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We study affine structures on a Lie groupoid, including affine k-vector fields, k-forms and (p, q)-tensors. We show that the space of affine structures is a 2-vector space over the space of multiplicative structures. Moreover, the space of affine multivector fields with the Schouten bracket and the space of affine vector-valued forms with the Frölicher–Nijenhuis bracket are graded strict Lie 2-algebras, and affine (1, 1)-tensors constitute a strict monoidal category. Such higher structures can be seen as the categorification of multiplicative structures on a Lie groupoid.

1. Introduction

Geometric structures on a Lie groupoid that are compatible with the groupoid multiplication are called multiplicative structures. They have been studied intensively and their infinitesimal correspondings have been developed. See [Iglesias-Ponte et al. 2012; Mackenzie and Xu 2000; Xu 1995] and [Bursztyn and Cabrera 2012; Bursztyn et al. 2009; Crainic et al. 2015] for multiplicative multivector fields and multiplicative forms, respectively, and see [Bursztyn and Drummond 2019] for the theory of multiplicative tensors. Beyond this, there are also multiplicative Dirac structures [Jotz Lean 2019; Ortiz 2013], multiplicative generalized complex structures [Jotz Lean et al. 2016], multiplicative contact and Jacobi structures [Crainic and Salazar 2015; Crainic and Zhu 2007; Iglesias-Ponte and Marrero 2003], multiplicative distributions [Jotz Lean and Ortiz 2014] and multiplicative Manin pairs [Li-Bland and Ševera 2011], etc. We refer to [Kosmann-Schwarzbach 2016] for a survey on this subject.

Our purpose is to study geometric structures that are compatible with the affinoid structure on a Lie groupoid. This is motivated by Weinstein's work [1990], where he studied Poisson manifolds also carrying affinoid structures. An affinoid structure on a space X is a subset of X^4 whose elements are called parallelograms, with axioms modeled on the properties of the quaternary relation $\{(g, h, l, k) : hg^{-1} = kl^{-1}\}$ on

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a group or a groupoid. Groupoids are affinoid spaces, but not every affinoid space arises in this way. Mackenzie [1992; 2000] regarded affinoid structures as a type of double groupoid. He gave the equivalence of affinoid structures, butterfly diagrams and generalized principal bundles and studied their infinitesimal invariants.

The multiplicativity condition for a k-vector field (a k-form) on a Lie groupoid is known as the graph $\{(g, h, gh) : s(g) = t(h)\}$ of the groupoid multiplication being a coisotropic (an isotropic) submanifold of $\mathcal{G}\times\mathcal{G}\times\mathcal{G}$. A Lie groupoid $\mathcal{G}$ carries an affinoid structure with the set of parallelograms given by $\{(g, h, l, hg^{-1}l)\}$ when $hg^{-1}l$ is well defined. So the affine condition is naturally defined to be that the set of parallelograms is a coisotropic or an isotropic submanifold of $\mathcal{G}\times\mathcal{G}\times\mathcal{G}\times\mathcal{G}$ for k-vector fields or k-forms respectively. This gives the notions of affine k-vector fields and affine k-forms on a Lie groupoid. This topic was first studied in [Weinstein 1990]. Then Lu [1990] studied the dressing transformation, Poisson cohomology and also the symplectic groupoids of affine Poisson structures on Lie groups. For more information on affine Poisson structures, see also [Dazord et al. 1991; Dazord and Sondaz 1991; Urbański 1994]. To define affine (p, q)-tensors on a Lie groupoid $\mathcal{G}$, we consider the tangent and cotangent groupoids of $\mathcal{G}$. A (p, q)-tensor on $\mathcal{G}$ can be viewed as a function on the Lie groupoid $\tilde{\mathcal{G}} := \oplus^q T\mathcal{G} \oplus^p T^*\mathcal{G} \rightrightarrows \oplus^q TM \oplus^p A^*$. Then a (p, q)-tensor is said to be affine if it is an affine function (0-form) on the Lie groupoid $\tilde{\mathcal{G}}$. This definition coincides with the previous definitions for affine k-vector fields and affine k-forms.

We shall first make clear the relations between affine and multiplicative structures. For Lie groups, Lu [1990] obtained two multiplicative bivector fields from an affine bivector field by using the right and left translations. We generalize this result to the case of Lie groupoids and obtain two multiplicative k-vector fields (k-forms, (p, q)-tensors) from an affine k-vector field (k-form, (p, q)-tensor). Furthermore, we show that the space of affine structures is a 2-vector space over the vector space of multiplicative structures. Thus affine structures can be viewed as the categorification of multiplicative structures and affine structures define an equivalence relation on multiplicative structures. For some cases, multiplicative structures are functors, as morphisms of Lie groupoids; then affine structures are natural transformations between these multiplicative structures. Moreover, for affine multivector fields, the Schouten bracket gives rise to a graded strict Lie 2-algebra structure on the aforementioned 2-vector space. This recovers the strict Lie 2-algebra structure on 1-vector fields in [Berwick-Evans and Lerman 2016] and is equivalent to the graded Lie 2-algebra in [Bonechi et al. 2018]; see also [Ortiz and Waldron 2019]. We give the geometric support of this graded Lie 2-algebra structure. We also prove that affine vector-valued forms are closed under the Frölicher–Nijenhuis bracket and thus constitute a graded strict Lie 2-algebra. For affine $(1, 1)$-tensors, the composition of affine $(1, 1)$-tensors defines a strict monoidal category structure on the aforementioned 2-vector space.

We remark that on Lie groups, affine (p, q)-tensors and multiplicative (p, q)-tensors are the same when $q \neq 0$. In particular, the affine k-forms and multiplicative k-forms are the same. An affine k-vector field differs from a multiplicative k-vector field by an element in $\wedge^k \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group. Affine k-vector fields, k-forms and (1, 1)-tensors on pair groupoids are also analyzed in detail.

The organization of this paper is as follows. In Section 2, we recall Lie 2-algebras, monoidal categories, tangent and cotangent Lie groupoids. In Section 3, we introduce the notion of affine k-vector fields and clarify the relation with multiplicative k-vector fields. We show that the space of affine k-vector fields is a 2-vector space. Moreover, affine multivector fields are closed under the Schouten bracket; we thus get a graded strict Lie 2-algebra structure on this 2-vector space. In Section 4, we introduce the notion of affine k-forms and study their properties analogously. In Section 5, affine tensors on a Lie groupoid are introduced. We obtain a graded strict Lie 2-algebra structure on the space of vector-valued forms and a strict monoidal category structure on the space of affine (1, 1)-tensors.

2. Preliminary

2A. *Strict Lie 2-algebras.* Lie 2-algebras are the categorification of Lie algebras, whose underlying spaces are 2-vector spaces. See [Baez and Crans 2004] for more details. Let Vect be the category of vector spaces.

Definition 2.1 [Baez and Crans 2004]. A 2-*vector space* is a category in Vect.

Explicitly, a 2-vector space is a category $V_1 \rightrightarrows V_0$ whose spaces of objects and arrows are both vector spaces, such that the source and target maps $s, t : V_1 \to V_0$, the identity-assigning map $\iota : V_0 \hookrightarrow V_1$, and the composition $\circ : V_1 \times_{V_0} V_1 \to V_1$ are all linear.

A 2-vector space is completely determined by the vector spaces V_0, V_1 with the source, target and the identity-assigning map. Actually, given $f \in V_1$, define its arrow part by $\vec{f} = f - \iota(s(f))$. Then $s(\vec{f}) = 0$ and $t(\vec{f}) = t(f) - s(f)$. So we can identify $f : x \to y$ with the pair $(x, \vec{f})$. With this notation, the composition of $f : x \to y$ and $g : y \to z$ is defined as $g \circ f = (x, \vec{f} + \vec{g})$. Any arrow $(x, \vec{f})$ has an inverse $(x + t(\vec{f}), -\vec{f})$, so a 2-vector space is always a Lie groupoid.

A 2-vector space is equivalent to a 2-term chain complex of vector spaces. On the one hand, given a 2-vector space $V_1 \rightrightarrows V_0$, the corresponding 2-term complex is $t : \ker s \to V_0$. On the other hand, given a chain complex $C_1 \to C_0$, the 2-vector space is $C_0 \oplus C_1 \to C_0$. We refer to [Baez and Crans 2004] for the details.

Definition 2.2 [Baez and Crans 2004]. A *strict Lie* 2-*algebra* is a 2-vector space V together with a skew-symmetric bilinear functor, the bracket, $[\cdot, \cdot] : V \times V \to V$ satisfying the Jacobi identity.

A strict Lie 2-algebra is equivalent to a strict 2-term L_∞-algebra. Namely, a 2-term complex $d : C_1 \to C_0$ with skew-symmetric brackets $[\cdot,\cdot] : C_0 \times C_0 \to C_0$ and $[\cdot,\cdot] : C_0 \times C_1 \to C_1$ satisfying the Jacobi identity and $[da, b] = [a, db]$ and $d[x, a] = [x, da]$ for $x \in C_0$ and $a, b \in C_1$. See [Baez and Crans 2004] for details.

When the spaces of objects and morphisms are graded vector spaces and the Lie bracket is a graded Lie bracket, we call it a *graded strict Lie* 2*-algebra*. We refer to [Bonechi et al. 2018] for the explicit definition.

2B. ***Strict monoidal categories.*** A monoidal category is a category $\mathcal{C}$ with a bifunctor $\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, its product, which is associative up to a natural isomorphism, and with an object which is a left unit and a right unit for the product up to natural isomorphisms. For our purpose, we only consider the category with a product which is strict associative and has a strict two-sided identity object.

Definition 2.3 [Mac Lane 1971]. A *strict monoidal category* $(\mathcal{C}, \circ, e)$ is a category $\mathcal{C}$ with a bifunctor $\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which is associative:

$$\circ(\circ \times 1) = \circ(1 \times \circ) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

and with an object e which is a left and right unit for $\circ$:

$$\circ(e \times 1) = \mathrm{id}_{\mathcal{C}} = \circ(1 \times e) : \mathcal{C} \to \mathcal{C}.$$

The bifunctor $\circ$ here assigns to each pair of objects x, y an object $x \circ y$ and to each pair of arrows $f : x \to x'$, $g : y \to y'$ an arrow $f \circ g : x \circ y \to x' \circ y'$. Thus $\circ$ being a bifunctor means that

$$1_x \circ 1_y = 1_{x \circ y}, \quad (f' \circ g') \cdot (f \circ g) = (f' \cdot f) \circ (g' \cdot g),$$

whenever f', f and g', g are composable. Here $\cdot$ is the multiplication in the category $\mathcal{C}$. The associative law and the unit law in the definition hold both for objects and arrows.

2C. ***Tangent and cotangent Lie groupoids.*** We recall the definition of the tangent and cotangent Lie groupoids of a Lie groupoid.

Denote the source and target maps for a Lie groupoid $\mathcal{G} \rightrightarrows M$ by $s, t : \mathcal{G} \to M$. Two elements $g, h \in \mathcal{G}$ are multiplicable or composable if and only if $s(g) = t(h)$ and their product is written as $g \cdot h$ or simply as gh. Such a pair is called a *multiplicable pair*. We denote the space of multiplicable pairs by $\mathcal{G}^{(2)}$. Let A be the Lie algebroid of $\mathcal{G}$. For $u \in \Gamma(A)$, the right and left translations $\overrightarrow{u}, \overleftarrow{u} \in \mathfrak{X}(\mathcal{G})$ are defined by

$$\overrightarrow{u}(g) = dR_g(u_{t(g)}), \quad \overleftarrow{u}(g) = -dL_g(d\,\mathrm{inv}(u_{s(g)})),$$

where R_g and L_g are the right and left multiplications on $\mathcal{G}$ and $\mathrm{inv} : \mathcal{G} \to \mathcal{G}$ is the inverse map in $\mathcal{G}$.

Throughout this paper, by abuse of notation, we use the same notations s and t

to denote the source and target of any Lie groupoid and we adopt the same notation to denote a map and its tangent map.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, its tangent bundle $T\mathcal{G} \rightrightarrows TM$ with the differentials of the structure maps of $\mathcal{G}$ is again a Lie groupoid.

