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EXCEPTIONAL GROUPS OF RELATIVE RANK ONE AND GALOIS INVOLUTIONS OF TITS QUADRANGLES

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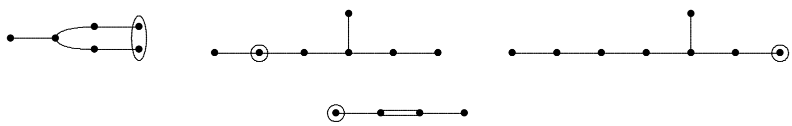
We show that every Moufang set associated with one of the Tits indices $^2E^{29}_{6,1}$, $E^{48}_{7,1}$, $E^{91}_{8,1}$ or $F^{21}_{4,1}$ in arbitrary characteristic can be obtained as the fixed point building of a Galois involution acting on a Tits quadrangle parametrized by a quadrangular algebra. This result is used to calculate an explicit formula for the structure map of an arbitrary Moufang set in this class.

MSC2010: 20E42, 51E12, 51E24.
Keywords: building, Moufang set, Tits polygon, exceptional group.

1. Introduction

Let $G(K)$ be the group of rational points of an absolutely simple algebraic group G defined over K and of relative rank 1. Let X be the set of parabolic subgroups P of G defined over K and for each $P \in X$, let U_P be the group of K-rational points of the unipotent radical of P. For each $P \in X$, the group U_P acts sharply transitively on $X \backslash \{P\}$. Thus the triple $(G(K), X, \{U_P\}_{P \in X})$ is a *Moufang set* as defined in Definition 4.1.

Let $\mathcal{T}_1$ denote the following set of Tits indices:



where the circled vertex in the last case corresponds to a short root. These are the Tits indices called ${}^2E_{6,1}^{29}$, $E_{7,1}^{48}$, $E_{8,1}^{91}$ and $F_{4,1}^{21}$ in [Tits 1966a]. The groups $G(K)$ that correspond to the Tits indices in $\mathcal{T}_1$ are precisely the groups of exceptional absolute type and relative rank 1 such that for each parabolic subgroup P defined over K, the unipotent radical of P is nonabelian but has a center of dimension greater than 1. We denote the class of Moufang sets associated with these groups by $\mathcal{M}$.

In this paper, we use the theory of *Tits polygons* introduced in [Mühlherr and Weiss ≥ 2020] to investigate the structure of the Moufang sets in $\mathcal{M}$. A Tits polygon is a bipartite graph Γ such that for each vertex v, the set of vertices adjacent to v is endowed with an "opposition relation" subject to certain axioms; see Section 3 for details. Moufang polygons (which were classified in [Tits and Weiss 2002]) are precisely the Tits polygons in which the opposition relations are all trivial.

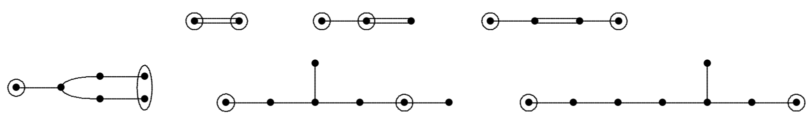
Let T be a spherical Tits index of relative rank 2. The *relative Coxeter group* associated with T (as defined in [Mühlherr et al. 2015, 20.32 and 20.34]; cf. [Tits 1966a, 2.5.2]) is a dihedral group of order $2n$ for $n = 3, 4, 6$ or 8. We call n the *polygon type* of T.

We say that a Tits index T is *linked* to a building Δ if the *absolute type of* T is also the Coxeter diagram of Δ.

Suppose that Δ is a spherical building of type (W, S) and that T is a Tits index of relative rank 2 linked to Δ. Let V be the set of residues of Δ whose type is the complement in S of either one of the two circles in T and let Γ be the bipartite graph with vertex set V, where two residues are adjacent whenever they have a nonempty intersection. If Δ is assumed to be Moufang as defined in Definition 6.5 (which is always the case if Δ is the spherical building associated to the group of rational points of an absolutely simple algebraic group of relative rank at least 2), then the graph Γ is endowed canonically with the structure of a Tits n-gon $X_{\Delta,T}$, where n is the polygon type of T.

We say that a building is *exceptional* if it is the spherical building associated to the group of rational points of an algebraic group of exceptional absolute type and relative rank at least 2. Let $\mathcal{E}$ be the set of pairs (Δ, T), where Δ is an exceptional building and T is a Tits index of relative rank 2 linked to Δ. An *exceptional Tits polygon* is a Tits polygon isomorphic to $X_{\Delta,T}$ for some pair (Δ, T) in $\mathcal{E}$. For each $n = 3$, 4, 6 and 8, let $\mathcal{E}_n$ be the subset consisting of all (Δ, T) in $\mathcal{E}$ such that n is the polygon type of T.

We have $\mathcal{E}_4 = \{(\Delta, T) \in \mathcal{E} \mid T \in \mathcal{T}_2\}$, where $\mathcal{T}_2$ denotes the following set of Tits indices:



Note that the Tits index in $\mathcal{T}_2$ of absolute type F_4 does not appear in [Tits 1966a]; it is, however, a Tits index as defined in [Mühlherr et al. 2015, 20.1]. Let

$$\mathcal{Q} = \{X_{\Delta,T} \mid (\Delta, T) \in \mathcal{E}_4\}.$$

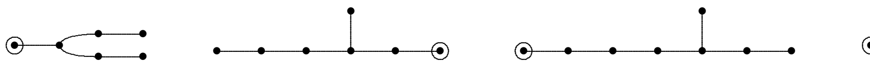
Quadrangular algebras are a class of algebras that arose in the classification of Moufang quadrangles in [Tits and Weiss 2002]. They were classified in [Mühlherr and Weiss 2019; Weiss 2006b]. In [Mühlherr and Weiss 2020], we showed that the Tits quadrangles in $\mathcal{Q}$ are classified by quadrangular algebras. More precisely, each Tits quadrangle in $\mathcal{Q}$ can be coordinatized by commutator relations parameterized by the corresponding quadrangular algebra and theses parametrizations give rise to a one-to-one correspondence between the Tits quadrangles in $\mathcal{Q}$ (up to isomorphism) and nondegenerate quadrangular algebras whose associated quadratic form is of dimension at least 5 (up to isotopy).

In Theorem 8.14, the main result of this paper, we show that every Moufang set in the class $\mathcal{M}$ defined above arises as the fixed point building of a Galois involution acting on a Tits quadrangle in $\mathcal{Q}$. We then use the corresponding quadrangular algebra to calculate explicitly the structure of a root group $\mathcal{U}$ and the *structure map* τ (as defined in Definition 10.8) of the fixed point building of a Galois involution acting on a Tits quadrangle in $\mathcal{Q}$ (modulo Conjecture 19.14). The product of these calculations can be found in Notation 9.18, (9.21) and Theorem 19.7.

Our result and our calculations are valid over arbitrary fields. In particular, we do not make any restriction on the characteristic of K.

In [Boelaert et al. 2019, 2.3.5 and 6.5.3], the structure maps for the Moufang sets in $\mathcal{M}$ were described in terms of the structurable division algebras attached to forms of tensor products of two composition algebras. The theory of structurable algebras requires, however, that $\operatorname{char}(K) \neq 2$ or 3. In [De Medts and Van Maldeghem 2010, 2.1; Callens and De Medts 2014, 5.3], the structure maps for the Moufang sets associated with the Tits indices $F_{4,1}^{21}$ and ${}^2E_{6,1}^{29}$ over an arbitrary field are given in terms of an octonion division algebras by exploiting a connection to the corresponding projective plane, but this method does not extend to the cases $E_{7,1}^{48}$ and $E_{8,1}^{91}$.

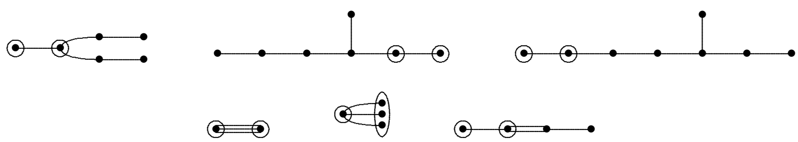
Let $\hat{\mathcal{T}}_1$ denote the following set of Tits indices:



These are the Tits indices called ${}^2E_{6,1}^{35}$, $E_{7,1}^{66}$, $E_{8,1}^{133}$ and ${}^{3,6}D_{4,1}^{9}$ in [Tits 1966a]. Let $\hat{\mathcal{M}}$ denote the class of Moufang sets associated with the Tits indices in $\hat{\mathcal{T}}_1$. These are the Moufang sets arising from an exceptional group of relative rank 1 such that the unipotent radical of a parabolic subgroup defined over K has a 1-dimensional center. We have

$$\{(\Delta, T) \in \mathcal{E} \mid T \in \hat{\mathcal{T}}_2\} \subset \mathcal{E}_6,$$

where $\hat{\mathcal{T}}_2$ denotes the following set of Tits indices:

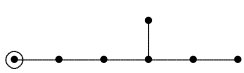


and the circled vertices in the last case corresponds to long roots. Note that the Tits index in $\hat{\mathcal{T}}_2$ of absolute type F_4 does not appear in [Tits 1966a]; it is, however, a Tits index as defined in [Mühlherr et al. 2015, 20.1]. See also [Mühlherr and Weiss $\geq$ 2020, 2.4.6]. Let

$$\mathcal{H} = \{X_{\Delta,T} \in \mathcal{E}_6 \mid T \in \hat{\mathcal{T}}_2\}.$$

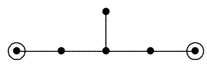
In [Mühlherr and Weiss $\geq$ 2020, 3.8.13], we showed that every Moufang set in $\hat{\mathcal{M}}$ arises as the fixed point building of a Galois involution acting on a Tits hexagon in $\mathcal{H}$. As in the previous case, we then calculated $\mathcal{U}$ and τ for an arbitrary fixed point building of this kind. In this case, the role of quadrangular algebras was played by *cubic norm structures*; see [Mühlherr and Weiss $\geq$ 2020, 3.4.7 and 3.7.1]. The class $\hat{\mathcal{M}}$ is also treated in [Boelaert et al. 2019], but again with the restriction that the characteristic of K is not 2 or 3.

The only exceptional Tits index of relative rank 1 in [Tits 1966a] that is not in $\mathcal{T}_1$ or $\hat{\mathcal{T}}_1$ is $E_{7,1}^{78}$:

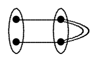


The Moufang sets associated with this index have abelian root groups. They are classified by Albert division algebras and the structure map of one of them is the simply the inverse map of the corresponding Albert algebra. See also Remark 6.14.

We mention, too, that the set $\mathcal{E}_3$ consists of the pairs (Δ, T) in $\mathcal{E}$ such that T is the Tits index



and the set $\mathcal{E}_8$ consists of the pairs (Δ, T) in $\mathcal{E}$ such that T is



(a Tits index in the sense of [Mühlherr et al. 2015, 20.1]).

Organization 1.1. This paper is organized as follows. In Sections 2–5 we introduce quadrangular algebras, Tits polygons, Moufang sets and Tits indices and describe the construction of the Tits polygon $X_{\Delta,T}$. In Section 6 we assemble material about descent, Galois groups and the construction of Moufang sets as fixed point buildings. In Section 7 we describe the connection between the exceptional quadrangles and quadrangular algebras and we prove our main theorem about Moufang sets in Theorem 8.14. This is the result that says that every Moufang set in $\mathcal{M}$ can be found as the fixed point building of a Galois involution of an exceptional Tits quadrangle.

In Section 9 we introduce an arbitrary Galois involution ω of an exceptional Tits quadrangle Δ and choose suitable coordinates for Δ. In Section 10 we introduce the structure map of the Moufang set $\Delta^{\langle\omega\rangle}$ associated with ω. In Sections 11–19 we calculate. These calculations result in the formula for τ in Proposition 19.6 and Theorem 19.7; see also Conjecture 19.14 and Proposition 19.15.

Conventions 1.2. Let G be a group. As in [Tits and Weiss 2002], we set $a^b = b^{-1}ab$ and

$$[a, b] = a^{-1}b^{-1}ab$$

for all $a, b \in G$. We compose permutations from left to right. Other functions will be written on the left and composed from right to left.

2. Quadrangular algebras

The notion of an anisotropic quadrangular algebra was introduced in [Weiss 2006b] and the following more general notion was introduced in [Mühlherr and Weiss 2019].

Definition 2.1. A *quadrangular algebra* is an ordered set

$$\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta),$$

where K is a field, L is a vector space over K, q is a nondegenerate quadratic form on L, f is the bilinear form associated with q, ε is an element of L such that $q(\varepsilon) = 1$, $\mathcal{X}$ is a nontrivial vector space over K, $(a, v) \mapsto a \cdot v$ is a map from $\mathcal{X} \times L$

to $\mathcal{X}$ (usually denoted simply by juxtaposition), h is a map from $\mathcal{X} \times \mathcal{X}$ to L and θ a map from $\mathcal{X} \times L$ to L satisfying the following axioms:

(A1) The map $\cdot$ is bilinear (over K).

(A2) $a \cdot \varepsilon = a$ for all $a \in \mathcal{X}$.

(A3) $(av)\bar{v} = q(v)a$ for all $a \in \mathcal{X}$ and all $v \in L$, where $\bar{v} = f(v, \varepsilon)\varepsilon - v$.

(B1) h is bilinear (over K).

(B2) $h(a, bv) = h(b, av) + f(h(a, b), \varepsilon)v$ for all $a, b \in \mathcal{X}$ and all $v \in L$.

(B3) $f(h(av, b), \varepsilon) = f(h(a, b), v)$ for all $a, b \in \mathcal{X}$ and all $v \in L$.

(C1) For each $a \in \mathcal{X}$, the map $v \mapsto \theta(a, v)$ is linear (over K).

(C2) $\theta(ta, v) = t^2\theta(a, v)$ for all $t \in K$, all $a \in \mathcal{X}$ and all $v \in L$.

(C3) There exists a function g from $\mathcal{X} \times \mathcal{X}$ to K such that

$$\theta(a + b, v) = \theta(a, v) + \theta(b, v) + h(a, bv) - g(a, b)v$$

for all $a, b \in \mathcal{X}$ and all $v \in L$.

(C4) There exists a function ϕ from $\mathcal{X} \times L$ to K such that

$$\theta(av, w) = q(v)\overline{\theta(a, \bar{w})} - f(w, \bar{v})\overline{\theta(a, v)} + f(\theta(a, v), \bar{w})\bar{v} + \phi(a, v)w$$

for all $a \in \mathcal{X}$ and $v, w \in L$, where $\bar{u}$ for $u \in L$ is as in (A3).

(D1) Let $\pi(a) = \theta(a, \varepsilon)$ for all $a \in \mathcal{X}$. Then

$$a\theta(a, v) = (a\pi(a))v$$

for all $a \in \mathcal{X}$ and all $v \in L$.

Notation 2.2. Let $\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$ be a quadrangular algebra. We say that Ξ is a quadrangular algebra *over* K if we want to specify the field K. The map h is *nondegenerate* if for each nonzero $a \in \mathcal{X}$ there exists $b \in \mathcal{X}$ such that $h(a, b) \neq 0$. We say that Ξ is δ-*standard* for some $\delta \in L$ if $f(\pi(a), \delta) = 0$ for all $a \in \mathcal{X}$ and either $\operatorname{char}(K) \neq 2$ and $\delta = \frac{1}{2}\varepsilon$ or $\operatorname{char}(K) = 2$, $f(\varepsilon, \delta) = 1$ and $q(\delta) \neq 0$. We say that Ξ is *anisotropic* if the quadratic form q is anisotropic *and*

$$\pi(a) = t\varepsilon \text{ for some } t \in K \text{ only if } a = 0,$$

where π is as in (D1). If either of these conditions fails to hold, we say that Ξ is *isotropic*.

The classification of δ-standard nondegenerate quadrangular algebras is given in [Mühlherr and Weiss 2019]. This classification extended the classification of anisotropic quadrangular algebras given in [Weiss 2006b]. We will say more about this in Remark 7.10

Notation 2.3. Let $\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$ be a quadrangular algebra. We set

$$Q(a) = f(\pi(a), \varepsilon)$$

for all $a \in \mathcal{X}$.

Almost all of the results in [Weiss 2006b, Chapters 3 and 4] (where it is assumed that Ξ is anisotropic) remain valid under the assumptions that Ξ is δ-standard for some $\delta \in L$ and $|K| > 2$. This is shown in [Mühlherr and Weiss 2019, Section 7, see, in particular, Conclusion 7.5]; We note that the hypothesis $|K| > 2$ is required for the result [Weiss 2006b, 3.22].

We will assume that the reader has access to [Weiss 2006b, Chapters 3 and 4] and will frequently cite results from these two chapters (keeping the caveats in [Mühlherr and Weiss 2019, Conclusion 7.5] in mind). Note that the map $v \mapsto \bar{v}$ in (A3) is called σ in [Weiss 2006b]. We refer to reader also to the remarks (i)–(viii) on page 7 of [Weiss 2006b], especially the remark (iv).

If $a \in \mathcal{X}$ and $u, v \in L$, we will write auv in place of $(au)v$. Since we are not assuming the existence of a multiplication on L, this convention should not cause any confusion.

By [Weiss 2006b, 4.1 and 4.5(iii)], the functions ϕ in (C4) and Q in Notation 2.2 are both identically zero if $\operatorname{char}(K) \neq 2$. We draw the reader's attention also to [Weiss 2006b, 1.4 and 3.6–3.8]. We will make particularly frequent use of these identities.

We have assembled a few additional identities, which are not in [Weiss 2006b], in the Appendix; see, in particular, Remark A.5.

We close this section with a definition which will not be used until Theorem 7.4.

Notation 2.4. Let $\mathcal{R}_\Xi$ denote the set $\mathcal{X} \times K$ endowed with the multiplication given by

$$(a, t) \cdot (b, r) = \big(a + b, t + r + g(b, a)\big)$$

for all $(a, t), (b, r)$. By [Weiss 2006b, 4.4], g is bilinear. It follows that $\mathcal{R}_\Xi$ is a group in which

$$(a, t)^{-1} = \big(-a, -t + g(a, a)\big)$$

for all (a, t).

3. Tits polygons

The notion of a Tits polygon was introduced in [Mühlherr and Weiss $\geq$ 2020]. In this section we give the definition and assemble just a few of their basic properties.

Definition 3.1. A *dewolla* is a triple

$$X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V}),$$

where:

(i) Γ is a bipartite graph with vertex set V and $|\Gamma_v| \geq 3$ for each $v \in V$, where Γ_v denotes the set of vertices adjacent to v.

(ii) For each $v \in V$, $\equiv_v$ is an antireflexive symmetric relation on Γ_v. We say that vertices $u, w \in V$ are *opposite at* v if $u, w \in \Gamma_v$ and $u \equiv_v w$ and a path $(w_0, w_1, \ldots, w_m)$ in Γ is called *straight* if w_{i-1} and w_{i+1} are opposite at w_i for all $i \in [1, m-1]$.

(iii) $\mathcal{A}$ is a set of connected subgraphs γ containing $2n$ vertices for some $n \geq 3$ such that for each vertex v on γ, Γ_v contains exactly two vertices in γ and these two vertices are opposite at v.

The parameter n is called the *level* of X. The automorphism group $\mathrm{Aut}(X)$ is the subgroup of $\mathrm{Aut}(\Gamma)$ consisting of all $g \in \mathrm{Aut}(\Gamma)$ such that $\gamma^g \in \mathcal{A}$ for all $\gamma \in \mathcal{A}$ and for all $u, v, w \in V$ such that u and w are opposite at v, u^g and w^g are opposite at v^g. A *root* of X is a straight path of length n.

Definition 3.2. A *Tits n-gon* is a dewolla

$$X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$$

of level n for some $n \geq 3$ such that Γ is connected and the following axioms hold:

(i) For all $v \in V$ and all $u, w \in \Gamma_v$, there exists $z \in \Gamma_v$ that is opposite both u and w at v.

(ii) For each straight path $\delta = (w_0, \ldots, w_k)$ of length $k \leq n-1$, δ is the unique straight path of length at most k from w_0 to w_k.

(iii) For $G = \mathrm{Aut}(X)$ and for each root $\alpha = (w_0, \ldots, w_n)$ of X, the group U_α acts transitively on the set of vertices opposite w_{n-1} at w_n, where U_α is the pointwise stabilizer of

$$\Gamma_{w_1} \cup \Gamma_{w_2} \cup \cdots \cup \Gamma_{w_{n-1}}$$

in $\mathrm{Aut}(X)$. The group U_α is called the *root group* associated with the root α.

A *Tits polygon* is a Tits n-gon for some $n \geq 3$. A Tits n-gon is called a *Tits triangle* if $n = 3$, a *Tits quadrangle* if $n = 4$, etc.

Definition 3.3. A *Moufang n-gon* is a Tits n-gon $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$ for some $n \geq 3$ in which for all vertices v, the relation $\equiv_v$ is trivial, i.e., every two vertices adjacent to v are opposite at v. In this case, $\mathcal{A}$ is the set of all circuits of Γ of length $2n$, so X is uniquely determined by Γ alone (and it is usual to refer to the Moufang n-gon as Γ rather than X). See [Mühlherr and Weiss $\geq$ 2020, 1.2.2 and 1.2.3]. Moufang polygons were classified in [Tits and Weiss 2002, 17.1–17.8].

Notation 3.4. Let $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ be a Tits n-gon. A *coordinate system* for X is a pair $(\gamma, i \mapsto w_i)$ where γ is an element of $\mathcal{A}$ and $i \mapsto w_i$ is a surjection from $\mathbb{Z}$ to the vertex set of γ such that w_{i-1} is adjacent to w_i for each i. For each coordinate system $(\gamma, i \mapsto w_i)$, we denote by U_i the root group associated with the root $(w_i, w_{i+1}, \ldots, w_{i+n})$ for each $i \in \mathbb{Z}$ and call the map $i \mapsto U_i$ the associated *root group labeling*. Thus $w_i = w_j$ and $U_i = U_j$ whenever i and j have the same image in $\mathbb{Z}_{2n}$. If $i \mapsto w_i$ is a coordinate system, then so is $i \mapsto w_{n+1-i}$. These two coordinate systems are called *opposite*.

For the rest of this section, we fix a Tits n-gon $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ and a coordinate system $(\gamma, i \mapsto w_i)$ of X and let $i \mapsto U_i$ be the corresponding root group labeling.

Proposition 3.5. *The following hold*:

(i) $[U_i, U_j] \subset U_{[i+1,j-1]}$ *for all* i, j *such that* $i < j < i+n$, *where*

$$U_{[k,m]} = \begin{cases} U_k U_{k+1} \cdots U_m & \text{if } k \le m, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

In particular, $[U_i, U_{i+1}] = 1$ *for all* i.

(ii) *The product map* $U_1 \times U_2 \times \cdots \times U_n \to U_{[1,n]}$ *is bijective.*

Proof. This holds by [Mühlherr and Weiss $\geq$ 2020, 1.3.38]. □

Proposition 3.6. *For each* i, U_i *acts sharply transitively on the set of vertices that are opposite* w_{i+1} *at* w_i *and on the set of vertices that are opposite* w_{i+n-1} *at* w_{i+n}.

Proof. This holds by [Mühlherr and Weiss $\geq$ 2020, 1.3.25]. □

Notation 3.7. Let

$$U_i^\sharp = \{a \in U_i \mid w_{i+n+1}^a \text{ is opposite } w_{i+n+1} \text{ at } w_{i+n}\}$$

for each i. By [Mühlherr and Weiss $\geq$ 2020, 1.4.8], we have

$$U_i^\sharp = \{a \in U_i \mid w_{i-1}^a \text{ is opposite } w_{i-1} \text{ at } w_i\}$$

for each i.

Proposition 3.8. *For each* $i \in \mathbb{Z}$, *there exist unique maps* κ_γ *and* λ_γ *from* $U_i^\sharp$ *to* $U_{i+n}^\sharp$ *such that for each* $a \in U_i^\sharp$, *the product*

$$\mu_\gamma(a) := \kappa_\gamma(a) \cdot a \cdot \lambda_\gamma(a) \tag{3.9}$$

interchanges the vertices w_{i+n-1} *and* w_{i+n+1}. *For each* $a \in U_i^\sharp$, *the element* $\mu_\gamma(a)$ *fixes the vertices* w_i *and* w_{i+n} *and interchanges the vertices* w_{i+j} *and* w_{i-j} *for all* $j \in \mathbb{Z}$ *and*

$$U_k^{\mu_\gamma(a)} = U_{2i+n-k} \tag{3.10}$$

for all $k \in \mathbb{Z}$.

Proof. This holds by [Mühlherr and Weiss $\geq$ 2020, 1.4.4]. □

4. Moufang sets

The notion of a Moufang set was introduced in [Tits 1992].

