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We prove a version of a trace Paley–Wiener theorem for tempered representations of a reductive p-adic group. This is applied to complete certain investigations of Shahidi on the proof that a Plancherel measure is an invariant of an L-packet of discrete series.

1. Introduction

Let G be a reductive p-adic group. Let $\mathrm{Rep}(G)$ be the category of smooth admissible complex representations of G of finite length, and let $R(G)$ be the corresponding Grothendieck group. We write $\Psi(G)$ (resp., $\Psi^u(G)$) for the group (resp., unitary group) of unramified characters of G. The group $\Psi(G)$ has a structure of an algebraic variety (a complex tours). The corresponding algebra of regular functions $\mathbb{C}[\Psi(G)]$ is generated by evaluations on elements of G as a $\mathbb{C}$-algebra. The subgroup $\Psi^u(G)$ is Zariski dense in $\Psi(G)$. We say that a complex function is regular on $\Psi^u(G)$ if it is a restriction of a regular function on $\Psi(G)$. We observe that the restriction map from $\mathbb{C}[\Psi(G)]$ into functions on $\Psi^u(G)$ is injective since $\Psi^u(G)$ is Zariski dense in $\Psi(G)$.

We fix a minimal parabolic subgroup P_0, its Levi decomposition $P_0 = M_0U_0$, and, as usual related to these choices, we fix a set of standard parabolic subgroups $P = MU$, where $M_0 \subset M$, $P = MP_0$. Since the standard parabolic subgroup is determined by the choice of Levi subgroup, the normalized parabolic induction $\mathrm{Ind}_P^G(\sigma)$, where σ is a smooth representation of M, we write as usual $i_{GM}(\sigma)$.

In [Bernstein et al. 1986], Bernstein, Deligne, and Kazhdan proved a trace Paley–Wiener theorem for category $\mathrm{Rep}(G)$. We consider a full subcategory $\mathrm{Rep}_t(G)$ of $\mathrm{Rep}(G)$ consisting of representations having all irreducible subquotients tempered. Let $R_t(G)$ be the corresponding Grothendieck group. We write $R_t^i(G)$ for the subgroup of $R_t(G)$ generated by $i_{GM}(\sigma)$, where M ranges over all standard Levi subgroups of G (including G), and σ ranges over a set of square-integrable modulo center irreducible representations of M. We warn the reader that this notion is not an analogue of the notion of strictly induced modules from

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[Bernstein et al. 1986, §3.1]. An analogue would be the subgroup of $R_t(G)$ generated by $i_{GM}(\tau)$, where M ranges over all *proper* standard Levi subgroups of G, and τ ranges over irreducible tempered representations of M. But this is not useful for us in the present paper.

The main result of the present paper is the following version of a trace Paley–Wiener theorem:

Theorem 1.1. *Let $f : R_t(G) \to \mathbb{C}$ be a $\mathbb{Z}$-linear form such that the following hold*:

(i) *There exists an open compact subgroup $K \subset G$ which dominates f (i.e., f is nonzero only on those irreducible tempered representations which have a nontrivial space of K-invariant vectors).*

(ii) *For each standard maximal Levi subgroup M, or $M = G$, and a square-integrable modulo center representation σ of M, the function $\psi \mapsto f(i_{GM}(\psi\sigma))$ is regular on $\Psi^u(M)$, and for any other proper standard Levi subgroup N, and a square-integrable modulo center representation τ of N, we have $f(i_{GN}(\tau)) = 0$.*

Then, there exists $F \in C_c^\infty(G)$ such that

$$f(\pi) = \operatorname{tr}(\pi(F)) \quad \textit{for all } \pi \in R_t^i(G).$$

Theorem 1.1 is a proved by reduction to the main result of [Bernstein et al. 1986] using the Harish-Chandra theory of tempered representations [Waldspurger 2003] and some standard considerations related to the Langlands classification [Renard 2010, Chapter VII]. The proof is given in Section 3. It is a consequence of its effective version given by Proposition 3.4. Proposition 3.4 constructs a correct function needed in the proof of [Shahidi 1990, Proposition 9.3 2] in the case when M (see notation there) is a Levi subgroup of a maximal parabolic subgroup. We remark that since Plancherel factors are multiplicative, it is enough to prove [Shahidi 1990, Proposition 9.3 2] for a maximal Levi subgroup.

