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**SHIFT OPERATORS, RESIDUE FAMILIES
AND DEGENERATE LAPLACIANS**

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In this paper, we introduce new aspects in conformal geometry of some very natural second-order differential operators. These operators are termed shift operators. In the flat space, they are intertwining operators which are closely related to symmetry breaking differential operators. In the curved case, they are closely connected with ideas of holography and the works of Fefferman–Graham, Gover–Waldron and one of the authors. In particular, we obtain an alternative description of the so-called residue families in conformal geometry in terms of compositions of shift operators. This relation allows easy new proofs of some of their basic properties. In addition, we derive new holographic formulas for Q -curvatures in even dimension. Since these turn out to be equivalent to earlier holographic formulas, the novelty here is their conceptually very natural proof. The overall discussion leads to a unification of constructions in representation theory and conformal geometry.

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1. Introduction and formulation of the main results

Conformal differential geometry has seen spectacular developments in recent years, both from a perspective of pure mathematics, and from a mathematical physics point of view. The construction of ambient metrics and Poincaré metrics by Fefferman and Graham [2012] gave rise to many important applications and fundamental insights. This is closely connected to the idea of holography.

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Attempts to extend these ideas to a conformal submanifold theory were one source for the notion of symmetry breaking operators. This recent notion in representation theory is central in studies of, for instance, the interplay between the representation theory of the conformal group (the Möbius group) of Euclidean space and the corresponding group for a hyperplane. In particular, it plays a basic role in the study of branching laws of representations. The works [Kobayashi et al. 2015; 2016; Kobayashi and Speh 2015; 2018; Kobayashi and Pevzner 2016; Fischmann et al. 2016; Frahm and Ørsted 2019] reflect recent progress in this area. Curved analogs of symmetry breaking operators in conformal geometry deal with conformally covariant differential operators $C^\infty(X) \rightarrow C^\infty(M)$, where M is a hypersurface of a Riemannian manifold X (see [Juhl 2009]). Residue families (introduced in [Juhl 2009]) are curved versions of symmetry breaking operators which are defined in a setting where X is a tubular neighborhood of M and where the metric on X is determined by the metric on M . For recent substantial progress in the general case we refer to [Gover and Peterson 2018].

In this paper, we shall develop a theory of some second-order differential operators, originally found via representation theory as *shift operators* between symmetry breaking operators. These operators turn out to be very natural and admit generalizations within a framework defined by Riemannian metrics. Remarkably, these generalizations did appear in earlier work from a number of different perspectives, in particular from the point of view of tractor calculus, a powerful tool in conformal geometry.

Shift operators recently appeared in the theory of symmetry breaking operators. The latter operators are generalizations of Knapp–Stein intertwining operators which intertwine principal series representations of semisimple Lie group. Symmetry breaking operators map between functional spaces on a given flag variety to functional spaces on a subvariety and are equivariant only with respect to the symmetry group of the subvariety. This loss of symmetry is the origin of the notion. A typical situation is that of the round sphere S^{n+1} with an equatorially embedded subsphere $S^n \hookrightarrow S^{n+1}$. The noncompact model of that situation is a standard embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$. In these cases, the relevant groups are the conformal groups of the respective submanifolds. Conformal symmetry breaking *differential* operators acting on functions in that setting are of particular importance for conformal differential geometry [Juhl 2009].

Shift operators shift the spectral parameter in the distributional Schwartz kernels of conformal symmetry breaking operators. Such results in the setting $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ first appeared in [Fischmann et al. 2019] and will be recalled in Section 2E. The basic shift operator in that theory is given by the 1-parameter family [Fischmann et al. 2019, (3.5)]

$$(1-1) \quad P(\lambda) = r \Delta - (2\lambda - n - 3) \partial_r : C^\infty(\mathbb{R}^{n+1}) \rightarrow C^\infty(\mathbb{R}^{n+1}), \quad \lambda \in \mathbb{C}$$

of second-order differential operators on \mathbb{R}^{n+1} . Here Δ denotes the nonpositive Laplacian of the flat metric on the space \mathbb{R}^{n+1} with coordinates (r, x) . We regard r as a defining function of the subspace \mathbb{R}^n . For any $\lambda \in \mathbb{C}$, the operator $P(\lambda)$ is equivariant with respect to principal series representations restricted to the conformal group $\text{Conf}(\mathbb{R}^n)$ of the subspace \mathbb{R}^n with the flat metric regarded as the subgroup of the conformal group $\text{Conf}(\mathbb{R}^{n+1})$ of \mathbb{R}^{n+1} leaving \mathbb{R}^n invariant. More precisely, $P(\lambda)$ satisfies the intertwining relation

$$(1-2) \quad \left(\frac{\gamma_*(r)}{r}\right)^{n-\lambda+2} \circ \gamma_* \circ P(\lambda) = P(\lambda) \circ \left(\frac{\gamma_*(r)}{r}\right)^{n-\lambda+1} \circ \gamma_*$$

for all $\gamma \in \text{Conf}(\mathbb{R}^n) \subset \text{Conf}(\mathbb{R}^{n+1})$. Here $\gamma_* = (\gamma^{-1})^*$ denotes the push-forward operator on functions induced by γ .

In the present paper, we generalize these results to a framework defined by Riemannian metrics and discuss some applications. One of the main applications concerns the residue families of [Juhl 2009]. As noted above, they can be regarded as curved analogs of symmetry breaking differential operators. More precisely, residue families are 1-parameter families of conformally covariant differential operators

$$(1-3) \quad D_N^{\text{res}}(h; \lambda) : C^\infty(M_+) \rightarrow C^\infty(M), \quad \lambda \in \mathbb{C}$$

of order $N \in \mathbb{N}$. These are defined in the following setting. We consider a general Riemannian manifold (M, h) . Let M_+ be an open neighborhood of $\{0\} \times M$ in $[0, \infty) \times M$. On the open interior $M_+^\circ = (0, \varepsilon) \times M$ of M_+ , let $g_+ = r^{-2}(dr^2 + h_r)$ be an even Poincaré metric in normal form relative to h . Here h_r is a 1-parameter family of metrics on M with $h_0 = h$. The metric $\bar{g} = dr^2 + h_r$ is a conformal compactification of g_+ . The relevant concepts were developed in [Fefferman and Graham 2012] and will be recalled in Section 2. Although the Poincaré metric g_+ is not completely determined by the metric h , residue families only depend on the Taylor coefficients of the family h_r at $r = 0$ which are uniquely determined by h . More precisely, the even-order family $D_{2N}^{\text{res}}(h; \lambda)$ involves $2N$ derivatives by the variable r and depends on the Taylor coefficients of h_r of order $\leq 2N$. The conformal covariance of residue families describes their behavior under conformal changes $h \rightarrow e^{2\varphi}h$ of the metric on the submanifold $M \hookrightarrow M_+ = [0, \varepsilon) \times M$.

Our generalizations of the shift operator $P(\lambda)$ are differential operators which act on smooth functions on M_+ and are defined in terms of an even Poincaré metric g_+ on M_+° . In fact, we define a curved version of the shift operator $P(\lambda)$ by the formula¹

$$(1-4) \quad S(g_+; \lambda) = r \Delta_{\bar{g}} - (2\lambda - n + 1) \partial_r - \frac{1}{2}(\lambda - n + 1) \text{tr}(h_r^{-1} \dot{h}_r).$$

¹The shift by 2 in the parameter λ is a matter of conventions.

Here the dot denotes derivatives with respect to r and $\bar{g} = r^2 g_+$. This definition can also be written in the form

$$S(g_+; \lambda) = r \Delta_{\bar{g}} - (2\lambda - n + 1) \partial_r - (\lambda - n + 1) \dot{v}(r)/v(r),$$

where the function $v(r, \cdot) \in C^\infty(M)$ is defined by the relation

$$d\text{vol}(h_r) = v(r) d\text{vol}(h)$$

of volume forms. We note that, in contrast to residue families, the shift operators are not defined only by the Taylor coefficients of h_r at $r = 0$.²

The operator $S(g_+; \lambda)$ is a second-order differential operator which degenerates for $r = 0$, i.e., on the submanifold M . Theorem 3.7 establishes the shift property of $S(g_+; \lambda)$. This property describes its action on functions of the form $r^\lambda u \in C^\infty(M_+^\circ)$, where u is an eigenfunction of the Laplacian Δ_{g_+} of the Poincaré metric g_+ on M_+° .

The following result describes the behavior of $S(g_+; \lambda)$ under conformal changes of the boundary metric h (Proposition 3.9).

Theorem 1. *Assume that (M^n, h) is a manifold of dimension n . Let $\hat{h} = e^{2\varphi} h$ be a metric in the conformal class of h . Let g_+ be an even Poincaré metric in normal form relative to h on M_+° . Let κ be a diffeomorphism of M_+ which restricts to the identity on M and for which the Poincaré metric $\hat{g}_+ = \kappa^*(g_+)$ is in normal form relative to \hat{h} . In these terms, we have*

$$(1-5) \quad S(\hat{g}_+; \lambda) = \kappa^* \circ \left(\frac{\kappa_*(r)}{r} \right)^{\lambda-n} \circ S(g_+; \lambda) \circ \left(\frac{\kappa_*(r)}{r} \right)^{n-\lambda-1} \circ \kappa_*.$$

This result may be regarded as a version of conformal covariance. Although the transformation law (1-5) formally resembles the equivariance property (1-2), the former law is *not* a generalization of the latter one. In fact, the diffeomorphisms κ should not be confused with the conformal maps γ : κ leaves the submanifold M pointwise fixed. However, the formal similarity between the intertwining property (1-2) and the conformal transformation law (1-5) can be explained by recognizing $P(\lambda)$ and $S(g_+; \lambda)$ (for Einstein g_+) both as special cases of the degenerate Laplacian $I \cdot D$ introduced in [Gover and Waldron 2014] (this concept will be recalled in Section 2G). Indeed, we note that

$$P(\lambda) = S(g_{\text{hyp}}; \lambda - 2)$$

and

$$S(g_{\text{hyp}}; \lambda) = (I \cdot D)[r^2 g_{\text{hyp}}; r, \lambda - n + 1] \quad \text{and} \quad S(g_+; \lambda) = -(I \cdot D)[r^2 g_+; r, \lambda - n + 1],$$

²However, in the analytic category, the family h_r is completely determined by h .

where g_{hyp} denotes the hyperbolic metric in the upper half-space and g_+ is Einstein (see (2-38) and (3-4)). Since γ preserves the hyperbolic metric g_{hyp} , we have

$$\gamma^*(r^2 g_{\text{hyp}}) = \left(\frac{\gamma^*(r)}{r} \right)^2 (r^2 g_{\text{hyp}})$$

and the conformal transformation law for $I \cdot D$ (Proposition 2.2) implies

$$\begin{aligned} & \left(\frac{r}{\gamma^*(r)} \right)^{\lambda-n-2} \circ \gamma^* \circ P(\lambda) \circ \gamma_* \circ \left(\frac{r}{\gamma^*(r)} \right)^{n-\lambda+1} \\ &= \left(\frac{r}{\gamma^*(r)} \right)^{\lambda-n-2} \circ \gamma^* \circ (I \cdot D)[r^2 g_{\text{hyp}}; r; \lambda-n-1] \circ \gamma_* \circ \left(\frac{r}{\gamma^*(r)} \right)^{n-\lambda+1} \\ &= \left(\frac{r}{\gamma^*(r)} \right)^{\lambda-n-2} \circ \gamma^* \circ (I \cdot D)[\gamma^*(r^2 g_{\text{hyp}}); \gamma^*(r); \lambda-n-1] \circ \gamma_* \circ \left(\frac{r}{\gamma^*(r)} \right)^{n-\lambda+1} \\ &= (I \cdot D)[r^2 g_{\text{hyp}}; r, \lambda-n-1] = P(\lambda). \end{aligned}$$

This proves (1-2). A similar calculation gives (1-5) (Remark 3.10).

For $N \in \mathbb{N}$, we define the compositions

$$(1-6) \quad S_N(g_+; \lambda) \stackrel{\text{def}}{=} \underbrace{S(g_+; \lambda) \circ \cdots \circ S(g_+; \lambda+N-1)}_{N \text{ factors}}.$$

We shall refer to these operators as *iterated shift operators* or simply also as *shift operators*. Theorem 1 implies that all iterated shift operators $S_N(g_+; \lambda)$ are conformally covariant (in the sense as in (1-5)). The following result states that residue families (1-3) can be written in terms of iterated shift operators (Corollary 4.2). Its proof rests on the shift property of the shift operators. Let the embedding $\iota: M \hookrightarrow M_+$ be defined by $m \mapsto (0, m)$.

Theorem 2. *Assume that (M^n, h) is a Riemannian manifold of dimension n . Let $N \in \mathbb{N}$ so that $2N \leq n$ if n is even. Then the residue family $D_{2N}^{\text{res}}(h; \lambda)$ of order $2N$ is proportional to the composition of the family $\lambda \mapsto S_{2N}(g_+; \lambda+n-2N)$ with the restriction ι^* to M . More precisely, we have*

$$(1-7) \quad (-2N)_N \left(\lambda + \frac{n+1}{2} - 2N \right)_N D_{2N}^{\text{res}}(h; \lambda) = \iota^* S_{2N}(g_+; \lambda+n-2N).$$

A similar formula holds true for odd-order residue families.

Some comments on this result are in order.

By construction, the family $S_{2N}(g_+; \lambda)$ involves $4N$ derivatives in the variable r and depends on all Taylor coefficients of h_r . The identity (1-7) shows that its composition with the restriction operator ι^* actually involves only $2N$ derivatives in r and depends only on the Taylor coefficients of h_r up to order $2N$. In particular, the compositions $\iota^* S_{2N}(g_+; \lambda)$ are completely determined by h . In the following,

we shall denote these compositions by $\Sigma_{2N}(h; \lambda)$. For $N \in \mathbb{N}$ not satisfying the assumptions in Theorem 2, the compositions $t^* S_{2N}(g_+; \lambda)$ in general are not determined only by h .

The product formula (1-7) yields a new expression for the residue families. It extends a result of [Clerc 2017; Fischmann et al. 2019] in the flat case. The description of residue families in terms of restrictions to M of "powers" of a universal shift operator living in a neighborhood of M resembles the construction of the conformally covariant powers of the Laplacian (GJMS operators) of h by powers of the Laplacian of an ambient metric associated to h [Graham et al. 1992].

Theorem 2 can be used to deduce properties of residue families from properties of shift operators and vice versa. In particular, the conformal covariance of $S_N(g_+; \lambda)$ implies a conformal covariance law for residue families. This reproves [Juhl 2009, Theorem 6.6.3].

In addition, Theorem 2 enables us to give easy proofs of the systems of factorization identities of residue families which play an important role in [Juhl 2009; Juhl 2013] in connection with the description of recursive structures for GJMS operators and Q -curvatures. Our new proofs of these factorization identities rest on two basic facts. The first one (Theorem 5.1) is also of independent interest.

Theorem 3. *Assume that $N \in \mathbb{N}$ with $2N \leq n$ if n is even. Then*

$$(1-8) \quad S_N\left(g_+; \frac{n-1}{2}\right) = r^N P_{2N}(\bar{g}),$$

up to an error term in $O(r^\infty)$ for n odd and $o(r^{n-N})$ for n even. Moreover, the equality holds true without an error term if g_+ is Einstein.

Here $P_{2N}(\bar{g})$ is a GJMS operator of the conformal compactification \bar{g} of the Poincaré metric g_+ in normal form relative to h .³ The second basic fact is the identity

$$(1-9) \quad \Sigma_{2N}\left(h; \frac{n}{2} - N\right) = ((2N-1)!)^2 P_{2N}(h) t^*$$

(Theorem 5.6). This formula reproves a special case of a result of [Gover and Waldron 2014]. Together with

$$(1-10) \quad \Sigma_{2N}\left(h; \frac{n-1}{2} - N\right) = (2N)! t^* P_{2N}(\bar{g})$$

it shows that the operators $\Sigma_{2N}(h; \lambda)$ interpolate between GJMS operators for the metrics h and \bar{g} .

The coefficients of the families $S_N(g_+; \lambda)$ depend on the parameters r and λ . A closer study of both dependencies seems to be of interest. Theorem 3 may be

³For odd n , the operators $P_{2N}(\bar{g})$ are well-defined for all $N \in \mathbb{N}$. See the comments at the beginning of Section 5A.

regarded as a result in that direction. More results in this direction are discussed in Section 6F.

Finally, through the relation between residue families and iterated shift operators, we derive a new formula for the critical Q -curvature $Q_n(h)$ of a manifold (M^n, h) of even dimension n (Theorem 5.12).

Theorem 4. *Let n be even. Then*

$$(1-11) \quad Q_n(h) = c_n \Sigma_{n-1}(h; 0) \partial_r(\log v),$$

where $c_n = (-1)^{n/2} 2^{n-2} (\Gamma(\frac{n}{2}) / \Gamma(n))^2$.

There is an interesting formal resemblance of the latter formula for the critical Q -curvature with a formula of Fefferman and Hirachi [2003].

Theorem 4 extends to all subcritical Q -curvatures $Q_{2N}(h)$ for $2N < n$ in the form

$$(1-12) \quad Q_{2N}(h) = c_{2N} \Sigma_{2N-1}\left(h; \frac{n}{2} - N\right) \partial_r(\log v),$$

where $c_{2N} = (-1)^N 2^{2N-2} (\Gamma(N) / \Gamma(2N))^2$ (Theorem 5.14).

Combining (1-11) and (1-12) with Theorem 2, yields formulas for Q -curvatures in terms of residue families. These turn out to be equivalent to the holographic formulas proved in [Graham and Juhl 2007; Juhl 2011]. In other words, these holographic formulas for Q -curvatures can be viewed as natural consequences of Theorem 2.

The operator $S(g_+; \lambda)$ appeared in the literature in different contexts. The construction of asymptotic expansions of eigenfunctions for the Laplacian Δ_{g_+} of a Poincaré metric in [Graham and Zworski 2003] involved a second-order operator \mathcal{D}_s . In [Gover and Waldron 2014], the operator \mathcal{D}_s was interpreted and generalized within tractor calculus. This led to the definition of the so-called degenerate Laplacian $I \cdot D$ which played a role in the discussion after Theorem 1. Compositions as in (1-6) of these operators were used in [Gover and Waldron 2014] in connection with the construction of asymptotic expansions in a more general eigenfunction problem. The relations among these construction will be described in Section 2. In [Clerc 2017], Clerc gave a representation theoretical alternative construction of a family of symmetry breaking differential operators introduced in [Juhl 2009] in terms of compositions of shifted operators $P(\lambda)$. Theorem 2 is a generalization of his result to the curved setting.

The paper is organized as follows. After a collection of background material, we use Section 3 to introduce the shift operator $S(g_+; \lambda)$ and prove basic properties. In Section 4, we establish the connection between residue families and iterated shift operators. Section 5 is devoted to various applications. Here we provide easy new proofs of the factorization identities of residue families and discuss holographic

formulas for Q -curvatures. In Section 6, we illustrate the main results in low-order cases. In the final section, we speculate on the role of iterated shift operators in the theory of the building block operators \mathcal{M}_{2N} [Juhl 2013; 2016] of GJMS operators. In this connection, we derive a new formula for the so-called holographic Laplacian [Juhl 2016] for the metric \bar{g} .

2. Preliminaries

In the present section, we fix notation and describe the general setting. We recall basic facts on GJMS operators, Q -curvatures, residue families, shift operators and the degenerate Laplacian.

2A. General notation. \mathbb{N} is the set of natural numbers and \mathbb{N}_0 the set of nonnegative integers. For a complex number $a \in \mathbb{C}$ and an integer $N \in \mathbb{N}$, the Pochhammer symbol $(a)_N$ is defined by $(a)_N \stackrel{\text{def}}{=} a(a+1) \cdots (a+N-1)$. We also set $(a)_0 \stackrel{\text{def}}{=} 1$. $C^\infty(M)$ is the space of smooth functions on the manifold M and $C_c^\infty(M)$ denotes the subspace of functions with compact support. The Laplacian of a Riemannian metric g on a manifold M is denoted Δ_g ; it acts on $C^\infty(M)$. Here we use the convention that $-\Delta_g$ is nonnegative, i.e., $-\Delta_g = \delta_g d$, where δ_g is the formal adjoint of the differential d . $\text{Ric}(g)$ and $\tau(g)$ denote the Ricci tensor and the scalar curvature of g . On a manifold (M^n, g) of dimension n , we set $J(g) = \frac{1}{2(n-1)} \tau(g)$ and define the Schouten tensor of g by $P(g) = \frac{1}{n-2} (\text{Ric}(g) - J(g)g)$. We shall also write simply P and J if the metric is clear by context. The symbol \circ denotes compositions of operators.

2B. GJMS operators and Q -curvatures. Let (M^n, h) be a Riemannian manifold of dimension $n \geq 3$. For $N \in \mathbb{N}$ if n is odd and $\mathbb{N} \ni N \leq \frac{n}{2}$ if n is even, the GJMS operators are conformally covariant differential operators

$$P_{2N}(h) : C^\infty(M) \rightarrow C^\infty(M)$$

of order $2N$ which are of the form $P_{2N}(h) = \Delta_h^N + \text{LOT}$, where LOT denotes lower-order terms. These lower-order terms only depend on covariant derivatives of the curvature of h . Under conformal changes $\hat{h} = e^{2\varphi} h$ with $\varphi \in C^\infty(M)$ of the metric, the GJMS operators satisfy

$$(2-1) \quad P_{2N}(\hat{h}) = e^{-(n/2+N)\varphi} \circ P_{2N}(h) \circ e^{(n/2-N)\varphi}.$$

In [Graham et al. 1992], these operators were constructed in terms of powers of the Laplacian of an ambient metric associated to h .

