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Let (M, F) be a geodesically forward complete Finsler manifold and $p \in M$. We observe how the preimage of a curve in M under exponential map at p can behave in the tangent space T_pM at p, when the curve approaches a conjugate cut point of p without crossing the cut locus of p. After this investigation, we may regard the internal region of a tangent cut locus of $p \in M$ as the development of M. We deal with isometry problems of Finsler manifolds and differentiability conditions of cut loci.

1. Introduction

Let (M, F) be a geodesically forward complete Finsler manifold without boundary and $\exp_p : T_pM \to M$ the exponential map at $p \in M$. Then $\exp_p$ is C^∞ on $T_pM \setminus \{0\}$ and C^1 at $0 \in T_pM$ (see [Shen 2001, Theorem 11.1.1]). Let $d(p, q)$ denote the distance from p to q induced by F and $S_pM := \{v \mid v \in T_pM \text{ with } F(p, v) = 1\}$.

For a tangent vector $v \in S_pM$ we define numbers $\rho(v), \lambda(v) \in (0, \infty]$ as follows:

$$\rho(v) = \sup\{s > 0 \mid d(p, \exp_p(tv)) = t \text{ for any } t \in (0, s)\},$$
$$\lambda(v) = \sup\{s > 0 \mid d\exp_p|_{tv} \text{ is nonsingular for any } t \in (0, s)\}.$$

It is well known that ρ and λ are continuous on the domain in S_pM where they are bounded. Let $\nu(v)$ denote the dimension of the kernel $N(v)$ of $d\exp_p|_{\lambda(v)v}$. It follows from the implicit function theorem that if ν is constant in an open set $U \subset S_pM$, then λ is C^∞ on U. In particular, if $\dim M = 2$, then λ is C^∞ on the domain U in S_pM where λ is bounded.

We call $\widetilde{C}(p) := \{\rho(v)v \mid v \in S_pM\}$ the *tangent cut locus* of p, and $C(p) := \exp_p(\widetilde{C}(p))$ the *cut locus* of p. In a similar way we define the *first tangent conjugate locus* $\widetilde{Q}(p) := \{\lambda(v)v \mid v \in S_pM\}$ and the *first conjugate locus* $Q(p) := \exp_p(\widetilde{Q}(p))$ of p. We call a point $q \in \exp_p(\widetilde{C}(p) \cap \widetilde{Q}(p))$ a *conjugate cut point* of p. We say

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that a point $q \in C(p)$ is a *nonconjugate cut point* of p if q is not a conjugate cut point of p.

Weinstein [1968] has proved that any compact differentiable manifold M not homeomorphic to the 2-sphere has a Riemannian metric on M such that there exists a point $p \in M$ satisfying $\widetilde{C}(p) \cap \widetilde{Q}(p) = \varnothing$. On the other hand, Innami, Shiohama and Soga [Innami et al. 2012] have proved that if a complete Riemannian manifold M has a pole p, i.e., $C(p) = \varnothing$, then $\widetilde{C}(q) \cap \widetilde{Q}(q) \neq \varnothing$ for any point $q \in M$ with $C(q) \neq \varnothing$. Ozols [1974] has given a description of $C(p)$ locally around a nonconjugate cut point $q \in C(p)$ as an intersection of a finite number of smooth $(n-1)$-dimensional manifolds and finitely many open sets given by smooth inequalities ($n = \dim M$). Ozols' structure theorem is applicable to all cut points of p in a manifold with Weinstein metric, i.e., $\widetilde{C}(p) \cap \widetilde{Q}(p) = \varnothing$ for the point $p \in M$. Itoh and Sakai [2007] have given the topological structure theorem of a compact manifold with Weinstein metric, using the distance function from p as a Morse function.

The structure of cut loci has been studied in [Warner 1965; 1967; Weinstein 1968; Ozols 1974; Itoh and Tanaka 1998; Itoh and Sakai 2007] and so on. The differentiability and Lipschitz continuity properties of cut loci and the distance functions are studied in [Castelpietra and Rifford 2010; Figalli et al. 2011; Hebda 1994; Itoh 1996; Itoh and Tanaka 2001a; 2001b; Rifford 2004; Tanaka 2003] and so on. As was seen in [Gluck and Singer 1978; Hebda 1983; Itoh and Sabau 2016; Myers 1935; Sabau and Tanaka 2016] and so on, the structure of a cut locus is very complicated.

If $\widetilde{U}_p = \{rv \mid v \in S_pM, 0 \leq r < \rho(v)\}$, then $\exp_p : \widetilde{U}_p \to U_p := M \setminus C(p)$ is a diffeomorphism. Moreover, Ozols [1976] has given a direction preserving diffeomorphism from $\widetilde{U}_p$ onto the open unit ball $B(1)$ in $\mathbb{R}^n$, $n = \dim M$, when M is a compact Riemannian manifold. Then, $M \setminus C(p)$ is diffeomorphic to $B(1)$. Obviously, $\partial\widetilde{U}_p = \widetilde{C}(p)$ where $\partial\widetilde{U}_p$ is the boundary of $\widetilde{U}_p$ in T_pM. Set $\widetilde{U}_p^c = \widetilde{U}_p \cup \widetilde{C}(p)$. We think the *covering* domain $\exp_p : \widetilde{U}_p^c \to M$ to be the development of a Finsler manifold M. When those points in the set $\exp_p{}^{-1}(q)$ are identified for any point $q \in M$, we regard the quotient space $\widetilde{U}_p^c / \exp_p$ as the Finsler manifold M.

We study how to draw $\tilde{c}(s) := \exp_p{}^{-1}(c(s))$ in $\widetilde{U}_p^c$ for a curve $c : [0, 1] \to M$. The problem arises in the case that $c(s_0) \in C(p)$ for some s_0, because $\exp_p{}^{-1}(c(s_0))$ may not be one point. The distribution of $\exp_p{}^{-1}(q)$ in $\widetilde{C}(p)$ for points $q \in C(p)$ is the key to investigate the topological and metrical structure of M.

Ozols' theorem [1974] and Itoh and Sakai's method [2007] show us that, under the condition $\widetilde{C}(p) \cap \widetilde{Q}(p) = \varnothing$, a certain neighborhood of $q \in C(p)$ is decomposed into finitely many sets through each of which there passes a unique minimal geodesic from p to q, and each of those sets are distributed isometrically at corresponding points $\tilde{q}$ contained in the inverse image $\exp_p{}^{-1}(q)$. Here we note that the unit speed minimal geodesics from p to q in M are denoted by $\gamma_v(t) = \exp_p(tv)$, $v = \tilde{q}/F(p, \tilde{q})$, $\tilde{q} \in \exp_p{}^{-1}(q)$. Namely, the Voronoi diagram D

of the negative tangent vectors of all minimal geodesics from p to q in T_qM makes the arrangement of those regions to the points $\tilde{q}$ with $q = \exp_p(\tilde{q})$ in $\widetilde{C}(p) \subset T_pM$. After arrangement, the vectors in those regions are pointing to the inside of $\widetilde{U}_p^c$ at those points $\tilde{q}$.

If $q \in \exp_p(\widetilde{C}(p) \cap \widetilde{Q}(p))$, then these decomposition and arrangement may be more complicated. To describe what happens around q, we use the notion of "limiting tangent line". Let $c'(s)$ be the tangent vector of c at s and $\mathrm{Span}(c'(s))$ the one-dimensional subspace spanned by $c'(s)$ of $T_{c(s)}M$, which is called a *tangent line* of c at s. We say that c has a *limiting tangent line* T at $s = s_0$ if $\mathrm{Span}(c'(s))$ converges to the one-dimensional subspace T of $T_{c(s_0)}M$ as $s \to s_0$.

The north pole p and the south pole q in the unit sphere $S^2(1)$ give a simple example (see Example 7.5): the tangent space at q is decomposed into individual tangent vectors and arranged in each inward normal vector of the circle $S^1(\pi)$ with center 0 and radius π in $T_pS^2(1)$. It is natural to ask how the curves approaching a point $\tilde{q}$ in $S^1(\pi)$ are mapped by $\exp_p$, if their limiting tangent lines are not orthogonal to $S^1(\pi)$ at $\tilde{q}$.

If $q \in \exp_p(\widetilde{C}(p) \cap \widetilde{Q}(p))$ is an end cut point in a surface M, then at q the tangent vector of the minimal geodesic γ_{v_0}, $v_0 \in S_pM$, from p to q is often the limiting tangent line of $C(p)$ at q. Hartman [1964] has stated without proof that $\lambda'(v_0) = 0$ (for the proof, see [Shiohama et al. 2003, Lemma 4.2.3, p. 142]). These facts imply that the half plane bounded by the orthogonal line to v_0 through ${\exp_p}^{-1}(q) =: t_0v_0$ is mapped to the whole tangent space T_qM except for ${\gamma_{v_0}}'(t_0)$ at q by $d\exp_p|_{t_0v_0}$ as the limiting tangent vectors.

Let $\tilde{q} \in \widetilde{C}(p) \cap \widetilde{Q}(p)$ and $q := \exp_p(\tilde{q})$. What happens on the curves approaching q and $\tilde{q}$? Let $\tilde{c}(s)$ and $c(s)$ be curves such that $\tilde{c}(0) = \tilde{q}$, $c(0) = q$ and $\exp_p(\tilde{c}(s)) = c(s)$. In this paper we study how the behaviors of $\tilde{c}$ and c are related. In Section 2, using the first variation formula, we show how to find a converging point $\lim_{s\to 0} {\exp_p}^{-1}(c(s))$ for a curve c with $c(0) \in C(p)$. In Section 3, we study the relation between the tangent vectors of $\tilde{c}(s)$ and $c(s)$ at $s = 0$. Theorem 1.1 is the two-dimensional case of our investigation. We say (see [Gibson 2001, p. 91]) that a function $\varphi(s)$ has a *zero of finite multiplicity m* at s_0 when

$$\varphi(s_0) = 0, \quad \varphi'(s_0) = 0, \quad \dots, \quad \varphi^{(m-1)}(s_0) = 0, \quad \varphi^{(m)}(s_0) \neq 0.$$

Theorem 1.1. *Let M be a geodesically forward complete Finsler surface and $p \in M$. Let $q \in C(p) \cap Q(p)$ be conjugate to p along a minimal geodesic $\gamma_{\theta_0} : [0, d(p,q)] \to M$ from p to q where (t, θ) is a polar coordinate system of T_pM. Assume that $\varphi(\theta) = \lambda(\theta) - \lambda(\theta_0)$ has a zero of finite multiplicity m at $\theta = \theta_0$. Let $c(s)$ be a curve emanating from q such that $c(s) \in M \setminus C(p)$ except for $q = c(0)$ and $\tilde{c}(s)$ the curve satisfying $c(s) = \exp_p(\tilde{c}(s))$. Then the following are true.*

(1) *If the limiting tangent line of* $\tilde{c}(s)$ *at* $s = 0$ *is not tangent to the circle with center* 0 *and radius* $d(p, q)$, *then the limiting tangent line of* $c(s)$ *is tangent to* γ_{θ_0} *at* q.

(2) *If the limiting tangent line of* $c(s)$ *at* $s = 0$ *is not tangent to* γ_{θ_0} *at* q, *then the limiting tangent line of* $\tilde{c}(s)$ *at* $s = 0$ *is tangent to the circle with center* 0 *and radius* $d(p, q)$.

Moreover, if $\tilde{c}(\theta) = (t(\theta), \theta)$ *and* $c'(\theta_0) \notin \mathrm{Span}(\gamma_{\theta_0}{}'(\lambda(\theta_0)))$, *then* $t(\theta) - t(\theta_0)$ *has a zero of multiplicity* $1 + m$ *at* θ_0.

The north pole p and the south pole q in the unit sphere do not satisfy the assumption in Theorem 1.1. However, an end cut point in a surface may satisfy the assumption. We will prove this theorem under a more detailed calculation of the high dimensional case. After we prepare some notations to be used in our discussion, the result will be stated (see Theorem 3.4).

