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YAEL KARSHON AND SUSAN TOLMAN

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TOPOLOGY OF COMPLEXITY ONE QUOTIENTS

YAEL KARSHON AND SUSAN TOLMAN

We describe of the topology of the geometric quotients of $2n$-dimensional compact connected symplectic manifolds with $(n-1)$-dimensional torus actions. When the isotropy weights at each fixed point are in general position, the quotient is homeomorphic to a sphere.

1. Introduction

This paper is a byproduct of our work on the classification of complexity one Hamiltonian torus actions [Karshon 1999; Karshon and Tolman 2001; 2003; 2014; $\geq$ 2020], but, in fact, it relies only on elementary aspects of such actions. It is motivated by a number of recent works by toric topologists (specifically, the papers by Buchstaber and Terzić [2016; 2019a; 2019b] and by Ayzenberg [2018]) that explore the topology of the geometric quotients of manifolds with certain torus actions. Our purpose in this paper is to highlight topological aspects of related works in equivariant symplectic geometry and to illustrate how equivariant symplectic methods reproduce some of the recent results in toric topology and yield new examples.

Similar results were recently obtained by Hendrik Süß [2018] from the point of view of algebraic geometry.

The examples Buchstaber and Terzić studied include the quotient of the Grassmannian of complex 2-planes in $\mathbb{C}^4$ by its standard torus action, which they showed is homeomorphic to a five-dimensional sphere, and the quotient of the manifold of complete flags in $\mathbb{C}^3$ by its standard torus action, which they showed is homeomorphic to a four-dimensional sphere. We exhibit these examples as special cases of a more general phenomenon: for any Hamiltonian action of a torus T on a compact symplectic manifold M, if the reduced spaces over the interior of the momentum polytope are two-dimensional and those over the boundary are single points — this condition holds if and only if the dimension of the torus is one less than the dimension of the manifold and at each fixed point the isotropy weights

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are in general position — then the geometric quotient M/T is homeomorphic to a sphere.

2. Background and main result

Let T be a torus and $\mathfrak{t}^*$ the dual to its Lie algebra.

Let (M, ω) be a symplectic manifold with a T action and with a momentum map $\mu \colon M \to \mathfrak{t}^*$. Such an action is called *Hamiltonian*. We recall the definitions and properties of Hamiltonian torus actions in Appendix A. In particular, the momentum map μ is constant on T orbits, so it induces a map, which is sometimes called the *orbital momentum map*, on the geometric quotient,

$$\bar{\mu} \colon M/T \to \mathfrak{t}^*.$$

In this paper we always assume that M is compact[1] and connected. Since M is compact, the fixed set M^T is not empty. To see this, fix a vector $\xi \in \mathfrak{t}$ that generates a dense one-parameter subgroup. Any point $p \in M$ on which the function $\langle \mu(\cdot), \xi \rangle \colon M \to \mathbb{R}$ achieves its minimal value is a fixed point for the one-parameter subgroup, and hence for T.

Local normal form and the convexity package. The local structure of a Hamiltonian torus action is governed by the local normal form, which describes a neighbourhood of an orbit up to an equivariant symplectomorphism that preserves momentum maps. We recall the statement of the local normal form in an appendix.

We denote

$$\Delta := \operatorname{image} \mu.$$

We will need the following theorem and corollary.

Theorem 2.1 (convexity package). *Δ is a rational*[2] *convex polytope, and the map $\mu \colon M \to \Delta$ is open and has connected fibres.*

Corollary 2.2. *For any convex subset C of $\mathfrak{t}^*$, the preimage $\mu^{-1}(C)$ is connected.*

The local normal form is due to Guillemin and Sternberg [1984] and Marle [1985].

The convexity package is due to Guillemin–Sternberg and Atiyah. Relevant references include the papers [Guillemin and Sternberg 1982; Atiyah 1982; Condevaux et al. 1988; Lerman and Tolman 1997; Hilgert et al. 1993; Lerman et al. 1998; Bjorndahl and Karshon 2010; Birtea et al. 2008; 2009]. The corollary follows from the (convexity of C and Δ, hence) connectedness of $C \cap \Delta$ by the following

[1] Many of the results in this paper remain true when M is not necessarily compact but μ is proper as a map to some convex subset of $\mathfrak{t}^*$.

[2] "Rational" means that the facets have rational conormal vectors.

exercise in point set topology: Given an continuous open map with connected fibres, the preimage of any connected subset of the image is connected.