The GJMS operators generalize the well-known Yamabe operator

$$(2-2) \quad P_2 = \Delta - \left(\frac{n}{2} - 1\right)J$$

and the Paneitz operator

$$(2-3) \quad P_4 = \Delta^2 + \delta((n-2)Jh - 4P)\#d + \left(\frac{n}{2}-2\right)\left(\frac{n}{2}J^2-2|P|^2-\Delta J\right),$$

where $|P|^2 = P_{ij}P^{ij}$ and $\#$ indicates the natural action of symmetric 2-tensors on $\Omega^1(M)$.

In odd dimension n , we have GJMS operators P_{2N} of any order $2N$, $N \in \mathbb{N}$. But, for general metrics h in even dimension n , the restriction $2N \leq n$ is necessary both for the definition of $P_{2N}(h)$ and for the existence of conformally covariant differential operators with leading term Δ_h^N [Graham 1992; Gover and Hirachi 2004].

Explicit formulas for GJMS operators for general metrics are very complicated [Juhl 2013]. But for some special metrics, they may be given by closed formulas. In particular, for Einstein manifolds (M^n, h) they are given by the formula

$$(2-4) \quad P_{2N}(h) = \prod_{l=1}^N \left(\Delta_h - 2\mu \left(\frac{n}{2} + l - 1 \right) \left(\frac{n}{2} - l \right) \right)$$

for all $N \in \mathbb{N}$, where the constant $\mu \in \mathbb{R}$ is defined by $\text{Ric}(h) = 2\mu(n-1)h$ [Fefferman and Graham 2012]. Here the above restriction on their order is irrelevant.

It is a basic observation [Branson 1995] that

$$(2-5) \quad P_{2N}(h)(1) = (-1)^N \left(\frac{n}{2} - N \right) Q_{2N}(h)$$

for a scalar curvature invariant $Q_{2N}(h) \in C^\infty(M^n)$ of order $2N$. The quantities $Q_{2N}(h)$ are well-defined by (2-5) as long as $2N < n$. These curvature quantities are called the subcritical Q -curvatures. Their analogs of even order n can be defined by analytic continuation in dimension n through the subcritical Q -curvatures. The quantity $Q_n(h)$ is called the critical Q -curvature of (M^n, h) . Under the respective conditions $n > 2$ and $n > 4$, (2-2) and (2-3) yield the subcritical Q -curvatures

$$(2-6) \quad Q_2 = J \quad \text{and} \quad Q_4 = \frac{n}{2}J^2 - 2|P|^2 - \Delta J$$

of order 2 and 4. Here we suppress the obvious dependence of constructions on the metric h . By continuation in dimension n , we define the respective critical Q -curvatures

$$(2-7) \quad Q_2 = J \quad \text{and} \quad Q_4 = 2J^2 - 2|P|^2 - \Delta J$$

in dimension $n = 2$ and $n = 4$.

In the following, we shall often simplify notation by omitting the composition sign \circ in compositions with multiplication operators. It also will often lead to simplifications to suppress the obvious dependence of constructions on h .

2C. Poincaré metrics, eigenfunction expansions and GJMS operators. In the present section, we briefly recall basic definitions concerning Poincaré metrics in the sense of Fefferman and Graham [2012] and recall a description of GJMS operators of (M, h) in terms of eigenfunctions of the Laplacian of an associated Poincaré metric on M_+° [Graham and Zworski 2003]. This description will be of central importance for all later constructions.

Let M be a manifold of dimension $n \geq 3$. Let M_+ be an open neighborhood of $\{0\} \times M$ in $[0, \infty) \times M$, i.e., $M_+ = [0, \varepsilon) \times M$ for some $\varepsilon > 0$. We use the coordinate r on the first factor. We define the embedding $\iota : M \rightarrow M_+$ by $\iota(m) = (0, m)$. Let $M_+^\circ = (0, \varepsilon) \times M$. A smooth metric

$$(2-8) \quad g_+ = r^{-2}(dr^2 + h_r)$$

on M_+° is called a Poincaré metric in normal form relative to a metric h on M if $\bar{g} = r^2 g_+$ extends to M_+ , \bar{g} restricts to h , i.e., $\iota^*(\bar{g}) = h$, and the Ricci tensor of g_+ satisfies the Einstein condition

$$(2-9) \quad \text{Ric}(g_+) + n g_+ = O(r^\infty)$$

for odd $n \geq 3$ and the Einstein condition

$$(2-10) \quad \text{Ric}(g_+) + n g_+ = O(r^{n-2})$$

together with the vanishing trace condition

$$(2-11) \quad \text{tr}_h(\iota^*(r^{-n+2}(\text{Ric}(g_+) + n g_+))) = 0$$

for even $n \geq 4$. The metric $\bar{g} = r^2 g_+$ on M_+ is called a conformal compactification of g_+ . The family h_r in (2-8) is a smooth 1-parameter family of metrics on M .

If, for odd n , we also assume that h_r has an *even* expansion

$$(2-12) \quad h_r = h_0 + r^2 h_2 + r^4 h_4 + \dots,$$

then the condition (2-9) implies that the coefficients h_2, h_4, \dots are uniquely determined by $h_0 = h$. These metrics are conveniently referred to as *even* Poincaré metrics. One may consider the conformal compactification \bar{g} of an even Poincaré metric as a smooth metric on the larger space $(-\varepsilon, \varepsilon) \times M$. In the real analytic category, h_r converges and $\text{Ric}(g_+) + n g_+ = 0$ in a neighborhood of $\{0\} \times M$.

For even n , the situation is more complicated. In that case, the family h_r has an expansion of the form

$$(2-13) \quad h_r = \underbrace{h_0 + r^2 h_2 + \dots + r^{n-2} h_{n-2}}_{\text{even powers}} + r^n (h_n + \log r h_0^{(1)}) + \dots$$

The condition (2-10) uniquely determines the coefficients h_2, \dots, h_{n-2} by $h_0 = h$. Moreover, the vanishing trace condition (2-11) can be satisfied and determines the h -trace of h_n . However, the trace-free part of h_n is *not* determined by h .

In general, the higher-order solutions of the Einstein condition contain $\log r$ terms. The first $\log r$ coefficient $h_0^{(1)}$ (Fefferman–Graham obstruction tensor) is uniquely determined by h and trace-free. For specific choices of the trace-free part of h_n , the condition

$$\text{Ric}(g_+) + ng_+ = O(r^\infty)$$

may be satisfied by solutions with expansions of the form

$$(2-14) \quad h_r = \underbrace{h_0 + r^2 h_2 + \dots + r^n h_n + \dots}_{\text{even powers}} + \sum_{j=1}^{\infty} (r^n \log r)^j h_r^{(j)}$$

with even families $h_r^{(j)}$. The $\log r$ terms in these expansion vanish if and only if the obstruction tensor vanishes.

Now assume that $g_+ = r^{-2}(dr^2 + h_r)$ is a Poincaré metric in normal form relative to h . Hence $h_0 = h$. Let $\hat{h} = e^{2\varphi} h$ be a metric in the same conformal class as h . Then a suitable change of coordinates brings g_+ into normal form relative to \hat{h} . Following [Graham and Lee 1991, Section 5; Fefferman and Graham 2012, Proposition 4.3], we briefly recall the arguments proving this basic observation. The metric g_+ is asymptotically hyperbolic since $|dr/r|_{g_+} = 1$ on $r = 0$. The latter property suffices to prove the existence of $u \in C^\infty(M_+)$ so that for $\rho = re^u \in C^\infty(M_+)$ we have

$$|d\rho|_{\rho^2 g_+}^2 = 1 \quad \text{near } M.$$

Here the restriction of u to $r = 0$ can be arbitrarily chosen. Now let $\mathfrak{X} = \text{grad}_{\rho^2 g_+}(\rho)$ be the gradient field of ρ with respect to the conformal compactification $\rho^2 g_+$ of g_+ . Let $\Phi_{\mathfrak{X}}^t$ be the flow of \mathfrak{X} . In these terms, we define the map

$$\kappa : [0, \varepsilon) \times M \ni (\lambda, x) \mapsto (\Phi_{\mathfrak{X}}^\lambda)(x) \in [0, \varepsilon) \times M$$

for sufficiently small ε . Then $\kappa(0, x) = x$ and $\kappa^*(\rho)(\lambda, x) = \lambda$. The gradient field \mathfrak{X} is orthogonal to the slices $\rho^{-1}(\lambda)$. It follows that

$$\kappa^*(\rho^2 g_+) = d\lambda^2 + k_\lambda$$

for some 1-parameter family k_λ . Hence

$$\kappa^*(g_+) = \frac{1}{\kappa^*(\rho)} \kappa^*(\rho^2 g_+) = \lambda^{-2}(d\lambda^2 + k_\lambda).$$

Finally, we note that

$$k_0 = \iota^*(d\lambda^2 + k_\lambda) = \iota^* \kappa^*(\rho^2 g_+) = (\kappa \iota)^*(\rho^2 g_+) = \iota^*(\rho^2 g_+) = \iota^*\left(\frac{\rho}{r}\right)^2 h_0 = e^{2\iota^*(u)} h_0$$

(with obvious embeddings ι). In other words, for the choice $\iota^*(u) = \varphi$, $\kappa^*(g_+)$ is a Poincaré metric in normal form relative to \hat{h} .

The volume function $v(r) \in C^\infty(M^n)$ is defined by the relation

$$(2-15) \quad d\text{vol}(h_r) = v(r) d\text{vol}(h)$$

of volume forms. For odd n and even Poincaré metrics, the function $v(r)$ has an even Taylor series $v(r) = 1 + r^2 v_2 + r^4 v_4 + \dots$. Similarly, for even n , we have

$$v(r) = 1 + r^2 v_2 + \dots + r^n v_n + \dots$$

The indicated coefficients in these expansions are locally determined by h . The coefficients v_{2j} are called the renormalized volume coefficients of (M^n, h) [Graham 2000]. Note that

$$(2-16) \quad \dot{v}(r)/v(r) = \frac{1}{2} \text{tr}(h_r^{-1} \dot{h}_r).$$

In particular, for even n , the coefficient v_n only depends on the trace of h_n . We also set

$$(2-17) \quad w(r) \stackrel{\text{def}}{=} \sqrt{v(r)}.$$

Then $w(r) = \sum_{j \geq 0} r^{2j} w_{2j}$. The coefficients w_{2N} are polynomials in v_{2k} for $k \leq N$.

A routine calculation shows that the Laplacian of g_+ takes the form

$$(2-18) \quad \Delta_{g_+} = r^2 \Delta_{h_r} + r^2 \partial_r^2 - (n-1)r \partial_r + \frac{1}{2} \text{tr}(h_r^{-1} \dot{h}_r) r^2 \partial_r,$$

where Δ_{h_r} is the Laplacian of h_r and \dot{h}_r denotes the derivative of h_r with respect to r . Since \bar{g} and g_+ are conformally equivalent, another calculation shows that

$$(2-19) \quad \Delta_{\bar{g}} = r^{-2} (\Delta_{g_+} + (n-1)r \partial_r) = \Delta_{h_r} + \partial_r^2 + \frac{1}{2} \text{tr}(h_r^{-1} \dot{h}_r) \partial_r.$$

For even h_r , this formula is well-defined on $(-\varepsilon, \varepsilon) \times M$.

We continue with the discussion of GJMS operators of (M^n, h) . These operators can be described in terms of asymptotic expansions of the solutions of the equation

$$\Delta_{g_+} u + \lambda(n - \lambda)u = 0, \quad \lambda \in \mathbb{C}.$$

In the case of the hyperbolic ball such solutions can be represented as Helgason–Poisson transforms of distributions (or even hyperfunctions) on the boundary S^n [Helgason 1984]. Here we consider solutions with *smooth* boundary values in the curved setting. A corresponding Poisson transform was constructed in [Graham and Zworski 2003, Proposition 3.5]. Its definition rests on a local (near the boundary) asymptotic analysis of the expansions of eigenfunctions and global mapping properties of the resolvent of Δ_{g_+} acting on appropriate functional spaces.

More precisely, we consider eigenfunctions u with asymptotic expansions of the form

$$(2-20) \quad u(r, x) \sim \sum_{j \geq 0} r^{\lambda+2j} a_{2j}(\lambda)(x) + \sum_{j \geq 0} r^{n-\lambda+2j} b_{2j}(\lambda)(x)$$

with coefficients $a_{2j}(\lambda), b_{2j}(\lambda) \in C^\infty(M)$. The coefficients in both sums in (2-20) are determined by the respective leading coefficients $a_0(\lambda)$ and $b_0(\lambda)$ through a recursive algorithm. Moreover, for a global eigenfunction u , both leading coefficients are related by a scattering operator $\mathcal{S}(\lambda)$.

We recall these constructions in some more detail. First, a local asymptotic analysis yields a map

$$\Phi(\lambda) : C^\infty(M) \rightarrow r^{n-\lambda} C^\infty(M_+), \quad \Re(\lambda) > n/2$$

so that

$$(\Delta_{g_+} + \lambda(n - \lambda))\Phi(\lambda)f = O(r^\infty).$$

For even Poincaré metrics, it has the form

$$\Phi(\lambda)f = r^{n-\lambda}f + \sum_{j \geq 1} r^{n-\lambda+2j} \mathcal{T}_{2j}(n - \lambda)f,$$

where $\mathcal{T}_{2j}(\lambda)$ are meromorphic families of differential operators on M of order $2j$. Next, the resolvent $R(\lambda) = (\Delta_{g_+} + \lambda(n - \lambda))^{-1} : L^2(M_+) \rightarrow L^2(M_+)$ is holomorphic for $\Re(\lambda) > n$. Moreover, its restriction to the space of smooth functions which vanish of infinite order on the boundary, admits a meromorphic continuation to \mathbb{C} . The range of that restriction of $R(\lambda)$ is contained in $r^\lambda C^\infty(M_+)$. Then the family

$$(2-21) \quad \mathcal{P}(\lambda) \stackrel{\text{def}}{=} \Phi(\lambda) - R(\lambda)(\Delta_{g_+} + \lambda(n - \lambda))\Phi(\lambda)$$

of *Poisson transforms* is meromorphic on $\Re(\lambda) > n/2$ with poles only for real λ with $\lambda(n - \lambda) \in \sigma_d(-\Delta_{g_+}) \subset (0, (n/2)^2)$.⁴ $\mathcal{P}(\lambda)$ is continuous up to $\Re(\lambda) = n/2$, $\lambda \neq n/2$. It satisfies

$$(\Delta_{g_+} + \lambda(n - \lambda))\mathcal{P}(\lambda) = 0.$$

Away from the real poles in $\Re(\lambda) > n/2$ and $\lambda \notin n/2 + \mathbb{N}_0$, we have

$$(2-22) \quad \mathcal{P}(\lambda)f = r^\lambda G + r^{n-\lambda} F$$

for $F, G \in C^\infty(M_+)$ with $\iota^*(F) = f$.⁵ f is viewed as the *boundary value* of the eigenfunction $u = \mathcal{P}(\lambda)f$. For the details see [Graham and Zworski 2003, Proposition 3.5]. Later we shall use (2-22) for $\Re(\lambda) = n/2$, $\lambda \neq n/2$.

⁴The poles of $\Phi(\lambda)$ in $n/2 + \mathbb{N}$ cancel in the sum (2-21).

⁵For $\lambda \in n/2 + \mathbb{N}$ the corresponding expansion contains a $\log r$ -term.

The scattering operator $\mathcal{S}(\lambda) : C^\infty(M) \rightarrow C^\infty(M)$ is defined by

$$\mathcal{S}(\lambda) : f \mapsto \iota^*(G)$$

for G as in (2-22). It follows that in the asymptotic expansion (2-20) of $u = \mathcal{P}(\lambda)f$ the coefficients are given by

$$b_{2j}(\lambda) = \mathcal{T}_{2j}(n - \lambda)f \quad \text{and} \quad a_{2j}(\lambda) = \mathcal{T}_{2j}(\lambda)\mathcal{S}(\lambda)f.$$

Note that $\mathcal{T}_0(\lambda) = \text{Id}$.

The families $\mathcal{T}_{2j}(\lambda)$ only depend on the Taylor series of h_r . For odd n and even Poincaré metrics, these are determined by h . Therefore, we write $\mathcal{T}_{2j}(\lambda) = \mathcal{T}_{2j}(h; \lambda)$. For even n , only the families $\mathcal{T}_{2j}(\lambda)$ with $2j \leq n$ are determined by h and we indicate that dependence accordingly. However, the scattering operator $\mathcal{S}(\lambda)$ is a global object which depends on the chosen metric on M_+° .

The families $\mathcal{T}_{2j}(h; \lambda)$ are meromorphic in λ , with simple poles at $\lambda = \frac{n}{2} - k$ for $k = 1, \dots, j$. Under the restriction $2j \leq n$ for even n , the residue of $\mathcal{T}_{2j}(h; \lambda)$ at $\lambda = \frac{n}{2} - j$ is proportional to the GJMS operator $P_{2j}(h)$ on (M, h) . More precisely, we have the basic residue formula

$$(2-23) \quad \text{Res}_{\lambda=n/2-j}(\mathcal{T}_{2j}(h; \lambda)) = \frac{1}{2^{2j} j!(j-1)!} P_{2j}(h)$$

describing GJMS operators in terms of asymptotic expansions of eigenfunctions of Δ_{g_+} . In [Graham and Zworski 2003, Section 4] this formula is derived from the original ambient metric definition of the GJMS operators.

The residue at $\lambda = \frac{n}{2} + N$ of the right-hand side of the expansion (2-20) yields the contribution

$$r^{n/2+N} (\text{Res}_{n/2+N}(\mathcal{S}(\lambda)) + \text{Res}_{n/2-N}(\mathcal{T}_{2N}(\lambda))).$$

Under mild assumptions, this residue vanishes. Hence (2-23) implies the residue formula

$$(2-24) \quad \text{Res}_{\lambda=n/2+j}(\mathcal{S}(\lambda)) = -\frac{1}{2^{2j} j!(j-1)!} P_{2j}(h)$$

for the poles of the scattering operator [Graham and Zworski 2003, Theorem 1].

We emphasize that the formula (2-23) only rests on the local analysis of eigenfunctions near the boundary $r = 0$. However, the definition of the scattering operator and the construction of *exact* eigenfunctions involves the *global* resolvent $R(\lambda)$ on an asymptotically hyperbolic manifold M_+° . For a given closed M , a simple choice for M_+° is $M_+^\circ = (0, 1) \times M$. In that case, the metric on M_+° is a Poincaré metric near *both* boundary components at $r = 0$ and $r = 1$. The resulting scattering operator then acts on smooth functions on the disjoint union of both copies of M .

A simple special case of the latter situation is the scattering operator of the hyperbolic cylinder $M_+^\circ = \Gamma \backslash \mathbb{H}^{n+1}$ by a cocompact discrete subgroup of $\mathrm{SO}(1, n)^\circ$ regarded as a subgroup of $\mathrm{SO}(1, n+1)^\circ$ (using a trivial embedding). Then M_+° can be identified with a cylinder $(-\infty, \infty) \times M$ with compact cross-section $M = \Gamma \backslash \mathbb{H}^n$. The boundary consists of two copies of M . The scattering operator of the cylinder acts on $C^\infty(M) \oplus C^\infty(M)$. It decomposes into the direct sum of endomorphisms on the spaces $E(\mu) \oplus E(\mu)$ generated by the eigenspaces

$$E(\mu) = \{u \in C^\infty(M) \mid -\Delta u = \mu(n-1-\mu)u\},$$

where Δ is the Laplacian of the hyperbolic metric on M . The restriction $\mathcal{S}(\lambda; \mu)$ of $\mathcal{S}(\lambda)$ to this space is given by [Patterson and Perry 2001, Appendix B]

$$\mathcal{S}(\lambda; \mu) = 2^{n-2\lambda} \frac{1}{\pi} \frac{\Gamma(\frac{n}{2}-\lambda)}{\Gamma(\lambda-\frac{n}{2})} \Gamma(\lambda-\mu) \Gamma(\lambda-(n-1-\mu)) \begin{pmatrix} \sin \pi(\frac{n}{2}-\mu) & \sin \pi(\frac{n}{2}-\lambda) \\ \sin \pi(\frac{n}{2}-\lambda) & \sin \pi(\frac{n}{2}-\mu) \end{pmatrix}.$$

Although $\mathcal{S}(\lambda; \mu)$ contains off-diagonal terms, its residues at $\frac{n}{2} + N$ are diagonal. More precisely, we find

$$\mathrm{Res}_{\lambda=n/2+N}(\mathcal{S}(\lambda; \mu)) = -\frac{1}{2^{2N} N!(N-1)!} \prod_{j=n/2}^{n/2+N-1} (-\mu(n-1-\mu) + j(n-1-j)) \mathrm{Id}.$$

This result implies the residue formula

$$\begin{aligned} \mathrm{Res}_{\lambda=n/2+N}(\mathcal{S}(\lambda)) &= -\frac{1}{2^{2N} N!(N-1)!} \prod_{j=n/2}^{n/2+N-1} (\Delta_M + j(n-1-j)) \\ &= -\frac{1}{2^{2N} N!(N-1)!} P_{2N}(M, g_{\mathrm{hyp}}), \end{aligned}$$

which confirms the residue formula (2-24) of [Graham and Zworski 2003].