The set of tangent vectors pointing to the interior of $\widetilde{U}^c_p$ at a point $\tilde{q} \in \widetilde{C}(p)$ is considered to be a part of the tangent space $T_{\exp_p(\tilde{q})}M$. In Section 4, for a cut point q, we see how to take $T_q M$ to pieces of convex cones with vertex 0 and find how to arrange those cones at points of $\exp_p{}^{-1}(q)$. After these investigations, for a curve $c(s)$ crossing $C(p)$, we consider what $\tilde{c}(s)$ should be. We propose the notion of *pull back* curves $\tilde{c}(s)$ which satisfy $\exp_p(\tilde{c}(s)) = c(s)$. They are not continuous, in general.

We consider the *pseudo-Finsler metric* F^* on $T_p M$ which is the pullback of F by $\exp_p$, i.e., $F^*(x, y) = F(\exp_p(x), d\exp_p|_x(y))$ for any $y \in T_x T_p M$ and $x \in T_p M$. It follows that $F^*(x, y) = 0$ if and only if $d\exp_p|_x(y) = 0$. There exists a nonzero vector $y \in T_x T_p M$ such that $F^*(x, y) = 0$ if and only if $x \in \widetilde{C}(p) \cap \widetilde{Q}(p)$. Let d^* denote the pseudodistance induced by F^*. We study the relation between d and d^*. In Section 5, we show the relation between M and $\widetilde{U}^c_p / \exp_p$ as distance spaces, using *pull back* curves and d^*.

We say that $C(p)$ is *differentiable* at $q \in C(p)$ if the tangent vector space $T_q C(p)$ of $C(p)$ is defined at q. In Section 6, we study how the set $\exp_p{}^{-1}(q)$ lies in $\widetilde{C}(p)$ when the cut locus $C(p)$ is differentiable at $q \in C(p)$. The Klingenberg lemma and the generalized Berger–Omori theorem proved in [Innami et al. 2019] (see [Berger 1960; 1961; Klingenberg 1959; Nakagawa and Shiohama 1970a; 1970b; Omori 1968] also) suggest us that it is homeomorphic to a sphere (see Theorem 6.5).

In Section 7, we give some examples which help us to understand the discussions and results in this paper.

2. Directional differentiation of a distance function

Let (M, F) be a geodesically forward complete Finsler manifold without boundary. For $y \in T_x M$ let ω_y denote a co-vector in $T_x M^*$ such that $\omega_y(v) = g_y(y, v)$ for any vector $v \in T_x M$ where g_y is the *fundamental tensor* induced by F. Let $f(q) :=$

$d(p, q)$ for all $q \in M$. Then f^2 is C^∞ nearby p and C^1 at p. Actually, $d(q, \cdot)^2$ are C^2 on M for all $q \in M$ if and only if F is Riemannian (see [Shen 2001, Proposition 11.3.3]).

Let $A_p(q)$ be the set of all tangent vectors at q which are the tangent vectors of all constant speed minimal geodesics from p to q and let $A_p(q)^s = A_p(q) \cap S_q M$ where $S_q M$ is the unit sphere with center 0 in $T_q M$. From the first variation formula, the distance function from a point is directionally differentiable. Sabau and Tanaka [2016] have proved this fact of the Finsler manifold version, using second order remainder term. Here we give a slightly modified proof, replacing the second order remainder term by the mean value theorem. Then we use only C^1 differentiability without second order derivatives.

Lemma 2.1 [Sabau and Tanaka 2016]. *Let $c : [0, 1] \to M$ be a curve of class C^1 such that $c(0) = q$ and $c'(0) = w \in T_q M$. We then have*

$$\left. \frac{d(f \circ c)}{dt} \right|_{t=0} = \min_{v \in A_p(q)^s} \omega_v(w).$$

Proof. Let $\gamma : [0, f(q)] \to M$ be a unit speed minimal geodesic from p to q, and let $H : (-\varepsilon, \varepsilon) \times [0, f(q)] \to M$ be a variation of γ through piecewise smooth curves such that $H(0, s) = \gamma(s)$, $H(t, 0) = p$, $H(t, f(q)) = c(t)$. Then $\partial H(0, f(q))/\partial t = w$. If $\gamma_t(s) := H(t, s)$ and $L(\gamma_t)$ is the length of γ_t, then $L(\gamma_t) \geq f \circ c(t)$. Hence, it follows from the first variation formula (see [Shen 2001, equation (5.6)]) that

$$g_{\gamma'(f(q))}(\gamma'(f(q)), w) \geq \limsup_{t \to 0} \frac{f \circ c(t) - f(q)}{t},$$

meaning that

$$\min_{v \in A_p(q)} \omega_v(w) \geq \limsup_{t \to 0} \frac{f \circ c(t) - f(q)}{t}. \tag{1}$$

Assume that t_j is a sequence converging to 0 such that

$$\lim_{j \to \infty} \frac{f \circ c(t_j) - f(q)}{t_j} = \liminf_{t \to 0} \frac{f \circ c(t) - f(q)}{t}$$

and a sequence of minimal geodesics $\gamma_j : [0, f(c(t_j))] \to M$ from p to $c(t_j)$ converges to a minimal geodesic γ from p to q. For a sufficiently small $\delta > 0$, we have

$$f(q) \leq f(\gamma_j(f(c(t_j)) - \delta)) + d(\gamma_j(f(c(t_j)) - \delta), q),$$

and, hence,

$$-d(\gamma_j(f(c(t_j)) - \delta), q) \leq f(\gamma_j(f(c(t_j)) - \delta)) - f(q).$$

Thus we have

$$\begin{aligned}
&d(\gamma_j(f(c(t_j))-\delta),\gamma_j(f(c(t_j))))-d(\gamma_j(f(c(t_j))-\delta),q)\\
&\qquad\le d(\gamma_j(f(c(t_j))-\delta),\gamma_j(f(c(t_j))))+f(\gamma_j(f(c(t_j))-\delta))-f(q)\\
&\qquad= f\circ c(t_j)-f(q).
\end{aligned}$$

Let $\alpha_j : [0, d(q, c(t_j))] \to M$ be the unique minimal geodesic from q to $c(t_j)$ for every j. Since $c(t_j)$ converges to q as $j \to \infty$, we may assume that all $\alpha_j([0, d(q, c(t_j))])$ are contained in a convex ball around $\gamma_j(f(c(t_j)) - \delta)$. If $h(t) := d(\gamma_j(f(c(t_j)) - \delta), \alpha_j(t))$ for $t \in [0, d(q, c(t_j))]$, we then have $h(0) = d(\gamma_j(f(c(t_j)) - \delta), q)$ and $h(d(q, c(t_j))) = d(\gamma_j(f(c(t_j)) - \delta), \gamma_j(f(c(t_j))))$. It follows from the first variation formula that there exists a number $s \in (0, d(q, c(t_j)))$ such that

$$h(d(q,c(t_j)))-h(0)=g_{v_j}(v_j,{\alpha_j}'(s))d(q,c(t_j)),$$

where v_j denotes the tangent vector of the unit speed minimal geodesic from $\gamma_j(f(c(t_j)) - \delta)$ to $\alpha_j(s)$. Thus we have

$$g_{v_j}(v_j,{\alpha_j}'(s))d(q,c(t_j))\le f\circ c(t_j)-f(q).$$

If $c_0(t) = {\exp_q}^{-1}(c(t))$ for sufficiently small $t > 0$, then $c_0(0) = 0$, $d(q, c(t)) = F(q, c_0(t))$ and we have

$$\begin{aligned}
\lim_{t\to+0}\frac{d(q,c(t))}{t}&=\lim_{t\to+0}\frac{F(q,c_0(t)-0)}{t}\\
&=F\left(q,\lim_{t\to+0}\frac{c_0(t)-0}{t}\right)\\
&=F(q,{c_0}'(0))=F(q,c'(0)),
\end{aligned}$$

because $d\exp_q|_0$ is the identity map. Since $\lim_{j\to\infty}{\alpha_j}'(s) = c'(0)/F(q, c'(0))$ and $\lim_{j\to\infty} v_j = \gamma'(f(q))$, we have

$$\lim_{j\to\infty}\frac{g_{v_j}(v_j,{\alpha_j}'(s))d(q,c(t_j))}{t_j}=g_{\gamma'(f(q))}(\gamma'(f(q)),c'(0)).$$

Therefore we conclude that

$$\min_{v\in A_p(q)^s}\omega_v(w)\le\omega_{\gamma'(f(q))}(w)\le\liminf_{t\to0}\frac{f\circ c(t)-f(q)}{t}.$$

Combining this inequality with (1), we complete the proof of the lemma. □

Let $X_p(w) := \{y \in S_pM \mid \omega_{\gamma_y'(d(p,q))}(w) = \min_{v \in A_p(q)^s} \omega_v(w)\}$. From Lemma 2.1 we see how to map $c(s)$ into T_pM by $\exp_p{}^{-1}$.

Theorem 2.2. *Let $c : [0, 1] \to M$ be a curve of class C^1 such that $c(0) = q \in C(p)$, $c'(0) = w \in T_qM$ and $c(s) \notin C(p)$ for all $s \in (0, 1]$. Let $\tilde{c}(s)$ be the curve in T_pM such that $\exp_p(\tilde{c}(s)) = c(s)$ for all $s \in (0, 1]$. We then have $\lim_{s\to 0} \tilde{c}(s) \in d(p, q)X_p(w)$.*

3. Curves approaching a conjugate cut point

Let (M, F) be a geodesically forward complete Finsler manifold without boundary. Let $Y(t) := d\exp_p|_{tv}(tw)$ for $v \in S_pM$ and $w \in T_pM$. Then, from Lemma 11.2.2 in [Shen 2001], $Y(t)$, $t \in [0, a)$, is the Jacobi field along $\gamma_v(t) = \exp_p(tv)$ with initial condition $Y(0) = 0$ and $D_{\gamma_v'(0)}Y = w$, where D_v is the covariant derivative at p in the direction v (see (5.33) in [Shen 2001]). From Lemma 6.1.1 in [Shen 2001], it satisfies

$$D_{\gamma_v'} D_{\gamma_v'} Y + R_{\gamma_v'}(Y) = 0$$

where $R_{\gamma_v'}(Y) = R(Y, \gamma_v')\gamma_v'$ is the Riemann curvature which is self-adjoint with respect to the fundamental tensor $g_{\gamma_v'}$ induced by F.

Let $e_1, \dots, e_n$ be an orthonormal basis of T_pM with respect to g_v such that $e_n = v$ and let $E_i(t)$ be the parallel vector field along γ_v with $E_i(0) = e_i$ for each $i = 1, \dots, n$. If $Y(t) = \sum_{j=1}^n y_j(t)E_j(t)$, then $Y(t)$ is identified with the column vector

$$Y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

Under this notation, the covariant derivative $D_{\gamma_v(t)}Y$ is identified with the differential $Y'(t)$ of its column vector $Y(t)$ with respect to t.

Let $Y_i(t) = d\exp_p|_{tv}(te_i) = \sum_{j=1}^n y_{ji}(t)E_j(t)$. If we set

$$Y_i(t) = \begin{pmatrix} y_{1i}(t) \\ \vdots \\ y_{ni}(t) \end{pmatrix} \quad \text{for } i = 1, \dots, n,$$

then the matrix value function $J(t) = (Y_1(t), \dots, Y_n(t))$ satisfies the differential equation of Jacobi type:

$$J''(t) + R(t)J(t) = 0$$

where $R(t) = \big(g_{\gamma_v'(t)}(R_{\gamma_v'(t)}(E_i(t)), E_j(t))\big)$ is a symmetric $n \times n$ matrix.

Lemma 3.1. *With respect to the orthonormal bases* $\{e_1, \dots, e_n\}$ *for* T_pM *and* $\{E_1(t), \dots, E_{n-1}(t), E_n(t) = \gamma_v{}'(t)\}$ *for* $T_{\gamma_v(t)}M$, *the Jacobi field* $Y(t)$ *along* γ_v *is denoted by*

$$Y(t) = (E_1(t) \cdots E_{n-1}(t)\, \gamma_v{}'(t)) J(v,t) \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{pmatrix}$$

$$= (E_1(t) \cdots E_{n-1}(t)\, \gamma_v{}'(t)) \begin{pmatrix} J_0(v,t) & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{pmatrix},$$

for $w = w_1e_1 + \cdots + w_{n-1}e_{n-1} + w_ne_n$, *where* $J_0(v,t)$ *is the matrix value function satisfying that* $J_0(v,0) = 0$ *and* $J_0{}'(v,0) = I$ (I *is the* $(n-1) \times (n-1)$ *identity matrix*).