Principal orbit types over faces and in level sets; the complexity. We continue to assume that M is compact and connected. Let T_{eff} be the quotient of T by the kernel of the action. Because M is connected, it has a connected open dense subset where the action of T_{eff} is free. The formula for the momentum map implies that the affine span of the momentum image of M is a translation of the annihilator in $\mathfrak{t}^*$ of the Lie algebra of the kernel of the action. In particular,

$$\dim T_{\text{eff}} = \dim \Delta. \tag{2.3}$$

The action is *toric* if $\dim T_{\text{eff}} = \frac{1}{2} \dim M$. More generally, the *complexity* of the action is $\frac{1}{2} \dim M - \dim T_{\text{eff}}$; it measures how far the action is from being toric.

Lemma 2.4. *For every face*[3] F *of* Δ, *its preimage* M_F *in* M, *with the structures induced from* M, *is a compact connected symplectic manifold with a Hamiltonian* T *action.*

Proof. By the definition of "face", there exist $\xi \in \mathfrak{t}$ and $a \in \mathbb{R}$ such that $\langle \mu(p), \xi \rangle \geq a$ for all $p \in M$, with equality exactly if $p \in M_F$.

Given any $p \in M_F$, let $\mathfrak{h}$ be the Lie algebra of its stabilizer, and let $\eta_j \in \mathfrak{h}^*$ be the isotropy weights at p (see Appendix B). By the local normal form theorem, the fact that $\langle \mu(q), \xi \rangle \geq \langle \mu(p), \xi \rangle$ for all q near p implies that $\xi \in \mathfrak{h}$ and that $\langle \eta_j, \xi \rangle \geq 0$ for all j. The local normal form theorem then implies that the intersection of M_F with a neighbourhood of the orbit of p is a T invariant symplectic submanifold.

By Corollary 2.2, M_F is connected. □

Remark 2.5. Let K be the identity component of the kernel of the T action on M_F. By the definition of the momentum map, the affine span of F is a translation of the annihilator in $\mathfrak{t}^*$ of the Lie algebra of K. Moreover, M_F is a connected component of M^K, the set of points fixed by K, because the component of M^K containing M_F must lie in the preimage of the affine span of F. In particular, the preimage in M of any vertex of Δ is a component of the fixed point set M^T.

Lemma 2.6. *Given any face* F *and any fixed point* p *in the preimage* M_F, *the complexity of the* T *action on* M_F *is the number of isotropy weights at* p *that are parallel to* F *minus the dimension of* F. *Moreover, the linear span of the weights that are parallel to* F *is a translation of the affine span of* F.

Proof. By Lemma 2.4, the preimage M_F of F in M is a compact connected symplectic manifold with a Hamiltonian T action. Let K be the identity component

[3]Because the convex set Δ is locally polyhedral, a subset F of Δ is a face if and only if it is equal either to Δ or to the intersection of Δ with a supporting hyperplane (a hyperplane that meets Δ and such that one of the two closed half-spaces that it bounds contains Δ).

of the kernel of the T action on M_F. By Remark 2.5, the affine span of F is a translation of the annihilator in $\mathfrak{t}^*$ of the Lie algebra of K, and M_F is a connected component of M^K. Hence, the dimension of T/K is the dimension of F, and the weights for the action on $T_p M_F$ are those weights for the action on $T_p M$ that annihilate the Lie algebra of K, or equivalently, are parallel to F. Therefore, the dimension of M_F is twice the number of such weights.

Finally, by the local normal form theorem, there is a neighbourhood of p in M_F that is equivariantly symplectomorphic to $T_p M_F$. Since K is the identity component of the stabilizer of an open dense set of points in M_F, the identity component of the kernel of the isotropy representation on $T_p M_F$ is also K. Hence, the isotropy weights at p span the annihilator in $\mathfrak{t}^*$ of the Lie algebra of K. □

Corollary 2.7. *Let M_F and $M_{F'}$ be the preimage of faces F and F' of Δ, respectively. If $F \subseteq F'$, then the complexity of M_F is less than or equal to the complexity of $M_{F'}$.*

Proof. By Lemma 2.4, M_F and $M_{F'}$ are compact connected symplectic manifolds with Hamiltonian T actions. Consider a fixed point $p \in M_F$. Since the linear span of the isotropy weights at p that are parallel to F' is a translation of the affine span of F', the number of weights that are parallel to F' but not F must be greater than or equal to the codimension of F in F'. □

Given a point $\beta \in \mathfrak{t}^*$, let $M_\beta := \bar{\mu}^{-1}(\{\beta\}) = \mu^{-1}(\{\beta\})/T$ be the *reduced space* at β. If T_{eff} acts freely on $\mu^{-1}(\{\beta\})$, then M_β is naturally a manifold. More generally, the following holds.