2D. Residue families. We recall the concept of residue families introduced in [Juhl 2009]. We assume that g_+ is an even Poincaré metric relative to h . For $N \in \mathbb{N}_0$ with $2N \leq n$ for even n , we define a polynomial 1-parameter family of differential operators $C^\infty(M_+) \rightarrow C^\infty(M)$ through

$$(2-25) \quad D_{2N}^{\mathrm{res}}(h; \nu) \stackrel{\mathrm{def}}{=} 2^{2N} N! \left(-\frac{n}{2} - \nu + N\right)_N \delta_{2N}(h; \nu + n - 2N),$$

$$(2-26) \quad D_{2N+1}^{\mathrm{res}}(h; \nu) \stackrel{\mathrm{def}}{=} 2^{2N} N! \left(-\frac{n}{2} - \nu + N + 1\right)_N \delta_{2N+1}(h; \nu + n - 2N - 1),$$

where the family $\delta_N(h, \nu) : C^\infty(M_+) \rightarrow C^\infty(M)$ is defined by the residue formula

$$(2-27) \quad \mathrm{Res}_{\lambda=-\nu-1-N} \left(\int_{M_+} r^\lambda u \varphi \, d\mathrm{vol}(\bar{g}) \right) = \int_M f \delta_N(h; \nu) \varphi \, d\mathrm{vol}(h).$$

Here u is an eigenfunction of $-\Delta_{g_+}$ on M_+° with eigenvalue $\nu(n - \nu)$ and boundary value f (Section 2C), and we use test functions $\varphi \in C_c^\infty(M_+)$ ([Juhl 2009, (6.6.11)]). Note that $D_0^{\text{res}}(\lambda) = \iota^*$. The above definitions yield the formula

$$(2-28) \quad \delta_N(h; \nu) = \sum_{j=0}^N \frac{1}{(N-j)!} [\mathcal{T}_j^*(h; \nu)v_0 + \cdots + \mathcal{T}_0^*(h; \nu)v_j] \iota^* \partial_r^{N-j}$$

([Juhl 2009, Definition 6.6.2]) in terms of *solution operators* $\mathcal{T}_j(h; \nu)$ and renormalized volume coefficients v_{2j} . Here the operator $\mathcal{T}_j^*(h; \nu)$ denotes the formal adjoint of $\mathcal{T}_j(h; \nu)$ with respect to the scalar product on $C^\infty(M)$ defined by h . Since g_+ is assumed to be even, solution operators and renormalized volume coefficients with odd indices vanish. Although the residue families $D_N^{\text{res}}(h; \nu)$ are defined in terms of asymptotic expansions of eigenfunctions of Δ_{g_+} , the assumptions on N guarantee that they only depend on the Taylor coefficients of h_r which are determined by h . This justifies the notation. For even n , the family $D_n^{\text{res}}(h; \nu)$ sometimes is called the *critical residue family*. The formula (2-28) shows that, for even n , also the odd-order residue family $D_{n+1}^{\text{res}}(h; \nu)$ is determined by h .

In later sections, we shall prefer to use the notation $D_N^{\text{res}}(h; \lambda)$ instead of $D_N^{\text{res}}(h; \nu)$.

2E. Shift operators and distributional kernels. We recall some results of [Fischmann et al. 2019]. We regard \mathbb{R}^n as a subspace of \mathbb{R}^{n+1} using the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ defined by $x \mapsto (0, x)$. Elements of \mathbb{R}^{n+1} are written in the form (r, x) . We regard \mathbb{R}^{n+1} as a Riemannian manifold with the flat Euclidean metric g_0 . The metric g_0 is the conformal compactification of the upper half-space $(\mathbb{R}_+^{n+1}, g_{\text{hyp}})$ with the hyperbolic metric $g_{\text{hyp}} \stackrel{\text{def}}{=} r^{-2}g_0$. The hyperbolic metric g_{hyp} is a Poincaré metric g_+ in normal form relative to the flat metric h_0 on the boundary \mathbb{R}^n of \mathbb{R}_+^{n+1} .

Let

$$(2-29) \quad K_{\lambda, \nu}^+(r, x) \stackrel{\text{def}}{=} |r|^{\lambda+\nu-n-1} (|x|^2 + r^2)^{-\nu} \quad \text{and} \quad K_{\lambda, \nu}^-(r, x) \stackrel{\text{def}}{=} r K_{\lambda-1, \nu}^+(r, x)$$

be the distributional Schwartz kernels studied in [Kobayashi and Spohn 2015; Frahm and Ørsted 2019]. The maps

$$\varphi \mapsto \int_{\mathbb{R}^{n+1}} K_{\lambda, \nu}^+(r, x - y) \varphi(r, x) \, dr \, dx$$

define operators $C_c^\infty(\mathbb{R}^{n+1}) \rightarrow C^\infty(\mathbb{R}^n)$ which are equivariant with respect to principal series representations of the conformal group of the subspace $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$. Sometimes they are referred to as *symmetry breaking operators* [Kobayashi and Spohn 2015]. Moreover, we set

$$(2-30) \quad P(\lambda) \stackrel{\text{def}}{=} r \Delta - (2\lambda - n - 3) \partial_r : C^\infty(\mathbb{R}^{n+1}) \rightarrow C^\infty(\mathbb{R}^{n+1})$$

(see [Fischmann et al. 2019, (3.5)]), where Δ is the Laplacian of the flat metric g_0 on \mathbb{R}^{n+1} . The operator $P(\lambda)$ is equivariant with respect to principal series representations for the conformal group of \mathbb{R}^n (see (1-2)). The following result motivates us to refer to this operator as a *shift operator* for the kernel $K_{\lambda,\nu}^{\pm}(r, x)$ [Fischmann et al. 2019, Theorem 3.5].

Proposition 2.1. *The operator $P(\lambda)$ shifts the λ -parameter of the kernels $K_{\lambda,\nu}^{\pm}(r, x)$, i.e.,*

$$P(\lambda)K_{\lambda,\nu}^{\pm}(r, x) = (\lambda + \nu - n - 1)(\nu - \lambda + 1)K_{\lambda-1,\nu}^{\mp}(r, x).$$

It was proven in [Fischmann et al. 2019, Theorem 4.2] that composition of shifted versions of $P(\lambda)$ recover the conformal symmetry breaking differential operators $D_N(\lambda)$ studied in [Juhl 2009; Kobayashi et al. 2015; Kobayashi and Speh 2015]. For a related representation theoretical proof of this fact we refer to [Clerc 2017].⁶

2F. The operator $\mathcal{D}_\lambda(g_+)$. Assume we are in the setting of Section 2B. In particular, (M^n, h) is a Riemannian manifold and g_+ is a Poincaré metric on M_+^o in normal form relative to h . Following Graham and Zworski [2003, (4.4)], we introduce a differential operator $\mathcal{D}_\lambda(g_+)$ on $C^\infty(M_+^o)$ which plays a basic role in the construction of the solution operators $\mathcal{T}_{2j}(h; \lambda)$ (see Section 2C). This operator is defined by the equation

$$(2-31) \quad r^{-(n-\lambda+1)}[\Delta_{g_+} + \lambda(n-\lambda)]r^{n-\lambda} = -\mathcal{D}_\lambda(g_+), \quad \lambda \in \mathbb{C}$$

and has the explicit form

$$(2-32) \quad \mathcal{D}_\lambda(g_+) = -r\partial_r^2 + \left(2\lambda - n - 1 - \frac{r}{2} \operatorname{tr}(h_r^{-1}\dot{h}_r)\right)\partial_r - \frac{n-\lambda}{2} \operatorname{tr}(h_r^{-1}\dot{h}_r) - r\Delta_{h_r}.$$

In the case of the hyperbolic upper-half space $(\mathbb{R}_+^{n+1}, g_{\text{hyp}})$, it is given by

$$\mathcal{D}_\lambda(g_{\text{hyp}}) = -r\partial_r^2 + (2\lambda - n - 1)\partial_r - r\Delta_{h_0} = -r\Delta_{g_0} + (2\lambda - n - 1)\partial_r.$$

Comparing this formula with (2-30), yields the relation

$$(2-33) \quad P(\lambda) = -\mathcal{D}_{\lambda-1}(g_{\text{hyp}}).$$

2G. The degenerate Laplacian. We recall the definition of the degenerate Laplacian introduced by Gover and Waldron [2014]. Let (X, c) be an $n+1$ -dimensional conformal Riemannian manifold equipped with a scale $\sigma \in C^\infty(X)$. Associated to these data, we define the operator

$$(2-34) \quad u \mapsto -\sigma\Delta_g u + (n+2\omega-1)\left[g(d\sigma, du) - \frac{\omega}{n+1}\Delta_g(\sigma)u\right] - \frac{2\omega}{n+1}(n+\omega)\sigma J(g)u$$

⁶The operator $P(\lambda)$ appears as (4.6) in [Clerc 2017].

on $C^\infty(X)$ (see [Gover and Waldron 2014, (2.9)]). Here g is a metric in the conformal class c and $\omega \in \mathbb{C}$. The above operator will be denoted by $(I \cdot D)[g, \sigma; \omega]$. The notation $I \cdot D$ for the degenerate Laplacian reflects its definition as a scalar product (in a tractor bundle) of a scale tractor I and the tractor operator D mapping functions to tractors. We shall not go here into the definitions of the relevant concepts of tractor calculus. Let $M = \sigma^{-1}(0)$ be the zero-locus of σ . We assume that σ is a defining function for the hypersurface M . Let $\iota : M \hookrightarrow X$ denote the embedding of M .

It follows from the definition that the operator $\iota^*(I \cdot D)[g, \sigma; \omega]$ degenerates to the first-order operator

$$u \mapsto (n+2\omega-1)\iota^* \left[g(d\sigma, du) - \frac{\omega}{n+1} \Delta_g(\sigma)u \right].$$

If $\sigma^{-2}g$ has constant scalar curvature $-n(n-1)$, it follows that $|\text{grad}_g(\sigma)|^2 = 1$ on M and this operator reduces to the conformally covariant Robin type boundary operator

$$u \mapsto (n+2\omega-1)\iota^*(\nabla_{\text{grad}_g(\sigma)} - \omega H_g)u,$$

where H_g is the mean curvature of M [Gover 2010, Section 3.1; Juhl 2009, Section 6.2].

By its very definition in terms of tractor calculus, the operator $I \cdot D$ satisfies a conformal covariance property. For the convenience of the reader, we provide an independent proof of that basic property.

Proposition 2.2. *The degenerate Laplacian satisfies*

$$(2-35) \quad (I \cdot D)[e^{2\varphi}g, e^\varphi\sigma; \omega] \circ e^{\omega\varphi} = e^{(\omega-1)\varphi} \circ (I \cdot D)[g, \sigma; \omega], \quad \omega \in \mathbb{C}$$

for all metrics g , scales $\sigma \in C^\infty(X)$ and $\varphi \in C^\infty(X)$. Moreover,

$$(2-36) \quad (I \cdot D)[\kappa^*(g), \kappa^*(\sigma); \omega] = \kappa^* \circ (I \cdot D)[g; \sigma; \omega] \circ \kappa_*$$

for any diffeomorphism κ of X . Here $\kappa_* \stackrel{\text{def}}{=} (\kappa^{-1})^*$.

Proof. The first claim follows by a straightforward computation involving the standard identities

$$(2-37) \quad \begin{aligned} \Delta_{e^{2\varphi}g} &= e^{-2\varphi} \left[\Delta_g + (n-1)g(\text{grad}_g(\varphi), \text{grad}_g(\cdot)) \right], \\ J(e^{2\varphi}g) &= e^{-2\varphi} \left[J(g) - \Delta_g\varphi - \frac{n-1}{2} |d\varphi|_g^2 \right], \\ \Delta_g(e^{\omega\varphi}) &= \omega e^{\omega\varphi} \Delta_g\varphi + \omega^2 e^{\omega\varphi} |d\varphi|_g^2, \\ \text{grad}_{e^{2\varphi}g}(f) &= e^{-2\varphi} \text{grad}_g(f), \\ \Delta_g(f_1 f_2) &= \Delta_g(f_1) f_2 + f_1 \Delta_g(f_2) + 2g(\text{grad}_g(f_1), \text{grad}_g(f_2)), \\ \text{grad}_g(f_1 f_2) &= \text{grad}_g(f_1) f_2 + f_1 \text{grad}_g(f_2) \end{aligned}$$

for any $f, f_1, f_2 \in C^\infty(X)$. These relations imply

$$\begin{aligned} & e^{-(\omega-1)\varphi} \circ (I \cdot D)[e^{2\varphi}g, e^\varphi\sigma; \omega] \circ e^{\omega\varphi} \\ &= (I \cdot D)[g, \sigma; \omega] + \left[-\omega - \frac{\omega}{n+1}(2\omega+n-1) + \frac{2\omega}{n+1}(n+\omega)\right] \sigma \Delta_g \varphi \\ &\quad + \left[-(2\omega+n-1) + (2\omega+n-1)\right] \sigma g(\text{grad}_g(\varphi), \text{grad}_g(\cdot)) \\ &\quad + \left[-\omega(\omega+n-1) + \omega(2\omega+n-1) - \frac{n\omega(2\omega+n-1)}{n+1} + \frac{\omega(n-1)(n+\omega)}{n+1}\right] \sigma |d\varphi|_g^2 \\ &\quad + \left[\omega(2\omega+n-1) - (n+1)\frac{\omega(2\omega+n-1)}{n+1}\right] g(\text{grad}_g(\varphi), \text{grad}_g(\sigma)) \\ &= (I \cdot D)[g, \sigma; \omega]. \end{aligned}$$

The second claim is immediate by construction. The proof is complete. \square

For (X, g) being the flat Euclidean space (\mathbb{R}^{n+1}, g_0) and the scale r with zero-locus \mathbb{R}^n (as in Section 2E), the degenerate Laplacian $I \cdot D$ reduces to

$$(I \cdot D)[g_0, r; \lambda-n-1] = -r \Delta_{g_0} + (2\lambda-n-3)\partial_r.$$

Hence

$$(2-38) \quad P(\lambda) = -(I \cdot D)[g_0, r, \lambda-n-1],$$

i.e., the shift operator $P(\lambda)$ is a special case of $I \cdot D$.

3. A curved version of the shift operator

Now we extend the definition of the shift operators $P(\lambda)$ (in Section 2E) to the setting of Section 2C. Thus, we assume that (M^n, h) is a Riemannian manifold of dimension $n \geq 3$ and we let $g_+ = r^{-2}(dr^2 + h_r)$ be an even Poincaré metric in normal form relative to h on $M_+^\circ = (0, \varepsilon) \times M$ (for some $\varepsilon > 0$). In particular, that means that, for odd n , h_r has an expansion

$$h_r = h + r^2 h_2 + r^4 h_4 + \dots,$$

where all coefficients h_{2j} are determined by h , and for even n in the expansion

$$h_r = h + r^2 h_2 + \dots + r^n h_n + r^{n+2} h_{n+2} + \dots$$

the coefficients $h_{\leq n-2}$ and $\text{tr}_h(h_n)$ are determined by h . We recall the notation $\bar{g} = r^2 g_+$. The following definition corresponds to (1-4).

Definition 3.1. The second-order differential operator

$$(3-1) \quad S(g_+; \lambda) \stackrel{\text{def}}{=} r \Delta_{\bar{g}} - (2\lambda-n+1)\partial_r - \frac{1}{2}(\lambda-n+1) \text{tr}(h_r^{-1} \dot{h}_r)$$

is called a *shift operator*. Here the dot denotes the derivative with respect to r .

The notion *shift operator* is motivated by Theorem 3.7.

We shall regard $S(g_+; \lambda)$ as an operator $C^\infty(M_+^\circ) \rightarrow C^\infty(M_+^\circ)$. Since \bar{g} is a smooth metric on $M_+ = [0, \varepsilon) \times M$, the shift operators may also be regarded as operators on $C^\infty(M_+)$, and mapping properties on M_+° naturally extend to M_+ .

We emphasize that the shift operator $S(g_+; \lambda)$ is not completely determined by the Taylor coefficients of h_r . The composition $\iota^*S(g_+; \lambda)$ degenerates to a first-order operator.

By (2-16), the shift operator can also be written in the form

$$(3-2) \quad S(g_+; \lambda) \stackrel{\text{def}}{=} r \Delta_{\bar{g}} - (2\lambda - n + 1)\partial_r - (\lambda - n + 1)\dot{v}(r)/v(r).$$

The following result describes the relations among the three operators \mathcal{D}_λ , $(I \cdot D)[\cdot; \lambda]$ and $S(\cdot; \lambda)$.

Proposition 3.2. *It holds that*

$$(3-3) \quad S(g_+; \lambda) = -\mathcal{D}_{\lambda+1}(g_+).$$

Moreover, we have

$$(3-4) \quad S(g_+; \lambda) = -(I \cdot D)[\bar{g}; r, \lambda - n + 1]$$

if g_+ is Einstein.

Proof. We recall the expressions (2-18) and (2-32) for Δ_{g_+} and $\mathcal{D}_\lambda(g_+)$. Moreover, by (2-19), we have

$$\Delta_{\bar{g}} = r^{-2}(\Delta_{g_+} + (n-1)r\partial_r).$$

Combining these formulas, we obtain

$$\begin{aligned} S(g_+; \lambda) &= r^{-1}(\Delta_{g_+} + (n-1)r\partial_r) - (2\lambda - n + 1)\partial_r - \frac{1}{2}(\lambda - n + 1)\text{tr}(h_r^{-1}\dot{h}_r) \\ &= r\Delta_{h_r} + r\partial_r^2 + \frac{1}{2}\text{tr}(h_r^{-1}\dot{h}_r)r\partial_r - (2\lambda - n + 1)\partial_r - \frac{1}{2}(\lambda - n + 1)\text{tr}(h_r^{-1}\dot{h}_r) \\ &= r\Delta_{h_r} + r\partial_r^2 - \left(2\lambda - n + 1 - \frac{r}{2}\text{tr}(h_r^{-1}\dot{h}_r)\right)\partial_r - \frac{1}{2}(\lambda - n + 1)\text{tr}(h_r^{-1}\dot{h}_r) \\ &= -\mathcal{D}_{\lambda+1}(g_+). \end{aligned}$$

This proves the first claim. The second claim follows by evaluating $I \cdot D$ for $g = \bar{g}$ and $\sigma = r$. In particular, using

$$\Delta_{\bar{g}}(r) = \frac{1}{2}\text{tr}(h_r^{-1}\dot{h}_r)$$

and the relation

$$(3-5) \quad J(\bar{g}) = -\frac{1}{2r}\text{tr}(h_r^{-1}\dot{h}_r)$$

[Juhl 2009, (6.11.8)], we find

$$\begin{aligned} (I \cdot D)[\bar{g}; r, \omega] &= -r \Delta_{\bar{g}} + (n+2\omega-1) \left(\partial_r - \frac{\omega}{n+1} \Delta_{\bar{g}}(r) \right) - \frac{2\omega}{n+1} (n+\omega)r J(\bar{g}) \\ &= -r \Delta_{\bar{g}} + (n+2\omega-1) \partial_r + \frac{\omega}{2} \text{tr}(h_r^{-1} \dot{h}_r). \end{aligned}$$

Hence $S(g_+; \lambda) = -(I \cdot D)[\bar{g}; r, \lambda - n + 1]$. The proof is complete. □

Remark 3.3. The identity (3-5) is not valid for general Poincaré metrics g_+ since these are only asymptotically Einstein. For the convenience of the reader, we insert a proof of (3-5) for Einstein g_+ which also clarifies the modification in the general case. We recall the well-known transformation formula

$$(3-6) \quad \tau(\hat{g}) = e^{-2\varphi} \tau(g) - 2n \Delta_{\hat{g}}(\varphi) + n(n-1) |d\varphi|_{\hat{g}}^2, \quad \hat{g} = e^{2\varphi} g$$

for the scalar curvature. We apply (3-6) to $\hat{g} = dr^2 + h_r$, $g = r^{-2}(dr^2 + h_r)$ and $\varphi = \log r$. For Einstein g , i.e., $\text{Ric}(g) + ng = 0$, we have $\tau(g) = -n(n+1)$. Hence we find

$$\begin{aligned} \tau(dr^2 + h_r) &= -\frac{n(n+1)}{r^2} - 2n \Delta_{dr^2 + h_r}(\log r) + n(n-1) |d \log r|_{dr^2 + h_r}^2 \\ &= -\frac{n(n+1)}{r^2} - 2n \left(\partial_r^2(\log r) + \frac{1}{2} \text{tr}(h_r^{-1} \dot{h}_r) \partial_r(\log r) \right) + \frac{n(n-1)}{r^2} \\ &= -n \text{tr}(h_r^{-1} \dot{h}_r) \frac{1}{r}. \end{aligned}$$

Hence

$$J(dr^2 + h_r) = -\frac{1}{2r} \text{tr}(h_r^{-1} \dot{h}_r).$$

We also note that, in [Gover and Waldron 2014], the authors deal with Einstein metrics outside hypersurfaces in Riemannian manifolds. In particular, the calculation at the end of [Gover and Waldron 2014, Section 5] employs the formula (3-5).