Lemma 3.2. *Assume that* $Y_1(t_0) = \cdots = Y_k(t_0) = 0$, $0 \le k \le n-1$, *and that* rank $J(v,t_0) = n-k$, *i.e.*, rank $d\exp_p|_{t_0v} = n-k$. *Then the set of tangent vectors* $\{Y_1{}'(t_0), \dots, Y_k{}'(t_0), Y_{k+1}(t_0), \dots, Y_{n-1}(t_0)\}$ *spans the orthogonal complement of* $\gamma_v{}'(t_0)$ *in* $T_{\gamma_v(t_0)}M$. *Furthermore*, $Y_i{}'(t_0)$ *and* $Y_j(t_0)$ *are orthogonal for* $i = 1, \dots, k$ *and* $j = k+1, \dots, n-1$.

Proof. For each $m = 1, \dots, n-1$, if $f(t) = g_{\gamma_v{}'(t)}(\gamma_v{}'(t), Y_m(t)) = 0$ for all $t \ge 0$, then $f(0) = 0$, $f'(0) = 0$ and $f''(t) = 0$ for all $t > 0$, and, hence, $g_{\gamma_v{}'(t)}(\gamma_v{}'(t), Y_m(t)) = 0$ for all $t \ge 0$. Therefore, $Y_m(t_0)$ is contained in the orthogonal compliment of $\gamma_v{}'(t_0)$.

We prove that $\{Y_1{}'(t_0), \dots, Y_k{}'(t_0)\}$ is linearly independent. Suppose for indirect proof that $\sum_{i=1}^k a_iY_i{}'(t_0) = 0$ where at least one of a_i's is not zero. Let $e = \sum_{i=1}^k a_ie_i$ and $Y(t)$ the Jacobi field along γ_v such that $Y(0) = 0$ and $Y'(0) = e$. Obviously, Y is not identically zero. However, since a Jacobi field Y satisfies $Y(t_0) = 0$ and $Y'(t_0) = 0$, we have $Y(t) = 0$ identically, a contradiction.

From this and

$$\mathrm{Span}(Y_1(t), \dots, Y_k(t)) = \mathrm{Span}\left(\frac{Y_1(t)}{t-t_0}, \dots, \frac{Y_k(t)}{t-t_0}\right), \quad 0 < t < t_0,$$

we see that $\mathrm{Span}(Y_1(t), \dots, Y_k(t))$ converges to $\mathrm{Span}(Y_1{}'(t_0), \dots, Y_k{}'(t_0))$ as $t \to t_0$.

Since rank $J(v,t_0) = n-k$, we see that $\{Y_{k+1}(t_0), \dots, Y_n(t_0)\}$ is linearly independent.

We next prove that $Y_i{}'(t_0)$ and $Y_j(t_0)$ are orthogonal for $i = 1, \dots, k$ and $j = k+1, \dots, n-1$. Since both Y_i and Y_j are Jacobi fields along γ_v with

$Y_i(0) = Y_j(0) = 0$, we see that $g_{\gamma_v'(t)}(Y_i'(t), Y_j(t)) - g_{\gamma_v'(t)}(Y_i(t), Y_j'(t))$ is constant for t and zero at $t = 0$. From this, we have

$$g_{\gamma_v'(t_0)}(Y_i'(t_0), Y_j(t_0)) = g_{\gamma_v'(t_0)}(Y_i(t_0), Y_j'(t_0)) = 0$$

because $Y_i(t_0) = 0$. This proves Lemma 3.2. □

Remark 3.3. As a simple application of Lemma 3.2, we get the following well-known fact: If $\gamma_v(t_0)$ is conjugate to $\gamma_v(0)$ along a geodesic γ_v, then there exists a $\delta > 0$ such that no point $\gamma_v(t)$ with $0 < |t - t_0| < \delta$ is conjugate to $\gamma_v(0)$. Because we have $\det(Y_1'(t_0) \cdots Y_k'(t_0)\, Y_{k+1}(t_0) \cdots Y_{n-1}(t_0)) \neq 0$ and

$$\det J(t) = (t - t_0)^k \det(Y_1'(t_1) \cdots Y_k'(t_k)\, Y_{k+1}(t) \cdots Y_{n-1}(t))$$

where, from the mean value theorem,

$$Y_i'(t_i) = \begin{pmatrix} y_{1i}'(t_{1i}) \\ \vdots \\ y_{ni}'(t_{ni}) \end{pmatrix} \quad \text{for } i = 1, \dots, k$$

for some t_{ji} with $|t_{ji} - t_0| < |t - t_0|$, $j = 1, \dots n$.

We investigate how the preimage $\exp_p{}^{-1}(c(s))$ behaves for a curve $c(s)$ if $q = c(0)$ is a conjugate cut point along a geodesic γ_v and $c(s) \in M \setminus C(p)$ for $s \neq 0$.

Let $v_0 = e_n \in S_pM$ and $\{e_1, \dots, e_{n-1}\}$ an orthonormal basis of the orthogonal complement $v_0{}^{\perp}$ of v_0 with respect to g_{v_0} in T_pM. We use this basis to have a coordinate system for T_pM. Let $(V, \tau^{-1} = (v_1, \dots, v_{n-1}))$ be a local coordinate system of S_pM around v_0, where $\tau : W = \tau^{-1}(V) \subset \mathbb{R}^{n-1} \to V \subset S_pM$, such that $\tau(0) = v_0$ and $d\tau_0$ is an isometry from $T_0\mathbb{R}^{n-1}$ to $T_{v_0}S_pM$ with respect to $g_{v_0}|_{T_{v_0}S_pM}$ and let $(\mathbb{R}_+V, \psi^{-1} = (v_1, \dots, v_{n-1}, t))$ be a polar coordinate system of T_pM such that $\psi(v, t) = t\tau(v)$ for $t > 0$ and $v \in W = \tau^{-1}(V)$. Then we have

$$d\psi_{(0,t)} = \begin{pmatrix} t\,d\tau_0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Assume that $q = \gamma_{\tau(0)}(t_0)$ for $\tau(0) = v_0$. We make a local coordinate system $(U, x_1, \dots, x_n)$ in a tubular neighborhood U around $\gamma_{\tau(0)}$ at q as follows:

(1) $x_1(q) = x_2(q) = \cdots = x_n(q) = 0$.

(2) $x_1(\exp_p(\psi(0, t))) = 0, \dots, x_{n-1}(\exp_p(\psi(0, t))) = 0$ and $x_n(\exp_p(\psi(0, t))) = t_0 - t$.

(3) If $z = \gamma_w(s)$ for $w = \sum_{i=1}^{n-1} a_i E_i(t)$ where γ_w is the geodesic with $\gamma_w'(0) = w$ and $\sum_{i=1}^{n-1} a_i{}^2 = 1$, then

$$(x_1(z), \dots, x_{n-1}(z), x_n(z)) = (sa_1, \dots, sa_{n-1}, t_0 - t).$$

Since $c(s) \in M \setminus C(p)$ for $s \neq 0$, we have $\tilde{c}(s) = \psi^{-1}(\exp_p{}^{-1}(c(s))) = (v(s), t(s))$ such that $t(0) = t_0$ and $v(0) = 0$, i.e., $\tau(v(0)) = v_0$. We study how $t'(s)$ and $v'(s)$ behave as $s \to 0+0$. In the coordinate systems defined as above, let $\exp_p(t\tau(v)) = \exp_p(\psi(v,t)) = (x_1(v,t), \dots, x_n(v,t))$. If the partial derivative of x_i with respect to v_j is written by $x_{i,j}$, then $d\exp_p|_{t\tau(v)} \circ d\psi_{(v,t)}$ is expressed by

$$d\exp_p|_{t\tau(v)} \circ d\psi_{(v,t)} = \begin{pmatrix} x_{1,1}(v,t) & \cdots & x_{1,n}(v,t) \\ \vdots & & \vdots \\ x_{n,1}(v,t) & \cdots & x_{n,n}(v,t) \end{pmatrix}$$

with respect to the bases $\left\{\frac{\partial}{\partial v_j}, \dots, \frac{\partial}{\partial v_{n-1}}, \frac{\partial}{\partial t}\right\}$ and $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$. Let the column vectors

$$\begin{pmatrix} y_{1i}(v,t) \\ \vdots \\ y_{ni}(v,t) \end{pmatrix}, \quad i = 1, \dots, n-1,$$

denote the Jacobi fields

$$Y_i(v,t) = \frac{\partial \exp_p(t\tau(v))}{\partial v_i}$$

along $\gamma_{\tau(v)}$ with respect to the basis $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$. Then, we have

$$d\exp_p|_{t\tau(v)} \circ d\psi_{(v,t)} = \begin{pmatrix} y_{11}(v,t) & \cdots & y_{1n-1}(v,t) & x_{1,n}(v,t) \\ \vdots & & \vdots & \vdots \\ y_{n1}(v,t) & \cdots & y_{nn-1}(v,t) & x_{n,n}(v,t) \end{pmatrix}.$$

Here the n-th column vector is the coordinate of $\gamma_{\tau(v)}{}'(t)$. When we assume that there exists a positive integer $k := \nu(v) = \dim\ker d\exp_p|_{\lambda(v)\tau(v)} > 0$ for all v in a neighborhood U of 0, it follows from the implicit function theorem that $\lambda(v) := \lambda(\tau(v))$ is smooth in U. Furthermore, we can choose a coordinate system around q such that

$$d\exp_p|_{\lambda(v)\tau(v)} \circ d\psi_{(v,\lambda(v))}$$
$$= \begin{pmatrix} 0 & \cdots & 0 & y_{1k+1}(v,\lambda(v)) & \cdots & y_{1n-1}(v,\lambda(v)) & x_{1,n}(v,\lambda(v)) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & y_{nk+1}(v,\lambda(v)) & \cdots & y_{nn-1}(v,\lambda(v)) & x_{n,n}(v,\lambda(v)) \end{pmatrix}.$$

From the mean value theorem, there exists a number $t_{ij}(v,t)$ for $i = 1, \dots, n$ and $j = 1, \dots, n-1$ between $\lambda(v)$ and t such that

$$x_{i,j}(v,t) = y_{ij}(v,\lambda(v)) + (t - \lambda(v)) y_{ij}{}'(v, t_{ij}(v,t)).$$

It should be noted that the covariant derivatives $D_{\gamma_{\tau(v)}'(t)}Y_i$ do not equal the differential $Y_i'(v,t)$ with respect to t, in general. However, because $Y_i(v,\lambda(v))=0$, we have $D_{\gamma_{\tau(v)}'(\lambda(v))}Y_i = Y_i'(v,\lambda(v))$.

We assume that $e_i = d\tau_0\big(\frac{\partial}{\partial v_i}\big)$ for $i=1,\dots,n-1$. Let $w_1 \in \mathrm{Span}(e_1,\dots,e_k)$ and $w_2 \in \mathrm{Span}(e_{k+1},\dots,e_{n-1})$ and $w_3 \in \mathbb{R}$. We briefly write $w=(w_1,w_2)=(w_1,\dots,w_{n-1})$, $w_1=(w_{11},w_{21},\dots,w_{k1})$ and $w_2=(w_{k+12},w_{k+22},\dots,w_{n-12})$.