Definition 4.1. A *Moufang set* is a triple $M = (G, X, \{U_x\}_{x\in X})$, where X is a set such that $|X| \geq 3$, G is a group acting on X and $\{U_x\}_{x\in X}$ is a conjugacy class of subgroups of G such that for each $x \in X$, U_x fixes x and acts sharply transitively on $X\backslash\{x\}$. The groups U_x for $x \in X$ are called the *root groups* of the Moufang set. We call X the *underlying set* of M. The action of G on X is not necessarily faithful; see Definition 4.8 below. If in an example X and the root groups of M are clear from the context, we sometimes refer to G itself as a Moufang set.

Example 4.2. The group $G = \mathrm{PGL}_2(D)$ is a Moufang set for each skew field D. In this example, the underlying set is the projective line $\mathrm{PG}(D)$, i.e., the set of 1-dimensional subspaces of a 2-dimensional right vector space V over D. Choose a basis v_1, v_2 of V and use it to identify V with $D \oplus D$ and G with the group of 2×2 invertible matrices over D acting on V on the left. Then $U_{\langle v_1\rangle}$ is the subgroup of G consisting of all matrices with 1s on the diagonal and 0 below the diagonal.

Example 4.3. Let Δ be a Moufang quadrangle or hexagon. If ρ is a polarity of Δ (i.e., a non-type-preserving involution), then the set of edges of Δ fixed by ρ has the structure of a Moufang set. See [De Medts et al. 2017; Tits 1966b] for explicit examples.

Example 4.4. Let Γ be a Moufang polygon, let $D = \mathrm{Aut}(\Gamma)$ and let v be a vertex of Γ. Let X be the set Γ_v of vertices adjacent to v, let G be subgroup of $\mathrm{Sym}(X)$ induced by the stabilizer D_v and for each $x \in X$, let U_x be the subgroup of G induced by the root group U_α, where α is a path $(z_0, z_1, \ldots, z_n)$ of length n in Γ such that $z_0 = v$ and $z_1 = x$. By [Tits and Weiss 2002, 4.6 and 4.8], the triple $(G, X, \{U_x\}_{x\in X})$ is a Moufang set.

Example 4.5. Let $G(K)$ be the group of rational points of an absolutely simple algebraic group G defined over K and of relative rank 1. Let X be the set of parabolic subgroups of G defined over K and for each P in X, let U_P be the group of K-rational points of the unipotent radical of P. As already observed in the introduction, the triple $(G(K), X, \{U_P\}_{P\in X})$ is a Moufang set.

A fifth source of Moufang sets will be described in Theorem 6.11(v) below. See [De Medts and Segev 2009] for more information about Moufang sets.

Remark 4.6. A building of rank 1 is simply a set of chambers without any additional structure. See, for example, [Weiss 2003, 1.4 and 7.1]. The buildings of rank 1 that arise "in nature", however, always arise in conjunction with a permutation group with respect to which the building has the structure of a Moufang set.

Definition 4.7. Let $M = (G, X, \{U_x\}_{x\in X})$ and $M' = (G', X, \{U'_x\}_{x\in X'})$ be two Moufang sets. We call a bijection π from X to X' such that $\pi U_x \pi^{-1} = U'_{\pi(x)}$ for all $x \in X$ an *equivalence map* from M to M'. An equivalence map that carries G to G' is called an *isomorphism*. We say that M and M' are *equivalent* if there exists an equivalence map from M to M'.

Definition 4.8. Let $M = (G, X, \{U_x\}_{x\in X})$ be a Moufang set. We say that M is faithful if G acts faithfully on X. Let $G^\dagger = \langle U_x \mid x \in X\rangle$. Then $G^\dagger$ acts transitively on X, hence $\{U_x\}_{x\in X}$ is a single conjugacy class of $G^\dagger$ and thus $(G^\dagger, X, \{U_x\}_{x\in X})$ is also a Moufang set.

Observation 4.9. Every Moufang set M is equivalent to a unique faithful Moufang set $M_0 = (G, X, \{U_x\}_{x\in X})$ such that $G = G^\dagger$.

5. Tits indices

In this section, we explain how Tits polygons arise "in nature."

Notation 5.1. Let Π be a Coxeter diagram with vertex set S and let (W, S) denote the corresponding Coxeter system. For each subset $J \subset S$, we denote by W_J the subgroup of W generated by J. A subset J is *spherical* if W_J is finite. If J is a spherical subset of S, we denote by w_J the longest element in the Coxeter system (W_J, J). For each spherical subset J of S, $s \mapsto w_J s w_J$ is an automorphism of the subdiagram Π_J spanned by J; see, for example, [Mühlherr et al. 2015, 19.6].

Definition 5.2. A *Tits index* is a triple

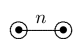
$$T = (\Pi, \Theta, A),$$

where Π is a Coxeter diagram with vertex set S, Θ is a subgroup of $\mathrm{Aut}(\Pi)$ and A is a Θ-invariant subset of S such that for each $s \in S\backslash A$, the subset $J_s := A \cup s^\Theta$ is spherical, and A is stabilized by the automorphism $s \mapsto w_{J_s} s w_{J_s}$ of Π_{J_s}, where s^Θ is the Θ-orbit containing s. The Coxeter diagram Π is called the *absolute type* of T and $|S|$ is called the *absolute rank* of T. For each $s \in S\backslash A$, we denote by $\tilde{s}$ the product $w_A w_{J_s}$. There is one element $\tilde{s}$ for each Θ-orbit disjoint from A. Let $\tilde{S}$ denote the set of all these elements and let $\widetilde{W} = \langle \tilde{S}\rangle$. By [Mühlherr et al. 2015, 20.4, 20.32 and 20.34], $(\widetilde{W}, \tilde{S})$ is also a Coxeter system. It is called the *relative Coxeter system* of T and $|\tilde{S}|$ is called the *relative rank* of T. The *relative type* $\widetilde{\Pi}$ of T is the Coxeter diagram corresponding to its relative Coxeter system.

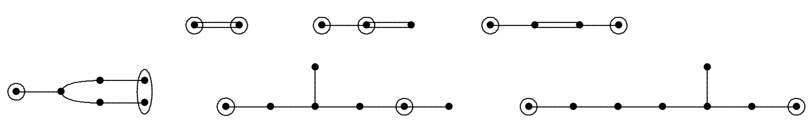
Definition 5.3. A Tits index (Π, Θ, A) is *split* if Θ is trivial and $A = \varnothing$.

Basic Construction 5.4. Let $T = (\Pi, \Theta, A)$ be a Tits index as in Definition 5.2 whose absolute type Π is an irreducible spherical Coxeter diagram and whose relative rank is 2. Thus $\widetilde{W}$ is a dihedral group, where $\widetilde{W}$ is as in Definition 5.2. Let $n = |\widetilde{W}|/2$ and let $J_i = A \cup s_i^\Theta$ for $i = 1$ and 2, where s_1 and s_2 are representatives of the two Θ-orbits disjoint from A. By [Mühlherr et al. 2015, 20.39–20.40], $n \geq 3$. Let Δ be a building of type Π. We assume that Δ is Moufang as defined in [Mühlherr and Weiss 2017, 2.7]. (This is automatic if the rank of T is greater than 2.) Let V_i be the set of J_i-residues of Δ for $i = 1$ and 2 and let Γ be the bipartite graph whose vertex set is $V := V_1 \cup V_2$, where a residue in V_1 is adjacent to a residue in V_2 whenever their intersection is an A-residue of Δ. We declare that two residues $u = R_1$ and $w = R_2$ in V are opposite at a residue $v = R_0$ in V (and we write $u \equiv_v w$) if u and w are adjacent to v and $R_0 \cap R_1$ and $R_0 \cap R_2$ are opposite residues of R_0. Let $\mathcal{A}$ be the set of $2n$ circuits γ in Γ such that for some apartment Σ of Δ, every vertex of γ contains chambers of Σ and for every vertex v of γ, the two vertices of γ adjacent to v are opposite at v. By [Mühlherr and Weiss $\geq$ 2020, 1.2.12 and 1.2.28(i)], the triple $(\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$ is a Tits polygon. We denote this Tits polygon by $X_{\Delta,T}$ and we call the Tits polygons that arise in this way Tits polygons *of index type*.

Remark 5.5. Let Δ be a Moufang n-gon for some $n \geq 3$. In other words, Δ is a building of type $\bullet\overset{n}{\text{—}}\bullet$ satisfying the Moufang condition defined in [Weiss 2003, 11.2]. By [Mühlherr and Weiss $\geq$ 2020, 1.2.2], Δ is naturally endowed with the structure of a Tits n-gon and, in fact, we have $\Delta = X_{\Delta,T}$ as defined in Basic Construction 5.4, where T is the Tits index



Remark 5.6. Tits indices are often represented figures drawn according to the conventions described in [Mühlherr et al. 2015, 34.2]. Drawn according to these conventions, the Tits indices of relative type B_2 (which is the same as relative type C_2) whose absolute type is an exceptional Coxeter diagram are



The indices of absolute type C_2 and C_3 are included here because of the existence of the exceptional Moufang quadrangles (which are buildings of type C_2 whose automorphism groups are forms of groups of type E_6, E_7, E_8 and F_4) and because of the existence of a family of buildings of type C_3 associated with a form of E_7. These are the buildings that appear in Definition 7.1(i) and (ii) below.

The following result should have been included in [Mühlherr and Weiss $\geq$ 2020].

Theorem 5.7. *Let* Δ, $T = (\Pi, \Theta, A)$ *and* $X_{\Delta,T} = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$ *be as in Basic Construction 5.4. Then every automorphism of X has a unique extension to an automorphism of* Δ.

Proof. Let $X = X_{\Delta,T}$ and let J_1, J_2 and R be as in Basic Construction 5.4. Thus $A = J_1 \cap J_2$ and R is an A-residue. Let G be stabilizer of the set $\{J_1, J_2\}$ in $\mathrm{Aut}(\Delta)$, let $G_0 = \mathrm{Aut}(X)$, let $G^\dagger$ denote the subgroup of G generated by all the root groups of Δ, let $G_0^\dagger$ denote the subgroup of G_0 generated by all the root groups of X as defined in Definition 3.2(iii) and let χ denote the restriction map from G to G_0. By [Mühlherr and Weiss $\geq$ 2020, 1.2.10], χ is injective. It will suffice, therefore, to show that χ is surjective.

Let $\mathcal{P}_i$ be the set stabilizers in $G^\dagger$ of all the J_i-residues of Δ for $i = 1$ and 2. Let N denote stabilizer of the set $\{\mathcal{P}_1, \mathcal{P}_2\}$ in $\mathrm{Aut}(G^\dagger)$. Since $G^\dagger$ is normal in $\mathrm{Aut}(\Delta)$, there is a canonical map from $\mathrm{Aut}(\Delta)$ to $\mathrm{Aut}(G^\dagger)$. By [Tits 1974, 5.8–5.10, 6.6, 6.13, 8.6, 9.3 and 10.4] (i.e., the solution to "Problem B" as described in [Tits 1974, Introduction]), this map is an isomorphism. Its restriction to G is an isomorphism from G to N. By [Mühlherr and Weiss $\geq$ 2020, 1.5.3], $\chi(G^\dagger) = G_0^\dagger$. The group $G_0^\dagger$ is normal in G_0 and G_0 stabilizes the subset $\chi(\{\mathcal{P}_1, \mathcal{P}_2\})$ of conjugacy classes of $G_0^\dagger$. It follows that $G_0 = E \cdot \chi(G)$, where $E = C_{G_0}(G_0^\dagger)$.

Let D denote the stabilizer of the A-residue R in $G^\dagger$, let Y denote the set of A-residues that are fixed by D and let M denote the stabilizer of Y in $G^\dagger$. Then M acts transitively on Y (since $G^\dagger$ acts transitively on the set of A-residues of Δ) and $D \subset M$. By [Weiss 2003, 11.16], it follows that M is the stabilizer in $G^\dagger$ of a residue R' of Δ containing R. Suppose $|Y| > 1$. Then $R \neq R'$. Hence we can choose j-adjacent chambers $c \in R$ and $d \in R' \backslash R$ for some $j \notin A$. Let Σ be an apartment containing c and d and let α be the unique root of Σ containing c but not d. Choose $g \in U_\alpha^*$ and let R_0 be the A-residue containing d. Then $g \in D$. By [Weiss 2003, 11.4], d and d^g are unequal but j-adjacent. Thus by [Weiss 2003, 7.21], $R_0^g \neq R_0$. Thus the residues R_0 and R_0^g both lie in R' and hence are in the same M-orbit as R. Thus R_0 and R_0^g both lie in Y. This contradicts the fact that D acts trivially on Y. We conclude that $|Y| = 1$. Thus $\chi(D)$ fixes only one edge of Γ. We have $\chi(D) = \chi(D)^g$ for all $g \in E$. Since $E \subset G_0$, the elements of E map edges to edges. It follows that E fixes the unique edge fixed by $\chi(D)$. Since $G_0^\dagger$ acts transitively on the edge set Γ (by [Mühlherr and Weiss $\geq$ 2020, 1.3.6 and 1.3.40]), it follows that E fixes every edge of Γ. Hence $E = 1$. Thus $G_0 = \chi(G)$. □

6. Descent

In this section we assemble the results in [Mühlherr et al. 2015] on descent in buildings that we will require.

Definition 6.1. Let Δ be a building and let G_0 be a subgroup of $\mathrm{Aut}(\Delta)$. A G_0-*residue* is a residue of Δ stabilized by G_0. A G_0-*chamber* is a G_0-residue which is minimal with respect to inclusion. A G_0-*panel* is a G_0-residue P such that for some G_0-chamber C, P is minimal in the set of all G_0-residues containing C properly.

Definition 6.2. Let Δ and G_0 be as in Definition 6.1. The group G_0 is *anisotropic* if Δ itself is the unique G_0-chamber and *isotropic* if this is not the case. Thus G_0 is isotropic if and only if there exist G_0-panels (equivalently, if there exist G_0-residues other than Δ itself).

Notation 6.3. Let Δ be a building and let G_0 be an isotropic subgroup of $\mathrm{Aut}(\Delta)$. We denote by Δ^{G_0} the graph with vertex set the set of all G_0-chambers, where two G_0-chambers are joined by an edge of Δ^{G_0} if and only if there is a G_0-panel containing them both.

Definition 6.4. Let Δ be a building. A *descent group* of Δ is an isotropic subgroup G_0 of $\mathrm{Aut}(\Delta)$ such that each G_0-panel contains at least three G_0-chambers.

Definition 6.5. Let Δ be a building (assumed to be thick). As in [Weiss 2003, 11.7] and [Mühlherr and Weiss 2017, 2.7], we say that Δ is *Moufang* if Δ is spherical, irreducible of rank at least 2 and for each root α of Δ, the root group U_α acts transitively on the set of apartments of Δ containing α. The spherical building associated with the group of rational points of an absolutely simple algebraic group of relative rank at least 2 is always Moufang, but there are a few other families as well. For a summary of the classification of Moufang buildings, see [Weiss 2009, 3.14–3.15].

Remark 6.6. A building of rank 1 is simply a set with no further structure. Thus the notion of a Moufang set is an extension of the notion of a Moufang building to buildings rank 1. (Buildings of rank 1 are automatically spherical and irreducible.)

Theorem 6.7. *Let Δ be a spherical building which is Moufang and let G_0 be a Galois subgroup of* $\mathrm{Aut}(\Delta)$ *as defined in [Mühlherr and Weiss 2017,* 4.5*]. Suppose that G_0 acts with finite orbits on the chamber set of Δ. Then G_0 is a descent group of Δ.*

Proof. This holds by [Mühlherr and Weiss 2017, 12.2(ii)]. □

Definition 6.8. Let Δ be as in Theorem 6.11. A *Galois involution* of Δ is an element ω of $\mathrm{Aut}(\Delta)$ of order 2 such that $\langle\omega\rangle$ is a Galois group as defined in [Mühlherr and Weiss 2017, 4.5].

Proposition 6.9. *Suppose that R is a residue of a Moufang building Δ. Let Σ be an apartment containing chambers of R and let U_R denote the subgroup generated by the root groups U_α for all roots α of Σ containing $R\cap\Sigma$. Then U_R is independent of the choice of Σ.*

Proof. This holds by [Mühlherr et al. 2015, 24.17]. □

Definition 6.10. The group U_R in Proposition 6.9 is called the *unipotent radical* of the residue R.

The following is a special case of the main results of [Mühlherr et al. 2015, Part 3].

Theorem 6.11. *Let G_0 be a descent group of a spherical building Δ. Let Π be the Coxeter diagram of Δ, let S denote the vertex set of Π and let Θ denote the subgroup of* $\mathrm{Aut}(\Pi)$ *induced by G_0. Then the following hold*:

(i) *The graph Δ^{G_0} is a building with respect to a canonical coloring of its edges.*

(ii) *All G_0-chambers are residues of Δ of the same type $A \subset S$, the set A is Θ-invariant and the rank k of Δ^{G_0} is the number of Θ-orbits in S disjoint from A.*

(iii) *The triple $T := (\Pi, \Theta, A)$ is a Tits index and Δ^{G_0} is a building of type $\widetilde{\Pi}$, where $\widetilde{\Pi}$ is the relative Coxeter diagram of T.*

(iv) *If Δ is Moufang and $k \geq 2$, then Δ^{G_0} is also Moufang.*

(v) *Suppose that Δ is Moufang and that $k = 1$ and let X denote the set of all G_0-chambers. Let G be the subgroup of* $\mathrm{Sym}(X)$ *induced by* $C_{\mathrm{Aut}(\Delta)}(G_0)$ *and for each $R \in X$, let $\widetilde{U}_R$ denote the subgroup of* $\mathrm{Sym}(X)$ *induced by* $C_{U_R}(G_0)$, *where U_R is as in Proposition 6.9. Then*

$$(G, X, \{\widetilde{U}_R\}_{R\in X})$$

is a Moufang set as defined in Definition 4.1.

Proof. Assertions (i) and (ii) hold by [Mühlherr et al. 2015, 22.20(v) and (viii)], assertion (iii) holds by [Mühlherr et al. 2015, 22.20(iv) and (viii)] and the remaining two assertions hold by [Mühlherr et al. 2015, 24.31]. □

Definition 6.12. Let G_0 and Δ be as in Theorem 6.11. We refer to the triple T in Theorem 6.11(iii) as the *Tits index of* G_0. (In fact, the Tits index of a descent group G_0 is defined also when Δ is not assumed to be spherical; see [Mühlherr et al. 2015, 22.20 and 22.22].)

Definition 6.13. A *fixed point building* is a building of the form Δ^{G_0} for some pair (Δ, G_0) as in Theorem 6.11. If the rank of Δ^{G_0} is 1 and Δ is Moufang, we interpret Δ^{G_0} to mean the Moufang set described in Theorem 6.11(v).

Remark 6.14. Apart from the Moufang sets described in Examples 4.2 and 4.3 and the Moufang sets described in Example 4.4 in the case that Γ is an indifferent quadrangle as defined in [Tits and Weiss 2002, 16.4], all other Moufang sets known to us (including those described in Example 4.5) arise (up to equivalence as defined in Definition 4.7) as the fixed point buildings Δ^{G_0} for some Moufang building Δ and some descent group with finite orbits G_0 whose Tits index is of relative rank 1.

7. The exceptional Tits quadrangles

In the following definition, we refer the reader to Remark 5.5 for (i) and we make use of the notation for spherical buildings described in [Weiss 2009, 30.15] (corresponding to the cases (vi), (xii) and (xiii) of [Weiss 2009, 30.14]) in (ii), (iii) and (iv).

Definition 7.1. We will say that a Tits quadrangle is *exceptional* if it is isomorphic to $X_{\Delta,T}$, where

(i) Δ is a Moufang quadrangle of type E_6, E_7, E_8 or F_4 as defined in [Tits and Weiss 2002, 16.6 and 16.7] and T is the split Tits index of absolute type C_2, or

(ii) Δ is the building $C_3^{\mathcal{I}}(C, K, \sigma)$ for some octonion division algebra C with center K and standard involution σ and T is the Tits index of absolute type C_3 displayed in Remark 5.6, or

(iii) Δ is the building $E_\ell(K)$ for some field K and $\ell = 6$, 7 or 8 and T is the Tits index of absolute type E_ℓ displayed in Remark 5.6, or

(iv) Δ is the building $F_4(C, K)$ for some composition division algebra (C, K) and T is the Tits index of absolute type F_4 displayed in Remark 5.6.

We will call a building *exceptional* if it is isomorphic to one of the buildings in (i)–(iv).

Notation 7.2. Let the pair (Δ, T) be as Definition 7.1 and let Π denote the Coxeter diagram associated with Δ. Thus $\Pi = C_2$, C_3, E_ℓ, F_4 in cases (i), (ii), (iii), (iv), respectively. If $\Pi = F_4$ but C/K is not a field extension in characteristic 2, we number the vertices of Π from left to right with the integers 1, 2, 3, 4 and choose the map Typ defined in [Mühlherr and Weiss $\geq$ 2020, 1.2.13] so that the $\{2, 3, 4\}$-residues are isomorphic to $\mathcal{B}_3^{\mathcal{Q}}(K, C, q)$, where q is the norm of the composition algebra (C, K). If $\Pi = C_2$ and X is a Moufang quadrangle of type E_6, E_7 or E_8, we number the vertices of Π from left to right with the integers 1, 2 and choose Typ so that a root group U_α is nonabelian if and only if the root α starts and ends at a vertex of type 2. Note that if $\Pi = F_4$ and C/K is a field extension in characteristic 2, then (C, K) is not necessarily an invariant of X and our choice of Typ is not uniquely determined. If $\Pi = C_2$ and X is a Moufang quadrangle of type F_4 or $\Pi = F_4$ and C/K is a field extension in characteristic 2 or $\Pi = E_6$, we choose Typ (between the two possible choices) arbitrarily. In the remaining cases, Π has no nontrivial symmetry and there is only one choice for Typ.

Remark 7.3. Since the Tits indices in Remark 5.6 are all invariant under $\mathrm{Aut}(\Pi)$, the Tits quadrangles in Definition 7.1 do not depend on the choice of Typ in Notation 7.2; see [Mühlherr and Weiss $\geq$ 2020, 1.2.13].

The exceptional Tits quadrangles were investigated in [Mühlherr and Weiss 2020]. Here is a reformulation of some of the results in that paper; we will cite others in Proposition 7.12 below.

Theorem 7.4. *Let X be an exceptional Tits quadrangle, let $(\gamma, i \mapsto w_i)$ be a coordinate system of X and let $i \mapsto U_i$ be the corresponding root group labeling. Let K be the field that appears in [Tits and Weiss 2002,* 16.6 *or* 16.7*] in the case that X is as in Definition 7.1*(i); *in the other cases, let K be as in Definition 7.1. If X is in case Definition 7.1*(iii) *or* (iv), *suppose that $|K| > 4$. Then the following hold*:

(i) *After replacing $(\gamma, i \mapsto w_i)$ by its opposite if necessary, there exists a δ-standard quadrangular algebra*

$$\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$$

over K and isomorphisms x_i from the group $\mathcal{R}_\Xi$ defined in Notation 2.4 to U_i for $i = 1$ and 3 and from the additive group of L to U_i for $i = 2$ and 4 such that

$$\begin{aligned} [x_1(a,t), x_4(v)^{-1}] &= x_2(\theta(a,v)+tv)x_3(av, tq(v)+\phi(a,v)),\\ [x_1(a,t), x_3(b,s)^{-1}] &= x_2(h(a,b)), \quad \textit{and} \\ [x_2(u), x_4(v)^{-1}] &= x_3(0, f(u,v)) \end{aligned} \tag{7.5}$$

for all $(a,t), (b,s) \in \mathcal{R}$ and all $u, v \in L$, where ϕ is as in (C4).

(ii) *If X is a Moufang polygon of type E_ℓ for $\ell = 6, 7$ or* 8, *then q is a quadratic form of type E_ℓ as defined in [Weiss 2006b,* 2.13*].*

(iii) *If X is a Moufang polygon of type F_4, then q is a quadratic form of type F_4 as defined in [Weiss 2006b,* 2.15*].*

(iv) *If X and C are as in Definition 7.1*(ii), *then q is the reduced norm of C.*

(v) *If X and ℓ are as in Definition 7.1*(iii), *then q is a split quadratic form of dimension $4 + 2^{\ell-5}$.*

(vi) *If X and (C, K) are as in Definition 7.1*(iv), *then q is the orthogonal sum of the reduced norm of (C, K) and a hyperbolic quadratic form of dimension* 4.

(vii) *X satisfies the hypothesis [Mühlherr and Weiss 2020,* 8.2(b)*].*

Proof. This holds by [Mühlherr and Weiss 2020, 7.4, 8.2, 10.4]. □

Remark 7.6. The quadratic forms in Theorem 7.4(ii)–(vi) are all of dimension at least 5.