Remark 3.4. By the first part of Proposition 3.2, we have

$$(3-7) \quad S(g_+; \lambda) = r^{\lambda-n} \circ (\Delta_{g_+} + (\lambda+1)(n-\lambda-1)) \circ r^{n-\lambda-1}$$

as an identity of operators acting on $C^\infty(M_+^o)$. Sometimes, this formula for $S(g_+; \lambda)$ will be more convenient to work with than the original definition (3-1). However, the original definition has the advantage that it clearly shows that $S(g_+; \lambda)$ acts on smooth functions on M_+ . In order to illustrate the convenience of the conjugation formula (3-7), we note that it yields a slightly more conceptual proof of (3-5). We use the fact that for Einstein g_+ ,

$$P_2(g_+) = \Delta_{g_+} + m(m-1)$$

with $m = \frac{n+1}{2}$. Hence

$$P_2(\bar{g}) = r^{-m-1} P_2(g_+) r^{m-1} = r^{-m-1} (\Delta_{g_+} + m(m-1)) r^{m-1} = r^{-1} S\left(g_+; \frac{n-1}{2}\right).$$

This identity is the special case $N = 1$ of Theorem 3. On the function 1 it yields (3-5).

Remark 3.5. For even n , the condition $\text{Ric}(g_+) + ng_+ = O(r^{n-2})$ implies $\tau(g_+) = -n(n+1) + O(r^n)$. Hence an extension of the arguments in Remark 3.4 gives

$$P_2(g_+) = \Delta_{g_+} + m(m-1) + O(r^n) \quad \text{and} \quad P_2(\bar{g}) = r^{-1}S\left(g_+; \frac{n-1}{2}\right) + O(r^{n-2}).$$

The trace condition (2-11) improves the estimate of the scalar curvature to $\tau(g_+) = -n(n+1) + o(r^n)$ and we obtain

$$rP_2(\bar{g}) = S\left(g_+; \frac{n-1}{2}\right) + o(r^{n-1}).$$

This is the special case $N = 1$ of Theorem 3 for Poincaré metrics.

We continue with the discussion of the *shift property* of the shift operators. Its description requires some more notation. We recall that $M_+^\circ = (0, \varepsilon) \times M$ and $u \in C^\infty(M_+^\circ)$ be a solution to the equation

$$(3-8) \quad \Delta_{g_+} u + \nu(n-\nu)u = 0 \quad \text{for } \nu \in n/2 + i\mathbb{R}, \nu \neq n/2$$

with *boundary value* $f \in C^\infty(M)$ (see Section 2C). Such an *exact* eigenfunction can be constructed as described in Section 2C by regarding M_+ as part of a conformally compact manifold. Then u gives rise to the family (see [Juhl 2009, Section 6.6])

$$(3-9) \quad \lambda \mapsto M_u(r; \lambda) \in C^\infty(M_+^\circ), \quad M_u(r; \lambda) \stackrel{\text{def}}{=} r^{\lambda-n+1}u, \quad \lambda \in \mathbb{C}.$$

We shall consider $M_u(r; \lambda)$ as a family of functions on M_+° as well as a family of distributions on M_+ . In the latter case, we have

$$\langle M_u(r; \lambda), \varphi \rangle = \int_{M_+} r^{\lambda-n+1}u\varphi \, d\text{vol}(\bar{g}), \quad \Re(\lambda) > n/2 - 2,$$

where the test functions φ are in $C_c^\infty(M_+)$ and the condition $\Re(\lambda) > \frac{n}{2} - 2$ guarantees the convergence of the integral. Then $M_u(r; \lambda)$ will be understood as a meromorphic family of distributions. For simplicity, we shall denote that family of distributions also by $M_u(r; \lambda)$. Finally, we introduce the multiplication operators

$$(3-10) \quad M_r \stackrel{\text{def}}{=} r \cdot .$$

Remark 3.6. We consider the distribution $M_u(r; \lambda)$ in the flat case. Let $u \in C^\infty(\mathbb{H}^{n+1})$ be an eigenfunction of the Laplacian on the hyperbolic upper-half space \mathbb{H}^{n+1} . We assume that u can be written as Helgason’s Poisson transform [1984],

$$(3-11) \quad u(r, x) = \int_{\mathbb{R}^n} \left(\frac{r}{|x-y|^2 + r^2} \right)^\nu f(y) \, dy$$

of $f \in C_0^\infty(\mathbb{R}^n)$, say. Then

$$\begin{aligned} \langle M_u(r; \lambda), \varphi \rangle &= \int_{\mathbb{H}^{n+1}} r^{\lambda-n+1} u(r, x) \varphi(r, x) dr dx \\ &= \int_{\mathbb{H}^{n+1}} \int_{\mathbb{R}^n} r^{\lambda-n+1} \left(\frac{r}{|x-y|^2+r^2} \right)^v f(y) \varphi(r, x) dy dr dx \\ &= \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{H}^{n+1}} K_{\lambda-2, v}^+(r, x-y) \varphi(r, x) dx dr \right) dy. \end{aligned}$$

The latter formula shows in which sense (3-9) can be regarded as a generalization of the distributional kernels $K_{\lambda, v}^\pm(r, x)$ (see (2-29)). We also note that the asymptotic expansion of u given by (3-11) is of the form

$$r^v \sum_{j \geq 0} r^{2j} a_{2j}(x) + r^{n-v} c(\lambda) \sum_{j \geq 0} r^{2j} b_{2j}(x), \quad r \rightarrow 0,$$

where $b_0 = f$ and $c(\lambda)$ is Harish-Chandra’s c -function. In particular, the boundary value f appears in the coefficient of $r^{n-\lambda}$. Note that this notion of boundary value slightly differs from that used in Section 2C by the coefficient $c(\lambda)$.

Now we are ready to state and prove the *shift property* of the operators $S(g_+; \lambda)$. The following result generalizes [Fischmann et al. 2019, Theorem 3.5]. Following the terminology of [Fischmann et al. 2019], it may be referred to as a Bernstein–Sato identity.

Theorem 3.7. *Let (M^n, h) be a Riemannian manifold. Let u be a solution of (3-8). Then the multiplication operator M_r and the shift operator $S(g_+; \lambda)$ shift the λ -parameter when acting on the function $M_u(r; \lambda)$, i.e.,*

$$\begin{aligned} M_r(M_u(r; \lambda)) &= M_u(r; \lambda + 1), \\ S(g_+; \lambda)(M_u(r; \lambda)) &= (\lambda + v - n + 1)(v - \lambda - 1)M_u(r; \lambda - 1). \end{aligned}$$

Proof. The first claim is obvious from the definitions. Using (3-7) and the fact that u satisfies (3-8), we compute

$$\begin{aligned} S(g_+; \lambda)(M_u(r; \lambda)) &= r^{\lambda-n} \Delta_{g_+}(r^{n-\lambda-1} r^{\lambda-n+1} u) + (\lambda+1)(n-\lambda-1)r^{\lambda-n} u \\ &= -v(n-v)r^{\lambda-n} u + (\lambda-n+1)(-\lambda-1)r^{\lambda-n} u \\ &= (\lambda+v-n+1)(v-\lambda-1)M_u(r; \lambda-1). \end{aligned}$$

The proof is complete. □

From Theorem 3.7 we easily deduce the poles of the meromorphic continuation of the holomorphic family of distributions

$$C_c^\infty(M_+) \ni \varphi \mapsto \int_{M_+} M_u(r; \lambda) \varphi d\text{vol}(\bar{g}), \quad \Re(\lambda) > \frac{n}{2} - 2.$$

Corollary 3.8. *The family $\lambda \mapsto M_u(r; \lambda)$ is meromorphic with generically simple poles at⁷*

$$(3-12) \quad \lambda = -v+n-2-N \quad \text{and} \quad \lambda = v-2-N$$

for $N \in \mathbb{N}_0$.

Proof. Indeed, Theorem 3.7 implies

$$M_u(r; \lambda) = \frac{1}{(\lambda+v-n+2)(v-\lambda-2)} S(g_+; \lambda+1)(M_u(r; \lambda+1)).$$

The distribution on the left-hand side of this identity is holomorphic in the half-plane $\Re(\lambda) > \frac{n}{2} - 2$. The right-hand side provides a meromorphic continuation to $\Re(\lambda) > \frac{n}{2} - 3$ with simple poles in $\lambda = -v + n + 2$ and $\lambda = v - 2$. The assertion follows by a repeated application of that argument. \square

Alternatively, Corollary 3.8 can be proved by directly inserting the asymptotic expansion of u into the integral which defines the distribution $M_u(r; \lambda)$ for $\Re(\lambda) > \frac{n}{2} - 2$. The above argument using Theorem 3.7 is more conceptual, however.

Next, we discuss a conformal transformation law for shift operators. Let $\hat{h} = e^{2\varphi}h$ be conformally equivalent to h . We choose an even Poincaré metric g_+ in normal form relative to h and let $\hat{g}_+ = \kappa^*(g_+)$ be a related even Poincaré metric in normal form relative to \hat{h} ; for the construction of the diffeomorphism κ we refer to Section 2C. In these terms, we have the following result.

Proposition 3.9. $S(\hat{g}_+; \lambda) = \left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n} \circ \kappa^* \circ S(g_+; \lambda) \circ \kappa_* \circ \left(\frac{r}{\kappa^*(r)}\right)^{n-\lambda-1}$.

Proof. By (3-7), we have

$$S(\hat{g}_+; \lambda) = r^{\lambda-n} \Delta_{\hat{g}_+} r^{n-\lambda-1} + (\lambda+1)(n-\lambda-1) \frac{1}{r}.$$

But $\kappa^*(g_+) = \hat{g}_+$ implies $\Delta_{\hat{g}_+} = \kappa^* \Delta_{g_+} \kappa_*$. Hence

$$\begin{aligned} S(\hat{g}_+; \lambda) &= r^{\lambda-n} \kappa^* r^{-\lambda+n} r^{\lambda-n} \Delta_{g_+} r^{n-\lambda-1} r^{\lambda-n+1} \kappa_*(r^{n-\lambda-1} \cdot) + (\lambda+1)(n-\lambda-1) \frac{1}{r} \\ &= r^{\lambda-n} \kappa^*(r^{-\lambda+n}) \kappa^* r^{\lambda-n} \Delta_{g_+} r^{n-\lambda-1} \kappa_* \kappa^*(r^{\lambda-n+1}) r^{n-\lambda-1} + (\lambda+1)(n-\lambda-1) \frac{1}{r} \\ &= \left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n} \kappa^*(r^{\lambda-n} \Delta_{g_+} r^{n-\lambda-1}) \kappa_* \left(\frac{r}{\kappa^*(r)}\right)^{n-\lambda-1} + (\lambda+1)(n-\lambda-1) \frac{1}{r}. \end{aligned}$$

Now the obvious identity

$$\frac{1}{r} = \left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n} \circ \kappa^* \circ \frac{1}{r} \circ \kappa_* \circ \left(\frac{r}{\kappa^*(r)}\right)^{n-\lambda-1}$$

⁷Here v is generic if both ladders of poles in (3-12) do not intersect. For $v \in \frac{n}{2} + i\mathbb{R}$ being generic it suffices to assume that $v \neq \frac{n}{2}$.

implies

$$S(\hat{g}_+, r; \lambda) = \left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n} \kappa^* S(g_+; \lambda) \kappa_* \left(\frac{r}{\kappa^*(r)}\right)^{n-\lambda-1}.$$

This completes the proof. \square

Theorem 1 is a direct consequence of this result.

Remark 3.10. For Einstein g_+ , Proposition 3.9 also follows by combining the conformal covariance of $I \cdot D$ (Proposition 2.2) with the identification of $S(g_+; \lambda)$ as a degenerate Laplacian (Proposition 3.2). Indeed, for $\hat{h} = e^{2\varphi} h$, we write $g_+ = r^{-2}(dr^2 + h_r)$ and $\hat{g}_+ = r^{-2}(dr^2 + \hat{h}_r)$ with $\hat{g}_+ = \kappa^*(g_+)$ and $h_0 = h$, $\hat{h}_0 = \hat{h}$. Then

$$(3-13) \quad \kappa^*(dr^2 + h_r) = \left(\frac{\kappa^*(r)}{r}\right)^2 (dr^2 + \hat{h}_r).$$

Now assume that g_+ is Einstein. Then

$$\begin{aligned} S(\hat{g}_+; \lambda) &= -(I \cdot D)[dr^2 + \hat{h}_r, r; \lambda - n + 1] && \text{(by (3-4))} \\ &= -(I \cdot D)\left[\left(\frac{\kappa^*(r)}{r}\right)^{-2} \kappa^*(dr^2 + h_r), \left(\frac{\kappa^*(r)}{r}\right)^{-1} \kappa^*(r); \lambda - n + 1\right] && \text{(by (3-13))} \\ &= -\left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n} (I \cdot D)[\kappa^*(dr^2 + h_r), \kappa^*(r); \lambda - n + 1] \left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n+1} && \text{(by (2-35))} \\ &= -\left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n} \kappa^*(I \cdot D)[dr^2 + h_r, r; \lambda - n + 1] \kappa_* \left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n+1} && \text{(by (2-36))} \\ &= \left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n} \kappa^* S(g_+; \lambda) \kappa_* \left(\frac{r}{\kappa^*(r)}\right)^{\lambda-n+1}. && \text{(by (3-4))} \end{aligned}$$

A general Poincaré metric g_+ is only approximate Einstein. For such a metric, the identification of $S(g_+; \lambda)$ with a degenerate Laplacian holds true only up to an error term. This leads to error terms in the above calculation and in the resulting conformal transformation law.

In Section 4, we shall prove a relation between compositions of shift operators and residue families. For that purpose, we need a relation among the operator $S(g_+; \lambda)$ and its formal adjoint $S^*(g_+; \lambda)$ with respect to the scalar product defined by \bar{g} .

Proposition 3.11. *The formal adjoint of the operator $S(g_+; \lambda)$ acting on $C_c^\infty(M_+^\circ)$ with respect to the scalar product defined by \bar{g} is given by*

$$S^*(g_+; \lambda) = S(g_+; n - \lambda - 2).$$

Proof. Let $f_1, f_2 \in C_c^\infty(M_+^\circ)$. Then, using $d\text{vol}(\bar{g}) = r^{n+1} d\text{vol}(g_+)$ and (3-7), we have

$$\begin{aligned} \int_{M_+} (S(g_+; \lambda) f_1) f_2 d\text{vol}(\bar{g}) &= \int_{M_+} r^{\lambda-n} [\Delta_{g_+} + (\lambda+1)(n-\lambda-1)] (r^{n-\lambda-1} f_1) f_2 r^{n+1} d\text{vol}(g_+) \\ &= \int_{M_+} [\Delta_{g_+} + (\lambda+1)(n-\lambda-1)] (r^{n-\lambda-1} f_1) r^{\lambda+1} f_2 d\text{vol}(g_+) \\ &= \int_{M_+} (r^{n-\lambda-1} f_1) [\Delta_{g_+} + (\lambda+1)(n-\lambda-1)] (r^{\lambda+1} f_2) d\text{vol}(g_+) \\ &= \int_{M_+} f_1 (r^{-\lambda-2} [\Delta_{g_+} + (\lambda+1)(n-\lambda-1)] (r^{\lambda+1} f_2)) d\text{vol}(\bar{g}). \end{aligned}$$

Thus

$$S^*(g_+; \cdot, \lambda) = r^{-\lambda-2} \circ (\Delta_{g_+} + (\lambda+1)(n-\lambda-1)) \circ r^{\lambda+1}.$$

Comparing this relation with (3-7) completes the proof. \square

In the proof of Theorem 4.1, we shall actually need the following improved version of the latter result.

Remark 3.12. The relation

$$\int_{M_+} S(g_+; \lambda)(f_1) f_2 d\text{vol}(\bar{g}) = \int_{M_+} f_1 S(g_+; n-\lambda-2)(f_2) d\text{vol}(\bar{g})$$

continues to be true for $f_2 \in C_c^\infty(M_+)$ and $f_1 \in C^\infty(M_+^\circ)$ with an asymptotic expansion of the form

$$f_1(r, x) \sim \sum_{j \geq 0} r^{\nu+j} a_j(x), \quad r \rightarrow 0, \quad a_j \in C^\infty(M)$$

with $\Re(\nu) > 0$.

Proof. It suffices to prove the relations

$$\int_{M_+} \partial_r(f_1) f_2 d\text{vol}(\bar{g}) = - \int_{M_+} f_1 \partial_r(f_2) d\text{vol}(\bar{g}) - \frac{1}{2} \int_{M_+} f_1 \text{tr}(h_r^{-1} \dot{h}_r) f_2 d\text{vol}(\bar{g})$$

and

$$\int_{M_+} (r \Delta_{\bar{g}})(f_1) f_2 d\text{vol}(\bar{g}) = \int_{M_+} f_1 (r \Delta_{\bar{g}} + 2\partial_r + \frac{1}{2} \text{tr}(h_r^{-1} \dot{h}_r))(f_2) d\text{vol}(\bar{g}).$$

For the proof of the first relation we use (2-16) and the observation that the restriction of f_1 to the boundary vanishes. Green's formula and (2-19) imply the second relation. Here the boundary contributions vanish since the restrictions of f_1 and $r \partial_r(f_1)$ to the boundary both vanish. \square

For later purpose, we need to understand the behavior of the composition of $S(g_+; \lambda)$ with the multiplication by powers of r .

Lemma 3.13. *Let $f \in C^\infty(M_+^\circ)$ and $a \in \mathbb{N}$. Then*

$$S(g_+; \lambda)(r^a f) = r^a S(g_+; \lambda - a) f - a(2\lambda - n + 2 - a)r^{a-1} f.$$

Proof. The proof is straightforward. The definition (3-1) and the formula (2-19) for $\Delta_{\bar{g}}$ imply

$$\begin{aligned} S(g_+; \lambda)(r^a f) &= r \Delta_{\bar{g}}(r^a f) - (2\lambda - n + 1) \partial_r(r^a f) - \frac{1}{2}(\lambda - n + 1) \operatorname{tr}(h_r^{-1} \dot{h}_r) r^a f \\ &= r^a S(g_+; \lambda - a) f + [a(a - 1) - (2\lambda - n + 1)] r^{a-1} f \\ &= r^a S(g_+; \lambda - a) f - a(2\lambda - n + 2 - a) r^{a-1} f. \end{aligned}$$

The proof is complete. □

Of course, this result is also true for $f \in C^\infty(M_+)$. In view of the second part of Proposition 3.2, it should be viewed as an analog of the $\mathfrak{sl}(2)$ -structure for the degenerate Laplacian $I \cdot D$ proved in [Gover and Waldron 2014, Lemma 3.1, Proposition 3.4].

4. A new formula for residue families

We recall that residue families $D_N^{\text{res}}(h; \lambda)$ are defined by normalizations of the families $\delta_N(h; \lambda)$ (see (2-27)). In the present section, we express these families in terms of shift operators $S(g_+; \lambda)$ (see (3-1)) and discuss some consequences.

For $N \in \mathbb{N}$, we define the family $S_N(g_+; \lambda)$ of shift operators on M_+° by

$$(4-1) \quad S_N(g_+; \lambda) \stackrel{\text{def}}{=} \underbrace{S(g_+; \lambda) \circ \cdots \circ S(g_+; \lambda + N - 1)}_{N \text{ factors}}.$$

We also set $S_0(g_+; \lambda) = \text{Id}$.

Similarly, we define

$$(4-2) \quad (I \cdot D)_N[\bar{g}, r; \lambda] \stackrel{\text{def}}{=} \underbrace{(I \cdot D)[\bar{g}, r; \lambda] \circ \cdots \circ (I \cdot D)[\bar{g}, r; \lambda + N - 1]}_{N \text{ factors}}.$$

Again, we shall regard $S_N(g_+; \lambda)$ as an operator $C^\infty(M_+^\circ) \rightarrow C^\infty(M_+^\circ)$ and also as an operator on $C^\infty(M_+)$. By definition, $S_N(g_+; \lambda)$ is a differential operator of order $2N$ with polynomial coefficients in λ . As a polynomial in λ it is of degree N .

We also recall that, for $N \in \mathbb{N}$ with $N \leq n + 1$ for even n , the families $\delta_N(h; \lambda)$ are determined by h .

Theorem 4.1. *Let $N \in \mathbb{N}$ with $N \leq n + 1$ for even n . Then*

$$(4-3) \quad \delta_N(h; \lambda) = \frac{1}{(-N)_N(2\lambda - n + 1)_N} \iota^* S_N(g_+; \lambda)$$

as an identity of meromorphic functions in λ with values in operators $C^\infty(M_+) \rightarrow C^\infty(M)$.