It follows from Taylor's theorem with integral form of the remainder (see [Warner 1971, Lemma on p. 13]) that

$$\begin{aligned}\lambda(w_1,w_2)-\lambda(0,0) &= \sum_{i_1=1}^{n-1} w_{i_1}\lambda_{i_1}(0,0)+\cdots+\frac{1}{\ell!}\sum_{i_1,\dots,i_\ell=1}^{n-1} w_{i_1}\cdots w_{i_\ell}\lambda_{i_1\cdots i_\ell}(0,0)\\ &\qquad +\frac{1}{\ell!}\sum_{i_1,\dots,i_{\ell+1}=1}^{n-1} w_{i_1}\cdots w_{i_\ell+1}\int_0^1(1-s)^\ell\lambda_{i_1\cdots i_{\ell+1}}(sw)\,ds.\end{aligned}$$

Define an integer $m(w)>0$ by

$$m(w)=\min\Big\{\ell>0 \,\Big|\, \sum_{i_1,\dots,i_\ell} w_{i_1}\cdots w_{i_\ell}\lambda_{i_1\cdots i_\ell}(0,0)\neq 0,\quad i_j\in\{1,\dots,k\},\ j=1,\dots,\ell\Big\}.$$

We then define a function g by, if $m=m(w)\neq\infty$,

$$g(w_1)=\frac{1}{m!}\sum_{i_1,\dots,i_m} w_{i_1 1}\cdots w_{i_m 1}\lambda_{i_1\cdots i_m}(0,0),$$

where $i_j\in\{1,\dots,k\}$, $j=1,\dots,m$, and $g(w_1)=0$ if $m(w)=\infty$. Then g is a homogeneous function with degree m. Further we define a function f by

$$f(w_1,w_2,s,u)=\lambda((sw_1,uw_2))-\lambda(0,0)-s^m g(w_1)$$

for any $(w_1,w_2)\in T_0\mathbb{R}^{n-1}=\mathbb{R}^k\times\mathbb{R}^{n-k-1}$. From the definition of f, each term contains the parameter u as a factor or the order of s is greater than m. In particular,

$$\lim_{s\to 0}\frac{f(w_1,w_2,s,s^{1+m})}{s^m}=0,$$

or, in other words,

$$\lim_{s\to 0}\frac{\lambda((sw_1,s^{1+m}w_2))-\lambda(0,0)}{s^m}=g(w_1).$$

Hence we have

$$\begin{aligned}
&x_{i,j}(sw_1, uw_2, t)\\
&= y_{ij}(sw_1, uw_2, \lambda(sw_1, uw_2))\\
&\quad + (t - \lambda(sw_1, uw_2)) {y_{ij}}'(sw_1, uw_2, t_{ij}(sw_1, uw_2, t))\\
&= y_{ij}(sw_1, uw_2, \lambda(sw_1, uw_2))\\
&\quad + (t - \lambda(0,0) - s^m g(w_1) - f(w_1, w_2, s, u)) {y_{ij}}'(sw_1, uw_2, t_{ij}(sw_1, uw_2, t)).
\end{aligned}$$

Recall for the next step that $\lambda(0,0) = t_0$ and $y_{ij}(v, \lambda(v)) = 0$ for $i = 1, \dots, n$ and $j = 1, \dots, k$ where v are near 0.

We first consider a curve

$$\tilde{c}(s) = \psi((sw_1, sw_2, t_0 + sw_3))$$

and set $c(s) = \exp_p(\tilde{c}(s))$. Since $\tilde{c}(0) = \psi((0, 0, t_0))$ and $\tilde{c}'(0) = d\psi((w_1, w_2, w_3))$, we have

$$c'(0) = \sum_{j=k+1}^{n-1} w_{j2} Y_j(0, 0, t_0) - w_3 E_n(t_0).$$

Therefore, we see that

$$c'(0) \in \operatorname{Span}\big(Y_{k+1}(0, 0, t_0), \dots, Y_{n-1}(0, 0, t_0), E_n(0, 0, t_0)\big).$$

Next, we consider a curve $\tilde{c}(s) = \psi((sw_1, s^{1+m} w_2, t_0 + s^{1+m} w_3))$ for $w = (w_1, w_2)$ with $m = m(w) > 0$ and set $c(s) = \exp_p(\tilde{c}(s))$. Then, we have $\tilde{c}(0) = \psi((0, 0, t_0))$ and

$$\tilde{c}'(s) = d\psi((w_1, (1+m)s^m w_2, (1+m)s^m w_3)).$$

Moreover, we have

$$\begin{aligned}
&\lim_{s\to 0} \frac{c'(s)}{s^m}\\
&= -g(w_1) \sum_{j=1}^{k} w_{j1} {Y_j}'(0,0,t_0) + (1+m) \sum_{j=k+1}^{n-1} w_{j2} Y_j(0,0,t_0) - (1+m) w_3 E_n(t_0).
\end{aligned}$$

Since $\{{Y_1}'(0, 0, t_0), \dots, {Y_k}'(0, 0, t_0), Y_{k+1}(0, 0, t_0), \dots, Y_{n-1}(0, 0, t_0)\}$ spans the orthogonal complement of ${\gamma_{\tau(0)}}'(t_0)$ in $T_{\gamma_{\tau(0)}(t_0)} M$, the above vector may become any tangent vector in $T_{\gamma_{v_0}(t_0)} M$. After changing the parameter, we have a curve

$$\tilde{c}(s) = \psi\big(((1+m)s)^{1/(1+m)} w_1, (1+m)sw_2, t_0 + (1+m)sw_3\big).$$

The image of this curve is the same as the previous one. We note that $\tilde{c}(s)$ is not differentiable at $s = 0$, but $c(s)$ is differentiable at $s = 0$.

Summarizing the discussion so far, we get the following theorem.

Theorem 3.4. *Let M be a geodesically forward complete Finsler manifold without boundary and $p \in M$. Let $\gamma(t) = \exp_p(tv)$ be a minimal geodesic from p to q such that $q = \gamma(t_0)$ is conjugate to p along γ. Suppose $k = \dim\ker d\exp_p|_{t_0 v} \geq 1$ is constant around v in S_pM. Let Jacobi vector fields $Y_1, \ldots, Y_{n-1}$ be defined as above such that $Y_1(t_0) = \cdots = Y_k(t_0) = 0$. Then we have an orthogonal sum $T_qM = W_1 + W_2 + \mathrm{Span}(\gamma'(t_0))$, where $W_1 = \mathrm{Span}(Y_1{}'(t_0), \ldots, Y_k{}'(t_0))$ and $W_2 = \mathrm{Span}(Y_{k+1}(t_0), \ldots, Y_{n-1}(t_0))$. Let $c(s)$ and $\tilde{c}(s)$, $s \in (0, 1)$, be smooth curves such that $c(s) = \exp_p(\tilde{c}(s)) \in M \setminus C(p)$ for all $s \in (0, 1)$, and $\lim_{s\to 0} c(s) = q$ and $\lim_{s\to 0} \tilde{c}(s) = t_0 v$. Then the following are true.*

(1) *If $\tilde{c}(s)$ is differentiable at $s = 0$, then $c'(0) \in W_2 + \mathrm{Span}(\gamma'(t_0))$.*

(2) *Suppose that $c'(0)$ exists and $w \neq 0$ is the projection of $c'(0)$ to W_1. If there exists a vector $w_1 \in \mathbb{R}^k$ such that $w = -g(w_1)\sum_{j=1}^{k} w_{j1} Y_j{}'(t_0)$, then $\lim_{s\to 0} \mathrm{Span}(\tilde{c}'(s)) \in \ker d\exp_p|_{t_0 v}$.*

4. Sending curves into the tangent space by $\exp_p{}^{-1}$

Let (M, F) be a geodesically forward complete Finsler manifold without boundary. For $q \in C(p)$ let $T_qC(p)$ be the *tangent cone* of $C(p)$ at q, i.e., $v \in T_qC(p) \setminus \{0\}$ if and only if there exists a sequence of vectors $v_j \in T_qM$ converging to 0 with $\exp_q(v_j) \in C(p)$ such that $v/F(q, v) = \lim_{j\to\infty} v_j/F(q, v_j)$. We do not know whether $\{-v \mid v \in A_p(q)\} \cap T_qC(p) = \varnothing$ is true or not.

The *Voronoi region* $V(v)$ for $v \in A_p(q)^s$ in T_qM is defined by

$$V(v) = \{w \in T_qM \mid \omega_v(w) < \omega_u(w) \text{ for all } u \in A_p(q)^s \text{ with } u \neq v\}.$$

Suppose $A_p(q)^s$ consists of more than one vector. If

$$H_v(u) := \{w \in T_qM \mid \omega_v(w) - \omega_u(w) < 0\}$$

for any $u \in A_p(q)^s$ with $u \neq v$, then $V(v) = \bigcap_{u\in A_p(q)^s, u\neq v} H_v(u)$. If $A_p(q)^s = \{v\}$, we then set $V(v) = T_qM \setminus \{0\}$. It follows that $w \in V(v)$ if and only if there exists the unique vector $v \in A_p(q)^s$ such that $\omega_v(w) = \min\{\omega_u(w) \mid u \in A_p(q)^s\}$.

Lemma 4.1. *The following are true.*

(1) *$-v \in V(v)$ for any $v \in A_p(q)^s$. If $v \in A_p(q)^s$ and $A_p(q)^s \neq \{v\}$, then $v \notin V(v)$.*

(2) *$V(v)$ is a cone for any $v \in A_p(q)^s$, i.e., $\mu w \in V(v)$ for any $w \in V(v)$ and any $\mu > 0$.*

(3) *$V(v) \cup \{0\}$ is convex, i.e., $\mu w_1 + (1-\mu)w_2 \in V(v) \cup \{0\}$ for any $\mu \in [0, 1]$ if $w_1, w_2 \in V(v)$.*

(4) *$V(v) \cap V(u) = \varnothing$ and $\overline{V}(v) \cap \overline{V}(u) \subset \ker(\omega_v - \omega_u)$ for any $u, v \in A_p(q)^s$ with $u \neq v$.*

(5) $\bigcup_{v\in A_p(q)^s} V(v)$ *is dense in* T_qM. *In particular*, $\bigcup_{v\in A_p(q)^s} \overline{V(v)} = T_qM$.

Proof. From the Cauchy–Schwarz inequality (see [Shen 2001, Lemma 1.2.6]: $\omega_u(w) \le F(u)F(w)$ for all $w \in T_qM$ with equality holding if and only if $w = \mu u$ for some $\mu \ge 0$), we have

$$\omega_u(-v) \ge -F(v)F(u) \ge -1 = \omega_v(-v)$$

for all $u \in A_p(q)^s$. Suppose the equality $\omega_u(-v) = \omega_v(-v)$ holds. Then we have $\omega_u(v) = \omega_v(v) = F(v)F(u)$. Hence $v = \mu u$ for some $\mu \ge 0$. Then, $1 = \omega_u(v) = g_u(u, \mu u) = \mu$, meaning $u = v$. Therefore, the equality does not hold if $u \ne v$.

If there exists a vector $u \in A_p(q)^s$ with $u \ne v$, then $\omega_v(v) = 1 = F(v)F(u) > \omega_u(v)$, meaning that $v \notin V(v)$. These prove (1).

Since ω_v is the homogeneous with degree 1 for every $v \in A_p(q)^s$, we have (2).

We prove (3). If $A_p(q)^s = \{v\}$, we then have nothing to prove because $V(v) = T_qM \setminus \{0\}$. Assume that $v \in A_p(q)^s$ and $A_p(q)^s \ne \{v\}$. Let $w_1, w_2 \in V(v)$. We then have $\omega_v(w_1) < \omega_u(w_1)$ and $\omega_v(w_2) < \omega_u(w_2)$ for any $u \in A_p(q)^s$ with $u \ne v$. We may assume that $\mu w_1 + (1-\mu)w_2 \ne 0$ for $\mu \in [0, 1]$. Then, we have

$$\begin{aligned}\omega_v(\mu w_1 + (1-\mu)w_2) &= \mu\omega_v(w_1) + (1-\mu)\omega_v(w_2)\\ &< \mu\omega_u(w_1) + (1-\mu)\omega_u(w_2)\\ &= \omega_u(\mu w_1 + (1-\mu)w_2)\end{aligned}$$

because of (2), proving that $\mu w_1 + (1-\mu)w_2 \in V(v)$.

From the definition of $V(v)$, property (4) is a direct consequence.

We prove (5). Suppose that there exists $w \in T_qM \setminus \bigcup_{v\in A_p(q)^s} V(v)$. It follows from (1) that $w \ne -v$ for any $v \in A_p(q)^s$. We assume that $\omega_v(w) = \inf\{\omega_u(w) \mid u \in A_p(q)^s\}$ for some $v \in A_p(q)^s$. Set $w(\varepsilon) = w - \varepsilon v$ for any $\varepsilon > 0$. Then, for any $u \in A_p(q)^s$ with $u \ne v$, we have

$$\begin{aligned}\omega_u(w(\varepsilon)) &= \omega_u(w) + \varepsilon\omega_u(-v)\\ &> \omega_v(w) - \varepsilon\\ &= \omega_v(w) + \varepsilon\omega_v(-v)\\ &= \omega_v(w(\varepsilon)).\end{aligned}$$

From this, we see that $w(\varepsilon) \in V(v)$, and, hence, $w \in \overline{V(v)} \subset \overline{\bigcup_{v\in A_p(q)^s} V(v)}$. This implies that $\overline{\bigcup_{v\in A_p(q)^s} V(v)} = T_qM$. Thus, $\bigcup_{v\in A_p(q)^s} \overline{V(v)} = T_qM$. □

Let $W_q := \bigcup_{v\in A_p(q)^s} V(v)$ which is dense in T_qM at $q \in C(p)$.

