ZEROS OF p-ADIC HYPERGEOMETRIC FUNCTIONS, p-ADIC ANALOGUES OF KUMMER'S AND PFAFF'S IDENTITIES

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ZEROS OF *p*-ADIC HYPERGEOMETRIC FUNCTIONS, *p*-ADIC ANALOGUES OF KUMMER'S AND PFAFF'S IDENTITIES

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We classify all the zeros and nonzero values of a family of hypergeometric series in the *p*-adic setting. These values of hypergeometric series in the *p*-adic setting lead to transformations of hypergeometric series in the *p*-adic setting which can be described as *p*-adic analogues of Kummer's and Pfaff's linear transformations on classical hypergeometric series. We also evaluate certain summation identities for hypergeometric series in the *p*-adic setting as well as Gaussian hypergeometric series.

1. Introduction and statement of results

The main goal of this paper is to study zeros of hypergeometric series in the *p*-adic setting introduced by D. McCarthy [2012a; 2013]. We also establish analogues of classical hypergeometric series transformations, particularly very special cases of Kummer's and Pfaff's linear transformations, for hypergeometric series in the *p*-adic setting. These types of questions were posed by McCarthy [2013]. We now begin with the definition of classical hypergeometric series. For a complex number a and a nonnegative integer k the rising factorial denoted by $(a)_k$ is defined by $(a)_k := a(a+1)(a+2)\cdots(a+k-1)$ for $k > 0$ and $(a)_0 := 1$. Then for $a_i, b_i, \lambda \in \mathbb{C}$ with $b_i \notin \{\ldots, -3, -2, -1, 0\}$, the classical hypergeometric series $_{r+1}F_r$ is defined by

$$_{r+1}F_r\left(\begin{array}{cccc} a_1, & a_2, & \ldots, & a_{r+1} \\ & b_1, & \ldots, & b_r \end{array} \,\middle|\, \lambda\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{r+1})_k}{(b_1)_k \cdots (b_r)_k} \cdot \frac{\lambda^k}{k!}.$$

This series converges for $|\lambda| < 1$. Classical hypergeometric series play important role in different areas of mathematics. For example, they have significant applications in modular forms, elliptic curves, representation theory, differential equations etc. [McCarthy 2012b; 2010; Mortenson 2005]. J. Greene [1987] introduced the notion

of hypergeometric series over finite fields which are finite field analogues of classical hypergeometric series. Let p be an odd prime and $\mathbb{F}_p$ denote a finite field with p elements. Let $\widehat{\mathbb{F}_p^\times}$ denote the group of all multiplicative characters of $\mathbb{F}_p^\times$ and $\bar{\chi}$ denote the inverse of a multiplicative character χ. We extend the domain of each $\chi \in \widehat{\mathbb{F}_p^\times}$ to $\mathbb{F}_p$ by simply setting $\chi(0) := 0$ including the trivial character ε. For multiplicative characters χ and ψ of $\mathbb{F}_p$ the Jacobi sum is defined by

$$
(1\text{-}1) \qquad\qquad J(\chi, \psi) := \sum_{y \in \mathbb{F}_p} \chi(y)\psi(1-y),
$$

and the normalized Jacobi sum known as binomial is defined by

$$
(1\text{-}2) \qquad\qquad \binom{\chi}{\psi} := \frac{\psi(-1)}{p} J(\chi, \bar{\psi}).
$$

Let n be a nonnegative integer. For multiplicative characters $A_1, A_2, \ldots, A_{n+1}$, and $B_1, B_2, \ldots, B_n$ of $\mathbb{F}_p$ with $t \in \mathbb{F}_p$, Greene [1987] defined the $_{n+1}F_n(\cdots)$ hypergeometric function over finite field $\mathbb{F}_p$ by

$$
_{n+1}F_n\left(\begin{array}{cccc} A_1, & A_2, & \ldots, & A_{n+1} \\ & B_1, & \ldots, & B_n \end{array} \,\middle|\, t \right) := \frac{p}{p-1} \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \binom{A_1\chi}{\chi} \cdots \binom{A_{n+1}\chi}{B_n\chi} \chi(t).
$$

This function is also known as Gaussian hypergeometric function. These functions were developed to have a parallel study of character sums analogous to special functions. Gaussian hypergeometric functions satisfy many identities which are often analogues of classical hypergeometric series identities. For more details, see [Greene 1987]. Since the entries of the Gaussian hypergeometric function are multiplicative characters so results involving Gaussian hypergeometric functions often be restricted to primes in certain congruence classes for the existence of characters of specific orders, see for example [Evans and Greene 2009; Fuselier 2010; Lennon 2011a; 2011b]. To overcome these limitations, McCarthy [2012a; 2013] defined a function $_nG_n[\cdots]$ in terms of quotients of p-adic gamma functions which can be best described as an analogue of classical hypergeometric series in the p-adic setting. Let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. Let $\Gamma_p(\cdot)$ denote the Morita's p-adic gamma function. Let ω denote the Teichmüller character of $\mathbb{F}_p$, satisfying $\omega(a) \equiv a \pmod{p}$, and $\bar{\omega}$ denote the character inverse of ω. For $x \in \mathbb{Q}$ let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote the fractional part of x, satisfying $0 \le \langle x \rangle < 1$. We now recall the McCarthy hypergeometric function $_nG_n[\cdots]$ in the p-adic setting.

Definition 1.1 [McCarthy 2013, Definition 5.1]. Let p be an odd prime and $t \in \mathbb{F}_p$. For positive integer n and $1 \le k \le n$, let $a_k, b_k \in \mathbb{Q} \cap \mathbb{Z}_p$. Then the function $_nG_n[\cdots]$

is defined as

$$
{}_nG_n\begin{bmatrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{bmatrix} t \Big] := \frac{-1}{p-1} \sum_{a=0}^{p-2} (-1)^{an}\ \bar{\omega}^a(t)
$$

$$
\times \prod_{k=1}^{n} (-p)^{-\lfloor \langle a_k \rangle - a/(p-1) \rfloor - \lfloor \langle -b_k \rangle + a/(p-1) \rfloor} \frac{\Gamma_p\big(\langle a_k - \frac{a}{p-1}\rangle\big)\,\Gamma_p\big(\langle -b_k + \frac{a}{p-1}\rangle\big)}{\Gamma_p(\langle a_k \rangle)} \frac{}{\Gamma_p(\langle -b_k \rangle)}.
$$

This function is also known as p-adic hypergeometric function. It is clear from the Definition 1.1 that the value of the ${}_nG_n[\cdots]$ function depends only on the fractional part of the parameters a_k and b_k. Therefore, we may assume that $0 \le a_k, b_k < 1$. Gaussian hypergeometric functions satisfy many powerful transformation formulas that are often mirror symmetrical to their classical counterparts, for details see [Greene 1987]. Note that these results can be converted into identities involving ${}_nG_n[\cdots]$ via the transformations [McCarthy 2013, Lemma 3.3; 2012c, Proposition 2.5] between finite field hypergeometric function and p-adic hypergeometric series. However, the new identities involving ${}_nG_n[\cdots]$ will be valid for the primes p, where the original characters existed over $\mathbb{F}_p$. Therefore, it will be interesting to extend such results to almost all primes. Fuselier and McCarthy [2016] established certain transformation identities for p-adic hypergeometric series in full generality. In particular, they proved a transformation result analogous to a Whipple's result for ${}_3F_2$-classical hypergeometric series. These transformations eventually leaded to settle one supercongruence conjecture of Rodriguez-Villegas between a truncated ${}_4F_3$-classical hypergeometric series and the Fourier coefficients of a certain weight four modular form. This is one of the motivation to study transformation formulas with the expectation that transformation formulas will lead to new identities. Let χ_4 be a multiplicative character of $\mathbb{F}_p$ of order 4. Also, let φ be the quadratic character of $\mathbb{F}_p$. Consider the classical hypergeometric series

$$
{}_2F_1\!\left(\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ & \frac{1}{2} \end{matrix} \,\Big|\, t \right).
$$

