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NORMAL p -COMPLEMENTS

HUNG P. TONG-VIET

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We study the structure of finite groups with a large number of conjugacy classes of p -elements for some prime p . As a consequence, we obtain some new criteria for the existence of normal p -complements in finite groups.

1. Introduction

Let p be a prime. Let G be a finite group and let P be a Sylow p -subgroup of G . Denote by $k(G)$ and $k_p(G)$ the number of conjugacy classes of G and the number of conjugacy classes of p -elements of G , respectively. By Sylow's theorem, we can choose a complete set Γ of representatives for the conjugacy classes of p -elements of G in such a way that $\Gamma \subseteq P$. This yields that $k_p(G) \leq k(P)$. Also $k_p(G) \geq 2$ unless G is a p' -group. Hence if p divides $|G|$, then $2 \leq k_p(G) \leq k(P) \leq |P|$. In [Külshammer et al. 2014], the authors study finite groups G with $k_p(G) = 2$. They show that the Sylow p -subgroup P of such a group G must be either elementary abelian or extra-special of order p^3 . In this paper, we will look at the case when $k_p(G)$ is large in comparison to $|P|$.

Recall that a subgroup N of a finite group G is called a normal p -complement of G if N is a normal subgroup of G whose order is relatively prime to p and whose index is a power of p . A finite group G is said to be p -nilpotent if it has a normal p -complement. A classical result in group theory states that a finite group G is p -nilpotent if and only if P controls its own fusion in G . (See [Isaacs 2008, Theorem 5.25] and the definitions in Section 2). The latter condition is equivalent to $x^G \cap P = x^P$ for every $x \in P$, which is equivalent to the condition $k_p(G) = k(P)$. Thus G is p -nilpotent if and only if $k_p(G) = k(P)$. If we assume that $k_p(G) = |P|$, then $k_p(G) = k(P) = |P|$; hence G is p -nilpotent and has an abelian Sylow p -subgroup. So, we may ask whether G is still p -nilpotent, if $k_p(G)/|P|$ is close to 1.

It turns out that the fraction $k_p(G)/|P|$ is related to the commuting probability $d(G)$ of a finite group G , which is defined to be the probability that two randomly chosen elements of G commute. Gustafson [1973] shows that $d(G) = k(G)/|G|$. The invariant $d(G)$ is also called the commutativity degree of G .

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Here is our first result for the prime $p = 2$.

Theorem A. *Let G be a finite group and let P be a Sylow 2-subgroup of G . Then $k_2(G) > |P|/2$ if and only if G has a normal 2-complement and $k(P) > |P|/2$.*

Clearly, any finite group with a normal 2-complement is solvable by the Feit–Thompson theorem. Also, the Sylow 2-subgroup P in [Theorem A](#) is nilpotent of class at most 2. (See [Lemma 2.7](#).) [Theorem A](#) does not hold if we allow equality. For example, if $G = A_4$ and $P \in \text{Syl}_2(G)$, then $k_2(G) = 2$ and $|P| = 4$, so $k_2(G) = |P|/2$ but G is not 2-nilpotent. Also, we cannot replace 2 by an odd prime. Indeed, if $G = A_5$ and $P \in \text{Syl}_3(G)$, then $k_3(G) = 2$ and $|P| = 3$; hence $k_3(G) = \frac{2}{3}|P| > \frac{1}{2}|P|$ but G is not 3-nilpotent.

In view of [Lemma 2.8](#), to investigate the structure of finite groups G with $k_p(G)/|P|$ a specified constant, we may assume that $\mathbf{O}_{p'}(G) = 1$.

Theorem B. *Let G be a finite group and let $P \in \text{Syl}_2(G)$. Suppose that $\mathbf{O}_{2'}(G) = 1$ and $k_2(G) = |P|/2$. Then*

- (1) $G/\mathbf{Z}(G) \cong A_4$ or S_4 ; or
- (2) $G/\mathbf{Z}(G)$ is an almost simple group with a nonabelian simple socle isomorphic to $\text{PSL}_2(q)$ with $3 < q \equiv 3, 5 \pmod{8}$.

Let π be a set of primes. Let $k_\pi(G)$ be the number of conjugacy classes of π -elements of G . Let $|G|_\pi$ be the π -part of the order $|G|$ of G . Define $d_\pi(G)$ to be $k_\pi(G)/|G|_\pi$. If $\pi = \{p\}$, then we write $d_p(G)$ and $k_p(G)$ instead of $d_{\{p\}}(G)$ or $k_{\{p\}}(G)$. We now investigate the structure of finite groups G with $d_\pi(G) > \frac{1}{2}$, where π is a set of primes containing 2.

Theorem C. *Let G be a finite group and let π be a set of primes with $2 \in \pi$. Let $\sigma = \pi \setminus \{2\}$. Suppose that $d_\pi(G) > \frac{1}{2}$. Then G has a normal π -complement and an abelian Hall σ -subgroup.*

We should point out that our proofs of Theorems [A–C](#) do not depend on the classification of finite simple groups.

For odd primes p , we obtain the following result. Here our proof depends on the odd version of Glauberman’s Z^* -theorem and thus depends on the classification of finite simple groups.

Theorem D. *Let G be a finite group and let p be an odd prime. Then $d_p(G) > (p+1)/2p$ if and only if G has a normal p -complement and an abelian Sylow p -subgroup.*

This bound cannot be improved since $d_p(D_{2p}) = (p+1)/2p$ but D_{2p} is not p -nilpotent, where p is an odd prime. For nonsolvable examples, let $f \geq 2$ be an integer and p be a prime such that $4^f - 1$ is divisible by p but not by p^2 . Then $d_p(\text{PSL}_2(2^f)) = (p+1)/2p$.

Theorem E. *Let G be a finite group and let π be a set of odd primes. Let p be the smallest prime in π . Suppose that $d_\pi(G) > (p+1)/2p$. Then G has a normal π -complement and an abelian Hall π -subgroup.*

In [Maróti and Nguyen 2014], the authors show that if $d_\pi(G) > \frac{5}{8}$, then $d_\pi(G) = 1$ or $\frac{2}{3}$. They also study the structure of finite groups G such that $\mathbf{O}_{3'}(G) = 1$ and $d_3(G) = \frac{2}{3}$. Thus if $p = 3$ in Theorem D, then our results follow immediately from their results. However, if $p \geq 5$, then $(p+1)/2p < \frac{5}{8}$. Hence our Theorems C and E above improve their Theorem 1 and finally our last result includes Theorem 2 in [Maróti and Nguyen 2014].

Theorem F. *Let G be a finite group and let p be an odd prime. Let P be a Sylow p -subgroup of G . Suppose that $\mathbf{O}_p(G) = 1$ and $d_p(G) = (p+1)/2p$. Then P is abelian, $N_G(P)/C_G(P)$ has order 2, $[P, N_G(P)]$ has order p and $G \cong A \times B$, where B is an abelian p -group and A is either a dihedral group of order $2p$ or an almost simple group with a Sylow p -subgroup of order p contained in the socle of A .*

The paper is organized as follows. We collect some results needed for the proofs of the main theorems in Section 2. We prove Theorems A–C in Section 3 and prove Theorems D–F in Section 4.