Remark 4.2. We do not know whether $W_q \cap T_qC(p) = \varnothing$ is true or not.

Let $B_f(q, \varepsilon)$ be the forward distance ball with center q and radius ε.

Lemma 4.3. *Let $q \in C(p)$. If q is not a conjugate cut point, then $W_q \cap T_q C(p) = \varnothing$. In particular, for any smooth curve $c : [0, 1] \to M$ with $c(0) = q$ and $c'(0) \in W_q$, there exists a number $\delta > 0$ such that $c(s) \in M \setminus C(p)$ for all $s \in (0, \delta)$.*

Proof. Since q is not a conjugate cut point, there exists a number $\varepsilon > 0$ such that $B_f(q, \varepsilon) \cap C(p)$ is a union of smooth hypersurfaces with boundaries (see [Ozols 1974]). This implies that $w \in W_q$ if and only if $w \notin T_q C(p)$. □

Lemma 4.4. *If $w \in W_q \setminus T_q C(p)$, then there exist the unique point $x \in \widetilde{U}_p^c$ and a curve $\tilde{c} : [0, \varepsilon) \to T_p M$ such that $\tilde{c}(0) = x$, $\tilde{c}$ is of class C^∞ on $(0, \varepsilon)$ and*

$$\left.\frac{d \exp_p \circ \tilde{c}}{ds}\right|_{s=0} = w.$$

Proof. Assume that $w \in V(v)$ for $v \in A_p(q)^s$. Let $\gamma : [0, d(p, q)] \to M$ be the minimal geodesic from p to q such that $\gamma'(d(p, q)) = v$ and let $c : [0, \varepsilon) \to M$ be a curve such that $c'(0) = w$. From Lemma 2.1, it follows that

$$\left.\frac{dd(p, c(s))}{ds}\right|_{s=0} = \min\{\omega_u(w) \mid u \in A_p(q)^s\} = \omega_v(w).$$

This implies that a sequence of minimal geodesics $T(p, c(s))$ from p to $c(s)$ converges to γ as $s \to 0+0$. Then $x = d(p, q)\gamma'(0)$. Since $w \notin T_q C(p)$, there exists a unique minimal geodesic $T(p, c(s))$ from p to $c(s)$ for a sufficiently small $s > 0$. If $\tilde{c}(s) = d(p, c(s))y(s)$, where $y(s)$ are the initial tangent vectors of $T(p, c(s))$ at p, then $c(s) = \exp_p(\tilde{c}(s))$ for $s \in [0, \varepsilon)$. □

Remark 4.5. As was seen in Theorem 3.4, $\tilde{c}'(0)$ may not exist: let M be a surface. Then we know from [Shiohama and Tanaka 1996] that $C(p)$ is locally a tree. Suppose there exists an end point q of $C(p)$ such that the sufficiently short edge e ending at q is smooth. Then q is a point conjugate to p along a minimal geodesic γ_v from p to q. If γ_v is the unique minimal geodesic, then the edge e has two lifts $\tilde{e}_1$ and $\tilde{e}_2$ in $T_p M$ by $\exp_p$ such that $\tilde{e}_1 \cap \tilde{e}_2 = \{\lambda(v)v\}$ where $\exp_p(\lambda(v)v) = q$. They are tangent at $\lambda(v)v$, i.e., they are linearly dependent because $\lambda'(v) = 0$ as mentioned in Section 1. Let N be the tangent space of the circle with center origin and radius $d(p, q)$ in $T_p M$. Then $\ker d \exp_p |_x = N$. Therefore, we can not find any $\tilde{w}$ such that $d \exp_p |_x(\tilde{w}) = w$ if w is a tangent vector at q orthogonal to ${\gamma_v}'(d(p, q))$.

Theorem 4.6. *Let $c(s)$, $s \in [a, b]$, be a curve of class C^1 in M such that $c'(a) \neq 0$ and $c(s) \notin C(p)$ for all $s \in (a, b]$. Then there exists the unique curve $\tilde{c}(s)$, $s \in [a, b]$, in $\widetilde{U}_p^c$ such that $\exp_p(\tilde{c}(s)) = c(s)$ for all $s \in [a, b]$.*

Proof. Since $\exp_p : \widetilde{U}_p \to U_p$ is a diffeomorphism, $\tilde{c}(s) = (\exp_p |_{\widetilde{U}_p})^{-1}(c(s))$ satisfies the required condition in $s \in (a, b]$. When $c(a) \notin C(p)$, $\tilde{c}(a) = (\exp_p |_{\widetilde{U}_p})^{-1}(c(a))$ is also defined. Therefore, $\tilde{c}(s)$, $s \in [a, b]$, is the unique curve mentioned in the theorem.

Assume that $q := c(a) \in C(p)$. Let $\gamma_s : [0, d(p, c(s))] \to M$ be minimal geodesics from p to $c(s)$ for all $s \in (a, b]$. Assume that $v \in A_p(q)^s$ is an accumulation tangent vector of $\gamma_s{}'(d(p, c(s)))$ as $s \to 0$. If $w := c'(a) \in V(v) \setminus T_qC(p)$, then we can have $x \in T_pM$ and a curve $\tilde{c}$ mentioned in this theorem as was seen in Lemma 4.4 and $\gamma_s{}'(d(p, c(s)))$ converges to v as $s \to a + 0$.

Assume that $w \in \overline{V}(v) \setminus V(v)$. For any sufficiently small $\varepsilon > 0$, let $c_\varepsilon(s) = \gamma_s(d(p, c(s)) - \varepsilon s)$ for $s \in (a, b]$. Then we have $c_\varepsilon(s) \notin C(p)$ and $c_\varepsilon{}'(a) = w - \varepsilon v$. Since $c_\varepsilon{}'(a) \in V(v) \setminus T_qC(p)$, we have a curve $\tilde{c}_\varepsilon(s)$ such that $\exp_p(\tilde{c}_\varepsilon(s)) = c_\varepsilon(s)$. Thus, $\exp_p(\tilde{c}(s)) = c(s)$ as $\varepsilon \to 0 + 0$. □

We call a map $\tilde{c} : [a, b] \to \widetilde{U}_p^c$ a *pull back curve* through $\exp_p$ (briefly, *pb-curve*) if there exists a set of numbers $\{a_\lambda \mid \lambda \in \Lambda\} \cup \{b_\lambda \mid \lambda \in \Lambda\}$ in $[a, b]$ satisfying the following conditions:

(1) $(a_\lambda, b_\lambda) \cap (a_{\lambda'}, b_{\lambda'}) = \varnothing$ for $\lambda, \lambda' \in \Lambda$ with $\lambda \neq \lambda'$ and $\tilde{c}^{-1}(\widetilde{U}_p^c \setminus \widetilde{C}(p)) \setminus \{a, b\} = \bigcup_{\lambda \in \Lambda}(a_\lambda, b_\lambda)$. In particular, we have $\tilde{c}\big((a, b) \setminus \bigcup_{\lambda \in \Lambda}(a_\lambda, b_\lambda)\big) \subset \widetilde{C}(p)$.

(2) Let $\tilde{c}_\lambda := \tilde{c}|_{[a_\lambda+0, b_\lambda-0]}$ for every $\lambda \in \Lambda$. Then $\tilde{c}_\lambda$ is continuous on $[a_\lambda, b_\lambda]$ and piecewise smooth on (a_λ, b_λ) for each $\lambda \in \Lambda$.

(3) Let $c := \exp_p \circ \tilde{c}$. Then $c : [a, b] \to M$ is a piecewise smooth curve.

Since c is piecewise smooth, condition (3) implies that $c(b_\lambda - 0) = c(a_{\lambda'} + 0)$ if $b_\lambda = a_{\lambda'}$. Since the pb-curve $\tilde{c}$ is not assumed to be continuous, it may happen that $\tilde{c}(b_\lambda - 0) \neq \tilde{c}(a_{\lambda'} + 0)$ even if $c(b_\lambda - 0) = c(a_{\lambda'} + 0) \in C(p)$.

We say that a nonconjugate cut point q of p is *normal* if exactly two minimal geodesics from p to q exist in M. It follows from the implicit function theorem that the set of all normal cut points of p makes a smooth hypersurface of M. Furthermore, Itoh and Tanaka [1998] proved that the Hausdorff dimensions of the sets of all conjugate cut points of p and all nonnormal cut points of p are not greater than $\dim M - 2$.

Lemma 4.7. *Let $c : [a, b] \to M$ be a piecewise smooth curve. Assume that c does not pass through any conjugate cut point and any nonnormal cut point of p. Then there exists a pb-curve $\tilde{c} : [a, b] \to \widetilde{U}_p^c$ such that $\exp_p(\tilde{c}(t)) = c(t)$ and $\tilde{c}$ is a union of piecewise smooth curves.*

Proof. Since $M \setminus C(p)$ is an open set, there exists a set of numbers

$$\{a_\lambda \mid \lambda \in \Lambda\} \cup \{b_\lambda \mid \lambda \in \Lambda\}$$

such that

$$c^{-1}(M \setminus C(p)) \setminus \{a, b\} = \bigcup_{\lambda \in \Lambda}(a_\lambda, b_\lambda).$$

Then we have curves $\tilde{c}_\lambda : (a_\lambda, b_\lambda) \to \widetilde{U}_p^c \setminus \widetilde{C}(p)$ such that $\exp_p \circ \tilde{c}_\lambda = c|_{(a_\lambda, b_\lambda)}$ for all $\lambda \in \Lambda$. The domains of these curves are extended to their closures because of Lemma 4.3 and Theorem 4.6.

Each connected component ℓ of $c([a,b]) \cap C(p)$ has two curves $\tilde{g}$ and $\tilde{h}$ in $\widetilde{C}(p)$, i.e., $\ell = \exp_p \circ \tilde{g} = \exp_p \circ \tilde{h}$, since ℓ is a piecewise smooth curve in a smooth hypersurface in M consisting of normal cut points of p. We choose one of $\tilde{g}$ and $\tilde{h}$ for a pb-curve of ℓ. Thus the union of those curves and $\tilde{c}_\lambda$, $\lambda \in \Lambda$, makes a pb-curve $\tilde{c}$ of c. □

5. The relation between the distances induced by F and F^*

Let (M, F) be a geodesically forward complete Finsler manifold without boundary. For a piecewise smooth curve $\tilde{c}(t)$, $t \in [a, b]$, in $\widetilde{U}_p^c$ the length $L(\tilde{c})$ of $\tilde{c}$ is defined by

$$L(\tilde{c}) = \int_a^b F^*(\tilde{c}(t), \tilde{c}'(t))\, dt.$$

It follows that $L(\tilde{c}) = L(\exp_p \circ \tilde{c})$. Let $\Omega(x, y)$ denote the set of all piecewise smooth curves in $\widetilde{U}_p^c$ from x to y for $x, y \in \widetilde{U}_p^c$ and $\Omega_0(x, y)$ the set of all pb-curves in $\widetilde{U}_p^c$ whose image by $\exp_p$ connects $\exp_p(x)$ to $\exp_p(y)$ in M. Obviously, we have $\Omega(x, y) \subset \Omega_0(x, y)$. For $\tilde{c} \in \Omega_0(x, y)$, we have

$$L(c) = L(\tilde{c}) = \sum_{\lambda \in \Lambda} \int_{a_\lambda}^{b_\lambda} F^*(\tilde{c}_\lambda(t), {\tilde{c}_\lambda}'(t))\, dt + \int_{[a,b] \setminus \cup_{\lambda \in \Lambda}(a_\lambda, b_\lambda)} F^*(\tilde{c}(t), \tilde{c}'(t))\, dt,$$

from the definition of a pb-curve.