Lemma 2.8. *Given a point β in the relative interior of Δ, the set of free orbits in the reduced space M_β is a connected open dense subset of M_β; moreover, it is naturally*[4] *a $2k$-dimensional manifold, where k is the complexity of the T action on M.*

Proof. This consequence of the local normal form theorem and the convexity package is proved by Sjamaar and Lerman [1991]. □

The *dimension* of a reduced space M_β is the dimension of an open dense subset of M_β that is a manifold; it is well defined, by Lemmas 2.4 and 2.8. For any nonnegative integer k, denote by Δ_k the set of points β in Δ such that $\dim M_\beta = 2k$, and denote $\Delta_{\leq k} := \Delta_0 \cup \cdots \cup \Delta_k$. By the connectedness of the momentum map fibres, Δ_0 is the set of points β in Δ such that the reduced space M_β consists of a single orbit.

[4]Explicitly, there exists a unique manifold structure on the set of free orbits in M_β such that a real valued function on this set is smooth if and only if its pullback to the preimage in $\mu^{-1}(\{\beta\})$ extends to a smooth function on an open subset of M.

Lemma 2.9. *For any nonnegative integer k, the set $\Delta_{\leq k}$ is a union of faces of Δ. Consequently, there exists an open convex subset U of $\mathfrak{t}^*$ such that $\Delta \smallsetminus \Delta_{\leq k} = \Delta \cap U$.*

Proof. By Lemma 2.4, the preimage $M_F := \mu^{-1}(F)$ of each face F of Δ is a compact connected symplectic manifold with a Hamiltonian T action. Hence, by Lemma 2.8, each Δ_k is the union of the relative interiors of those faces F for which the complexity of M_F is equal to k. The first claim then follows from Corollary 2.7.

To prove the second claim, for each face F in $\Delta_{\leq k}$ choose a supporting hyperplane H_F of Δ such that $F = H_F \cap \Delta$. Then the intersection U of the appropriate open half-spaces bound by these hyperplanes is an open convex set. □

Remark 2.10 (toric manifolds). If we assume that the T action on M is toric, then the quotient M/T is homeomorphic to the disk D^n, where $n = \frac{1}{2} \dim M$. To see this, first note that Lemma 2.4 and Corollary 2.7 together show that the preimage $M_F := \mu^{-1}(F)$ of each face F of Δ is a symplectic toric manifold. Hence, by Lemma 2.8, the reduced space M_β is a point for all $\beta \in \Delta$, that is, $\Delta_0 = \Delta$. Thus, the orbital momentum map $\bar{\mu}\colon M/T \to \Delta$ is a bijection; since it is proper and continuous, this implies that it is a homeomorphism. Since Δ is a convex polytope, this proves the claim.

More generally, consider a complete unimodular fan in $\mathbb{R}^n$. Even if the fan does not correspond to any convex polytope, we can construct a complex toric manifold M from the fan, as described by Audin [2004]. The geometric quotient M/T is still homeomorphic to a sphere; see [Karshon and Tolman 1993, Lemma 3.2].

A collection of vectors in the vector space $\mathfrak{t}^*$ is *in general position* if every subcollection of size $< \dim \mathfrak{t}^*$ is linearly independent.

Lemma 2.11. *Assume that M is compact.*

(1) *Assume that there exists an isolated fixed point in M whose momentum image is a vertex of Δ; in particular, this holds if the fixed points in M are isolated. Then $\Delta_0 \neq \varnothing$.*

(2) *Assume that the T action on M has complexity ≥ 1 and that the isotropy weights at every fixed point are in general position. Then $\Delta_0 = \partial\Delta$.*

Proof. Part (1) is a consequence of the following two facts. First, since M is compact, its momentum image Δ has a vertex. Second, by Remark 2.5, the preimage of any vertex of Δ is a connected component of the fixed point set M^T.

We now prove Part (2). First, consider $\beta \in \partial\Delta$. Let $F \subsetneq \Delta$ be the face whose relative interior contains β. By Lemma 2.4, the preimage M_F of F in M is a compact connected symplectic T manifold with a Hamiltonian T action. So it has a fixed point p. Since $\dim F < \dim \mathfrak{t}^*$ and the isotropy weights at p are in general position, Lemma 2.6 implies that M_F is toric. Therefore, by Lemma 2.8, $\beta \in \Delta_0$.

In contrast, if β is in the relative interior of Δ then, since the action of T on M is not toric, Lemma 2.8 implies that β is not in Δ_0. $\square$

Remark 2.12. In Part (2) of Lemma 2.11, if the complexity of the T action on M is *equal* to one, then the converse is true too, so $\Delta_0 = \partial\Delta$ if and only if the isotropy weights at every fixed point are in general position.