Its cotangent bundle $T^*\mathcal{G}$ is also equipped with a Lie groupoid structure which is over A^*, written as $T^*\mathcal{G} \rightrightarrows A^*$. First we have the inclusion $A^* \hookrightarrow T^*\mathcal{G}$ since $T^*\mathcal{G}|_M \cong T^*M \oplus A^*$. The source and target maps of an element $\xi \in T^*_g\mathcal{G}$ with $g \in \mathcal{G}$ are

$$\langle s(\xi), u\rangle = \langle \xi, \overleftarrow{u}\rangle, \quad \langle t(\xi), v\rangle = \langle \xi, \overrightarrow{v}\rangle \qquad \text{for all } u \in A_{s(g)}, v \in A_{t(g)},$$

So for any $u \in \Gamma(A)$, seen as a function on the base manifold A^*, we get the formulas

$$s^*u = \overleftarrow{u}, \qquad t^*u = \overrightarrow{u}. \tag{1}$$

For a multiplicable pair $(g, h) \in \mathcal{G}^{(2)}$, if $\xi \in T^*_g\mathcal{G}$ and $\eta \in T^*_h\mathcal{G}$ are multiplicable, the product is the element $\xi \cdot \eta \in T^*_{gh}\mathcal{G}$ such that

$$(\xi \cdot \eta)(X \cdot Y) = \xi(X) + \eta(Y) \quad \text{for all } (X, Y) \in T\mathcal{G}^{(2)},$$

where $X \cdot Y \in T_{gh}\mathcal{G}$ is the product of $X \in T_g\mathcal{G}$ and $Y \in T_h\mathcal{G}$ in the Lie groupoid $T\mathcal{G}$. See, for example, [Kosmann-Schwarzbach 2016; Lang and Liu 2018] for more explanation of the cotangent groupoid.

3. Affine k-vector fields on a Lie groupoid

An affinoid structure on a space X is a subset of X^4 whose elements are seen as parallelograms, with axioms modeled on the properties of the quaternary relation $\{(g, h, l, k) \mid hg^{-1} = kl^{-1}\}$ on a group or a groupoid [Weinstein 1990]. In particular, a groupoid has an affinoid structure with parallelograms given by the relation

$$\{(g, h, l, hg^{-1}l) : s(g) = s(h), t(g) = t(l)\}.$$

A k-vector field on a Lie groupoid is called affine when it is compatible with the affinoid structure in the sense that the submanifold of parallelograms is coisotropic. While a k-vector field is multiplicative when the graph $\{(g, h, gh) : s(g) = t(h)\}$ of the multiplication, or space of triangles, is coisotropic.

Let V be a vector space and $\Pi \in \wedge^k V$. A subspace $W \subset V$ is coisotropic with respect to Π if

$$\Pi(\xi_1, \dots, \xi_k) = 0 \quad \text{for all } \xi_1, \dots, \xi_k \in W^0,$$

where W^0 is the annihilator space of W, namely,

$$W^0 = \{\xi \in V^* : \xi(w) = 0 \text{ for all } w \in W\}.$$

More generally, for a manifold M and $\Pi \in \mathfrak{X}^k(M)$, a submanifold S of M is coisotropic with respect to Π if T_xS is coisotropic with respect to Π_x for all $x \in S$.

The following definition is motivated by Weinstein's [1990] definition for affine Poisson structures on a Lie groupoid.

Definition 3.1. A k-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is called *affine* if the submanifold

$$S := \{(g, h, l, hg^{-1}l) : s(g) = s(h), t(g) = t(l)\} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$$

is coisotropic with respect to $\Pi \oplus (-1)^{k+1}\Pi \oplus (-1)^{k+1}\Pi \oplus \Pi$.

Comparatively, a k-vector field on a Lie groupoid $\mathcal{G}$ is multiplicative [Iglesias-Ponte et al. 2012; Mackenzie and Xu 2000; Xu 1995] if it satisfies that the submanifold $\{(g, h, gh) : s(g) = t(h)\} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ is coisotropic relative to $\Pi \oplus \Pi \oplus (-1)^{k+1}\Pi$.

It is shown in [Chen et al. 2013; Iglesias-Ponte et al. 2012; Weinstein 1990] that a k-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ is multiplicative if and only if it is affine and the base manifold M is coisotropic with respect to Π. We refer the readers to [Chen et al. 2013, Lemma 2.3], where the authors pointed out that some of the conditions listed in [Iglesias-Ponte et al. 2012, Theorem 2.19] are redundant.

As shown in [Iglesias-Ponte et al. 2012] for the $k = 2$ case, for any $\mu \in T^*_{gh}\mathcal{G}$, the covector $(-\mu, L^*_{\mathcal{X}}\mu, R^*_{\mathcal{Y}}\mu, -L^*_{\mathcal{X}}R^*_{\mathcal{Y}}\mu)$ is conormal to S, that is, an element in the annihilator space of TS at $(gh, h, g, s(g))$, where $\mathcal{X}$ and $\mathcal{Y}$ are bisections passing through g and h. Another two classes of vectors conormal to S are $(-t^*\eta, t^*\eta, 0, 0)$ and $(-s^*\xi, s^*\xi, 0, 0)$ for $\eta \in T^*_{t(g)}M$ and $\xi \in T^*_{s(g)}M$. We thus get an explicit description of affine k-vector fields:

Lemma 3.2 [Chen et al. 2013, Lemma 2.3]. *A k-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is affine if and only if the following two conditions hold*:

(i) *For any $(g, h) \in \mathcal{G}^{(2)}$,*

$$\Pi(gh) = L_{\mathcal{X}}\Pi(h) + R_{\mathcal{Y}}\Pi(g) - L_{\mathcal{X}} \circ R_{\mathcal{Y}}(\Pi(s(g))), \tag{2}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are any two local bisections passing through g and h respectively.

(ii) *For any $\xi \in \Omega^1(M)$, $\iota_{t^*\xi}\Pi$ is right-invariant.*

An equivalent description of (2) is that $[\Pi, \overrightarrow{X}]$ is right-invariant for $X \in \Gamma(A)$ [Mackenzie and Xu 2000].

Remark 3.3. We emphasize that our definition of affine multivector fields is different from that in [Iglesias-Ponte et al. 2012; Xu 1995], where they call multivector fields satisfying (2) *affine multivector fields*. We have an extra condition.

On the other hand, we shall see that one affine k-vector field defines two multiplicative k-vector fields.

The restriction on M for a k-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ has $k+1$ components:

$$\Pi|_M \in \Gamma(\wedge^k T\mathcal{G}|_M) \cong \Gamma(\wedge^k(TM\oplus A)) = \Gamma(\wedge^k TM \oplus (\wedge^{k-1}TM\otimes A)\oplus\cdots\oplus\wedge^k A).$$

We denote by π the $\wedge^k A$-component:

$$\pi = \mathrm{pr}_{\wedge^k A}\Pi|_M \in \Gamma(\wedge^k A).$$

So the base manifold M is coisotropic with respect to $\Pi \in \mathfrak{X}^k(\mathcal{G})$ if $\pi = 0$.

Proposition 3.4. *Let Π be a k-vector field on the Lie groupoid $\mathcal{G}$ with $\pi = \mathrm{pr}_{\wedge^k A}\Pi|_M$. Define*

$$\Pi_r = \Pi - \overrightarrow{\pi}, \quad \Pi_l = \Pi - \overleftarrow{\pi}. \tag{3}$$

Then Π is affine if and only if Π_l or Π_r is a multiplicative k-vector field on $\mathcal{G}$. Here the right and left translations are

$$\overrightarrow{\pi}(g) := R_g(\pi_{t(g)}), \quad \overleftarrow{\pi}(g) := -L_g(\mathrm{inv}(\pi_{s(g)})) \qquad \textit{for all } g \in \mathcal{G}.$$

Proof. It is known from [Mackenzie and Xu 2000] that a k-vector field Π on $\mathcal{G}$ satisfying $\Pi(gh) = L_{\mathcal{X}}\Pi(h) + R_{\mathcal{Y}}\Pi(g) - L_{\mathcal{X}}\circ R_{\mathcal{Y}}(\Pi(s(g)))$ is equivalent to saying that $[\Pi, \overrightarrow{X}]$ is right-invariant for any $X \in \Gamma(A)$.

For any $X \in \Gamma(A)$, $[\Pi_r, \overrightarrow{X}]$ is right-invariant if and only if $[\Pi, \overrightarrow{X}]$ is right-invariant. So Π satisfies (2) if and only if Π_r satisfies (2). Besides, since $t\circ R_g = t$, it is clear that $\iota_{t^*\xi}\overrightarrow{\pi}$ is right-invariant for $\xi \in \Omega^1(M)$. Also it is obvious that M is coisotropic with respect to Π_r. We conclude that Π_r is multiplicative if and only if Π is affine.

For Π_l, since $[\Pi_l, \overrightarrow{X}] = [\Pi, \overrightarrow{X}]$, $\iota_{t^*\xi}\overleftarrow{\pi} = 0$ and M is coisotropic with respect to Π_l, we obtain the conclusion that Π is affine if and only if Π_l is multiplicative. □

Example 3.5. Multiplicative k-vector fields on $\mathbb{R}^n$ are linear k-vector fields. An affine k-vector field is of the form

$$\sum f^{i_1,\dots,i_k}(x)\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}} + \sum c^{i_1,\dots,i_k}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_k}},$$

where $f^{i_1,\dots,i_k}(x)$ is a linear function on $\mathbb{R}^n$ and $c^{i_1,\dots,i_k}$ is a constant. Namely, an affine k-vector field is a sum of a linear k-vector field and a constant k-vector field.

Example 3.6. Multiplicative k-vector fields on the pair groupoid $M\times M \rightrightarrows M$ all have the form $(\Pi, -\Pi)$ for $\Pi \in \mathfrak{X}^k(M)$ and affine k-vector fields on $M\times M$ are of the form (Π, Π') for two k-vector fields $\Pi, \Pi' \in \mathfrak{X}^k(M)$.

Example 3.7. Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid A. For any $\pi \in \Gamma(\wedge^k A)$, the k-vector field $\Pi = \overrightarrow{\pi}$ is affine and the associated two multiplicative k-vector fields are $\Pi_r = 0$ and $\Pi_l = \overrightarrow{\pi} - \overleftarrow{\pi}$.

The space $\mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$ of affine k-vector fields is a vector space with the space $\mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$ of multiplicative k-vector fields as a linear subspace.

Theorem 3.8. *We have a* 2*-vector space*

$$\mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G}) \rightrightarrows \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}),$$

where the groupoid structure is as follows: *the source and target maps are given by* $s(\Pi) = \Pi_r$ *and* $t(\Pi) = \Pi_l$ *as defined in* (3), *and the multiplication* $*$ *is*

$$\Pi * \Pi' = \Pi + \overleftarrow{\pi'},$$

for a pair Π, Π' *of affine* k*-vector fields such that* $\Pi_r = \Pi'_l$. *Here* $\pi' = \mathrm{pr}_{\wedge^k A}\Pi'|_M$ *is the* $\wedge^k A$*-component of* $\Pi'|_M$.

Proof. We first verify the groupoid structure. It follows from $\Pi_r = \Pi'_l$ that $\Pi - \overrightarrow{\pi} = \Pi' - \overleftarrow{\pi'}$. Then

$$s(\Pi * \Pi') = \Pi + \overleftarrow{\pi'} - \overrightarrow{\pi} - \overrightarrow{\pi'} = \Pi' - \overrightarrow{\pi'} = \Pi'_r = s(\Pi'),$$

and

$$t(\Pi * \Pi') = \Pi + \overleftarrow{\pi'} - \overleftarrow{\pi} - \overleftarrow{\pi'} = \Pi_l = t(\Pi).$$

Here we have used the fact that

$$\mathrm{pr}_{\wedge^k A}\overleftarrow{\pi'}|_M = (-1)^k \mathrm{pr}_{\wedge^k A}\mathrm{inv}(\pi') = \mathrm{pr}_{\wedge^k A}(\pi' - \rho(\pi')) = \pi',$$

where if $\pi' = X_1 \wedge \cdots \wedge X_k$, we have

$$(-1)^k \mathrm{inv}(\pi') = -\mathrm{inv}(X_1) \wedge \cdots \wedge (-\mathrm{inv}(X_k)) = (X_1 - \rho(X_1)) \wedge \cdots (X_k - \rho(X_k)).$$

For the associativity of this multiplication, let Π'' be another affine k-vector field such that $\Pi'_r = \Pi''_l$. We see

$$(\Pi * \Pi') * \Pi'' = \Pi + \overleftarrow{\pi'} + \overleftarrow{\pi''} = \Pi * (\Pi' * \Pi'').$$

Also, it is immediate that all the groupoid structures are linear. This gives a 2-vector space structure on $\mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$. □

The inverse of $\Pi \in \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$ in this 2-vector space is

$$\Pi^{-1} = \Pi - (\overrightarrow{\pi} + \overleftarrow{\pi}), \qquad \pi = \mathrm{pr}_{\wedge^k A}\Pi|_M. \tag{4}$$

Remark 3.9. Lu [1990] considered the case when $\Pi \in \mathfrak{X}^2(\mathcal{G})$ is an affine Poisson vector field on a Lie group. This affine vector field Π^{-1} is also Poisson and is called the *opposite affine Poisson structure* of Π. We see here that it is actually the inverse of Π in the 2-vector space given above.