Remark 7.7. If X and ℓ are as in Definition 7.1(iii), let (C, K) denote the split composition algebra of dimension $2^{\ell-5}$. Thus $q = q_C$ if q is as in Theorem 7.4(v) or (vi), where q_C is as defined in [Mühlherr and Weiss 2020, 7.3].

Remark 7.8. The quadratic forms in Theorem 7.4(ii)–(iv) are anisotropic. Over a finite field, there are no anisotropic quadratic forms of dimension greater than 2; see, for example, [Tits and Weiss 2002, 34.3]. By Remark 7.6, therefore, the field K must be infinite in cases Definition 7.1(i) and (ii).

Remark 7.9. The quadrangular algebra Ξ in Theorem 7.4 is anisotropic when X is as in Definition 7.1(i) but not when X is as in one of the other cases.

Remark 7.10. The quadrangular algebras that appear in Theorem 7.4 are, up to isotopy as defined in [Mühlherr and Weiss 2019, 5.3], the only δ-standard nondegenerate quadrangular algebras (as defined in Notation 2.2) satisfying $|K| > 4$ and $\dim_K L > 4$. This holds by [Weiss 2006b, 6.42 and 7.57] in the anisotropic case and by [Mühlherr and Weiss 2019, 8.16, 9.8 and 10.16] in the isotropic case.

Hypothesis 7.11. For the rest of this section, let X, $(\gamma, i \mapsto w_i)$, $i \mapsto U_i$, Ξ and $x_1, \ldots, x_4$ be as in Theorem 7.4.

Proposition 7.12. *Let $Y_i = \{x_i(0,t) \mid t \in K\}$ for all odd i and let β_i denote the root $(w_i, w_{i+1}, \ldots, w_{i+4})$ for all i. There exists a Tits quadrangle $\hat{X} = (\hat{\Gamma}, \hat{\mathcal{A}}, \{\hat{\equiv}_v\}_{v\in\hat{V}})$ such that the following hold*:

(i) *$\hat{\Gamma}$ is a subgraph of Γ containing γ and $\gamma \in \hat{\mathcal{A}}$.*

(ii) *For all even i, $\hat{\Gamma}$ is normalized by U_i and the natural embedding of U_i into* $\mathrm{Aut}(\hat{X})$ *is an injective isomorphism from U_i to the root group $\hat{U}_i$ of $\hat{X}$ corresponding to the root β_i.*

(iii) *For all odd i, $\hat{\Gamma}$ is normalized by Y_i and the natural embedding of Y_i into* $\mathrm{Aut}(\hat{X})$ *is an injective isomorphism from Y_i to the root group $\hat{U}_i$ of $\hat{X}$ corresponding to the root β_i.*

(iv) *The isomorphisms in* (ii) *and* (iii) *extend to an isomorphism from the subgroup $Y_1U_2Y_3U_4$ of* $\mathrm{Aut}(X)$ *to the subgroup $\hat{U}_{[1,4]} := \hat{U}_1\hat{U}_2\hat{U}_3\hat{U}_4$ of* $\mathrm{Aut}(\hat{X})$.

Proof. This holds by [Mühlherr and Weiss 2020, 5.1 and 10.4(iii)]. □

Proposition 7.13. *Let G_γ denote the pointwise stabilizer of γ in $G :=$* $\mathrm{Aut}(X)$ *and let δ be as in Notation 2.2 and Theorem 7.4. Then the following hold*:

(i) *Let $\varphi \in G_\gamma$. Then there exists σ-linear automorphisms ξ and ψ of L, respectively $\mathcal{X}$, for some $\sigma \in$* $\mathrm{Aut}(K)$ *and an element $\eta \in K^*$ such that the following hold*:

  (a) $q(\xi(v)) = q(\lambda)q(v)^\sigma$, *where* $\lambda := \xi(\varepsilon)$,

  (b) $\psi(av)\lambda = \psi(a)\xi(v)$,

  (c) $h(\psi(a), \psi(b)\lambda) = \eta\xi(h(a,b))$ *and*

  (d) $\theta(\psi(a), \xi(v)) = \eta\xi(\theta(a,v)) + M(a)\xi(v)$, *where*

$$M(a) = \begin{cases} 0 & \text{if } \mathrm{char}(K) \neq 2, \\ f\big(\theta(\psi(a),\lambda), \xi(\delta)\big)q(\lambda)^{-1} & \text{if } \mathrm{char}(K) = 2 \end{cases}$$

for all $a, b \in \mathcal{X}$ and all $v \in L$ and

$$(7.14)\qquad \begin{aligned} x_1(a,t)^\varphi &= x_1(\psi(a), \eta t^\sigma + M(a)),\\ x_2(v)^\varphi &= x_2(\eta\xi(v)),\\ x_3(a,t)^\varphi &= x_3\big(\psi(a)\lambda, (\eta t^\sigma + M(a))q(\lambda) + \phi(\psi(a),\lambda)\big),\\ x_4(v)^\varphi &= x_4(\xi(v)) \end{aligned}$$

for all $(a, t) \in S$ and all $v \in L$. If $\dim_K L > 1$, *then η is uniquely determined by ψ and ξ.*

(ii) *Suppose that* $\dim_K L > 4$. *Let $\sigma \in$* Aut(K), *let ξ and ψ be σ-linear automorphisms of L, respectively $\mathcal{X}$, and let η be an element of K^* such that* (a)–(d) *hold. Then there exists $\varphi \in G_\gamma$ satisfying the identities* (7.14).

Proof. This holds by [Mühlherr and Weiss 2020, 13.1]. □

Notation 7.15. By [Mühlherr and Weiss 2020, 6.4(i) and (ii)], $x_1(0, 1) \in U_1^\sharp$ and $x_4(\varepsilon) \in U_4^\sharp$, where ε is as in Definition 2.1 and $U_1^\sharp$ and $U_4^\sharp$ as in Notation 3.7; see also [Mühlherr and Weiss 2020, 5.1(iii) and 10.4(iii)]. We can thus set $m_1 = \mu_\gamma(x_1(0, 1))$ and $m_4 = \mu_\gamma(x_4(\varepsilon))$, where μ_γ is as in (3.9).

Proposition 7.16. *Let m_1 and m_4 be as in Notation 7.15. Then the following hold*:

(i) $\big(x_1(a,t)x_2(u)x_3(b,s)\big)^{m_4} = x_1(b,s)x_2(-h(b,a)-\bar{u})x_3(-a,t)$, *and*

(ii) $\big(x_2(u)x_3(b,s)x_4(v)\big)^{m_1} = x_2(v)x_3(b, s - f(u,v))x_4(-u)$

for all $(a, t), (b, s) \in \mathcal{R}$ and $u, v \in L$.

Proof. Choose $(a, t), (b, s) \in \mathcal{R}$ and $u, v \in L$. By [Mühlherr and Weiss 2020, 6.4(iv), 8.19, 8.25 and the first display in 8.59], we have $x_1(a,t)^{m_4} = x_3(-a,t)$ and $x_2(u)^{m_4} = x_2(-\bar{u})$ and $x_3(b,s)^{m_4} = x_1(b,s)$ and by [loc. cit., 6.4(v) and 8.31], we have $x_2(u)^{m_1} = x_4(-u)$, $x_3(a,t)^{m_1} = x_3(a,t)$ and $x_4(v)^{m_1} = x_2(v)$. The claims hold, therefore, by Conventions 1.2 and (7.5). □

8. Galois groups

The main result of this section is Theorem 8.14, in which we show that the Moufang sets described in Example 4.5 associated with one of the Tits indices displayed in Hypothesis 8.4 can all be obtained as the fixed point building of a Galois involution acting on one of the Tits quadrangles described in Notation 7.2.

Let Δ, T, $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$,

$$\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$$

and $x_1, \dots, x_4$ be as in Theorem 7.4. Thus the pair (Δ, X) is as in one of the four cases of Definition 7.1. Let (C, K) be as in Definition 7.1(ii) or (iv) if X is in one of these two cases and let (C, K) be as in Remark 7.7 if X is in the case Definition 7.1(iii).

Proposition 8.1. *If K is separably closed, then X is as in Definition 7.1(iii) or* (iv) *and one of the following holds*:

(i) (C, K) *is split and* $\Delta = F_4(K, K)$, $E_6(K)$, $E_7(K)$ *or* $E_8(K)$.

(ii) C/K *is an inseparable field extension*, $\mathrm{char}(K) = 2$, $C^2 \subset K$ *and* $\Delta = F_4(C, K)$.

Proof. A quadratic form of type E_6, E_7, E_8 or F_4 has a subform that is similar to the norm of a separable quadratic extension. Since the only composition algebras over a separably closed fields are those that appear in (i) and (ii), the claims hold. □

Notation 8.2. Let $\hat{X}$ be as in Proposition 7.12, let $\hat{Q}$ be the orthogonal sum of q with a hyperbolic quadratic form of dimension 4, let $\hat{V} = K^4 \oplus L$ be the underlying vector space. Let $\hat{\Delta} = \mathcal{B}(\hat{Q})$ as defined in [Mühlherr et al. 2015, 35.5] if $\hat{Q}$ is not hyperbolic and let $\hat{\Delta} = \mathcal{D}(\hat{Q})$ as defined in [loc. cit., 35.9] if $\hat{Q}$ is hyperbolic. Thus $\hat{\Delta}$ is a building of type $\hat{\Pi} = B_{m+2}$ or D_{m+2}, where m is the Witt index of q. By [loc. cit., 35.10], we can identify $\mathrm{Aut}(\hat{\Delta})$ with $\mathrm{P\Gamma O}(\hat{Q})$. Let $\hat{T}$ be the Tits index $(\hat{\Pi}, \hat{A}, \hat{\Theta})$, where $\hat{A}$ is the unique subset of the vertex set of $\hat{\Pi}$ that spans a subdiagram of type B_m or D_m and $\hat{\Theta} = 1$. By [Mühlherr and Weiss 2020, 6.1, 6.3 and 6.10], $\hat{X}$ is isomorphic to the Tits quadrangle $X_{\hat{\Delta},\hat{T}}$. By Theorem 5.7, therefore, we can identify $\mathrm{Aut}(\hat{X})$ with $\mathrm{Aut}(\hat{\Delta})$. Let ψ denote the map from $\mathrm{Aut}(\hat{X})$ to $\mathrm{Aut}(K)$ such that $\psi(g) = \rho$ whenever g lifts to a ρ-linear element of $\mathrm{\Gamma O}(\hat{Q})$.

Proposition 8.3. *Let ψ be as in Notation 8.2, let G, G_γ and $\sigma_\varphi = \sigma \in$* $\mathrm{Aut}(K)$ *for each $\varphi \in G_\gamma$ be as Proposition 7.13. Then the following hold*:

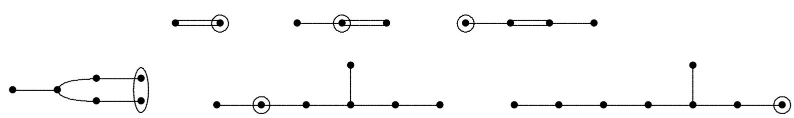
(i) $\psi(\varphi) = \sigma_\varphi$ *for all* $\varphi \in G_\gamma$.

(ii) *A subgroup G_0 of G is Galois as defined in [Mühlherr and Weiss 2017,* 4.5*] if and only if the restriction of ψ to G_0 is injective.*

Proof. In [Mühlherr et al. 2015, 3.20], a description of the Moufang quadrangle $B_2^{\mathcal{I}}(K, L, q)$ and its root groups associated to a fixed apartment is given. If we assume in this description that q is merely nondegenerate rather than anisotropic, the calculations in [loc. cit., 3.20] yield a description of $\hat{X}$, a coordinate system $(\gamma, i \mapsto w_i)$ (where γ is the circuit called Σ in [loc. cit., 3.20]), root groups $U_1, \dots, U_4$ and isomorphisms $x_1, \dots, x_4$ (called $\dot{\delta}$, $\dot{\beta}$, $\dot{\gamma}$, $\dot{\alpha}$ in [loc. cit., 3.20]). In this description, the vertices of $\hat{X}$ are the 1- and 2-dimensional totally isotropic subspaces of the quadratic space $(K, \hat{V}, \hat{Q})$ described in Notation 8.2 and two 2-dimensional totally isotropic subspaces W_1, W_2 of $\hat{V}$ that intersect in a 1-dimensional subspace Y are opposite at Y if and only if the subspace spanned by W_1 and W_2 is not totally isotropic. Given this description of the isomorphisms $x_1, \dots, x_4$, we can observe that (i) holds.

Let $g \in G$. The root groups of $\hat{\Delta}$ are all in the kernel of ψ. Let W be a 3-dimensional totally isotropic subgroup of V, let e_1, e_2, e_3 be a basis of W, and let h be an element in the subgroup of $\mathrm{Aut}(\hat{X})$ generated by all its root groups such that the product gh fixes the three 1-dimensional subspaces spanned by e_1, e_2 and e_3. Let $x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}$ be as in [Mühlherr and Weiss 2016, §7] with W in place of V. The projective plane associated with W is a residue of $\hat{\Delta}$ which we denote by R and there exist $\lambda_1, \lambda_2, \lambda_3 \in E^*$ such that $x_{\alpha_i}(t)^{gh} = x_{\alpha_i}(\lambda_i t^{\sigma_g})$ for $i = 1, 2$ and 3 and all $t \in E$. By [Mühlherr et al. 2015, 29.15], it follows that ψ is a Galois map of $\hat{\Delta}$ as defined in [loc. cit., 29.25]. Thus (ii) holds. □

Hypothesis 8.4. Let G_0 be an arbitrary Galois group of Δ (as defined in [Mühlherr and Weiss 2017, 4.5]) that acts on the chamber set of Δ with finite orbits. By Theorem 6.7, G_0 is a descent group of Δ. Let $T_0 = (\Pi, A_0, \Theta_0)$ be its Tits index and let S be the vertex set of Π. The Coxeter diagram Π is as in Notation 7.2. If $\Pi = C_2$ or F_4, we assume that the type function of Δ is chosen as described in Notation 7.2. We also suppose that T_0 is one of the diagrams



Remark 8.5. The group G_0 acts on $X = X_{\Delta,T}$. Let R_1 and R_1' be two A_0-residues stabilized by G_0. By [Mühlherr et al. 2015, 22.10(i)], R_1 and R_1' are opposite in Δ. By [Mühlherr et al. 2015, 24.21], $\mathrm{Aut}(\Delta)$ acts transitively on ordered pairs of opposite A_0-residues. Replacing G_0 by a conjugate subgroup of $\mathrm{Aut}(\Delta)$, we can assume from now on that R_1 and R_1' are the A_0-residues corresponding to the vertices w_4 and w_0 of Γ.

Proposition 8.6. *The group G_0 stabilizes $\hat{X}$, where $\hat{X}$ is as in Proposition 7.12.*

Proof. Since G_0 fixes w_0 and w_4, it normalizes $E := \langle Z(U_{w_0}), Z(U_{w_4})\rangle$, where U_{w_i} denotes the maximal unipotent subgroup of the stabilizer of the residue w_i in $\mathrm{Aut}(\Delta)$. By (7.5) and [Mühlherr and Weiss $\geq$ 2020, 1.2.25], we have $Z(U_{w_4}) = Y_3U_4Y_5$ and $Z(U_{w_0}) = Y_7U_0Y_1$. By Proposition 7.12(ii)–(iii), E stabilizes $\hat{X}$. By [Mühlherr and Weiss $\geq$ 2020, 1.3.4], the group $\langle U_0, U_4\rangle$ acts transitively on $\hat{\Gamma}_{w_4} = \Gamma_{w_4}$ and by [Mühlherr and Weiss 2020, 8.2(b) and 10.4(ii)], the group $\langle Y_1, Y_5\rangle$ acts transitively on $\hat{\Gamma}_{w_5}$. By [Mühlherr and Weiss $\geq$ 2020, 1.2.21], $\hat{\Gamma}$ is connected. It follows that E acts transitively on the set of edges of $\hat{X}$. Since G_0 normalizes E and fixes the edge $\{w_4, w_5\}$, we conclude that G_0 stabilizes $\hat{X}$. □

Remark 8.7. We have $\hat{X} = X_{\hat{\Delta},\hat{T}}$, where $\hat{\Delta}$ and $\hat{T}$ are as in Notation 8.2. By Theorem 5.7 and Proposition 8.6, G_0 acts on $\hat{\Delta}$. Let $\hat{G}_0$ denote its image in $\mathrm{Aut}(\hat{\Delta})$.

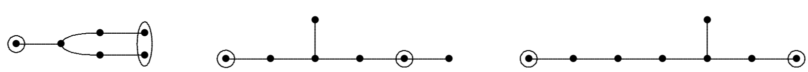
We now want to apply [Mühlherr et al. 2015, 35.13] with $\hat{\Delta}$ in place of Δ and $\hat{G}_0$ in place of Γ. By Proposition 8.3(ii), $\hat{G}_0$ is a Galois group of $\hat{\Delta}$. It is not necessarily finite. In the proof of [loc. cit., 35.13], however, the finiteness of the Galois group is used only in the proof of [loc. cit., 2.41] which is, in turn, a corollary of [loc. cit., 2.40] and thus of [Springer 1998, 11.1.6]. For this result to apply, it suffices to know that $\hat{G}_0$ acts with finite orbits on the chamber set of Δ. This follows from our assumption in Hypothesis 8.4 that G_0 acts with finite orbits on the chamber set of Δ.

Notation 8.8. For each subgroup G_1 of G_0, G_1 is also a Galois group of Δ and hence, by Theorem 6.7, has a Tits index as defined in Definition 6.12. We denote this Tits index by T_{G_1}.

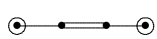
Proposition 8.9. *Suppose that K is separably closed. Then there exists a subgroup G_1 of index 2 in G_0 such that the Tits index T_{G_1} defined in Notation 8.8 has relative rank at least 2.*

Proof. Let ψ be as in Notation 8.2 and let $\hat{G}_0$ be as in Remark 8.7. By [Mühlherr et al. 2015, 35.13] and Remark 8.7, $\hat{\Delta}^{\hat{G}_0} \cong B_1^{\mathcal{Q}}(F, M, Q_0)$ for some anisotropic quadratic space (F, M, Q_0), where $\psi(G_0) = \mathrm{Gal}(K/F)$. Since the bilinear form associated with the quadratic form $\hat{Q}$ in Notation 8.2 is not identically zero, the same holds for Q_0. Since K is separably closed, there exists a subfield E of K over which Q_0 is not anisotropic such that E/F is a separable quadratic extension. Let $G_1 = \psi^{-1}(\mathrm{Gal}(K/E))$ and let $\hat{G}_1$ be the image of G_1 in $\hat{G}_0$. By [Mühlherr et al. 2015, 35.13], we have $\hat{\Delta}^{\hat{G}_1} \cong B_m^{\mathcal{Q}}(E, M_1, Q_1)$ for some $m \geq 1$, where $M_1 = M \otimes_F E$ and Q_1 is the anisotropic part of $Q \otimes_F E$. Thus $\dim_E Q_1 < \dim_F Q_0$. By [Mühlherr et al. 2015, 35.13(i)–(ii)], it follows that $m > 1$. Thus $\hat{G}_1$ acts isotropically (as defined in Definition 6.2) on the residue $\hat{R}_1$ of $\hat{\Delta}$ that corresponds to the vertex w_4. Let R_1 be as in Remark 8.5. If $\Pi \neq E_7$, then $R_1 = \hat{R}_1$ and hence G_1 acts isotropically on R_1. If $\Pi \neq E_7$, then $R_1 = \hat{R}_0 \times \hat{R}_1$, where $\hat{R}_0$ is a building of type A_1. Since $\hat{G}_1$ acts isotropically on $\hat{R}_1$, it follows by [Mühlherr et al. 2015, 21.37(iii)] that G_1 acts isotropically on R_1 also in this case. Thus the relative rank of T_{G_1} is at least 2. □

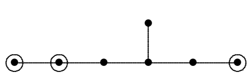
Proposition 8.10. *Let G_1 be as in Proposition 8.9 and let $T_1 := T_{G_1}$ be as in Notation 8.8. Then T_1 is one of the Tits indices*



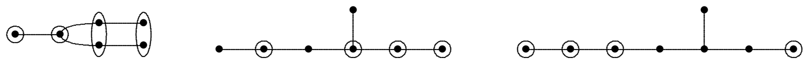
or



of relative rank 2 or the Tits index



of relative rank 3 *or one of the Tits indices*



of relative rank 4 *or* T_1 *is split as defined in Definition 5.3.*

Proof. The Tits index T_1 is as in Hypothesis 8.4 with at least one more circle. The only such Tits indices are those indicated. □

Proposition 8.11. *Suppose* K *is separably closed, let* $\hat{G}_1$ *be as in Proposition 8.9 and let* $\Delta_1 = \Delta^{\hat{G}_1}$. *Then the following hold*:

(i) *If the relative rank of* T_1 *is* 2, *then* Δ_1 *is an exceptional Moufang quadrangle.*

(ii) *If the relative rank of* T_1 *is* 3, *then* $\Delta_1 \cong C_3^{\mathcal{I}}(C, K, \sigma)$ *for some octonion division algebra* (C, K) *with standard involution* σ.

(iii) *If the relative rank of* T_1 *is* 4 *and* $\Pi = E_\ell$ *for* $\ell = 6, 7$ *or* 8, *then* $\Delta_1 \cong F_4(D, E)$ *for some composition division algebra* (D, E) *of dimension* $2^{\ell-5}$ *with nonzero trace.*

(iv) *If* T_1 *is split and* $\Pi = F_4$, *then either* Δ_1 *is the split building of type* F_4 *over* E *or* $\Delta_1 \cong F_4(D, E)$ *for some inseparable field extension* D/E *such that* $\mathrm{char}(E) = 2$ *and* $D^2 \subset E$.

(v) *If* T_1 *is split and* $\Pi = E_6$, E_7 *or* E_8, *then* Δ_1 *is the split building of type* Π *over* E.

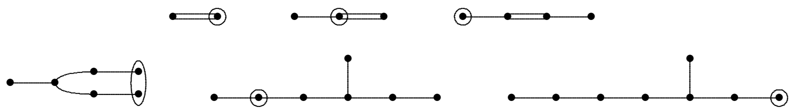
Proof. The assertion (i) holds by [Tits and Weiss 2002, 42.6], (ii) holds by [Tits 1974, 9.1–9.2], (iii) and (iv) hold by [Tits 1974, 10.2–10.3] and (v) holds by [Tits 1974, 6.13]. □

Proposition 8.12. *Suppose* K *is separably closed and* Δ_1 *is as in Proposition 8.11. Then there is a Galois group* $\overline{G}$ *of* Δ_1 *of order* 2 *such the fixed point building* $\Delta_1^{\overline{G}}$ *is isomorphic to* Δ^G.

Proof. Let $\overline{G}$ denote the image of G in $\mathrm{Aut}(\Delta_1)$. Then $\overline{G}$ is a Galois group and hence, by Theorem 6.7, $\overline{G}$ is a descent group of Δ_1. The claim holds, therefore, by [Mühlherr et al. 2015, 22.47(i)]. □

Remark 8.13. Suppose that K is separably closed and let Δ_1 and $\overline{G}$ be as in Proposition 8.12 and let Π_1 be the Coxeter diagram of Δ_1. If $\Pi_1 = C_2$ or F_4, we assume that the type function of Δ_1 is chosen as in Notation 7.2. Let $\overline{T} = (\Pi_1, \overline{A}, \overline{\Theta})$ be the Tits index of $\overline{G}$ as defined in Definition 6.12. Then $\overline{T}$ is the unique Tits index of absolute type Π_1 in Hypothesis 8.4. To see this we observe that in each case the A-residues of Δ stabilized by G_0 are precisely the $\overline{A}$-residues of Δ_1 stabilized by $\overline{G}$.

Theorem 8.14. *Let M be an arbitrary Moufang set associated with one of the Tits indices ${}^2E_{6,1}^{29}$, $E_{7,1}^{48}$, $E_{8,1}^{91}$ or $F_{4,1}^{21}$ as described in [Tits 1966a]. Then there exists an exceptional building Δ as defined in Definition 7.1 of type Π with type function chosen as in Notation 7.2 and a Galois group G_0 of order 2 of Δ whose Tits index is the unique Tits index of absolute type Π among the following*:



such that Δ^{G_0} is isomorphic to M.

Proof. The Tits indices ${}^2E_{6,1}^{29}$, $E_{7,1}^{48}$, $E_{8,1}^{91}$ or $F_{4,1}^{21}$ are precisely the indices that appear in Hypothesis 8.4. Thus the claims are simply restatements of Proposition 8.12 and Remark 8.13. □

Remark 8.15. Note that Proposition 8.12 and Remark 8.13 are actually slightly more general than Theorem 8.14 since they include the case Proposition 8.1(ii) which is not considered in [Tits 1966a].