We define the pseudodistances $d^*(x, y)$ and $d_0^*(x, y)$ from x to y by

$$d^*(x, y) = \inf\{L(\tilde{c}) \mid \tilde{c} \in \Omega(x, y)\}$$
$$d_0^*(x, y) = \inf\{L(\tilde{c}) \mid \tilde{c} \in \Omega_0(x, y)\}.$$

It follows that $d^*(x, y) \geq d_0^*(x, y) \geq d(\exp_p(x), \exp_p(y))$ for any $x, y \in \widetilde{U}_p^c$. It may happen that $d^*(x, y) = 0$ for $x \neq y$ when there exists a curve $c(t)$, $t \in [a, b]$, from x to y in $\widetilde{C}(p)$ such that $d\exp_p(\tilde{c}'(t)) = 0$ for $t \in [a, b]$.

Lemma 5.1. *For $d_0^*(x, y)$ as above, we have $d_0^*(x, y) = d(\exp_p(x), \exp_p(y))$ for any $x, y \in \widetilde{U}_p^c$.*

Proof. We prove that

$$d_0^*(x, y) \leq d(\exp_p(x), \exp_p(y)).$$

For any $\varepsilon > 0$ let $c : [a, b] \to M$ be a piecewise smooth curve from $\exp_p(x)$ to $\exp_p(y)$ such that $L(c) < d(\exp_p(x), \exp_p(y)) + \varepsilon$. Since the Hausdorff dimensions of the sets of all conjugate cut points of p and all nonnormal cut points of p are not greater than $\dim M - 2$ (see [Itoh and Tanaka 1998, Lemmas 2 and 3; Federer

1969]), we may assume that c does not pass those points. Then we can apply Lemma 4.7 to obtain a pb-curve $\tilde{c}$ such that $\exp_p \circ \tilde{c} = c$. Note that if this pb-curve $\tilde{c}$ does not satisfy $\tilde{c}(a) = x$ and $\tilde{c}(b) = y$, then those end points are replaced by x and y, because $\lim_{t \to a} \exp_p(\tilde{c}(t)) = \exp_p(x)$ and $\lim_{t \to b} \exp_p(\tilde{c}(t)) = \exp_p(y)$. The resulting curve $\tilde{c}$ after this change is a pb-curve as well and connects from x to y such that $\exp_p(\tilde{c}(t)) = c(t)$ for all $t \in [a, b]$. Therefore, we have

$$d_0^*(x, y) \leq L(\tilde{c}) = L(c) < d(\exp_p(x), \exp_p(y)) + \varepsilon,$$

and, hence, $d_0^*(x, y) \leq d(\exp_p(x), \exp_p(y))$. □

We define an equivalence relation $\sim$ in $\widetilde{U}_p^c$ as follows: $x \sim y$ if and only if $d_0^*(x, y) = 0$. Let $[x]$ denote the equivalence class of this relation $\sim$ containing $x \in \widetilde{U}_p^c$ and $\widetilde{U}_p^c/\!\sim\, = \{[x] \mid x \in \widetilde{U}_p^c\}$. It follows from Lemma 5.1 that $[x] = [y]$ if and only if $\exp_p(x) = \exp_p(y)$ for any $x, y \in \widetilde{U}_p^c$. We define a metric $d_1^*([x], [y])$ on $\widetilde{U}_p^c/\!\sim$ by $d_1^*([x], [y]) = d_0^*(x, y)$ for any $x, y \in \widetilde{U}_p^c$.

Lemma 5.2. *Let d_1^* be the distance defined as above. Then (M, d) is isometric to $(\widetilde{U}_p^c/\!\sim, d_1^*)$ where d is the distance induced by F on M.*

The following theorem is a direct consequence of Lemma 5.2.

Theorem 5.3. *Let (M, F_M) and (N, F_N) be geodesically forward complete Finsler manifolds and $p_M \in M$, $p_N \in N$. Assume that there exists a linear isomorphism $I : T_{p_M}M \to T_{p_N}N$ such that $F_M(p_M, x) = F_N(p_N, I(x))$ for all $x \in T_{p_M}M$ and $\exp_{p_N}(I(x)) = \exp_{p_N}(I(y))$ for all $x, y \in \widetilde{C}(p_M)$ with $\exp_{p_M}(x) = \exp_{p_M}(y)$. If $(\widetilde{U}_{p_M}^c, d_M^*)$ and $(\widetilde{U}_{p_N}^c, d_N^*)$ are isometric under the map I, then (M, F_M) and (N, F_N) are isometric.*

Proof. From the assumption, we see that $\widetilde{C}(p_N) = I(\widetilde{C}(p_M))$ and $[I(x)] = [I(y)]$ for all $x, y \in \widetilde{C}(p_M)$ with $[x] = [y]$. Therefore, $(\widetilde{U}_{p_M}^c/\!\sim, d_{p_M}{}_1^*)$ is isometric to $(\widetilde{U}_{p_N}^c/\!\sim, d_{p_N}{}_1^*)$. This theorem follows from Lemma 5.2. □

Without assuming the invariance of the equivalence classes under the map I, we do not have an isometry from M to N. In fact, there exist surfaces (M, F_M) and (N, F_N) such that they are not isometric, although $(\widetilde{U}_{p_M}^c, d_M^*)$ and $(\widetilde{U}_{p_N}^c, d_N^*)$ are isometric.

Example 5.4. Let $a > b > 0$. Let T^2 be a torus defined by

$$T^2 := \big\{\big((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u\big) \bigm| 0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi\big\}.$$

Let $M = T^2$. We make a surface N in the following way. Cut and open T^2 along the meridian circle $v = \pi$, and then glue the boundary as follows:

$$((a + b\cos u)\cos(\pi - 0), (a + b\cos u)\sin(\pi - 0), b\sin u)$$

and

$$((a + b\cos(-u))\cos(\pi + 0), (a + b\cos(-u))\sin(\pi + 0), b\sin(-u))$$

are identified for $-\pi \le u \le \pi$. The resulting surface N is a Klein bottle. Hence M is not isometric to N. However, if $p_M \in M$ and $p_N \in N$ are the points corresponding to $(a+b, 0, 0)$, then $(\widetilde{U}^c_{p_M}, d^*_M)$ is isometric to $(\widetilde{U}^c_{p_N}, d^*_N)$.

We next see the relation between $(\widetilde{U}^c_p, d^*)$ and (M, d). We say that a critical point $q \in C(p)$ is *jointing* if there exists a number $\delta > 0$ such that $B_b(q, \varepsilon) \cap B_f(p, d(p, q))$ is not connected for any $\varepsilon \in (0, \delta)$, where $B_b(x, \tau)$ and $B_f(x, \tau)$ are backward and forward open distance balls with center $x \in M$ and radius $\tau > 0$, respectively. It follows from Theorem 4.3 in [Innami et al. 2019] that if a nonconjugate cut point $q \in C(p)$ is a local minimum point of the distance function $d(p, \cdot)|_{C(p)}$, then q is jointing. Let $\mathrm{Join}(p)$ denote the set of all jointing cut points of p in M.

Lemma 5.5. *Let $q \in C(p)$ be jointing. Then $A_p(q)^s$ consists exactly two elements, say v and w, satisfying that $\omega_v + k\omega_w = 0$ for some number $k > 0$. In particular, q is a local minimum point of the function $d(p, \cdot)|_{C(p)}$.*

Proof. Since q is jointing, there exist at least two elements v and w in $A_p(q)^s$. Let $N(v) = \{z \in T_q M \mid \omega_v(z) = 0\}$. If $N(v) \ne N(w)$ for $v \ne w$, then there exists a curve $z(t)$, $t \in [0, 1]$, connecting $-v$ and $-w$ such that $\omega_v(z(t)) < 0$ and $\omega_w(z(t)) < 0$ for all $t \in [0, 1]$. This contradicts the fact that q is a jointing cut point. This implies that $\omega_v + k\omega_w = 0$ for some number $k > 0$ and there is no vector other than v and w in $A_p(q)^s$. □

Lemma 5.6. *Let $q \in C(p)$ be not jointing. Then, for any $\varepsilon > 0$, there exists a curve $c : [0, 1] \to M$ such that $c(0) = c(1) = q$, $d(p, c(t)) < d(p, q)$ for all $t \in (0, 1)$ and $L(c) < \varepsilon$.*

When $\dim M = 2$, we can see the detailed structure of the cut locus $C(p)$ in [Sabau and Tanaka 2016; Shiohama and Tanaka 1996]. Let $S_f(p, t)$ be the forward distance sphere with center p and radius t (the distance is measured from p) and $C_e(p)$ the set of all end cut points.

Lemma 5.7 [Shiohama and Tanaka 1996, Theorems A and B; Sabau and Tanaka 2016, Theorems B and C]. *Let* $\dim M = 2$. *Then the following are true.*

(1) *There exists a class $\mathscr{M} = \{m_1, \ldots\}$ of countably many rectifiable Jordan arcs $m_i : I_i \to C(p)$, $i = 1, \ldots,$ such that I_i is an open or closed interval and such that*

$$C(p) \setminus C_e(p) = \bigcup_{i=1}^{\infty} m_i(I_i), \quad \textit{disjoint union}$$

where each m_i has at most countably many branch points such that there are at most countably many members in $\mathscr{M}$ emanating from each of them.

(2) *There exists a set $\mathscr{E}_M \subset (0, \infty)$ of measure zero with the following properties. For every $t \notin \mathscr{E}_M$ with $t > 0$, there exist at most two minimal geodesics from p to every point $x \in S_f(p, t) \cap C(p)$. Furthermore, if $x \in S_f(p, t) \cap C(p)$ is joined from p by a unique minimal geodesic, then x is an end point of $C(p)$. There exists at most countably many points in $S_f(p, t) \cap C(p)$ which are joined from p by two distinct minimal geodesics.*

Some claims of this lemma can be translated into the tangent space T_pM under the exponential map $\exp_p$ as follows.

Lemma 5.8. *Let* $\dim M = 2$. *There exists a set $\mathscr{E}_M \subset (0, \infty)$ of measure zero with the following properties. For every $t \notin \mathscr{E}_M$ with $t > 0$, for every point $x \in S_f(p, t) \cap C(p)$, we have that ${\exp_p}^{-1}(x)$ consists of at most two points. Furthermore, if ${\exp_p}^{-1}(x)$ consists of a unique point, then x is an end point of $C(p)$. There exists at most countably many pairs of points $\tilde{x}$ and $\tilde{y}$ in $\widetilde{S}(0, t) \cap \widetilde{C}(p)$ such that $\exp_p(\tilde{x}) = \exp_p(\tilde{y}) =: z$, and, moreover, we may assume that $\tilde{x}$ and $\tilde{y}$ are interior points of ${\exp_p}^{-1}(m_i)$ for some members m_i in $\mathscr{M}$. In particular, there exist subarcs $\ell_{\tilde{x}}$ and $\ell_{\tilde{y}}$ of ${\exp_p}^{-1}(m_i)$ containing $\tilde{x}$ and $\tilde{y}$, respectively, with* $\exp_p(\ell_{\tilde{x}}) = \exp_p(\ell_{\ell_{\tilde{y}}}) \subset m_i$.

From the construction of m_i, we remark that, for every i, $d_i : I_i \to \mathbb{R}$ defined by $d_i(t) = d(p, m_i(t))$ for all $t \in I_i$ is strictly monotone.

Theorem 5.9. *Let (M, F_M) and (N, F_N) be geodesically forward complete Finsler orientable surfaces, that is,* $\dim M = 2$, *and $p_M \in M$, $p_N \in N$. Assume that $d(p_M, \cdot)^{-1}(a) \cap \mathrm{Join}(p_M)$ has at most one element for all $a \in \mathbb{R}$. If $(\widetilde{U}^c_{p_M}, d^*_M)$ and $(\widetilde{U}^c_{p_N}, d^*_N)$ are isometric under some linear map I from $T_{p_M}M$ to $T_{p_N}N$, then M and N are isometric.*

Proof. Let $\psi : M \setminus C(p_M) \to N \setminus C(p_N)$ be given by $\psi(x) = \exp_{p_N} \circ I \circ {\exp_{p_M}}^{-1}(x)$ for all $x \in M \setminus C(p_M)$. From Lemma 5.1, $B_f(p_M, r)$ is isometric to $B_f(p_N, r)$ for a sufficiently small $r > 0$. Let $R = \sup\{r \mid B_f(p_M, r)$ is isometric to $B_f(p_N, r)\}$. We prove $R = \infty$. Suppose for indirect proof that $R < \infty$.