Corollary 3.10. *The associated* 2*-term chain complex of vector spaces for the* 2*-vector space in the above theorem is*

$$\Gamma(\wedge^k A) \to \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}), \quad \pi \mapsto \overrightarrow{\pi} - \overleftarrow{\pi}.$$

In addition to this, since affine multivector fields are closed under the Schouten bracket [Iglesias-Ponte et al. 2012], we further obtain a graded strict Lie 2-algebra on this 2-vector space. See [Baez and Crans 2004] for the details of Lie 2-algebras.

Theorem 3.11. *We have a graded strict Lie* 2*-algebra structure on*

$$\oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G}) \rightrightarrows \oplus_k \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}),$$

where the bracket is the Schouten bracket.

Proof. The Schouten bracket defines a graded Lie algebra structure on $\oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$. It suffices to check that it is a functor. Let $\Pi_1, \Pi_1' \in \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$ and $\Pi_2, \Pi_2' \in \mathfrak{X}^l_{\mathrm{aff}}(\mathcal{G})$ be two multiplicable pairs, that is $(\Pi_1)_r = (\Pi_1')_l$ and $(\Pi_2)_r = (\Pi_2')_l$. The Schouten bracket $[\,\cdot\,,\cdot\,] : \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G}) \times \mathfrak{X}^l_{\mathrm{aff}}(\mathcal{G}) \to \mathfrak{X}^{k+l-1}_{\mathrm{aff}}(\mathcal{G})$ being a functor means

$$[(\Pi_1, \Pi_2) * (\Pi_1', \Pi_2')] = [\Pi_1, \Pi_2] * [\Pi_1', \Pi_2']. \tag{5}$$

Actually, by Theorem 3.8, the left-hand side of (5) is equal to

$$[\Pi_1 + \overleftarrow{\pi_1'}, \Pi_2 + \overleftarrow{\pi_2'}] = [\Pi_1, \Pi_2] + [\Pi_1, \overleftarrow{\pi_2'}] + [\overleftarrow{\pi_1'}, \Pi_2] - \overleftarrow{[\pi_1', \pi_2']},$$

where $\pi_1' = \mathrm{pr}_{\wedge^k A} \Pi_1'|_M$ and $\pi_2' = \mathrm{pr}_{\wedge^l A} \Pi_2'|_M$. And the right-hand side of (5) amounts to

$$[\Pi_1, \Pi_2] + \overleftarrow{\mathrm{pr}_{\wedge^{k+l-1} A}[\Pi_1', \Pi_2']|_M}.$$

By straightforward calculation, we have

$$\begin{aligned}
[\Pi_1', \Pi_2'] &= [(\Pi_1')_l, (\Pi_2')_l] + [\overleftarrow{\pi_1'}, (\Pi_2')_l] + [(\Pi_1')_l, \overleftarrow{\pi_2'}] + [\overleftarrow{\pi_1'}, \overleftarrow{\pi_2'}] \\
&= [(\Pi_1')_l, (\Pi_2')_l] + [\overleftarrow{\pi_1'}, (\Pi_2)_r] + [(\Pi_1)_r, \overleftarrow{\pi_2'}] + [\overleftarrow{\pi_1'}, \overleftarrow{\pi_2'}] \\
&= [(\Pi_1')_l, (\Pi_2')_l] + [\overleftarrow{\pi_1'}, \Pi_2] + [\Pi_1, \overleftarrow{\pi_2'}] - \overleftarrow{[\pi_1', \pi_2']},
\end{aligned}$$

where we have used the fact that $[\overleftarrow{X}, \overrightarrow{Y}] = 0$ for any $X, Y \in \Gamma(A)$. Moreover, Π_1 is affine so $[\Pi_1, \overleftarrow{\pi_2'}]$ is left-invariant and so is $[\overleftarrow{\pi_1'}, \Pi_2]$. From this and the fact that $(\Pi_1')_l$ ($(\Pi_2')_l$) has no component in $\wedge^k A$ ($\wedge^l A$), we see

$$\overleftarrow{\mathrm{pr}_{\wedge^{k+l-1} A}[\Pi_1', \Pi_2']|_M} = [\overleftarrow{\pi_1'}, \Pi_2] + [\Pi_1, \overleftarrow{\pi_2'}] - \overleftarrow{[\pi_1', \pi_2']}.$$

Thus we get (5). This finishes the proof. □

Remark 3.12. In [Berwick-Evans and Lerman 2016; Ortiz and Waldron 2019], the authors constructed a strict Lie 2-algebra on the multiplicative 1-vector fields and their natural transformations. They proved the Morita invariance of this construction and obtained a strict Lie 2-algebra structure on the differentiable stack. Actually, this is our case for $k = 1$ when writing the strict Lie 2-algebra as a Lie algebra crossed module. Another remark is that our graded Lie 2-algebra is actually the same as the one in [Bonechi et al. 2018], where they wrote it in the 2-term L_∞-algebra form $\Gamma(\wedge^\bullet A) \to \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G})$. Moreover, this Lie 2-algebra is Morita invariant and is used to define multivector fields on a differentiable stack in [Bonechi et al. 2018]. Here we see affine multivector fields as the geometric support of this graded Lie 2-algebra structure.

Now we move to consider the infinitesimal of affine k-vector fields.

For an affine k-vector field $\Pi \in \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$, by Lemma 3.2, define $\delta_\Pi f \in \Gamma(\wedge^{k-1} A)$ and $\delta_\Pi X \in \Gamma(\wedge^k A)$ for any $f \in C^\infty(M)$ and $X \in \Gamma(A)$, such that

$$\overrightarrow{\delta_\Pi f} = [\Pi, t^* f], \quad \overrightarrow{\delta_\Pi X} = [\Pi, \overrightarrow{X}]. \tag{6}$$

Recall that a *k-differential* [Iglesias-Ponte et al. 2012] on a Lie algebroid A is a pair of maps

$$\delta_0 : C^\infty(M) \to \Gamma(\wedge^{k-1} A), \quad \delta_1 : \Gamma(A) \to \Gamma(\wedge^k A),$$

satisfying

$$\delta_0(fg) = \delta_0(f)g + f\delta_0(g), \quad \delta_1(fX) = \delta_0(f)X + f\delta_1(X)$$

for all $f, g \in C^\infty(M)$, $X \in \Gamma(A)$, and

$$\delta_1[X, Y] = [\delta_1(X), Y] + [X, \delta_1(Y)], \quad X, Y \in \Gamma(A).$$

Denote by $\oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$ (resp. $\oplus_k \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$) and $\oplus_k \mathcal{A}_k$ the spaces of affine (resp. multiplicative) vector fields on $\mathcal{G}$ and k-differentials on A. It is straightforward to check that they are graded Lie algebras with the Schouten bracket and the commutator Lie bracket.

With these notions, by (6), we have a map

$$\delta : \oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G}) \to \oplus_k \mathcal{A}_k, \quad \Pi \mapsto \delta_\Pi. \tag{7}$$

The universal lifting theorem says that

$$\delta|_{\oplus_k \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})} : \oplus_k \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}) \to \oplus_k \mathcal{A}_k$$

is an isomorphism of graded Lie algebras when $\mathcal{G}$ is s-connected and s-simply connected [Iglesias-Ponte et al. 2012].

As a direct consequence of Proposition 3.4, we have the following isomorphism of graded Lie algebras.

Proposition 3.13. *We have an isomorphism*

$$\oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G}) \to \oplus_k \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}) \rhd (\oplus_k \Gamma(\wedge^k A)), \quad \Pi \mapsto (\Pi - \overrightarrow{\pi}, \pi), \quad \pi = \mathrm{pr}_{\wedge^k A} \Pi|_M,$$

of graded Lie algebras, where the brackets on $\oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$ and $\oplus_k \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$ are the Schouten bracket, the bracket on $\oplus_k \Gamma(\wedge^k A)$ is the graded Lie bracket induced by the Lie bracket on A, and the mixed bracket is

$$[\Gamma, \pi] = \delta_\Gamma(\pi) \in \Gamma(\wedge^{k+l-1} A), \quad \Gamma \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}), \pi \in \Gamma(\wedge^l A).$$

Proof. By Proposition 3.4, this map is an isomorphism of graded vector spaces whose inverse is $(\Gamma, \pi) \mapsto \Gamma + \overrightarrow{\pi}$. Identifying an element $\pi \in \Gamma(\wedge^k A)$ with the affine k-vector field $\overrightarrow{\pi}$, we see that the Lie bracket on the right-hand side is actually induced by the Schouten bracket on the left-hand side under the isomorphism. Hence the right-hand side is a graded Lie algebra and this map is an isomorphism of graded Lie algebras.

We could also check directly that this map is a morphism of graded Lie algebras. The key point is

$$\text{(8)} \quad \begin{aligned} &\mathrm{pr}_{\wedge^{k+l-1}A}[\Pi, \Pi']|_M \\ &\qquad = \delta_\Pi(\pi') - (-1)^{(k-1)(l-1)}\delta_{\Pi'}(\pi) - [\pi, \pi'], \qquad \Pi \in \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G}), \Pi' \in \mathfrak{X}^l_{\mathrm{aff}}(\mathcal{G}). \end{aligned}$$

The proof of this is similar to the proof of Theorem 3.11 for the right translation. □

The map δ defined in (7) is not a bijection on $\oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$. In fact, for $\Pi \in \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$, we have

$$\text{(9)} \qquad \delta_{\Pi - \overrightarrow{\pi}} = \delta_\Pi - [\pi, \cdot], \quad \delta_{\Pi - \overleftarrow{\pi}} = \delta_\Pi, \quad \pi = \mathrm{pr}_{\wedge^k A}\Pi|_M.$$

Proposition 3.13 together with the universal lifting theorem for multiplicative multivector fields tells us that the kernel of the map δ is $\oplus_k \Gamma(\wedge^k A)$ and we obtain the universal lifting theorem for affine multivector fields.

Theorem 3.14. *Let $\mathcal{G}$ be an s-simply connected and s-connected Lie groupoid with Lie algebroid A. We have a graded Lie algebra isomorphism*

$$\oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G}) \cong \oplus_k \mathcal{A}_k \rhd (\oplus_k \Gamma(\wedge^k A)), \quad \Pi \mapsto (\delta_\Pi - [\pi, \cdot], \pi), \quad \pi = \mathrm{pr}_{\wedge^k A}\Pi|_M,$$

where the brackets on $\oplus_k \mathfrak{X}^k_{\mathrm{aff}}(\mathcal{G})$ and $\oplus_k \mathcal{A}_k$ are the Schouten bracket and the commutator bracket, the bracket on $\oplus_k \Gamma(\wedge^k A)$ is the graded Lie bracket induced by the Lie bracket on A and the mixed bracket is

$$[\delta, \pi] = \delta(\pi) \in \Gamma(\wedge^{k+l-1} A), \quad \delta \in \mathcal{A}_k, \pi \in \Gamma(\wedge^l A).$$

Here δ acts on π as a degree $k-1$ derivation.

Proof. This follows from (9), Proposition 3.13 and the universal lifting theorem for multiplicative multivector fields [Iglesias-Ponte et al. 2012]. □

Next, we consider the case when an affine bivector field Π on a Lie groupoid $\mathcal{G}$ is also Poisson. We shall generalize Lu's results for Lie groups [1990].

Proposition 3.15. *Let Π be an affine bivector field on a Lie groupoid $\mathcal{G}$, and Π_r, Π_l be the multiplicative bivector fields given by* (3). *Then*

(i) Π_r (*resp.* Π_l) *is Poisson if and only if* $[\Pi, \Pi]$ *is right* (*resp. left*)*-invariant*;

(ii) *if* Π_r *is Poisson*, *then* Π *is Poisson if and only if* $2\delta_{\Pi_r}\pi + [\pi, \pi] = 0$, *where* $\pi = \mathrm{pr}_{\wedge^2 A}\Pi|_M$.