Then the finite field analogue of this series can be considered as ${}_2F_1\big(\begin{smallmatrix} \chi_4, & \chi_4^3 \\ & \varphi \end{smallmatrix} \,|\, t\big)$. Using the transformations [McCarthy 2013, Lemma 3.3; 2012c, Proposition 2.5] the function

$$
{}_2G_2\begin{bmatrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{bmatrix} \frac{1}{t} \Big]
$$

can be described as the p-adic analogue of the classical hypergeometric series

$$
{}_2F_1\!\left(\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ & \frac{1}{2} \end{matrix} \,\Big|\, t \right).
$$

We know that classical hypergeometric series satisfy many powerful identities. For example, Gauss [1812], Kummer [1836], Whipple [Lidl and Niederreiter 1983, p. 54], Saalschütz [Slater 1966, p. 49], Dixon [Slater 1966, p. 51], and Watson [Slater 1966, p. 54] studied special values of classical hypergeometric series. For instance, the following evaluation of classical hypergeometric series in terms of quotients of classical gamma function was due to Gauss [1812]. If $R(c - a - b) > 0$ then

$$(1\text{-}3) \qquad {}_2F_1\left(\begin{array}{cc} a, & b \\ & c \end{array} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

If we put $a = \frac{1}{4}$, $b = \frac{3}{4}$ and $c = 1 + \frac{1}{2}$ in (1-3) then we have

$$(1\text{-}4) \qquad {}_2F_1\left(\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ & 1 + \frac{1}{2} \end{array} \middle| 1 \right) = \frac{\Gamma\left(1 + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1 + \frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}.$$

Also, consider Kummer's theorem [1836]:

$$(1\text{-}5) \qquad {}_2F_1\left(\begin{array}{cc} a, & b \\ & 1 + b - a \end{array} \middle| -1 \right) = \frac{\Gamma(1 + b - a)\Gamma\left(1 + \frac{b}{2}\right)}{\Gamma(1 + b)\Gamma\left(1 + \frac{b}{2} - a\right)}.$$

Putting $a = \frac{1}{4}$ and $b = \frac{3}{4}$ into (1-5) we have

$$(1\text{-}6) \qquad {}_2F_1\left(\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ & 1 + \frac{1}{2} \end{array} \middle| -1 \right) = \frac{\Gamma\left(1 + \frac{1}{2}\right)\Gamma\left(1 + \frac{3}{8}\right)}{\Gamma\left(1 + \frac{3}{4}\right)\Gamma\left(1 + \frac{1}{8}\right)}.$$

Classical hypergeometric series with dihedral monodromy group can be expressed as elementary functions as their hypergeometric equations can be reformulated to Fuchsian equations with cyclic monodromy groups. For example, two interesting cases that can be expressed as square roots inside powers are:

$$(1\text{-}7) \qquad {}_2F_1\left(\begin{array}{cc} \frac{a}{2}, & \frac{a+1}{2} \\ & a + 1 \end{array} \middle| z \right) = \left(\frac{1 + \sqrt{1 - z}}{2} \right)^{-a},$$

and

$$(1\text{-}8) \qquad {}_2F_1\left(\begin{array}{cc} \frac{a}{2}, & \frac{a+1}{2} \\ & \frac{1}{2} \end{array} \middle| z \right) = \frac{(1 - \sqrt{z})^{-a} + (1 + \sqrt{z})^{-a}}{2}.$$

All these evaluations of Gauss, Kummer etc. motivate us to study the special values of p-adic hypergeometric series ${}_2G_2[\cdots]$. Indeed, we completely determine all the possible zeros and nonzero values of a certain family of ${}_2G_2[\cdots]$. We first discuss a theorem that classify all the zeros and nonzero values of the function ${}_2G_2[\cdots]$. For brevity we write $a \neq \square$ if a is not square in $\mathbb{F}_p$.

Theorem 1.2. *Let $p \geq 3$ be a prime and $t \in \mathbb{F}_p^\times$. Then we have the following values.*

(1)

(1-9)
$$
{}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \,\middle|\, 1\right] = \begin{cases} 1 & \textit{if } p \equiv \pm 1 \ (\mathrm{mod}\ 8), \\ -1 & \textit{if } p \equiv \pm 3 \ (\mathrm{mod}\ 8). \end{cases}
$$

(2) *Let $t \neq 1$ and $\frac{t-1}{t}$ be a square in $\mathbb{F}_p^\times$ such that $\frac{t-1}{t} = a^2$ for some $a \in \mathbb{F}_p^\times$. Then we have*

(1-10)
$$
{}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \,\middle|\, t\right] = \varphi(2) \cdot (\varphi(1+a) + \varphi(1-a)).
$$

(3) *If $\frac{t-1}{t} \neq \square$ in $\mathbb{F}_p$ then*

(1-11)
$$
{}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \,\middle|\, t\right] = 0.
$$

From Theorem 1.2 we obtain the following corollary which explicitly states whenever the function

$$
{}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \,\middle|\, t\right]
$$

is zero and nonzero.

Corollary 1.3. *Let $p \geq 3$ be a prime and $t \in \mathbb{F}_p^\times$. Then*

$$
{}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \,\middle|\, t\right] = 0
$$

if $\frac{t-1}{t} = a^2$ for some $a \in \mathbb{F}_p^\times$ with $\varphi(1+a) = -\varphi(1-a)$ or if $\frac{t-1}{t} \neq \square$ in $\mathbb{F}_p^\times$. On the other hand,

$$
{}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \,\middle|\, t\right] \neq 0
$$

if $t = 1$ or if $\frac{t-1}{t} = a^2$ for some $a \in \mathbb{F}_p^\times$ with $\varphi(1+a) \neq -\varphi(1-a)$.

As a consequence of the Theorem 1.2 we state the following corollary.

Corollary 1.4. *Let $p \geq 3$ be a prime and $t \in \mathbb{F}_p^\times$. Then the only possible values the function*

$$
{}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \,\middle|\, t\right]
$$

can take are $0, \pm 1$ and ± 2.

Remark 1.5. Note that (1-9) can be described as a *p*-adic analogue of (1-4).
Theorem 1.2 provides a *p*-adic analogue of (1-6). The value of the function

$$ {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; -1 \right] $$

completely depends on the prime as if $p \equiv \pm 3 \pmod 8$ then it will be equal to
zero. However, (1-6) cannot be equal to zero. Theorem 1.2 can also be described
as *p*-adic analogue of (1-7) and (1-8) for $a = \frac{1}{2}$.