We prove that the map ψ can be extended to $B_f(p_M, R) \cup S_f(p_M, R)$ isometrically. It follows from Lemma 5.6 that $\exp_{p_M}(\tilde{x}) = \exp_{p_M}(\tilde{y})$ if and only if $\exp_{p_N}(I(\tilde{x})) = \exp_{p_N}(I(\tilde{y}))$ for all $\tilde{x}, \tilde{y} \in S(0, R) \cap \widetilde{C}(p_M)$ which are not tangent jointing cut points. If $\exp_{p_M}(\tilde{x}) = \exp_{p_M}(\tilde{y}) =: x$ is jointing, then x is a local minimum point of the function $d(p_M, \cdot)|_{C(p_M)}$ because of Lemma 5.5. From the assumptions on $\mathrm{Join}(p_M)$, $(\widetilde{U}^c_{p_M}, d^*_M)$ and $(\widetilde{U}^c_{p_N}, d^*_N)$, we see that $\exp_{p_N}(I(\tilde{x}))$ is a local minimum point of $d(p_N, \cdot)|_{C(p_N)}$ and, moreover,

$$d(p_N, \cdot)^{-1}(d_M(p, x)) \cap \mathrm{Join}(p_N)$$

has exactly one point. Thus, from Lemma 5.5, $\exp_{p_N}(I(\tilde{x})) = \exp_{p_N}(I(\tilde{y})) =: x_N$, since $A_{p_N}(x_N)^s$ consists of exactly two elements.

Since we suppose $R < \infty$, there exists a sequence of numbers $t_j \notin \mathscr{E}_M \cap \mathscr{E}_N$, $t_j > R$, such that there exist points $\tilde{x}_j$ and $\tilde{y}_j$ with $\exp_{p_M}(\tilde{x}_j) = \exp_{p_M}(\tilde{y}_j) =: x_j \in S(p_M, t_j) \cap C(p_M)$ but $\exp_{p_N}(I(\tilde{x}_j)) \neq \exp_{p_N}(I(\tilde{y}_j))$ for every j and they converge to points $\tilde{x}$ and $\tilde{y}$ with $\exp_{p_M}(\tilde{x}) = \exp_{p_M}(\tilde{y}) =: x \in S(p_M, R) \cap C(p_M)$, respectively. However this cannot happen. In fact, we may assume that x_j are contained in one member m_x in $\mathscr{M}_M$ emanating from $x \in C(p_M)$. Let $\ell_{\tilde{x}}$ and $\ell_{\tilde{y}}$ be subarcs of $\widetilde{C}(p_M)$ containing all $\tilde{x}_j$ and $\tilde{y}_j$, respectively. Since $t_j \notin \mathscr{E}_M \cap \mathscr{E}_N$ and the functions $t \mapsto d(p, I(\ell_{\tilde{x}}(t)))$ and $t \mapsto d(p, I(\ell_{\tilde{y}}(t)))$ are strictly monotone, we see that $I(\ell_{\tilde{x}})$ and $I(\ell_{\tilde{y}})$ are identified and members in $\mathscr{M}_N$. This implies that $\exp_{p_N}(I(\tilde{x}_j)) = \exp_{p_N}(I(\tilde{y}_j))$, a contradiction. □

6. Differentiable points of the cut locus

Let (M, F) be a geodesically forward complete Finsler manifold without boundary. Let $P(v, w)$ denote the vector subspace of T_qM spanned by $v, w \in T_qM$. Note that if $\{v, w\}$ is linearly dependent and $w \neq 0$, then the dimension of $P(v, w)$ is one.

Lemma 6.1. *Let $q \in C(p)$ and $w \in T_qM \setminus T_qC(p)$. Then there exists a tangent vector $v \in T_qC(p)$ such that $P(v, w) \cap A_p(q) \setminus \{0\} \neq \emptyset$.*

Proof. Let $c(s)$, $s \in [0, \delta)$, be a smooth curve such that $c(0) = q$, $c'(0) = w$ and $c(s) \notin C(p)$ for all $s \in (0, \delta)$. Let $\gamma_s : [0, \ell_s] \to M$ be maximal minimal geodesics from p through $c(s)$. Then γ_s converges a minimal geodesic $\gamma : [0, d(p, q)] \to M$ from p to q because of Theorem 4.6 and $\gamma_s(\ell_s) \in C(p)$ converges to q as $s \to 0$. If v_s denotes the initial tangent vector of the unit speed minimal geodesics from q to $\gamma_s(\ell_s)$ at q, then there exists a subsequence v_{s_j} converging to a vector v and $\gamma'(d(p, q)) \in P(v, w) \cap A_p(q)$. □

Lemma 6.2. *Let X, Y and Z be vector spaces such that X is the direct sum of Y and Z. For $z_1 \in Z$ (resp. $z_2 \in Z$), let Z_1 (resp. Z_2) be the vector subspace spanned by Y and z_1 (resp. z_2) in X. Then either $Z_1 = Z_2$ or $Z_1 \cap Z_2 = Y$. Furthermore, if $\{z_1, z_2\}$ is linearly independent, we then have $Z_1 \cap Z_2 = Y$.*

Proof. We prove that $\{z_1, z_2\}$ is linearly dependent if $Z_1 \cap Z_2 \neq Y$. Suppose $Z_1 \cap Z_2 \neq Y$. Then there exists a vector $z \in Z_1 \cap Z_2 \setminus Y$. Hence $z = ay_1 + bz_1 = cy_2 + dz_2$ for some $y_1, y_2 \in Y$ and $a, b, c, d \in \mathbb{R}$ with $b \neq 0$, $d \neq 0$. Since $ay_1 - cy_2 = -bz_1 + dz_2$ and $Y \cap Z = \{0\}$, we have $z_2 = (b/d)z_1$. Therefore $Z_1 \cap Z_2 \neq Y$ implies that $Z_1 = Z_2$. □

Let $H(T_qC(p))$ and $L(T_qC(p))$ be the convex hull of $T_qC(p)$ in T_qM and the vector subspace generated by $H(T_qC(p))$ in T_qM, respectively. Obviously, $T_qC(p) \subset H(T_qC(p)) \subset L(T_qC(p))$. We say that $q \in C(p)$ is an *end cut point* of $C(p)$ if $\gamma'(d(p, q)) \in T_qC(p)$ for some minimal geodesic γ from p to q.

Lemma 6.3. *Let $q \in C(p)$ be not an end cut point and let a vector subspace V be a complement of $L(T_qC(p))$ in T_qM. Suppose* $\dim V \geq 1$. *Then $A_p(q)$ contains a cone which is a graph over V in $T_qM = V + L(T_qC(p))$. Namely, there exists a function $f : V \to T_qC(p)$ such that f is positively homogeneous and $A_p(q) = \{(w, f(w)) \mid w \in V\}$.*

Proof. Let $w \in V \setminus \{0\}$. As was seen in the proof of Lemma 6.1, we have a tangent vector $v \in T_qC(p)$ and a minimal geodesic segment γ from p to q such that $\gamma'(d(p,q)) \in P(v, w) \cap A_p(q)$ i.e., $\gamma'(d(p,q)) = aw + bv$ for certain numbers a, $b \in \mathbb{R}$. Since q is not an end cut point of $C(p)$, we have $a \neq 0$. Moreover, from Lemma 6.2, a map from $V \setminus \{0\}$ to T_qM, $aw \mapsto aw + bv$, is injective. In fact, if $aw_1 + bv_1 = cw_2 + dv_2$ for some $a, b, c, d \in \mathbb{R}$ with $a \neq 0$, $c \neq 0$ and $w_1, w_2 \in V$, $v_1, v_2 \in T_qC(p)$, then $\mathrm{Span}(w_1, L(T_qC(p))) = \mathrm{Span}(w_2, L(T_qC(p))) \neq L(T_qC(p))$. Therefore w_1 and w_2 are linearly dependent and, moreover, $aw_1 = cw_2$. Thus, we can define a map $f : V \to T_qC(p)$ by $f(w) = (b/a)v$ and $f(0) = 0$; then $A_p(q)$ is the graph of f over V in T_qM. From the construction of f, we see that f is positively homogeneous on V. □

Remark 6.4. Because there are various selection methods of the complement of $L(T_qC(p))$ in T_qM, this cone may not be determined uniquely (see Examples 7.2 and 7.4).

We say that $C(p)$ is *differentiable* at $q \in C(p)$ if $T_qC(p)$ is a vector subspace of T_qM, i.e., $L(T_qC(p)) = T_qC(p)$. From the definition, $C(p)$ is not differentiable at any end cut point of $C(p)$, because there exists a minimal geodesic γ from p to q such that $\gamma'(d(p,q)) \in T_qC(p)$ and $-\gamma'(d(p,q)) \notin T_qC(p)$, implying that $T_qC(p)$ is not a vector subspace.

The following theorem is a direct consequence of Lemma 6.3.

Theorem 6.5. *If $C(p)$ is differentiable at $q \in C(p)$, then $A_p(q)$ contains a cone which is a graph over the complement V of $T_qC(p)$ in T_qM. In particular, the union S of minimal geodesics $T(p,q)$ from p to q contains a sphere with dimension* $\dim V = n - \dim T_qC(p)$.

Corollary 6.6. *If $C(p)$ is differentiable at $q \in C(p)$, there is an $n - \dim T_qC(p) - 1$ sphere S in $\widetilde{C}(p)$ such that $\exp_p(T(0, \tilde{q}))$ is a minimal geodesic from p to q in M for the line segment $T(0, \tilde{q})$ from origin and ending any point $\tilde{q} \in S$.*

We define a distance function from p restricted to $C(p)$ by $d_{C(p)}(q) = d(p,q)$ for any $q \in C(p)$ and a distance function on M to $C(p)$ by ${d_{C(p)}}^b(x) = d(x, C(p))$ for any point $x \in M$. Let $q \in C(p)$. We say that a minimal geodesic $T(x,q)$ from x to q is a *perpendicular* to $C(p)$ with *foot* q if $d(y,q) = {d_{C(p)}}^b(y)$ for all $y \in T(x,q)$.

Corollary 6.7. *Let $q \in C(p)$ be a local minimum point of $d_{C(p)}$. Then there exists a number $\delta > 0$ such that $\gamma_v : [-\delta, 0] \to M$ is a perpendicular to $C(p)$ with foot q for any $v \in A_p(q)^s$. In particular, if $C(p)$ is differentiable at q, then the set of all perpendiculars to q is homeomorphic to a disk containing 0 in the complement V of $T_qC(p)$ for sufficiently small $\varepsilon > 0$.*

Proof. Let $\delta > 0$ be a number such that, for all $x \in B_b(q, 2\delta) \cap C(p)$, we have $d_{C(p)}(q) \le d_{C(p)}(x)$. For $v \in A_p(q)^s$, if $\gamma_v : [-d(p, q), 0] \to M$ is a geodesic with $\gamma_v{}'(0) = v$, then γ_v is a minimal geodesic from p to q. Let $p_1 = \gamma_v(-\delta)$. Then q is a foot of p_1 on $C(p)$. In fact, otherwise, there exists a point $q' \in B_b(q, 2\delta) \cap C(p)$ such that $d_{C(p)}(q') < d_{C(p)}(q)$, contradicting the choice of δ. This proves the corollary. □

7. Examples

It is well known that the cut loci of the compact rank one symmetric spaces (two point homogeneous spaces) are smooth. We construct other cut loci in product spaces such that they have differentiable points.