2. Control of fusion and Glauberman's Z^* -theorem

Let G be a finite group and let $K \leq H \leq G$ be subgroups of G . We say that H controls G -fusion in K if and only if every pair of G -conjugate elements of K are H -conjugate; that is, if $x, x^g \in K$ for some $g \in G$, then $x^g = x^h$ for some $h \in H$. Let p be a prime and let H be a subgroup of G . We say that H controls p -fusion in G if H contains a Sylow p -subgroup P of G and H controls G -fusion in P . We first recall some classical results on the existence of normal p -complements as well as the control of fusion in finite groups.

Lemma 2.1. *Let G be a finite group and let P be a Sylow p -subgroup of G for some prime p .*

- (1) $N_G(P)$ controls G -fusion in $C_G(P)$.
- (2) If $P \subseteq \mathbf{Z}(N_G(P))$, then G has a normal p -complement.
- (3) G has a normal p -complement if and only if P controls its own fusion in G .

Proof. These are well known results; for proofs, see Lemma 5.12 and Theorems 5.13 and 5.25 in [Isaacs 2008]. □

Parts (1) and (2) above are known as Burnside's lemma and Burnside's normal p -complement theorem, respectively. Here are some obvious consequences of the lemma.

Corollary 2.2. *Let G be a finite group and let P be a Sylow p -subgroup of G for some prime p .*

- (1) $k_p(G) \leq k(P)$, and equality holds if and only if G has a normal p -complement.
- (2) $k_p(G) = |P|$, or equivalently $d_p(G) = 1$, if and only if G has a normal p -complement and an abelian Sylow p -subgroup.

Note that part (1) of the corollary is equivalent to the statement that $d_p(G) \leq d(P)$ and equality holds if and only if G has a normal p -complement. The following result is a consequence of the definitions above and Sylow's theorem.

Lemma 2.3. *Let G be a finite group and let P be a Sylow p -subgroup of G for some prime p . Let $x \in P$. Then $x^G \cap P = \{x\}$ if and only if $C_G(x)$ controls p -fusion in G .*

Proof. Let $x \in P$. Assume that $x^G \cap P = \{x\}$. We claim that $C_G(x)$ controls p -fusion in G . Since $x^P \subseteq x^G \cap P = \{x\}$, we see that $x \in Z(P)$ and thus $P \leq C_G(x)$. Now assume that $y, y^g \in P$ for some $g \in G$. We need to show that $y^g = y^h$ for some $h \in C_G(x)$. We have that $\{y, y^g\} \subseteq P \subseteq C_G(x)$, which implies that $\{x, x^{g^{-1}}\} \subseteq C_G(y)$. Let U be a Sylow p -subgroup of $C_G(y)$ containing x . By Sylow's theorem, $U \leq P^t$ for some $t \in G$. It follows that $x^{t^{-1}} \in P \cap x^G = \{x\}$; hence $x^{t^{-1}} = x$, so $t \in C_G(x)$. Now $x^{g^{-1}} \in U^c$ for some $c \in C_G(y)$ as $x^{g^{-1}} \in C_G(y)$ is a p -element. We now have that $x^{g^{-1}c^{-1}t^{-1}} \in P$ and thus $x^{g^{-1}c^{-1}t^{-1}} = x$, which implies that $g^{-1}c^{-1} \in C_G(x)$. Therefore $cg = h \in C_G(x)$. Now $y^g = y^{cg} = y^h$ as wanted.

For the converse, let $P_1 \in \text{Syl}_p(G)$ and assume that $P_1 \subseteq C_G(x)$ and that $C_G(x)$ controls G -fusion in P_1 . It follows that $x \in P_1$. By Sylow's theorem, $P = P_1^t$ for some $t \in G$. Since $x \in P$, $x^{t^{-1}} \in P_1 \leq C_G(x)$. As $C_G(x)$ controls G -fusion in P_1 , it follows that $x^{t^{-1}} = x^h$ for some $h \in C_G(x)$. Hence $x^{t^{-1}} = x^h = x$ and so $t \in C_G(x)$. In particular, $P = P_1^t \subseteq C_G(x)$. Finally, if $x^g \in P$ for some $g \in G$, then $x^g = x^h$ for some $h \in C_G(x)$ and so $x^g = x^h = x$. Therefore $x^G \cap P = \{x\}$. $\square$

For a finite group G and a prime p , we define $Z_p^*(G)$ to be the normal subgroup of G such that $Z_p^*(G)/O_{p'}(G) = Z(G/O_{p'}(G))$.

We first state the original form of Glauberman's Z^* -theorem, whose proof does not depend on the classification of finite simple groups.

Lemma 2.4 (Glauberman's Z^* -theorem). *Let G be a finite group and let P be a Sylow 2-subgroup of G . If $x \in P$ and $x^G \cap P = \{x\}$, then $x \in Z_2^*(G)$.*

Proof. This is a restatement of Theorem 3 in [Glauberman 1966]. $\square$

The odd version of Glauberman's Z^* -theorem, which is called Glauberman's Z_p^* -theorem, says that if $x \in P$ is an element of order p and $x^G \cap P = \{x\}$, then $x \in Z_p^*(G)$. The proof of this theorem depends on the classification (for a sketch proof, see [Guralnick and Robinson 1993, Theorem 4.1]). By Sylow's theorem, it

is easy to see that if $x^G \cap P = \{x\}$ then x does not commute with any G -conjugate $x^g \neq x$ of x . Finally, we note that the conclusion of Glauberman's Z_p^* -theorem can be written as $G = C_G(x)O_{p'}(G)$.

To use Glauberman's Z_p^* -theorem for an arbitrary p -element $x \in P$ which is not of prime order satisfying $x^G \cap P = \{x\}$, we need the following lemma.

Lemma 2.5. *Let G be a finite group and let P be a Sylow p -subgroup of G for some prime p . Let $x \in P$. If $x^G \cap P = \{x\}$, then $y^G \cap P = \{y\}$ for every $y \in \langle x \rangle$.*

Proof. Suppose that $x^G \cap P = \{x\}$ and $y \in \langle x \rangle$. Then $P \leq C_G(x) \leq C_G(y)$. By Lemma 2.3, we need to show that $C_G(y)$ controls G -fusion in P . Let $z, z^g \in P$ for some $g \in G$. By Lemma 2.3, $C_G(x)$ controls p -fusion in G so $z^g = z^t$ for some $t \in C_G(x)$. As $C_G(x) \leq C_G(y)$, we have $t \in C_G(y)$ and the claim follows. $\square$

We will need the following results.

Lemma 2.6. *Let G be a finite group and let π be a nonempty set of primes.*

- (1) *If $\mu \subseteq \pi$ is a nonempty subset, then $d_\pi(G) \leq d_\mu(G) \leq 1$.*
- (2) *If $N \trianglelefteq G$, then $d_\pi(G) \leq d_\pi(G/N)d_\pi(N)$.*
- (3) *If G is a nonabelian p -group for some prime p , then $d(G) < (p+1)/p^2$.*
- (4) *If G does not have a normal Sylow p -subgroup for some prime p , then $d(G) \leq 1/p$.*

Proof. Part (1) can be found in [Maróti and Nguyen 2014, Proposition 5] and Part (2) is Lemma 2.3 in [Fulman and Guralnick 2012]. Finally, the last two parts can be found in Lemma 2 in [Guralnick and Robinson 2006]. $\square$

Finite groups G with $d(G) \geq \frac{1}{2}$ were classified by Lescot [1995; 2001]. To state the result, we need the following notation. For any integer $m \geq 1$, denote by G_m the group defined by

$$G_m = \langle a, b : a^3 = b^{2^m} = 1, a^b = a^{-1} \rangle.$$

Note that $G_1 \cong S_3$. We have that $|G_m| = 3 \cdot 2^m$, $Z(G_m) = \langle b^2 \rangle$, $G'_m = \langle a \rangle$, and $G_m/Z(G_m) \cong S_3$.