9. The Galois involution ω

In light of Theorem 8.14, we now make once and for all the following assumptions.

Hypothesis 9.1. We assume that Δ, T, $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$, $(\gamma, i \mapsto w_i)$, $i \mapsto U_i$, $x_1, \dots, x_4$ and

$$\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$$

are as in Notation 7.2 and Theorem 7.4. Let $v \mapsto \bar{v}$, ϕ, g and π be as in (A3), (C3), (C4) and (D1), let Q be as in Notation 2.3, let G and G_γ be as in Proposition 7.13 and let G_0 and T_0 be as in Hypothesis 8.4. We assume that $|G_0| = 2$ and let ω be the nontrivial element in G_0. Thus ω is a Galois involution of Δ as defined in Definition 6.8. Since the relative rank of T_0 is 1, the fixed point building Δ^{G_0} is the Moufang set described in Theorem 6.11(v). We denote this Moufang set by

$$M(\Delta, \omega).$$

Its underlying set (as defined in Definition 4.1) is the set of vertices of Γ fixed by ω. We also assume that $|K| > 4$.

Proposition 9.2. *The coordinate system $(\gamma, i \mapsto w_i)$ of X can be chosen so that $\omega = m_4\varphi$ for some $\varphi \in G_\gamma$, where m_4 is as in Notation 7.15.*

Proof. The fixed points of ω are all vertices of Γ at even distance from w_4. Let v, v_1 be two of them. By [Mühlherr et al. 2015, 25.2; Mühlherr and Weiss ≥ 2020, 1.2.12], u and u^ω are opposite at v for all $u \in \Gamma_v$. By [Mühlherr et al. 2015, 22.10(i)],

v and v_1 are opposite residues of Δ. Therefore there exist $u \in \Gamma$ and $u_1 \in \Gamma_{v_1}$ such that the edges $\{u, v\}$ and $\{u_1, v_1\}$ are opposite residues of Δ. By [Mühlherr and Weiss $\geq$ 2020, 1.2.19], it follows that there exists a straight 4-path from u to v. We conclude that $(\gamma, i \mapsto w_i)$ can be chosen so that w_0 and w_4 are fixed by ω and ω interchanges w_3 and w_5. By Definition 3.2(ii), it follows that ω maps w_i to w_{-i} for all i. By Proposition 3.8, m_4 also maps w_i to w_{-i} for all i. Thus $\omega = m_4\varphi$ for some $\varphi \in G_\gamma$. □

Notation 9.3. Let φ be as in Proposition 9.2 and let $\xi \in \mathrm{Aut}_K(L)$, $\psi \in \mathrm{Aut}_K(\mathcal{X})$, $\sigma \in \mathrm{Aut}(K)$, $\eta \in K^*$, $\lambda \in L$ and $M \colon \mathcal{X} \to K$ be as in Proposition 7.13 applied to φ. Note that since ξ is a bijection, it follows from Proposition 7.13(a) that $q(\lambda) \neq 0$.

Proposition 9.4. $\quad f(\xi(u), \xi(v)) = q(\lambda) f(u, v)^\sigma$ *for all* $u, v \in L$.

Proof. This is a consequence of Proposition 7.13(a). □

Proposition 9.5. *Let ω, λ, σ, ξ and M be as in Notation 9.3. Then the following hold*:

(i) $\psi^2(a)\lambda = -a$ *for all* $a \in \mathcal{X}$.

(ii) σ *has order* 2.

(iii) $\eta^{\sigma+1} = \beta$, *where* $\beta^{-1} = q(\lambda)$.

(iv) $M(\psi(a)) = \eta M(a)^\sigma + \phi(a, \lambda^{-1})$ *for all* $a \in \mathcal{X}$.

(v) $\xi(\overline{\xi(\bar{u})}) = q(\lambda)u$ *for all* $u \in L$.

Proof. By Proposition 7.16(i), we have $x_3(a,t)^{m_4} = x_1(a,t)$ and $x_1(a,t)^{m_4} = x_3(-a,t)$ for all $(a,t) \in \mathcal{R}$. Thus

$$\begin{aligned}(9.6)\quad x_3(a,t) &= x_3(a,t)^{\omega^2}\\ &= x_3\big(-\psi^2(a)\lambda,\ \big(\eta(\eta t^\sigma + M(a))^\sigma + M(\psi(a))\big)q(\lambda) + \phi(\psi^2(a),\lambda)\big)\end{aligned}$$

for all $(a,t) \in \mathcal{R}$ by (7.14). Comparing first coordinates, we obtain (i). Since ψ is σ-linear, it follows that $\sigma^2 = 1$. By Proposition 8.3 and Hypothesis 9.1, on the other hand, σ is nontrivial. Thus (ii) holds. Comparing second coordinates, we have

$$\eta^{\sigma+1}t + \eta M(a)^\sigma + M(\psi(a)) = \beta\big(\phi(\psi^2(a),\lambda) + t\big)$$

for all $(a,t) \in \mathcal{R}$, where β is as in (iii). Setting $a = 0$ and $t = 1$ in this equation, we obtain (iii) and thus

$$(9.7)\qquad \eta M(a)^\sigma + M(\psi(a)) = \beta\phi(\psi^2(a),\lambda).$$

By Proposition 7.13(d) and [Weiss 2006b, 4.5(iii)], M and ϕ are both identically zero if $\mathrm{char}(K) \neq 2$. If $\mathrm{char}(K) = 2$, then by [Mühlherr and Weiss 2019, (7.10)] and (i), we have

$$(9.8)\quad \phi(\psi^2(a),\lambda) = \phi(-a\lambda^{-1},\lambda) = \phi(a\bar{\lambda},\lambda)\beta^2 = \phi(a,\bar{\lambda})\beta = \phi(a,\lambda^{-1})q(\lambda)$$

for all $a \in \mathcal{X}$. By (9.7), therefore, (iv) holds. By [Weiss 2006b, 6.4(iv)], we have $x_2(u)^{m_4} = x_2(-\bar{u})$ for all $u \in L$ and by Proposition 7.13(i), ξ is σ-linear. Thus

$$x_2(u) = x_2(u)^{\omega^2} = x_2\big(\eta^{\sigma+1}\xi(\overline{\xi(\bar{u})})\big) \tag{9.9}$$

for all u by (7.14). By (iii), therefore, (v) holds. □

Proposition 9.10. *Let $\zeta(u) = \eta\xi(\bar{u})$ for all $u \in L$. Then ζ is a σ-linear automorphism of L of order 2 and $q(\zeta(u)) = \eta^{1-\sigma}q(u)^\sigma$ for all $u \in L$.*

Proof. Since ξ is σ-linear, so is ζ. By Proposition 9.5(ii), (iii) and (v), ζ is of order 2. We have $q(\bar{u}) = q(u)$ for all $u \in L$. Thus the second claim holds by Propositions 7.13(a) and 9.5(iii). □

Proposition 9.11. *Let $v \in L$ and let ζ be as in Proposition 9.10. Then v is fixed by ζ if and only if $v = u + \zeta(u)$ for some $u \in L$.*

Proof. Since ζ is of order 2, $v = u + \zeta(u)$ implies $\zeta(v) = v$. Suppose, conversely, that $\zeta(v) = v$. Since σ is of order 2, we can choose $\delta_0 \in K$ such that $\delta_0 + \delta_0^\sigma = 1$. Then $u + \zeta(u) = (\delta_0 + \delta_0^\sigma)v = v$ for $u = \delta_0 v$. □

Remark 9.12. Let $F = K^\sigma := \{t \in K \mid t^\sigma = t\}$ and let $L_\zeta = \{v \in L \mid \zeta(v) = -v\}$. Then L_ζ is a vector space over F and by [Mühlherr et al. 2015, 2.40] applied to the group generated by $v \mapsto -\zeta(v)$, $\dim_F L_\zeta = \dim_K L$.

Remark 9.13. By Proposition 9.10, $q(u) \in \eta F$ for all $u \in L_\zeta$, where F and L_ζ are as in Remark 9.12. Thus $\eta^{-1}q$ is a quadratic form on L_ζ as a vector space over F.

Proposition 9.14. *For each $b \in \mathcal{X}$, there exists $u \in L$ such that*

$$u + \zeta(u) = h(\psi(b), b),$$

where ζ is as in Proposition 9.10.

Proof. Let $b \in \mathcal{X}$. Then

$$\zeta(h(\psi(b), b)) = \eta\xi(\overline{h(\psi(b), b)}) = -\eta\xi\big(h(b, \psi(b))\big) = -h(\psi(b), \psi^2(b)\lambda)$$

by Proposition 7.13(c). By Proposition 9.5(i), it follows that $h(\psi(b), b)$ is fixed by ζ. The claim holds, therefore, by Proposition 9.11. □

Proposition 9.15. *Let $(a, t), (b, s) \in \mathcal{R}$ and $u \in L$. Then $x_1(a, t)x_2(u)x_3(b, s)$ is centralized by ω if and only if $a = \psi(b)$, $t = \eta s^\sigma + M(b)$ and $u + \zeta(u) = h(a, b)$.*

Proof. By Proposition 7.16(i), we have

$$\begin{aligned}\big(x_1(a, t)x_2(u)x_3(b, s)\big)^{m_4} &= x_1(b, s)x_2(-\bar{u} - h(b, a))x_3(-a, t)\\ &= x_1(b, s)x_2(\overline{h(a, b) - u})x_3(-a, t).\end{aligned}$$

By (7.14) and Propositions 9.2 and 9.10, therefore,

$$\big(x_1(a,t)x_2(u)x_3(b,s)\big)^{\omega} = x_1\big(\psi(b),\ \eta s^{\sigma} + M(b)\big)x_2\big(\zeta(h(a,b)-u)\big)\\ \cdot x_3\big(-\psi(a)\lambda,\ (\eta t^{\sigma} + M(a))q(\lambda) + \phi(\psi(a),\lambda)\big).$$

Thus $x_1(a,t)x_2(u)x_3(b,s)$ is centralized by ω if and only if

$$\psi(b) = a, \quad t = \eta s^{\sigma} + M(b) \quad \text{and} \quad u + \zeta(u) = h(a,b) \tag{9.16}$$

as well as

$$-\psi(a)\lambda = b \quad \text{and} \quad s = (\eta t^{\sigma} + M(a))q(\lambda) + \phi(\psi(a),\lambda). \tag{9.17}$$

By Proposition 9.5(i)–(iv) and [Mühlherr and Weiss 2019, (7.10)], (9.16) implies that $-\psi(a)\lambda = b$ and

$$\begin{aligned}(\eta t^{\sigma} + M(a))q(\lambda) &= \big(\eta^{\sigma+1}s + \eta M(b)^{\sigma} + M(\psi(b))\big)q(\lambda)\\ &= s + \phi(b,\lambda^{-1})q(\lambda)\\ &= s + \phi(\psi(a)\lambda,\bar{\lambda})q(\lambda)^{-1}\\ &= s + \phi(\psi(a),\lambda).\end{aligned}$$

Therefore (9.16) implies (9.17). □

Notation 9.18. Let

$$\mathcal{U} = \big\{[b,s,u] \mid (b,s,u) \in \mathcal{R} \times L \text{ and } u + \zeta(u) = h(\psi(b),b)\big\},$$

where

$$[b,s,u] = x_1\big(\psi(b),\eta s^{\sigma} + M(b)\big)x_2(u)x_3(b,s).$$

Thus $\mathcal{U} = C_{U_{[1,3]}}(\omega)$ by Proposition 9.15.

Remark 9.19. By Definition 6.10 and [Mühlherr and Weiss ≥ 2020, 1.2.25], $U_{[1,3]}$ is the unipotent radical of the residue of Δ corresponding to w_4. Hence the group $\mathcal{U}$ is the root group U_{w_4} of the Moufang set $M(\Delta,\omega)$ defined in Hypothesis 9.1.

Remark 9.20. Let $[b_1,u_1,s_1], [b_2,u_2,s_2] \in \mathcal{U}$. Then

$$[b_1,s_1,u_1]\cdot[b_2,s_2,u_2] = x_1\big(\psi(b_1),\ \eta s_1^{\sigma} + M(b_1)\big)x_1\big(\psi(b_2),\ \eta s_2^{\sigma} + M(b_2)\big)\\ \cdot x_2\big(u_1 + u_2 + h(\psi(b_2),b_1)\big)x_3(b_1,s_1)x_3(b_2,s_2)$$

by (7.5). By Notation 2.4, we have

$$x_1\big(\psi(b_1),\ \eta s_1^{\sigma} + M(b_1)\big)x_1\big(\psi(b_2),\ \eta s_2^{\sigma} + M(b_2)\big)\\ = x_1\big(\psi(b_1+b_2),\ \eta(s_1+s_2)^{\sigma} + M(b_1) + M(b_2) + g(\psi(b_2),\psi(b_1))\big)$$

and

$$x_3(b_1,s_1)x_3(b_2,s_2) = x_3\big(b_1+b_2,\ s_1+s_2+g(b_2,b_1)\big).$$

Since $\mathcal{U} = C_{U_{[1,3]}}$ is closed under multiplication, we conclude by Notation 9.18 that

$$(9.21)\quad [b_1, s_1, u_1]\cdot[b_2, s_2, u_2] = \big[b_1 + b_2,\ s_1 + s_2 + g(b_2, b_1),\ u_1 + u_2 + h(\psi(b_2), b_1)\big]$$

and

$$(9.22)\qquad M(b_1 + b_2) + \eta g(b_2, b_1)^\sigma = M(b_1) + M(b_2) + g(\psi(b_2), \psi(b_1)).$$

for all $[b_1, s_1, u_1], [b_2, s_2, u_2] \in \mathcal{U}$. It follows from (9.21) that

$$(9.23)\qquad [b, s, u]^{-1} = \big[-b,\ -s + g(b, b),\ -u + h(\psi(b), b)\big]$$

for all $[b, s, u] \in \mathcal{U}$.

10. The structure map τ

The goal of this section is to introduce the structure map of the Moufang set $M(\Delta, \omega)$ defined in Hypothesis 9.1. See Notation 10.7 and Definition 10.8.

Proposition 10.1. *Let m_1 and m_4 be as in Notation 7.15, let ω and φ be as in Proposition 9.2, let $r_0 = (m_1m_4)^2$ and let $h_0 \in G$. Then*

$$[\omega, r_0h_0] = 1$$

if and only if

$$h_0^{m_4}\varphi = \varphi^{r_0}h_0.$$

Proof. By [Mühlherr and Weiss $\geq$ 2020, 1.4.15], $r_0 = m_4m_1m_4m_1$ and hence $m_4r_0 = m_4m_1m_4m_1m_4 = r_0m_4$. Thus $[\omega, r_0h_0] = 1$ if and only if

$$r_0m_4 \cdot h_0^{m_4}\varphi = r_0h_0 \cdot m_4\varphi = m_4\varphi \cdot r_0h_0 = m_4r_0 \cdot \varphi^{r_0}h_0 = r_0m_4 \cdot \varphi^{r_0}h_0. \qquad \square$$

Proposition 10.2. *Let $\omega = m_4\varphi$ be as in Proposition 9.2, let ξ, ψ, σ, η and λ be as Notation 9.3 and let r_0 be as Proposition 10.1. Then conjugation by φ^{r_0} induces the maps*

$$x_2(u) \mapsto x_2(\eta^\sigma\xi(u))$$

and

$$x_3(b, s) \mapsto x_3\big(\eta^\sigma\psi(b)\lambda,\ \eta^\sigma s^\sigma + \eta^{\sigma-1}M(b) + \eta^{2\sigma}\phi(\psi(b), \lambda)\big)$$

on U_2 and U_3.

Proof. By (7.14) and Proposition 7.16(ii), conjugation by φ^{m_1} induces the maps

$$\begin{aligned} x_2(u) &\mapsto x_2(\xi(u)) \\ x_3(b,s) &\mapsto x_3\big(\psi(b)\lambda,\ (\eta s^\sigma + M(b))q(\lambda) + \phi(\psi(b),\lambda)\big) \\ x_4(v) &\mapsto x_4(\eta\xi(v)) \end{aligned}$$

on U_2, U_3 and U_4. We apply now the notation in [Tits and Weiss 2002, 5.10]. Conjugating both sides of the identity

$$[x_1(a,t), x_4(\varepsilon)^{-1}]_3 = x_3(a,t) \tag{10.3}$$

by $h_1 := \varphi^{m_1}$, we thus obtain

$$\begin{aligned} x_3\big(\psi(a)\lambda,\ (\eta t^\sigma + M(a))q(\lambda) + \phi(\psi(a),\lambda)\big) &= [x_1(a_1,t_1), x_4(\eta\lambda)^{-1}]_3 \\ &= x_3\big(a_1\cdot\eta\lambda,\ t_1 q(\eta\lambda) + \phi(a_1,\eta\lambda)\big), \end{aligned}$$

where $x_1(a_1,t_1) = x_1(a,t)^{h_1}$. By Proposition 9.5(iii), $\eta^{\sigma+1} = q(\lambda)^{-1}$. Solving for a_1 and then for t_1, we conclude that h_1 induces the map

$$x_1(a,t) \mapsto x_1\big(\eta^{-1}\psi(a),\ \eta^{-1}t^\sigma + \eta^{-2}M(a)\big)$$

on U_1. By Proposition 7.16(i), it follows that conjugation by $\varphi^{m_1m_4}$ induces the maps

$$\begin{aligned} x_1(a,t) &\mapsto x_1\big(\psi(a)\lambda,\ (\eta t^\sigma + M(a))q(\lambda) + \phi(\psi(a),\lambda)\big) \\ x_2(u) &\mapsto x_2(\overline{\xi(\bar{u})}) \\ x_3(b,s) &\mapsto x_3\big(\eta^{-1}\psi(b),\ \eta^{-1}s^\sigma + \eta^{-2}M(b)\big) \end{aligned}$$

on U_1, U_2 and U_3. Conjugating both sides of the identity

$$[x_1(0,1), x_4(v)^{-1}]_2 = x_2(v)$$

by $h_2 := \varphi^{m_1m_4}$, we obtain

$$[x_1(0,\eta q(\lambda)), x_4(v_1)^{-1}]_2 = x_2(\overline{\xi(\bar{v})}),$$

where $x_4(v_1) = x_4(v)^{h_2}$. It follows that $\varphi^{m_1m_4}$ induces the map

$$x_4(v) \mapsto x_4(\eta^\sigma\overline{\xi(\bar{v})})$$

on U_4. By Proposition 7.16(i), therefore, conjugation by $\varphi^{m_1m_4m_1}$ induces the maps

$$\begin{aligned} x_2(u) &\mapsto x_2(\eta^\sigma\overline{\xi(\bar{u})}) \\ x_3(b,s) &\mapsto x_3\big(\eta^{-1}\psi(b),\ \eta^{-1}s^\sigma + \eta^{-2}M(b)\big) \\ x_4(v) &\mapsto x_4(\overline{\xi(\bar{v})}) \end{aligned}$$

on U_2, U_3 and U_4. By [Mühlherr and Weiss 2019, (7.10)], we have $\phi(bv, \bar{v}) = \phi(b, v)q(v)$ for all $(b, v) \in \mathcal{R}$. Using (10.3) again, we can thus conclude that conjugation by $\varphi^{m_1 m_4 m_1}$ induces the map

$$x_1(a, t) \mapsto x_1\big(\eta^\sigma \psi(a)\lambda,\ \eta^\sigma t^\sigma + \eta^{\sigma-1}M(a) + \eta^{2\sigma}\phi(\psi(a), \lambda)\big)$$

on U_1. The claims follow now by one more application of Proposition 7.16(i). □

Notation 10.4. Let η be as in Notation 9.3, let $\psi_0(a) = \eta a$ for all $a \in \mathcal{X}$, let $\xi_0(v) = \eta^{-1}v$ for all $v \in L$, let σ_0 be the identity automorphism of K and let $\eta_0 = \eta^2$. Then ψ_0, ξ_0, σ_0 and η_0 fulfill the identities Proposition 7.13(a)–(d) with $\lambda = \eta^{-1}$; note that by [Weiss 2006b, 4.1], $M(a) = 0$ in (d). Let h_0 be the element of G_γ obtained by applying Proposition 7.13(ii) to ψ_0, ξ_0, σ_0 and η_0. By [Weiss 2006b, (4.13)], $\phi(a, \varepsilon) = 0$ for all $a \in \mathcal{X}$. Thus conjugation by h_0 induces the maps

$$\begin{aligned} x_1(a, t) &\mapsto x_1(\eta a, \eta^2 t) && \text{on } U_1, \\ x_2(v) &\mapsto x_2(\eta v) && \text{on } U_2, \quad \text{and} \\ x_3(b, s) &\mapsto x_3(b, s) && \text{on } U_3. \end{aligned}$$

Let $\nu = r_0 h_0$, where r_0 is as in Proposition 10.1.

Proposition 10.5. *Let h_0 and η be as in Notation 10.4. Then conjugation by $h_0^{m_4}$ induces the maps*

$$x_2(u) \mapsto x_2(\eta u) \quad \textit{and} \quad x_3(b, s) \mapsto x_3(\eta b, \eta^2 s)$$

on U_2 and U_3.

Proof. This holds by Proposition 7.16(i). □

Proposition 10.6. $[\omega, \nu] = 1$, *where ω is as in Proposition 9.2 and ν is as in Notation 10.4.*

Proof. Note that by Proposition 7.13(d), $M(tb) = t^{2\sigma}M(b)$ for all $b \in \mathcal{X}$ and all $t \in K$. By (7.14), Propositions 10.2, 10.5 and Notation 10.4, therefore, conjugation by $h_0^{m_4}\varphi$ and conjugation by $\varphi^{r_0}h_0$ both induce the maps

$$\begin{aligned} x_2(u) &\mapsto x_2(\eta^{\sigma+1}\xi(u)) && \text{on } U_2, \quad \text{and} \\ x_3(b, s) &\mapsto x_3\big(\eta^\sigma \psi(b)\lambda,\ \eta^\sigma s^\sigma + \eta^{\sigma-1}M(b) + \eta^{2\sigma}\phi(\psi(b), \lambda)\big) && \text{on } U_3. \end{aligned}$$

By [Mühlherr and Weiss ≥ 2020, 1.4.17(ii)], it follows that $h_0^{m_4}\varphi = \varphi^{r_0}h_0$. By Proposition 10.1, therefore, $[\omega, \nu] = 1$. □

Notation 10.7. Let $\mathcal{U}$ be as in Notation 9.18 and let $\nu = (m_1 m_4)^2 h_0$ be as in Notation 10.4. Thus ν interchanges w_4 and w_0 and by Proposition 10.6, ν commutes with ω. By [Mühlherr et al. 2015, 24.10(i)], the proper residues fixed by ω are pairwise opposite in Δ and by [Mühlherr et al. 2015, 24.21], $\mathcal{U}$ acts sharply transitively on the set of vertices fixed by ω that are distinct from w_4. Choose $g \in \mathcal{U}^*$. Then w_0^g is opposite both w_4 and w_0. Hence $w_0^{g\nu}$ is also opposite both w_4 and w_0. Thus

$$w_0^{g\nu} = w_0^{\tau(g)}$$

for a unique element $\tau(g)$ of $\mathcal{U}^*$. Since ν stabilizes the set $w_0^{\mathcal{U}^*}$, the map τ is a permutation of $\mathcal{U}^*$.

Definition 10.8. The map τ defined in Notation 10.7 is the map called f in [Thompson 1972, (3.1)], which Thompson called the *structure equation* of the Moufang set he was investigating. For this reason, we call the permutation τ the *structure map* of the Moufang set $M(\Delta, \omega)$. By [De Medts and Weiss 2006, 3.1], the Moufang set $M(\Delta, \omega)$ is uniquely determined by the pair $(\mathcal{U}, \tau)$.

11. The subgraph Λ

We have described the group $\mathcal{U}$ in Notation 9.18 and (9.21) and defined τ in Notation 10.7. Our goal in the remainder of this paper is to determine τ explicitly. In this section we begin to investigate how the product m_1m_4 operates on Γ.