Proof. For (i), direct calculation shows that

$$[\Pi_r, \Pi_r] = [\Pi - \overrightarrow{\pi}, \Pi - \overrightarrow{\pi}] = [\Pi, \Pi] - 2[\Pi, \overrightarrow{\pi}] + [\overrightarrow{\pi}, \overrightarrow{\pi}]$$
$$= [\Pi, \Pi] - 2\overrightarrow{\delta_\Pi \pi} + \overrightarrow{[\pi, \pi]}.$$

Therefore if Π_r is Poisson, $[\Pi, \Pi]$ is right-invariant. Conversely, by (8), we have $\mathrm{pr}_{\wedge^3 A}[\Pi, \Pi]|_M = 2\delta_\Pi \pi - [\pi, \pi]$. If $[\Pi, \Pi]$ is right-invariant, we must have

$$[\Pi, \Pi] = \overrightarrow{2\delta_\Pi \pi - [\pi, \pi]}$$

and hence $[\Pi_r, \Pi_r] = 0$.

For (ii), following from

$$[\Pi, \Pi] = [\Pi_r + \overrightarrow{\pi}, \Pi_r + \overrightarrow{\pi}] = [\Pi_r, \Pi_r] + 2\overrightarrow{\delta_{\Pi_r} \pi} + \overrightarrow{[\pi, \pi]},$$

we get the result. □

As a corollary, if an affine bivector field Π is Poisson, the associated two multiplicative vector fields Π_r and Π_l are also Poisson.

Corollary 3.16. *Let Π be an affine Poisson structure on a Lie groupoid $\mathcal{G}$ with $\pi = \mathrm{pr}_{\wedge^2 A}\Pi|_M$. Then its inverse as introduced in* (4),

$$\Pi^{-1} = \Pi - (\overrightarrow{\pi} + \overleftarrow{\pi}),$$

is also an affine Poisson structure on $\mathcal{G}$.

Proof. Π^{-1} is obviously affine. To see that it is Poisson, we have

$$[\Pi^{-1}, \Pi^{-1}] = [\Pi - (\overrightarrow{\pi} + \overleftarrow{\pi}), \Pi - (\overrightarrow{\pi} + \overleftarrow{\pi})]$$
$$= -2[\Pi, \overrightarrow{\pi}] - 2[\Pi, \overleftarrow{\pi}] + [\overleftarrow{\pi}, \overleftarrow{\pi}] + [\overrightarrow{\pi}, \overrightarrow{\pi}] = [\Pi_r, \Pi_r] + [\Pi_l, \Pi_l].$$

Therefore, by Proposition 3.15, Π^{-1} is Poisson. □

Example 3.17. Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid A. For $\pi \in \Gamma(\wedge^2 A)$, the bivector field $\Pi = \overrightarrow{\pi}$ is affine. It is Poisson if and only if π satisfies the classical Yang–Baxter equation $[\pi, \pi] = 0$. Moreover, we have $\Pi_r = 0$, $\Pi_l = \overrightarrow{\pi} - \overleftarrow{\pi}$ and $\Pi^{-1} = -\overleftarrow{\pi}$.

Besides, given any $\gamma \in \Gamma(\wedge^2 A)$, define $\Pi = \overrightarrow{\pi} + \overleftarrow{\gamma}$. Then we get $\Pi_r = \overleftarrow{\gamma} - \overrightarrow{\gamma}$ and $\Pi_l = \overrightarrow{\pi} - \overleftarrow{\pi}$ and $\Pi^{-1} = -\overleftarrow{\pi} - \overrightarrow{\gamma}$. Furthermore, direct calculation shows that Π is Poisson if and only if

$$\overrightarrow{[\pi, \pi]} = \overleftarrow{[\gamma, \gamma]},$$

which implies that $[\pi, \pi] = [\gamma, \gamma] \in \wedge^3 \ker\rho$ and both of them are Ad-invariant.

Affine Poisson structures give rise to a natural equivalence relation between multiplicative Poisson structures on a Lie groupoid (Poisson groupoids), which further give an equivalence relation on Lie bialgebroids.

4. Affine k-forms on a Lie groupoid

A k-form $\Theta \in \Omega^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is multiplicative if the graph of multiplication $\{(g, h, gh) : s(g) = t(h)\}$, or space of triangles, is an isotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ with respect to $\Theta \oplus \Theta \oplus -\Theta$. Algebraically, a k-form Θ on $\mathcal{G} \rightrightarrows M$ is multiplicative [Bursztyn and Cabrera 2012; Bursztyn et al. 2009; Crainic et al. 2015] if it satisfies

$$m^*\Theta = \mathrm{pr}_1^*\Theta + \mathrm{pr}_2^*\Theta,$$

where m and $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ are the groupoid multiplication and the projections to the first and second components, respectively.

One consequence of the multiplicativity condition is that Θ is isotropic on M, namely, $\iota^*\Theta = 0$, where $\iota : M \hookrightarrow \mathcal{G}$ is the natural inclusion. In other words, the restriction of Θ on M has no component in $\wedge^k T^*M$. Relaxing this condition, we shall get the notion of affine k-forms.

The restriction of a k-form $\Theta \in \Omega^k(\mathcal{G})$ on M has $k+1$ components:

$$\begin{aligned}\Theta|_M \in \Gamma(\wedge^k T^*\mathcal{G}|_M) &= \Gamma(\wedge^k(A^* \oplus T^*M)) \\ &= \Gamma(\wedge^k A^* \oplus (\wedge^{k-1} A^* \otimes T^*M) \oplus \cdots \oplus \wedge^k T^*M).\end{aligned}$$

Denote by θ the $\wedge^k T^*M$-component: $\theta = \mathrm{pr}_{\wedge^k T^*M}\Theta|_M$. In other words, $\theta = \iota^*\Theta$ for $\iota : M \hookrightarrow \mathcal{G}$.

Definition 4.1. A k-form $\Theta \in \Omega^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is *affine* if it satisfies

$$m^*\Theta = \mathrm{pr}_1^*\Theta + \mathrm{pr}_2^*\Theta - \mathrm{pr}_1{}^* s^*\theta, \tag{10}$$

where $\theta := \mathrm{pr}_{\wedge^k T^*M}\Theta|_M$.

Since $s \circ \mathrm{pr}_1 = t \circ \mathrm{pr}_2 : \mathcal{G}^{(2)} \to \mathcal{G}$, the affine condition has another expression:

$$m^*\Theta = \mathrm{pr}_1^*\Theta + \mathrm{pr}_2^*\Theta - \mathrm{pr}_2{}^* t^*\theta. \tag{11}$$

One direct consequence of the definition is that the de Rham differential of an affine k-form on $\mathcal{G}$ is an affine $(k+1)$-form.

Unlike the multiplicative case, it is not obvious from (10) that a k-form is affine if the submanifold of parallelograms is isotropic in $\mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$.

Proposition 4.2. *A k-form Θ on $\mathcal{G}$ is affine if and only if the space of parallelograms*

$$\Gamma = \{(g, h, l, hg^{-1}l) : s(g) = s(h), t(g) = t(l)\}$$

is an isotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ with respect to $\Theta \oplus -\Theta \oplus -\Theta \oplus \Theta$, that is,

$$\iota^*(\mathrm{pr}_1^*\Theta - \mathrm{pr}_2^*\Theta - \mathrm{pr}_3^*\Theta + \mathrm{pr}_4^*\Theta) = 0, \tag{12}$$

where $\mathrm{pr}_i : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is the projection to the i-th component and $\iota : \Gamma \hookrightarrow \mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ is the inclusion.

Proof. The tangent space of Γ at $(g, h, l, hg^{-1}l)$ consists of 4-tuples

$$(X_g, Y_h, Z_l, Y_h \cdot \mathrm{inv}(X)_{g^{-1}} \cdot Z_l)$$

of tangent vectors, where $Y_h \cdot \mathrm{inv}(X)_{g^{-1}} \cdot Z_l$ means the multiplication of three tangent vectors in $T\mathcal{G}$. Applying (12) to k such vectors, we have

$$\tag{13} \begin{aligned}\Theta(Y_h^1 \cdot \mathrm{inv}(X)^1_{g^{-1}} \cdot Z_l^1, \dots, Y_h^k \cdot \mathrm{inv}(X)^k_{g^{-1}} \cdot Z_l^k)\\ = -\Theta(X_g^1, \dots, X_g^k) + \Theta(Y_h^1, \dots, Y_h^k) + \Theta(Z_l^1, \dots, Z_l^k),\end{aligned}$$

where $\mathrm{inv}(X)^i_{g^{-1}} := \mathrm{inv}(X_g^i)$. In particular, we choose $(h, l) \in \mathcal{G}^{(2)}$ and $g = 1_{t(l)} = 1_{s(h)}$. Moreover, each (X_g^i, Y_h^i, Z_l^i) is chosen to satisfy $t(Z_l^i) = s(Y_h^i) = X_g^i$. Then the equation becomes

$$\tag{14} \begin{aligned}\Theta(Y_h^1 \cdot Z_l^1, \dots, Y_h^k \cdot Z_l^k)\\ = -\Theta(s(Y_h^1), \dots, s(Y_h^k)) + \Theta(Y_h^1, \dots, Y_h^k) + \Theta(Z_l^1, \dots, Z_l^k).\end{aligned}$$

This is exactly (10).

Conversely, if Θ is affine, by setting $l = h^{-1}$ and $Z_l^i = \mathrm{inv}(Y)_l^i$ in (14), we get

$$\tag{15} \begin{aligned}\Theta(\mathrm{inv}(Y)_l^1, \dots, \mathrm{inv}(Y)_l^k)\\ = \Theta(s(Y_h^1), \dots, s(Y_h^k)) - \Theta(Y_h^1, \dots, Y_h^k) + \Theta(t(Y_h^1), \dots, t(Y_h^k)).\end{aligned}$$

Applying (14) twice to the left-hand side of (13), we obtain

$$\begin{aligned}&\Theta(Y_h^1 \cdot \mathrm{inv}(X)^1_{g^{-1}} \cdot Z_l^1, \dots, Y_h^k \cdot \mathrm{inv}(X)^k_{g^{-1}} \cdot Z_l^k)\\ &\quad = \Theta(Y_h^1, \dots, Y_h^k) + \Theta(\mathrm{inv}(X)^1_{g^{-1}} \cdot Z_l^1, \dots, \mathrm{inv}(X)^k_{g^{-1}} \cdot Z_l^k) - \Theta(s(X_g^1), \dots, s(X_g^k))\\ &\quad = \Theta(Y_h^1, \dots, Y_h^k) + \Theta(\mathrm{inv}(X)^1_{g^{-1}}, \dots, \mathrm{inv}(X)^k_{g^{-1}}) + \Theta(Z_l^1, \dots, Z_l^k)\\ &\qquad\qquad - \Theta(t(X_g^1), \dots, t(X_g^k)) - \Theta(s(X_g^1), \dots, s(X_g^k))\\ &\quad = \Theta(Y_h^1, \dots, Y_h^k) - \Theta(X_g^1, \dots, X_g^k) + \Theta(Z_l^1, \dots, Z_l^k),\end{aligned}$$

where we have used (15) in the last step. Hence, we get (12). □

Regarding the relation between multiplicative and affine k-forms, we have already seen that a multiplicative k-form is an affine k-form which is isotropic on M. On the other hand, an affine k-form is associated with two multiplicative k-forms.

Proposition 4.3. *Let $\Theta \in \Omega^k(\mathcal{G})$ be a k-form on $\mathcal{G}$ with $\theta = \mathrm{pr}_{\wedge^k T^* M} \Theta|_M \in \Omega^k(M)$. Define two k-forms on $\mathcal{G}$:*

$$\tag{16} \Theta_l := \Theta - s^*\theta, \qquad \Theta_r := \Theta - t^*\theta.$$

Then Θ is affine if and only if Θ_l (resp. Θ_r) is a multiplicative k-form.

Proof. By straightforward calculation, we have

$$m^*\Theta_l = m^*\Theta - m^*s^*\theta, \quad \mathrm{pr}_1^*\Theta_l = \mathrm{pr}_1^*\Theta - \mathrm{pr}_1^*s^*\theta, \quad \mathrm{pr}_2^*\Theta_l = \mathrm{pr}_2^*\Theta - \mathrm{pr}_2^*s^*\theta.$$

Following from $s \circ \mathrm{pr}_2 = s \circ m : \mathcal{G}^{(2)} \to \mathcal{G}$, we see that the equation $m^*\Theta_l = \mathrm{pr}_1^*\Theta_l + \mathrm{pr}_2^*\Theta_l$ holds if and only if (10) holds. Similarly, noticing that $t \circ \mathrm{pr}_1 = t \circ m$, we get that Θ_r is multiplicative if and only if Θ satisfies (11), that is, Θ is affine. □

Denote by $\Omega^k_{\mathrm{aff}}(\mathcal{G})$ and $\Omega^k_{\mathrm{mult}}(\mathcal{G})$ the spaces of affine and multiplicative k-forms, respectively. It is immediate that $\Omega^k_{\mathrm{aff}}(\mathcal{G})$ is a vector space with $\Omega^k_{\mathrm{mult}}(\mathcal{G})$ being a linear subspace.