Lemma 2.7. *Let G be a finite group. Then $d(G) \geq \frac{1}{2}$ if and only if one of the following holds:*

- (i) *G is abelian and $d(G) = 1$.*
- (ii) *$G \cong P \times A$, where A is abelian of odd order and P is a Sylow 2-subgroup of G with $|G'| = |P'| = 2$ and $d(G) = d(P) = (1 + 4^{-m})/2$ and $G/Z(G)$ is elementary abelian of order 4^m for some integer $m \geq 1$. Further, $\frac{1}{2} < d(G) \leq \frac{5}{8}$.*
- (iii) *$G \cong G_m \times A$ and $d(G) = \frac{1}{2}$, where A is abelian and $m \geq 1$.*

Proof. This is a combination of Theorem 3.1 in [Lescot 2001] and Corollary 3.2 in [Lescot 1995]. $\square$

It follows from Lemma 2.7 that there is no 2-group G with $d(G) = \frac{1}{2}$. Also, if G is of odd order with $d(G) \geq \frac{1}{2}$, then G is abelian and $d(G) = 1$.

Lemma 2.8. *Let G be a finite group and let p be a prime. If $N \trianglelefteq G$ is a p' -subgroup, then $k_p(G) = k_p(G/N)$ and so $d_p(G) = d_p(G/N)$.*

Proof. Let N be a normal p' -subgroup of G and let $P \in \text{Syl}_p(G)$. Write $\bar{G} = G/N$. Since $p \nmid |N|$, the Sylow p -subgroups of G and $\bar{G}$ have the same order, thus it suffices to show that $k_p(G) = k_p(\bar{G})$. By Lemma 2.6(2), we have $d_p(G) \leq d_p(\bar{G})$ since $d_p(N) \leq 1$. It follows that $k_p(G) \leq k_p(\bar{G})$. The reverse inequality is obvious. $\square$

3. Conjugacy classes of 2-elements

We will prove Theorems A, B and C in this section. Recall that a p -group P is said to be extra-special if $P' = \Phi(P) = Z(P)$ and $|Z(P)| = p$. The following lemma is key to our proofs.

Lemma 3.1. *Let G be a finite group and let P be a Sylow 2-subgroup of G . Suppose that $O_2(G) = Z(G) = 1$ and $|P| > 1$. Then $k_2(G) \leq |P|/2$ and $d_2(G) \leq \frac{1}{2}$. Moreover, if $k_2(G) = |P|/2$, then one of the following holds:*

- (1) $k_2(G) = 2$ and P is elementary abelian of order 4.
- (2) $k_2(G) > 2$ and P is an extra-special group of order 2^{1+2m} for some integer $m \geq 1$ with $Z(P) = \langle z \rangle$ a cyclic group of order 2. Moreover, $|z^G \cap P| = 3$ and for any $1 \neq y \in P$ with $y \notin z^G$, $|y^G \cap P| = 2$.

Proof. The hypothesis of the lemma implies that $Z_2^*(G) = 1$. Let P be a Sylow 2-subgroup of G . By Lemma 2.4, if $1 \neq x \in P$, then $|x^G \cap P| \geq 2$. Let $k = k_2(G) - 1$ be the number of nontrivial conjugacy classes of 2-elements in G . Clearly, we can choose a complete set $\Gamma = \{x_i\}_{i=1}^k$ of representatives for all nontrivial conjugacy classes of 2-elements in G such that $\Gamma \subseteq P \setminus \{1\}$. Notice that $k \geq 1$ as otherwise P is trivial.

Observe that $|x_i^G \cap P| \geq 2$ for all i with $1 \leq i \leq k$, and $P \setminus \{1\} = \bigcup_{i=1}^k x_i^G \cap P$. We have that

$$|P \setminus \{1\}| = \sum_{i=1}^k |x_i^G \cap P| \geq \sum_{i=1}^k 2 = 2k.$$

Hence $|P| - 1 \geq 2k$ and so $2k \leq |P| - 2$ since $|P| - 1$ is odd. Therefore $k_2(G) = k + 1 \leq |P|/2$ and $d_2(G) \leq \frac{1}{2}$ as wanted.

Next, assume that $k_2(G) = |P|/2$. We know that $|x_i^G \cap P| \geq 2$ for all $i = 1, 2, \dots, k$. If $k = 1$, then $k_2(G) = 2$ and $|P| = 4$, so $|x_1^G \cap P| = 3$ and P is elementary abelian of order 4. Thus part (1) holds. Assume that $k \geq 2$. Since $|x_i^G \cap P| \geq 2$

for every i and $k = |P|/2 - 1$, we obtain that $|x_j^G \cap P| = 3$ for a unique index j and $|x_i^G \cap P| = 2$ for all $1 \leq i \neq j \leq k$. So we may assume that $|x_1^G \cap P| = 3$ and $|x_i^G \cap P| = 2$ for $2 \leq i \leq k$.

By [Corollary 2.2](#), we have $\frac{1}{2} = d_2(G) \leq d(P)$ and thus either P is abelian or $|P'| = 2$, $d(P) = (1 + 4^{-m})/2$ and $P/Z(P)$ is elementary abelian of order 4^m by [Lemma 2.7](#). We claim that P is nonabelian. By way of contradiction, assume that P is abelian. Let $H = N_G(P)$. By [Lemma 2.1\(1\)](#), H controls G -fusion in P (since P is abelian) and so $x_k^G \cap P = x_k^H$. Moreover, as P is abelian, $P \leq C_H(x_k) \leq H$ and hence $|x_k^H| \geq 1$ is odd. However $|x_k^H| = |x_k^G \cap P| = 2$ by the result in the previous paragraph, which is a contradiction.

Next, we claim that P is extra-special. It suffices to show that $Z(P) = P'$. Write $P' = \langle z \rangle$. Since $|P'| = 2$, we have $P' \leq Z(H) \cap Z(P)$. Let $1 \neq u \in Z(P)$. We claim that $|u^G \cap P| = 3$. Assume by contradiction that $u^G \cap P = \{u, v\}$, where $u \neq v \in P$. If $v \in Z(P)$, then $v = u^h$ for some $h \in H$ by [Lemma 2.1\(1\)](#). It follows that $|u^H| = 2$, which is impossible as $P \leq C_H(u) \leq H$. Thus $v \notin Z(P)$. Now $u^G \cap P = v^G \cap P = \{u, v\}$. Since $v \in P \setminus Z(P)$, we have $|v^P| > 1$ whence $v^P = \{u, v\}$. In particular $u = v^t$ for some $t \in P$. Hence $v = u^{t^{-1}} = u$ as $u \in Z(P)$. This contradiction shows that $|u^G \cap P| = 3$ for every $1 \neq u \in Z(P)$. In particular, $|z^G \cap P| = 3$. Now if $Z(P) \neq P'$, then we can choose $u \in Z(P) \setminus P'$ and by our previous claim, $|u^G \cap P| = 3$. It follows that u and z are G -conjugate as there is only one class of 2-elements satisfying the previous condition. Again this is a contradiction by using [Lemma 2.1\(1\)](#) and the fact that $z \in Z(H)$. The proof is now complete. $\square$

We are now ready to prove our first theorem.