Notation 11.1. We continue with all the notation and assumptions in Hypothesis 9.1. Let $\mathcal{R}_\Xi$ as in Notation 2.4. We set $\mathcal{R} = \mathcal{R}_\Xi$ and

$$\mathcal{R}^\times = \{(a, t) \in \mathcal{R} \mid q(\pi(a) + t\varepsilon) \neq 0\}.$$

Observations 11.2. By Conventions 1.2 and (7.5),

$$x_4(v)x_1(a, t) = x_1(a, t)x_2(\theta(a, v) + tv)x_3\big(av, tq(v) + \phi(a, v)\big)x_4(v)$$

for all $(a, t) \in \mathcal{R}$ and all $v \in L$. Replacing v by $-v$ in this identity and applying (C1) and [Weiss 2006b, 4.14], we thus conclude that

$$x_1(a, t)x_4(v) = x_4(v)x_1(a, t)x_2(-\theta(a, v) - tv)x_3\big(-av, tq(v) + \phi(a, v)\big)$$

for all $(a, t) \in \mathcal{R}$ and all $v \in L$. By Conventions 1.2 and (7.5), we also have

$$x_3(b, s)x_1(a, t) = x_1(a, t)x_3(b, s)x_2(h(a, b))$$

and

$$x_1(a, t)x_3(b, s) = x_3(b, s)x_1(a, t)x_2(-h(a, b))$$

for all $(a, t), (b, s) \in \mathcal{R}$ and

$$x_4(v)x_2(u) = x_2(u)x_4(v)x_3(0, f(u, v))$$

and

$$x_2(u)x_4(v) = x_4(v)x_2(u)x_3(0, -f(u, v))$$

for all $u, v \in L$.

Using these observations, we now calculate that

$$(11.3)\quad \begin{aligned} &x_1(a,t)x_2(u)x_3(b,s)x_4(v)\\ &= x_4(v)x_1(a,t)x_2(-\theta(a,v)-tv+u)\\ &\qquad\qquad \cdot x_3\big(b-av,\ tq(v)+\phi(a,v)+s-f(u,v)-g(b,av)\big) \end{aligned}$$

and

$$(11.4)\quad \begin{aligned} &x_4(v)x_1(a,t)x_2(u)x_3(b,s)\\ &= x_1(a,t)x_2(\theta(a,v)+tv+u)\\ &\qquad\qquad \cdot x_3\big(b+av,\ tq(v)+\phi(a,v)+f(u,v)+s+g(b,av)\big)x_4(v) \end{aligned}$$

for all (a,t), $(b,s)\in\mathcal{R}$ and for all $u, v\in L$.

Notation 11.5. Let Γ and $\mathcal{A}$ be as in Hypothesis 9.1, let Λ be the subgraph of Γ defined in [Mühlherr and Weiss $\geq$ 2020, 1.5.4] and let $U_+ = U_{[1,4]}$ as defined in Proposition 3.5(i). Thus Λ is the subgraph consisting of all the vertices and edges of Γ that lie on an element of $\mathcal{A}$ containing the vertices w_4 and w_5.

Let Ψ be the graph whose vertex set is the disjoint union of the set $\{w_4, w_5\}$, the set of right cosets in U_+ of the subgroups U_4, $U_{[3,4]}$ and $U_{[2,4]}$ and the set of right cosets in U_+ of the subgroups U_1, $U_{[1,2]}$ and $U_{[1,3]}$ and whose edge set consists of the unordered pairs $\{w_4, w_5\}$ as well as $\{U_1 g, U_4 g\}$,

$$\begin{aligned} &\{w_4, U_{[1,3]}g\},\quad \{U_{[1,3]}g, U_{[1,2]}g\},\quad \{U_{[1,2]}g, U_1 g\}\\ \text{and}\quad &\{w_5, U_{[2,4]}g\},\quad \{U_{[2,4]}g, U_{[3,4]}g\},\quad \{U_{[3,4]}g, U_4 g\} \end{aligned}$$

for all $g\in U_+$. By [Mühlherr and Weiss $\geq$ 2020, 1.5.8], there is a unique isomorphism from Ψ to Λ that sends w_4 to w_4, w_5 to w_5, w_i^g to $U_{[1,i]}g$ and w_{5+i}^g to $U_{[1+i,4]}g$ for $i = 1, 2, 3$ and for all $g\in U_+$. We identify Λ with Ψ via this isomorphism. Note that with this identification, the set $w_0^{\mathcal{U}}$ that appears in Notation 10.7 is now $\{U_4 g \mid g\in\mathcal{U}\}$.

Proposition 11.6. *Let $g\in U_+$. The following hold*:

(i) *For all $(b,s)\in\mathcal{R}$ and all $u, v\in L$, the vertex*

$$U_1 x_2(u)x_3(b,s)x_4(v)$$

of Λ is adjacent to the vertex $U_4 g$ if and only if $U_4 g$ equals

$$U_4 x_1(a,t)x_2\big(-\theta(a,v)-tv+u\big)x_3\big(b-av,\ tq(v)+\phi(a,v)+s-f(u,v)-g(b,av)\big)$$

for some $(a,t)\in\mathcal{R}$.

(ii) *For all (a,t), $(b,s)\in\mathcal{R}$ and all $u\in L$, the vertex*

$$U_4 x_1(a,t)x_2(u)x_3(b,s)$$

of Λ is adjacent to the vertex $U_1 g$ if and only if $U_1 g$ equals

$$U_1 x_2\big(\theta(a,v)+tv+u\big)x_3\big(b+av,\ tq(v)+\phi(a,v)+f(u,v)+s+g(b,av)\big)x_4(v)$$

for some $v \in L$.

Proof. Let $e \in U_+$. The vertices of Λ adjacent to $U_1 e$ are $U_{[1,2]}e$ and $U_4 x_1(a,t)e$ for all $(a,t) \in \mathcal{R}$ and the vertices of Λ adjacent to $U_4 e$ are $U_{[3,4]}e$ and $U_1 x_4(v)e$ for all $v \in L$. The two claims hold, therefore, by (11.3) and (11.4). □

Proposition 11.7. *Let z be a vertex of Γ. There exists a root $(v_0, v_1, v_2, v_3, v_4)$ such that $v_0 = w_4$, $v_1 = w_5$ and $v_4 = z$ if and only if z is a vertex of Λ of the form $U_4 g$ for some $g \in U_{[1,3]}$.*

Proof. It suffices to observe that if $z = U_4 g$ for some $g \in U_{[1,3]}$, then

$$\big(w_4, w_5, U_{[2,4]}g, U_{[3,4]}g, U_4 g\big)$$

is a root. □

Proposition 11.8. *Let $\mathcal{R}^\times$ be as in Notation 11.1, let*

$$\mathcal{O} = \big\{U_4 x_1(a,t)x_2(u)x_3(b,s) \mid (a,t), (b,s) \in \mathcal{R},\ u \in L\big\},$$
$$\mathcal{B} = \big\{U_4 x_1(a,t)x_2(u)x_3(b,s) \in \mathcal{O} \mid (a,t) \in \mathcal{R}^\times\big\},$$

and let $p = U_4 x_1(a,t)x_2(u)x_3(b,s) \in \mathcal{O}$. Then $p^{m_1 m_4} \in \mathcal{O}$ if and only if $p \in \mathcal{B}$.

Proof. Let α denote the sequence of vertices

$$\big(U_{[2,4]},\ w_5,\ U_{[2,4]}x_1(a,t),\ U_{[3,4]}x_1(a,t)x_2(u),\ U_4 x_1(a,t)x_2(u)x_3(b,s)\big).$$

The sequence α is a root (i.e., a straight path of length 4 in Γ) if and only if the vertices $U_{[2,4]}$ and $U_{[2,4]}x_1(a,t)$ are opposite at w_5. By [Mühlherr and Weiss 2020, 9.12], $U_{[2,4]}$ and $U_{[2,4]}x_1(a,t)$ are opposite at w_5 if and only if $p \in \mathcal{B}$. Thus the sequence $\alpha^{m_1 m_4}$ (which starts at w_4 and ends at $p^{m_1 m_4}$) is a root if and only if $p \in \mathcal{B}$. Every root starting at w_4 ends at a vertex in $\mathcal{O}$. Thus if $p \in \mathcal{B}$, then $p^{m_1 m_4} \in \mathcal{O}$. By [Mühlherr and Weiss $\geq$ 2020, 1.2.28(ii) and 1.3.18], every path of length at most 4 from w_4 to a vertex in $\mathcal{O}$ is a root. Thus if $p^{m_1 m_4} \in \mathcal{O}$, then $p \in \mathcal{B}$. □

12. The action of $m_1 m_4$ on the set $\mathcal{B}$

The main result of this section is Proposition 12.15.

Proposition 12.1. *Let $(a,t) \in \mathcal{R}^\times$, $u \in L$ and $(b,s) \in \mathcal{R}$. Then the image of the vertex*

$$p := U_4 x_1(a,t)x_2(u)x_3(b,s)$$

of Λ under m_1 is the vertex

$$U_4 x_1(a_0,t_0)x_2(u_0)x_3(b_0,s_0),$$

where

(i) $a_0 = q(\pi(a)+t\varepsilon)^{-1}a\big(\pi(a)+(Q(a)-t)\varepsilon\big)$,

(ii) $t_0 = q(\pi(a)+t\varepsilon)^{-1}(Q(a)-t)$,

(iii) $u_0 = q(\pi(a)+t\varepsilon)^{-1}\theta(a,u)+t_0u$,

(iv) $b_0 = b+a_0u$, *and*

(v) $s_0 = t_0q(u)+\phi(a_0,u)+g(b,a_0u)+s$.

Proof. By Proposition 11.8, there exist elements (a_0, t_0), $(b_0, s_0) \in \mathcal{R}$ and $u_0 \in L$ such that $p^{m_1} = U_4x_1(a_0,t_0)x_2(u_0)x_3(b_0,s_0)$. By Proposition 11.6(ii), p is adjacent to the vertex

$$q_v := U_1x_2(\hat{z}_v)x_3(b+av,\hat{s}_v)x_4(v)$$

for all $v \in L$, where

$$\hat{z}_v = \theta(a,v)+tv+u \quad \text{and} \quad \hat{s}_v = tq(v)+\phi(a,v)+f(u,v)+s+g(b,av).$$

By Proposition 7.16(ii), the identity $f(\theta(a,v),v) = Q(a)q(v)$ and the fact that Q is identically zero if $\operatorname{char}(K) \neq 2$ (by [Weiss 2006b, 4.1(i)]), the element m_1 maps q_v to

$$U_1x_2(v)x_3(b+av,\hat{t}_v)x_4(-\hat{z}_v)$$

for all $v \in L$, where

$$\hat{t}_v = -tq(v)+\phi(a,v)+s+g(b,av)+Q(a)q(v).$$

Since p and q_v are adjacent for all v, so are their images under m_1. Therefore, by Proposition 11.6(i),

$$u_0 = \theta(a_0,\hat{z}_v)+t_0\hat{z}_v+v \tag{12.2}$$

as well as

$$b_0 = b+av+a_0\hat{z}_v \tag{12.3}$$

and

$$s_0 = t_0q(\hat{z}_v)+\phi(a_0,\hat{z}_v)+\hat{t}_v+f(\hat{z}_v,v)+g(b+av,\ a_0\hat{z}_v) \tag{12.4}$$

for all $v \in L$. Setting $v = 0$ in (12.3) and (12.4), we conclude that (iv) and (v) hold. Setting $v = \varepsilon$ in (12.3), it follows from (iv) that

$$a+a_0(\pi(a)+t\varepsilon) = 0.$$

Thus (i) holds.

We have $q(\pi(a)+(Q(a)-t)\varepsilon)=q(\pi(a)+t\varepsilon)$ and hence

$$\begin{aligned}\theta(a_0,\hat{z}_v)&=q(\pi(a)+t\varepsilon)^{-2}\theta\big(a\pi(a)+(Q(a)-t)a,\ \hat{z}_v\big)\\&=q(\pi(a)+t\varepsilon)^{-1}\theta(a,\hat{z}_v)\end{aligned}$$

for all $v\in L$ by (i) and Proposition A.4(i). By (12.2), therefore,

$$u_0=q(\pi(a)+t\varepsilon)^{-1}\theta(a,\hat{z}_v)+t_0\hat{z}_v+v \tag{12.5}$$

for all $v\in L$. Setting $v=0$ in (12.5), we conclude that (iii) holds. Setting $v=\varepsilon$ in (12.5), we obtain

$$u_0=q(\pi(a)+t\varepsilon)^{-1}\theta(a,\pi(a)+t\varepsilon+u)+t_0(\pi(a)+t\varepsilon+u)+\varepsilon.$$

By [Weiss 2006b, 4.21], we have $\theta(a,\pi(a))=Q(a)\pi(a)-q(\pi(a))\varepsilon$. By (iii), therefore,

$$\begin{aligned}-q(\pi(a)+t\varepsilon)t_0(\pi(a)+t\varepsilon)&=\theta(a,\pi(a))+t\pi(a)+q(\pi(a)+t\varepsilon)\varepsilon\\&=(t+Q(a))(\pi(a)+t\varepsilon).\end{aligned}$$

Since $\pi(a)+t\varepsilon\neq 0$, we conclude that (ii) holds. □

Corollary 12.6. *Let $(a,t)\in\mathcal{R}^\times$, $u\in L$ and $(b,s)\in\mathcal{R}$. Then the image of the vertex*

$$p:=U_4x_1(a,t)x_2(u)x_3(b,s)$$

of Λ under m_1m_4 is the vertex

$$U_4x_1(a_1,t_1)x_2(u_1)x_3(b_1,s_1),$$

where

(i) $a_1=b+a_0u$,

(ii) $t_1=-q(\pi(a)+t\varepsilon)^{-1}(t+Q(a))q(u)+\phi(a_0,u)+g(b,a_0u)+s$,

(iii) $u_1=-h(b+a_0u,a_0)+q(\pi(a)+t\varepsilon)^{-1}\big((t+Q(a))\bar{u}-\overline{\theta(a,u)}\big)$,

(iv) $b_1=-a_0$,

(v) $s_1=-q(\pi(a)+t\varepsilon)^{-1}(t+Q(a))$,

and $a_0=q(\pi(a)+t\varepsilon)^{-1}a\big(\pi(a)+(Q(a)-t)\varepsilon\big)$.

Proof. First apply Proposition 12.1 and then apply Proposition 7.16(i). □

Notation 12.7. Let $(a,t),(b,s)\in\mathcal{R}$ and $u\in L$. We set

$$P=\pi(a)+r\varepsilon\quad\text{and}\quad\rho=q(P),\quad\text{where } r=Q(a)-t.$$

Note that $\overline{P}=-(\pi(a)+t\varepsilon)$ and hence

$$\rho=q(\pi(a)+r\varepsilon)=q(\pi(a)+t\varepsilon). \tag{12.8}$$

We then set

$$v = \theta(a, u) + ru. \tag{12.9}$$

Thus $av = aPu$ by (D1) and

$$\begin{aligned} \theta(a, v) &= \theta(a, \theta(a, u)) + r\theta(a, u) \\ &= -q(\pi(a))u + (Q(a) + r)\theta(a, u) \\ &= -q(\pi(a))u - t\theta(a, u) \end{aligned} \tag{12.10}$$

by (12.9) and [Weiss 2006b, 4.1(i) and 4.21].

Proposition 12.11. *Let (a, t), u, P be as in Notation 12.7. Then*

$$h(aP, au) = \big(Q(a)t + 2q(\pi(a))\big)u + (Q(a) - 2t)\theta(a, u).$$

Proof. We have

$$\begin{aligned} h(aP, au) &= h(a, aPu) + f(h(aP, a), \varepsilon)u \\ &= h(a, av) + f(h(a, a), P)u, \end{aligned}$$

where v is as in Notation 12.7. Thus

$$\begin{aligned} h(aP, au) &= 2\big(\theta(a, v) + f(\pi(a), P)u\big) \\ &= 2\big(-q(\pi(a))u - t\theta(a, u) + 2q(\pi(a))u\big) \\ &= 2\big(q(\pi(a))u - t\theta(a, u)\big) \end{aligned}$$

by [Weiss 2006b, 4.1(i) and (iii) and 4.5(i)] if $\mathrm{char}(K) \neq 2$ and

$$\begin{aligned} h(aP, au) &= Q(a)(v + Q(a)u) \\ &= Q(a)(\theta(a, u) + tu) \end{aligned}$$

by [Weiss 2006b, 3.15 and 3.16] if $\mathrm{char}(K) = 2$. □

Proposition 12.12. *Let (a, t), u, P, v be as in Notation 12.7. Then*

$$h(aP, av) = \rho\big(Q(a)u + 2\theta(a, u)\big) = \rho h(a, au).$$

Proof. Replacing u by v in Proposition 12.11, we obtain the first equality. The second equality follows by [Weiss 2006b, 3.15, 3.16 and 4.5(i)]. □

Lemma 12.13. *Let (a, t), u, b, u_1 be as in Corollary 12.6. Then*

$$u_1 = \rho^{-1}\big(\overline{h(aP, b)} + t\bar{u} + \overline{\theta(a, u)}\big),$$

where P is as in Notation 12.7.

Proof. By Corollary 12.6(iii), we have

$$\bar{u}_1 = \rho^{-2}h(aP, \rho b + av) + \rho^{-1}\big((t + Q(a))u - \theta(a, u)\big),$$

where v is as in Notation 12.7. The claim holds, therefore, by Proposition 12.12. □

Lemma 12.14. $\phi(aP, u) = \phi(a, v) = \rho\phi(a, u).$

Proof. This holds by Proposition A.4(ii) and the last display on page 119 of [Tits and Weiss 2002]. □

Proposition 12.15. *Let* (a, t), u, (b, s), (a_1, t_1), u_1, (b_1, s_1) *be as in Corollary 12.6 and let* $r = Q(a) - t$, $P = \pi(a) + r\varepsilon$, $\rho = q(P)$ *and* $v = \theta(a, u) + ru$ *as in Notation 12.7. Then the following hold*:

(i) $a_1 = \rho^{-1}(\rho b + av)$.

(ii) $t_1 = \rho^{-1}\big(rq(u) + \phi(a, u) + g(b, av) + \rho s\big)$.

(iii) $\bar{u}_1 = \rho^{-1}\big(h(aP, b) + tu + \theta(a, u)\big)$.

(iv) $b_1 = -\rho^{-1}aP$.

(v) $s_1 = \rho^{-1}r$.

Proof. This holds by Lemma 12.13 and Lemma 12.14. □

13. Some more identities

In this section, we suppose that (a, t), u, (b, s), r, P, ρ, v are as in Notation 12.7 and Proposition 12.15 and assemble a number of identities we will need in the next few sections.

Proposition 13.1. $\theta(a, v) + tv = -\rho u.$

Proof. This holds by (12.9) and (12.10). □

Proposition 13.2. $q(h(b, av)) = \rho q(h(a, bu)) = \rho q(h(b, au)).$

Proof. By Proposition A.4(iii), we have

$$\rho h(a, bu) = \theta(a, h(aP, bu)) + rh(aP, bu).$$

We have $\rho = q(P)$. Hence

$$\begin{aligned}\rho^2 q(h(a, bu)) &= q\big(\theta(a, h(aP, bu))\big) + rf\big(\theta(a, h(aP, bu)),\ h(aP, bu)\big)\\ &\qquad + r^2 q(h(aP, bu))\\ &= q(\pi(a))q(h(aP, bu)) + rQ(a)q(h(aP, bu)) + r^2 q(h(aP, bu))\\ &= \rho q(h(aP, bu))\end{aligned}$$

by [Weiss 2006b, 4.9(i) and 4.22]. Thus $q(h(aP, bu)) = \rho q(h(a, bu))$. By (D1), therefore, $q(h(b, av)) = q(h(b, aPu) = q(h(aP, bu)) = \rho q(h(a, bu))$. The other equality holds by Proposition A.7. □

Proposition 13.3. $\phi(a, P) = 0.$

Proof. By Proposition A.4(ii) and [Weiss 2006b, (4.14)], $\phi(a, P) = \phi(aP, \varepsilon)$ and by [Weiss 2006b, 4.5(iii) and (4.13)], $\phi(aP, \varepsilon) = 0$. ☐

Proposition 13.4. $q(v) = \rho q(u).$

Proof. This holds by Proposition A.3(i). ☐

Proposition 13.5. $Q(au) = Q(a)q(u).$

Proof. By [Weiss 2006b, 4.1(i)], Q is identically zero if $\mathrm{char}(K) \neq 2$. The claim holds, therefore, by [Weiss 2006b, 3.21]. ☐

Proposition 13.6. $Q(av) = \rho Q(a)q(u).$

Proof. This holds by Propositions 13.4 and 13.5. ☐

Lemma 13.7.

$$f(\theta(a, v), w)v - f(w, v)\theta(a, v) = \rho\big(f(\theta(a, u), w)u - f(w, u)\theta(a, u)\big) \quad \text{for all } w \in L.$$

Proof. This holds by (12.9) and (12.10). ☐

Lemma 13.8. $\theta(av, w) = \rho\theta(au, w)$ *for all* $w \in L$.

Proof. By (C4), Lemma 12.14, Proposition 13.4 and Lemma 13.7, we have

$$\begin{aligned}\overline{\theta(av, w)} &= \rho\big(q(u)\theta(a, \bar{w}) + \phi(a, u)\bar{w}\big) + f(\theta(a, v), \bar{w})v - f(v, \bar{w})\theta(a, v)\\ &= \rho\big(q(u)\theta(a, \bar{w}) + \phi(a, u)\bar{w} + f(\theta(a, u), \bar{w})u - f(v, \bar{w})\theta(a, u)\big)\\ &= \rho\overline{\theta(au, w)}\end{aligned}$$

for all $w \in L$. ☐

Proposition 13.9. *Let* $\omega = f\big(h(b, av), \pi(au)\big) + f(h(b, av), \varepsilon)\phi(a, u)$. *Then*

$$\omega = q(u)\big(f\big(h(a, b), \theta(a, v)\big) + Q(a)f(h(a, b), v)\big).$$

Proof. By [Weiss 2006b, 3.6] and Proposition A.1, we have

$$\begin{aligned}f\big(h(b, av), \pi(au)\big) &= -f\big(h(av, b), \overline{\pi(au)}\big)\\ &= -f\Big(h(av, b),\ q(u)\pi(a) - f(u, \varepsilon)\theta(a, u) + f(\theta(a, u), \varepsilon)v + \phi(a, u)\varepsilon\Big)\end{aligned}$$

and thus

$$\begin{aligned}\omega &= -q(u)f(h(av, b), \pi(a)) + f(u, \varepsilon)f(h(av, b), \theta(a, u)) - f(\theta(a, u), \varepsilon)f(h(av, b), u)\\ &= -q(u)f(h(av\pi(a), b), \varepsilon) + f(u, \varepsilon)f(h(av\theta(a, u), b), \varepsilon) - f(\theta(a, u), \varepsilon)f(h(avu, b), \varepsilon).\end{aligned} \tag{13.10}$$

By (A3), (D1) and [Weiss 2006b, 3.8 and 4.9(i)], we have

$$
\begin{aligned}
(13.11) \qquad av\pi(a) &= a\big(f(v,\varepsilon)\varepsilon - \bar{v}\big)\pi(a)\\
&= f(v,\varepsilon)a\pi(a) - a\bar{v}\pi(a)\\
&= f(v,\varepsilon)a\pi(a) + a\overline{\pi(a)}v - f(\pi(a),v)a\\
&= f(v,\varepsilon)a\pi(a) - a\pi(a)v + Q(a)av - f(\pi(a),v)a\\
&= f(v,\varepsilon)a\pi(a) - a\theta(a,v) + Q(a)av - f(\pi(a),v)a
\end{aligned}
$$

and

$$
\begin{aligned}
(13.12) \qquad avu &= -av\bar{u} + f(u,\varepsilon)av\\
&= -q(u)aP + f(u,\varepsilon)av
\end{aligned}
$$

as well as

$$
\begin{aligned}
av\theta(a,u) &= f\big(\theta(a,u),\varepsilon\big)av - av\overline{\theta(a,u)}\\
&= f\big(\theta(a,u),\varepsilon\big)av + a\theta(a,u)\bar{v} - f\big(\theta(a,u),v\big)a\\
&= f\big(\theta(a,u),\varepsilon\big)av + a\theta(a,u)\big(\overline{\theta(a,u)} + r\bar{u}\big) - f\big(\theta(a,u),v\big)a\\
&= f\big(\theta(a,u),\varepsilon\big)av + q(\theta(a,u))a + ra\theta(a,u)\bar{u} - f\big(\theta(a,u),v\big)a\\
&= f\big(\theta(a,u),\varepsilon\big)av + q(\pi(a))q(u)a + rq(u)a\pi(a) - f\big(\theta(a,u),v\big)a.
\end{aligned}
$$

Since

$$
\begin{aligned}
f\big(\theta(a,u),v\big) &= f\big(\theta(a,u),\theta(a,u)\big) + rf\big(\theta(a,u),u\big)\\
&= 2q(\theta(a,u)) + Q(a)q(u)r = q(u)\big(2q(\pi(a)) + Q(a)\big),
\end{aligned}
$$

it follows that

$$
(13.13) \qquad av\theta(a,u) = f\big(\theta(a,u),\varepsilon\big)av - q(\pi(a))q(u)a - ra\overline{\pi(a)}.
$$