When the complexity of the Hamiltonian T action is equal to one, we denote by Δ_{short} the set of points in Δ whose reduced space contains a single orbit and by Δ_{tall} the set of points in Δ whose reduced space is two-dimensional. Thus,

$$\Delta_{\text{short}} = \Delta_0 \quad \text{and} \quad \Delta = \Delta_{\text{short}} \sqcup \Delta_{\text{tall}}.$$

By Lemma 2.9, Δ_{short} is closed,

Proposition 2.13. *Let T be a torus and $\mathfrak{t}^*$ the dual to its Lie algebra. Let M be a compact connected symplectic manifold with a T action and with a momentum map $\mu\colon M \to \mathfrak{t}^*$ with image Δ. Assume that the action has complexity one.*

Then there exists a connected closed oriented surface Σ and a homeomorphism

$$(M/T)_{\text{tall}} \to \Delta_{\text{tall}} \times \Sigma$$

that intertwines the orbital momentum map $\bar{\mu}$ with the projection map to Δ_{tall}.

If Δ_{short} is nonempty, then Σ is a two-sphere.

Proof. By Lemma 2.9, there exists a convex open subset U of $\mathfrak{t}^*$ such that $\Delta_{\text{tall}} = \Delta \cap U$. The first part of Proposition 2.2 of [Karshon and Tolman 2003] then implies that there is a homeomorphism $(M/T)_{\text{tall}} \to \Delta_{\text{tall}} \times \Sigma$ as required. By [Karshon and Tolman 2001, Lemma 5.7], if Δ_{short} is nonempty, then Σ is a sphere. $\square$

We now state our main theorem.

Theorem 2.14. *Let T be a torus and $\mathfrak{t}^*$ the dual to its Lie algebra. Let M be a $2n$-dimensional compact connected symplectic manifold with a T action and with a momentum map $\mu\colon M \to \mathfrak{t}^*$ with image Δ. Assume that the action has complexity one. Then there exist a connected closed oriented surface Σ and a homeomorphism*

$$M/T \to (\Delta \times \Sigma)/\sim,$$

where $\sim$ is the finest equivalence relation with $(x, y) \sim (x, y')$ if $x \in \Delta_{\text{short}}$. Moreover,

(i) *If Δ_{short} is nonempty, then Σ is a two-sphere.*

(ii) *If $\Delta_{\text{short}} = \partial\Delta$, then M/T is homeomorphic to the $(n+1)$-sphere.*

Proof. By Proposition 2.13, there exists a connected closed oriented surface Σ and a homeomorphism

$$(M/T)_{\text{tall}} \to \Delta_{\text{tall}} \times \Sigma$$

that intertwines the orbital momentum map $\bar{\mu}$ and the projection map to Δ_{tall}. Since $\Delta = \Delta_{\text{short}} \sqcup \Delta_{\text{tall}}$ and Δ_{short} consists of those β such that M_β consists of a single orbit, this homeomorphism extends to a unique bijection

$$f\colon M/T \to (\Delta \times \Sigma)/{\sim}$$

that intertwines the orbital momentum map $\bar{\mu}$ with the map $\pi : (\Delta \times \Sigma)/{\sim} \to \mathfrak{t}^*$ induced by the projection to Δ. Since $(M/T)_{\text{tall}}$ is open in M/T and Δ_{tall} is open in Δ, the map f is continuous and open at every point of $(M/T)_{\text{tall}}$.

Since M and Σ are compact, the maps $\bar{\mu}\colon M/T \to \mathfrak{t}^*$ and $\pi : (\Delta \times \Sigma)/{\sim} \to \mathfrak{t}^*$ are proper. Since $\mathfrak{t}^*$ is a locally compact Hausdorff space, the proper maps $\bar{\mu}$ and π to $\mathfrak{t}^*$ are closed. Since π is closed, f is continuous at every point of $(M/T)_{\text{short}}$. Since $\bar{\mu}$ is closed and f is onto, f is open at every point of $(M/T)_{\text{short}}$.

Part (i) follows from the last claim of Proposition 2.13.

We now prove Part (ii). Since M is compact and connected, Δ is a convex polytope; hence, it is homeomorphic to D^{n-1}, where $\dim M = 2n$. Therefore, the map from $D^{n-1} \times S^2$ that sends (x, z) to $(x, \sqrt{1 - |x|^2}\, z)$ induces a continuous proper map from $(\Delta \times \Sigma)/{\sim}$ to S^{n+1}. If $\Delta_{\text{short}} = \partial\Delta$, this map is a bijection. Since S^{n+1} is a locally compact Hausdorff space, being a continuous proper bijection implies that this map is a homeomorphism. □

Remark 2.15. Part (ii) of Theorem 2.14 can be rephrased as follows: If $\Delta_{\text{short}} = \partial\Delta$, then M/T is homeomorphic to the join $\partial\Delta * S^2$. To see this, recall that the join $A * B$ of two topological spaces A and B is the quotient of $A \times B \times [0, 1]$ under the identifications $(a, b, 0) \sim (a', b, 0)$ and $(a, b, 1) \sim (a, b', 1)$ for all $a, a' \in A$ and $b, b' \in B$. We may assume without loss of generality that $0 \in \text{interior}\,\Delta$. Then, since Δ is convex, the map $\partial\Delta \times B \times [0, 1] \to \Delta \times B$ that is defined by $(a, b, t) \mapsto (ta, b)$ descends to a continuous proper bijection $\partial\Delta \times B \to (\Delta \times B)/\sim$, where here $\sim$ is the finest equivalence relation with $(x, y) \sim (x, y')$ if $x \in \partial\Delta$. When B is a locally compact Hausdorff space, this bijection is a homeomorphism.