Proof of Theorem A. Let G be a finite group. Assume first that G has a normal 2-complement and $d(P) > \frac{1}{2}$ for some Sylow 2-subgroup P of G . By [Corollary 2.2\(1\)](#), $d_2(G) = d(P) > \frac{1}{2}$. Conversely, assume that $d_2(G) > \frac{1}{2}$. Let P be a Sylow 2-subgroup of G . If G has a normal 2-complement, then $d(P) = d_2(G) > \frac{1}{2}$. Thus we only need to show that G has a normal 2-complement. We proceed by induction on $|G|$. Observe that if N is a proper nontrivial normal subgroup of G , then $d_2(N)$ and $d_2(G/N)$ are strictly larger than $\frac{1}{2}$ by [Lemma 2.6\(2\)](#). By induction, both N and G/N have normal 2-complements. Hence if N is of odd order or G/N is a 2-group then G has a normal 2-complement and we are done. Therefore, we may assume that $\mathcal{O}_{2'}(G) = 1$ and $G = \mathcal{O}^2(G)$.

Suppose that $Z(G)$ is nontrivial. As $\mathcal{O}_{2'}(G) = 1$, $Z(G)$ must be a 2-group. Now $G/Z(G)$ has a normal 2-complement, say $K/Z(G) \trianglelefteq G/Z(G)$, for some normal subgroup K of G with $Z(G) \leq K$. Hence $Z(G) \trianglelefteq K \trianglelefteq G$ and G/K is a 2-group. Since $G = \mathcal{O}^2(G)$, we obtain that $G = K$. We now see that $Z(G)$ is a normal Sylow 2-subgroup of G and thus G has a normal 2-complement by [Lemma 2.1\(2\)](#). So we may assume that $Z(G) = 1$. Now [Lemma 3.1](#) yields a contradiction. $\square$

We now study the structure of finite groups G with $d_2(G) = \frac{1}{2}$. We first consider the solvable case.

Lemma 3.2. *Let G be a finite solvable group. Suppose that $\mathbf{O}_{2'}(G) = 1$ and $G = \mathbf{O}^2(G)$. Then $d_2(G) = \frac{1}{2}$ if and only if $G \cong A_4$.*

Proof. If $G \cong A_4$, then $d_2(G) = \frac{1}{2}$ as $k_2(G) = 2$ and $|P| = 4$. Conversely, assume that $d_2(G) = \frac{1}{2}$. We proceed by induction on $|G|$. By [Corollary 2.2](#), we have $\frac{1}{2} = d_2(G) \leq d(P)$. [Lemma 2.7](#) yields that either P is abelian or $|P'| = 2$ and $P/\mathbf{Z}(P)$ is elementary abelian of order 4^m . In both cases, $d(P) > \frac{1}{2}$. It follows that G is not a 2-group and so P is noncyclic by [Corollary 5.14](#) in [\[Isaacs 2008\]](#). As $\mathbf{O}_{2'}(G) = 1$, $\mathbf{C}_G(\mathbf{O}_2(G)) \subseteq \mathbf{O}_2(G)$ by [\[Isaacs 2008, Theorem 3.21\]](#), hence $\mathbf{Z}(G) \leq P$.

We claim that $G/\mathbf{Z}$ satisfies the hypothesis of the lemma for any central subgroup $Z \leq \mathbf{Z}(G)$. Clearly, $\mathbf{O}^2(G/Z) = G/Z$ since $G = \mathbf{O}^2(G)$. Next, assume that $K/Z = \mathbf{O}_{2'}(G/Z)$, where $Z \leq K \trianglelefteq G$. Then K has a central Sylow 2-subgroup Z and so by [Lemma 2.1\(2\)](#), K has a normal 2-complement $\mathbf{O}_{2'}(K)$. Since $K \trianglelefteq G$, $\mathbf{O}_{2'}(K) \leq \mathbf{O}_{2'}(G) = 1$. Hence $K = Z$ and so $\mathbf{O}_{2'}(G/Z) = 1$.

By [Lemma 2.6\(2\)](#), we have $\frac{1}{2} = d_2(G) \leq d_2(G/Z)d_2(Z) = d_2(G/Z)$. If $d_2(G/Z) > \frac{1}{2}$, then G/Z has a normal 2-complement by [Theorem A](#) but this would imply that G/Z is a 2-group and so G is a 2-group, a contradiction. Hence $d_2(G/Z) = \frac{1}{2}$. Therefore, by using induction on $|G|$, if Z is nontrivial, then $G/Z \cong A_4$. We now consider two cases separately, according to whether P is abelian or not.

Case 1: P is abelian. As $\mathbf{C}_G(\mathbf{O}_2(G)) \subseteq \mathbf{O}_2(G)$, we have $P = \mathbf{O}_2(G)$ and so $P = \mathbf{C}_G(P) \trianglelefteq G$. Clearly, $P \neq \mathbf{Z}(G)$, as otherwise G is a 2-group by applying [Lemma 2.1\(2\)](#). Thus $|P : \mathbf{Z}(G)| \geq 2$.

Assume first that $\mathbf{Z}(G) = 1$. By [Lemma 3.1](#), P is elementary abelian of order 4 and $k_2(G) = 2$. As $P = \mathbf{C}_G(P) \trianglelefteq G$, G/P embeds into $\mathrm{GL}_2(2) \cong S_3$. Since G/P is of odd order and nontrivial, $G/P \cong C_3$. It is not hard to see that $G \cong A_4$.

Next, assume that $\mathbf{Z}(G)$ is nontrivial. Then $G/\mathbf{Z}(G) \cong A_4$. By [\[Isaacs 2008, Theorem 5.18\]](#), $G' \cap \mathbf{Z}(G) = G' \cap P \cap \mathbf{Z}(G) = 1$. Let R be a Sylow 3-subgroup of G . Then $G = PR$, $|R| = 3$ and R acts nontrivially and coprimely on P ; hence $\mathbf{Z}(G) = \mathbf{C}_P(R)$ and $[P, R] = G' \leq P$. Since R acts coprimely on P , we have $P = [P, R] \times \mathbf{C}_P(R) = G' \times \mathbf{Z}(G)$. Moreover, $G'R \trianglelefteq G$ and $|G/G'R|$ is a 2-power, so $G = G'R$ forcing $\mathbf{Z}(G) = 1$, a contradiction.

Case 2: P is nonabelian. We have $P' \leq \mathbf{Z}(P) \leq \mathbf{C}_G(\mathbf{O}_2(G)) \leq \mathbf{O}_2(G)$. Observe that $G/\mathbf{O}_{2,2'}(G)$ has an abelian Sylow 2-subgroup, so $G/\mathbf{O}_{2,2'}(G)$ has a normal Sylow 2-subgroup by using Hall–Higman Lemma 1.2.3 ([\[Isaacs 2008, Theorem 3.21\]](#)); hence $G = \mathbf{O}_{2,2',2,2'}(G)$. (For the definitions of $\mathbf{O}_{2,2'}(G)$ and $\mathbf{O}_{2,2',2,2'}(G)$, see [\[Gorenstein 1968, Section 6.3\]](#).)