2. Preliminaries

We continue with the notation introduced in the introduction. Let M be a standard Levi subgroup. Then, we write $\Psi(M)^r$ for the group of all unramified characters ψ which are $\mathbb{R}_{>0}$-valued. As we stated in the introduction, every standard Levi subgroup M determines a unique standard parabolic subgroup, say P. We denote by $\Psi(M)^{r,+}$ the set of all characters from $\Psi(M)^r$ which correspond to the points of the (open) Weyl chamber determined by the roots of the split component of M which belong to the unipotent radical of P in the usual description of unramified characters (see, for example, [Muić 2008, Section 2]). If $M = G$, then $\Psi(M)^{r,+} = \Psi(M)^r$.

For a standard Levi subgroup M, an irreducible tempered representation π of M, and $\psi \in \Psi(M)^{r,+}$, the module $i_{GM}(\psi\pi)$ is called a standard module; it has a unique (Langlands quotient) $L(i_{GM}(\psi\pi))$. The condition is empty if $M = G$. By the Langlands classification [Renard 2010, Theorem VII.4.2], every irreducible representation can be expressed in the form $L(i_{GM}(\psi\pi))$ for unique such datum (M, π, ψ). The following standard result will be used in the proof:

Lemma 2.1. *The standard modules of G form a $\mathbb{Z}$-basis of $R(G)$.*

Proof. The proof is as in [Clozel 1986, Proposition 1]. □

In analogy with [Bernstein et al. 1986, §2.1], we make the following definitions.

Let $\sigma \in \mathrm{Irr}(M)$ where M is a standard Levi subgroup of G. We define the usual affine variety attached to σ

$$\mathrm{Irr}(M) \supset D(\sigma) = \Psi(M)\sigma = \Psi(M)/\operatorname{Stab}_{\Psi(M)}(\sigma),$$

where $\operatorname{Stab}_{\Psi(M)}(\sigma)$ is a finite group consisting of all $\psi \in \Psi(M)$ such that $\psi\sigma \simeq \sigma$.

If A is a maximal split torus in the center of M, the restriction map $\Psi(M) \to \Psi(A)$ is surjective, and the kernel is a finite group. Therefore, by considering the restriction to A we find that

$$\operatorname{Stab}_{\Psi^u(M)}(\sigma) = \operatorname{Stab}_{\Psi(M)}(\sigma).$$

So, we may consider

$$D^u(\sigma) \overset{\text{def}}{=} \Psi^u(M)/\operatorname{Stab}_{\Psi^u(M)}(\sigma) \subset D(\sigma).$$

It is easy to see that $D^u(\sigma)$ is Zariski dense in $D(\sigma)$.

The action of the Weyl group

$$W(M) = N_G(M)/M$$

on $\Psi(M)$ is algebraic. Furthermore, $w \in W(M)$ transforms $\operatorname{Stab}_{\Psi(M)}(\sigma)$ onto $\operatorname{Stab}_{\Psi(M)}(w(\sigma))$, so it maps $D(\sigma)$ (resp., $D^u(\sigma)$) onto $D(w(\sigma))$ (resp., $D^u(w(\sigma))$).

Put $D = D(\sigma)$ and $D^u = D^u(\sigma)$. As usual, we consider the group $W(D)$ of all $w \in W(M)$ such that there exists $\psi_w \in \Psi(M)$ such that

$$w(\sigma) \simeq \psi_w\sigma. \tag{2.2}$$

The character ψ_w is determined uniquely modulo $\operatorname{Stab}_{\Psi(M)}(\sigma)$. The group $W(D)$ acts on the affine variety $D = \Psi(M)/\operatorname{Stab}_{\Psi(M)}(\sigma)$ as follows:

$$w.\psi \operatorname{Stab}_{\Psi(M)}(\sigma) = \psi_w w(\psi) \operatorname{Stab}_{\Psi(M)}(\sigma). \tag{2.3}$$

The resulting orbit space

$$D/W(D)$$

is again an affine variety with algebra of regular functions given as usual,

$$\mathbb{C}[D/W(D)] = \mathbb{C}[D]^{W(D)}.$$

One can construct a regular function $D/W(D)$ in the following way:

Lemma 2.4. *Let $F \in C_c^\infty(G)$. Then, the function $\psi \mapsto \mathrm{tr}(i_{GM}(\psi\sigma)(F))$ is a regular function on $D/W(D)$.*

Proof. It is standard that this function is regular on D. We show that it is $W(D)$-invariant. Let $w \in W(D)$. By [Bernstein et al. 1986, Lemma 5.4 (iii)], we have

$$\mathrm{tr}(i_{GM}(\psi\sigma)(F)) = \mathrm{tr}(i_{GM}(w(\psi\sigma))(F)),$$

which completes the proof. □

The above explicit description shows that the analogously defined group $W(D^u)$ is a subgroup of $W(D)$. In fact, we have the following lemma:

Lemma 2.5. *Assume that the central character $\omega_\sigma : A \to \mathbb{C}^\times$ of σ is unitary. Then, $W(D^u) = W(D)$. Moreover, $D^u/W(D)$ is Zariski dense in $D/W(D)$.*

Proof. As we remarked above, it is always $W(D^u) \subset W(D)$. Conversely, if $w \in W(D)$, then $w(\sigma) \simeq \psi_w\sigma$ by (2.2). Considering central characters, we find that

$$\omega_{w(\sigma)} = (\psi_w|_A)\omega_\sigma.$$

This implies that $\psi_w|_A$ is a unitary character. By the standard description of unramified characters of M, and its relation to unramified characters of A, this implies that $\psi_w \in \Psi^u(M)$ (see [Muić 2008, Section 2]). Hence, $w \in W(D^u)$. This completes the proof that $W(D^u) = W(D)$. The remaining claim is obvious from above considerations. □

The following lemma is a fundamental result of Harish-Chandra:

Lemma 2.6. *Assume that M and N are standard Levi subgroups of G, and σ and τ are square-integrable modulo center representations of M and N, respectively. Then, $i_{GM}(\sigma)$ and $i_{GN}(\tau)$ have a common irreducible subrepresentation if and only if there exists $w \in G$ such that $N = wMw^{-1}$ and $\tau \simeq w(\sigma)$, where $w(\sigma)$ is defined by $w(\sigma)(n) = \sigma(w^{-1}nw)$, $n \in N$. Moreover, if there exists $w \in G$ such that $N = wMw^{-1}$, then $i_{GM}(\sigma)$ and $i_{GM}(w(\sigma))$ are isomorphic, and in particular equal in $R_t(G)$.*

Proof. See [Waldspurger 2003]. □

Motivated by [Bernstein et al. 1986, §2.1], we proceed as follows. By the standard theory of tempered irreducible representations due to Harish-Chandra (see [Waldspurger 2003]), for an irreducible tempered representation $\pi \in \mathrm{Irr}(G)$, there exists a standard Levi subgroup M and a square-integrable modulo center representation σ of M such that $\pi \hookrightarrow i_{GM}(\sigma)$. The pair (M, σ) is unique up to a conjugation (see Lemma 2.6). We call the equivalence class $[M, \sigma]$ under conjugation of the pair (M, σ) the t-infinitesimal character of π. The set of equivalence of such pairs we denote by $\Theta_t(G)$.

For a pair (M, σ), we define a natural map $\Psi^u(M) \to \Theta_t(G)$ given by

$$\psi \mapsto [M, \psi\sigma].$$

The image is called a connected component of $\Theta_t(G)$. We denote it by $\Theta_t(M, \sigma)$. This map induces a bijection which enables us to identify

$$\Theta_t(M, \sigma) = D^u(\sigma)/W(D(\sigma)).$$

Thus, in view of Lemma 2.5, we may consider

$$\Theta_t(M, \sigma) \subset D(\sigma)/W(D(\sigma)).$$

This realizes $\Theta_t(M, \sigma)$ as a Zariski dense subset of the affine variety $D(\sigma)/W(D(\sigma))$.