By (13.10)–(13.13) and some calculation, we conclude that

$$
\omega = q(u)\Big(f\big(h(a\theta(a,v),b),\varepsilon\big) + Q(a)f(h(av,b),\varepsilon)\Big).
$$

The claim follows now by (B3). □

Lemma 13.14. $b\theta(a,v)\bar{v} = \rho b\theta(a,u)\bar{u}.$

Proof. We have

$$
\begin{aligned}
&b\theta(a,v)\bar{v}\\
&\quad= -b\big(q(\pi(a))u + t\theta(a,u)\big)\big(\overline{\theta(a,u)} + r\bar{u}\big)\\
&\quad= -\big(q(u)rq(\pi(a)) + tq(\theta(a,u))\big)b - q(\pi(a))bu\overline{\theta(a,u)} - rtb\theta(a,u)\bar{u}\\
&\quad= Q(a)q(u)q(\pi(a))b + \big(q(\pi(a)) - rt\big)b\theta(a,u)\bar{u} - q(\pi(a))f(\theta(a,u),u)b\\
&\quad= \big(q(\pi(a)) - rt\big)b\theta(a,u)\bar{u} = \rho\theta(a,u)\bar{u}
\end{aligned}
$$

by (12.9) and (12.10). □

Lemma 13.15. *The following holds*:

$$avh(av, b) = \rho\big(q(u)ah(a, b) + q(u)b\pi(a) - b\theta(a, u)\bar{u}\big).$$

Proof. By [Weiss 2006b, 3.22], we have

$$\begin{aligned} avh(av, b) &= -a\overline{h(av, b)}\bar{v} + f(\overline{h(av, b)}, v)a \\ &= ah(b, av)\bar{v} - f(h(b, av), v)a \\ &= ah(a, bv)\bar{v} + f(h(b, a), \varepsilon)av\bar{v} - f(h(b, av\bar{v}), \varepsilon)a \\ &= ah(a, bv)\bar{v} \\ &= ah(a, b)v\bar{v} + \big(b\pi(a)v - b\theta(a, v)\big)\bar{v} \\ &= q(v)ah(a, b) + q(v)b\pi(a) - b\theta(a, v)\bar{v}. \end{aligned}$$

The claim holds, therefore, by Proposition 13.4 and Lemma 13.14. □

Lemma 13.16. $f\big(\theta(a, u), h(aP, b)\big) = f\big(h(a, b), \theta(a, v)\big).$

Proof. By [Weiss 2006b, 4.20], we have

$$\begin{aligned} aP\theta(a, u) &= a\pi(a)\theta(a, u) + ra\theta(a, u) \\ &= -a\overline{\pi(a)}\theta(a, u) - ta\theta(a, u) \\ &= -q(\pi(a))au - ta\theta(a, u) = a\theta(a, v). \end{aligned}$$

The claim follows by (B3). □

14. The form Θ

In this section, we prove the following result:

Proposition 14.1. *Let* (a, t), u, (b, s), (a_1, t_1), ρ, r, v *be as in Proposition 12.15 and let*

(14.2) $$\Theta(a, t, u, b, s) = \rho q(\pi(a_1) + t_1\varepsilon).$$

Then

(14.3) $$\begin{aligned} \Theta(a, t, u, b, s) = \rho q(\pi(b) + s\varepsilon) &+ q(h(a, bu)) + q(u)^2 \\ &- q(u)f(h(a, b), u) + \big(sQ(a) + tQ(b)\big)q(u) \\ &+ f(u, \varepsilon)f\big(\theta(a, u), \pi(b)\big) - f(\theta(a, u), \varepsilon)f(u, \pi(b)) \\ &- f\big(h(a, b), \overline{\theta(b, \bar{v})} + sv\big) \\ &- q(u)f\big(\pi(a), \pi(b)\big) - 2stq(u). \end{aligned}$$

Proof. Let

$$z = \pi(\rho b + av) + \rho\big(rq(u) + \phi(a,u) + g(b,av) + \rho s\big)\varepsilon.$$

Thus

$$\rho^3\Theta(a,t,u,b,s) = \rho^4 q(\pi(a_1) + t_1\varepsilon) = q(z)$$

by Proposition 12.15. By (C3) and Lemma 13.8, we have

$$\begin{aligned} z &= \rho^2\pi(b) + \rho h(b,av) + \pi(av) + \rho\big(\phi(a,u) + \rho s + rq(u)\big)\varepsilon \\ &= \rho\big(\rho\pi(b) + h(b,av) + \pi(au) + \big(\phi(a,u) + \rho s + rq(u)\big)\varepsilon\big). \end{aligned}$$

By Proposition A.3(ii), we have

$$q\big(\pi(au) + rq(u) + \phi(a,u)\varepsilon\big) = \rho q(u)^2.$$

By [Weiss 2006b, 4.5(iii)], Proposition 13.2 and Proposition 13.6, it follows that

$$\begin{aligned} \rho^{-2}q(z) = {} & \rho^2 q(\pi(b)) + \rho q(h(b,au)) + \rho q(u)^2 + \rho^2 s^2 \\ & + \rho s Q(a)q(u) + 2\rho r s q(u) \\ & + \rho^2 Q(b)s + \rho f\big(\pi(b), h(b,av)\big) \\ & + \rho f\big(\pi(b), \pi(au)\big) + \rho Q(b)\big(\phi(a,u) + rq(u)\big) \\ & + f\big(h(b,av), \pi(au)\big) + f\big(h(b,av), \varepsilon\big)\phi(a,u) \\ & + \big(\rho s + rq(u)\big) f\big(h(b,av), \varepsilon\big). \end{aligned} \tag{14.4}$$

By Propositions 13.1 and 13.9, we have

$$\begin{aligned} & f\big(h(b,av), \pi(au)\big) + f\big(h(b,av), \varepsilon\big)\phi(a,u) + rq(u) f\big(h(b,av), \varepsilon\big) \\ & \qquad = q(u)\big(f\big(h(a,b), \theta(a,v)\big) + Q(a) f(h(a,b), v) - r f(h(a,b), v)\big) \\ & \qquad = -\rho q(u) f\big(h(a,b), u\big). \end{aligned}$$

Replacing $2\rho r s q(u)$ by $-2\rho s t q(u)$ in (14.4), we thus conclude that

$$\begin{aligned} \Theta = \rho^{-3}q(z) = {} & \rho q(\pi(b) + s\varepsilon) + q(h(a,bu)) + rq(u)Q(b) + Q(a)q(u)s \\ & - 2stq(u) + f\big(\pi(b), h(b,av)\big) + sf\big(h(b,av), \varepsilon\big) \\ & + f\big(\pi(b), \pi(au)\big) + \phi(a,u)Q(b) \\ & + q(u)^2 - q(u) f\big(h(a,b), u\big). \end{aligned} \tag{14.5}$$

By (B3), (D1) and [Weiss 2006b, 3.6 and 3.7], we have

$$\begin{aligned} f\big(\pi(b), h(b,av)\big) + sf\big(h(b,av), \varepsilon\big) &= f\big(h(b(\pi(b) + s\varepsilon)\bar{v}, a), \varepsilon\big) \\ &= -f\big(h(a,b), \overline{\theta(b,\bar{v})} + sv\big) \end{aligned}$$

and by (C4), we have

$$
\begin{aligned}
&f\bigl(\pi(b),\pi(au)\bigr)+Q(b)\phi(a,u)\\
&=f\bigl(\overline{\pi(b)},\ q(u)\pi(a)-f(u,\varepsilon)\theta(a,u)+f(\theta(a,u),\varepsilon)u\bigr)\\
&=-q(u)f(\pi(a),\pi(b))+q(u)Q(a)Q(b)+f(u,\varepsilon)f(\pi(b),\theta(a,u))\\
&\qquad\qquad -f(\theta(a,u),\varepsilon)f(\pi(b),u).
\end{aligned}
$$

By (14.5), therefore, $\Theta(a,t,u,b,s)$ is as in (14.3). □

Remark 14.6. Note that a_1 and t_1 are only defined under the assumption that ρ is nonzero. In particular, we cannot conclude from (14.2) that $\rho=0$ implies that the expression $\Theta(a,t,u,b,s)$ in (14.3) is zero (and, in fact, this is clearly not true).

15. The smallest F_4-cases

We interrupt our calculations to make a few remarks about the case that $\Delta = F_4(C, K)$ and either $\mathrm{char}(K)\neq 2$ and $C=K$ or $\mathrm{char}(K)=2$ and C/K is a field extension such that $C^2\subset K$. By Theorem 7.4(vi), $\dim_K L=5$ if $\mathrm{char}(K)\neq 2$ and f is degenerate if $\mathrm{char}(K)=2$. If $\mathrm{char}(K)\neq 2$, then by [Mühlherr and Weiss 2019, 5.10], Ξ is isotopic to the quadrangular algebra $\mathcal{Q}_4(C,K)$ defined in [loc. cit, 4.12] and hence by [loc. cit, 4.13 and 10.5], $q(\pi(a))=0$ and $a\pi(a)=0$ for all $a\in\mathcal{X}$. (If the composition of q and π is identically zero in one isotope of Ξ, then by [Mühlherr and Weiss 2019, 5.3; Weiss 2006b, 8.7], it is identically zero in all isotopes of Ξ.) By [Mühlherr and Weiss 2019, 8.1(i), 8.5 and 9.2], we have $Q(a)=0$ and $f(h(a,b),u)=0$ for all $a,b\in\mathcal{X}$ and all $u\in L$ if $\mathrm{char}(K)=2$.

Proposition 15.1. *Suppose that* $\mathrm{char}(K)\neq 2$ *and* $a\pi(a)=0$ *for all* $a\in\mathcal{X}$ *and let* $\Theta=\Theta(a,t,u,b,s)$ *be as in* (14.3). *Then*

$$\Theta=\bigl(st+\tfrac{1}{2}f(h(a,b),u)-q(u)\bigr)^2$$

for all $(a,t)\in\mathcal{R}^\times$, $u\in L$ *and* $(b,s)\in\mathcal{R}$.

Proof. Choose $(a,t)\in\mathcal{R}^\times$, $u\in L$ and $(b,s)\in\mathcal{R}$. By (14.2), we have $\Theta=(tt_1)^2$, where

$$t_1=t^{-1}\bigl(st-g(b,au)-q(u)\bigr).$$

The claim holds, therefore, since

$$
\begin{aligned}
g(b,au)&=\tfrac{1}{2}f\bigl(h(b,au),\varepsilon\bigr) && \text{by [Weiss 2006b, 4.1(i) and (4.3)]}\\
&=\tfrac{1}{2}f\bigl(\overline{h(b,au)},\bar{\varepsilon}\bigr) && \text{by [Weiss 2006b, 1.4]}\\
&=-\tfrac{1}{2}f\bigl(h(au,b),\varepsilon\bigr) && \text{by [Weiss 2006b, 3.6]}\\
&=-\tfrac{1}{2}f\bigl(h(a,b),u\bigr) && \text{by (B3).}
\end{aligned}
$$

□

Remark 15.2. Suppose that $\mathrm{char}(K) \neq 2$ and that $a\pi(a) = 0$ for all $a \in \mathcal{X}$. By (A3), we have $q(\pi(a)) = 0$ for all $a \in \mathcal{X}$. By (B3), we have $f\big(\pi(b), h(b, av)\big) = f((h(b\pi(b), av), \varepsilon\big) = 0$. Similarly, we have

$$f\big(\pi(au), h(au, b)\big) = f\big(h(au\pi(au), b), \varepsilon\big) = 0$$

and $f\big(\pi(b), h(au, b)\big) = -f\big(h(au, b\pi(b), \varepsilon\big) = 0$. By [Weiss 2006b, 4.1(i) and 4.3], we have

$$g(au, b) = \tfrac{1}{2} f\big(h(a, b), u\big)$$

and by Proposition A.7, $q(h(au, b)) = q(h(a, bu))$. Applying (C3) to $\pi(au + b)$ and then applying q to both sides of this equation, we find thus that

$$q\big(h(a, bu)\big) + f\big(\pi(au), \pi(b)\big) = \tfrac{1}{4} f\big(h(a, b), u\big)^2.$$

With these observations, Proposition 15.1 can be confirmed directly from (14.5).

Proposition 15.3. *Suppose that $f(h(a, b), X) = 0$ and $Q(a) = 0$ for all $a, b \in \mathcal{X}$ and that* $\mathrm{char}(K) = 2$ *and let $\Theta = \Theta(a, t, u, b, s)$ be as in* (14.3). *Then*

$$\Theta = q(\pi(a) + t\varepsilon)q(\pi(b) + s\varepsilon) + q(h(a, bu)) + f\big(\pi(au), \pi(b)\big) + q(u)^2$$

for all $(a, t) \in \mathcal{R}^\times$, $u \in L$ and $(b, s) \in \mathcal{R}$.

Proof. This holds by (14.5). ☐

16. The element $(m_1m_4)^2$

Let $(a, t), (a_1, t_1) \in \mathcal{R}^\times$, $u \in L$ and $(b, s) \in \mathcal{R}$ be as in Proposition 12.15 and let $\Theta = \Theta(a, t, u, b, s)$ be as in Proposition 14.1.

Notation 16.1. Suppose that $\Theta \neq 0$. Thus by (14.2), $q(\pi(a_1) + t_1\varepsilon) \neq 0$. By Proposition 12.15 applied to the image of the vertex p under m_1m_4, it follows that there exist $(a_2, t_2), (b_2, s_2) \in \mathcal{R}$ and $u_2 \in L$ such that

$$U_4x_1(a_2, t_2)x_2(u_2)x_3(b_2, s_2)$$

is the image of the vertex

$$p := U_4x_1(a, t)x_2(u)x_3(b, s)$$

of Λ under $(m_1m_4)^2$.

The expressions we obtain for $a_2, \ldots, s_2$ by applying Proposition 12.15 have $\rho^{-1}\Theta$ in the denominator and, in various places, negative powers of ρ appear in the numerator. Our goal for the rest of this section is to obtain expressions for s_2 and b_2 in which all the negative powers of ρ have been eliminated. In the following two sections, we do the same for u_2.

Notation 16.2. Let $r = Q(a) - t$, $P = \pi(a) + r\varepsilon$, $\rho = q(P)$ and $v = \theta(a, u) + ru$ as in Notation 12.7.

Proposition 16.3. *Let s_2 be as in Notation 16.1. Then*

$$s_2 = \frac{tq(u) - \rho s + g(av, b) + \phi(a, u) + \rho Q(b)}{\Theta}.$$

Proof. By Proposition 12.15 and (14.2), we have $s_2 = z/\Theta$, where

$$z = -rq(u) + \phi(a, u) - g(b, av) - \rho s + \rho^{-1} Q(\rho b + av).$$

If char(K) $\neq 2$, then ϕ and Q are identically zero and by [Weiss 2006b, 4.4 and 4.5(ii)], g is alternating, so the claim holds. We can assume, therefore, that char(K) $= 2$. Then

$$\begin{aligned}\rho^{-1} Q(\rho b + av) &= \rho Q(b) + \rho^{-1} Q(av) + f(h(b, av), \varepsilon)\\ &= \rho Q(b) + Q(a)q(u) + f(h(b, av), \varepsilon)\end{aligned}$$

by (C3) and Proposition 13.6. By Notation 2.2 and [Weiss 2006b, 4.3], we have

$$\begin{aligned}g(b, av) + f(h(b, av), \varepsilon) &= f(h(b, av), \delta + \varepsilon)\\ &= f(h(b, av), \bar{\delta})\\ &= f(h(av, b), \delta) = g(av, b),\end{aligned}$$

where δ is as in Theorem 7.4(i). Thus the claim holds also in this case. □

Proposition 16.4. *Let b_2 be as in Notation 16.1. Then*

$$b_2 = \frac{y}{\Theta},$$

where

$$y = q(u)a\big(u - h(a, b)\big) + (\rho b + av)\big(\overline{\pi(b)} + s\varepsilon\big) - b\bar{u}\theta(a, u) - bh(av, b) - tq(u)b$$

Proof. By Proposition 12.15 and (14.2), we have

$$b_2 = \frac{-(\rho b + av)z}{\rho^2\Theta}, \tag{16.5}$$

where

$$z = \pi(\rho b + av) + \big(Q(\rho b + av) - \rho rq(u) + \rho\phi(a, u) - \rho g(b, av) - \rho^2 s\big)\varepsilon.$$

By (C3), we have

$$\pi(\rho b + av) = \rho^2\pi(b) + \pi(av) + \rho\big(h(b, av) - g(b, av)\varepsilon\big).$$

If char(K) $\neq 2$, then Q is identically zero and

$$2g(b, av) = f(h(b, av), \varepsilon)$$

by [Weiss 2006b, 4.1(iii) and 4.3]. If $\mathrm{char}(K)=2$, then

$$Q(\rho b+av)=\rho^2 Q(b)+Q(av)+\rho f(h(b,av),\varepsilon).$$

Thus

$$\begin{aligned}\pi(\rho b+av)&+Q(\rho b+av)\varepsilon-\rho g(b,av)\varepsilon\\ &=\rho^2\pi(b)+\pi(av)+\rho h(b,av)+\big(\rho^2 Q(b)+Q(av)-\rho f(h(b,av),\varepsilon)\big)\varepsilon\\ &=\rho^2\pi(b)+\pi(av)-\rho\overline{h(b,av)}+\big(\rho^2 Q(b)+Q(av)\big)\varepsilon\\ &=\rho^2\pi(b)+\pi(av)+\rho h(av,b)+\big(\rho^2 Q(b)+Q(av)\big)\varepsilon\end{aligned}$$

in all characteristics. By Proposition 13.6, we have

$$Q(av)-\rho rq(u)=\rho(Q(a)-r)q(u)=\rho tq(u).$$

Thus

$$z=\rho^2\pi(b)+\pi(av)+\rho h(av,b)+\big(\rho^2 Q(b)+\rho tq(u)+\rho\phi(a,u)-\rho^2 s\big)\varepsilon. \tag{16.6}$$

Hence

$$\frac{\rho bz}{\rho^2\Theta}=\frac{bz_1}{\Theta}+\frac{b\pi(av)}{\rho\Theta}, \tag{16.7}$$

where

$$z_1=\rho\pi(b)+h(av,b)+\big(\rho Q(b)+tq(u)+\phi(a,u)-\rho s\big)\varepsilon.$$

By Lemma 12.14, and Propositions 13.4 and A.2, we have

$$\begin{aligned}av\pi(av)&=q(v)a\theta(a,v)+\phi(a,v)av\\ &=\rho\big(q(u)a\theta(a,v)+\phi(a,u)av\big)\end{aligned}$$

and by Proposition 13.1, we have

$$a\theta(a,v)+tav=-\rho au.$$

By (16.6), therefore,

$$\frac{avz}{\rho^2\Theta}=\frac{av\pi(b)+(Q(b)-s)av-q(u)au}{\Theta}+\frac{avh(av,b)}{\rho\Theta}. \tag{16.8}$$

By Lemma 13.8, we have $b\pi(av)=\rho b\pi(au)$. By Lemma 13.15, and Equations (16.5), (16.7) and (16.8), it follows that

$$b_2=-\frac{y_1}{\Theta},$$

where

$$y_1 = \rho b\pi(b) + bh(av, b) + \big(\rho Q(b) + tq(u) + \phi(a, u) - \rho s\big)b + b\pi(au)$$
$$+ av\pi(b) + (Q(b) - s)av - q(u)au + q(u)ah(a, b)$$
$$+ q(u)b\pi(a) - b\theta(a, u)\bar{u}.$$

Next note that

$$b\pi(au) + \phi(a, u)b = q(u)b\overline{\pi(a)} - f(u, \varepsilon)b\overline{\theta(a, u)} + f\big(\theta(a, u), \varepsilon\big)b\bar{u}$$

by Proposition A.1 and

$$-f(u, \varepsilon)b\overline{\theta(a, u)} + f\big(\theta(a, u), \varepsilon\big)b\bar{u} - b\theta(a, u)\bar{u}$$
$$= -f(u, \varepsilon)b\overline{\theta(a, u)} + b\overline{\theta(a, u)}\bar{u}$$
$$= -b\overline{\theta(a, u)}u.$$

Hence

$$b\pi(au) + \phi(a, u)b - b\theta(a, u)\bar{u} + q(u)b\pi(a) = q(u)Q(a)b - b\overline{\theta(a, u)}u$$
$$= b\bar{u}\theta(a, u)$$

by [Weiss 2006b, 4.9(i)]. Therefore

$$b_2 = -\frac{y_2}{\Theta},$$

where

$$y_2 = q(u)a\big(h(a, b) - u\big) + av\pi(b) + (Q(b) - s)av + b\bar{u}\theta(a, u)$$
$$+ \rho b\pi(b) + bh(av, b) + \big(\rho(Q(b) - s) + tq(u)\big)b$$

Since

$$b\pi(b) + Q(b)b - sb = -b(\overline{\pi(b)} + s\varepsilon)$$

and

$$av\pi(b) + Q(b)av - sav = -av(\overline{\pi(b)} + s\varepsilon),$$

the claim holds. □

17. The element u_2, part I

Our next goal is to prove Proposition 18.8. In this section, we begin the proof. We continue to assume that (a, t), (b, s), (a_1, t_1), u, $r = Q(a) - t$, $P = \pi(a) + r\varepsilon$, $\rho = q(P)$ and $v = \theta(a, u) + ru$ are as in Proposition 12.15, that $\Theta = \Theta(a, t, u, b, s)$ is as in Proposition 14.1, that $\Theta \neq 0$ and that u_2 is as in Notation 16.1.