Assume first that $G = \mathbf{O}_{2,2'}(G)$. Then $\mathbf{C}_G(P) \leq P \trianglelefteq G$. It follows that $P' \leq \mathbf{Z}(G)$ as $|P'| = 2$. Thus $G/P' \cong \mathbf{A}_4$ and $|G| = 24$. It is easy to check that $G \cong 2 \cdot \mathbf{A}_4 \cong \mathrm{SL}_2(3)$ as $G = \mathbf{O}^2(G)$. However, $d_2(\mathrm{SL}_2(3)) = \frac{3}{8} < \frac{1}{2}$.

Assume that $G/\mathbf{O}_{2,2'}(G)$ is nontrivial. Let $L = \mathbf{O}_{2,2'}(G) \trianglelefteq G$. Since $\mathbf{O}_{2'}(L) = 1$, by using [Theorem A](#) and [Lemma 2.6\(2\)](#), we see that $d_2(L) = \frac{1}{2}$. But then this forces $d_2(G/L) = 1$. By [Corollary 2.2\(2\)](#), G/L has a normal 2-complement and since $G = \mathbf{O}^2(G)$, we deduce that G/L is a $2'$ -group, forcing $G = L$, which is a contradiction. This completes our proof. $\square$

Lemma 3.3. *Let G be a finite solvable group. Suppose that $\mathbf{O}_{2'}(G) = 1$. If $d_2(G) = \frac{1}{2}$, then $G/\mathbf{Z}(G) \cong \mathbf{A}_4$ or $\mathbf{S}_4$.*

Proof. Suppose that G is a finite solvable group with $d_2(G) = \frac{1}{2}$ and $\mathbf{O}_{2'}(G) = 1$. Let $L = \mathbf{O}^2(G)$. Then $\mathbf{O}_{2'}(L) = 1$ and $L = \mathbf{O}^2(L)$. By [Lemma 2.6\(2\)](#), $d_2(L) \geq \frac{1}{2}$. If $d_2(L) > \frac{1}{2}$, then L has a normal 2-complement by [Theorem A](#). However, as $\mathbf{O}_{2'}(L) = 1$, L must be a 2-group and hence G is a 2-group with $d_2(G) = d(G) = \frac{1}{2}$, which is impossible by [Lemma 2.7](#). Therefore, $d_2(L) = \frac{1}{2}$. So $L \cong \mathbf{A}_4$ by [Lemma 3.2](#).

Let $C = \mathbf{C}_G(L) \trianglelefteq G$. As $\mathbf{Z}(L) = 1$, we have $C \cap L = 1$. Then $\mathbf{A}_4 \cong LC/C \trianglelefteq G/C \leq \mathrm{Aut}(\mathbf{A}_4) = \mathbf{S}_4$. Hence $G/C \cong \mathbf{A}_4$ or $\mathbf{S}_4$. It remains to show that $C = \mathbf{Z}(G)$. Let P be a Sylow 2-subgroup of G .

As $C \times L = CL \trianglelefteq G$, we have $\frac{1}{2} = d_2(G) \leq d_2(CL) \leq d_2(L)d_2(C) = d_2(C)/2$ and so $d_2(C) = 1$. Thus C has a normal 2-complement and an abelian Sylow 2-subgroup. However, as $\mathbf{O}_{2'}(C) \leq \mathbf{O}_{2'}(G) = 1$, C must be an abelian 2-group and $C \leq P$. We also have that $\frac{1}{2} = d_2(G) \leq d_2(G/L)d_2(L) = d_2(G/L)/2$, so $d_2(G/L) = 1$ where G/L is a 2-group. It follows that G/L is an abelian 2-group. In particular, $G' \leq L$. Thus $[P, C] \subseteq G' \cap C \subseteq L \cap C = 1$, so $[P, C] = 1$. As $[L, C] = 1$ and $G = PL$, we have $C \leq \mathbf{Z}(G)$. Since $\mathbf{Z}(G/C)$ is trivial, we must have $C = \mathbf{Z}(G)$ as wanted. $\square$

We next classify all finite nonabelian simple groups S such that $d_2(S) = \frac{1}{2}$.

Lemma 3.4. *Let S be a finite nonabelian simple group. Then $d_2(S) = \frac{1}{2}$ if and only if $S \cong \mathrm{PSL}_2(q)$ with $3 < q \equiv 3, 5 \pmod{8}$, where q is a prime power.*

Proof. Let S be a finite nonabelian simple with a Sylow 2-subgroup P . If $S \cong \mathrm{PSL}_2(q)$ with $3 < q \equiv 3, 5 \pmod{8}$, then P is elementary abelian of order 4 and S has only one class of involutions so $k_2(S) = 2$ and thus $d_2(S) = \frac{1}{2}$. Conversely, assume that S is a finite nonabelian simple group with $d_2(S) = \frac{1}{2}$. By [Lemma 3.1](#), either P is elementary abelian of order 4 with $k_2(P) = 2$ or P is extra-special of order 2^{1+2m} for some integer $m \geq 1$.

Assume first that P is elementary abelian of order 4. It follows from [[Walter 1969](#), Theorem I] that S is isomorphic to $\mathrm{PSL}_2(2^f)$, where $f \geq 2$, $\mathrm{PSL}_2(q)$ with $3 < q \equiv 3, 5 \pmod{8}$, ${}^2\mathrm{G}_2(3^{2n+1})$, where $n \geq 1$, or the first Janko group J_1 . Since $|P| = 4$, we deduce that $S \cong \mathrm{PSL}_2(q)$ with $3 < q \equiv 3, 5 \pmod{8}$. Note $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$.

Assume now that P is extra-special of order 2^{1+2m} , $m \geq 1$. In this case, P is nilpotent of class 2. It follows from the Main Theorem of [Gilman and Gorenstein 1975a; 1975b] that S is isomorphic to one of the groups $\mathrm{PSL}_2(q)$ with $q \equiv 7, 9 \pmod{16}$, A_7 , $\mathrm{Sz}(2^n)$, $\mathrm{PSU}_3(2^n)$, $\mathrm{PSL}_3(2^n)$ or $\mathrm{PSp}_4(2^n)$ with $n \geq 2$. However, except for the first two groups, the centers of the Sylow 2-subgroups of the remaining simple groups have order at least 4. For A_7 , we can check that $k_2(A_7) = 3$ so $d_2(A_7) = \frac{3}{8}$ as a Sylow 2-subgroup of A_7 is isomorphic to D_8 . Similarly, the Sylow 2-subgroup of $S = \mathrm{PSL}_2(q)$ with $q \equiv 7, 9 \pmod{16}$ is also isomorphic to D_8 . Again, except for the identity, S has two nontrivial classes of 2-elements, one consisting of all involutions in S and another consisting of elements of order 4. Thus these cases cannot occur. $\square$

For a finite group G , we denote by $\mathrm{Sol}(G)$ the solvable radical of G , that is, the largest solvable normal subgroup of G .

Lemma 3.5. *Let G be a finite group. Suppose that $\mathrm{Sol}(G) = 1$ and $d_2(G) = \frac{1}{2}$. Then G is a finite almost simple group.*

Proof. Let M be a minimal normal subgroup of G . As G has a trivial solvable radical, $M \cong S^k$, where S is a nonabelian simple group and $k \geq 1$ is an integer. By Lemma 2.6(2), we have $\frac{1}{2} = d_2(G) \leq d_2(G/M)d_2(M) \leq d_2(M)$. By applying this lemma repeatedly, we have $\frac{1}{2} \leq d_2(M) \leq d_2(S)^k$. By Lemma 3.1, $d_2(S) \leq \frac{1}{2}$; so $\frac{1}{2} \leq d_2(S)^k \leq (\frac{1}{2})^k$, forcing $k = 1$ and $d_2(S) = \frac{1}{2}$.