Corollary 2.16. *Let T be a torus and $\mathfrak{t}^*$ the dual to its Lie algebra. Let M be a compact connected symplectic manifold with a T action and a momentum map $\mu\colon M \to \mathfrak{t}^*$ with image Δ. Assume that the action has complexity one.*

(a) *Assume that there exists an isolated fixed point in M whose momentum image is a vertex of Δ; in particular, this holds if the fixed points in M are isolated. Then M/T is homeomorphic to $(\Delta \times S^2)/{\sim}$, where $\sim$ is the finest equivalence relation with $(x, y) \sim (x, y')$ if $x \in \Delta_{\text{short}}$.*

(b) *Assume that the isotropy weights at every fixed point are in general position. Then M/T is homeomorphic to a sphere.*

Proof. Part (a) follows from Part (1) of Lemma 2.11 and Part (i) of Theorem 2.14. Part (b) follows from Part (2) of Lemma 2.11 and Part (ii) of Theorem 2.14. □

In Part (b) of Corollary 2.16, the fact that M/T is a topological manifold already follows from a result of Ayzenberg [2018]. Ayzenberg's work also implies that if the action extends to a toric action then M/T is homeomorphic to a sphere.

Corollary 2.17. *Let the circle S^1 act on a compact connected symplectic four-manifold (M, ω) with momentum map $\mu\colon M \to \mathbb{R}$. Then exactly one of the following is true.*

(1) *The fixed point set is finite and M/T is homeomorphic to a three-sphere.*

(2) *The fixed point set contains one surface, which is a sphere, and M/T is homeomorphic to a three-disk.*

(3) *The fixed point set contains two surfaces that have the same genus g, and M/T is homeomorphic to $[0, 1] \times \Sigma$, where Σ is a surface of genus g.*

Proof. By rescaling ω if necessary, we may assume that the momentum image is the interval $[0, 1]$. Since 0 and 1 are vertices of $[0, 1]$, Lemma 2.4 and Remark 2.5 imply that each of $\mu^{-1}(\{0\})$ and $\mu^{-1}(\{1\})$ is a connected component of the fixed point set that is either a single point or a fixed surface. By the local normal form theorem, a fixed point that is not isolated is a local minimum or local maximum of the momentum map; since by the convexity package the momentum map is open as a map to its image $[0, 1]$, such a fixed point must be mapped to 0 or to 1. Hence, there are at most two components of the fixed point set that are not isolated fixed points, and each of them is mapped to 0 or to 1.

Assume first that the fixed point set contains no surfaces. Then the fixed points are isolated, and so none of the isotropy weights at any fixed point are zero. Hence, M/T is homeomorphic to a three-sphere by Part (b) of Corollary 2.16.

Assume now that the fixed point set contains exactly one surface Σ. By replacing ω by $-\omega$ if necessary, we may assume that $\mu(\Sigma) = 1$. Since Σ is the only fixed surface, $\mu^{-1}(\{0\})$ is an isolated fixed point. Hence, $\Delta_{\text{short}} = \{0\}$. By Part (a) of Corollary 2.16, this implies the M/S^1 is homeomorphic to $[0, 1] \times S^2/{\sim}$, where $\sim$ is the finest equivalence relation such that $(0, x) \sim (0, x')$. Define a map

$$[0, 1] \times S^2 \to \mathbb{R}^3, \quad (t, x) \mapsto tx,$$

where we identify S^2 with the unit sphere in $\mathbb{R}^3$. This induces a homeomorphism from $[0, 1] \times S^2/{\sim}$ to the three-disk D^3, and hence from M/T to D^3.

Finally, assume that the fixed point set contains two surfaces, Σ and Σ'. By the first paragraph, we may assume that $\mu^{-1}(\{0\}) = \Sigma$ and $\mu^{-1}(\{1\}) = \Sigma'$. Hence, Δ_{short} is empty, and so Theorem 2.14 implies that M/T is homeomorphic to $[0, 1] \times \Sigma_g$ for some oriented surface Σ_g. □