Another purpose of this paper is to establish *p*-adic analogous of the Kummer's
linear transformation [Bailey 1935, p. 4 Equation (1)]:

$$ (1\text{-}12) \quad {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \;\middle|\; z\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \cdot {}_2F_1\left(\begin{matrix} a, & b \\ & 1+a+b-c \end{matrix} \;\middle|\; 1-z\right) $$

$$ + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} \cdot {}_2F_1\left(\begin{matrix} c-a, & c-b \\ & 1+c-a-b \end{matrix} \;\middle|\; 1-z\right). $$

The next theorem provides transformations of *p*-adic hypergeometric series which
can be described as *p*-adic analogue of a particular case of Kummer's linear trans-
formation (1-12). This theorem is obtained as a consequence of Theorem 1.2.

Theorem 1.6. *Let $p \geq 3$ be a prime and $x \in \mathbb{F}_p$ be such that $x \neq 0, 1$. Then we
have the following*:

(1) *If x and $1 - x$ are not squares in $\mathbb{F}_p$ then*

$$ (1\text{-}13) \qquad {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; \frac{1}{x} \right] = {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; \frac{1}{1-x} \right]. $$

(2) *If $x = b^2$ for some $b \in \mathbb{F}_p$ and $1 - x$ is not a square in $\mathbb{F}_p$ then*

$$ (1\text{-}14) \quad {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; \frac{1}{x} \right] = {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; \frac{1}{1-x} \right] - \varphi(2)(\varphi(1+b)+\varphi(1-b)). $$

(3) *If x, and $1 - x$ are both squares such that $1 - x = a^2$ and $x = b^2$ for some
$a, b \in \mathbb{F}_p$ then*

$$ (1\text{-}15) \quad (\varphi(1+b)+\varphi(1-b)){}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; \frac{1}{x} \right] $$

$$ = (\varphi(1+a)+\varphi(1-a)){}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; \frac{1}{1-x} \right]. $$

(4) *If $1 - x = a^2$ for some $a \in \mathbb{F}_p$ and x is not a square in $\mathbb{F}_p$ then*

$$ (1\text{-}16) \quad {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; \frac{1}{x} \right] = {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \;\middle|\; \frac{1}{1-x} \right] + \varphi(2)(\varphi(1+a)+\varphi(1-a)). $$

Note that the finite field analogue of Theorem 1.6 involving characters of order 4 follows from Greene's evaluation [1987, Theorem 4.4(i)] and if we use this result of Greene along with the relations [McCarthy 2013, Lemma 3.3; 2012c, Proposition 2.5] then we also obtain a similar transformation for the p-adic hypergeometric series

$$_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| t\right]$$

under the condition that $p \equiv 1 \pmod 4$. However, Theorem 1.6 has no congruence condition on primes.

Fuselier and McCarthy [2016] evaluated certain summation identities for p-adic hypergeometric series. This motivates us to study summation identities of p-adic hypergeometric series.

Theorem 1.7. *Let $p \geq 3$ be a prime. Let $x \in \mathbb{F}_p^\times$. Then we have the following.*

(1)

(1-17) $$\sum_{t \in \mathbb{F}_p^\times} \varphi(t(t-1)) \, _2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{array} \middle| t\right] = -1 - p \cdot \varphi(2).$$

(2) *If $x \neq 1$ and $1-x$ is not a square in $\mathbb{F}_p$ then we have*

(1-18) $$\sum_{t \in \mathbb{F}_p^\times} \varphi(t(t-1)) \, _2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{array} \middle| \frac{t}{x}\right] = -1.$$

(3) *If $x \neq 1$ and $1-x = a^2$ for some $a \in \mathbb{F}_p$ then*

(1-19) $$\sum_{t \in \mathbb{F}_p^\times} \varphi(t(t-1)) \, _2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{array} \middle| \frac{t}{x}\right] = -1 - p\varphi(2)(\varphi(1+a) + \varphi(1-a)).$$

The following theorem gives a summation identity of Gaussian hypergeometric functions involving characters of order 4.

Theorem 1.8. *Let $p \geq 3$ be a prime such that $p \equiv 1 \pmod 4$. Let $x \in \mathbb{F}_p^\times$ and χ_4 be a multiplicative character of $\mathbb{F}_p$ of order 4. Then we have the following.*

(1)

(1-20) $$\sum_{t \in \mathbb{F}_p^\times} \varphi(1-t) \, _2F_1\left(\begin{array}{cc} \chi_4, & \chi_4^3 \\ & \varepsilon \end{array} \middle| t\right) = \frac{1}{p} + \varphi(2).$$

(2) *If $x \neq 1$ and $1-x$ is not a square in $\mathbb{F}_p$ then we have*

(1-21) $$\sum_{t \in \mathbb{F}_p^\times} \varphi(x-t) \, _2F_1\left(\begin{array}{cc} \chi_4, & \chi_4^3 \\ & \varepsilon \end{array} \middle| t\right) = \frac{\varphi(x)}{p}.$$

(3) *If $x \neq 1$ and $1 - x = a^2$ for some $a \in \mathbb{F}_p^\times$ then*

$$(1\text{-}22) \quad \sum_{t \in \mathbb{F}_p^\times} \varphi(x - t) \, {}_2F_1\left(\begin{array}{cc} \chi_4, & \chi_4^3 \\ & \varepsilon \end{array} \middle| t \right) = \frac{\varphi(x)}{p} + \varphi(2x)(\varphi(1 + a) + \varphi(1 - a)).$$

Apart from Kummer's transformation there are other interesting transformation formulas exist in the literature. For example, Euler [Slater 1966, p. 10], Whipple [1925], and Dixon [1903] studied transformation properties of classical hypergeometric series. In this paper, we are interested in the Pfaff's transformation [Slater 1966, p. 31]

$$(1\text{-}23) \qquad {}_2F_1\left(\begin{array}{cc} a, & b \\ & c \end{array} \middle| z \right) = (1 - z)^{-a} \, {}_2F_1\left(\begin{array}{cc} a, & c - b \\ & c \end{array} \middle| \frac{z}{z-1} \right).$$

In particular, if $a = \frac{1}{4}$, $b = \frac{3}{4}$, and $c = \frac{1}{2}$ then the above result gives

$$(1\text{-}24) \qquad {}_2F_1\left(\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ & \frac{1}{2} \end{array} \middle| z \right) = (1 - z)^{-1/4} \, {}_2F_1\left(\begin{array}{cc} \frac{1}{4}, & \frac{-1}{4} \\ & \frac{1}{2} \end{array} \middle| \frac{z}{z-1} \right).$$

We know that p-adic analogue of

$$ {}_2F_1\left(\begin{array}{cc} \frac{1}{4}, & \frac{-1}{4} \\ & \frac{1}{2} \end{array} \middle| z \right)$$

can be described as the function

$$ {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{-1}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{z} \right] = {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & 1 - \frac{1}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{z} \right].$$

Then a p-adic analogue of Pfaff's transformation (1-24) is described in the next theorem.