Let $C = C_G(M)$. Then $C \trianglelefteq G$ and $CM = C \times M \trianglelefteq G$. By Lemma 2.6(2),

$$\frac{1}{2} = d_2(G) \leq d_2(G/MC)d_2(MC) \leq d_2(MC) \leq d_2(M)d_2(C) = d_2(C)/2.$$

Hence $d_2(C) = 1$ and so C is solvable by Corollary 2.2(2) and the Feit–Thompson theorem. Since $\mathrm{Sol}(G) = 1$ and C is a solvable normal subgroup of G , we must have $C = 1$ so G is almost simple with simple socle M . $\square$

Lemma 3.6. *Let G be a finite perfect group. Suppose that $\mathbf{O}_{2'}(G) = 1$ and $d_2(G) = \frac{1}{2}$. Then $G \cong \mathrm{PSL}_2(q)$ with $3 < q \equiv 3, 5 \pmod{8}$, where q is a prime power.*

Proof. Let U be the solvable radical of G . Then G/U is nonsolvable. Since $\frac{1}{2} = d_2(G) \leq d_2(U)d_2(G/U)$ by Lemma 2.6, both $d_2(U)$ and $d_2(G/U)$ are at least $\frac{1}{2}$. By Theorem A, $d_2(G/U) = \frac{1}{2}$ as otherwise G/U is solvable. By Lemmas 3.4 and 3.5 and the fact that G is perfect, $G/U \cong \mathrm{PSL}_2(q)$, where $q \equiv 3, 5 \pmod{8}$. We have $d_2(U) = 1$ and since $\mathbf{O}_{2'}(G) = 1$, U is an abelian 2-group. We will show that G is nonabelian simple by induction on $|G|$.

If $U = 1$, then G is simple and we are done. Assume that U is nontrivial. Assume first that $Z := \mathbf{Z}(G)$ is nontrivial. Then Z must be a 2-group. Consider the quotient group G/Z . Observe that G/Z is perfect, $d_2(G/Z) = \frac{1}{2}$ and $\mathbf{O}_{2'}(G/Z) = 1$. Since $|G/Z| < |G|$, the inductive hypothesis implies that G/Z is nonabelian simple and

thus $Z = U$. It follows that $G \cong \text{SL}_2(q)$, the only Schur cover of $\text{PSL}_2(q)$ with $q \equiv 3, 5 \pmod{8}$. However, it is easy to see that $\text{SL}_2(q)$ has only two classes of nontrivial 2-elements and the Sylow 2-subgroup of $\text{SL}_2(q)$ with $q \equiv 3, 5 \pmod{8}$ has order 8, so $d_2(G) = \frac{3}{8} < \frac{1}{2}$, which is a contradiction. Hence we may assume that $\mathbf{Z}(G) = 1$. Since U is a normal abelian subgroup of G , we have $U \leq \mathbf{C}_G(U) \trianglelefteq G$. Since U is not central in G and G/U is nonabelian simple, we must have that $U = \mathbf{C}_G(U)$.

Let $P \in \text{Syl}_2(G)$. Note that the hypothesis of [Lemma 3.1](#) holds for G , that is, $\mathbf{O}_{2'}(G) = \mathbf{Z}(G) = 1$, $|P| > 1$ and that $d_2(G) = \frac{1}{2}$. We claim that $k_2(G) > 2$. Assume by contradiction that $k_2(G) = 2$. Then P is elementary abelian of order 4 by [Lemma 3.1](#). However the Sylow 2-subgroup of $\text{PSL}_2(q)$ with $q \equiv 3, 5 \pmod{8}$ has order 4. So $U = 1$, which is a contradiction. Therefore $k_2(G) > 2$ and so part (2) of [Lemma 3.1](#) holds. Obviously $|U| \geq 4$ and $P' = \langle z \rangle = \mathbf{Z}(P) < U$. If $z^G = U \setminus \{1\}$, then U is elementary abelian of order 4 and thus G/U embeds into $\text{GL}_2(2)$ which is impossible. Thus there exists $1 \neq y \in U \setminus z^G$ and so $|y^G \cap P| = 2$. Since $y \in U \trianglelefteq G$, we have $y^G \subseteq U \leq P$, so $|y^G \cap P| = |y^G| = 2$ which implies that $U \leq \mathbf{C}_G(y) < G$ and $|G : \mathbf{C}_G(y)| = 2$. Therefore, G/U has a subgroup of index 2 which is impossible as G/U is nonabelian simple. $\square$

Proof of Theorem B. Let G be a finite group and assume that $d_2(G) = \frac{1}{2}$ and $\mathbf{O}_{2'}(G) = 1$. If G is solvable, then $G/\mathbf{Z}(G) \cong A_4$ or S_4 by [Lemma 3.3](#). So part (1) of the theorem holds. Assume that G is nonsolvable. Let L be the last term of the derived series of G . By [Theorem A](#) and [Lemma 2.6\(2\)](#), $d_2(L) = \frac{1}{2}$. Moreover L is perfect and $\mathbf{O}_{2'}(L) = 1$. By [Lemma 3.6](#), $L \cong S$ where $S = \text{PSL}_2(q)$ $q \equiv 3, 5 \pmod{8}$. Write $q = p^f$, where p is a prime and $f \geq 1$ is an integer. We see that f must be odd and thus $\text{Out}(S) = C_2 \times C_f$.

Let $C = \mathbf{C}_G(L)$. Then $C \trianglelefteq G$, $C \cap L = 1$ and G/C is an almost simple group with socle isomorphic to S . Since $d_2(L) = \frac{1}{2}$, we see that $d_2(C) = 1$ and since $\mathbf{O}_{2'}(G) = 1$, C is a normal abelian 2-subgroup of G . We also have that

$$\frac{1}{2} = d_2(G) \leq d_2(G/L)d_2(L) = d_2(G/L)/2$$

so $d_2(G/L) = 1$ and so G/L has a normal 2-complement W/L and an abelian Sylow 2-subgroup PL/L by [Corollary 2.2](#), where P is any Sylow 2-subgroup of G containing C . Since CL/L and W/L are normal subgroups of G/L and have coprime orders, we deduce that $[C, W] \leq L$. As $C \trianglelefteq G$, we have $[C, W] \leq L \cap C = 1$. Thus $[C, W] = 1$. On the other hand, PL/L is abelian, thus $[C, P] \leq L$. With the same reasoning, we have $[C, P] \leq L \cap C = 1$. Since $G = PW$, we obtain that $[C, G] = 1$. In particular, $C \leq \mathbf{Z}(G)$ and since G/C is almost simple, we must have that $C = \mathbf{Z}(G)$. Therefore, we have shown that $G/\mathbf{Z}(G)$ is almost simple with socle S as required. $\square$

We will need the following result for our proof of [Theorem C](#).