As in [Bernstein et al. 1986, §2.1], we can decompose

$$R_t(G) = \oplus_\theta R_t(G)(\theta), \tag{2.7}$$

where θ ranges over connected components of $\Theta_t(G)$. Here

$$R_t(G)(\theta)$$

is generated with all tempered irreducible representations with t-infinitesimal characters belonging to θ. We denote by 1_θ the projector

$$R_t(G) \to R_t(G)(\theta),$$

for all $\theta \in \Theta_t(G)$.

We end this section with an analogue for $\mathrm{Rep}_t(G)$ of the decomposition theorem for the category of all smooth complex representations of G (see [Bernstein et al. 1986, §2.3]; [Bernstein 1984, §2.10]).

Lemma 2.8. *Let $K \subset G$ be an open compact subgroup. Then, there exists a finite set T_K consisting of connected components in $\Theta_t(G)$ such that for each irreducible tempered representation $\pi \in \mathrm{Rep}_t(G)$, having nonzero space of K-invariants, there exists $\theta \in T_k$ such that $\pi \in R_t(G)(\theta)$.*

Proof. By the decomposition theorem (see [Bernstein et al. 1986, §2.3]), there exists a finite set, say S, of pairs (N, ρ), where N is a standard Levi subgroup of G, and ρ are irreducible supercuspidal representations, such that for every irreducible representation π of G, having nonzero space of K-invariants, there exists $(N, \rho) \in S$, and an unramified character χ such that π is a subquotient of $i_{G,N}(\chi\rho)$.

Now, assume that π is as in the statement of the lemma. Then, there exist a standard Levi subgroup M and a square-integrable modulo center σ of M such that $\pi \hookrightarrow i_{GM}(\sigma)$. Moreover, there exist a standard Levi subgroup M' of M (and of G), and a supercuspidal irreducible representation ρ' such that σ is an irreducible subquotient of $i_{M,M'}(\rho')$. By induction in stages, π must be a subquotient of $i_{G,M'}(\rho')$.

By standard theory of induced representations [Bernstein and Zelevinsky 1977], the pair (M', ρ') must be G-conjugate to the one in S. Thus, we may assume that $(M', \rho') \in S$ already.

Thus, it is enough to prove that given $(N, \ \rho) \in S$ and given a standard Levi subgroup M of G such that $N \subset M$, there are finitely many $\Psi^u(M)$-orbits of square-integrable modulo center representations of M such they are subquotients of the induced representations in the family $i_{M,N}(\chi\rho)$ parametrized by $\chi \in \Psi(N)$. But that is easy. We can select a sufficiently small open compact subgroup $L \subset M$ such that every irreducible representation that appears as a subquotient of $i_{M,N}(\chi\rho)$ for some $\chi \in \Psi(N)$ has a nonzero space of L-invariants.

Hence, we need to prove that there are finitely many $\Psi^u(M)$-orbits of square-integrable modulo center representations of M having a nonzero space of L-invariants. This is proved in (iii) in the introduction of [Waldspurger 2003]. □

3. Proof of Theorem 1.1

We begin the proof of Theorem 1.1 with the following lemma:

Lemma 3.1. *Let f be as in the statement of Theorem 1.1. Then, there exists a finite set T_f consisting of connected components in $\Theta_t(G)$ such that for each irreducible tempered representation $\pi \in \mathrm{Rep}_t(G)$ such that $f(\pi) \neq 0$ there exists $\theta \in T_f$ such that $\pi \in R_t(G)(\theta)$.*

Proof. This follows from the assumption (i) in Theorem 1.1 combined with Lemma 2.8. □

By Lemma 3.1, we can decompose f into $\mathbb{Z}$-linear forms $f_\Theta : R_t(G) \to \mathbb{C}$, $\theta \in T_f$,

$$f = \sum_{\theta \in T_f} f_\theta,$$

where f_θ is defined as follows (see (2.7)):

$$f_\theta = f \circ 1_\theta.$$

Obviously, each f_θ satisfies the assumptions analogous to (i) and (ii) in Theorem 1.1.

Hence, in what follows we may assume that $f = f_\theta$ for some $\theta \in \Theta_t(G)$. By the assumption (ii) of Theorem 1.1, we may assume that θ has the form $\theta = \Theta_t(M, \sigma)$, where M is a standard maximal Levi subgroup of G, or $M = G$, and σ is a σ is a square-integrable modulo center representation of M. We observe that

$$\psi \in \Psi^u(M) \mapsto f(i_{GM}(\psi\sigma))$$

is a regular function by the assumption (ii) of Theorem 1.1. Thus, by definition this means that it is a restriction of a regular function, say a, on the affine variety $\Psi(M)$.