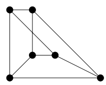
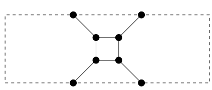
	M	$\dim_{\mathbb{R}} M$	T	complexity	$M/T \overset{\text{homeo}}{\cong}$
(1)	$G_2(\mathbb{C}^4) = \{E^2_{\mathbb{C}} \subset \mathbb{C}^4\}$	8	$(S^1)^4/\mathrm{diag}$	1	S^5
(2)	$F_3 = \{L^1_{\mathbb{C}} \subset E^2_{\mathbb{C}} \subset \mathbb{C}^3\}$	6	$(S^1)^3/\mathrm{diag}$	1	S^4
(3)	$G_2^+(\mathbb{R}^5) = \{E^2_{\text{oriented}} \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}\}$	6	$(S^1)^2$	1	S^4
(4)		6	$(S^1)^2$	1	S^4
(5)	$S^1 \circlearrowright (S^2)^2$ $a\cdot(u, v) = (a\cdot u, a\cdot v)$	4	S^1	1	S^3
(6)	$S^1 \circlearrowright \mathbb{CP}^2$ $a\cdot[z_0 : z_1 : z_2] = [az_0 : z_1 : z_2]$	4	S^1	1	D^3
(7)	$S^1 \circlearrowright S^2 \times \Sigma_g$	4	S^1	1	$I \times \Sigma_g$
(8)	$S^1 \times S^1 \circlearrowright (S^2)^3$ $(a, b)\cdot(u, v, w) = (a\cdot u, a\cdot v, b\cdot w)$	6	$S^1 \times S^1$	1	$S^3 \times I$
(9)		6	$(S^1)^2$	1	$I \times I \times \Sigma_g$
(10)	$\mathbb{CP}^5 = \mathbb{P}\big(\bigwedge^2 \mathbb{C}^4\big)$	10	$(S^1)^4/\mathrm{diag}$	2	$S^2 * \mathbb{CP}^2$ (†)
(11)	$G_2(\mathbb{C}^5) = \{E^2_{\mathbb{C}} \subset \mathbb{C}^5\}$	12	$(S^1)^5/\mathrm{diag}$	2	(‡)

Table 1. Examples of geometric quotients. For (†), see [Buchstaber and Terzić 2016]. For (‡), see [Buchstaber and Terzić 2019b; Süß 2019].

3. Examples

In Table 1 we list some examples of symplectic torus actions and their geometric quotients.

We now discuss these examples and give some references.

(1) Let M be the Grassmannian of complex 2-planes in $\mathbb{C}^4$, with the three-dimensional torus action induced from the standard action of $(S^1)^4$ on $\mathbb{C}^4$. Then M/T is homeomorphic to the sphere S^5; this is shown in [Buchstaber and Terzić 2016] and revisited in [Buchstaber and Terzić 2019a, Section 10]. Alternatively, we can identify M equivariantly with a coadjoint orbit in $\mathrm{SU}(4)$, where T is a maximal torus acting through the coadjoint action. There is a natural symplectic structure on every coadjoint orbit of any Lie group, and the coadjoint action is Hamiltonian.

Hence, since the isotropy weights at each fixed point are in general position, we can apply Corollary 2.16 and conclude that M/T is homeomorphic to the sphere S^5.

(2) Let M be the manifold of complete complex flags in $\mathbb{C}^3$, with the two-dimensional torus action that is induced from standard action of $(S^1)^3$ on $\mathbb{C}^3$. Then M/T is homeomorphic to the sphere S^4; this is shown in [Buchstaber and Terzić 2019a]. Alternatively, M is a coadjoint orbit of SU(3), and so — as in the previous example — M/T is homeomorphic to the sphere S^4 by Corollary 2.16.

(3) Let M be the Grassmannian of oriented (real) 2-planes in $\mathbb{R}^5 \cong (\mathbb{R}^2)^2 \times \mathbb{R}$, with the two-dimensional torus action that is induced from the standard action of $(S^1)^2$ on the $(\mathbb{R}^2)^2$ factor. By identifying M with a coadjoint orbit of SO(5), we obtain a symplectic form such that the action is Hamiltonian. Since the isotropy weights at each fixed point are in general position, M/T is homeomorphic to the sphere S^4 by Corollary 2.16. For more details, see, e.g., [Karshon and Tolman 2001, Section 14].

(4) Let (M, ω, μ) be the compact symplectic six manifold with Hamiltonian $(S^1)^2$ action constructed in [Tolman 1998]. The picture drawn in the table shows the momentum map images of the orbit type strata. The solid dots are the images of isolated fixed points, and the segments are the images of 2-spheres with circle stabilizer. As the second author showed in [Tolman 1998], M does not admit any invariant Kähler structure. As in the previous examples, M/T is homeomorphic to the sphere S^4 by Corollary 2.16.

(5) Let M be the product of the two-sphere S^2 with itself. There is a standard area form on S^2; the height function is a momentum map for the circle action that rotates the sphere around the vertical axis. Take the product symplectic form on M; then the momentum map for the diagonal circle action sends (u, v) to the sum $u_3 + v_3$. By [Ayzenberg 2018], M/T is homeomorphic to the sphere S^3. Alternatively, this follows from Corollary 2.17 or from Corollary 2.16.