Theorem 4.4. *We have a 2-vector space*

$$\Omega^k_{\mathrm{aff}}(\mathcal{G}) \rightrightarrows \Omega^k_{\mathrm{mult}}(\mathcal{G}),$$

where the groupoid structure is given as follows: the source and target maps are

$$s(\Theta) = \Theta_r, \quad t(\Theta) = \Theta_l \qquad \textit{for all } \Theta \in \Omega^k_{\mathrm{aff}}(\mathcal{G}),$$

where Θ_r and Θ_l are defined in (16), *and the multiplication is*

$$\Theta * \Theta' = \Theta + s^*\theta', \quad \theta' = \mathrm{pr}_{\wedge^k T^*M}\Theta'|_M$$

for a pair $\Theta, \Theta' \in \Omega^k_{\mathrm{aff}}(\mathcal{G})$ such that $\Theta_r = \Theta'_l$.

Proof. The proof is similar to that for Theorem 3.8. □

Corollary 4.5. *The 2-term chain complex of vector spaces associated to the above 2-vector space $\Omega^k_{\mathrm{aff}}(\mathcal{G}) \rightrightarrows \Omega^k_{\mathrm{mult}}(\mathcal{G})$ is*

$$\Omega^k(M) \to \Omega^k_{\mathrm{mult}}(\mathcal{G}), \quad \theta \mapsto t^*\theta - s^*\theta.$$

It is seen from the definition that the affine and multiplicative forms are closed under the de Rham differential. So we get two subcomplexes of the de Rham complex on $\mathcal{G}$:

$$(\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), d) \subset (\Omega^\bullet_{\mathrm{aff}}(\mathcal{G}), d) \subset (\Omega^\bullet(\mathcal{G}), d).$$

Proposition 4.6. *The map*

$$\Phi : \Omega^\bullet_{\mathrm{aff}}(\mathcal{G}) \to \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) \oplus \Omega^\bullet(M), \quad \Theta \mapsto (\Theta - t^*\theta, \theta), \quad \theta = \mathrm{pr}_{\wedge^\bullet T^*M}\Theta|_M,$$

is an isomorphism of cochain complexes, where the differentials are the de Rham differential. Thus we get an isomorphism on the cohomology

$$H^\bullet_{\mathrm{aff}}(\mathcal{G}) \cong H^\bullet_{\mathrm{mult}}(\mathcal{G}) \oplus H^\bullet(M).$$

Proof. The inverse of Φ can be defined by $(\Lambda, \lambda) \mapsto \Lambda + t^*\lambda$ for any $\Lambda \in \Omega^k_{\mathrm{mult}}(\mathcal{G})$ and $\lambda \in \Omega^k(M)$. So Φ is an isomorphism. Next, we check that it is a cochain map, namely, $d \circ \Phi = \Phi \circ d$. Since $\theta = \iota^*\Theta$ for $\iota : M \hookrightarrow \mathcal{G}$, we have $d\theta = \mathrm{pr}_{\wedge^{k+1}T^*M} d\Theta$. Then we have

$$d \circ \Phi(\Theta) = (d\Theta - t^* d\theta, d\theta) = \Phi \circ d(\Theta).$$

Thus it induces an isomorphism on the cohomology. □

Now we discuss the infinitesimal of affine k-forms. It is known from [Bursztyn and Cabrera 2012; Crainic et al. 2015] that there is a one-to-one correspondence between multiplicative k-forms on $\mathcal{G}$ and IM k-forms on its Lie algebroid A when $\mathcal{G}$ is s-connected and s-simply connected.

A pair (μ, ν) of bundle maps

$$\mu : A \to \wedge^{k-1}T^*M, \quad \nu : A \to \wedge^k T^*M, \quad k \geq 1,$$

is called an *IM k-form* on A if

$$\begin{aligned} &\iota_{\rho(X)}\mu(Y) = -\iota_{\rho(Y)}\mu(X), \\ &\mu([X, Y]) = \mathcal{L}_{\rho(X)}\mu(Y) - \iota_{\rho(Y)}d\mu(X) - \iota_{\rho(Y)}\nu(X), \\ &\nu([X, Y]) = \mathcal{L}_{\rho(X)}\nu(Y) - \iota_{\rho(Y)}d\nu(X) \quad \text{for all } X, Y \in \Gamma(A). \end{aligned}$$

The pair (μ, ν) determines a linear k-form on the vector bundle A. These conditions are described in such a way that the induced map $\oplus_A^k TA \to \mathbb{R}$ is a Lie algebroid morphism with the tangent Lie algebroid structure on $\oplus_A^k TA \to \oplus^k TM$ and the trivial Lie algebroid structure on $\mathbb{R} \to \{*\}$. See [Bursztyn and Cabrera 2012] for details.

By Proposition 4.6, the infinitesimals of affine k-forms on a Lie groupoid $\mathcal{G}$ are clear.

Proposition 4.7. *If $\mathcal{G}$ is an s-connected and s-simply connected Lie groupoid, there is a one-to-one correspondence between affine k-forms Θ on $\mathcal{G}$ and triples (μ, ν, θ) of IM k-forms (μ, ν) and $\theta \in \Omega^k(M)$. That is*

$$\Omega^k_{\mathrm{aff}}(\mathcal{G}) \cong \Omega^k_{\mathrm{mult}}(\mathcal{G}) \oplus \Omega^k(M) \cong \Omega^k_{\mathrm{IM}}(A) \oplus \Omega^k(M),$$

$$\Theta \mapsto (\Theta_r := \Theta - t^*\theta, \theta := \mathrm{pr}_{\wedge^k T^*M}\Theta|_M) \mapsto (\mu, \nu, \theta).$$

Example 4.8. For a Lie groupoid $\mathcal{G} \rightrightarrows M$, given any $\theta \in \Omega^k(M)$, then $s^*\theta$ and $t^*\theta$ are affine k-forms on $\mathcal{G}$ and $s^*\theta - t^*\theta$ is a multiplicative k-form on $\mathcal{G}$.

Example 4.9. On a Lie group G, affine k-forms are multiplicative k-forms. They are nonzero only when k is 0 and 1. This is because for any $k \geq 2$,

$$\begin{aligned} &\Theta((X_1, 0, X_3, \dots, X_k) \cdot (0, Y_2, Y_3, \dots, Y_k)) \\ &\qquad = \Theta(R_{h_1}X_1, L_{g_2}Y_2, X_3 \cdot Y_3, \dots, X_k \cdot Y_k) \\ &\qquad = \Theta(X_1, 0, X_3, \dots, X_k) + \Theta(0, Y_2, \dots, Y_k) = 0 \end{aligned}$$

for $X_i \in T_{g_i}G$, $Y_j \in T_{h_j}G$. So multiplicative 1-forms on a Lie group are always closed.

On the abelian group $\mathbb{R}^n$, multiplicative 1-forms are constant 1-forms. They have the form $\Theta = \sum_i c_i dx^i$, where c_i is a constant.

Example 4.10. For the pair Lie groupoid $M \times M \rightrightarrows M$, multiplicative k-forms all have the form $\mathrm{pr}_1^*\alpha - \mathrm{pr}_2^*\alpha$ for $\alpha \in \Omega^k(M)$, where $\mathrm{pr}_i : M \times M \to M$ is the projection to the i-th component. Affine k-forms are of the form $\mathrm{pr}_1^* \alpha + \mathrm{pr}_2^* \beta$ for any two k-forms $\alpha, \beta \in \Omega^k(M)$.

5. Affine tensors on a Lie groupoid

5A. ***Definition of affine tensors.*** A tensor on a Lie groupoid is said to be affine if it is affine as a function on a more complicated Lie groupoid. This is motivated by the notion of multiplicative tensors on a Lie groupoid introduced in [Bursztyn and Drummond 2019].

Affine functions on a Lie groupoid $\mathcal{G} \rightrightarrows M$ are naturally defined as affine 0-forms on $\mathcal{G}$.

Definition 5.1. A function $F \in C^\infty(\mathcal{G})$ is *affine* if it satisfies

$$F(gh) = F(g) + F(h) - F(s(g)) \quad \text{for all } (g, h) \in \mathcal{G}^{(2)}, \tag{17}$$

or $F(gh) = F(g) + F(h) - F(t(h))$ as $s(g) = t(h)$.

In particular, if an affine function F satisfies $F|_M = 0$, it is called a multiplicative function. By the definition, the space of affine functions is a vector space with the space of multiplicative functions as a subspace.

As a corollary of Proposition 4.3 for 0-forms, we have:

Lemma 5.2. *Let $F \in C^\infty(\mathcal{G})$ be a function on a Lie groupoid $\mathcal{G} \rightrightarrows M$ and $f = \iota^* F \in C^\infty(M)$ be the restriction of F on M, where $\iota : M \hookrightarrow \mathcal{G}$ is the natural inclusion. Define*

$$F_l = F - s^* f, \quad F_r = F - t^* f.$$

Then F is affine if and only if F_l or F_r is a multiplicative function on $\mathcal{G}$.

Example 5.3. For any function $f \in C^\infty(M)$, we see that $s^* f - t^* f \in C^\infty(\mathcal{G})$ is a multiplicative function on $\mathcal{G}$ and $s^* f$ and $t^* f$ are affine functions on $\mathcal{G}$.

Consider the Lie groupoid

$$\tilde{\mathcal{G}} : \oplus^q T\mathcal{G} \oplus^p T^*\mathcal{G} \rightrightarrows \oplus^q TM \oplus^p A^*.$$

A (p, q)-tensor $F \in \Gamma(\wedge^p T\mathcal{G} \otimes \wedge^q T^*\mathcal{G})$ on $\mathcal{G}$ can be viewed as a function on $\tilde{\mathcal{G}}$.

Definition 5.4. A (p, q)-tensor $F \in \Gamma(\wedge^p T\mathcal{G} \otimes \wedge^q T^*\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is called *affine* if it is an affine function on $\tilde{\mathcal{G}}$.

The following proposition ensures the consistence of this definition with Definitions 3.1 and 4.1 for the cases of affine k-vector fields and affine k-forms.

Proposition 5.5. (i) *An affine $(p, 0)$-tensor is an affine p-vector field as defined in Definition 3.1.*

(ii) *An affine $(0, q)$-tensor is an affine q-form as defined in Definition 4.1.*

Proof. Let $F \in \Gamma(\wedge^p T\mathcal{G})$ be an affine $(p, 0)$-tensor. Namely,

$$F(\xi_g^1 \cdot \eta_h^1, \dots, \xi_g^p \cdot \eta_h^p) = F(\xi_g^1, \dots, \xi_g^p) + F(\eta_h^1, \dots, \eta_h^p) - F(s(\xi_g^1), \dots, s(\xi_g^p)),$$

for $\xi_g^i \in T_g^*\mathcal{G}$ and $\eta_h^i \in T_h^*\mathcal{G}$ such that $s(\xi_g^i) = t(\eta_h^i)$, where s, t are the source and target maps in the Lie groupoid $T^*\mathcal{G} \rightrightarrows A^*$.

Let f equal $\mathrm{pr}_{\wedge^p A} F|_M$, the projection of F restricting on M to $\wedge^k A$. We claim that F is an affine $(p, 0)$-tensor if and only if $F - \overrightarrow{f}$ is a multiplicative $(p, 0)$-tensor, that is,

$$(F - \overrightarrow{f})(\xi_g^1 \cdot \eta_h^1, \dots, \xi_g^p \cdot \eta_h^p) = (F - \overrightarrow{f})(\xi_g^1, \dots, \xi_g^p) + (F - \overrightarrow{f})(\eta_h^1, \dots, \eta_h^p).$$

By (1), we have $t^* f = \overrightarrow{f}$, where t is the target map in $T^*\mathcal{G} \rightrightarrows A^*$. The assertion holds by Lemma 5.2.

By [Iglesias-Ponte et al. 2012, Proposition 2.7], $F - \overrightarrow{f}$ is a multiplicative function if and only if it is a multiplicative p-vector field on $\mathcal{G}$, which is further equivalent to F being an affine p-vector field by Proposition 3.4. So F is an affine $(p, 0)$-tensor if and only if F is an affine p-vector field.