Theorem 1.9. *Let $p \geq 3$ be a prime and $1 \neq x \in \mathbb{F}_p^\times$. Then we have the following.*

(1) *If $1 - x \neq \square$ then*

$$(1\text{-}25) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{x} \right] = {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{x-1}{x} \right].$$

(2) *If $1 - x = a^2$ for some $a \in \mathbb{F}_p^\times$ then*

$$(1\text{-}26) \quad \varphi(a)(\varphi(a + 1) + \varphi(a - 1)) {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{x} \right]$$

$$= (\varphi(1 + a) + \varphi(1 - a)) {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{x-1}{x} \right].$$

The rest of this paper is organized as follows. We introduce some basic definitions in Section 2 including Gauss sum and p-adic gamma function. In Section 2 we state some results including the Hasse–Davenport result, and the Gross–Koblitz formula. We give the proofs of the main theorems in Section 3.

2. Notation and preliminary results

Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$. Since each $\chi \in \widehat{\mathbb{F}_p^\times}$ takes values from μ_{p-1}, the group of $(p-1)$-th roots of unity in $\mathbb{C}^\times$, and $\mathbb{Z}_p^\times$ contains μ_{p-1}, so we may assume that the multiplicative characters of $\mathbb{F}_p^\times$ to be mapped $\chi : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$. Recall that $\omega : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ is the Teichmüller character. Also, $\widehat{\mathbb{F}_p^\times} = \{\omega^j : 0 \le j \le p - 2\}$ and $\overline{\omega}$ denotes the inverse of ω.

Preliminary results on multiplicative characters and Gauss sums. The following result gives the orthogonality relation of multiplicative characters.

Lemma 2.1 [Ireland and Rosen 2005, Chapter 8]. *Let p be an odd prime. Then we have*

$$(2\text{-}1) \qquad \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \chi(x) = \begin{cases} p - 1 & \text{if } x = 1, \\ 0 & \text{if } x \ne 1. \end{cases}$$

Let ζ_p be a fixed primitive p-th root of unity in $\overline{\mathbb{Q}}_p$. For $\chi \in \widehat{\mathbb{F}_p^\times}$, the Gauss sum is defined by

$$g(\chi) := \sum_{x \in \mathbb{F}_p} \chi(x)\, \zeta_p^x.$$

From the definition we can say that $g(\varepsilon) = -1$. For more details on Gauss sums, see [Berndt et al. 1998]. We now introduce some properties of Gauss sums. Let $\delta : \widehat{\mathbb{F}_p^\times} \to \{0, 1\}$ be defined by

$$(2\text{-}2) \qquad \delta(\chi) = \begin{cases} 1 & \text{if } \chi = \varepsilon, \\ 0 & \text{if } \chi \ne \varepsilon. \end{cases}$$

We start with a result that provides a formula for the multiplicative inverse of Gauss sum.

Lemma 2.2 [Greene 1987, Equation 1.12]. *Let $\chi \in \widehat{\mathbb{F}_p^\times}$. Then*

$$(2\text{-}3) \qquad g(\chi)g(\overline{\chi}) = p\chi(-1) - (p - 1)\delta(\chi).$$

Another important product formula for Gauss sums is the Hasse–Davenport formula.

Theorem 2.3 [Berndt et al. 1998, Hasse–Davenport relation, Theorem 11.3.5]. *Let χ be a character of order m of $\mathbb{F}_p$ for some positive integer m. For a multiplicative character ψ of $\mathbb{F}_p$ we have*

$$(2\text{-}4) \qquad \prod_{i=0}^{m-1} g(\psi \chi^i) = g(\psi^m)\psi^{-m}(m) \prod_{i=1}^{m-1} g(\chi^i).$$

The following lemma relates Gauss and Jacobi sums.

Lemma 2.4 [Greene 1987, Equation 1.14]. *Let $\chi_1, \chi_2 \in \widehat{\mathbb{F}_p^\times}$. Then*

$$(2\text{-}5) \qquad J(\chi_1, \chi_2) = \frac{g(\chi_1)g(\chi_2)}{g(\chi_1\chi_2)} + (p-1)\chi_2(-1)\delta(\chi_1\chi_2).$$

Let χ, ψ be multiplicative characters of $\mathbb{F}_p$. Then the following special values of binomials are very useful to prove our main results, for more details we refer [Greene 1987, Equations 2.12, 2.7]:

$$(2\text{-}6) \qquad \binom{\chi}{\varepsilon} = \binom{\chi}{\chi} = -\frac{1}{p} + \frac{p-1}{p}\delta(\chi),$$

$$(2\text{-}7) \qquad \binom{\chi}{\psi} = \binom{\psi\bar{\chi}}{\psi}\psi(-1).$$

p-adic preliminaries. We recall the p-adic gamma function, for further details see [Koblitz 1980]. For a positive integer n, the p-adic gamma function $\Gamma_p(n)$ is defined as

$$\Gamma_p(n) := (-1)^n \prod_{0<j<n,\; p\nmid j} j$$

and one can extend it to all $x \in \mathbb{Z}_p$ by setting $\Gamma_p(0) := 1$ and

$$\Gamma_p(x) := \lim_{x_n \to x} \Gamma_p(x_n)$$

for $x \neq 0$, where x_n runs through any sequence of positive integers p-adically approaching x. Two important product formulas of the p-adic gamma function from [Gross and Koblitz 1979] are as follows. If $x \in \mathbb{Z}_p$ then

$$(2\text{-}8) \qquad \Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)},$$

where $a_0(x) \equiv x \pmod{p}$ such that $a_0(x) \in \{1, 2, \ldots, p\}$. If $m \in \mathbb{Z}^+$, $p\nmid m$ and $x = \frac{r}{p-1}$ with $0 \le r \le p-1$ then

$$(2\text{-}9) \qquad \prod_{h=0}^{m-1} \Gamma_p\left(\frac{x+h}{m}\right) = \omega(m^{(1-x)(1-p)})\Gamma_p(x) \prod_{h=1}^{m-1} \Gamma_p\left(\frac{h}{m}\right).$$

Another interesting product formula of p-adic gamma function given in [McCarthy 2013] as a consequence of (2-9) described as follows. Let $t \in \mathbb{Z}^+$ and $p \nmid t$. Then for $0 \le j \le p-2$ we have

$$(2\text{-}10) \qquad \omega(t^{-tj})\Gamma_p\left(\left\langle \frac{-tj}{p-1} \right\rangle\right) \prod_{h=1}^{t-1} \Gamma_p\left(\frac{h}{t}\right) = \prod_{h=1}^{t} \Gamma_p\left(\left\langle \frac{h}{t} - \frac{j}{p-1} \right\rangle\right).$$

Let $\pi \in \mathbb{C}_p$ be the fixed root of the polynomial $x^{p-1} + p$, which satisfies the congruence condition $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$. The Gross–Koblitz formula relates the Gauss sum and p-adic gamma function as follows.

Theorem 2.5 Gross–Koblitz formula [1979]. *For $j \in \mathbb{Z}$,*

$$g(\bar{\omega}^j) = -\pi^{(p-1)\langle j/(p-1) \rangle} \Gamma_p\left(\left\langle \frac{j}{p-1} \right\rangle\right).$$

The next two lemmas are helpful in the proof of our main results. These two lemmas are direct applications of the Gross–Koblitz formula.