(6) Let $M = \mathbb{CP}^2$, with the Fubini–Study symplectic form, the circle action given by $a \cdot [z_0 : z_1 : z_2] = [az_0 : z_1 : z_2]$, and momentum map

$$[z_0 : z_1 : z_2] \mapsto \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

By Corollary 2.17, M/T is homeomorphic to the disc D^3.

(7) Let $M = \Sigma_g \times S^2$, where Σ_g is a surface of genus g, with the circle acting on the second factor, a product symplectic form, and momentum map $(u, v) \mapsto v_3$. Then M/T is homeomorphic to $I \times \Sigma_g$, where I is a closed interval. This follows from Corollary 2.17 and is also easy to see directly.

Let $\widehat{M}$ be an equivariant symplectic blowup of M at a fixed point. Then $\widehat{M}$ has one isolated fixed point and two fixed surfaces of genus g. The quotient $\widehat{M}/T$ is still homeomorphic to $I \times \Sigma_g$.

(8) Let $M = (S^2)^3$ with the product symplectic form, the $S^1 \times S^1$ action

$$(a, b) \cdot (u, v, w) = (a \cdot u, a \cdot v, b \cdot w),$$

and momentum map $(u, v, w) \mapsto (u_3 + v_3, w_3)$. The momentum image Δ is the rectangle $[-2, 2] \times [-1, 1]$, and $\Delta_{\text{short}} = \{-2, 2\} \times [-1, 1]$. By Theorem 2.14, this implies that M/T is homeomorphic to $S^3 \times I$. Alternatively, this follows from the facts that $S^2/S^1 \simeq I$ and, as we saw in (5), that $(S^2)^2/S^1 \simeq S^3$. Note that in this example $\varnothing \neq \Delta_{\text{short}} \subsetneq \partial\Delta$.

(9) Let Σ_g be a surface of genus g. Let (M, ω, μ) be any one of the compact symplectic six-manifolds with Hamiltonian $(S^1)^2$ action and reduced spaces homeomorphic to Σ_g that are described in [Karshon and Tolman 2014, Example 1.11]. (If $g > 0$, there is an infinite number of isomorphism classes of such manifolds even if we fix the Duistermaat–Heckman measure.) As is (4), the solid dots are the momentum map images of isolated fixed points, and the segments are the momentum map images of 2-spheres with circle stabilizer. The momentum image Δ is the closed rectangle whose boundary is marked by dashed lines, and Δ_{short} is empty. Theorem 2.14 implies that M/T is homeomorphic to $I \times I \times \Sigma_g$.

(10) Let M be the projective space $\mathbb{CP}^5$, with the three-dimensional torus action induced by the $(S^1)^4$ action on $\bigwedge^2\mathbb{C}^4 \cong \mathbb{C}^6$, which itself is induced by the standard action on $\mathbb{C}^4$. Corollary 12 in [Buchstaber and Terzić 2016, §10] states that the quotient M/T is homeomorphic to the join $S^2 * \mathbb{CP}^2$.

(11) Let M be the Grassmannian of two-planes in $\mathbb{C}^5$, with the four-dimensional torus action induced from the standard action of $(S^1)^5$ on $\mathbb{C}^5$. The quotient M/T was studied by Buchstaber and Terzić [2019b] and Süß [2019].

Appendix A: Hamiltonian *T* actions

A torus T is a Lie group that is isomorphic to $(S^1)^r$ for some nonnegative integer r. A symplectic manifold is a manifold M equipped with a differential two-form ω that is closed and nondegenerate. A momentum map is a map from the manifold to the dual of the Lie algebra of the torus such that, for every element ξ of the Lie algebra $\mathfrak{t}$ of the torus, the corresponding vector field ξ_M on M (whose value at a point $x \in M$ is $\xi_M|_x = \frac{d}{dt}|_{t=0} \exp(t\xi) \cdot x$) and the corresponding component of the momentum map $\mu^\xi : M \to \mathbb{R}$ (whose value at a point $x \in M$ is $\langle \mu(x), \xi \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$) are related by Hamilton's equations

$$d\mu^\xi = -\iota(\xi_M)\omega \quad \text{for all } \xi \in \mathfrak{t} \tag{A.1}$$

(where $\iota(\xi_M)\omega(v) = \omega(\xi_M, v)$ for any $v \in TM$). We then call the T action *Hamiltonian*.

If M is connected, then the affine span of the momentum image $\mu(M)$ is a translate of the annihilator of the Lie algebra of the kernel of the action. This is a consequence of Hamilton's equations (A.1).