Proof. Let u be an eigenfunction with boundary value f and satisfying (3-8) with $\Re(\nu) = n/2$, $\nu \neq n/2$. We derive the assertion from Theorem 3.7 and the identity

$$(4-4) \quad \text{Res}_{\lambda=-\nu-1} \left(\int_{M_+} M_u(r; \lambda+n-1) \varphi \, d\text{vol}(\bar{g}) \right) = \int_M f i^* \varphi \, d\text{vol}(h)$$

(see (2-27) for $N = 0$). In the following, it will be convenient to use, for any λ -dependent operator $A(\lambda)$, the notation

$$\begin{aligned} A((\lambda)_N) &\stackrel{\text{def}}{=} A(\lambda) \circ A(\lambda+1) \circ \cdots \circ A(\lambda+N-1), \\ A((\lambda)^N) &\stackrel{\text{def}}{=} A(\lambda) \circ A(\lambda-1) \circ \cdots \circ A(\lambda-N+1). \end{aligned}$$

On the one hand, (2-27) implies

$$(4-5) \quad \text{Res}_{\lambda=-\nu-1-N} \left(\int_{M_+} M_u(r; \lambda+n-1) \varphi \, d\text{vol}(\bar{g}) \right) = \int_M f \delta_N(h; \nu) \varphi \, d\text{vol}(h).$$

On the other hand, using Theorem 3.7, we obtain for $\Re(\lambda) \notin -\frac{n}{2} - \mathbb{N}$

$$\begin{aligned} M_u(r; \lambda+n-1) &= \frac{S(g_+; \lambda+n)(M_u(r; \lambda+n))}{(\lambda+\nu+1)(\nu-\lambda-n-1)} \\ &= \frac{S(g_+; \lambda+n)S(g_+; \lambda+n+1)(M_u(r; \lambda+n+1))}{(\lambda+\nu+1)(\lambda+\nu+2)(\nu-\lambda-n-2)(\nu-\lambda-n-1)} \\ &= \cdots = \frac{S(g_+; (\lambda+n)_N)(M_u(r; \lambda+n+N-1))}{(\lambda+\nu+1)_N(\nu-\lambda-n-N)_N}. \end{aligned}$$

Now we take adjoints using Remark 3.12. This gives

$$\begin{aligned} \int_{M_+} M_u(r; \lambda+n-1) \varphi \, d\text{vol}(\bar{g}) &= \frac{1}{(\lambda+\nu+1)_N(\nu-\lambda-n-N)_N} \\ &\quad \times \int_{M_+} M_u(r; \lambda+n+N-1) S^*(g_+; (\lambda+n+N-1)^N) \varphi \, d\text{vol}(\bar{g}) \end{aligned}$$

for $\Re(\lambda) > -\frac{n}{2} - 1$. By the assumptions, the zeros of $\nu \mapsto (\lambda+\nu+1)_N(\nu-\lambda-n-N)_N$ are simple for $\Re(\lambda) > -\frac{n}{2} - 1$. Hence (4-4) implies

$$\begin{aligned} \text{Res}_{\lambda=-\nu-1-N} \left(\int_{M_+} M_u(r; \lambda+n-1) \varphi \, d\text{vol}(\bar{g}) \right) &= \frac{1}{(-N)_N(2\nu-n+1)_N} \int_M f i^* S^*(g_+; (-\nu+n-2)^N) \varphi \, d\text{vol}(h) \\ &= \frac{1}{(-N)_N(2\nu-n+1)_N} \int_M f i^* S_N(g_+; \nu) \varphi \, d\text{vol}(h). \end{aligned}$$

Comparing this result with (4-5), completes the proof for $\Re(v) = n/2$, $v \neq n/2$. The assertion then follows by meromorphic continuation. \square

Since the operator $S_N(g_+; \lambda)$ involves $2N$ derivatives in r , it is a nontrivial observation that its composition with ι^* only depends on h for appropriate choices of N . But this is an immediate consequence of Theorem 4.1 as long as $N \leq n$ for even n . Therefore, it is justified to introduce the notation

$$(4-6) \quad \Sigma_N(h; \lambda) \stackrel{\text{def}}{=} \iota^* S_N(g_+; \lambda)$$

for such N . However, for even n and general $N \in \mathbb{N}$, the composition $\iota^* S_N(g_+; \lambda)$ does not only depend on h and the notation $\Sigma_N(h; \lambda)$ will not be used.

As a direct consequence of Theorem 4.1, we obtain the following identification of residue families with operators $\Sigma_N(h; \lambda)$.

Corollary 4.2. *Assume that $N \in \mathbb{N}$ so that $2N \leq n$ for even n . Then*

$$(4-7) \quad D_{2N}^{\text{res}}(h; \lambda) = \frac{1}{(-2N)_N \left(\lambda + \frac{n}{2} - 2N + \frac{1}{2}\right)_N} \Sigma_{2N}(h; \lambda + n - 2N)$$

and

$$(4-8) \quad D_{2N+1}^{\text{res}}(h; \lambda) = \frac{1}{2(-2N-1)_{N+1} \left(\lambda + \frac{n}{2} - 2N - \frac{1}{2}\right)_{N+1}} \Sigma_{2N+1}(h; \lambda + n - 2N - 1).$$

Some further comments on this result are in order.

We recall that all residue families $D_N^{\text{res}}(h; \lambda)$ are polynomials in λ . Hence Corollary 4.2 shows that the zeros of the denominators on the right-hand sides of (4-7) and (4-8) actually are zeros of the respective numerators. These formulas naturally reflect the degrees of the families on both sides. In fact, the degrees of the polynomials $D_{2N}^{\text{res}}(h; \lambda)$ and $\Sigma_{2N}(h; \lambda)$ are N and $2N$, respectively. Similarly, the degrees of $D_{2N+1}^{\text{res}}(h; \lambda)$ and $\Sigma_{2N+1}(h; \lambda)$ are N and $2N + 1$, respectively.

Corollary 4.2 also shows that the composition of the order $2N$ family $S_N(g_+; \lambda)$ with ι^* degenerates to a family of order N . Some of the properties of the families $S_N(g_+; \lambda)$ away from $r = 0$ will be discussed in Sections 5–6.

A direct consequence of Corollary 4.2 and Lemma 4.6 are factorization identities for residue families $D_N^{\text{res}}(h; \lambda)$ into compositions with the factors

$$S(g_+; \lambda + n - 1) \quad \text{and} \quad M_r.$$

Corollary 4.3. *Let $N \in \mathbb{N}$ so that $2N \leq n$ for even n . Then*

$$\begin{aligned} D_{2N}^{\text{res}}(h; \lambda) &= D_{2N-1}^{\text{res}}(h; \lambda - 1) S(g_+; \lambda + n - 1), \\ -(2N + 1)(2\lambda + n - 2N - 1) D_{2N+1}^{\text{res}}(h; \lambda) &= D_{2N}^{\text{res}}(h; \lambda - 1) S(g_+; \lambda + n - 1), \end{aligned}$$

and

$$\begin{aligned} D_{2N}^{\text{res}}(h; \lambda) &= D_{2N+1}^{\text{res}}(h; \lambda + 1)M_r, \\ -2N(2\lambda + n - 2N + 2)D_{2N-1}^{\text{res}}(h; \lambda) &= D_{2N}^{\text{res}}(h; \lambda + 1)M_r. \end{aligned}$$

Proof. The first two identities immediately follow from Corollary 4.2. The proofs of the last two identities also require Lemma 4.6. We omit the details. \square

The following result is a consequence of the conformal transformation law in Proposition 3.9.

Lemma 4.4. *Let $N \in \mathbb{N}$ and assume that $\hat{h} = e^{2\varphi}h$. Then*

$$S_N(\hat{g}_+; \lambda) = \left(\frac{\kappa^*(r)}{r}\right)^{n-\lambda} \kappa^* S_N(g_+; \lambda) \kappa_* \left(\frac{\kappa^*(r)}{r}\right)^{\lambda+N-n}.$$

Proof. Proposition 3.9 yields

$$S(\hat{g}_+; \lambda) = \left(\frac{\kappa^*(r)}{r}\right)^{n-\lambda} \kappa^* S(g_+; \lambda) \kappa_* \left(\frac{\kappa^*(r)}{r}\right)^{\lambda-n+1}.$$

An application of that identity to the composition $S_N(\hat{g}_+; \lambda)$ gives

$$S_N(\hat{g}_+; \lambda) = \left(\frac{\kappa^*(r)}{r}\right)^{n-\lambda} \kappa^* S(g_+; \lambda) \cdots S(g_+; \lambda + N - 1) \kappa_* \left(\frac{\kappa^*(r)}{r}\right)^{\lambda+N-n}.$$

The proof is complete. \square

Now by combining Theorem 4.1 with Lemma 4.4, we obtain an alternative proof of the conformal transformation law of residue families [Juhl 2009, Theorem 6.6.3].

Corollary 4.5. *Let $N \in \mathbb{N}$ so that $N \leq n + 1$ for even n . Assume that $\hat{h} = e^{2\varphi}h$. Then*

$$D_N^{\text{res}}(\hat{h}; \lambda) = e^{(\lambda-N)\varphi} D_N^{\text{res}}(h; \lambda) \kappa_* \left(\frac{\kappa^*(r)}{r}\right)^\lambda.$$

Proof. The assertion is a consequence of Theorem 4.1, Lemma 4.4, the limit formula [Juhl 2009, (6.6.16)]

$$\lim_{r \rightarrow 0} \left(\frac{\kappa^*(r)}{r}\right) = e^{-\varphi}$$

and the fact that κ acts as the identity on M . \square

Finally, we prove:

Lemma 4.6. *Let $N \in \mathbb{N}$ and $f \in C^\infty(M_+^\circ)$. Then*

$$(4-9) \quad S_N(g_+; \lambda)(rf) = rS_N(g_+; \lambda - 1)(f) - N(2\lambda - n + N)S_{N-1}(g_+; \lambda)(f).$$

Proof. We iteratively apply Lemma 3.13 to compute

$$\begin{aligned} S_N(g_+; \lambda)(rf) &= S(g_+; \lambda) \circ \cdots \circ S(g_+; \lambda + N - 1)(rf) \\ &= S_{N-1}(g_+; \lambda)rS(g_+; \lambda + N - 2)(f) - (2\lambda - n + 2N - 1)S_{N-1}(g_+; \lambda)(f) \\ &= \cdots = rS_N(g_+; \lambda - 1)(f) - \left[N(2\lambda - n + 2N + 1) - \sum_{j=1}^N 2j \right] S_{N-1}(g_+; \lambda)(f) \end{aligned}$$

Now the identity $\sum_{j=1}^N 2j = N(N + 1)$ completes the proof. □

The identity (4-9) obviously holds true also for all $f \in C^\infty(M_+)$.

5. Applications

In the present section, we further exploit the relation between families of shift operators and residue families. The flow of information will be in both directions, i.e., we use facts on families of shift operators to derive properties of residue families and also use properties of residue families to derive properties of families of shift operators.

It was shown in [Juhl 2009; 2013] that residue families $D_{2N}^{\text{res}}(h; \lambda)$ satisfy two systems of factorization identities. In Section 5A we provide new proofs of these identities. They rest on the identification of two special values of the families of shift operator in terms of GJMS operators. These are given in Theorem 5.1 and Theorem 5.6, respectively. Theorem 5.1 will be further exploited in Section 6F. In Section 5B, we shall describe a compressed formulation of the recursive algorithm for the solution operators $\mathcal{T}_{2j}(h; \lambda)$ in terms of the families $S_N(g_+; \lambda)$. In the last section, we derive a new formula for all Q -curvatures (critical and subcritical ones) in even dimension in terms of shift operators.

5A. Shift operators and GJMS operators. Juhl [2009; 2013] proved that, for $N \in \mathbb{N}$ with $2N \leq n$ for even n , the even-order residue families $D_{2N}^{\text{res}}(h; \nu)$ satisfy the identities

$$(5-1) \quad D_{2N}^{\text{res}}\left(h; -\frac{n}{2} + N\right) = P_{2N}(h)\iota^* \quad \text{and} \quad D_{2N}^{\text{res}}\left(h; -\frac{n+1}{2} + N\right) = \iota^* P_{2N}(\bar{g}).$$

We briefly comment on the well-definedness of the second identity. For odd n , the Taylor coefficients of h_r are determined by h . Hence, for any $N \in \mathbb{N}$, the left-hand side of the second identity is determined by h . The right-hand side of this identity involves a GJMS operator in even dimension $n + 1$. We recall that, for general metrics, these are only defined for subcritical orders $2N \leq n + 1$. But here they are defined for all $N \in \mathbb{N}$. In fact, there is an explicit formula for the Taylor coefficients of a Poincaré metric of \bar{g} [Juhl 2013]. These are determined by h_r , i.e., by h . Thus

the right-hand side is well-defined for all $N \in \mathbb{N}$. For even n , the left-hand side of the second identity is defined for $2N \leq n$. The right-hand side uses derivatives of a Poincaré metric for \bar{g} which are determined by h .

For even N , the identities (5-1) are the simplest respective special cases of the systems

$$(5-2) \quad D_N^{\text{res}}\left(h; -\frac{n}{2} + N - k\right) = P_{2k}(h) D_{N-2k}^{\text{res}}\left(h; -\frac{n}{2} + N - k\right), \quad 2 \leq 2k \leq N$$

and

$$(5-3) \quad D_N^{\text{res}}\left(h; -\frac{n+1}{2} + k\right) = D_{N-2k}^{\text{res}}\left(h; -\frac{n+1}{2} - k\right) P_{2k}(\bar{g}), \quad 2 \leq 2k \leq N$$

of factorization identities [Juhl 2013, Theorems 3.1–3.2]. They play an important role in connection with the description of recursive structures among GJMS operators and Q -curvatures. Here we shall give an *independent proof* of the second system. The arguments will also prove their counterparts for odd-order residue families. Moreover, we derive the first system from its special case $k = N$, i.e., from the first identity in (5-1). The new proofs completely differ from earlier arguments.

We start with the proof of system (5-3). The proof rests on Corollary 4.2 and two basic facts. The first of these is also of independent interest. It will be used in Section 6F.

Theorem 5.1. *Let $N \in \mathbb{N}$ so that $2N \leq n$ if n is even. Set $m = \frac{n+1}{2}$. Then*

$$S_N(g_+; m - 1) = r^N P_{2N}(\bar{g})$$

up to an error term in $O(r^\infty)$ for odd n and $o(r^{n-N})$ for even n . Moreover, the equality is true without an error term if g_+ is Einstein.

Proof. We first consider the case g_+ Einstein. The identity (2-4) shows that the $2N$ -th order GJMS operator of g_+ is given by the product

$$(5-4) \quad P_{2N}(g_+) = \prod_{l=1}^N (\Delta_{g_+} + (m+l-1)(m-l)).$$

The conformal covariance of GJMS operators implies that

$$P_{2N}(\bar{g}) = r^{-m-N} P_{2N}(g_+) r^{m-N}.$$

By the definition (4-1), we have

$$S_N(g_+; m - 1) = S(g_+; m - 1) \circ \cdots \circ S(g_+; m + N - 2).$$

Hence, using (3-7) and (5-4), we obtain

$$\begin{aligned} S_N(g_+; m-1) &= r^{-m} (\Delta_{g_+} + m(m-1)) \cdots (\Delta_{g_+} + (m+N-1)(m-N)) r^{m-N} \\ &= r^{-m} P_{2N}(g_+) r^{m-N} \\ &= r^N P_{2N}(\bar{g}). \end{aligned}$$

This completes the proof for g_+ Einstein. For general Poincaré metrics and odd n , the assertion follows by similar arguments using the generalization

$$(5-5) \quad P_{2N}(g_+) = \prod_{l=1}^N (\Delta_{g_+} + (m+l-1)(m-l)) + O(r^\infty)$$

of (5-4). The conformal covariance of P_{2N} show that

$$S_N(g_+; m-1) = r^N P_{2N}(\bar{g}) + O(r^\infty).$$

Similarly, for even n , the formula

$$(5-6) \quad P_{2N}(g_+) = \prod_{l=1}^N (\Delta_{g_+} + (m+l-1)(m-l)) + o(r^n)$$

(see Remark 3.5 for $N = 1$) and the conformal covariance of P_{2N} show that

$$S_N(g_+; m-1) = r^N P_{2N}(\bar{g}) + o(r^{n-N}).$$

The proof is complete. □

Theorem 5.1 directly implies the following N factorization identities.

Corollary 5.2. *Let $N \in \mathbb{N}$ so that $2N \leq n$ for even n . Let $0 \leq k \leq N-1$. Then*

$$S_N(g_+; m-k-1) = S_k(g_+; m-k-1) r^{N-k} P_{2N-2k}(\bar{g})$$

if g_+ is Einstein. For general Poincaré metrics g_+ , the identity holds true with an error term in $O(r^\infty)$ for odd n and

$$S_k(g_+; m-k-1) o(r^{n-N+k})$$

for even n .

The second basic fact is a generalization of Lemma 3.13 which states that

$$S(g_+; \lambda) r^j = r^j S(g_+; \lambda-j) - j(2\lambda-n-j+2) r^{j-1}$$

for $j \in \mathbb{N}$.

Lemma 5.3. *Let $k, j \in \mathbb{N}$. Then*

$$(5-7) \quad S_k(g_+; \lambda) r^j = \sum_{l=0}^k \binom{k}{l} (-j)_l (2\lambda-n-j+k+1)_l r^{j-l} S_{k-l}(g_+; \lambda-j+l).$$

Proof. We use induction over k . For $k = 1$, we have

$$\begin{aligned} S_1(g_+; \lambda)r^j &= S(g_+; \lambda)r^j = r^j S(g_+; \lambda - j) - j(2\lambda - n - j + 2)r^{j-1} \\ &= \sum_{l=0}^1 \binom{1}{l} (-j)_l (2\lambda - n - j + 2)_l r^{j-l} S_{1-l}(g_+; \lambda - j + l). \end{aligned}$$

Now assume that the assertion holds true for $k - 1$. Then we compute

$$\begin{aligned} S_k(g_+; \lambda)r^j &= S(g_+; \lambda) \circ S_{k-1}(g_+; \lambda + 1)r^j \\ &= S(g_+; \lambda) \sum_{l=0}^{k-1} \binom{k-1}{l} (-j)_l (2\lambda - n - j + k + 2)_l r^{j-l} S_{k-l-1}(g_+; \lambda - j + l + 1) \\ &= \sum_{l=0}^{k-1} \binom{k-1}{l} (-j)_l (2\lambda - n - j + k + 2)_l S(g_+; \lambda) r^{j-l} S_{k-l-1}(g_+; \lambda - j + l + 1). \end{aligned}$$

By Lemma 3.13, the last display equals

$$\begin{aligned} &\sum_{l=0}^{k-1} \binom{k-1}{l} (-j)_l (2\lambda - n - j + k + 2)_l r^{j-l} S(g_+; \lambda - j + l) S_{k-l-1}(g_+; \lambda - j + l + 1) \\ &\quad - \sum_{l=0}^{k-1} \binom{k-1}{l} (-j)_l (2\lambda - n - j + k + 2)_l \\ &\quad \quad \times (j - l) (2\lambda - n - j + l + 2) r^{j-l-1} S_{k-l-1}(g_+; \lambda - j + l + 1). \end{aligned}$$

Shifting the summation index in the second sum and simplification gives

$$\begin{aligned} &\sum_{l=0}^{k-1} \binom{k-1}{l} (-j)_l (2\lambda - n - j + k + 2)_l r^{j-l} S_{k-l}(g_+; \lambda - j + l) \\ &\quad + \sum_{l=1}^k \binom{k-1}{l-1} (-j)_l (2\lambda - n - j + k + 2)_{l-1} (2\lambda - n - j + l + 1) r^{j-l} S_{k-l}(g_+; \lambda - j + l). \end{aligned}$$

Now observe that

$$(2\lambda - n - j + k + 2)_l = (2\lambda - n - j + k + 1)_l + l(2\lambda - n - j + k + 2)_{l-1}$$

and

$$\begin{aligned} (2\lambda - n - j + k + 2)_{l-1} (2\lambda - n - j + l + 1) \\ = (2\lambda - n - j + k + 1)_l - (k - l) (2\lambda - n - j + k + 2)_{l-1} \end{aligned}$$

as well as

$$l \binom{k-1}{l} - (k-l) \binom{k-1}{l-1} = 0 \quad \text{and} \quad \binom{k-1}{l} - \binom{k-1}{l-1} = \binom{k}{l}.$$

Putting things together, we conclude

$$S_k(g_+; \lambda) r^j = \sum_{l=0}^k \binom{k}{l} (-j)_l (2\lambda - n - j + k + 1)_l r^{j-l} S_{k-l}(g_+; \lambda - j + l).$$

The proof is complete. \square

Now we are able to prove the system (5-3).

Theorem 5.4. *Let $N \in \mathbb{N}$ so that $N \leq n + 1$ for even n . Then*

$$(5-8) \quad D_N^{\text{res}}\left(h; -\frac{n+1}{2} + k\right) = D_{N-2k}^{\text{res}}\left(h; -\frac{n+1}{2} - k\right) P_{2k}(\bar{g})$$

for all $0 \leq 2k \leq N$.