Proposition 17.1. *The following holds*:

$$\begin{aligned}h\big(aP, bh(av,b)\big) &= \rho\big(h(b, ah(au,b)) + f(h(a,b),\varepsilon)h(au,b)\big)\\ &\quad + f\big(h(a,b),v\big)h(a\pi(a),b)\\ &\quad + \big(f\big(h(a,b),\theta(a,v)\big) + Q(a)f(h(a,b),v)\big)h(a,b).\end{aligned}$$

Proof. We have

$$\begin{aligned}&h(aP, bh(av,b))\\ &\qquad = h(b, aPh(av,b)) + f(h(aP,b),\varepsilon)h(av,b)\\ &\qquad = h\big(b, a\theta(a, h(av,b))\big) + rh\big(b, ah(av,b)\big) + f\big(h(a,b),P\big)h(av,b)\end{aligned}$$

and by Proposition A.6(iii), we have

$$\begin{aligned}h\big(b, a\theta(a,h(av,b))\big) &= -f\big(h(a,b),\varepsilon\big)h\big(b,a\theta(a,v)\big) - f\big(h(b,a),\pi(a)\big)h(b,av)\\ &\quad + Q(a)h\big(b, a\overline{h(av,b)}\big) + h\big(b, a\overline{h(a\theta(a,v),b)}\big)\\ &\quad + f\big(h(a,b),v\big)h\big(b,a\pi(a)\big).\end{aligned}$$

Thus by (12.9) and (12.10), we have

$$\begin{aligned}&h\big(b, a\theta(a,h(av,b))\big)\\ &= q(\pi(a))f\big(h(a,b),\varepsilon\big)h(b,au) + tf\big(h(a,b),\varepsilon\big)h\big(b,a\theta(a,u)\big)\\ &\qquad - f\big(h(b,a),\pi(a)\big)h\big(b,a\theta(a,u)\big) - rf\big(h(b,a),\pi(a)\big)h(b,au)\\ &\qquad\quad + Q(a)h\big(b, a\overline{h(a\theta(a,u),b)}\big) + Q(a)rh\big(b, a\overline{h(au,b)}\big)\\ &\quad - q(\pi(a))h\big(b, a\overline{h(au,b)}\big) - th\big(b, a\overline{h(a\theta(a,u),b)}\big) + f\big(h(a,b),v\big)h\big(b,a\pi(a)\big)\end{aligned}$$

as well as

$$rh\big(b, ah(av,b)\big) = rh\big(b, ah(a\theta(a,u),b)\big) + r^2h\big(b, ah(au,b)\big)$$

and

$$\begin{aligned}&f\big(h(a,b),P\big)h(av,b)\\ &\quad = N\big(h(a\theta(a,u),b) + rh(au,b)\big)\\ &\quad = N\big(h(b,a\theta(a,u)) + f\big(h(a\theta(a,u),b),\varepsilon\big)\varepsilon + rh(b,au) + rf(h(au,b),\varepsilon)\varepsilon\big)\\ &\quad = N\big(h(b,a\theta(a,u)) + f(h(a,b),v)\varepsilon + rh(b,au)\big),\end{aligned}$$

where

$$N := f\big(h(a,b),\pi(a)\big) + rf\big(h(a,b),\varepsilon\big).$$

Next we observe that

$$\begin{aligned}Q(a)h\big(b, a\overline{h(a\theta(a,u),b)}\big) - th\big(b, a\overline{h(a\theta(a,u),b)}\big) + rh\big(b, ah(a\theta(a,u),b)\big)\\ = rf\big(h(a\theta(a,u),b),\varepsilon\big)h(b,a)\\ = rf\big(h(a,b),\theta(a,u)\big)h(b,a)\\ = rf\big(h(a,b),\theta(a,u)\big)\big(h(a,b) - f(h(a,b),\varepsilon)\varepsilon\big)\end{aligned}$$

and

$$\begin{aligned}&\big(q(\pi(a))f(h(a,b),\varepsilon) - rf\big(h(b,a),\pi(a)\big) + Nr\big)h(b,au)\\ &\quad= \big(q(\pi(a))f(h(a,b),\varepsilon) + rf\big(h(a,b),\overline{\pi(a)}\big) + Nr\big)h(b,au)\\ &\quad= \big(q(\pi(a)) + rQ(a) + r^2\big)f\big(h(a,b),\varepsilon\big)h(b,au)\\ &\quad= \rho f(h(a,b),\varepsilon)h(b,au)\\ &\quad= \rho f(h(a,b),\varepsilon)\big(h(au,b) - f(h(au,b),\varepsilon)\varepsilon\big)\end{aligned}$$

as well as

$$\begin{aligned}&Q(a)rh\big(b, a\overline{h(au,b)}\big) - q(\pi(a))h\big(b, a\overline{h(au,b)}\big) + r^2h\big(b, ah(au,b)\big)\\ &= \big(Q(a)r + q(\pi(a)) + r^2\big)h\big(b, ah(au,b)\big)\\ &\qquad + \big(Q(a)r - q(\pi(a))\big)f\big(h(au,b),\varepsilon\big)h(b,a)\\ &= \rho h\big(b, ah(au,b)\big) + \big(Q(a)r - q(\pi(a))\big)f\big(h(a,b),u\big)h(a,b)\\ &\qquad - \big(Q(a)r - q(\pi(a))\big)f\big(h(a,b),u\big)f\big(h(a,b),\varepsilon\big)\varepsilon.\end{aligned}$$

We also have

$$\begin{aligned}&\big(tf\big(h(a,b),\varepsilon\big) - f\big(h(b,a),\pi(a)\big) + N\big)h\big(b, a\theta(a,u)\big)\\ &\quad= \big(tf\big(h(a,b),\varepsilon\big) + f\big(h(a,b),\overline{\pi(a)}\big) + N\big)h\big(b, a\theta(a,u)\big)\\ &\quad= \big(t + r + Q(a)\big)f\big(h(a,b),\varepsilon\big)h\big(b, a\theta(a,u)\big) = 0\end{aligned}$$

and

$$\begin{aligned}&f\big(h(a,b),v\big)h\big(b, a\pi(a)\big) + f\big(h(a,b),v\big)f\big(h(a,b),\pi(a)\big)\varepsilon\\ &\quad= f\big(h(a,b),v\big)\big(h\big(b, a\pi(a)\big) - f\big(h(b,a\pi(a)),\varepsilon\big)\varepsilon\big)\\ &\quad= f\big(h(a,b),v\big)h\big(a\pi(a),b\big).\end{aligned}$$

Finally, we observe that

$$rv - r\theta(a,u) + \big(Q(a)r + q(\pi(a))\big)u = \rho u$$

by (12.9) and (12.10) and hence

$$\begin{aligned} rf\big(h(a,b),v\big)f\big(h(a,b),\varepsilon\big) - rf\big(h(a,b),\theta(a,u)\big)f\big(h(a,b),\varepsilon\big) \\ + \big(Q(a)r + q(\pi(a))\big)f\big(h(a,b),u\big)f\big(h(a,b),\varepsilon\big) \\ = \rho f\big(h(a,b),u\big)f\big(h(a,b),\varepsilon\big). \end{aligned}$$

Assembling all these calculations, we obtain the desired formula. □

Proposition 17.2. *The following holds*:

$$h\big(aP, bh(b,av)\big) = \rho\big(h\big(b, ah(b,au)\big) + f\big(h(a,b),\varepsilon\big)h(b,au)\big).$$

Proof. We have

$$\begin{aligned} h\big(aP, bh(b,av)\big) &= h\big(aP, bh(av,b)\big) + f\big(h(b,av),\varepsilon\big)h(aP,b) \\ &= h\big(aP, bh(av,b)\big) - f\big(h(a,b),v\big)h(aP,b) \\ &= h\big(aP, bh(av,b)\big) - f\big(h(a,b),v\big)\big(h(a\pi(a),b) + rh(a,b)\big) \end{aligned}$$

and thus

$$\begin{aligned} h\big(aP, bh(b,av)\big) = \rho\big(h\big(b, ah(au,b)\big) + f\big(h(a,b),\varepsilon\big)h(au,b)\big) \\ + \big(f\big(h(a,b),\theta(a,v)\big) + tf\big(h(a,b),v\big)\big)h(a,b). \end{aligned}$$

by Proposition 17.1. By Proposition 13.1, we have

$$f\big(h(a,b),\theta(a,v)\big) + tf\big(h(a,b),v\big) = -\rho f\big(h(a,b),u\big).$$

Hence

$$\begin{aligned} &h\big(aP, bh(b,av)\big) \\ &\quad = \rho\big(h\big(b, ah(au,b)\big) + f\big(h(a,b),\varepsilon\big)h(au,b) - f\big(h(a,b),u\big)h(a,b)\big) \\ &\quad = \rho\big(h\big(b, ah(au,b)\big) + f\big(h(a,b),\varepsilon\big)h(b,au) - f\big(h(au,b),\varepsilon\big)h(b,a)\big) \\ &\quad = \rho\big(f\big(h(a,b),\varepsilon\big)h(b,au) - h\big(b, a\overline{h(au,b)}\big)\big) \end{aligned}$$

and so the claim holds. □

Proposition 17.3. *The following holds*:

$$\begin{aligned} h\big(aP, b\bar{u}\theta(a,u)\big) = -q(\pi(a))q(u)h(a,b) - f\big(h(a,b),\theta(a,v)\big)u \\ - tq(u)h\big(a\pi(a),b\big) + f\big(h(a,b),v\big)\theta(a,u). \end{aligned}$$

Proof. We have

$$\begin{aligned}
h\big(aP, b\bar{u}\theta(a,u)\big) &= h\big(b\bar{u}, aP\theta(a,u)\big) + f\big(h(aP,b\bar{u}),\varepsilon\big)\theta(a,u)\\
&= h\big(b\bar{u}, a\big(\theta(a,\theta(a,u)) + r\theta(a,u)\big)\big) + f\big(h(a,b),v\big)\theta(a,u)\\
&= h\big(b\bar{u}, a\big((Q(a)+r)\theta(a,u) - q(\pi(a))u\big)\big) + f\big(h(a,b),v\big)\theta(a,u)\\
&= (Q(a)+r)h\big(a\pi(a), b\bar{u}u\big) + (Q(a)+r)f\big(h(b\bar{u},a\pi(a)),\varepsilon\big)u\\
&\qquad - q(\pi(a))h(a,b\bar{u}u) - q(\pi(a))f(h(b\bar{u},a),\varepsilon)u\\
&\qquad + f\big(h(a,b),v\big)\theta(a,u)\\
&= -tq(u)h\big(a\pi(a),b\big) + tf\big(h(a\pi(a),b),u\big)u\\
&\qquad - q(u)q(\pi(a))h(a,b) + q(\pi(a))f(h(a,b),u)u\\
&\qquad + f\big(h(a,b),v\big)\theta(a,u)\\
&= -tq(u)h\big(a\pi(a),b\big) - f\big(h(a,b),\theta(a,v)\big)u\\
&\qquad - q(u)q(\pi(a))h(a,b) + f\big(h(a,b),v\big)\theta(a,u).
\end{aligned}$$

□

Proposition 17.4. *The following holds*:

$$h\big(aP, ah(a,b)\big) = \big(Q(a)+2t\big)h(a\pi(a),b) + \big(Q(a)r + 2q(\pi(a))\big)h(a,b).$$

Proof. Set $u = h(a,b)$ in Proposition 12.11 and then apply Proposition A.6(i). □

Proposition 17.5. *The following holds*:

$$h\big(aP, bh(b,aP)u\big) = \rho h\big(b\bar{u}, ah(a,b)\big).$$

Proof. We have

$$\begin{aligned}
h(av, b\bar{u}) = h(aPu, b\bar{u}) &= h(b, aPu\bar{u}) + f\big(h(aPu,b),\varepsilon\big)\bar{u}\\
&= q(u)h(b,aP) + f\big(h(a,b),v\big)\bar{u}
\end{aligned}$$

and thus

$$q(u)h(b,aP) = h(av,b\bar{u}) - f\big(h(a,b),v\big)\bar{u}.$$

Hence

$$\begin{aligned}
q(u)h\big(aP, bh(b,aP)u\big) &= h\big(aP, bh(av,b\bar{u})u\big) - q(u)f\big(h(a,b),v\big)h(aP,b)\\
&= -h\big(aP, b\bar{u}\overline{h(av,b\bar{u})}\big) + f\big(h(av,b\bar{u}),\bar{u}\big)h(aP,b)\\
&\qquad - q(u)f\big(h(a,b),v\big)h(aP,b).
\end{aligned}$$

Since

$$f\big(h(av,b\bar{u}),\bar{u}\big) = f\big(h(av,b\bar{u}\cdot u),\varepsilon\big) = q(u)f\big(h(a,b),v\big),$$

it follows that

$$q(u)h\big(aP, bh(b,aP)u\big) = h\big(aP, b\bar{u}h(b\bar{u},av)\big).$$

Replacing b by $b\bar{u}$ in Proposition 17.2, we thus obtain

$$\begin{aligned}q(u)h\big(aP, bh(b, aP)u\big) &= \rho\big(h\big(b\bar{u}, ah(b\bar{u}, au)\big) + f\big(h(a, b\bar{u}), \varepsilon\big)h(b\bar{u}, au)\big)\\ &= \rho\big(h\big(b\bar{u}, ah(b\bar{u}, au)\big) + f\big(h(a, b), u\big)h(b\bar{u}, au)\big).\end{aligned}$$

Since

$$\begin{aligned}h(b\bar{u}, au) &= h(a, b\bar{u}u) + f\big(h(b\bar{u}, a), \varepsilon\big)u\\ &= q(u)h(a, b) - f\big(h(a, b), u\big)u,\end{aligned}$$

it follows that

$$h\big(b\bar{u}, ah(b\bar{u}, au)\big) = q(u)h\big(b\bar{u}, ah(a, b)\big) - f\big(h(a, b), u\big)h(b\bar{u}, au).$$

Hence

$$q(u)h\big(aP, bh(b, aP)u\big) = \rho q(u)h\big(b\bar{u}, ah(a, b)\big).$$

Thus the claim holds for all u such that $q(u) \neq 0$. Since every element of L can be written as a sum $u_1 + u_2$ with $q(u_1)$ and $q(u_2)$ nonzero, it follows that the claim holds in general. □

Proposition 17.6. *The following holds*:

$$\begin{aligned}h\big(aP, av\overline{\pi(b)}\big) = \rho\Big(h\big(a, a\overline{\theta(b, \bar{u})}\big) + h\big(b, ah(b, au)\big) - h\big(b\bar{u}, ah(a, b)\big)\\ + f\big(h(a, b), \varepsilon\big)h(b, au)\Big).\end{aligned}$$

Proof. We have

$$\begin{aligned}&h\big(aP, av\overline{\pi(b)}\big)\\ &\quad= h\big(aP, aPu\overline{\pi(b)}\big)\\ &\quad= -h\big(aP, aP\bar{u}\overline{\pi(b)}\big) + f(u, \varepsilon)h\big(aP, aP\overline{\pi(b)}\big)\\ &\quad= h\big(aP, aP\pi(b)u\big) - f\big(u, \overline{\pi(b)}\big)h(aP, aP) + f(u, \varepsilon)h\big(aP, aP\overline{\pi(b)}\big)\\ &\quad= h\big(aP, aP\pi(b)u\big) + f\big(u, \pi(b)\big)h(aP, aP) - f(u, \varepsilon)h\big(aP, aP\pi(b)\big)\end{aligned}$$

By [Weiss 2006b, 3.22], it follows that

$$\begin{aligned}h\big(aP, av\overline{\pi(b)}\big) = h\big(aP, aP\theta(b, u)\big) + h\big(aP, bh(b, av)\big) - h\big(aP, bh(b, aP)u\big)\\ + f\big(u, \pi(b)\big)h(aP, aP) - f(u, \varepsilon)h\big(aP, aP\pi(b)\big).\end{aligned}$$

Let $N = \theta(b, u) - f(u, \varepsilon)\pi(b) + f(u, \pi(b))\varepsilon$, so that

$$h(aP,aPN) = h\big(aP,aP\theta(b,u)\big) + f\big(u,\pi(b)\big)h(aP,aP) - f(u,\varepsilon)h\big(aP,aP\pi(b)\big).$$

Since

$$\begin{aligned} N &= -\theta(b,\bar{u}) + f\big(u,\pi(b)\big)\varepsilon \\ &= -\theta(b,\bar{u}) + f\big(\bar{u},\overline{\pi(b)}\big)\varepsilon \\ &= -\theta(b,\bar{u}) - f\big(\bar{u},\pi(b)\big)\varepsilon + Q(b)f(u,\varepsilon)\varepsilon \\ &= -\theta(b,\bar{u}) + f\big(\theta(b,\bar{u}),\varepsilon\big)\varepsilon \quad \text{by [Weiss 2006b, 4.9(iii)]} \\ &= \overline{\theta(b,\bar{u})}, \end{aligned}$$

it follows by Proposition 12.12 that

$$h(aP, aPN) = \rho h\big(a, a\overline{\theta(b,\bar{u})}\big).$$

The claim holds, therefore, by Proposition 17.2 and Proposition 17.5. □

Proposition 17.7. *The following holds*:

$$h\big(b, ah(au,b)\big) - h\big(b, ah(b,au)\big) + f\big(h(a,b),\varepsilon\big)h(au,b) \\ - f\big(h(a,b),\varepsilon\big)h(b,au,b) = f\big(h(a,b),u\big)h(a,b).$$

Proof. The expression on the left-hand side equals

$$\begin{aligned} &h\big(b, ah(au,b) + a\overline{h(au,b)}\big) + f\big(h(a,b),\varepsilon\big)\big(h(au,b) + \overline{h(au,b)}\big) \\ &\qquad = f\big(h(au,b),\varepsilon\big)h(b,a) + f\big(h(a,b),\varepsilon\big)f\big(h(au,b),\varepsilon\big)\varepsilon \\ &\qquad = f\big(h(a,b),u\big)\big(f\big(h(a,b),\varepsilon\big)\varepsilon - \overline{h(a,b)}\big) \\ &\qquad = f\big(h(a,b),u)\big)h(a,b). \end{aligned}$$

□

Proposition 17.8. *We have* $h(b_1,b_2) = \frac{1}{\Theta}(\omega + \rho^{-1}\xi)$, *where*

$$\omega = h\big(b\bar{u}, ah(a,b)\big) - h\big(a, a\overline{\theta(b,\bar{u})}\big) + f\big(h(a,b),u\big)h(a,b) \\ - s\big(2\theta(a,u) + Q(a)u + h(aP,b)\big) - h\big(aP, b\overline{\pi(b)}\big)$$

and

$$\begin{aligned} \xi = \Big(&q(u)q(\pi(a)) - q(u)r^2 + f\big(h(a,b),\theta(a,v)\big) + Q(a)f\big(h(a,b),v\big)\Big)h(a,b) \\ &+ \big(q(u)Q(a) + 2q(u)t + f\big(h(a,b),v\big)\big)h\big(a\pi(a),b\big) \\ &\quad + \big(q(u)Q(a)t - 2q(u)q(\pi(a)) - f\big(h(a,b),\theta(a,v)\big)\big)u \\ &\qquad + \big(q(u)Q(a) + 2tq(u) + f\big(h(a,b),v\big)\big)\theta(a,u). \end{aligned}$$

Proof. We have

$$h(b_1,b_2) = -\frac{1}{\Theta}\rho^{-1}h(aP,y),$$

where y is as in Proposition 16.4. The claim holds, therefore, by Propositions 12.11, 12.12, 17.1, 17.3, 17.4, 17.6, 17.7 and some calculation. □

18. The element u_2, part II

We continue with all the notation and assumptions of the previous two sections.

Proposition 18.1. *We have* $\rho t_1\bar{u}_1 = \alpha + \rho^{-1}\beta$, *where*

$$\alpha = s\big(h(aP, b) + tu + \theta(a, u)\big),$$
$$\textit{and}\quad \beta = \big(rq(u) + \phi(a, u) + g(b, av)\big)\big(h(aP, b) + tu + \theta(a, u)\big).$$

Proof. This holds by Proposition 12.15(ii) and (iii). □

Lemma 18.2. *The following holds*:

$$\begin{aligned}\overline{\theta(av, u_1)} = \big(\phi(a, u)t + Q(a)tq(u) + q(\pi(a))q(u) + f\big(h(a, b), \theta(a, v)\big)\big)u \\ + \big(\phi(a, u) - q(u)t - f(h(a, b), v)\big)\theta(a, u) \\ + \big(\phi(a, u) - q(u)r\big)h\big(a\pi(a), b\big) \\ + \big(\phi(a, u)r - q(u)r^2 + \rho q(u)\big)h(a, b).\end{aligned}$$

Proof. By (C4), and Lemmas 12.13 and 13.8, we have

$$\begin{aligned}\overline{\theta(av, u_1)} &= \rho\overline{\theta(au, u_1)} \\ &= \overline{\theta\big(au, \overline{h(aP, b)} + t\bar{u} + \overline{\theta(a, u)}\big)} \\ &= q(u)\theta\big(a, h(aP, b) + tu + \theta(a, u)\big) \\ &\quad - f\big(u, h(aP, b) + tu + \theta(a, u)\big)\theta(a, u) \\ &\quad + f\big(\theta(a, u), h(aP, b) + tu + \theta(a, u)\big)u \\ &\quad + \phi(a, u)\big(h(aP, b) + tu + \theta(a, u)\big).\end{aligned}$$

By Proposition A.4(iii),

$$\theta\big(a, h(aP, b)\big) = \rho h(a, b) - rh(aP, b).$$

By Lemma 13.16 and [Weiss 2006b, 4.9(i), 4.21 and 4.22], therefore, the claim follows. □

Proposition 18.3. *The following hold*:

(i) $h(b, av\bar{u}) = q(u)h(b, aP)$.

(ii) $h(b, av\overline{\theta(a, u)}) = q(u)\big(\big(q(\pi(a)) + Q(a)r\big)h(b, a) - rh\big(b, a\pi(a)\big)\big)$.

(iii) $avh(b, aP) = \rho ah(a, b)\bar{u} - f\big(h(a, b), v\big)aP$.

(iv) $h\big(b, avh(b, aP)\big) = \rho h\big(b, ah(a, b)\bar{u}\big) - f\big(h(a, b), v\big)h(b, aP)$.

Proof. Since $av\bar{u} = aPu\bar{u} = q(u)aP$, (i) holds. By [Weiss 2006b, 4.9(i) and 4.20], we have

$$\begin{aligned} av\overline{\theta(a,u)} &= aPu\overline{\theta(a,u)} \\ &= -aP\theta(a,u)\bar{u} + f(\theta(a,u),u)aP \\ &= a(\overline{\pi(a)} + t\varepsilon)\theta(a,u)\bar{u} + f(\theta(a,u),u)aP \\ &= q(\pi(a))au\bar{u} + ta\pi(a)u\bar{u} + Q(a)q(u)aP \\ &= q(u)\big(q(\pi(a))a + ta\pi(a) + Q(a)a(\pi(a)+r\varepsilon)\big). \end{aligned}$$

Thus (ii) holds.

Next note that

$$\begin{aligned} (18.4)\qquad avh(b,aP) &= aPuh(b,aP) \\ &= -aP\overline{h(b,aP)}\bar{u} + f\big(\overline{h(b,aP)},u\big)aP \\ &= aPh(aP,b)\bar{u} - f\big(h(aP,b),u\big)aP \\ &= aPh(aP,b)\bar{u} - f\big(h(a,b),v\big)aP. \end{aligned}$$

By Proposition A.4(i) and two applications of [Weiss 2006b, 3.22], we have

$$\begin{aligned} (18.5)\qquad aPh(aP,b)\bar{u} &= aPh(aP,b\bar{u}) + b\theta(aP,\bar{u}) - b\pi(aP)\bar{u} \\ &= aPh(aP,b\bar{u}) + \rho\big(b\theta(a,\bar{u}) - b\pi(a)\bar{u}\big) \\ &= aPh(aP,b\bar{u}) + \rho\big(ah(a,b)\bar{u} - ah(a,b\bar{u})\big) \end{aligned}$$

and by Proposition A.4(iii), we have

$$\begin{aligned} aPh(aP,b\bar{u}) &= a(\pi(a)+r\varepsilon)h(aP,b\bar{u}) \\ &= a\theta\big(a,h(aP,b\bar{u})\big) + rah(aP,b\bar{u}) \\ &= \rho ah(a,b\bar{u}). \end{aligned}$$

By (18.4) and (18.5), therefore,

$$avh(b,aP) = \rho ah(a,b)\bar{u} - f\big(h(a,b),v\big)aP.$$

Thus (iii) holds. The assertion (iv) follows immediately. □

Corollary 18.6. *The following holds*:

$$\begin{aligned} \rho\overline{h\big(b,avu_1\big)} = \rho h\big(ah(a,b)\bar{u},b\big) &+ \big((r-t)q(u) - f(h(a,b),v)\big)h(a\pi(a),b) \\ &+ \big(r^2q(u) - q(\pi(a))q(u) - rf(h(a,b),v)\big)h(a,b). \end{aligned}$$

Proof. This holds by Lemma 12.13 and Proposition 18.3. □

Proposition 18.7. *We have* $\rho\overline{\theta(a_1,u_1)} = \nu + \rho^{-1}\zeta$, *where*

$$\nu = -\overline{\theta\big(b,h(b,aP)\big)} + t\overline{\theta(b,\bar{u})} + \overline{\theta\big(b,\overline{\theta(a,u)}\big)} + q(u)h(a,b) + h\big(ah(a,b)\bar{u},b\big)$$

and

$$\begin{aligned}\zeta = \big(\phi(a,u)t + Q(a)tq(u) + q(\pi(a))q(u) + f\big(h(a,b),\theta(a,v)\big)\big)u \\ + \big(\phi(a,u) - q(u)t - f(h(a,b),v)\big)\theta(a,u) \\ + \big(r\phi(a,u) - q(\pi(a))q(u) - rf(h(a,b),v)\big)h(a,b) \\ + \big(\phi(a,u) - q(u)t - f(h(a,b),v)\big)h(a\pi(a),b) \\ - g(b,av)\big(h(aP,b) + tu + \theta(a,u)\big).\end{aligned}$$

Proof. This follows from (C3), Proposition 12.15(i) and (iii), Lemma 18.2 and Corollary 18.6. ☐

Proposition 18.8. *Let u_2 be as in Notation 16.1. Then*

$$u_2 = \frac{\delta}{\Theta},$$

where

$$\begin{aligned}\delta = t\overline{\theta(b,\bar{u})} - q(u)u - h\big(a, a\overline{\theta(b,\bar{u})}\big) + \overline{\theta(b,\overline{\theta(a,u)})} + q(u)h(a,b) \\ + 2h\big(b\bar{u}, ah(a,b)\big) + 2q(h(a,b))u - sv\end{aligned}$$

Proof. By Proposition 12.15 and (14.2), we have

$$\begin{aligned}u_2 &= \overline{h(-b_2,b_1)} + \frac{\rho}{\Theta}\big(t_1\bar{u}_1 + \overline{\theta(a_1,u_1)}\big) \\ &= h(b_1,b_2) + \frac{\rho}{\Theta}\big(t_1\bar{u}_1 + \overline{\theta(a_1,u_1)}\big).\end{aligned}$$

The sum of the elements ξ in Proposition 17.8, β in Proposition 18.1 and ζ in Proposition 18.7 is

$$\iota := f\big(h(a,b),\theta(a,v) + tv\big)h(a,b) - \rho q(u)u.$$

By Proposition 13.1, we have

$$\rho^{-1}\iota = -f\big(h(a,b),u\big)h(a,b) - q(u)u$$

and thus the sum of $\rho^{-1}\iota$ with the elements ω in Proposition 17.8, α in Proposition 18.1 and ν in Proposition 18.7 is

$$\begin{aligned}t\overline{\theta(b,\bar{u})} - \overline{\theta(b,h(b,aP))} - q(u)u - h\big(a, a\overline{\theta(b,\bar{u})}\big) \\ + \overline{\theta(b,\overline{\theta(a,u)})} + q(u)h(a,b) + h\big(ah(a,b)\bar{u}, b\big) \\ + h\big(b\bar{u}, ah(a,b)\big) - sv - h\big(aP, b\overline{\pi(b)}\big).\end{aligned}$$

By Proposition A.6(i), we have

$$\begin{aligned}\overline{\theta\big(b,h(b,aP)\big)} &= -\overline{h(b\pi(b),aP)} + Q(b)\overline{h(b,aP)} \\ &= h\big(aP, b\pi(b)\big) + Q(b)h(aP,b) = -h\big(aP, b\overline{\pi(b)}\big)\end{aligned}$$

and by (B2) and [Weiss 2006b, 3.6], we have

$$\begin{aligned} h\big(ah(a,b)\bar{u},b\big) &= -\overline{h\big(b,ah(a,b)\bar{u}\big)} \\ &= -\overline{h\big(ah(a,b),b\bar{u}\big)} - f\big(h(b,ah(a,b)),\varepsilon\big)u \\ &= h\big(b\bar{u},ah(a,b)\big) + f\big(h(a,b),h(a,b)\big)u \\ &= h\big(b\bar{u},ah(a,b)\big) + 2q(h(a,b))u. \end{aligned}$$

Hence the claim holds. □

19. A formula for τ

We are now in a position to determine a formula for the structure map of the Moufang set $M(\Delta,\omega)$ defined in Definition 10.8 (modulo Conjecture 19.14). Our main result is Theorem 19.7.