By Lemma 2.6, we have

$$a \in \mathbb{C}[D]^{W(D)}, \tag{3.2}$$

where

$$D = \Psi(M)/\operatorname{Stab}_{\Psi(M)}(\sigma). \tag{3.3}$$

We refer to previous section for the notation.

Now, the following proposition completes the proof of the theorem.

Proposition 3.4. *Let M be a standard maximal Levi subgroup of G, or $M = G$. Assume that σ is a square-integrable modulo center representation of M. We define D by* (3.3), *and let a be any function in $\mathbb{C}[D]^{W(D)}$. Then, there exists $F \in C_c^\infty(G)$ such that*

$$tr(\pi(F)) = \begin{cases} a(\psi) & \text{for } \pi = i_{GM}(\psi\sigma),\ \psi \in \Psi^u(M), \\ 0 & \text{for } \pi = i_{GN}(\psi\tau),\ \psi \in \Psi^u(N), \end{cases}$$

for any other standard Levi subgroup N and a square-integrable modulo center representation τ such that $\Theta_t(N,\tau) \neq \Theta_t(M,\sigma)$.

Proof. The proof of Proposition 3.4 is a generalization of [Clozel 1986, §4.2, Proposition 1] where the proof of existence of pseudocoefficients for semisimple G is given based also on [Bernstein et al. 1986]. We consider only the case where M is a standard maximal Levi subgroup of G. The case of $M = G$ is about the construction of a specific pseudocoefficient of σ. The proof is on the same lines but considerably easier.

We remark that $\Psi^u(G)$ acts on $\Psi^u(M)$ in a usual way:

$$\psi \mapsto \chi|_M\psi, \quad \chi \in \Psi^u(G), \quad \psi \in \Psi^u(M).$$

For $\psi \in \Psi^u(M)$, the stabilizer

$$\operatorname{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$$

is the group of all $\chi \in \Psi^u(G)$ such that

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

We remind the reader that for all $\chi \in \Psi^u(G)$ we have

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\chi|_M\psi\sigma).$$

Lemma 3.5. *Assume that $\chi \in \Psi^u(G)$ and $\psi \in \Psi^u(M)$. Then, for each irreducible constituent π of $i_{GM}(\psi\sigma)$, the multiplicity of $\chi\pi$ in $\chi i_{GM}(\psi\sigma)$ is the same as that of π in $i_{GM}(\psi\sigma)$.*

Proof. This is obvious. □

Lemma 3.6. *Assume that for $\chi \in \Psi^u(G)$ and $\psi \in \Psi^u(M)$ there exists an irreducible constituent π of $i_{GM}(\psi\sigma)$ such that $\chi\pi$ is an irreducible constituent of $i_{GM}(\psi\sigma)$. Then, $\chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$. In particular, we have*

$$\mathrm{Stab}_{\Psi^u(G)}(\pi) \subset \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma)).$$

Proof. First, $\chi\pi$ is a common constituent of $i_{GM}(\psi\sigma)$ and $i_{GM}(\chi|_M\psi\sigma)$. So, by Lemma 2.6, there exists $w \in W(M)$ such that

$$\chi|_M\psi\sigma = w(\psi\sigma).$$

Then, again by Lemma 2.6, we obtain

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\chi|_M\psi\sigma) \simeq i_{GM}(\psi\sigma). \qquad \square$$

Lemma 3.7. *Let $\psi \in \Psi^u(M)$. Then, we have the following*:

(i) *If $\chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, then $a(\chi|_M\psi) = a(\psi)$.*

(ii) *For each $\eta \in \Psi(G)$ and $\chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, we have*

$$a(\chi|_M\eta|_M\psi) = a(\eta|_M\psi).$$

Proof. We prove (i). Since $\chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, we obtain

$$i_{GM}(\chi|_M\psi\sigma) \simeq \chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