Proof. We only discuss the case of even N . The proof in the odd-order case is analogous. By Corollary 4.2, we have

$$D_{2N}^{\text{res}}\left(h; -\frac{n+1}{2} + k\right) = \frac{1}{(-2N)_N (k-2N)_N} \iota^* S_{2N}\left(g_+; \frac{n-1}{2} + k - 2N\right).$$

Now we additionally assume that g_+ is Einstein. Then Corollary 5.2 implies the identity

$$(5-9) \quad S_{2N}\left(g_+; \frac{n-1}{2} + k - 2N\right) = S_{2N-k}\left(g_+; \frac{n-1}{2} + k - 2N\right) r^k P_{2k}(\bar{g})$$

We use Lemma 5.3 to conclude that

$$\begin{aligned} S_{2N-k}\left(g_+; \frac{n-1}{2} + k - 2N\right) r^k &= \sum_{l=0}^{2N-k} \binom{2N-k}{l} (-k)_l (-2N)_l r^{k-l} S_{2N-k-l}\left(g_+; \frac{n-1}{2} + l - 2N\right). \end{aligned}$$

Note that no summand with summation index $l \geq k + 1$ will contribute due to $(-k)_l = 0$. By restriction to $r = 0$, the last display yields

$$(5-10) \quad \begin{aligned} \iota^* S_{2N-k}\left(g_+; \frac{n-1}{2} + k - 2N\right) r^k &= \binom{2N-k}{k} (-k)_k (-2N)_k \iota^* S_{2N-2k}\left(g_+; \frac{n-1}{2} + k - 2N\right). \end{aligned}$$

Finally, by Corollary 4.2, we have

$$D_{2N-2k}^{\text{res}}\left(h; -\frac{n+1}{2}-k\right) = \frac{1}{(-2N+2k)_{N-k}(k-2N)_{N-k}} \iota^* S_{2N-2k}\left(g_+; \frac{n-1}{2}+k-2N\right).$$

Combining these observation, proves

$$D_{2N}^{\text{res}}\left(h; -\frac{n+1}{2}+k\right) = \frac{\binom{2N-k}{k}(-k)_k(-2N)_k(-2N+2k)_{N-k}(k-2N)_{N-k}}{(-2N)_N(k-2N)_N} \times D_{2N-2k}^{\text{res}}\left(h; -\frac{n+1}{2}-k\right) P_{2k}(\bar{g}).$$

Now the combinatorial identity

$$\frac{\binom{2N-k}{k}(-k)_k(-2N)_k(-2N+2k)_{N-k}(k-2N)_{N-k}}{(-2N)_N(k-2N)_N} = 1$$

completes the proof for Einstein g_+ . For general Poincaré metrics g_+ , we have to control the error terms coming from Corollary 5.2. For odd n , error terms obviously do not contribute to (5-10). For even n , the relation (5-9) contains an error term in

$$S_{2N-k}\left(g_+; \frac{n-1}{2}+k-2N\right) o(r^{n-k}).$$

By Lemma 5.3, this contribution is contained in $o(r^{n-2N})$. Hence its composition with ι^* vanishes if $2N \leq n$.⁸ □

Remark 5.5. The proof of the factorizations (5-8) for even N given in [Juhl 2013] assumes that g_+ is Einstein. A closer inspection of this proof shows that it can be refined to establish the assertion in full generality. The refinement rests on the factorizations (5-5) and (5-6) with remainder terms. The point is that the remainder terms do not contribute to the residue calculations in the refinement of that proof. Note also that, along these lines, again only the critical case $2N = n$ (for n even) requires the remainder term $o(r^n)$ in (5-6).

We continue with the discussion of the factorization identities (5-2). Their proof rests on Corollary 4.2 and the identification of the value

$$\iota^* S_{2N}\left(g_+; \frac{n}{2}-N\right) = \Sigma_{2N}\left(h, \frac{n}{2}-N\right)$$

as a tangential operator being a multiple of $P_{2N}(h)\iota^*$. This fact actually will be deduced from the first identity in (5-1). Unfortunately, we do not have an

⁸The arguments show that, for even n and $2N < n$, the weaker estimate $O(r^n)$ in (5-6) suffices. However, the critical case $2N = n$ requires the stronger estimates $o(r^n)$.

independent proof of this identity. The following result contains the relevant details and some further information.

Theorem 5.6. *Let $k \in \mathbb{N}$. The operator $\iota^* S_k(g_+; \frac{n-k}{2})$ defines a tangential differential operator $\mathcal{P}_k : C^\infty(M) \rightarrow C^\infty(M)$, i.e.,*

$$\iota^* S_k\left(g_+; \frac{n-k}{2}\right) = \mathcal{P}_k \iota^*.$$

For $k \in \mathbb{N}$ with $k \leq n$ for even n , the operator \mathcal{P}_k only depends on h and is conformally covariant, i.e.,

$$\mathcal{P}_k(\hat{h}) = e^{-((n+k)/2)\varphi} \circ \mathcal{P}_k(h) \circ e^{((n-k)/2)\varphi}$$

for $\hat{h} = e^{2\varphi}h$. For $k = 2N - 1$, the operator $\mathcal{P}_k(h)$ vanishes identically. For $k = 2N$, the operator $\mathcal{P}_k(h)$ is proportional to the GJMS operator $P_{2N}(h)$ of (M, h) :

$$(5-11) \quad \mathcal{P}_{2N}(h) = ((2N-1)!)^2 P_{2N}(h).$$

Finally, we have

$$(5-12) \quad \iota^* S_{2N}\left(g_+; \frac{n-1}{2} - N\right) = (2N)! \iota^* P_{2N}(\bar{g})$$

and

$$(5-13) \quad \iota^* S_{2N+1}\left(g_+; \frac{n-3}{2} - N\right) = (2N+2)! \iota^* \partial_r P_{2N}(\bar{g}).$$

Proof. We recall that a differential operator $D : C^\infty(M_+) \rightarrow C^\infty(M_+)$ restricts to a tangential operator with respect to M if and only if $D(rf) = rD'(f)$ for all $f \in C^\infty(M_+)$ and some differential operator $D' : C^\infty(M_+) \rightarrow C^\infty(M_+)$. By Lemma 4.6, we have

$$S_k(g_+; \lambda)(rf) = rS_k(g_+; \lambda - 1)f - k(2\lambda - n + k)S_{k-1}(g_+; \lambda - 1)f$$

for all $f \in C^\infty(M_+)$. Hence the Taylor series of $\iota^* S_k(g_+; \frac{n-k}{2})(rf)$ in the variable r has vanishing constant term. It follows that $\iota^* S_k(g_+; \frac{n-k}{2})$ defines a tangential operator, i.e., there is an operator

$$\mathcal{P}_k : C^\infty(M) \rightarrow C^\infty(M)$$

so that

$$\iota^* S_k\left(g_+; \frac{n-k}{2}\right) = \mathcal{P}_k \iota^*.$$

Now assume that $k \in \mathbb{N}$ with $k \leq n$ for even n . The further properties of \mathcal{P}_k follow from the relation between shift operators and residue families (Corollary 4.2). In particular, for these values of k , the operators \mathcal{P}_k are determined by h and the conformal transformation law for \mathcal{P}_k follows from Corollary 4.5. The fact that \mathcal{P}_k

vanishes identically for odd $k = 2N - 1$ is obvious. Indeed, by Corollary 4.2, we have

$$\begin{aligned} \mathcal{P}_{2N-1} &= \iota^* S_{2N-1} \left(g_+; \frac{n+1}{2} - N \right) \\ &= 2(-2N+1)_N (-N+1)_N D_{2N-1}^{\text{res}} \left(h; N - \frac{n+1}{2} \right) = 0 \end{aligned}$$

since $(-N+1)_N = 0$ and $D_{2N-1}^{\text{res}}(h; \lambda)$ is regular in λ . By Corollary 4.2 and the first factorization relation in (5-1), we conclude that

$$\begin{aligned} \mathcal{P}_{2N} \iota^* &= \iota^* S_{2N} \left(g_+; \frac{n}{2} - N \right) \\ &= (-2N)_N \left(-N + \frac{1}{2}\right)_N D_{2N}^{\text{res}} \left(h; N - \frac{n}{2} \right) \\ &= ((2N-1)!!)^2 P_{2N}(h) \iota^*. \end{aligned}$$

The identity (5-12) follows from Corollary 4.2 and the second factorization identity in (5-1). Similarly, the identity (5-13) follows from the factorization identity

$$D_{2N+1}^{\text{res}} \left(h; -\frac{n+1}{2} + N \right) = D_1^{\text{res}} \left(h; -\frac{n+1}{2} - N \right) P_{2N}(\bar{g})$$

for odd-order residue families (Theorem 5.4). By Corollary 4.2, this relation is equivalent to

$$\begin{aligned} \frac{1}{2(-2N-1)_{N+1} (-N-1)_{N+1}} \iota^* S_{2N+1} \left(g_+; -N + \frac{n-3}{2} \right) \\ = \frac{1}{2(N+1)} \iota^* S_1 \left(g_+; -N + \frac{n-3}{2} \right) P_{2N}(\bar{g}). \end{aligned}$$

Simplification proves the claim. The proof is complete. \square

Some comments on the latter results are in order.

Theorem 5.6 overlaps with [Gover and Waldron 2014, Theorems 4.1 and 4.5]. Indeed, assume that g_+ is Einstein. Then, by Proposition 3.2, $S_k(g_+; \lambda)$ can be regarded as a composition of k degenerate Laplacians $I \cdot D$. The fact that $\iota^* S_k(g_+; \frac{n-k}{2})$ is tangential, is a consequence of a basic $\mathfrak{sl}(2)$ -structure for the degenerate Laplacian [Gover and Waldron 2014, Section 3.1]. It holds true for compositions of degenerate Laplacians in a much more general setting. In the present situation, Lemmas 4.6 and 5.3 reflect that structure. In order to relate the tangential operator \mathcal{P}_{2N} to the GJMS operator $P_{2N}(h)$, we used the first relation in (5-1). Note that this relation is a consequence of the basic residue relation (2-23) (derived in [Graham and Zworski 2003] from the ambient metric construction of $P_{2N}(h)$). In [Gover and Waldron 2014], the identification (5-11) also rests on ambient metric arguments.

As a consequence of Theorem 5.6, we obtain a new proof of the first system (5-2) of factorization identities for residue families.

Corollary 5.7. *Let $N \in \mathbb{N}$ with $N \leq n+1$ for even n . Then we have the factorization identities*

$$(5-14) \quad D_N^{\text{res}}\left(h; -\frac{n}{2} + N - k\right) = P_{2k}(h) D_{N-2k}^{\text{res}}\left(h; -\frac{n}{2} + N - k\right)$$

for $0 \leq 2k \leq N$.

Proof. The obvious relation

$$S_N(g_+; \lambda) = S_{2k}(g_+; \lambda) S_{N-2k}(g_+; \lambda + 2k)$$

implies the identity

$$t^* S_N\left(g_+; \frac{n}{2} - k\right) = t^* S_{2k}\left(g_+; \frac{n}{2} - k\right) S_{N-2k}\left(g_+; \frac{n}{2} + 2k\right).$$

By Theorem 5.6, it is equivalent to

$$t^* S_N\left(g_+; \frac{n}{2} - k\right) = ((2k-1)!!)^2 P_{2k}(h) t^* S_{N-2k}\left(g_+; \frac{n}{2} + 2k\right).$$

Now the identity (4-7) shows that, for even N , this relation can be restated as

$$\begin{aligned} & (-2N)_N (-k + \frac{1}{2})_N D_{2N}^{\text{res}}\left(h; -\frac{n}{2} + 2N - k\right) \\ &= ((2k-1)!!)^2 (-2N+2k)_{N-k} (k + \frac{1}{2})_{N-k} P_{2k}(h) D_{2N-2k}^{\text{res}}\left(h; -\frac{n}{2} + 2N - k\right). \end{aligned}$$

Simplification proves the claim for even-order residues families. We omit the analogous proof for odd-order residue families which utilizes the identity (4-8). \square

Since the family $S_N(g_+; \lambda)$ is a polynomial of degree N in λ , the N identities in Corollary 5.2 do *not* suffice to determine $S_N(g_+; \lambda)$ in terms of lower-order $S_k(g_+; \lambda)$ and GJMS operators of \bar{g} . However, that is possible by combining Corollary 5.2 with the following formula for the leading coefficient of that polynomial. We also recall that $w(r) = \sqrt{v(r)}$.

Proposition 5.8. *Let $N \in \mathbb{N}$. Then*

$$(5-15) \quad \frac{1}{N!} \frac{d^N}{d\lambda^N} S_N(g_+; \lambda) = (-2)^N w^{-1} \partial_r^N (w \cdot).$$

Proof. The relations (2-16) and

$$w^{-1} \partial_r (w f) = \frac{1}{2} v^{-1} \partial_r (v) f + \partial_r f, \quad f \in C^\infty(M_+^o)$$

show that we can rewrite the operator $S(g_+; \lambda)$ as

$$S(g_+; \lambda) = r \Delta_{\bar{g}} + (n-1)(\partial_r + v^{-1} \partial_r(v)) - 2\lambda w^{-1} \partial_r (w \cdot).$$

Now the leading coefficient of the polynomial $\lambda \rightarrow S_N(g_+; \lambda)$ coincides with the product of the leading coefficients of the N factors. But these are all given by $-2w^{-1} \partial_r (w \cdot)$. This proves the assertion. \square

For later use, we introduce the notation $\partial_r^w \stackrel{\text{def}}{=} w^{-1} \partial_r (w \cdot)$. In these terms, the right-hand side of (5-15) equals $(-2)^N (\partial_r^w)^N$. In Section 6F, we shall discuss further consequences of Corollary 5.2 and Proposition 5.8.

Finally, we combine Proposition 5.8 with Corollary 4.2 to read off the leading coefficients of residue families. The definition of residue families implies formulas for these coefficients in terms of solution operators $\mathcal{T}_{2j}(h; \lambda)$ and renormalized volume coefficients v_{2j} . The following result shows how these can be simplified.

Corollary 5.9. *Let $N \in \mathbb{N}$ with $2N \leq n$ for even n . The leading coefficient of the even-order residue family $\lambda \mapsto D_{2N}^{\text{res}}(h; \lambda)$ equals*

$$(-1)^N 2^{2N} \frac{N!}{(2N)!} \iota^* \partial_r^{2N} (w \cdot).$$

Similarly, the leading coefficient of the odd-order residue family $\lambda \mapsto D_{2N+1}^{\text{res}}(h; \lambda)$ equals

$$(-1)^N 2^{2N} \frac{N!}{(2N+1)!} \iota^* \partial_r^{2N+1} (w \cdot).$$

Proof. The even-order residue family $D_{2N}^{\text{res}}(h; \lambda)$ is a polynomial of degree N . By Proposition 5.8, Corollary 4.2 and $\iota^* w(r) = 1$, the coefficient of λ^N equals

$$(-2)^{2N} \frac{1}{(-2N)_N} \iota^* \partial_r^{2N} (w \cdot).$$

This proves the assertion. The proof in the odd-order case is analogous. □

5B. Shift operators and solution operators. In the present section, we assume that g_+ is Einstein and that n is odd. By the latter assumption, all Taylor coefficients of h_r are determined by h . The obvious modifications for even n are left to the reader. We relate the solution operator $\mathcal{T}_{2N}(h; \lambda)$ (see Section 2B) to the coefficients in the formal power series

$$(5-16) \quad S(g_+; \lambda) = -(2\lambda - n + 1) \partial_r + r \sum_{k \geq 0} r^k S^{(k)}(h; \lambda)$$

following from the formal power series

$$\Delta_{\bar{g}} + (\lambda - n + 1) J(\bar{g}) = \sum_{k \geq 0} r^k S^{(k)}(h; \lambda).$$

Here we used the identity (3-5).⁹ Only the operator $S^{(0)}(h; \lambda)$ contains two derivatives in r . By (2-19) and [Juhl 2009, Lemma 6.11], the first few coefficients in the

⁹The assumptions guarantee that (3-5) is an identity of formal power series.

expansion (5-16) are given by the operators

$$\begin{aligned}
 S^{(0)}(h; \lambda) f &= \Delta_h f + \partial_r^2 f + (\lambda - n + 1)J(h) f, \\
 S^{(1)}(h; \lambda) f &= -J(h)\partial_r f, \\
 S^{(2)}(h; \lambda) f &= -\delta_h(P(h)\#df) - \frac{1}{2}h(dJ(h), df) + \frac{1}{2}(\lambda - n + 1)|P(h)|^2 f, \\
 S^{(3)}(h; \lambda) f &= -\frac{1}{2}|P(h)|^2 \partial_r f.
 \end{aligned}
 \tag{5-17}$$

We recall that, under the present assumptions, the solution operators $\mathcal{T}_{2N}(h; \lambda)$ are well-defined for all $N \in \mathbb{N}$. In the following, we shall regard functions in $C^\infty(M)$ as functions on M_+ that do not depend on r , i.e., ∂_r annihilates functions in $C^\infty(M)$.

Proposition 5.10. *Let $N \in \mathbb{N}$. Then*

$$(5-18) \quad -2N(2\lambda - n + 2N)\mathcal{T}_{2N}(h; \lambda) = \sum_{k=0}^{N-1} S^{(2N-2k-2)}(h; n - \lambda - 2k - 1)\mathcal{T}_{2k}(h; \lambda)$$

as an identity of operators acting on $C^\infty(M)$. In particular, the $2N$ -th GJMS operator on (M^n, h) is given by

$$\begin{aligned}
 (5-19) \quad P_{2N}(h) &= -2^{2N-2}((N-1)!)^2 \sum_{k=0}^{N-1} S^{(2N-2k-2)}\left(h; \frac{n}{2} + N - 2k - 1\right)\mathcal{T}_{2k}\left(h; \frac{n}{2} - N\right).
 \end{aligned}$$

Proof. In (3-9), we replace the eigenfunction u by its asymptotic expansion

$$\sum_{j \geq 0} r^{\nu+2j} a_{2j}(h; \nu) + \sum_{j \geq 0} r^{n-\nu+2j} b_{2j}(h; \nu)$$

(see (2-20)). In order to simplify the following equations, we shall suppress the second sum. Then Theorem 3.7 implies

$$\begin{aligned}
 S(g_+; \lambda) \left(\sum_{j \geq 0} r^{\lambda+\nu-n+2j+1} a_{2j}(h; \nu) \right) \\
 = (\lambda + \nu - n + 1)(\nu - \lambda - 1) \left(\sum_{j \geq 0} r^{\lambda+\nu-n+2j} a_{2j}(h; \lambda) \right).
 \end{aligned}$$

By Lemma 3.13, this relation is equivalent to

$$\begin{aligned}
 \sum_{j \geq 0} r^{\lambda+\nu-n+2j+1} S(g_+; n - \nu - 2j - 1) a_{2j}(h; \nu) \\
 = - \sum_{j \geq 0} 2j(2\nu - n + 2j) r^{\lambda+\nu-n+2j} a_{2j}(h; \nu).
 \end{aligned}$$

We rewrite this identity in terms of the expansion (5-16) and compare coefficients of powers of r . This gives

$$S^{(2j-2)}(h; n-v-1)a_0(h; v) + \dots + S^{(0)}(h; n-v-2j+1)a_{2j-2}(h; v) = -2j(2v-n+2j)a_{2j}(h; v)$$

for $j \geq 1$. Using $a_{2k}(h; v) = \mathcal{T}_{2k}(h; v)a_0(h; v)$, we obtain

$$-2j(2v-n+2j)\mathcal{T}_{2j}(h; v) = \sum_{k=0}^{j-1} S^{(2j-2k-2)}(h; n-v-2k-1)\mathcal{T}_{2k}(h; v).$$

Note that $\mathcal{T}_{2j}(h; v)$ has a simple pole at $v = \frac{n}{2} - j$ with residue given by $P_{2j}(h)$ (see (2-23)). Thus the last display implies

$$P_{2j}(h) = -2^{2j-2}((j-1)!)^2 \sum_{k=0}^{j-1} S^{(2j-2k-2)}\left(h; \frac{n}{2} + j - 2k - 1\right)\mathcal{T}_{2k}\left(h; \frac{n}{2} - j\right).$$

The proof is complete. □

Proposition 5.10 is a compressed version of the usual algorithm for the calculation of the solution operators. We shall illustrate it by low-order examples in Section 6.

5C. Holographic formulas for Q -curvatures. In the present section, we assume that n is even. We shall discuss new holographic formulas for the Q -curvatures $Q_{2N}(h)$ for $2N \leq n$.

We start with a simple proof of a result which is also of independent interest (see [Baum and Juhl 2010, Theorem 1.6.6]). It has been useful in connection with a discussion of the recursive structure of Q -curvatures [Juhl 2014].

Proposition 5.11. *Assume that n is even and let $N \in \mathbb{N}$ with $2N \leq n$. Then*

$$D_{2N}^{\text{res}}(h; 0)(1) = 0.$$

Proof. Corollary 4.2 implies that

$$D_{2N}^{\text{res}}(h; 0)(1) = \frac{1}{(-2N)_N \left(\frac{n+1}{2} - 2N\right)_N} \Sigma_{2N}(h; n - 2N)(1).$$

By definition, we have

$$S_{2N}(g_+; n - 2N) = S(g_+; n - 2N) \circ \dots \circ S(g_+; n - 1).$$

But (3-1) shows that

$$(5-20) \quad S(g_+; n - 1)(1) = 0.$$

Hence $\Sigma_{2N}(h; n - 2N)(1) = 0$. This completes the proof. □

The polynomial $\lambda \mapsto D_{2N}^{\text{res}}(h; \lambda)(1)$ is called the Q -curvature polynomial.