Example 7.1. Let $(M \times N, g = \alpha \times \beta)$ be the Riemannian product of two complete Riemannian manifolds (M, α) and (N, β). Let $p = (p_1, p_2) \in M \times N$. Then the cut locus $C(p)$ of p in $M \times N$ is given by $C(p) = C_M(p_1) \times N \cup M \times C_N(p_2)$ where $C_M(p_1)$ (resp. $C_N(p_2)$) is the cut locus of p_1 (resp. p_2) in M (resp. N). This follows from the fact: $\gamma(t) = (\mu(t), \nu(t))$ is a minimal geodesic from p to $q = (q_1, q_2)$ with unit speed for $t \in [0, d(p, q)]$ if and only if $\mu(t)$ (resp. $\nu(t)$) is the minimal geodesic from p_1 (resp. p_2) to q_1 (resp. q_2) with speed $d_M(p_1, q_1)/d(p, q)$ (resp. $d_N(p_2, q_2)/d(p, q)$). Therefore, we see, at $q = (q_1, q_2) \in C(p)$,

$$T_qC(p) = \begin{cases} T_{q_1}C_M(p_1) + T_{q_2}N =: S_1 & \text{if } q_1 \in C_M(p_1) \text{ and } q_2 \notin C_N(p_2), \\ T_{q_1}M + T_{q_2}C_N(p_2) =: S_2 & \text{if } q_1 \notin C_M(p_1) \text{ and } q_2 \in C_N(p_2), \\ S_1 \cup S_2 & \text{if } q_1 \in C_M(p_1) \text{ and } q_2 \in C_N(p_2). \end{cases}$$

Assume that $C_M(p_1)$ and $C_N(p_2)$ are differentiable. Then, $C(p)$ is differentiable at $q = (q_1, q_2) \in C(p)$ if $q_1 \notin C_M(p_1)$ or $q_2 \notin C_N(p_2)$, since $S_1 \cup S_2$ is a vector subspace only when one of S_1 and S_2 is the empty set, i.e., one of $C_M(p_1)$ and $C_N(p_2)$ is the empty set.

Let h be a smooth function on $M \times N$ such that $\|dh(v)\|_g < \|v\|_g$ for all $v \in T(M \times N)$. Then we have a Finsler metric F by $F(v) = \|v\|_g + dh(v)$ for all $v \in T(M \times N)$ which is a Randers change by an exact 1-form dh. The cut locus of p in $(M \times N, F)$ is the same as $(M \times N, g)$ (see [Innami et al. 2019]) and, hence, is smooth at the same points.

To be seen in the following example, the differentiability of a cut locus $C(p)$ in our sense is different from that of the distance function from p restricted to $C(p)$.

An example is like the cross shape of a thick long surface of revolution and a thin long surface of revolution.

Example 7.2. Let M_0 be a smooth surface of revolution whose Riemannian metric is given by

$$ds^2 = dt^2 + g(t)^2 d\theta^2, \quad 0 \le t \le \ell, \quad 0 \le \theta \le 2\pi,$$

where $g(t) > 0$ for all $t \in (0, \ell)$, $g(0) = g(\ell) = 0$, $g'(0) = -g'(\ell) = 1$. Let p and q be the south and north pole of M_0, i.e., the point with $t = 0$ and $t = \ell$, respectively. Then we have $C_{M_0}(p) = \{q\}$ and $A_p(q) = T_q M_0 \setminus \{0\}$. Assume that g is constant on some interval $[a, b] \subset (0, \ell)$. Let $B(x, r)$ and $B(y, r)$ be open distance balls around x and y, respectively, where $x = \left(\frac{1}{2}(a+b), 0\right)$ and $y = \left(\frac{1}{2}(a+b), \pi\right)$ are their coordinates.

Since $[a, b] \times [0, 2\pi]$ is a flat cylinder, for a sufficiently small $r > 0$, $B(x, r)$ and $B(y, r)$ are developed isometrically into the Euclid plane. The following surface S of revolution replaces $B(x, r)$ and $B(y, r)$ in M_0.

$$ds^2 = dt^2 + h(t)^2 d\varphi^2, \quad 0 \le t \le \ell_1, \quad 0 \le \varphi \le 2\pi,$$

where $h(t) > 0$ for all $t \in (0, \ell_1]$, $h(0) = 0$, $h'(0) = 1$ and $h(t) = r - (\ell_1 - t)$ for all $t \in (\ell_1 - \delta, \ell_1]$, for some $\delta > 0$. Let M be a surface such that two surfaces S_x and S_y isometric to S are glued to $M_0 \setminus B(x, r) \cup B(y, r)$ along each connected component of its boundary. Then $q \in C_M(p)$ and $C_M(p)$ is a smooth curve passing through q in a neighborhood of q. Furthermore, $C_M(p)$ near q is a geodesic through q because of symmetry of M with respect to meridians $\theta = \pi/2$ and $\theta = 3\pi/2$.

We see that $A_p(q)$ consists of two connected components with interior points. In fact, the set of all minimal geodesics from p to q consists of meridians defined by $\theta \in [-\theta_0, \theta_0] \cup [\pi - \theta_0, \pi + \theta_0]$ for some $\theta_0 > 0$. This implies that the differential of the distance function $d_p|_{C_M(p)}$ from p on $C_M(p)$ at q is not 0 and it is 0 for all initial tangent vectors of meridians with $\theta \in (-\theta_0, \theta_0) \cup (\pi - \theta_0, \pi + \theta_0)$. Therefore, $C_M(p)$ is differentiable in our sense but the distance function $d_p|_{C_M(p)}$ is not differentiable at the tangent vectors of the meridians with $\theta = -\theta_0, \theta_0, \pi - \theta_0, \pi + \theta_0$ (see Figure 1).

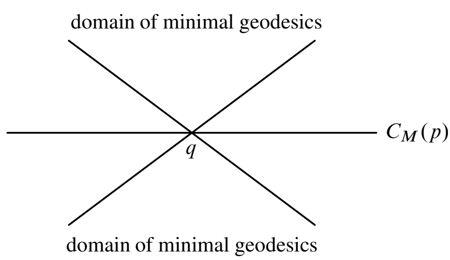

Figure 1. $C_M(p)$ around q.

We introduce the polar coordinate (r, ψ) in T_qM such that $\psi = 0$ contains the tangent vector of $C_M(p)$ at q and $A_p(q)$ is the cone $\psi \in [\psi_0, \pi - \psi_0] \cup [\pi + \psi_0, 2\pi - \psi_0]$. If we choose the vector subspace $V = \{u(\cos\psi, \sin\psi) \mid u \in \mathbb{R}\}$ for any $\psi \in [\psi_0, \pi - \psi_0] \cup [\pi + \psi_0, 2\pi - \psi_0]$ as a complement to $T_qC_M(p)$, then V itself is a graph over V contained in $A_p(q)$ and obtained with the method in Lemma 6.1. While, if $V = \{u(\cos\psi, \sin\psi) \mid u \in \mathbb{R}\}$ for $\psi \in (0, \psi_0)$, then $\{u(\cos\psi_0, \sin\psi_0) \mid u \in \mathbb{R}\} \subset A_p(q)$ is the graph over V mentioned in Lemma 6.1.

Using the following Weinstein's result, we may make many examples of cut points as above.

Proposition 7.3 [Weinstein 1968, Proposition C]. *Let D be an n-disk embedded in a Riemannian manifold M^n. Then there is a new metric on M agreeing with the original metric on a neighborhood of M-(interior of D) such that, for some point p in D, $\exp_p$ is a diffeomorphism of the unit disk about the origin in T_pM onto D.*

Example 7.4. Let S be a sphere having a flat domain Q. We first draw the smooth simple closed curve K in Q such that the cut locus $C(K)$ of K in the inside of K is like Figure 1. For example, let E be an ellipse in Q. We modify slightly arcs C_1 and C_2 near the end points of short axis of E in such a way that C_1 and C_2 are pieces of a circle with center at the center of the ellipse. We may assume that the resulting simple closed curve K is still symmetric with respect to the long axis and its cut locus $C(K)$ is like Figure 1. Then, we change the metric on the outside of K and find a point $p \in S$ as stated in Proposition 7.3. We have $C(p) = C(K)$ which is like Figure 1.

We check the exponential map and its differential map of the unit sphere in the Euclid space.

Example 7.5. Let (r, θ) be a geodesic polar coordinate about the north pole $p = (0, 0, 1)$ of the unit sphere

$$M = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\},$$

namely,

$$x = \sin r \cos\theta, \quad y = \sin r \sin\theta, \quad z = \cos r.$$

Let $D = \{(x, y) \mid x^2 + y^2 < 1\}$ be the unit disk. We use the orthogonal projection to xy-plane as a local coordinate system about the south half sphere of M. Then the exponential map $\exp_p$ at p is given by

$$\exp_p(r, \theta) = (\sin r \cos\theta, \sin r \sin\theta), \quad \frac{\pi}{2} < r < \frac{3\pi}{2}.$$

We see that $\exp_p(\pi, \theta) = q$ where $q = (0, 0, -1)$ is the south pole of M and q is the point conjugate to p along all minimal geodesics from p to q, since

$$d \exp_p |_{(r,\theta)} = \begin{pmatrix} \cos r \cos\theta & -\sin r \sin\theta \\ \cos r \sin\theta & \sin r \cos\theta \end{pmatrix}$$

and, hence,

$$d \exp_p |_{(\pi,\theta)} = \begin{pmatrix} -\cos\theta & 0 \\ -\sin\theta & 0 \end{pmatrix}.$$

Let $\tilde{c}(t) = (r(t), \theta(t))$ be a curve in T_pM. Then we have

$$c(t) = \exp_p(\tilde{c}(t)) = (\sin r(t) \cos\theta(t), \sin r(t) \sin\theta(t)).$$

If $c'(t) = (x'(t), y'(t))$, we then have the equation

$$\begin{cases} x'(t) = r'(t) \cos r(t) \cos\theta(t) - \theta'(t) \sin r(t) \sin\theta(t), \\ y'(t) = r'(t) \cos r(t) \sin\theta(t) + \theta'(t) \sin r(t) \cos\theta(t) \end{cases}$$

and, hence,

$$\begin{pmatrix} \cos r(t) \cos\theta(t) & -\sin r(t) \sin\theta(t) \\ \cos r(t) \sin\theta(t) & \sin r(t) \cos\theta(t) \end{pmatrix} \begin{pmatrix} r'(t) \\ \theta'(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix},$$

or

$$\begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \cos r(t) & 0 \\ 0 & \sin r(t) \end{pmatrix} \begin{pmatrix} r'(t) \\ \theta'(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

Hence, we have

$$r'(t) = \frac{1}{\cos r(t)} \big(x'(t) \cos\theta(t) + y'(t) \sin\theta(t) \big),$$
$$\theta'(t) = \frac{1}{\sin r(t)} \big(y'(t) \cos\theta(t) - x'(t) \sin\theta(t) \big).$$

Since $x^2 + y^2 = \sin^2 r$, $\cos\theta = x/\sqrt{x^2+y^2}$ and $\sin\theta = y/\sqrt{x^2+y^2}$, we have

$$r'(t) = \frac{x'(t)x(t) + y'(t)y(t)}{\sqrt{x(t)^2 + y(t)^2} \cos r(t)}, \qquad \theta'(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x(t)^2 + y(t)^2}.$$

If $c(t) = (t, 0)$, then $\tilde{c}(t) = (\arcsin t, 0)$ for $t \in [0, 1]$. We consider a curve $c(t) = (t, bt^a)$, $t \in [0, 1]$, for $a > 1$ and $b \neq 0$. Then we have $c'(0) = (1, 0)$ and $\tilde{c}(0) = (\pi, 0)$.

Hence, $\cos r(t) < 0$ near $t = 0$. For $\tilde{c}'(0)$, since

$$r'(t) = -\frac{t + ab^2 t^{2a-1}}{\sqrt{t^2 + b^2 t^{2a}}\sqrt{1 - (t^2 + b^2 t^{2a})}}, \qquad \theta'(t) = \frac{abt^a - bt^a}{t^2 + b^2 t^{2a}},$$

we have $r'(t) \to -1$ and

$$\theta'(t) \to \begin{cases} \infty & \text{for } 1 < a < 2, \\ b & \text{for } a = 2, \\ 0 & \text{for } a > 2, \end{cases}$$

as $t \to 0+0$. Therefore, $\tilde{c}$ is differentiable at $t = 0$ if $a \geq 2$. Otherwise, $\tilde{c}$ cannot be extended differentiably to $t = 0$. This example shows us that we are not allowed to express $d\exp_p(\tilde{c}'(0)) = c'(0)$ in Lemma 4.4 for $1 < a < 2$ and $b \neq 0$.

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