The symplectic form is T invariant. We recall why. For any $\xi \in \mathfrak{t}$, the Lie derivative of ω along ξ_M satisfies $L_{\xi_M}\omega = d\iota(\xi_M)\omega + \iota(\xi_M)\, d\omega$; the first summand vanishes because (by Hamilton's equation) $\iota(\xi_M)\omega$ is exact; the second summand vanishes because (by assumption) ω is closed.

The momentum map is constant on orbits. We recall why. For any $\xi, \eta, \zeta \in \mathfrak{t}$ we have $L_{\xi_M}(\omega(\eta_M, \zeta_M)) = (L_{\xi_M}\omega)(\eta_M, \zeta_M) + \omega([\xi_M, \eta_M], \zeta_M) + \omega(\eta_M, [\xi_M, \zeta_M])$; the first summand vanishes because the symplectic form is T invariant; the second and third summands vanish because T is abelian. Hence, $\omega(\eta_M, \zeta_M)$ is constant along T orbits. By Hamilton's equation, $\omega(\eta_M, \zeta_M) = L_{\eta_M}\mu^{\zeta_M}$; because for each T orbit the right hand vanishes at the point on the (compact) orbit where μ^{ζ_M} attains its maximum, $L_{\eta_M}\mu^{\zeta_M} = 0$. Because $\eta \in \mathfrak{t}$ is arbitrary, μ^{ζ_M} is constant along T orbits; because $\zeta \in \mathfrak{t}$ is arbitrary, μ is constant along T orbits.

Appendix B: Local normal form

A *Hamiltonian T model* is a Hamiltonian T-manifold (Y, ω_Y, μ_Y) that is obtained by the following construction. Let a closed subgroup H of T act on $\mathbb{C}^\ell$ through a homomorphism $H \to (S^1)^\ell$ followed by the standard action of $(S^1)^\ell$ on $\mathbb{C}^\ell$; the corresponding quadratic momentum map $\mu_H \colon \mathbb{C}^\ell \to \mathfrak{h}^*$ is

$$z \mapsto \sum_{j=1}^{\ell} \frac{|z_j|}{2}\eta_j,$$

where $\eta_1, \dots, \eta_\ell \in \mathfrak{h}^*_{\mathbb{Z}}$ are the weights for the H action on $\mathbb{C}^\ell$. Take Y to be the manifold $T \times_H (\mathfrak{h}^0 \times \mathbb{C}^\ell)$, where $\mathfrak{h}^0$ is the annihilator in $\mathfrak{t}^*$ of the Lie algebra of H. Here, we quotient by the antidiagonal action of H, in which $a \in H$ acts on T by right multiplication by a^{-1}, it acts on $\mathfrak{h}^0$ trivially, and it acts on $\mathbb{C}^\ell$ through the given action. The torus T acts on Y by left multiplication on the T factor. The *central orbit* in the model Y is the orbit $[a, 0, 0]$.

Equip $(T \times \mathfrak{t}^*) \times \mathbb{C}^\ell$ with the product of the standard symplectic form on $T \times \mathfrak{t}^*$, viewed as the cotangent bundle of T, and the standard symplectic form $\sum_{j=1}^{\ell} dx_j \wedge dy_j$ on $\mathbb{C}^\ell$. The pullback of this symplectic form under the inclusion map from $T \times \mathfrak{h}^0 \times \mathbb{C}^\ell$ to $(T \times \mathfrak{t}^*) \times \mathbb{C}^\ell$ taking (a, ν, z) to $(a, \nu + \Phi_H(z), z)$ is equal to the pullback of the symplectic form ω_Y under the quotient map $T \times \mathfrak{h}^0 \times \mathbb{C}^\ell \to T \times_H (\mathfrak{h}^0 \times \mathbb{C}^\ell)$; this determines ω_Y. Here, we have identified $\mathfrak{t}^*$ with $\mathfrak{h}^0 \oplus \mathfrak{h}^*$.

The momentum map is $\mu_Y([a, \nu, z]) = \alpha + \nu + \mu_H(z)$ for some $\alpha \in \mathfrak{t}^*$, where μ_H is the quadratic momentum map for the linear H action.

Theorem B.1 (local normal form). *Let M be a symplectic manifold with a Hamiltonian T action. Then for every orbit in M there exists an equivariant symplectomorphism that preserves the momentum maps from a neighbourhood of the orbit in M to a neighbourhood of the central orbit in some Hamiltonian T model.*

In Theorem B.1, the H that appears in the Hamiltonian T model that corresponds to the orbit of a point p is the stabilizer of p. Moreover, the weights η_j that appear in the model are unique up to permutation; we call them the *isotropy weights* at p.

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YAEL KARSHON
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
TORONTO, ON
CANADA
karshon@math.toronto.edu

SUSAN TOLMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
URBANA, IL
UNITED STATES
tolman@illinois.edu

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