So, by Lemma 2.6, there exists $w \in W(M)$ such that

$$\chi|_M\psi\sigma \simeq w(\psi\sigma) \simeq w(\psi)w(\sigma).$$

By definition of $W(D)$ (see (2.2)), this implies $w \in W(D)$, and the above relation can be written as

$$\chi|_M\psi\sigma \simeq \psi_w w(\psi)\sigma,$$

where

$$\psi_w = w(\psi)^{-1}\chi|_M\psi.$$

Consequently, by the definition of the action of $W(D)$ on D (see (2.3)) we obtain

$$\chi|_M\psi\,\mathrm{Stab}_{\Psi(M)}(\sigma) = \psi_w w(\psi)\,\mathrm{Stab}_{\Psi(M)}(\sigma) = w.\psi\,\mathrm{Stab}_{\Psi(M)}(\sigma).$$

This implies $a(\chi|_M\psi) = a(\psi)$. This proves (i).

To prove (ii), we may assume that η is unitary. Then, we obviously have

$$\mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\eta|_M\psi\sigma)) = \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma)).$$

Now, the claim follows from (i). $\square$

Now, in order to complete the proof of Proposition 3.4, we apply [Bernstein et al. 1986, Theorem 1.2]. We define a $\mathbb{Z}$-linear form $f : R(G) \to \mathbb{C}$ in several steps. We warn the reader that we use the same letter for a functional different than one from the statement of Theorem 1.1.

(1) For each $\Psi^u(G)$-orbit $\mathcal{O}$ in $\Psi^u(M)$, we fix a representative $\psi_{\mathcal{O}} \in \mathcal{O}$ and an irreducible constituent $\pi_{\mathcal{O}}$ in $i_{GM}(\psi_{\mathcal{O}}\sigma)$. By Lemma 3.6, we have

$$\mathrm{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}}) \subset \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)). \tag{3.8}$$

The quotient is finite and if χ ranges over representatives of the quotient, then $\chi\pi_{\mathcal{O}}$ ranges over the set of all mutually nonequivalent irreducible subrepresentations in $i_{GM}(\psi_{\mathcal{O}}\sigma)$ which are $\Psi^u(G)$-equivalent to $\pi_{\mathcal{O}}$. Any of those representations have the same multiplicity in $i_{GM}(\psi_{\mathcal{O}}\sigma)$. Let $m_{\mathcal{O}}$ be the sum of their multiplicities. We define

$$f(\chi\pi_{\mathcal{O}}) = \frac{a(\psi_{\mathcal{O}})}{m_{\mathcal{O}}}, \quad \chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)).$$

(2) For each $\chi \in \Psi^u(G)$, we obviously have

$$\mathrm{Stab}_{\Psi^u(G)}(\chi\pi_{\mathcal{O}}) = \mathrm{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}})$$

and

$$\mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\chi|_M\psi_{\mathcal{O}}\sigma)) = \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)).$$

By, Lemma 3.5 and these remarks, the sum of multiplicities of $\Psi^u(G)$-equivalent representations of $\pi_{\mathcal{O}}$ which belong to $i_{GM}(\chi|_M\psi_{\mathcal{O}}\sigma)$ is again $m_{\mathcal{O}}$. We let

$$f(\chi\pi_{\mathcal{O}}) = \frac{a(\chi|_M\psi_{\mathcal{O}})}{m_{\mathcal{O}}}, \quad \chi \in \Psi^u(G).$$

Lemma 3.7 (ii) shows that this is well-defined.

(3) For any other tempered irreducible representation (and, in particular, square-integrable modulo center representation) π of G we let

$$f(\pi) = 0.$$

(4) For any quasitempered irreducible representation π of G, we can write $\pi = \chi\pi^u$, where $\chi \in \Psi^r(G)$ and π^u is tempered. We let

$$f(\pi) = 0,$$

if π^u is not in $\Psi^u(G)\pi_{\mathcal{O}}$ for any orbit $\mathcal{O}$ described in (1). But, if $\pi^u \in \Psi^u(G)\pi_{\mathcal{O}}$, for some $\mathcal{O}$, then we can write $\pi^u = \psi\pi_{\mathcal{O}}$, for some $\psi \in \Psi^u(G)$ uniquely determined modulo $\mathrm{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}})$. We let

$$f(\pi) = \frac{a(\chi|_M\psi|_M\psi_{\mathcal{O}})}{m_{\mathcal{O}}}.$$

Using (3.8) and Lemma 3.7(ii) we see that this is well-defined.