If $F \in \Gamma(\wedge^q T^*\mathcal{G})$ is an affine $(0, q)$-tensor, then

$$F(X \cdot Y) = F(X) + F(Y) - F(s(X)),$$

where $X \in \wedge^q T_g\mathcal{G},\ Y \in \wedge^q T_h\mathcal{G},\ (g, h) \in \mathcal{G}^{(2)}$ and $s(X) = t(Y)$, which implies that

$$m^* F = \mathrm{pr}_1^* F + \mathrm{pr}_2^* F - \mathrm{pr}_1^* s^* F.$$

So F is an affine q-form as defined in Definition 4.1. □

Regarding the relation between affine and multiplicative (p, q)-tensors, we also have the assertion as for affine k-vector fields and affine k-forms.

Let $f \in \Gamma(\wedge^p A \otimes \wedge^q T^*M)$. View it as a function on the base manifold of the Lie groupoid

$$\tilde{\mathcal{G}} : \oplus^q T\mathcal{G} \oplus^p T^*\mathcal{G} \rightrightarrows \oplus^q TM \oplus^p A^*.$$

By Example 5.3, $s_{\tilde{\mathcal{G}}}^* f$ and $t_{\tilde{\mathcal{G}}}^* f$ are affine functions on $\tilde{\mathcal{G}}$ and hence affine (p, q)-tensors on $\mathcal{G}$, where $s_{\tilde{\mathcal{G}}}$ and $t_{\tilde{\mathcal{G}}}$ are the source and target maps of the Lie groupoid $\tilde{\mathcal{G}}$.

Lemma 5.6. *Let $f \in \Gamma(\wedge^p A \otimes \wedge^q T^*M)$. We denote $\overleftarrow{f} := s_{\tilde{\mathcal{G}}}^* f$ and $\overrightarrow{f} := t_{\tilde{\mathcal{G}}}^* f$, where $\overleftarrow{f}, \overrightarrow{f} \in \Gamma(\wedge^p T\mathcal{G} \otimes \wedge^q T^*\mathcal{G})$. We have*

$$\text{(18)} \quad \overleftarrow{f}(X_1, \dots, X_q) = \overleftarrow{f(sX_1, \dots, sX_q)}, \quad \overrightarrow{f}(X_1, \dots, X_q) = \overrightarrow{f(tX_1, \dots, tX_q)},$$

for $X_i \in T\mathcal{G}$.

Proof. This follows from the definition and (1). □

If assuming $f = u \otimes \beta$ for $u \in \Gamma(\wedge^p A)$ and $\beta \in \Omega^q(M)$, we get

$$\overleftarrow{f} = \overleftarrow{u} \otimes s^*\beta, \quad \overrightarrow{f} = \overrightarrow{u} \otimes t^*\beta.$$

The following result is a direct consequence of Lemma 5.2.

Proposition 5.7. *Let $F \in \Gamma(\wedge^p T\mathcal{G} \otimes \wedge^q T^*\mathcal{G})$ and $f = \mathrm{pr}_{\wedge^p A \otimes \wedge^q T^* M} F|_M$. Define*

$$F_l = F - \overleftarrow{f}, \quad F_r = F - \overrightarrow{f},$$

where $\overleftarrow{f}$ and $\overrightarrow{f}$ are defined in (18). *Then F is an affine (p, q)-tensor on $\mathcal{G}$ if and only if F_l or F_r is a multiplicative (p, q)-tensor.*

Denote by $T^{p,q}_{\mathrm{aff}}(\mathcal{G})$ and $T^{p,q}_{\mathrm{mult}}(\mathcal{G})$ the spaces of affine and multiplicative (p, q)-tensors on $\mathcal{G}$, respectively. It is immediate that $T^{p,q}_{\mathrm{aff}}(\mathcal{G})$ is a vector space with $T^{p,q}_{\mathrm{mult}}(\mathcal{G})$ being a linear subspace.

Theorem 5.8. *With the above notation, we have a* 2*-vector space*

$$T^{p,q}_{\mathrm{aff}}(\mathcal{G}) \rightrightarrows T^{p,q}_{\mathrm{mult}}(\mathcal{G}),$$

where the source and target maps of the groupoid structure are

$$s(F) = F_r, \quad t(F) = F_l \qquad \text{for all } F \in T^{p,q}_{\mathrm{aff}}(\mathcal{G}),$$

and for a pair $F_1, F_2 \in T^{p,q}_{\mathrm{aff}}(\mathcal{G})$ such that $(F_1)_r = (F_2)_l$, the multiplication is

$$F_1 * F_2 = F_1 + \overleftarrow{f_2}, \quad f_2 = \mathrm{pr}_{\wedge^p A \otimes \wedge^q T^* M} F_2|_M.$$

Proof. The proof is similar to that for Theorem 3.8. □

Corollary 5.9. *The* 2*-term complex of vector spaces of the above* 2*-vector space is*

$$\Gamma(\wedge^p A \otimes \wedge^q T^* M) \to T^{p,q}_{\mathrm{mult}}(\mathcal{G}), \quad f \mapsto \overrightarrow{f} - \overleftarrow{f}.$$

An *IM (p, q)-tensor* ([Bursztyn and Drummond 2019]) on a Lie algebroid A is a triple (D, l, r), where

$$l : A \to \wedge^p A \otimes \wedge^{q-1} T^* M$$

and

$$r : T^* M \to \wedge^{p-1} A \otimes \wedge^q T^* M$$

are bundle maps covering the identity, and

$$D : \Gamma(A) \to \Gamma(\wedge^p A \otimes \wedge^q T^* M)$$

is an $\mathbb{R}$-linear map satisfying

$$D(f X) = f D(X) + df \wedge l(X) - X \wedge r(df), \quad f \in C^\infty(M), \ X \in \Gamma(A).$$

The following equations hold:

$$
\begin{aligned}
D[X,Y] &= X\cdot D(Y) - Y\cdot D(X),\\
l[X,Y] &= X\cdot l(Y) - \iota_{\rho(Y)} D(X),\\
r(\mathcal{L}_{\rho(X)}\alpha) &= X\cdot r(\alpha) - \iota_{\rho^*(\alpha)} D(X),\\
\iota_{\rho(X)} l(Y) &= -\iota_{\rho(Y)} l(X),\\
\iota_{\rho^*(\alpha)} r(\beta) &= -\iota_{\rho^*(\beta)} r(\alpha),\\
\iota_{\rho(X)} r(\alpha) &= \iota_{\rho^*(\alpha)} l(X),
\end{aligned}
$$

for $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Omega^1(M)$. Here $\cdot$ denotes the action of $\Gamma(A)$ on

$$\Gamma(\wedge^p A \otimes \wedge^q T^*M)$$

by

$$X\cdot(Z\otimes\gamma) = [X,Z]\otimes\gamma + Z\otimes\mathcal{L}_{\rho(X)}\gamma, \quad \gamma\in\Omega^q(M),\ Z\in\Gamma(\wedge^p A).$$

Denote by $T^{p,q}_{\mathrm{IM}} A$ the space of IM (p,q)-tensors on A.

The universal lifting theorem for multiplicative (p,q)-tensors is given in [Bursztyn and Drummond 2019] as follows: If $\mathcal{G}$ is an s-simply connected and s-connected Lie groupoid, then there is a one-to-one correspondence between multiplicative (p,q)-tensors on $\mathcal{G}$ and IM (p,q)-tensors on the Lie algebroid A of $\mathcal{G}$.

Based on this result and Proposition 5.7, we have the universal lifting theorem for affine (p,q)-tensors.

Proposition 5.10. *If $\mathcal{G}$ is an s-simply connected and s-connected Lie groupoid, then we have the following isomorphisms of vector spaces*:

$$T^{p,q}_{\mathrm{aff}}(\mathcal{G}) \cong T^{p,q}_{\mathrm{mult}}(\mathcal{G}) \oplus \Gamma(\wedge^p A\otimes\wedge^q T^*M) \cong T^{p,q}_{\mathrm{IM}} A \oplus \Gamma(\wedge^p A\otimes\wedge^q T^*M),$$
$$F \mapsto (F - \overrightarrow{f}, f) \mapsto (D, l, r, f),$$

where $f = \mathrm{pr}_{\wedge^p A\otimes\wedge^q T^*M} F|_M$ and $\overrightarrow{f}$ is defined in (18).

5B. ***The Frölicher–Nijenhuis bracket on affine vector-valued forms.*** A vector-valued form on a manifold M is an element in $\Omega^\bullet(M, TM) = \Gamma(TM\otimes\wedge^\bullet T^*M)$. So a vector-valued q-form on M is actually a $(1,q)$-tensor. The space of vector-valued forms relative to the Frölicher–Nijenhuis bracket is a graded Lie algebra [Frölicher and Nijenhuis 1956].

In [Bursztyn and Drummond 2013], the authors proved that the multiplicative vector-valued forms on a Lie groupoid are closed under the Frölicher–Nijenhuis bracket. Thus they form a graded Lie algebra. We shall prove that affine vector-valued forms are also closed under the Frölicher–Nijenhuis bracket. Moreover, the space of affine vector-valued forms is a graded strict Lie 2-algebra over the space of multiplicative vector-valued forms.

One formula for the Frölicher–Nijenhuis bracket

$$[\,\cdot\,,\cdot\,]_{\mathrm{FN}} : \Omega^k(M, TM) \times \Omega^q(M, TM) \to \Omega^{k+q}(M, TM)$$

is as follows:

$$\begin{aligned}[X \otimes \phi, Y \otimes \psi]_{\mathrm{FN}} = {} & [X, Y] \otimes \phi \wedge \psi + Y \otimes \phi \wedge \mathcal{L}_X \psi - X \otimes \mathcal{L}_Y \phi \wedge \psi \\ & + (-1)^k (Y \otimes d\phi \wedge \iota_X \psi + X \otimes \iota_Y \phi \wedge d\psi),\end{aligned} \tag{19}$$

where $X, Y \in \mathfrak{X}(M)$, $\phi \in \Omega^k(M)$ and $\psi \in \Omega^q(M)$, and the bracket $[\,\cdot\,,\cdot\,]$ and d on the right-hand side are the Schouten bracket and the de Rham differential. When $k = q = 0$, this bracket agrees with the Schouten bracket on vector fields. We refer to [Bursztyn and Drummond 2013; 2019] for an intrinsic definition of this bracket.

Now we discuss the vector-valued forms on a Lie groupoid $\mathcal{G}$. Recall that $T^{1,q}_{\mathrm{aff}}(\mathcal{G})$ and $T^{1,q}_{\mathrm{mult}}(\mathcal{G})$ are spaces of affine and multiplicative $(1, q)$-tensors, respectively.

Theorem 5.11. *Let $F \in T^{1,k}_{\mathrm{aff}}(\mathcal{G})$ and $N \in T^{1,q}_{\mathrm{aff}}(\mathcal{G})$ be affine tensors. Then $[F, N]_{\mathrm{FN}}$ is an affine $(1, k+q)$-tensor on $\mathcal{G}$.*

Proof. By Proposition 5.7, from F and N, we get two multiplicative tensors:

$$F_r = F - \overrightarrow{f}, \quad N_r = N - \overrightarrow{n},$$

where $f = \mathrm{pr}_{A\otimes\wedge^k T^*M} F|_M$ and $n = \mathrm{pr}_{A\otimes\wedge^q T^*M} N|_M$ and $\overrightarrow{f}$ and $\overrightarrow{n}$ are defined in (18). Based on this, we have

$$\begin{aligned}[F, N]_{\mathrm{FN}} &= [F_r + \overrightarrow{f}, N_r + \overrightarrow{n}]_{\mathrm{FN}} \\ &= [F_r, N_r]_{\mathrm{FN}} + [F_r, \overrightarrow{n}]_{\mathrm{FN}} + [\overrightarrow{f}, N_r]_{\mathrm{FN}} + [\overrightarrow{f}, \overrightarrow{n}]_{\mathrm{FN}}.\end{aligned} \tag{20}$$