We continue with all the notation and assumptions in Hypothesis 9.1. Let ω be as in Proposition 9.2, let ξ, ψ, σ, η and λ be as in Notation 9.3, let ν be as in Notation 10.4 and let τ be as in Notation 10.7.

Proposition 19.1. $\varepsilon = \eta^{\sigma+1}\xi(\bar{\lambda}).$

Proof. By Proposition 9.5(v), $\xi(\bar{\lambda}) = \xi(\overline{\xi(\varepsilon)}) = \xi(\overline{\xi(\bar{\varepsilon})}) = q(\lambda)\varepsilon$. Hence by Proposition 9.5(iii), the claim holds. □

Proposition 19.2. *Let $x_1(a,t)x_2(u)x_3(b,s) \in C_{U_{[1,3]}}(\omega)$. Then*

(i) $\pi(a) = \eta^{\sigma+2}\xi(\theta(b,\bar{\lambda})) + M(b)\varepsilon.$

(ii) $\eta^{\sigma+1} f\big(\xi(\theta(b,\bar{\lambda})),\varepsilon\big) = Q(b)^{\sigma}.$

(iii) $q(\pi(a)+t\varepsilon) = \eta^2 q(\pi(b)+s\varepsilon)^{\sigma}.$

(iv) $Q(a) = \eta Q(b)^{\sigma}.$

Proof. By Proposition 9.15, we have $a = \psi(b)$ and

$$q(\pi(a)+t\varepsilon) = q\big(\pi(\psi(b)) + (\eta s^{\sigma} + M(b))\varepsilon\big) \tag{19.3}$$

and by Propositions 7.13(d) and 19.1, we have

$$\begin{aligned} \pi(\psi(b)) &= \theta(\psi(b),\varepsilon) \\ &= \eta^{\sigma+1}\theta(\psi(b),\xi(\bar{\lambda})) \\ &= \eta^{\sigma+2}\xi(\theta(b,\bar{\lambda})) + M(b)\varepsilon. \end{aligned}$$

Thus (i) holds. Since $M(b) = 0$ if $\mathrm{char}(K) \neq 2$, it follows that

$$\begin{aligned} &q\big(\pi(\psi(b)) + (\eta s^{\sigma} + M(b))\varepsilon\big) \\ &\quad = q\big(\eta^{\sigma+2}\xi(\theta(b,\bar{\lambda})) + \eta s^{\sigma}\varepsilon\big) \\ &\quad = \eta^{2\sigma+4} q(\xi(\theta(b,\bar{\lambda}))) + \eta^{\sigma+3}s^{\sigma} f\big(\xi(\theta(b,\bar{\lambda})),\varepsilon\big) + \eta^2 s^{2\sigma}. \end{aligned} \tag{19.4}$$

By Propositions 9.4, 9.5(iii) and 19.1, we have

$$\begin{aligned} f\big(\xi(\theta(b,\bar\lambda)),\varepsilon\big) &= \eta^{\sigma+1} f\big(\xi(\theta(b,\bar\lambda)),\xi(\bar\lambda)\big) \\ &= f(\theta(b,\bar\lambda),\bar\lambda)^{\sigma}. \end{aligned}$$

By [Weiss 2006b, 4.9], we have $f(\theta(b,\bar\lambda),\bar\lambda) = Q(b)q(\lambda)$. Therefore (ii) holds and thus

$$\eta^{\sigma+3} s^{\sigma} f\big(\xi(\theta(b,\bar\lambda)),\varepsilon\big) = \eta^2 s^{\sigma} Q(b)^{\sigma}. \tag{19.5}$$

By Propositions 7.13(i), 9.5(iii) and [Weiss 2006b, 4.22], we have

$$\begin{aligned} \eta^{2\sigma+4} q\big(\xi(\theta(b,\bar\lambda))\big) &= \eta^{2\sigma+4} q(\lambda) q(\theta(b,\bar\lambda))^{\sigma} \\ &= \eta^{2\sigma+4} q(\lambda)^{\sigma+1} q(\pi(b))^{\sigma} = \eta^2 q(\pi(b))^{\sigma}. \end{aligned}$$

By (19.3), (19.4) and (19.5), therefore, (iii) holds. By (i), we have

$$Q(a) = f(\pi(a),\varepsilon) = \eta^{\sigma+2} f\big(\xi(\theta(b,\bar\lambda)),\varepsilon\big).$$

By (ii), therefore, (iv) holds. □

Proposition 19.6. *Let* Θ *be as in* (14.3), *let* ψ, η, σ *and* M *be as in Notation 9.3, let* ζ *be as in Proposition 9.10 and let* $\mathcal{U}$ *be as in Notation 9.18. Let*

$$\Theta_0(b,s,u) = \Theta\big(\psi(b),\ \eta s^{\sigma} + M(b),\ u,\ b,\ s\big)$$

for all $[b,s,u] \in \mathcal{U}$. *Then*

$$\begin{aligned} \Theta_0(b,s,u) = \eta^2 q\big(\pi(b)+s\varepsilon\big)^{\sigma+1} &+ q\big(h(\psi(b),bu)\big) - q(u)^2 - q(u) f(u,\zeta(u)) \\ &- q(u) f\big(\pi(\psi(b)),\pi(b)\big) + f\big(\theta(b,\bar u),\overline{\theta(\psi(b),u)}\big) \\ &+ f(u,\varepsilon) f\big(\theta(\psi(b),u),\ \pi(b)\big) - f(\pi(b),u) f\big(\theta(\psi(b),u),\ \varepsilon\big) \\ &- f\big(\zeta(u),\overline{\theta(b,\overline{\theta(\psi(b),u)})} + s\theta(\psi(b),u)\big) \\ &+ \big(\eta(Q(b)+s)^{\sigma} + M(b)\big) f\big(\zeta(u),\overline{\theta(b,\bar u)} + su\big) \end{aligned}$$

for all $[b,s,u] \in \mathcal{U}$.

Proof. Recall that the map M in Proposition 7.13(d) is identically zero when $\mathrm{char}(K) \neq 2$ and that by Notation 9.18, $h(\psi(b),b) = \zeta(u) + u$ for all $[b,s,u] \in \mathcal{U}$. The claim holds, therefore, by Proposition 19.2 and a bit of calculation. □

Here, at last, is our formula:

Theorem 19.7. *Let* ω *be as in Proposition 9.2, let* ψ, η, σ *and* M *be as in Notation 9.3, let* ζ *be as in Proposition 9.10, let* $\mathcal{U}$ *be as in Notation 9.18 and let* τ *be as in Notation 10.7. Let* $[b,s,u] \in \mathcal{U}^*$ *and let* $\Theta_0 = \Theta_0(b,s,u)$ *be as in Proposition 19.6. Suppose that*

$$q(\pi(b)+s\varepsilon)\Theta_0(b,s,u) \neq 0. \tag{19.8}$$

Then

$$\tau([b,s,u]) = \Big[\frac{\hat{b}}{\Theta_0}, \frac{\hat{s}}{\Theta_0}, \frac{\hat{u}}{\Theta_0}\Big], \tag{19.9}$$

where

$$\begin{aligned}\hat{s} = \big(\eta s^\sigma + M(b)\big)q(u) + \eta^2 q(\pi(b)+s\varepsilon)^\sigma(Q(b)-s) + \phi(\psi(b),u)\\ + g\big(\psi(b)\theta(\psi(b),u),b\big) + \big(\eta(Q(b)-s)^\sigma + M(b)\big)g(\psi(b)u,b),\end{aligned}$$

$$\begin{aligned}\hat{b} = -q(u)\psi(b)\zeta(u) + \eta^2 q\big(\pi(b)+s\varepsilon\big)^\sigma b\big(\overline{\pi(b)}+s\varepsilon\big)\\ + \big(\eta(Q(b)-s)^\sigma + M(b)\big)\big(\psi(b)u(\overline{\pi(b)}+s\varepsilon) - bh(\psi(b)u,b)\big)\\ + \psi(b)\theta(\psi(b),u)\big(\overline{\pi(b)}+s\varepsilon\big) - bh\big(\psi(b)\theta(\psi(b),u),b\big)\\ - b\bar{u}\theta(\psi(b),u) - \big(\eta s^\sigma + M(b)\big)q(u)b.\end{aligned}$$

and

$$\begin{aligned}\hat{u} = \eta\big(\eta s^\sigma + M(b)\big)\overline{\theta(b,\bar{u})}\\ - \eta h\big(\psi(b),\psi(b)\overline{\theta(b,\bar{u})}\big) + \eta\overline{\theta\big(b,\overline{\theta(\psi(b),u)}\big)}\\ + 3\eta q(u)\zeta(u) + 2\eta^{2-\sigma}q(u)^\sigma u\\ + 2\eta h\big(b\bar{u},\psi(b)\zeta(u)\big)\\ - \eta s\theta(\psi(b),u) + \eta\big(\eta(Q(b)+s)^\sigma + M(b)\big)su.\end{aligned}$$

Proof. By Proposition 19.2(iii), $q(\pi(a)+t\varepsilon) \neq 0$, where $a = \psi(b)$ and $t = \eta s^\sigma + M(b)$ and by Notation 9.18, $h(\psi(b),b) = \zeta(u) + u$ for all $[b,s,u] \in \mathcal{U}$. By Proposition 9.10, we have $q(\zeta(u)) = \eta^{1-\sigma}q(u)^\sigma$. The claim holds, therefore, by Notation 10.4, 16.1, and Propositions 16.3, 16.4, 18.8. □

Proposition 19.10. *Let $\mathcal{U}$ be as in Notation 9.18 and Θ_0 as in Proposition 19.6. Let $(b,s,u) \in \mathcal{U}$ and suppose that either $q(\pi(b)+s\varepsilon) \neq 0$ or $(b,s) = (0,0)$ but $u \neq 0$. Then $\Theta_0(b,s,u) \neq 0$.*

Proof. Let $\nu = (m_1m_4)^2h_0$ be as in Notation 10.4, let

$$p = U_4x_1(a,t)x_2(u)x_3(b,s),$$

where $a = \psi(b)$ and $t = \eta s^\sigma + M(b)$, and let $\mathcal{O}$ and $\mathcal{B}$ be as in Proposition 11.8. By Notation 10.7,

$$p^\nu = U_4\tau(b,s,u).$$

Since h_0 stabilizes the set $U_4\mathcal{U}$, we conclude that

$$p^{(m_1m_4)^2} \in U_4\mathcal{U} \subset \mathcal{O}. \tag{19.11}$$

Suppose now that $q(\pi(b)+s\varepsilon)\neq 0$. By Proposition 19.2(iii), $q(\pi(a)+t\varepsilon)\neq 0$. By Proposition 11.8, therefore, $p^{m_1m_4}\in\mathcal{O}$. By (19.11) and a second application of Proposition 11.8, it follows that $p^{m_1m_4}\in\mathcal{B}$. Thus $q(\pi(a_1)+t_1\varepsilon)\neq 0$, where a_1 and t_1 are as in Proposition 12.15. By (14.2), therefore, $\Theta_0(b,s,u)=\Theta(a,t,u,b,s)\neq 0$.

Suppose, instead, that $(b,s)=(0,0)$ but $u\neq 0$. Thus $p=U_4x_2(u)$ and

$$\alpha := \big(U_4x_2(u),\, U_{[3,4]}x_2(u),\, U_{[2,4]},\, U_{[3,4]},\, U_4\big)$$

is a path of length 4 from p to U_4. By Theorem 6.11(v), the pair (p, U_4) is in the same G-orbit as the pair (w_4, U_4), where G is as in Hypothesis 9.1. Since

$$(w_4, w_5, U_{[2,4]}, U_{[3,4]}, U_4)$$

is a straight 4-path (i.e., a root) from w_4 to U_4, it follows that there exists a root from (w_4, U_4). By [Mühlherr and Weiss $\geq$ 2020, 1.2.28(ii) and 1.3.18], every path of length at most 4 from w_4 to U_4 is a root. Hence α is a root. In particular, α is straight and thus the vertices $U_{[3,4]}x_2(u)$ and $U_{[3,4]}$ are opposite at $U_{[2,4]}$. By [Mühlherr and Weiss 2020, 6.4(ii)], therefore, $q(u)\neq 0$. By (14.3), we have $\Theta(0,0,u,0,0)=q(u)^2\neq 0$ and hence $\Theta_0(b,s,u)\neq 0$ also in this case. □

Corollary 19.12. *Let F be as in Remark 9.12, let $\mathcal{V}=\{[b,s,u]\in\mathcal{U}\mid b=0\}$ and let*

$$q_0(0,s,u)=q(u)-\eta s^{\sigma+1}$$

for all $(0,s,u)\in\mathcal{V}$. Then q_0 is an anisotropic quadratic form on $\mathcal{V}$ as a vector space over F.

Proof. By Proposition 9.15, $[0,s,u]\in\mathcal{U}$ for $s\in K$ and $u\in L$ if and only if $\zeta(u)=-u$. Thus if $[0,s,u]\in\mathcal{V}$, then $f(u,\zeta(u))=-2q(u)$. By Remark 9.13, q_0 is a quadratic form on $\mathcal{V}$ as a vector space over F and by Proposition 19.6,

$$\Theta_0(0,s,u)=(q(u)-\eta s^{\sigma+1})^2$$

for all $(0,s,u)\in\mathcal{V}$. By Proposition 19.10, therefore, q_0 is anisotropic. □

Corollary 19.13. *The field K is infinite.*

Proof. By Remarks 7.6 and 9.12, $\dim_F q_0\geq 7$ and K/F is a quadratic extension. Over a finite field, there are no anisotropic quadratic forms of dimension greater than 2; see, for example, [Tits and Weiss 2002, 34.3]. By Corollary 19.12, therefore, F is infinite. Hence K is infinite. □

Conjecture 19.14. We conjecture that $\Theta_0(b,s,t)\neq 0$ for all $[b,s,u]\in\mathcal{U}$ such that $(b,s)\neq 0$ but $q(\pi(b)+s\varepsilon)=0$. This holds vacuously if the quadrangular algebra Ξ is anisotropic; see Notation 2.2. By Proposition 19.10, this conjecture would imply that Θ_0 is anisotropic.

Proposition 19.15. *Suppose that Conjecture 19.14 holds. Then the hypothesis* (19.8) *in Theorem 19.7 is superfluous.*

Proof. Exactly as at the end of [Mühlherr and Weiss ≥ 2020, Section 3.7], we observe that the function τ defined in Notation 10.7 is a regular map from $\mathcal{U}^*$ to itself. Since Θ_0 is anisotropic, we can let $\hat{\tau}$ denote the map from $\mathcal{U}^*$ to $U_{[1,3]}$ defined by the expression on the right-hand side of (19.9). The map $\hat{\tau}$ is also regular and by Theorem 19.7, τ and $\hat{\tau}$ agree on a Zariski dense subset of $\mathcal{U}^*$. By Corollary 19.13, K is infinite. It follows that $\tau = \hat{\tau}$. □

Conjecture 19.16. Suppose only that ω is as Proposition 9.2. We conjecture, but with less confidence, that the hypothesis that $G_0 := \langle\omega\rangle$ is as in Hypothesis 8.4 is, in fact, equivalent with the assumption that the restriction of Θ to $C_{U_{[1,4]}}(\omega)$ is anisotropic.

Conjecture 19.17. We conjecture that the form Θ_0 is, up to similarity, an invariant of the Moufang set $M(\Delta, \omega)$.

Appendix

In this appendix we assemble a few elementary properties of quadrangular algebras that cannot be found in [Weiss 2006b].

Let $\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$ be a quadrangular algebra as defined in Definition 2.1. Let $v \mapsto \bar{v}$, ϕ and π be as in (A1), (C4) and (D1).

Proposition A.1. *Let $a \in \mathcal{X}$ and $u \in L$. Then*

$$\pi(au) = q(u)\overline{\pi(a)} - f(u, \varepsilon)\overline{\theta(a, u)} + f(\theta(a, u), \varepsilon)\bar{u} + \phi(a, u)\varepsilon.$$

Proof. Set $w = \varepsilon$ in (C4). □

Proposition A.2. *For all $a \in \mathcal{X}$ and all $u \in L$,*

$$au\pi(au) = q(u)a\theta(a, u) + \phi(a, u)au.$$

Proof. This holds by [Weiss 2006a, 3.18]. (In [Weiss 2006a], it is assumed that the quadrangular algebra is anisotropic, but given [Mühlherr and Weiss 2019, Conclusion 7.5], it is straightforward to check that the proof remains valid in the present context.) □

Proposition A.3. *Let $a \in \mathcal{X}$, $t \in K$ and $u \in L$. Then the following hold*:

(i) $$q(\theta(a, u) + tu) = q(\pi(a) + t\varepsilon)q(u).$$

(ii) $$q\big(\pi(au) + tq(u) + \phi(a, u)\big) = q(\pi(a) + t\varepsilon)q(u)^2.$$

Proof. These assertions hold by [Weiss 2009, 21.10]. (In [Weiss 2009], it is assumed that the quadrangular algebra is anisotropic, but given [Mühlherr and Weiss 2019, Conclusion 7.5], it is straightforward to check that the proof remains valid in the present context.) □

Proposition A.4. *Let $a \in \mathcal{X}$, $r \in K$ and $u \in L$ and let $P = \pi(a) + r\varepsilon$. Then the following hold*:

(i) $\theta(aP, u) = q(P)\theta(a, u)$.

(ii) $\phi\big(a, \theta(a, u) + ru\big) = \phi(aP, u)$.

(iii) $q(P)h(a, b) = \theta\big(a, h(aP, b)\big) + rh(aP, b)$.

Proof. These assertions hold by [Tits and Weiss 2002, 13.67(i)–(iii)]. (Given [Mühlherr and Weiss 2019, Conclusion 7.5], it is straightforward to check that the proofs in [Tits and Weiss 2002] remain valid in the present context.) □

Remark A.5. The notion of a quadrangular algebra had not yet been formulated when [Tits and Weiss 2002] was written. The axioms defining a quadrangular algebra, however, can all be found in [Tits and Weiss 2002, Chapter 13] with the same notion as in Definition 2.1 with one small exception: If we call the function g introduced in [Tits and Weiss 2002, 13.26] for the moment g' and let g be as in (C3), then $g'(a, b) = g(b, a)$ for all $a, b \in \mathcal{X}$. See, in particular, [Tits and Weiss 2002, 13.37] and the remark (viii) on page 7 of [Weiss 2006b].

Proposition A.6. *The following hold for all $a, b \in \mathcal{X}$ and all $u \in L$, where Q is as in Notation 2.3.*

(i) $\theta(a, h(a, b)) = Q(a)h(a, b) - h(a\pi(a), b)$.

(ii) $\theta\big(a, h(a\pi(a), b)\big) = q(\pi(a))h(a, b)$.

(iii) $$\theta(a, h(au, b)) = -f(h(a, b), \varepsilon)\theta(a, u) - f\big(h(b, a), \pi(a)\big)u + Q(a)\overline{h(au, b)} + \overline{h(a\theta(a, u), b)} + f(h(a, b), u)\pi(a).$$

Proof. The assertions (i) and (ii) hold by (e) and (f) at the bottom of page 120 of [Tits and Weiss 2002] (in the proof of [Tits and Weiss 2002, 13.67](iii)). Choose $a, b \in \mathcal{X}$ and $u \in L$. First note that by [Weiss 2006b, 4.9(i)],

$$\begin{aligned} f\big(\theta(a, u), h(b, au)\big) &= f\big(h(b, au\overline{\theta(a, u)}), \varepsilon\big) \\ &= -f\big(h(b, a\theta(a, u)\bar{u}), \varepsilon\big) + f\big(h(b, a), \varepsilon\big) f\big(\theta(a, u), u\big) \\ &= -f\big(h(b, a\pi(a)u\bar{u}), \varepsilon\big) + Q(a)q(u) f\big(h(b, a), \varepsilon\big) \\ &= -q(u) f\big(h(b, a\pi(a)), \varepsilon\big) + Q(a)q(u) f\big(h(b, a), \varepsilon\big) \\ &= q(u) f\big(h(b, a), -\overline{\pi(a)} + Q(a)\varepsilon\big) \\ &= q(u) f\big(h(b, a), \pi(a)\big). \end{aligned}$$

By (C4), therefore, we have

$$\begin{aligned}\overline{\theta\big(au, h(au,b)\big)} &= q(u)\theta\big(a, \overline{h(au,b)}\big) - f\big(h(au,b), \bar{u}\big)\theta(a,u) \\ &\qquad + f\big(\theta(a,u), \overline{h(au,b)}\big)u + \phi(a,u)\overline{h(au,b)} \\ &= q(u)\theta\big(a, \overline{h(au,b)}\big) - f\big(h(au\bar{u},b), \varepsilon)\theta(a,u) \\ &\qquad - f\big(\theta(a,u), h(b,au)\big)u + \phi(a,u)\overline{h(au,b)} \\ &= q(u)f\big(h(au,b),\varepsilon)\pi(a) - q(u)\theta(a, h(au,b)) \\ &\qquad - q(u)f\big(h(a,b),\varepsilon\big)\theta(a,u) - q(u)f\big(h(b,a),\pi(a)\big)u \\ &\qquad + \phi(a,u)\overline{h(au,b)}.\end{aligned}$$

By Propositions 13.5, A.2 and (i), on the other hand, we have

$$\begin{aligned}\overline{\theta\big(au, h(au,b)\big)} &= q(u)Q(a)\overline{h(au,b)} - \overline{h(au\pi(au),b)} \\ &= q(u)Q(a)\overline{h(au,b)} - q(u)\overline{h(a\theta(a,u),b)} + \phi(a,u)\overline{h(au,b)}.\end{aligned}$$

We set these two expressions equal and delete the term $\phi(a,u)\overline{h(au,b)}$ that appears on both sides. All the remaining terms have $q(u)$ as a factor. If we assume that $q(u) \neq 0$ and delete all these factors, we obtain (iii). Thus (iii) holds under the assumption that $q(u) \neq 0$. Since every term in (iii) is linear in u and every element of L can be written as a sum $u_1 + u_2$ with $q(u_1)$ and $q(u_2)$ nonzero, we conclude that (iii) holds in general. (This is the same argument we used at the end of the proof of Proposition 17.5.) □

We leave it to the reader to confirm that if we set $u = \varepsilon$ and $u = \pi(a)$ in Proposition A.6(iii), we obtain Proposition A.6(i) and (ii), respectively.

Proposition A.7. *Let $a, b \in \mathcal{X}$ and $u \in L$. Then $q(h(a,bu)) = q(h(b,au))$.*

Proof. By (B2), we have

$$q(h(b,au)) = q(h(a,bu)) + f(h(b,a),\varepsilon)f(h(a,bu),u) + f(h(b,a),\varepsilon)^2 q(u).$$

By (A3) and [Weiss 2006b, 3.7],

$$f(h(a,bu),u) = f(h(a,bu\bar{u}),\varepsilon) = f(h(a,b),\varepsilon)q(u)$$

and by [Weiss 2006b, 3.6],

$$f(h(a,b),\varepsilon) = f(\overline{h(a,b)},\bar{\varepsilon}) = -f(h(b,a),\varepsilon).$$

Hence $f(h(b,a),\varepsilon)f(h(a,bu),u) + f(h(b,a),\varepsilon)^2 q(u) = 0$. □

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