible to generalize the results above. For the formulation we need some additional notation. Consider a smooth real-valued function p on $T^*\Omega$. The *Hamiltonian vector field* H_p of p is defined by

$$H_p = \sum_{j=1}^{n} \left(\frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

An integral curve of H_p, i.e., a solution of the Hamilton–Jacobi equations

$$\frac{dx_j}{dt} = \frac{\partial p}{\partial \xi_j}(x, \xi),$$
$$\frac{d\xi_j}{dt} = -\frac{\partial p}{\partial x_j}(x, \xi),$$

$j = 1, \dots, n$, is called a *bicharacteristic* if p vanishes on it. If q is another smooth real-valued function on $T^*\Omega$ then the *Poisson bracket* is defined by $\{p, q\} := H_p(q)$ or, in coordinates,

$$\{p, q\} = \sum_{j=1}^{n} \left(\frac{\partial p}{\partial \xi_j} \frac{\partial q}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial q}{\partial \xi_j} \right).$$

See [Grigis and Sjöstrand 1994] or [Hörmander 1983] for more details.

We continue by recalling a result of Sjöstrand [1982] (see also [Hörmander 1983]).

Theorem 7.9. *Let F be a closed subset of Ω and suppose that $p \in \mathcal{E}(T^*\Omega \setminus \{0\})$ is real-valued and vanishes on $N_e(F)$. If $(x_0, \xi_0) \in N_e(F)$ then the bicharacteristic $t \mapsto (x(t), \xi(t))$ with $(x(0), \xi(0)) = (x_0, \xi_0)$ stays for $|t|$ small in $N_e(F)$.*

The analogous statement is of course also true for $N_i(F)$ replacing $N_e(F)$.

Corollary 7.10 [Bony 1976]. *Let F be a closed subset of Ω and set*

$$\mathcal{N}_F := \{p \in \mathcal{E}(T^*\Omega \setminus \{0\}) \mid p \equiv 0 \text{ on } N(F)\}.$$

Then $\mathcal{N}_F$ is an ideal in $\mathcal{E}(T^*\Omega \setminus \{0\})$ that is closed under Poisson brackets.

We obtain the quasianalytic version of a result of Bony [1969; 1976].

Theorem 7.11. *Let P a differential operator with $\mathcal{E}_{\mathcal{M}}$-coefficients on Ω and Π the Poisson algebra that is generated by all functions $f \in \mathcal{E}(T^*\Omega \setminus \{0\})$ that vanish on* Char *P*.

If $u \in \mathcal{D}'(\Omega)$ is a solution of the homogeneous equation $Pu = 0$ then all functions in Π have to vanish on $N(\operatorname{supp} u)$.

Corollary 7.12. *If the elements of Π have no common zeros and u vanishes in a neighborhood of a point $p_0 \in \Omega$ then u must vanish in the connected component of Ω that contains p_0.*

We continue by taking a closer look at Theorem 7.9. Let $\pi : T^*\Omega \to \Omega$ be the canonical projection and $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. If q is a smooth function on $T^*\Omega\setminus\{0\}$ that vanishes on $N(F)$, $F \subseteq \Omega$ closed, and $\lambda(t)$ is the bicharacteristic through (x_0, ξ_0) then we conclude that the bicharacteristic curve $\gamma(t) = \pi \circ \lambda$ must stay in ∂F for small t in view of the remarks before Theorem 7.4.

Now suppose that Q is a real vector field on Ω and q its symbol. If we denote by γ the integral curve of Q through x_0 and by λ the bicharacteristic of q through (x_0, ξ_0) where (x_0, ξ_0) then it is trivial that $\gamma = \pi \circ \lambda$.

Definition 7.13. We say that a partial differential operator P on Ω with $\mathcal{E}_{\mathcal{M}}$-coefficients is $\mathcal{M}$-admissible if and only if there are ultradifferentiable real-valued vector fields $Q_1, \dots, Q_d$ with symbols $q_1, \dots, q_d$ such that each q_j vanishes on $\operatorname{Char} P$.

Following the approach of Sjöstrand [1982] we can generalize results of Zachmanoglou [1972] (see also [Bony 1976]) to the quasianalytic setting.

Proposition 7.14. *Let P be an $\mathcal{M}$-admissible operator. If $\mathcal{L} = \mathcal{L}(Q_1, \dots, Q_d)$ is the Lie algebra generated by the vector fields Q_j, $j = 1, \dots, d$, $\varphi \in \mathcal{C}^1(\Omega, \mathbb{R})$ near a point $x_0 \in \Omega$ such that $(x_0, \varphi'(x_0)) \in \operatorname{Char} P$ and $u \in \mathcal{D}'(\Omega)$ is a solution of $Pu = 0$ such that near x_0 we have $x_0 \in \operatorname{supp} u \subseteq \{\varphi \geq 0\}$. Then each $Q \in \mathcal{L}$ is tangent to $\{\varphi = 0\}$ at x_0 and the local Nagano leaf $\gamma_{x_0}(\mathcal{L})$ is contained in $\operatorname{supp} u$.*

Proof. By assumption all $Q_1, \dots, Q_d$ are tangent to $\{\varphi = 0\}$ at x_0 and hence also all $Q \in \mathcal{L}$. From the remarks before Definition 7.13 and Theorem 7.4 we see that all integral curves of the vector fields in $\mathcal{L}$ must be contained in $\partial \operatorname{supp} u$ for a small neighborhood of x_0. Inspecting the construction of the representative of the local Nagano leaf in the proof of Theorem 2.17 we see that $\gamma_{x_0}(\mathcal{L}) \subseteq \operatorname{supp} u$ near x_0. $\square$

In fact, we have the following global theorem (for the analytic case see [Zachmanoglou 1972; Bony 1976, Theorem 2.4.]).