Lemma 2.6. *For $1 \le j \le p-2$*

$$(2\text{-}11) \qquad \Gamma_p\left(\left\langle 1 - \frac{j}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle \frac{j}{p-1} \right\rangle\right) = -\omega^j(-1).$$

Proof. Applying the Gross–Koblitz formula (Theorem 2.5) on the left-hand side of (2-11) and then using (2-3) it is straightforward to verify the lemma. $\qquad\square$

Lemma 2.7. *For $1 \le j \le p-2$ we have*

$$(2\text{-}12) \qquad \frac{(-p)^{-\lfloor 1/2 + j/(p-1) \rfloor}}{\Gamma_p\left(\frac{1}{2}\right)} \Gamma_p\left(\left\langle \frac{1}{2} + \frac{j}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle 1 - \frac{j}{p-1} \right\rangle\right)$$

$$= \frac{1}{p} \sum_{t \in \mathbb{F}_p^\times} \bar{\omega}^j(-t)\varphi(t(t-1)).$$

Proof. Let

$$U = \frac{(-p)^{-\lfloor 1/2 + j/(p-1) \rfloor}}{\Gamma_p\left(\frac{1}{2}\right)} \Gamma_p\left(\left\langle \frac{1}{2} + \frac{j}{p-1} \right\rangle\right) \Gamma_p\left(\left\langle 1 - \frac{j}{p-1} \right\rangle\right).$$

Using the Gross–Koblitz formula (Theorem 2.5), (2-3), and (2-5) we obtain

$$U = \frac{\varphi\omega^j(-1)}{p} J(\varphi\bar{\omega}^j, \varphi) = \frac{1}{p} \sum_{t \in \mathbb{F}_p^\times} \bar{\omega}^j(-t)\varphi(t(t-1)).$$

This completes the proof of the lemma. $\qquad\square$

3. Proof of the theorems

We begin with a proposition which explicitly determines the value of a character sum. We use this proposition to prove Theorems 1.2 and 1.7.

Proposition 3.1. *For $x \in \mathbb{F}_p^{\times}$ we have*

$$\sum_{\chi \in \widehat{\mathbb{F}_p^{\times}}} g(\varphi\chi^2)g(\varphi\bar{\chi})g(\bar{\chi})\chi\left(\frac{x}{4}\right)$$

$$= \begin{cases} 0 & \text{if } 1-x \neq \square, \\ p(p-1)\varphi(-2) & \text{if } x = 1, \\ p(p-1)\varphi(-2)(\varphi(1+a)+\varphi(1-a)) & \text{if } x \neq 1 \text{ and } 1-x = a^2. \end{cases}$$

Proof. Let

$$A = \sum_{\chi \in \widehat{\mathbb{F}_p^{\times}}} g(\varphi\chi^2)g(\varphi\bar{\chi})g(\bar{\chi})\chi\left(\frac{x}{4}\right).$$

Multiplying both numerator and denominator by $g(\varphi\chi)$ we can write

$$(3\text{-}1) \qquad A = \sum_{\chi \in \widehat{\mathbb{F}_q^{\times}}} \frac{g(\varphi\chi^2)g(\bar{\chi})}{g(\varphi\chi)} g(\varphi\chi)g(\varphi\bar{\chi})\chi\left(\frac{x}{4}\right).$$

Applying (2-5) we have

$$(3\text{-}2) \qquad \frac{g(\varphi\chi^2)g(\bar{\chi})}{g(\varphi\chi)} = J(\varphi\chi^2, \bar{\chi}) - (p-1)\chi(-1)\delta(\varphi\chi).$$

Also, applying (2-3) we have

$$(3\text{-}3) \qquad g(\varphi\chi)g(\varphi\bar{\chi}) = p\varphi\chi(-1) - (p-1)\delta(\varphi\chi).$$

Substituting (3-2) and (3-3) into (3-1) we obtain

$$(3\text{-}4) \qquad A = A_1 + A_2 + A_3 + A_4,$$

where

$$(3\text{-}5) \qquad A_1 = p\varphi(-1) \sum_{\chi \in \widehat{\mathbb{F}_p^{\times}}} J(\varphi\chi^2, \bar{\chi})\chi\left(\frac{-x}{4}\right),$$

$$(3\text{-}6) \qquad A_2 = -(p-1) \sum_{\chi \in \widehat{\mathbb{F}_p^{\times}}} \delta(\varphi\chi)J(\varphi\chi^2, \bar{\chi})\chi\left(\frac{x}{4}\right)$$

$$= -(p-1)\varphi(x)J(\varphi, \varphi),$$

$$(3\text{-}7) \qquad A_3 = -p(p-1)\varphi(-1) \sum_{\chi \in \widehat{\mathbb{F}_p^{\times}}} \delta(\varphi\chi)\chi\left(\frac{x}{4}\right)$$

$$= -p(p-1)\varphi(-x),$$

(3-8)
$$A_4 = (p-1)^2 \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \delta(\varphi\chi)\chi\left(\frac{-x}{4}\right)$$

$$= (p-1)^2 \varphi(-x).$$

Using (1-2), and (2-6) in (3-6) we have

(3-9)
$$A_2 = -p(p-1)\varphi(-x)\binom{\varphi}{\varphi}$$

$$= (p-1)\varphi(-x).$$

Adding (3-7), (3-8), and (3-9) we obtain

(3-10)
$$A_2 + A_3 + A_4 = 0.$$

Substituting (3-10) into (3-4) we have

$$A = A_1 = p\varphi(-1) \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} J(\varphi\chi^2, \bar{\chi})\chi\left(\frac{-x}{4}\right).$$

(1-2) gives

$$A = p^2\varphi(-1) \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \binom{\varphi\chi^2}{\chi}\chi\left(\frac{x}{4}\right).$$

If we use (2-7) then we have $\binom{\varphi\chi^2}{\chi} = \chi(-1)\binom{\varphi\bar{\chi}}{\chi}$. This yields

(3-11)
$$A = p^2\varphi(-1) \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \binom{\varphi\bar{\chi}}{\chi}\chi\left(\frac{-x}{4}\right)$$

$$= p\varphi(-1) \sum_{\chi \in \widehat{\mathbb{F}_p^\times}, \, y \in \mathbb{F}_p} \varphi(y)\bar{\chi}(y)\bar{\chi}(1-y)\chi\left(\frac{x}{4}\right).$$

Replacing χ by $\bar{\chi}$ in (3-11) we obtain

(3-12)
$$A = p\varphi(-1) \sum_{y \in \mathbb{F}_p} \varphi(y) \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} \chi\left(\frac{4y(1-y)}{x}\right).$$

Using the orthogonality relation (2-1) we can say that second sum present in (3-12) is nonzero if and only if the equation $4y^2 - 4y + x = 0$ admits a solution. We know that $4y^2 - 4y + x = 0$ is solvable if and only if $1 - x$ is a square in $\mathbb{F}_p$. Let $1 - x = a^2$ for some $a \in \mathbb{F}_p$. Then $\frac{1}{2}(1 \pm a)$ are solutions of $4y^2 - 4y + x = 0$. Hence, we obtain

$$A = \begin{cases} p(p-1)\varphi(-2) & \text{if } x = 1, \\ p(p-1)\varphi(-2)(\varphi(1+a) + \varphi(1-a)) & \text{if } x \neq 1 \text{ and } 1 - x = a^2, \\ 0 & \text{if } 1 - x \neq \square. \end{cases}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the next proposition, we again consider the same character sum as considered in Proposition 3.1 and express the sum as a special value of p-adic hypergeometric series. We use this proposition to prove Theorem 1.2.