Lemma 3.7. *Let G be a finite group of odd order and let σ be a nonempty set of primes. If $d_\sigma(G) \geq \frac{1}{2}$, then G has a normal σ -complement and an abelian Hall σ -subgroup.*

Proof. By the Feit–Thompson theorem, we know G is solvable. By Lemma 2.6(2), if $N \triangleleft G$, then $d_\sigma(N) \geq \frac{1}{2}$ and $d_\sigma(G/N) \geq \frac{1}{2}$. Assume that G has a normal σ -complement K . Let H be a Hall σ -subgroup of G . We claim that H is abelian. As $G/K \cong H$, we have $\frac{1}{2} \leq d_\sigma(H) = d(H)$, where the last equality holds as H is a σ -group. Thus $d(H) \geq \frac{1}{2}$ where H is a group of odd order. By Lemma 2.7, H must be abelian as wanted. Therefore, it suffices to show that G has a normal σ -complement. We will prove this claim by induction on $|G|$.

Let N be a minimal normal subgroup of G . Then N is an elementary abelian p -subgroup for some odd prime p . As $d_\sigma(G/N) \geq \frac{1}{2}$, by induction on $|G|$, G/N has a normal σ -complement, say M/N . If $p \notin \sigma$, then M is also a normal σ -complement of G , and we are done. Thus we may assume that $\mathbf{O}_{\sigma'}(G) = 1$ and $p \in \sigma$. We have $M \triangleleft G$ and $d_\sigma(M) \geq \frac{1}{2}$. Therefore, by induction again, M has a normal σ -complement whenever $M < G$; but then this would imply that M is a σ -subgroup since $\mathbf{O}_{\sigma'}(M) \subseteq \mathbf{O}_{\sigma'}(G) = 1$ and hence G is a σ -group. So, we can assume $M = G$, hence G/N is a σ' -group.

Since G/N is solvable, let T/N be a maximal normal subgroup of G/N of prime index $r \notin \sigma$. Since $T \triangleleft G$, we have $d_\sigma(T) \geq \frac{1}{2}$ and again by induction, T has a normal σ -complement which implies that $T = N$. Thus N is a maximal normal subgroup of G and $|G/N| = r$ is a prime different from p .

If $\mathbf{C}_G(x) = G$ for some $1 \neq x \in N$, then $\langle x \rangle = N \leq \mathbf{Z}(G)$ and $G \cong C_p \times C_r$ by Lemma 2.1(2), which is a contradiction as $\mathbf{O}_{\sigma'}(G) = 1$. So, we may assume that $\mathbf{C}_G(x) < G$ for all $1 \neq x \in N$. Since N is maximal in G , $\mathbf{C}_G(x) = N$ for every $1 \neq x \in N$. Thus G is a Frobenius group with Frobenius kernel N and Frobenius complement isomorphic to C_r . Set $|N| = p^k$ for some integer $k \geq 1$. We can see that $\sigma = \{p\}$ and that $k_p(G) = (p^k - 1)/r + 1$ and so $d_p(G) = 1/r + (r - 1)/(rp^k) \geq \frac{1}{2}$. Notice that $r \neq p \geq 3$ and $r \mid p^k - 1$. We consider the following cases:

(1) $r = 3$ and $p \geq 5$. In this case, we have

$$d_p(G) \leq \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{5} = \frac{7}{15} < \frac{1}{2}.$$

(2) $r \geq 5$ and $p = 3$. Since $r > p$, $k \geq 2$. We have

$$d_p(G) \leq \frac{1}{5} + \frac{1}{9} = \frac{14}{45} < \frac{1}{2}.$$

(3) $r \geq 5$ and $p \geq 5$. Clearly, we have

$$d_p(G) < \frac{1}{5} + \frac{1}{5} = \frac{2}{5} < \frac{1}{2}.$$

Thus we have shown that G has a normal σ -complement as wanted. $\square$

We are now ready to prove [Theorem C](#).

Proof of Theorem C. Let G be a finite group and let π be a set of primes containing 2 and let $\sigma = \pi \setminus \{2\}$. Suppose $d_\pi(G) > \frac{1}{2}$. By [Lemma 2.6\(1\)](#), $\frac{1}{2} < d_\pi(G) \leq d_2(G)$ and thus by [Theorem A](#), G has a normal 2-complement K and by [Lemma 2.6\(2\)](#), we have

$$\frac{1}{2} < d_\pi(G) \leq d_\pi(K)d_\pi(G/K) \leq d_\pi(K) = d_\sigma(K).$$

By [Lemma 3.7](#), K has a normal σ -complement, say N , and an abelian Hall σ -subgroup T . It follows that $G = PTN$, where N is also a normal π -complement of G . $\square$

4. Conjugacy classes of p -elements with p odd

We now consider odd primes. We start with the following easy result.

Lemma 4.1. *Let p be an odd prime. Let G be a finite group and let P be a Sylow p -subgroup of G . If $d_p(G) \geq (p+1)/2p$, then P is abelian.*

Proof. Let G be a finite group such that $d_p(G) \geq (p+1)/2p$. By [Corollary 2.2](#), we have $d_p(G) \leq d(P)$ which implies that $d(P) \geq (p+1)/2p$. If P is abelian, then we are done. So, assume that P is nonabelian. By [Lemma 2.6\(3\)](#), we have $d(P) < (p+1)/p^2$. Since p is odd, we can check that $(p+1)/2p > (p+1)/p^2$ and so $d(P) < (p+1)/p^2 < (p+1)/2p \leq d(P)$, which is a contradiction. $\square$

Proof of Theorem D. Let p be an odd prime. Let G be a finite group. Assume that G has a normal p -complement and an abelian Sylow p -subgroup P . By [Corollary 2.2\(2\)](#), we have $d_p(G) = d(P) = 1 > (p+1)/2p$. Conversely, assume that $d_p(G) > (p+1)/2p$. By [Lemma 4.1](#), P is abelian. It remains to show that G has a normal p -complement. We proceed by using induction on $|G|$.

We first claim that $\mathbf{O}_{p'}(G) = 1$. Assume by contradiction that $\mathbf{O}_{p'}(G)$ is nontrivial. By [Lemma 2.6\(2\)](#), $d_p(G) \leq d_p(G/\mathbf{O}_{p'}(G))d_p(\mathbf{O}_{p'}(G)) \leq d_p(G/\mathbf{O}_{p'}(G))$, so by induction $G/\mathbf{O}_{p'}(G)$ has a normal p -complement; hence G will have a normal p -complement. Thus we may assume that $\mathbf{O}_{p'}(G) = 1$.

We next claim that $G = \mathbf{O}^p(G)$. Indeed, if $N = \mathbf{O}^p(G)$ is a proper subgroup of G , then $(p+1)/2p < d_p(G) \leq d_p(N)$; thus by induction again, N has a normal p -complement $\mathbf{O}_{p'}(N)$. Clearly, this is also a normal p -complement of G .

We now show that G is p -solvable. In fact, suppose that G is not p -solvable and let M/N be a nonabelian chief factor of G with p dividing $|M/N|$. There exists a nonabelian simple group S and an integer $k \geq 1$ such that $M/N \cong S^k$. By applying [Lemma 2.6\(2\)](#) repeatedly, we have $(p+1)/2p < d_p(S)^k \leq d_p(S)$. (Note that p divides $|S|$.) Let $T \in \text{Syl}_p(S)$ and let $H = N_S(T)$. Clearly T is abelian, so by [Lemma 2.1\(1\)](#), H controls S -fusion in T . Thus $x^S \cap T = x^H \subseteq T$ for every $x \in T$. Since S is nonabelian simple, $\mathbf{Z}_p^*(S) = 1$. Now [Lemmas 2.3](#) and [2.5](#) together with

Glauberman's Z_p^* -theorem imply that $|x^S \cap T| \geq 2$ for all $1 \neq x \in T$. It follows that $|T| - 1 \geq 2(k_p(S) - 1)$. This implies that $k_p(S) \leq (|T| + 1)/2$ and hence

$$(p + 1)/2p < d_p(S) \leq (|T| + 1)/(2|T|) \leq (p + 1)/2p$$

as $|T| \geq p$. This contradiction shows that G is p -solvable.