The Q -curvature polynomial $D_{2N}^{\text{res}}(h; \lambda)(1)$ also vanishes in odd dimensions n . The above arguments prove this fact if $(\frac{n+1}{2} - 2N)_N \neq 0$.¹⁰

In the critical case $2N = n$, Proposition 5.11 states that $D_n^{\text{res}}(h; 0)(1) = 0$. Of course, this result also follows from $D_n^{\text{res}}(h; 0) = P_n(h)\iota^*$ (see (5-1)) and $P_n(h)(1) = 0$.

We continue with the discussion of the critical Q -curvature $Q_n(h)$ and recall the holographic formula [Juhl 2009, Theorem 6.6.1]

$$(5-21) \quad Q_n(h) = -(-1)^{n/2} \dot{D}_n^{\text{res}}(h; 0)(1).$$

The following result is a consequence of this identity.

Theorem 5.12 (holographic formula for critical Q -curvature). *Let n be even. Then*

$$(5-22) \quad Q_n(h) = (-1)^n / 2 D_{n-1}^{\text{res}}(h; -1) \partial_r(\log v)$$

or, equivalently,

$$(5-23) \quad Q_n(h) = c_n \Sigma_{n-1}(h; 0) \partial_r(\log v)$$

with $c_n = (-1)^{n/2} 2^{n-2} (\Gamma(\frac{n}{2}) / \Gamma(n))^2$.

We recall that the composition $\Sigma_{n-1}(h; 0) = \iota^* S_{n-1}(g_+; 0)$ only depends on h .

Proof. The critical special case $2N = n$ of the first factorization identity in Corollary 4.3 reads

$$D_n^{\text{res}}(h; \lambda) = D_{n-1}^{\text{res}}(h; \lambda - 1) S(g_+; \lambda + n - 1).$$

Now we differentiate this relation at $\lambda = 0$ and use the vanishing result (5-20). We obtain

$$\dot{D}_n^{\text{res}}(h; 0)(1) = D_{n-1}^{\text{res}}(h; -1) \dot{S}(g_+; n - 1)(1)$$

But Proposition 5.8 shows that $\dot{S}(g_+; n - 1) = -2w^{-1} \partial_r(w \cdot)$. Hence

$$\dot{S}(g_+; n - 1)(1) = -2w^{-1} \partial_r(w) = -2\partial_r(\log w) = -\partial_r(\log v).$$

The relation (5-22) follows by combining these facts with the holographic formula (5-21). The second relation (5-23) follows by combining the first relation with Corollary 4.2. □

Remark 5.13. Formula (5-23) should be compared with the special case

$$(5-24) \quad ((n-1)!!)^2 Q_n(h) = \iota^*(I \cdot D)_{n-1}[-n+1] \circ (I \cdot D)_L[1] \log(1)$$

¹⁰In the remaining cases, the polynomial $\Sigma_{2N}(h; \lambda)(1)$ has a double zero at $\lambda = n - 2N$.

of [Gover and Waldron 2014, Theorem 4.7], where we set $(I \cdot D)[\lambda] = (I \cdot D)[\bar{g}, r; \lambda]$ and likewise for $(I \cdot D)_L$. Here the factor $(I \cdot D)_L$ is defined to act on the log density $\log(\mu)$ (μ any positive smooth function) according to

$$(I \cdot D)_L[g, \sigma; \omega] \log(\mu) \stackrel{\text{def}}{=} [-\sigma \Delta_g + (n-1)g(d\sigma, d\cdot)] \log(\mu) - \frac{\omega}{n+1} [(n-1)\Delta_g(\sigma) + 2n\sigma J(g)]$$

(see [Gover and Waldron 2014, Section 2]). For $g = \bar{g}$, $\sigma = r$ and $\mu = 1$, we find

$$(5-25) \quad \begin{aligned} (I \cdot D)_L[\bar{g}, r; \omega] \log(1) &= -\frac{\omega}{n+1} [(n-1)\Delta_{\bar{g}}(r) + 2nrJ(\bar{g})] \\ &= -\frac{\omega}{n+1} \left[\frac{n-1}{2} \text{tr}(h_r^{-1} \dot{h}_r) - n \text{tr}(h_r^{-1} \dot{h}_r) \right] \\ &= \frac{\omega}{2} \text{tr}(h_r^{-1} \dot{h}_r) = \omega \partial_r(\log v). \end{aligned}$$

Hence the formula (5-24) reduces to (5-23), up to a sign due to conventions. Now the transformation law $\mu^n Q_n(\mu^2 h) = P_n(h) \log(\mu) + Q_n(h)$ and (5-11) imply

$$\begin{aligned} &((n-1)!!)^2 \mu^n Q_n(\mu^2 h) \\ &= t^*(I \cdot D)_{n-1}[-n+1] \circ (I \cdot D)[0] \log(\mu) + t^*(I \cdot D)_{n-1}[-n+1] \circ (I \cdot D)_L[1] \log(1). \end{aligned}$$

But the relation

$$(I \cdot D)_L[1]a - (I \cdot D)_L[1]b = (I \cdot D)[0](a - b)$$

yields

$$(I \cdot D)[0] \log(\mu) + (I \cdot D)_L[1] \log(1) = (I \cdot D)_L[1] \log(\mu).$$

Hence

$$((n-1)!!)^2 \mu^n Q_n(\mu^2 h) = t^*(I \cdot D)_{n-1}[-n+1] (I \cdot D)_L[1] \log(\mu).$$

This proves [Gover and Waldron 2014, Theorem 4.7].¹¹

The formula (5-23) resembles the well-known beautiful formula

$$Q_n(h) = (-1)^{n/2-1} \Delta^{n/2}(\log t)|_{\rho=0, t=1}$$

of Fefferman and Hirachi [2003]. Here Δ denotes the Laplacian of the ambient metric in normal form relative to h , and t is the homogeneous coordinate on the ambient space $\mathbb{R}^+ \times M \times (-\varepsilon, \varepsilon)$ with coordinates (t, x, ρ) .

Next, we establish a generalization of Theorem 5.12 to all subcritical Q -curvatures.

¹¹Strictly speaking, μ on the right-hand side is a function on M_+ and the formula for Q_n does not depend on the restriction of μ to the hypersurface M .

Theorem 5.14 (holographic formula for subcritical Q -curvatures). *Let n be even and assume that $2N < n$. Then*

$$(5-26) \quad Q_{2N}(h) = c_{2N} \Sigma_{2N-1} \left(h; \frac{n}{2} - N \right) \partial_r(\log v),$$

where $c_{2N} = (-1)^N 2^{2N-2} (\Gamma(N) / \Gamma(2N))^2$. Equivalently,

$$(5-27) \quad Q_{2N}(h) = (-1)^N D_{2N-1}^{\text{res}} \left(h; -\frac{n}{2} + N - 1 \right) \partial_r(\log v).$$

Proof. On the one hand, we have

$$D_{2N}^{\text{res}} \left(h; -\frac{n}{2} + N \right) (1) = P_{2N}(h)(1) = (-1)^N \left(\frac{n}{2} - N \right) Q_{2N}(h)$$

using (2-5) and (5-1). On the other hand, Corollary 4.2 implies

$$D_{2N}^{\text{res}} \left(h; -\frac{n}{2} + N \right) (1) = \frac{1}{(-2N)_N \left(\frac{1}{2} - N \right)_N} \Sigma_{2N} \left(h; \frac{n}{2} - N \right) (1).$$

But

$$\Sigma_{2N} \left(h; \frac{n}{2} - N \right) = \Sigma_{2N-1} \left(h; \frac{n}{2} - N \right) S \left(g_+; \frac{n}{2} + N - 1 \right)$$

and

$$S \left(g_+; \frac{n}{2} + N - 1 \right) (1) = \left(\frac{n}{2} - N \right) \partial_r(\log v).$$

By comparing both expressions for $D_{2N}^{\text{res}} \left(h; -\frac{n}{2} + N \right) (1)$, we obtain

$$Q_{2N}(h) = (-1)^N \frac{1}{(-2N)_N \left(\frac{1}{2} - N \right)_N} \Sigma_{2N-1} \left(h; \frac{n}{2} - N \right) \partial_r(\log v).$$

Now simplification proves the first assertion. The second follows from this result using the second relation in Corollary 4.2. \square

Theorem 5.12 obviously follows from Theorem 5.14 by analytic continuation in the dimension n . However, the above proof avoids this argument.

Finally, we prove that the formulas (5-23) and (5-27) are equivalent to the well-known holographic formulas for Q -curvatures proved in [Graham and Juhl 2007; Juhl 2011]. In fact, the definitions imply that (5-27) is equivalent to

$$\begin{aligned} (-1)^N Q_{2N} &= 2^{2N-2} (N-1)!^2 \\ &\times \left(\sum_{j=0}^{2N-1} \mathcal{T}_j^* \left(\frac{n}{2} - N \right) v_0 + \cdots + \mathcal{T}_0^* \left(\frac{n}{2} - N \right) v_j \right) \frac{1}{(2N-1-j)!} t^* \partial_r^{2N-1-j} \left(\frac{\dot{v}}{v} \right). \end{aligned}$$

In the latter sum, the operator $\mathcal{T}_{2k}^* \left(\frac{n}{2} - N \right)$ acts on

$$\left(v_0 \frac{1}{(2N-1-2k)!} t^* \partial_r^{2N-1-2k} + \cdots + v_{2N-2k-2} t^* \partial_r \right) \left(\frac{\dot{v}}{v} \right).$$

But this sum equals the $(2N - 1 - 2k)$ -th Taylor coefficient of $v(\dot{v}/v) = \dot{v}$, i.e., equals $(2N - 2k)v_{2N-2k}$. Hence the above formula simplifies to

$$(-1)^N Q_{2N} = 2^{2N-2}(N - 1)!^2 \sum_{k=0}^{N-1} (2N - 2k) \mathcal{T}_{2k}^*\left(\frac{n}{2} - N\right) (v_{2N-2k}).$$

This formula is equivalent to [Juhl 2011, Theorem 1.1]. The same arguments also apply in the critical case. This completes the proof.

In summary, the above discussion provides reformulations and easy new proofs of well-known holographic formulas for Q -curvatures (in even dimension). The key arguments here are the second identity in Corollary 4.2 and/or the first identity in Corollary 4.3.

6. A panorama of examples

In the present section, we illustrate the main results (Theorems 4.1, 5.1, 5.6, 5.12 and Proposition 5.10) by low-order examples. Furthermore, we discuss certain remarkable expansion of the families $S_N(g_+; \lambda)$ with respect to the parameter r .

We shall often simplify notation by omitting the obvious metrics. In particular, we shall write Δ for Δ_h , J for $J(h)$, P for $P(h)$, $S(\lambda)$ for $S(g_+; \lambda)$ etc.

We also recall the expansion [Juhl 2009, Section 6.11]

$$(6-1) \quad \Delta_{\bar{g}} f = [\Delta f + \partial_r^2 f] - r J \partial_r f + r^2 [-\delta(P\#df) - \frac{1}{2}(dJ, df)] + O(r^3).$$

By $\dot{v}/v = 2rv_2 + O(r^3)$ with $v_2 = -\frac{1}{2}J(h)$, we have

$$S(g_+; \lambda) = -(2\lambda - n + 1)\partial_r + r[\Delta_h + \partial_r^2 + (\lambda - n + 1)J(h)] + O(r^2).$$

6A. Theorem 4.1 for $N \leq 3$. The first-order family $\delta_1(\lambda)$ is given by [Juhl 2009, Section 6.2]

$$\delta_1(\lambda) = \iota^* \partial_r.$$

On the other hand, we have

$$(6-2) \quad \Sigma_1(\lambda) = \iota^* S_1(\lambda) = -(2\lambda - n + 1)\iota^* \partial_r = -(2\lambda - n + 1)\delta_1(\lambda).$$

This confirms Theorem 4.1 in the case $N = 1$.

The second-order family $\delta_2(\lambda)$ is given by [Juhl 2009, Section 6.7]

$$\delta_2(\lambda) = \frac{1}{2}\iota^* \partial_r^2 + \frac{1}{2(n-2-2\lambda)}(\Delta + (\lambda - n + 2)J)\iota^*.$$

On the other hand, by definition, we have

$$\begin{aligned} \Sigma_2(\lambda) &= \iota^* S_2(\lambda) = \iota^* S(\lambda)S(\lambda + 1) \\ &= -(2\lambda - n + 1)\iota^* \partial_r [r\Delta_{\bar{g}} + (-2\lambda + n - 3)\partial_r + (\lambda - n + 2)rJ]. \end{aligned}$$

By (6-1), this formula simplifies to

$$\begin{aligned}
 (6-3) \quad \Sigma_2(\lambda) &= -(2\lambda-n+1)[\Delta\iota^* - (2\lambda-n+2)\iota^*\partial_r^2 + (\lambda-n+2)J\iota^*] \\
 &= (-2)_2(2\lambda-n+1)_2\left[\frac{1}{2}\iota^*\partial_r^2 - \frac{1}{2(2\lambda-n+2)}(\Delta\iota^* + (\lambda-n+2)J\iota^*)\right] \\
 &= (-2)_2(2\lambda-n+1)_2\delta_2(\lambda).
 \end{aligned}$$

This identity confirms Theorem 4.1 for $N = 2$.

Note that, for $n = 3$, the latter formula gives

$$\Sigma_2(\lambda)(1) = -(-2)_2(2\lambda-2)_2\frac{(\lambda-1)}{2(2\lambda-1)}J = -2(\lambda-1)^2J.$$

This confirms the double zero mentioned after Proposition 5.11. In this case, both factors in the definition of $\Sigma_2(\lambda)$ contribute a zero.

The third-order family $\delta_3(\lambda)$ is given by [Juhl 2009, Section 6.8]

$$\delta_3(\lambda) = \frac{1}{6}\iota^*\partial_r^3 + \frac{1}{2(n-2-2\lambda)}(\Delta + (\lambda-n+2)J)\iota^*\partial_r.$$

On the other hand, by definition, we have

$$\begin{aligned}
 \Sigma_3(\lambda) &= \iota^*S_3(\lambda) = \iota^*S(\lambda)S(\lambda+1)S(\lambda+2) \\
 &= -(2\lambda-n+1)\iota^*\partial_r[r\Delta_{\bar{g}} - (2\lambda-n+3)\partial_r + (\lambda-n+2)rJ] \\
 &\quad \circ [r\Delta_{\bar{g}} - (\lambda-n+5)\partial_r + (\lambda-n+3)rJ].
 \end{aligned}$$

By (6-1), the last display simplifies to

$$\begin{aligned}
 (6-4) \quad &-(2\lambda-n+1)\left[(-\lambda-n+5) + 2 - 2(2\lambda-n+3)\right]\Delta\iota^*\partial_r \\
 &\quad + (2 - (2\lambda-n+5) - 2(2\lambda-n+3) + (2\lambda-n+3)(2\lambda-n+5))\iota^*\partial_r^3 \\
 &\quad + (-2 + 2(\lambda-n+3) + 2(2\lambda-n+3) - 2(2\lambda-n+3)(\lambda-n+3) \\
 &\quad \quad - (\lambda-n+2)(2\lambda-n+5))J\iota^*\partial_r] \\
 &= (-3)_3(2\lambda-n+1)_3\left[\frac{1}{6}\iota^*\partial_r^3 - \frac{1}{2(2\lambda-n+2)}[\Delta\iota^*\partial_r + (\lambda-n+2)J\iota^*\partial_r]\right] \\
 &= (-3)_3(2\lambda-n+1)_3\delta_3(\lambda).
 \end{aligned}$$

This identity confirms Theorem 4.1 for $N = 3$.

6B. Theorem 5.1 for $N = 1$. We have shown that the family $S_N(g_+; \lambda)$ contains the GJMS operator $P_{2N}(\bar{g})$ if g_+ is an Einstein metric on M_+^o . Now we give a direct proof for the first example. We recall that $m = \frac{n+1}{2}$. By formula (3-5), we compute

$$S_1(g_+; m-1) = S(g_+; m-1) = r\left(\Delta_{\bar{g}} - \frac{n-1}{2}J(\bar{g})\right)$$

if g_+ is Einstein. By (2-2), the right-hand side coincides with $rP_2(\bar{g})$. For general Poincaré metrics, the identity holds with error terms (see Remark 3.5).

6C. Theorem 5.6 for $N = 1$. It shows that the family $S_{2N}(g_+; \lambda)$ induces a tangential operator which is proportional to the GJMS operators for (M, h) . Now we compute

$$\begin{aligned} \iota^* S_2\left(\frac{n}{2} - 1\right) &= \iota^* S\left(\frac{n}{2} - 1\right) S\left(\frac{n}{2}\right) \\ &= \iota^* \partial_r \left[-\partial_r + r \left[\Delta + \partial_r^2 - \left(\frac{n}{2} - 1\right) J \right] \right] \\ &= \Delta \iota^* - \left(\frac{n-2}{2}\right) J \iota^*, \end{aligned}$$

which coincides with $P_2 \iota^*$ (see (2-2)).

Note also that

$$\iota^* S_2\left(g_+; \frac{n-1}{2} - 1\right) = 2\iota^* \partial_r \left[r \Delta_{\bar{g}} - \left(\frac{n-1}{2}\right) r J \right] = 2\iota^* P_2(\bar{g})$$

and

$$\iota^* S_3\left(g_+; \frac{n-3}{2} - 1\right) = 4\iota^* \partial_r P_2(\bar{g})$$

by (6-3) and (6-4) (for Einstein g_+ see also (6-8) and (6-11)). The latter two identities are special cases of (5-12) and (5-13).

6D. Proposition 5.10 for $N \leq 2$. We recall that [Juhl 2009, Section 6.7]

$$(6-5) \quad \mathcal{T}_2(\lambda) = \frac{1}{2(n-2\lambda-2)} [\Delta - \lambda J].$$

By (5-16), we have

$$S^{(0)}(n-\lambda-1) = \Delta + \partial_r^2 - \lambda J.$$

Hence we obtain

$$-2(2\lambda-n+2)\mathcal{T}_2(\lambda) = S^{(0)}(n-\lambda-1)\mathcal{T}_0(\lambda)$$

as an identity of operators on $C^\infty(M)$. This coincides with (5-18) for $N = 1$.

We recall that [Juhl 2009, Theorem 6.9.4]

$$\begin{aligned} \mathcal{T}_4(\lambda) &= \frac{1}{4(n-4-2\lambda)} \left[\frac{1}{2(n-2-2\lambda)} [\Delta - (\lambda+2)J][\Delta - \lambda J] - \frac{1}{2}\lambda|P|^2 - \delta(P\#d) - \frac{1}{2}(dJ, d) \right]. \end{aligned}$$

Now, using (5-17) and (6-5), we compute

$$\begin{aligned} S^{(2)}(n-\lambda-1)\mathcal{T}_0(\lambda) + S^{(0)}(n-\lambda-3)\mathcal{T}_2(\lambda) &= -\delta(P\#d) - \frac{1}{2}(dJ, d) - \frac{1}{2}|P|^2 + \frac{1}{2(n-2\lambda-2)} [\Delta - (\lambda+2)J][\Delta - \lambda J]. \end{aligned}$$

Hence

$$-4(2\lambda - n + 4)\mathcal{T}_4(\lambda) = S^{(2)}(n - \lambda - 1)\mathcal{T}_0(\lambda) + S^{(0)}(n - \lambda - 3)\mathcal{T}_2(\lambda).$$

This coincides with (5-18) for $N = 2$.

6E. Theorem 5.12 for $n \leq 4$ and Theorem 5.14 for $N \leq 2$. We first confirm (5-23) for $n = 2$ and $n = 4$. As a preparation, we note that

$$\partial_r(\log v) = \dot{v}(r)/v(r) = 2rv_2 + r^3(4v_4 - 2v_2^2) + \dots$$

For $n = 2$, (5-23) claims that

$$Q_2 = -\Sigma_1(0)\partial_r(\log v).$$

Now, using (6-2), this identity simplifies to

$$Q_2 = -2v_2.$$

For $n = 4$, (5-23) claims that

$$Q_4 = \frac{4}{36}\Sigma_3(0)\partial_r(\log v).$$

Using (6-4), this formula reads

$$Q_4 = 4\left[\frac{1}{6}t^*\partial_r^3 + \frac{1}{4}(\Delta t^*\partial_r - 2Jt^*\partial_r)\right]\partial_r(\log v).$$

The latter expression simplifies to

$$16v_4 - 8v_2^2 + 2(\Delta - 2J)v_2.$$

Now the standard formulas

$$v_2 = -\frac{1}{2}J \quad \text{and} \quad v_4 = \frac{1}{8}(J^2 - |P|^2)$$

confirm that the above formulas are equivalent to the well-known expressions

$$Q_2 = J \quad \text{and} \quad Q_4 = 2J^2 - 2|P|^2 - \Delta J.$$

Similar calculations confirm (5-26) for $N = 1$ and $N = 2$. In fact, by (6-2), the assertion

$$Q_2 = -\Sigma_1\left(\frac{n}{2} - 1\right)\partial_r(\log v)$$

is equivalent $Q_2 = -2v_2$. Similarly, by (6-4), the assertion

$$Q_4 = \frac{4}{36}\Sigma_3\left(\frac{n}{2} - 2\right)\partial_r(\log v)$$

is equivalent to

$$Q_4 = 16v_4 - 8v_2^2 + 2\Delta v_2 - nJv_2 = \frac{n}{2}J^2 - 2|P|^2 - \Delta J.$$

These results reproduce the formulas in (2-6).