Proposition 3.2. *For $x \in \mathbb{F}_p^\times$ we have*

$$\sum_{\chi \in \widehat{\mathbb{F}_p^\times}} g(\varphi\chi^2)g(\varphi\overline{\chi})g(\overline{\chi})\chi\left(\frac{x}{4}\right) = p(1-p)\varphi(2)\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right) {}_2G_2\!\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \Big| \; \frac{1}{x} \right].$$

Proof. Let

$$A = \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} g(\varphi\chi^2)g(\varphi\overline{\chi})g(\overline{\chi})\chi\left(\frac{x}{4}\right).$$

Since $\widehat{\mathbb{F}_p^\times} = \{\omega^j : 0 \le j \le p-2\}$, replacing χ by ω^j and applying the Gross–Koblitz formula we obtain

$$(3\text{-}13) \quad A = -\sum_{j=0}^{p-2} \omega^j\left(\frac{x}{4}\right)\pi^{(p-1)\ell_j}\Gamma_p\left(\left\langle\frac{1}{2} - \frac{2j}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{1}{2} + \frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\frac{j}{p-1}\right),$$

$$\text{where} \quad \ell_j = \left\langle\frac{1}{2} - \frac{2j}{p-1}\right\rangle + \left\langle\frac{1}{2} + \frac{j}{p-1}\right\rangle + \left(\frac{j}{p-1}\right).$$

Applying (2-9) with $x = \left\langle\frac{1}{2} - \frac{2j}{p-1}\right\rangle$ and $m = 2$ we obtain

$$\Gamma_p\left(\left\langle\frac{1}{2} - \frac{2j}{p-1}\right\rangle\right) = \frac{\Gamma_p\left(\frac{1}{2}\langle\frac{1}{2} - \frac{2j}{p-1}\rangle\right)\Gamma_p\left(\frac{1}{2}\langle\frac{1}{2} - \frac{2j}{p-1}\rangle + \frac{1}{2}\right)}{\Gamma_p\left(\frac{1}{2}\right)\omega\left(2^{(1-p)(1-\langle 1/2 - 2j/(p-1)\rangle)}\right)}.$$

Taking j in the intervals $\left[0, \lfloor\frac{p-1}{4}\rfloor\right]$, $\left(\lfloor\frac{p-1}{4}\rfloor, \lfloor\frac{3(p-1)}{4}\rfloor\right]$ and $\left(\lfloor\frac{3(p-1)}{4}\rfloor, p-2\right]$ we verify that

$$\Gamma_p\left(\frac{1}{2}\left\langle\frac{1}{2} - \frac{2j}{p-1}\right\rangle\right)\Gamma_p\left(\frac{1}{2}\left\langle\frac{1}{2} - \frac{2j}{p-1}\right\rangle + \frac{1}{2}\right)$$
$$= \Gamma_p\left(\left\langle\frac{1}{4} - \frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3}{4} - \frac{j}{p-1}\right\rangle\right)$$

and $\omega\left(2^{(1-p)(1-\langle 1/2 - 2j/(p-1)\rangle)}\right) = \varphi(2)\,\overline{\omega}^j(4)$. Therefore, we can write

$$(3\text{-}14) \qquad \Gamma_p\left(\left\langle\frac{1}{2} - \frac{2j}{p-1}\right\rangle\right) = \frac{\Gamma_p\left(\langle\frac{1}{4} - \frac{j}{p-1}\rangle\right)\Gamma_p\left(\langle\frac{3}{4} - \frac{j}{p-1}\rangle\right)}{\Gamma_p\left(\frac{1}{2}\right)\varphi(2)\overline{\omega}^j(4)}.$$

Substituting (3-14) into (3-13) we obtain

$$(3\text{-}15) \quad A = -\frac{\varphi(2)}{\Gamma_p\left(\frac{1}{2}\right)}\sum_{j=0}^{p-2}\omega^j(x)\,\pi^{(p-1)\ell_j}\Gamma_p\left(\left\langle\frac{1}{4} - \frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3}{4} - \frac{j}{p-1}\right\rangle\right)$$
$$\times \Gamma_p\left(\left\langle\frac{1}{2} + \frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\frac{j}{p-1}\right).$$

Now,

$$\ell_j = \left\langle \frac{1}{2} - \frac{2j}{p-1} \right\rangle + \left\langle \frac{1}{2} + \frac{j}{p-1} \right\rangle + \left(\frac{j}{p-1} \right)$$

$$= 1 - \left\lfloor \frac{1}{2} - \frac{2j}{p-1} \right\rfloor - \left\lfloor \frac{1}{2} + \frac{j}{p-1} \right\rfloor.$$

By considering $\left\lfloor \frac{1}{2} - \frac{2j}{p-1} \right\rfloor = 2k+s$ for some $k \in \mathbb{Z}$ and $s = 0, 1$ it is straight forward to verify that

$$\left\lfloor \frac{1}{2} - \frac{2j}{p-1} \right\rfloor = \left\lfloor \frac{1}{4} - \frac{j}{p-1} \right\rfloor + \left\lfloor \frac{3}{4} - \frac{j}{p-1} \right\rfloor.$$

This gives

(3-16) $$\ell_j = 1 - \left\lfloor \frac{1}{4} - \frac{j}{p-1} \right\rfloor - \left\lfloor \frac{3}{4} - \frac{j}{p-1} \right\rfloor - \left\lfloor \frac{1}{2} + \frac{j}{p-1} \right\rfloor.$$

Substituting (3-16) into (3-15) we obtain

$$A = p(1-p)\varphi(2)\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right) {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \, \middle| \, \frac{1}{x} \right].$$

This completes the proof. $\square$

Proof of Theorem 1.2. Comparing Propositions 3.1 and 3.2 for $x = 1$ we have

(3-17) $${}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \, \middle| \, 1 \right] = \frac{-\varphi(-1)}{\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right)}.$$

Applying (2-9) (with $x = \frac{1}{2}$ and $m = 2$) we obtain

(3-18) $$\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right) = \varphi(2)\Gamma_p\left(\tfrac{1}{2}\right)^2 = -\varphi(-2).$$

Note that the last equality is obtained by using (2-8) (with $x = \frac{1}{2}$). Substituting (3-18) into (3-17) and using the fact that

$$\varphi(2) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \ (\text{mod } 8), \\ -1 & \text{if } p \equiv \pm 3 \ (\text{mod } 8), \end{cases}$$

we prove (1-9). Now, letting $x \neq 0, 1$, and $1 - x = a^2$ for some $a \in \mathbb{F}_p^\times$ and then comparing Propositions 3.1 and 3.2, we obtain

(3-19) $${}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \, \middle| \, \frac{1}{x} \right] = \frac{-\varphi(-1)}{\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right)} (\varphi(1+a) + \varphi(1-a)).$$

Then substituting (3-18) into (3-19) we have

(3-20) $${}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{matrix} \, \middle| \, \frac{1}{x} \right] = \varphi(2) \cdot (\varphi(1+a) + \varphi(1-a)).$$

Replacing x by $\frac{1}{t}$ in (3-20) we obtain (1-10). Finally, if $1 - x$ is not a square in $\mathbb{F}_p$ then again, comparing Propositions 3.1 and 3.2 we obtain

$$(3\text{-}21) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{x}\right] = 0.$$

Replacing x by $\frac{1}{t}$ in (3-21) we derive (1-11). This completes the proof of the theorem. $\qquad\square$