By [Bursztyn and Drummond 2013, Theorem 4.3], we have $[F_r, N_r]_{\mathrm{FN}} \in T^{1,k+p}_{\mathrm{mult}}(\mathcal{G})$, a multiplicative $(1, k+p)$-tensor. By [Bursztyn and Drummond 2019, Lemma 5.3], we have

$$[\overrightarrow{f}, N_r]_{\mathrm{FN}} = \overrightarrow{D_N(f)},$$

where $D_N : \Gamma(A \otimes \wedge^k T^*M) \to \Gamma(A \otimes \wedge^{k+q} T^*M)$ is determined by the IM $(1, q)$-tensor (D, l, r) of the multiplicative $(1, q)$-tensor N_r. We refer to [Bursztyn and Drummond 2019] for details. Now it suffices to check that

$$[\overrightarrow{f}, \overrightarrow{n}]_{\mathrm{FN}} = \overrightarrow{s}, \tag{21}$$

for some $s \in \Gamma(A \otimes \wedge^{k+q} T^*M)$. Assume $f = u \otimes \alpha$, $n = v \otimes \beta$ for $u, v \in \Gamma(A)$ and $\alpha \in \Omega^k(M)$, $\beta \in \Omega^q(M)$. Then by (19),

$$\begin{aligned}[\overrightarrow{f}, \overrightarrow{n}]_{\mathrm{FN}} &= [\overrightarrow{u} \otimes t^*\alpha, \overrightarrow{v} \otimes t^*\beta] \\ &= \overrightarrow{[u, v]} \otimes t^*(\alpha \wedge \beta) + \overrightarrow{v} \otimes t^*(\alpha \wedge \mathcal{L}_{\rho(u)}\beta) - \overrightarrow{u} \otimes t^*(\mathcal{L}_{\rho(v)}\alpha \wedge \beta) \\ &\quad + (-1)^k (\overrightarrow{v} \otimes t^*(d\alpha \wedge \iota_{\rho(u)}\beta) + \overrightarrow{u} \otimes t^*(\iota_{\rho(v)}\alpha \wedge d\beta)),\end{aligned}$$

where we have used the relations $\iota_{\overrightarrow{u}} t^*\beta = t^*\iota_{\rho(u)}\beta$ and $d \circ t^* = t^* \circ d$. Write

$$s = [u, v] \otimes \alpha \wedge \beta + v \otimes \alpha \wedge \mathcal{L}_{\rho(u)}\beta - u \otimes \mathcal{L}_{\rho(v)}\alpha \wedge \beta + (-1)^k (v \otimes d\alpha \wedge \iota_{\rho(u)}\beta + u \otimes \iota_{\rho(v)}\alpha \wedge d\beta).$$

We get (21). Following (20), we have shown that

$$[F, N]_{\mathrm{FN}} = [F_r, N_r]_{\mathrm{FN}} - (-1)^{kq}\overrightarrow{D_F(n)} + \overrightarrow{D_N(f)} + \overrightarrow{s},$$

where $D_F : \Gamma(A \otimes \wedge^q T^*M) \to \Gamma(A \otimes \wedge^{k+q} T^*M)$ is determined by the IM $(1, k)$-tensor (D', l', r') of the multiplicative $(1, k)$-tensor F_r. Thus $[F, N]_{\mathrm{FN}}$ is affine. □

Proposition 5.12. *We have a graded strict Lie* 2*-algebra structure on*

$$\oplus_k T^{1,k}_{\mathrm{aff}}(\mathcal{G}) \rightrightarrows \oplus_k T^{1,k}_{\mathrm{mult}}(\mathcal{G}),$$

where the bracket is the Frölicher–Nijenhuis bracket.

Proof. By Theorems 5.8 and 5.11, we only need to show that the Frölicher–Nijenhuis bracket is a functor. This is similar to the proof in Theorem 3.11. We omit the detail. □

5C. ***The strict monoidal category of affine*** **(1, 1)*-tensors.*** Another important case is affine (1, 1)-tensors, which can be used to define affine Nijenhuis operators on a Lie groupoid. On the space of affine (1, 1)-tensors, in addition to the 2-vector space structure proposed in Theorem 5.8, we shall also construct a strict monoidal category structure in this subsection.

A (1, 1)-tensor on a Lie groupoid $\mathcal{G}$ is *multiplicative* if the induced bundle map $T\mathcal{G} \to T\mathcal{G}$ is a Lie groupoid morphism [Laurent-Gengoux et al. 2009], which amounts to saying that the corresponding function on $T^*\mathcal{G} \oplus T\mathcal{G} \rightrightarrows A^* \oplus TM$ is multiplicative by [Bursztyn and Drummond 2019, Proposition 3.9]. Thus the composition of two multiplicative (1, 1)-tensors is still a multiplicative (1, 1)-tensor. We shall show that the composition of two affine (1, 1)-tensors is also an affine (1, 1)-tensor.

By (17), a (1, 1)-tensor $N \in \Gamma(T\mathcal{G} \otimes T^*\mathcal{G})$ is affine if it satisfies

$$N(X \cdot Y, \xi \cdot \eta) = N(X, \xi) + N(Y, \eta) - n(s_{T\mathcal{G}}X, s_{T^*\mathcal{G}}\xi), \quad n = \mathrm{pr}_{A \otimes T^*M} N|_M,$$

where $(X, Y) \in T\mathcal{G}^{(2)}$ and $(\xi, \eta) \in T^*\mathcal{G}^{(2)}$ are multiplicable pairs in $T\mathcal{G}$ and $T^*\mathcal{G}$ covering the same pair $(g, h) \in \mathcal{G}^{(2)}$. Here $s_{T\mathcal{G}}$ and $s_{T^*\mathcal{G}}$ are the source maps of the tangent and cotangent groupoids, respectively.

A multiplicative (1, 1)-tensor N corresponds to a Lie groupoid morphism

$$(N, n_{TM}) : T\mathcal{G} \to T\mathcal{G},$$

where $n_{TM} : TM \to TM$ is the map on the base manifold. Since N preserves the

s-fibers, it induces a bundle map $n_A : A \to A$. Then we have $N|_M = n_{TM} + n_A$. For an affine (1, 1)-tensor N, from the difference of affine and multiplicative (1, 1)-tensors, we have that the restriction of N on M is

$$N|_M = \begin{pmatrix} n_{TM} & 0 \\ n & n_A \end{pmatrix} : TM \oplus A \to TM \oplus A,$$

where $n_{TM} : TM \to TM$ and $n_A : A \to A$ and $n = \mathrm{pr}_{A\otimes T^*M} N|_M$.

Lemma 5.13. *The composition $N \circ N'$ of two affine* (1, 1)*-tensors N and N' is still an affine* (1, 1)*-tensor with*

$$(N \circ N')_l = N_l \circ N'_l, \quad (N \circ N')_r = N_r \circ N'_r \quad \textit{for all } N, N' \in T^{1,1}_{\mathrm{aff}}(\mathcal{G}).$$

Moreover, the $A \otimes T^*M$-component of $N \circ N'|_M$ is

$$\mathrm{pr}_{A\otimes T^*M} N \circ N'|_M = n_A \circ n' + n \circ n'_{TM} + n \circ \rho \circ n',$$

where

$$N|_M = \begin{pmatrix} n_{TM} & 0 \\ n & n_A \end{pmatrix}, \quad N'|_M = \begin{pmatrix} n'_{TM} & 0 \\ n' & n'_A \end{pmatrix} : TM \oplus A \to TM \oplus A$$

are the decompositions of N and N' restricting on M and $\rho : A \to TM$ is the anchor map.

Proof. Write $N = N_r + \overrightarrow{n}$ and $N' = N'_r + \overrightarrow{n'}$, where n and n' are the $A \otimes T^*M$-components of $N|_M$ and $N'|_M$, respectively, and $\overrightarrow{n}$, $\overrightarrow{n'}$ are defined in (18). Then

$$N \circ N' = N_r \circ N'_r + N_r \circ \overrightarrow{n'} + \overrightarrow{n} \circ N'_r + \overrightarrow{n} \circ \overrightarrow{n'}.$$

By Proposition 5.7, $(N_r, n_{TM}), (N'_r, n'_{TM}) : T\mathcal{G} \to T\mathcal{G}$ are morphisms of Lie groupoids. So we have the formulas

$$N_r(\overrightarrow{u}) = \overrightarrow{n_A(u)} \quad \text{for all } u \in \Gamma(A)$$

and $t \circ N'_r = n'_{TM} \circ t$. Applying this to $X \in T_g\mathcal{G}$, we get

$$\begin{aligned}
N_r \circ \overrightarrow{n'}(X) &= N_r \overrightarrow{n'(tX)} = \overrightarrow{n_A(n'(tX))} = \overrightarrow{n_A \circ n'}(X),\\
\overrightarrow{n} \circ N'_r(X) &= \overrightarrow{n(tN'_r(X))} = \overrightarrow{n(n'_{TM}(tX))} = \overrightarrow{n \circ n'_{TM}}(X),\\
\overrightarrow{n} \circ \overrightarrow{n'}(X) &= \overrightarrow{n}\, \overrightarrow{n'(tX)} = \overrightarrow{n(\rho(n'(tX)))} = \overrightarrow{n \circ \rho \circ n'}(X).
\end{aligned}$$

Thus we proved that

$$N \circ N' = N_r \circ N'_r + \overrightarrow{n_A \circ n' + n \circ n'_{TM} + n \circ \rho \circ n'}.$$

Notice that the composition $N_r \circ N'_r$ of two multiplicative (1, 1)-tensors is still multiplicative. Then by Proposition 5.7, we obtain that $N \circ N'$ is an affine (1, 1)-tensor with the properties as desired. □

Actually, the 2-vector space $T^{1,1}_{\mathrm{aff}}(\mathcal{G}) \rightrightarrows T^{1,1}_{\mathrm{mult}}(\mathcal{G})$ from Theorem 5.8 with the composition is a strict monoidal category.

Theorem 5.14. *We have a strict monoidal category structure on the* 2*-vector space*

$$T^{1,1}_{\mathrm{aff}}(\mathcal{G}) \rightrightarrows T^{1,1}_{\mathrm{mult}}(\mathcal{G}),$$

with the product being the composition of two affine (1, 1)*-tensors and the unit given by the identity* $I : T\mathcal{G} \to T\mathcal{G}$.

Proof. It is obvious that the identity $I : T\mathcal{G} \to T\mathcal{G}$, as a multiplicative (1, 1)-tensor, is a left and right unit for the composition.

It suffices to verify that the composition $\circ : T^{1,1}_{\mathrm{aff}}(\mathcal{G}) \times T^{1,1}_{\mathrm{aff}}(\mathcal{G}) \to T^{1,1}_{\mathrm{aff}}(\mathcal{G})$ is a bifunctor. Let N_1, N_2, N_3, N_4 be four affine (1, 1)-tensors such that $(N_1)_r = (N_3)_l$ and $(N_2)_r = (N_4)_l$. That is,

$$N_1 - \overrightarrow{n_1} = N_3 - \overleftarrow{n_3} \quad \text{and} \quad N_2 - \overrightarrow{n_2} = N_4 - \overleftarrow{n_4}.$$

By Lemma 5.13, we see $(N_1 \circ N_2)_r = (N_3 \circ N_4)_l$. Then we prove

$$(N_1 * N_3) \circ (N_2 * N_4) = (N_1 \circ N_2) * (N_3 \circ N_4). \tag{22}$$

The left-hand side of (22) is equal to

$$(N_1 + \overleftarrow{n_3}) \circ (N_2 + \overleftarrow{n_4}) = N_1 \circ N_2 + N_1 \circ \overleftarrow{n_4} + \overleftarrow{n_3} \circ N_2 + \overleftarrow{n_3} \circ \overleftarrow{n_4},$$

and by Lemma 5.13, the right-hand side of (22) amounts to

$$N_1 \circ N_2 + \overleftarrow{(n_{3A} \circ n_4 + n_3 \circ n_{4TM} + n_3 \circ \rho \circ n_4)}.$$

By the same calculation in Lemma 5.13 for the left translation instead of the right translation, we get

$$N_1 \circ \overleftarrow{n_4} = (N_3 - \overleftarrow{n_3} + \overrightarrow{n_1}) \circ \overleftarrow{n_4} = \overleftarrow{n_{3A} \circ n_4} + \overrightarrow{n_1} \circ \overleftarrow{n_4} = \overleftarrow{n_{3A} \circ n_4},$$

which follows from

$$\overrightarrow{n_1} \circ \overleftarrow{n_4}(X) = \overrightarrow{n_1}(\overleftarrow{n_4(sX)} = \overrightarrow{n_1(t\overleftarrow{n_4}(sX))} = 0, \quad X \in T_g\mathcal{G}.$$

Similarly, we have

$$\overleftarrow{n_3} \circ N_2 = \overleftarrow{n_3 \circ n_{4TM}}.$$

Observe that

$$\overleftarrow{n_3} \circ \overleftarrow{n_4} = \overleftarrow{n_3 \circ \rho \circ n_4}$$

for the same reason as for the right translation proved in Lemma 5.13. This proves (22). □

Remark 5.15. This strict monoidal category from affine (1, 1)-tensors is related to the 2-vector spaces constructed from affine 2-vector fields and 2-forms as in Theorems 3.8 and 4.4 if we take into consideration the generalized tangent bundle $T\mathcal{G} \oplus T^*\mathcal{G} \rightrightarrows TM \oplus A^*$. Identify an affine 2-vector field $\Pi \in \mathfrak{X}^2(\mathcal{G})$ with a matrix

$$\begin{pmatrix} I & \Pi \\ 0 & I \end{pmatrix} : T\mathcal{G} \oplus T^*\mathcal{G} \to T\mathcal{G} \oplus T^*\mathcal{G}.$$

Then the addition in the vector space $\mathfrak{X}^2_{\text{aff}}(\mathcal{G})$ is actually the composition of two affine 2-vector fields as matrices. Likewise, viewing an affine 2-form $\Theta \in \Omega^2(\mathcal{G})$ as a matrix

$$\begin{pmatrix} I & 0 \\ \Theta & I \end{pmatrix} : T\mathcal{G} \oplus T^*\mathcal{G} \to T\mathcal{G} \oplus T^*\mathcal{G},$$

we get that the addition in $\Omega^2_{\text{aff}}(\mathcal{G})$ is the composition of two affine 2-forms as matrices.