6F. Some interesting expansions of $S_N(g_+; \lambda)$. We finish this section with a discussion of some interesting expansions of the families $S_N(g_+; \lambda)$. Here we restrict to the low-order examples $N \leq 3$. In order to simplify the presentation, we shall also assume that g_+ is Einstein. The expansions in question describe the families $S_N(g_+; \lambda)$ as polynomials in r with coefficients that are polynomials in the GJMS operators $P_{2k}(\bar{g})$ for $k \leq N$ and the operators $\partial_r^w = w^{-1} \partial_r(w \cdot)$. The existence of such expansions follows from Theorem 5.1, Corollary 5.2 and Proposition 5.8. Of particular interest will be the coefficient of r^N in the expansion of $S_N(h; \lambda)$. This coefficient is a polynomial in λ the leading coefficient of which is related to the second-order operators $\mathcal{M}_2(\bar{g})$, $\mathcal{M}_4(\bar{g})$ and $\mathcal{M}_6(\bar{g})$ (see (6-9), (6-12)) associated to (M_+, \bar{g}) . We recall that these operators are the first coefficients in the r -expansion of the holographic Laplacian introduced in [Juhl 2013]. These experiments provide evidence for a general property of these expansions which will be formulated at the end.

As in Theorem 5.1, we shall use the notation $m = \frac{n+1}{2}$.

Example 6.1. The second-order family $S_1(g_+; \lambda)$ is linear in the variable λ and satisfies two identities:

$$\begin{aligned} S_1(g_+; m-1) &= r P_2(\bar{g}), \\ \frac{d}{d\lambda} S_1(g_+; \lambda) &= -2 \partial_r^w. \end{aligned}$$

These identities imply the representation

$$(6-6) \quad S_1(g_+; \lambda) = -2(\lambda - m + 1) \partial_r^w + r P_2(\bar{g}).$$

In particular, the coefficient of r is given by $\mathcal{M}_2(\bar{g}) = P_2(\bar{g})$.

Example 6.2. The fourth-order family $S_2(g_+; \lambda)$ is quadratic in the variable λ and satisfies three identities:

$$\begin{aligned} S_2(g_+; m-1) &= r^2 P_4(\bar{g}), \\ S_2(g_+; m-2) &= S_1(g_+; m-2) r P_2(\bar{g}), \\ \frac{1}{2!} \frac{d^2}{d\lambda^2} S_2(g_+; \lambda) &= 4(\partial_r^w)^2. \end{aligned}$$

Hence $S_2(g_+; \lambda)$ can be written in the form

$$(6-7) \quad S_2(g_+; \lambda) = 4(\lambda - m + 1)_2 (\partial_r^w)^2 - (\lambda - m + 1) S_1(g_+; m-2) r P_2(\bar{g}) + (\lambda - m + 2) r^2 P_4(\bar{g}).$$

Now applying Lemma 4.6 to the middle summand gives

$$S_1(g_+; m-2) r P_2(\bar{g}) = r S_1(g_+; m-3) P_2(\bar{g}) + 2 P_2(\bar{g}).$$

Thus, by combination with (6-6), we can rewrite (6-7) as

$$(6-8) \quad S_2(g_+; \lambda) = 4(\lambda - m + 1)_2 (\partial_r^w)^2 - 2(\lambda - m + 1) P_2(\bar{g}) - 4(\lambda - m + 1) r \partial_r^w P_2(\bar{g}) + r^2 [(\lambda - m + 2) P_4(\bar{g}) - (\lambda - m + 1) P_2(\bar{g})^2].$$

In particular, the coefficient of r^2 is a linear polynomial in λ the leading coefficient of which is given by the operator

$$(6-9) \quad \mathcal{M}_4(\bar{g}) = P_4(\bar{g}) - P_2(\bar{g})^2.$$

Example 6.3. The sixth-order family $S_3(g_+; \lambda)$ is cubic in the variable λ and satisfies four identities:

$$\begin{aligned} S_3(g_+; m-1) &= r^3 P_6(\bar{g}), \\ S_3(g_+; m-2) &= S_1(g_+; m-2)r^2 P_4(\bar{g}), \\ S_3(g_+; m-3) &= S_2(g_+; m-3)r P_2(\bar{g}), \\ \frac{1}{3!} \frac{d^3}{d\lambda^3} S_3(g_+; \lambda) &= -8(\partial_r^w)^3. \end{aligned}$$

Hence $S_3(g_+; \lambda)$ can be represented in the form

$$(6-10) \quad \begin{aligned} S_3(g_+; \lambda) &= -8(\lambda-m+1)_3(\partial_r^w)^3 + \frac{1}{2}(\lambda-m+1)_2 S_2(g_+; m-3)r P_2(\bar{g}) \\ &\quad - (\lambda-m+1)(\lambda-m+3)S_1(g_+; m-2)r^2 P_4(\bar{g}) \\ &\quad + \frac{1}{2}(\lambda-m+2)_2 r^3 P_6(\bar{g}). \end{aligned}$$

Applying Lemma 4.6 to the middle two summands yields the relations

$$\begin{aligned} S_2(g_+; m-3)(r P_2(\bar{g})) &= r S_2(g_+; m-4)P_2(\bar{g}) + 6S_1(g_+; m-3)P_2(\bar{g}), \\ S_1(g_+; m-2)(r^2 P_4(\bar{g})) &= r^2 S_1(g_+; m-4)P_4(\bar{g}) + 6r P_4(\bar{g}). \end{aligned}$$

Hence, by combination with the respective representations (6-6) and (6-8) of $S_1(g_+; \lambda)$ and $S_2(g_+; \lambda)$, we can rewrite $S_3(g_+; \lambda)$ in the form

$$(6-11) \quad \begin{aligned} &(2\lambda-n+1)(2\lambda-n+3)(3\partial_r^w P_2(\bar{g}) - (2\lambda-n+5)(\partial_r^w)^3) \\ &\quad - \frac{3}{2}(2\lambda-n+1)r \left[(2\lambda-n+5)P_4(\bar{g}) - (2\lambda-n+3)P_2(\bar{g})^2 \right. \\ &\quad \quad \quad \left. - 2(2\lambda-n+3)(\partial_r^w)^2 P_2(\bar{g}) \right] \\ &\quad - \frac{3}{2}(2\lambda-n+1)r^2 \left[(2\lambda-n+5)\partial_r^w P_4(\bar{g}) - (2\lambda-n+3)\partial_r^w P_2(\bar{g})^2 \right] \\ &\quad + \frac{1}{8}r^3 \left[-2(2\lambda-n+1)(2\lambda-n+5)P_2(\bar{g})P_4(\bar{g}) + (2\lambda-n+3)(2\lambda-n+5)P_6(\bar{g}) \right. \\ &\quad \quad \left. - 2(2\lambda-n+1)(2\lambda-n+3)P_4(\bar{g})P_2(\bar{g}) + 3(2\lambda-n+1)(2\lambda-n+3)P_2(\bar{g})^3 \right]. \end{aligned}$$

In particular, the coefficient of r^3 is a quadratic polynomial in λ the leading coefficient of which is a constant multiple of

$$(6-12) \quad \mathcal{M}_6(\bar{g}) = P_6(\bar{g}) - 2P_2(\bar{g})P_4(\bar{g}) - 2P_4(\bar{g})P_2(\bar{g}) + 3P_2(\bar{g})^3.$$

The above examples suggest that in the analogous representation of $S_N(g_+; \lambda)$ the coefficient of r^N is a constant multiple of the degree $N - 1$ polynomial

$$(6-13) \quad \sum_{|I|=N} m_I \frac{(\lambda - \frac{n-1}{2})_N}{\lambda - \frac{n-1}{2} + N - I_r} P_{2I}(\bar{g}).$$

Here the sum runs over all partitions $I = (I_1, \dots, I_r)$ of size $|I| = N$ and, for any I , we set $P_{2I} = P_{2I_1} \cdots P_{2I_r}$; for more details on the coefficients m_I we refer to [Juhl 2013]. The expression (6-13) resembles the polynomials in [Juhl 2013, Theorem 4.1] which describes residue families in terms of GJMS operators. Note that the leading coefficient of the polynomial (6-13) is the building block operator

$$\mathcal{M}_{2N}(\bar{g}) = \sum_{|I|=N} m_I P_{2I}(\bar{g})$$

(see Section 7). We will return to that problem in later work.

7. Epilogue

In the present section, we sketch some further developments and indicate some interesting future developments.

We first prove a consequence of the expansions discussed in Section 6F. Then we generalize this result to arbitrary order. We show that the result naturally follows from a basic property of an operator which in [Juhl 2016] was termed the holographic Laplacian. We expect that these generalizations play a similar role in the study of the families $S_N(g_+; \lambda)$ of arbitrary order $N \in \mathbb{N}$.

In order to simplify the presentations as much as possible, we assume that M is an analytic manifold of odd dimension n . For an analytic metric h on M , we let g_+ be a Poincaré metric in normal form relative to h . It satisfies $\text{Ric}(g_+) + ng_+ = 0$ and, for any $N \in \mathbb{N}$, there is a well-defined GJMS operator $P_{2N}(h)$.

We also recall the notation $\partial_r^w = w^{-1} \partial_r(w \cdot)$ and

$$\mathcal{M}_2(h) = P_2(h), \quad \mathcal{M}_4(h) = P_4(h) - P_2(h)^2.$$

As usual, let $\bar{g} = r^2 g_+$. The analogous definitions yield $\mathcal{M}_2(\bar{g})$ and $\mathcal{M}_4(\bar{g})$.

Theorem 7.1. $r \mathcal{M}_4(\bar{g}) = 2[\partial_r^w, \mathcal{M}_2(\bar{g})].$

Proof. Using (6-6), we compute

$$\begin{aligned} S_2(g_+; \lambda) &= S_1(g_+; \lambda) S_1(g_+; \lambda + 1) \\ &= S_1(g_+; \lambda) [-2(\lambda - m + 2) \partial_r^w + r P_2(\bar{g})] \\ &= -2(\lambda - m + 2) S_1(g_+; \lambda) \partial_r^w + S_1(g_+; \lambda) r P_2(\bar{g}). \end{aligned}$$

Now we apply Lemma 3.13 to move the variable r in the last term to the left, i.e.,

$$S_2(g_+; \lambda) = -2(\lambda - m + 2)S_1(g_+; \lambda)\partial_r^w + rS_1(g_+; \lambda)P_2(\bar{g}) - 2(\lambda - m + 1)P_2(\bar{g}).$$

By another application of (6-6), we conclude

$$\begin{aligned} S_2(g_+; \lambda) &= 2(\lambda - m + 2)\left[-2(\lambda - m + 1)(\partial_r^w)^2 + rP_2(\bar{g})\partial_r^w\right] \\ &\quad + r\left[-2(\lambda - m)\partial_r^w P_2(\bar{g}) + P_2(\bar{g})^2 - 2(\lambda - m + 1)P_2(\bar{g})\right] \\ &= 4(\lambda - m + 1)_2(\partial_r^w)^2 - 2(\lambda - m + 1)P_2(\bar{g}) \\ &\quad - 2r\left[(\lambda - m + 2)P_2(\bar{g})\partial_r^w + (\lambda - m)\partial_r^w P_2(\bar{g})\right] + r^2 P_2(\bar{g})^2. \end{aligned}$$

The difference between (6-8) and the last display gives the relation

$$0 = (\lambda - m + 2)\left[r^2 \mathcal{M}_4(\bar{g}) - 2r[\partial_r^w, P_2(\bar{g})]\right].$$

The proof is complete. □

Now we describe an alternative proof of the commutator relation in Theorem 7.1. The proof rests on a basic property of the *holographic Laplacian* $\mathcal{H}(h)(r)$ introduced in [Juhl 2013; 2016]. We recall that this operator is the Schrödinger-type operator

$$(7-1) \quad \mathcal{H}(h)(r) \stackrel{\text{def}}{=} -\delta_h(h_r^{-1}d) + \mathcal{U}(h)(r)$$

with the potential

$$(7-2) \quad \mathcal{U}(h)(r) \stackrel{\text{def}}{=} -w(r)^{-1}(\partial^2/\partial r^2 - (n - 1)r^{-1}\partial/\partial r - \delta(h_r^{-1}d))(w(r)).$$

The operator $\mathcal{H}(h)(r)$ should be viewed as a 1-parameter deformation of the Yamabe operator $\mathcal{H}(h)(0) = P_2(h)$. It is a key fact (see [Fefferman and Graham 2013; Juhl 2013]) that (the Taylor series of) this operator coincides with

$$\mathcal{G}(h)\left(\frac{r^2}{4}\right),$$

where

$$\mathcal{G}(h)(\rho) \stackrel{\text{def}}{=} \sum_{N \geq 1} \mathcal{M}_{2N}(h) \frac{\rho^{N-1}}{(N-1)!^2}$$

is a generating function of the so-called building block operators $\mathcal{M}_{2N}(h)$ of the GJMS operators of the metric h . In fact, any GJMS operator $P_{2N}(h)$ can be written as a linear combination

$$(7-3) \quad P_{2N}(h) = \sum_{|I|=N} n_I \mathcal{M}_{2I}(h), \quad n_I \in \mathbb{Z}$$

of compositions $\mathcal{M}_{2I} = \mathcal{M}_{2I_1} \cdots \mathcal{M}_{2I_r}$ for $I = (I_1, \dots, I_r)$. For the details we refer to [Juhl 2013]. Now we consider the generating function $\mathcal{G}(\bar{g}(r))(\eta)$. It satisfies

the basic relation

$$(7-4) \quad \mathcal{G}(\bar{g}(r))(\eta) = w(r)^{-1} \mathcal{G}(h) \left(\frac{r^2}{4} + \eta \right) w(r) + (\partial_r^w)^2$$

of second-order operators acting on functions in (r, x) . Note that the operator in the first term on the right-hand side only differentiates along M . The identity (7-4) follows by combining the relation between $\mathcal{G}(\bar{g})$ and the holographic Laplacian of \bar{g} with an explicit formula [Juhl 2013, Theorem 7.2] for the Poincaré metric in normal form relative to \bar{g} in terms of the Poincaré metric in normal form relative to h ; for the details we refer to [Juhl 2012]. By expansion into powers of η , the relation (7-4) implies the identities

$$(7-5) \quad \mathcal{M}_{2N}(\bar{g}(r)) = \sum_{k \geq N} w(r)^{-1} \mathcal{M}_{2k}(h) w(r) \frac{(N-1)!}{(k-1)!(k-N)!} \left(\frac{r^2}{4} \right)^{k-N}$$

for $N \geq 2$. In turn, the latter relation yields the following commutator relations.

Theorem 7.2. *Let $N \geq 2$. Then*

$$(7-6) \quad r \mathcal{M}_{2N}(\bar{g}) = 2(N-1) [\partial_r^w, \mathcal{M}_{2N-2}(\bar{g})].$$

Proof. Let $N \geq 3$. We use (7-5) and its relative

$$(7-7) \quad \mathcal{M}_{2N-2}(\bar{g}(r)) = \sum_{k \geq N-1} w^{-1} \mathcal{M}_{2k}(h) w \frac{(N-2)!}{(k-1)!(k-N+1)!} \left(\frac{r^2}{4} \right)^{k-N+1}$$

to calculate

$$\begin{aligned} [\partial_r^w, \mathcal{M}_{2N-2}(\bar{g})] &= \partial_r^w \mathcal{M}_{2N-2}(\bar{g}) - \mathcal{M}_{2N-2}(\bar{g}) \partial_r^w \\ &= w^{-1} \partial_r w \mathcal{M}_{2N-2}(\bar{g}) - \mathcal{M}_{2N-2}(\bar{g}) w^{-1} \partial_r w \\ &= w^{-1} \partial_r \left(\sum_{k \geq N-1} \mathcal{M}_{2k}(h) w \frac{(N-2)!}{(k-1)!(k-N+1)!} \left(\frac{r^2}{4} \right)^{k-1} \cdot \right) \\ &\quad - \sum_{k \geq N-1} w^{-1} \mathcal{M}_{2k}(h) \frac{(N-2)!}{(k-1)!(k-N+1)!} \left(\frac{r^2}{4} \right)^{k-1} \partial_r (w \cdot) \\ &= \frac{r}{2} \sum_{k \geq N-2} w^{-1} \mathcal{M}_{2k}(h) w \frac{(N-2)!}{(k-1)!(k-N)!} \left(\frac{r^2}{4} \right)^{k-2} \partial_r \\ &= \frac{r}{2(N-1)} \mathcal{M}_{2N}(\bar{g}). \end{aligned}$$

This proves the assertion for $N \geq 3$. For $N = 2$, the identity (7-7) is to be replaced by

$$\mathcal{M}_2(\bar{g}) = \sum_{k \geq 1} w^{-1} \mathcal{M}_{2k}(h) w \frac{1}{(k-1)!(k-1)!} \left(\frac{r^2}{4} \right)^{k-1} + (\partial_r^w)^2.$$

Since the additional term $(\partial_r^w)^2$ commutes with ∂_r^w , the above arguments extend to that case. \square

Theorem 7.1 is the special case $N = 2$ of Theorem 7.2.

Example 7.3. Let M be the sphere S^n with the round metric g_{S^n} . Then

$$\bar{g}(r) = dr^2 + (1 - r^2/4)^2 g_{S^n}.$$

But $\mathcal{M}_2(\bar{g}) = P_2(\bar{g})$ and

$$\mathcal{M}_{2N}(\bar{g}) = (N - 1)!N!(1 - r^2/4)^{-N-1} P_2(g_{S^n})$$

for $N \geq 2$. These results can be derived from the identification of the holographic Laplacian $\mathcal{H}(\bar{g})$ with the generating series $\mathcal{G}(\bar{g})$ of the operators $\mathcal{M}_{2N}(\bar{g})$. For a direct proof see [Juhl 2013, Section 11.10]. Moreover, we have

$$\partial_r^w = \partial_r - \frac{n}{4}r(1 - r^2/4)^{-1}.$$

Hence we calculate

$$\begin{aligned} 2(N - 1)[\partial_r^w, \mathcal{M}_{2N-2}(\bar{g})] &= 2(N - 1)(N - 1)!\partial_r((1 - r^2/4)^{-N})P_2(g_{S^n}) \\ &= N!(N - 1)!(1 - r^2/4)^{-N-1}rP_2(g_{S^n}) \\ &= r\mathcal{M}_{2N}(\bar{g}) \end{aligned}$$

for $N \geq 3$. This proves (7-6) for $N \geq 3$. A direct calculation also confirms the case $N = 2$.

In turn, Theorem 7.2 leads to a simple compressed formula for the generating series $\mathcal{G}(\bar{g})$ and thus for the holographic Laplacian of \bar{g} . In order to formulate the result, we introduce the following notation. Let

$$R \circ \text{ad}(\partial_r^w)(\cdot) \stackrel{\text{def}}{=} \frac{1}{r} \circ [\partial_r^w, \cdot].$$

Theorem 7.4. Assume that (M^n, h) is real analytic of odd dimension n . Then

$$(7-8) \quad \mathcal{H}(\bar{g})(\eta) = \mathcal{G}(\bar{g})\left(\frac{\eta^2}{4}\right) = \exp\left(R \circ \text{ad}(\partial_r^w)\frac{\eta^2}{2}\right)(\mathcal{M}_2(\bar{g})).$$

Proof. By a repeated application of the identity (7-6) we obtain

$$\mathcal{M}_{2N}(\bar{g}) = 2^{N-1}(N - 1)!(R \circ \text{ad}(\partial_r^w))^{N-1}(\mathcal{M}_2(\bar{g}))$$

for $N \geq 2$. Using the natural convention $(R \circ \text{ad}(\partial_r^w))^0 = \text{Id}$, the latter identity also holds for $N = 1$. Hence

$$\begin{aligned} \mathcal{H}(\bar{g})(\eta) &= \sum_{N \geq 1} \mathcal{M}_{2N}(\bar{g}) \frac{1}{(N-1)!^2} \left(\frac{\eta^2}{4}\right)^{N-1} \\ &= \sum_{N \geq 1} (R \circ \text{ad}(\partial_r^w))^{N-1} (\mathcal{M}_2(\bar{g})) \frac{1}{(N-1)!} \left(\frac{\eta^2}{2}\right)^{N-1} \\ &= \sum_{N \geq 0} (R \circ \text{ad}(\partial_r^w))^N (\mathcal{M}_2(\bar{g})) \frac{1}{N!} \left(\frac{\eta^2}{2}\right)^N \\ &= \exp\left(R \circ \text{ad}(\partial_r^w) \frac{\eta^2}{2}\right) (\mathcal{M}_2(\bar{g})). \end{aligned}$$

This completes the proof. \square

Theorem 7.4 again clearly shows that $\mathcal{H}(\bar{g})(\eta)$ is a deformation of $\mathcal{M}_2(\bar{g}) = P_2(\bar{g})$.

We finish with brief comments on generalizations of the present theory to differential forms. The theory of differential symmetry breaking operators on functions has a natural extension to differential forms [Fischmann et al. 2016; Kobayashi et al. 2016]. Curved versions of that theory deal with residue families acting on differential forms. Their theory will be developed elsewhere. Residue families on differential forms are expected to have analogous descriptions in terms of compositions of shift operators on differential forms. Results in [Fischmann et al. 2019] confirm that picture in the flat case. Curved analogs of the degenerate Laplacian acting on differential forms were studied in [Gover et al. 2015] in terms of tractor calculus.

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
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