Proof of Theorem 1.6. If $t \neq 0, 1$ and $1 - \frac{1}{t} = \frac{t-1}{t} \neq \square$ then from (1-11) we have

$$(3\text{-}22) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| t\right] = 0.$$

Replacing t by $\frac{1}{x}$ in (3-22) we obtain that if $1 - x \neq \square$ then

$$(3\text{-}23) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{x}\right] = 0.$$

Similarly, if $t \neq 0, 1$ and $1 - \frac{1}{1-t} = \frac{t}{t-1} \neq \square$ then (1-11) gives

$$(3\text{-}24) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| 1 - t\right] = 0.$$

Replacing $1 - t$ by $\frac{1}{1-x}$ in (3-24) we can write that if $x \neq \square$ then

$$(3\text{-}25) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{1-x}\right] = 0.$$

Combining (3-23) and (3-25) we obtain (1-13). Now, let $x = b^2$. Putting $1 - x = \frac{1}{t}$ we have $1 - \frac{1}{t} = b^2$. Applying (1-10) we have

$$ {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| t\right] = \varphi(2) \cdot (\varphi(1+b) + \varphi(1-b)).$$

Therefore, if $x = b^2$ then

$$(3\text{-}26) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{1-x}\right] = \varphi(2) \cdot (\varphi(1+b) + \varphi(1-b)).$$

Combining (3-23) and (3-26) we deduce (1-14). Let $1 - x = a^2$. Also, let $x = \frac{1}{t}$. Then $1 - \frac{1}{t} = a^2$. Using (1-10) we obtain that

$$ {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| t\right] = \varphi(2) \cdot (\varphi(1+a) + \varphi(1-a)).$$

Therefore, if $1 - x = a^2$ then

$$(3\text{-}27) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array} \middle| \frac{1}{x}\right] = \varphi(2) \cdot (\varphi(1+a) + \varphi(1-a)).$$

Combining (3-26) and (3-27) we derive (1-15). Finally, comparing (3-25) and (3-27) we obtain (1-16). This completes the proof of the theorem. $\qquad\square$

Proof of Theorem 1.7. Again, consider the sum

$$A = \sum_{\chi \in \widehat{\mathbb{F}_p^\times}} g(\varphi\chi^2)g(\varphi\bar{\chi})g(\bar{\chi})\chi\left(\frac{x}{4}\right).$$

Then from (3-15), and (3-16) we have

$$(3\text{-}28) \quad A = -\frac{\varphi(2)}{\Gamma_p\left(\frac{1}{2}\right)}\sum_{j=0}^{p-2}\omega^j(x)\pi^{(p-1)\ell_j}\Gamma_p\left(\left\langle\frac{1}{4}-\frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3}{4}-\frac{j}{p-1}\right\rangle\right)$$

$$\times \Gamma_p\left(\left\langle\frac{1}{2}+\frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\frac{j}{p-1}\right),$$

$$\text{where} \quad \ell_j = 1 - \left\lfloor\frac{1}{4}-\frac{j}{p-1}\right\rfloor - \left\lfloor\frac{3}{4}-\frac{j}{p-1}\right\rfloor - \left\lfloor\frac{1}{2}+\frac{j}{p-1}\right\rfloor.$$

Now, the term for $j = 0$ present in (3-28) is equal to $p\varphi(2)\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)$. Therefore, we have

$$(3\text{-}29) \quad A = -\frac{\varphi(2)}{\Gamma_p\left(\frac{1}{2}\right)}\sum_{j=1}^{p-2}\omega^j(x)\pi^{(p-1)\ell_j}\Gamma_p\left(\left\langle\frac{1}{4}-\frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{3}{4}-\frac{j}{p-1}\right\rangle\right)$$

$$\times \Gamma_p\left(\left\langle\frac{1}{2}+\frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\frac{j}{p-1}\right) + p\varphi(2)\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right).$$

Using (2-11) we can write

$$A = p\varphi(2)\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right) + \frac{\varphi(2)}{\Gamma_p\left(\frac{1}{2}\right)}\sum_{j=1}^{p-2}\omega^j(-x)\pi^{(p-1)\ell_j}\Gamma_p\left(\left\langle\frac{1}{4}-\frac{j}{p-1}\right\rangle\right)$$

$$\times \Gamma_p\left(\left\langle\frac{3}{4}-\frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle\frac{1}{2}+\frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\left\langle1-\frac{j}{p-1}\right\rangle\right)\Gamma_p\left(\frac{j}{p-1}\right)^2$$

$$= -p\varphi(2)\sum_{j=1}^{p-2}\omega^j(-x)\cdot\frac{(-p)^{(-\lfloor 1/2+j/(p-1)\rfloor)}\Gamma_p\left(\left\langle\frac{1}{2}+\frac{j}{p-1}\right\rangle\right)}{\Gamma_p\left(\frac{1}{2}\right)}\Gamma_p\left(\left\langle1-\frac{j}{p-1}\right\rangle\right)$$

$$\times (-p)^{-\lfloor 1/4-j/(p-1)\rfloor-\lfloor 3/4-j/(p-1)\rfloor}\Gamma_p\left(\frac{j}{p-1}\right)^2\Gamma_p\left(\left\langle\frac{1}{4}-\frac{j}{p-1}\right\rangle\right)$$

$$\times \Gamma_p\left(\left\langle\frac{3}{4}-\frac{j}{p-1}\right\rangle\right) + p\varphi(2)\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right).$$

Also, applying (2-12) we obtain

$$(3\text{-}30) \quad A = -\varphi(2) \sum_{t \in \mathbb{F}_p^\times} \varphi(t(t-1)) \sum_{j=1}^{p-2} \omega^j \left(\frac{x}{t}\right) (-p)^{-\lfloor 1/4 - j/(p-1)\rfloor - \lfloor 3/4 - j/(p-1)\rfloor}$$

$$\times \Gamma_p \left(\left\langle \frac{1}{4} - \frac{j}{p-1} \right\rangle\right) \Gamma_p \left(\left\langle \frac{3}{4} - \frac{j}{p-1} \right\rangle\right) \Gamma_p \left(\frac{j}{p-1}\right)^2$$

$$+ p\varphi(2)\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right).$$

The term under summation for $j = 0$ is equal to

$$-\varphi(2) \sum_{t \in \mathbb{F}_p^\times} \varphi(t(t-1))\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right) = -\varphi(-2)\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right)J(\varphi, \varphi)$$

$$= -p\varphi(2)\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right)\binom{\varphi}{\varphi}$$

$$= \varphi(2)\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right).$$

Note that the last equality is obtained by applying (2-6). Using this value in (3-30) we obtain

$$(3\text{-}31) \quad A = (p-1)\varphi(2)\Gamma_p\left(\tfrac{1}{4}\right)\Gamma_p\left(\tfrac{3}{4}\right)\left(1 + \sum_{t \in \mathbb{F}_p^\times} \varphi(t(t-1)) \, {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{matrix} \,\middle|\, \frac{t}{x}\right]\right).$$

Now, from Proposition 3.1 comparing the values of A and using (3-18) we deduce (1-17), (1-18), and (1-19). This completes the proof of the theorem. $\square$

Proof of Theorem 1.8. Applying the transformations [McCarthy 2013, Lemma 3.3; 2012c, Proposition 2.5] for $x, t \neq 0$ we obtain

$$(3\text{-}32) \quad {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{matrix} \,\middle|\, \frac{t}{x}\right] = \left(\frac{\chi_4^3}{\varepsilon}\right)^{-1} {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \,\middle|\, \frac{x}{t}\right)$$

$$= -p \cdot {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \,\middle|\, \frac{x}{t}\right).$$

Note that we obtain the last equality by using (2-6).