Theorem 7.15. *Let P an $\mathcal{M}$-admissable differential operator. If $u \in \mathcal{D}'(\Omega)$ is a solution of $Pu = 0$ and $p_0 \notin \operatorname{supp} u$ then every integral curve of the vector fields $Q_1, \dots, Q_d$ through p_0 stays in $\Omega \setminus \operatorname{supp} u$.*

Proof. Let $\Gamma = \Gamma_{p_0}(\mathcal{L})$ be the global Nagano leaf of $\mathcal{L} = \mathcal{L}(Q_1, \dots, Q_d)$ through p_0 and suppose that $\partial \operatorname{supp} u \cap \Gamma \neq \varnothing$. Then there has to be a point $q_0 \in \Gamma \cap \partial \operatorname{supp} u$

such that for all neighborhoods $V \subseteq \Omega$ of x_0 we have

$$(\Gamma \cap V) \cap (\Omega \setminus \operatorname{supp} u) \neq \varnothing.$$

Let V be small enough such that $\Gamma \cap V$ is the representative of the local Nagano leaf of $\mathcal{L}$ at q_0 constructed in the proof of Theorem 2.17. Then

$$\Gamma \setminus \operatorname{supp} u \cap V \neq \varnothing.$$

Thence there is a vector field $X \in \mathcal{L}$ such that if $\gamma(t) = \exp tX$ is the integral curve of X through q_0 then $\gamma(0) = q_0$ and $\gamma(1) = q_1 \in V \setminus \operatorname{supp} u$. Possibly shrinking V and applying an ultradifferentiable coordinate change in V we may assume that $q_0 = 0$, $q_1 = (0, \dots, 0, 1)$ and

$$X = \frac{\partial}{\partial x_n}.$$

We note that in these new coordinates the assumption on P can be stated in the following way. Let $\xi \in \mathbb{R}^n$ with $\xi_n \neq 0$ then $p_m(x, \xi) \neq 0$ for all $x \in V$. There is also a neighborhood $V_1 \subseteq V$ of q_1 such that u vanishes on V_1.

We adapt the proof of [Zachmanoglou 1969, Theorem 1]. Let $r > 0$ and $\delta > 0$ be small enough so that

$$U = \{x \in \mathbb{R}^n \mid |x'| < r,\ -\delta < x_n < 1\}$$

is contained in V and

$$\{x \in \mathbb{R}^n \mid |x'| < r,\ x_n = 1\} \subseteq V_1.$$

We consider the real-analytic function

$$F(x) = (1 + \delta)\frac{|x'|^2}{r^2} - \delta - x_n.$$

The normals of the level hypersurfaces of F are always nonzero in the direction of the n-th unit vector. It follows that the level hypersurfaces are everywhere noncharacteristic with respect to P in V. Set

$$U_1 = \left\{ x \in U : F(x) < -\frac{\delta}{2} \right\}$$

and note that if $x \in U_1$ then $x_n > -\delta/2$. It is easy to see that $U_1 \cap \operatorname{supp} u$ is relatively compact in U. We conclude that $u = 0$ in U_1 by Corollary 7.7. That is a contradiction to the assumption $q_0 \in \partial \operatorname{supp} u$. □

If $Q_1, \dots, Q_d$ are real-valued vector fields with $\mathcal{E}_{\mathcal{M}}$-coefficients, the operators

$$P_0 = Q_1 + i Q_2, \qquad P_k = \sum_{j=1}^{d} Q_j^{2k}, \quad k \in \mathbb{N},$$

are $\mathcal{M}$-admissible.

For our last result we need to recall the notion of finite type which was introduced by Hörmander [1967]. We say a collection of smooth real vector fields $X_1, \dots, X_d$ on Ω is of finite type (of length at most r) if at any point $p \in \Omega$ the tangent space $T_p\Omega$ is generated by $X_j(p)$ and some iterated commutators $[X_{i_1}, [X_{i_2}, [\dots, [X_{i_{q-1}}, X_{i_q}] \dots]]](p)$, where $q \le r$.

A straightforward application of Theorem 7.15 gives the following corollary.

Corollary 7.16. *Let Ω be connected and assume the real vector fields $X_1, \dots, X_d$ are of class $\{\mathcal{M}\}$ and of finite type and let $u \in \mathcal{D}'(\Omega)$ be a solution of $P_k u = 0$. If u vanishes on an open subset of Ω then $u \equiv 0$ in Ω.*

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