Remark 5.16. As a multiplicative (1, 1)-tensor can be characterized as a Lie groupoid morphism from $T\mathcal{G}$ to $T\mathcal{G}$, a multiplicative p-vector field defines a morphism of Lie groupoids from $\oplus^{p-1}T^*\mathcal{G}$ to $T\mathcal{G}$ and a multiplicative p-form defines a morphism of Lie groupoids from $\oplus^{p-1}T\mathcal{G}$ to $T^*\mathcal{G}$ [Bursztyn and Drummond 2019]. In these cases, viewing multiplicative structures as functors, we see that affine structures are actually natural transformations between these multiplicative structures.

Example 5.17. One example of multiplicative (1, 1)-tensors on a Lie groupoid $\mathcal{G}$ is $\overrightarrow{n} - \overleftarrow{n}$ for any $n \in \Gamma(A \otimes T^*M)$. And $\overrightarrow{n}$ and $\overleftarrow{n}$ are both affine (1, 1)-tensors on $\mathcal{G}$.

Example 5.18. An affine (1, 1)-tensor on a Lie group G is a multiplicative (1, 1)-tensor on G, which is a G-equivariant linear map from $\mathfrak{g} := \text{Lie}(G)$ to $\mathfrak{g}$. Namely,

$$T^{1,1}_{\text{aff}}(G) = \{N \in \text{End}(\mathfrak{g}) \mid N(\text{Ad}_g u) = \text{Ad}_g N(u),\ g \in G, u \in \mathfrak{g}\}.$$

The product of two affine (1, 1)-tensors in the strict monoidal category structure is the composition of linear maps and the groupoid multiplication is trivial, i.e., $N * N = N$, meaning that an affine (1, 1)-tensor can only multiply itself, which results in itself.

Example 5.19. For the pair groupoid $\mathcal{G} = M \times M \rightrightarrows M$, a multiplicative (1, 1)-tensor on $\mathcal{G}$ is always of the form $\overrightarrow{N} - \overleftarrow{N}$ for a bundle map $N : TM \to TM$, where by definition,

$$(\overrightarrow{N} - \overleftarrow{N})(X, Y) = \overrightarrow{N(X)} - \overleftarrow{N(Y)} = (N(X), 0) + (0, N(Y)) = (N(X), N(Y)),$$

for all $(X, Y) \in T_xM \times T_yM$. In fact, for any $u \in \mathfrak{X}(M)$, we get $\overrightarrow{u}(x, y) = \frac{d}{dt}|_{t=0}(\phi^u_t(x), x)(x, y) = (u, 0)$, where $\phi^u_t(x)$ is a flow of u such that $\phi^u_0(x) = x$.

And $\overleftarrow{u}(x, y) = -\frac{d}{dt}|_{t=0}(x, y)(y, \phi_t^u(y)) = -(0, u)$, where the minus sign comes from the convention that $\overleftarrow{u}(\cdot) = -L_{(\cdot)}\mathrm{inv}(u)$.

By the relation between affine and multiplicative $(1, 1)$-tensors, an affine $(1, 1)$-tensor is of the form $\overrightarrow{N} + \overleftarrow{N'}$ for two bundle maps $N, N' \in \mathrm{End}(TM)$. For simplicity, we write an affine $(1, 1)$-tensor as (N, N'). The product in the strict monoidal category structure is the composition of two $(1, 1)$-tensors:

$$(N_1, N_2) \circ (N_3, N_4) = (N_1 \circ N_3, -N_2 \circ N_4) \quad \text{for all } N_i \in \mathrm{End}(TM),$$

which follows from

$$\begin{aligned}(\overrightarrow{N_1} + \overleftarrow{N_2}) \circ (\overrightarrow{N_3} + \overleftarrow{N_4})(X, Y) &= (\overrightarrow{N_1} + \overleftarrow{N_2})(N_3(X), -N_4(Y)) \\ &= (N_1 \circ N_3(X), N_2 \circ N_4(Y)).\end{aligned}$$

For the groupoid multiplication, two affine $(1, 1)$-tensors (N_1, N_2) and (N_3, N_4) are multiplicable if and only if $N_2 = -N_3$ and the multiplication is

$$(N_1, N_2) * (N_3, N_4) = \overrightarrow{N_1} + \overleftarrow{N_2} + \overleftarrow{N_3} + \overleftarrow{N_4} = \overrightarrow{N_1} + \overleftarrow{N_4} = (N_1, N_4).$$

The next example we are interested in is the direct sum of the pair groupoid and a Lie group: $M \times M \times G \rightrightarrows M$. The following proposition tells us that this case is just the direct sum of Examples 5.18 and 5.19. So the strict monoidal category structure for this case is also clear.

Proposition 5.20. *An affine* $(1, 1)$*-tensor N on $M \times M \times G \rightrightarrows M$ is of the form*

$$N(X,Y,g,u)=(N_1(X),N_2(Y),g,L(u)) \quad \textit{for all } N_1, N_2 \in \mathrm{End}(TM),\ L \in \mathrm{End}(\mathfrak{g})^G,$$

for $X \in T_xM$, $Y \in T_yM$, $g \in G$ and $u \in \mathfrak{g}$. It is multiplicative if and only if $N_1 = N_2$.

Proof. A multiplicable $(1, 1)$-tensor field on $\mathcal{G}$ is a bundle map

$$N = (N_1, N_2, N_3) : TM \times TM \times TG \to TM \times TM \times TG$$

over the base manifold $M \times M \times G$ and also a groupoid morphism over TM to itself. We claim that a multiplicative $(1, 1)$-tensor field can only be of the form

$$N(X, Y, g, u) = (N(X), N(Y), g, L(u)) \quad \text{for all } X \in T_xM,\ Y \in T_yM,\ g \in G,\ u \in \mathfrak{g},$$

for some $N \in \mathrm{End}(TM)$ and a G-equivariant linear map $L \in \mathrm{End}(\mathfrak{g})^G$. In fact, since N preserves the s and t-fibers, the first and second components N_1, N_2 have to be the same. Then N being a groupoid morphism requires that

$$N_3(X, Y, g, u) + \mathrm{Ad}_g\, N_3(Y, Z, h, v) = N_3(X, Z, gh, u + \mathrm{Ad}_g\, v).$$

Since N_3 is a bundle map, it follows that $N_3(X, Y, g, u)$ is independent of X and Y and it is determined by a G-equivariant linear map $L \in \mathrm{End}(\mathfrak{g})$. We thus easily get that an affine (1, 1)-tensor field N on $M \times M \times G \rightrightarrows M$ is of the form

$$N(X, Y, g, u) = (N_1(X), N_2(Y), g, L(u)) \quad \text{for all } N_1, N_2 \in \mathrm{End}(TM),\ L \in \mathrm{End}(\mathfrak{g}).$$

Hence the strict monoidal structure for this case is clear. □

At the end of this section, we provide a class of affine (1, 1)-tensors coming from the composition of affine 2-vector fields and affine 2-forms.

Proposition 5.21. *The composition $\Pi \circ \Theta : T\mathcal{G} \to T\mathcal{G}$ of an affine 2-vector field $\Pi \in \mathfrak{X}^2(\mathcal{G})$ and an affine 2-form $\Theta \in \Omega^2(\mathcal{G})$ is an affine (1, 1)-tensor with*

$$\mathrm{pr}_{A \otimes T^*M}(\Pi \circ \Theta)|_M = \pi_{A^*} \circ \theta + \pi \circ \theta_{TM} + \pi \circ \rho^* \circ \theta,$$

where $\pi \in \Gamma(\wedge^2 A)$, $\pi_{A^*} \in \Gamma(A \otimes TM)$ and $\theta_{TM} \in \Gamma(T^*M \otimes A^*)$ and $\theta \in \Omega^2(M)$ are the corresponding components of Π and Θ restricting on M, respectively.

Moreover, the associated two multiplicative (1, 1)-tensors are

$$(\Pi \circ \Theta)_l = \Pi_l \circ \Theta_l, \quad (\Pi \circ \Theta)_r = \Pi_r \circ \Theta_r.$$

Proof. Denote $\Pi = \Pi_r + \overrightarrow{\pi}$ and $\Theta = \Theta_r + t^*\theta$, where Π_r and Θ_r are the associated multiplicative 2-vector field and 2-form. Then

$$\Pi \circ \Theta = \Pi_r \circ \Theta_r + \overrightarrow{\pi} \circ \Theta_r + \Pi_r \circ t^*\theta + \overrightarrow{\pi} \circ t^*\theta. \tag{23}$$

Acting on $X \in T_g\mathcal{G}$ and pairing with $\xi \in T^*_g\mathcal{G}$, we find

$$\langle \overrightarrow{\pi} \circ \Theta_r(X), \xi \rangle = \langle \pi, t\Theta_r(X) \wedge t\xi \rangle = \langle \pi, \theta_{TM}(tX) \wedge t\xi \rangle = \langle \overrightarrow{\pi \circ \theta_{TM}(tX)}, \xi \rangle,$$

where we have used (1) and the fact that (Θ_r, θ_{TM}) is a Lie groupoid morphism from $T\mathcal{G}$ to $T^*\mathcal{G}$. This implies that

$$\overrightarrow{\pi} \circ \Theta_r = \overrightarrow{\pi \circ \theta_{TM}}.$$

On the other hand, using (1) again and the fact that $(\Pi_r, \pi_{A^*}) : T^*\mathcal{G} \to T\mathcal{G}$ is a Lie groupoid morphism, we have

$$\langle \Pi_r \circ t^*\theta(X), \xi \rangle = -\langle \theta, tX \wedge t\Pi_r(\xi) \rangle = -\langle \theta(tX), \pi_{A^*}t(\xi) \rangle = \langle \overrightarrow{\pi_{A^*} \circ \theta(tX)}, \xi \rangle.$$

Thus,

$$\Pi_r \circ t^*\theta = \overrightarrow{\pi_{A^*} \circ \theta}.$$

At the end, observe that $t\overrightarrow{\pi}(\xi) = \rho\pi(t\xi)$ since

$$\langle t\overrightarrow{\pi}(\xi), \alpha \rangle = \langle \pi(t\xi), t^*\alpha \rangle = \langle \pi(t\xi), \rho^*\alpha \rangle \quad \text{for all } \alpha \in \Omega^1(M).$$

We obtain

$$\langle \overrightarrow{\pi} \circ t^*\theta(X), \xi \rangle = -\langle \theta(tX), t\overrightarrow{\pi}(\xi) \rangle = -\langle \theta(tX), \rho\pi(t\xi) \rangle = \langle \overrightarrow{\pi \circ \rho^* \circ \theta(tX)}, \xi \rangle.$$

Hence,

$$\overrightarrow{\pi} \circ t^*\theta = \overrightarrow{\pi \circ \rho^* \circ \theta}.$$

Coupled with (23), we get

$$\Pi \circ \Theta = \Pi_r \circ \Theta_r + \overrightarrow{(\pi \circ \theta_{TM} + \pi_{A^*} \circ \theta + \pi \circ \rho^* \circ \theta)}.$$

Since $\Pi_r \circ \Theta_r : T\mathcal{G} \to T\mathcal{G}$ is a Lie groupoid morphism and thus gives a multiplicative $(1,1)$-tensor, we have proved that $\Pi \circ \Theta$ is an affine $(1,1)$-tensor. The other assertions are also clear. □

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