Let $x = 1$. Then substituting (3-32) into (1-17) we have

$$(3\text{-}33) \quad -p \sum_{t \in \mathbb{F}_p^\times} \varphi(t(t-1)) \, {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \,\middle|\, \frac{1}{t}\right) = -1 - p\varphi(2).$$

Replacing t by $1/t$ we derive (1-20). Similarly, if $x \neq 1$ and $1 - x$ is not a square then substituting (3-32) into (1-18) and replacing t by x/t we obtain (1-21). Finally, if $x \neq 1$ and $1 - x = a^2$ then substituting (3-32) into (1-19) and replacing t by x/t we deduce (1-22). This completes the proof. $\square$

Proof of Theorem 1.9. If $1 - x \neq \square$ then applying Propositions 3.1 and 3.2 we have

$$
{}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array}\Big| \frac{1}{x}\right] = {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array}\Big| \frac{x-1}{x}\right] = 0.
$$

This proves (1-25). Now, let $1 - x = a^2$ for some $a \in \mathbb{F}_p^\times$. Let a^{-1} denote the inverse of a in $\mathbb{F}_p^\times$. Then again applying Propositions 3.1 and 3.2 we have

$$
(3\text{-}34) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array}\Big| \frac{1}{x}\right] = -\frac{\varphi(-1)}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)}(\varphi(1+a) + \varphi(1-a)),
$$

and

$$
(3\text{-}35) \qquad {}_2G_2\left[\begin{array}{cc} \frac{1}{4}, & \frac{3}{4} \\ 0, & \frac{1}{2} \end{array}\Big| \frac{x-1}{x}\right] = -\frac{\varphi(-1)}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)}(\varphi(1+a^{-1}) + \varphi(1-a^{-1})).
$$

Comparing (3-34) and (3-35) we prove (1-26). This completes the proof. $\square$

Concluding Remarks. Let $\mathbb{F}_q$ be a finite field with q elements, where $q = p^r$. We note that all the transformations and special values of p-adic hypergeometric series that are proved in this paper can also be extended to the q-version of the p-adic hypergeometric series ${}_nG_n[\cdots \mid t]_q$ with $t \in \mathbb{F}_q$ using [McCarthy 2013, Definition 5.1]. We avoid this case here for brevity. We also make the same comment for Gaussian hypergeometric functions over $\mathbb{F}_q$. We believe that using this method we can settle many other transformation formulas for p-adic hypergeometric series that are analogous to classical hypergeometric series transformations. This is considered as the subject of forthcoming work.

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References

[Bailey 1935] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics **32**, Cambridge University Press, London, 1935. JFM

[Berndt et al. 1998] B. C. Berndt, R. J. Evans, and K. S. Williams, *Gauss and Jacobi sums*, Wiley, New York, 1998. MR Zbl

[Dixon 1903] A. C. Dixon, "Summation of a certain series", *Proc. Lond. Math. Soc.* **35** (1903), 284–289. MR Zbl

[Evans and Greene 2009] R. Evans and J. Greene, "Clausen's theorem and hypergeometric functions over finite fields", *Finite Fields Appl.* **15**:1 (2009), 97–109. MR Zbl

[Fuselier 2010] J. G. Fuselier, "Hypergeometric functions over $\mathbb{F}_p$ and relations to elliptic curves and modular forms", *Proc. Amer. Math. Soc.* **138**:1 (2010), 109–123. MR Zbl

[Fuselier and McCarthy 2016] J. G. Fuselier and D. McCarthy, "Hypergeometric type identities in the p-adic setting and modular forms", *Proc. Amer. Math. Soc.* **144**:4 (2016), 1493–1508. MR Zbl

[Gauss 1812] C. F. Gauss, "Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 +$ etc.", pp. 233–279 in *Commentationes societatis regiae scientiarum Gottingensis recentiores*, vol. II, 1812. Reprinted as pp. 123–163, 207–229 in *Carl Friedrich Gauss Werke*, Band III, Königlichen Gesellschaft der Wissenschaften, Göttingen, 1866.

[Greene 1987] J. Greene, "Hypergeometric functions over finite fields", *Trans. Amer. Math. Soc.* **301**:1 (1987), 77–101. MR Zbl

[Gross and Koblitz 1979] B. H. Gross and N. Koblitz, "Gauss sums and the p-adic Γ-function", *Ann. of Math.* (2) **109**:3 (1979), 569–581. MR Zbl

[Ireland and Rosen 2005] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, International ed., Springer, 2005.

[Koblitz 1980] N. Koblitz, *p-adic analysis: a short course on recent work*, London Mathematical Society Lecture Note Series **46**, Cambridge University Press, 1980. MR Zbl

[Kummer 1836] E. E. Kummer, "Über die hypergeometrische Reihe $1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$", *J. Reine Angew. Math.* **15** (1836), 39–83. MR Zbl

[Lennon 2011a] C. Lennon, "Gaussian hypergeometric evaluations of traces of Frobenius for elliptic curves", *Proc. Amer. Math. Soc.* **139**:6 (2011), 1931–1938. MR Zbl

[Lennon 2011b] C. Lennon, "Trace formulas for Hecke operators, Gaussian hypergeometric functions, and the modularity of a threefold", *J. Number Theory* **131**:12 (2011), 2320–2351. MR Zbl

[Lidl and Niederreiter 1983] R. Lidl and H. Niederreiter, *Finite fields*, Encyclopedia of Mathematics and its Applications **20**, Addison-Wesley Publishing Company, Reading, MA, 1983. MR Zbl

[McCarthy 2010] D. McCarthy, "$_3F_2$ hypergeometric series and periods of elliptic curves", *Int. J. Number Theory* **6**:3 (2010), 461–470. MR Zbl

[McCarthy 2012a] D. McCarthy, "Extending Gaussian hypergeometric series to the p-adic setting", *Int. J. Number Theory* **8**:7 (2012), 1581–1612. MR Zbl

[McCarthy 2012b] D. McCarthy, "On a supercongruence conjecture of Rodriguez-Villegas", *Proc. Amer. Math. Soc.* **140**:7 (2012), 2241–2254. MR Zbl

[McCarthy 2012c] D. McCarthy, "Transformations of well-poised hypergeometric functions over finite fields", *Finite Fields Appl.* **18**:6 (2012), 1133–1147. MR Zbl

[McCarthy 2013] D. McCarthy, "The trace of Frobenius of elliptic curves and the p-adic gamma function", *Pacific J. Math.* **261**:1 (2013), 219–236. MR Zbl

[Mortenson 2005] E. Mortenson, "Supercongruences for truncated $_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms", *Proc. Amer. Math. Soc.* **133**:2 (2005), 321–330. MR Zbl

[Slater 1966] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966. MR Zbl

[Whipple 1925] F. J. W. Whipple, "On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum", *Proc. London Math. Soc.* (2) **24**:4 (1925), 247–263. MR Zbl

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