By Hall–Higman Lemma 1.2.3 ([Isaacs 2008, Lemma 3.21]) and the fact that P is abelian, we have $P \leq C_G(O_p(G)) \leq O_p(G)$, so

$$P = O_p(G) \trianglelefteq G.$$

Let P/N be a chief factor of G . Assume that N is nontrivial. Then

$$(p + 1)/2p < d_p(G) \leq d_p(G/N)$$

and so by induction G/N has a normal p -complement K/N . However, as $G = O^p(G)$, $G = K$ which is impossible. So, we can assume that P is an elementary abelian minimal normal p -subgroup of G .

If $|x^G| = 1$ for some $1 \neq x \in P$, then $x \in Z(G) \cap P$ which forces $P = \langle x \rangle \subseteq Z(G)$. In this case, G has a normal p -complement by Lemma 2.1(2). Hence we can also assume that $|x^G| \geq 2$ for all $1 \neq x \in P$ whence $k_p(G) \leq (|P| + 1)/2$. Since $d_p(G) > (p + 1)/2p$,

$$|P|(p + 1)/2p < (|P| + 1)/2.$$

However, this inequality cannot occur as $|P| \geq p$. □

Proof of Theorem E. Let π be a nonempty set of odd primes and let p be the smallest member in π . Let G be a finite group with $d_\pi(G) > (p + 1)/2p$. For every $r \in \pi$, we see that

$$(r + 1)/2r \leq (p + 1)/2p < d_\pi(G) \leq d_r(G)$$

by Lemma 2.6(1), so $d_r(G) > (r + 1)/2r$. By Theorem D, G has a normal r -complement and an abelian Sylow r -subgroup. It follows that G has a normal π -complement $N = O_{\pi'}(G) \trianglelefteq G$ and G is π -solvable. By [Isaacs 2008, Theorem 3.20], G has a Hall π -subgroup H . Clearly $G = HN$ and $G/N \cong H$. Since

$$(p + 1)/2p < d_\pi(G) \leq d_\pi(H)d_\pi(N) \leq d_\pi(H) = d(H),$$

we deduce that

$$d(H) > (p + 1)/2p > 1/p \geq 1/r$$

and so by Lemma 2.6(4), H has a normal Sylow r -subgroup. It follows that H is nilpotent and thus H is abelian. □

Proof of Theorem F. Let G be a finite group with a Sylow p -subgroup P , where p is an odd prime. Suppose that $d_p(G) = (p + 1)/2p$ and $O_{p'}(G) = 1$. By Lemma 4.1,

P is abelian and thus by Lemma 2.1(1), $d_p(N_G(P)) = d_p(G) = (p+1)/2p$. By Theorem 7.4.4 in [Gorenstein 1968], we have $P = P \cap N' \times P \cap Z(N)$, where $N = N_G(P)$. Set $Z = P \cap Z(N)$ and $U = P \cap N'$. We have that $k_p(N) \leq (|P| + |Z|)/2$ as for any $x \in P \setminus Z$, $|x^N| \geq 2$. It follows that

$$(p+1)/2p \leq (|P| + |Z|)/2|P|$$

and thus $|P : Z| \leq p$. Clearly, $|P : Z| > 1$ as otherwise $P \subseteq Z(N)$ and thus G has a normal p -complement which forces $G = P$ since $\mathcal{O}_{p'}(G) = 1$. But then $d_p(G) = 1$, a contradiction. Thus $|P : Z| = p$; hence $|U| = p$ and $|Z| = |P|/p$. Moreover $|x^N| = 2$ for every $x \in P \setminus Z$. Set $U = \langle y \rangle$. Then $C_G(P) = C_N(y) \trianglelefteq N$. As $|y^N| = 2$, we have $|N : C_G(P)| = 2$. Since $U = P \cap N'$ has order p , we see that $U = [P, N]$. Furthermore, by Theorem 7.4.4 in [Gorenstein 1968], $P \cap G' = P \cap N' = U$ is cyclic of order p and by Theorem 5.18 in [Isaacs 2008], $G' \cap Z = 1$.

Let $F^*(G)$ be the generalized Fitting subgroup of G . Then $F^*(G) = F(G)E(G)$ is the central product of the Fitting subgroup $F(G)$ and the layer $E(G)$ of G , which is the product of all components of G , that is, subnormal quasisimple subgroups of G . Bender's theorem ([Isaacs 2008, Theorem 9.8]) says that $C_G(F^*(G)) \subseteq F^*(G)$.

Assume first that $E(G) = 1$. Then $F^*(G) = F(G)$ is a p -group since $\mathcal{O}_{p'}(G) = 1$. As $C_G(F(G)) \subseteq F(G)$ and P is abelian, $P = F(G)$ and thus $P = C_G(P) \trianglelefteq G$. It follows that $|G : P| = 2$ and $G' = [G, P] = U$ is cyclic of order p ; moreover $Z = Z(G) \cap P = Z(G)$ as P is self-centralizing. Now G/G' is an abelian group of order $2|Z|$. Hence G/G' has a normal Sylow 2-subgroup A/G' and a normal Sylow p -subgroup $P/G' = ZG'/G'$, so $G/G' = A/G' \times ZG'/G'$ which implies that $G = Z \times A$, where A is a nonabelian group of order $2p$ and it has a normal cyclic Sylow p -subgroup of order p . It is easy to see that $A \cong D_{2p}$, the dihedral group of order $2p$.

Assume now that $E := E(G)$ is nontrivial. Since $G' \cap P$ is cyclic of order p , the center of E is either trivial or cyclic of order p . If $|Z(E)| = p$, then $E/Z(E)$ is a p' -group which is impossible by Corollary 5.4 in [Isaacs 2008]. Thus $Z(E) = 1$. Hence E has a Sylow p -subgroup of order p which forces E to be a nonabelian simple group. Now we have that $F^*(G) = E \times F$ where $F = F(G) \leq P$ is a p -subgroup. Since $E \cap P \leq G' \cap P = U$ is of order p , we have $U = E \cap P = G' \cap P$. Therefore, $F \cap G' \leq F \cap G' \cap P = E \cap P \cap F = 1$, so $F \cap G' = 1$, whence $F \leq Z(G)$. Hence $C_G(E) = F = Z(G)$ and so G/F is an almost simple group with socle isomorphic to E . Since E has a cyclic Sylow p -subgroup of order p , we deduce from Lemma 2.3 in [Tong-Viet 2018] that $|\text{Out}(E)|$ is prime to p . In particular, G/EF is a solvable p' -group. Thus G/E has a central Sylow p -subgroup $EF/E \cong F$. By Lemma 2.1(2), G/E has a normal p -complement A/E and $G/E = EF/E \times A/E$. Since $E \cap F = 1$, we have $G = A \times F$ and so A is almost simple with socle E . $\square$

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HUNG P. TONG-VIET
 DEPARTMENT OF MATHEMATICAL SCIENCES
 BINGHAMTON UNIVERSITY
 BINGHAMTON, NY 13902-6000
 UNITED STATES
tongviet@math.binghamton.edu

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Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

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qing@cats.ucsc.edu

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