(5) Finally, we define f on nontempered Langlands quotients (see Lemma 2.1). Let f be equal to zero on all standard modules induced from proper parabolic subgroups except in the following two obvious cases:

(a) The standard module $i_{GM}(\chi\psi\sigma)$, where $\chi \in \Psi(M)^{r,+}$ and $\psi \in \Psi^u(M)$. In this case, we let

$$f(i_{GM}(\chi\psi\sigma)) = a(\chi\psi).$$

(b) It is also possible that $\chi \in \Psi(M)^r$ belongs to the positive Weyl chamber for the opposite parabolic $\overline{P}$ (see the beginning of the previous section). Then, there exists a unique standard maximal parabolic subgroup Q with standard Levi N, and $w \in G$ such that $N = wMw^{-1}$. Now, by [Bernstein et al. 1986, Lemma 5.3(iii)], we have

$$i_{GM}(\chi\psi\sigma) = i_{GN}(w(\chi)w(\psi)w(\sigma))$$

in $R(G)$. Also, $w(\chi) \in \Psi(N)^{r,+}$. On the standard module $i_{GN}(w(\chi)w(\psi)w(\sigma))$ we let

$$f(i_{GN}(w(\chi)w(\psi)w(\sigma))) = a(\chi\psi).$$

Thus, we have

$$f(i_{GM}(\chi\psi\sigma)) = f(i_{GN}(w(\chi)w(\psi)w(\sigma))) = a(\chi\psi),$$

for $\chi \in \Psi(M)^r$ such that $w(\chi) \in \Psi(N)^{r,+}$.

The third case is that $\chi \in \Psi(M)^r$ is in neither chamber. Then, $\chi \in \Psi(G)^r$, by standard description of unramified characters [Muić 2008, Section 2]. In this case

$$i_{GM}(\chi\psi\sigma) = \chi i_{GM}(\psi\sigma)$$

is a quasitempered representation, and, by

$$f(i_{GM}(\chi\psi\sigma)) = f(\chi i_{GM}(\psi\sigma)) = a(\chi\psi),$$

by (1)–(4).

This completes the construction of $\mathbb{Z}$-linear form $f : R(G) \to \mathbb{C}$. In order to complete the proof of Proposition 3.4, we just need to check that it satisfies the assumptions of [Bernstein et al. 1986, Theorem 1.2]. First, let N be a standard Levi subgroup of G contained in M, and ρ an irreducible supercuspidal representation of N such that σ is an irreducible subquotient of $i_{M,N}(\rho)$. Then, by construction, f is zero on irreducible representations which are not irreducible subquotients of members of the family $i_{M,N}(\chi\rho)$ parametrized by $\chi \in \Psi(N)$. Then, as in the proof of Lemma 2.8, there exists an open compact subgroup K such that f is zero on all irreducible representations which do not have a nonzero K-invariant vector. This is (ii) in [Bernstein et al. 1986, §1.2]. It remains to check (i) in [Bernstein et al. 1986, §1.2]. We need to check that for an arbitrary standard Levi subgroup N of G and an irreducible representation τ of N, the function $\chi \mapsto f(i_{G,N}(\chi\tau))$ is regular on $\Psi(N)$. By Lemma 2.1 applied to N, τ is a $\mathbb{Z}$-linear combination of standard modules for N. So, instead of being irreducible, we may assume that τ is a standard module for N, i.e.,

$$\tau = i_{NN'}(\chi'\tau'),$$

N' is a standard Levi subgroup, τ' is an irreducible tempered representation of N' and $\chi' \in \Psi^{r,+}(N', N)$. Here, by definition $\Psi^{r,+}(N', N)$ is an analogue of $\Psi^{r,+}(N', G) \overset{\text{def}}{=} \Psi^{r,+}(N')$ defined in the previous section. Now, by induction in stages, we have

$$i_{G,N}(\chi\tau) = i_{G,N'}(\chi|_{N'}\chi'\tau').$$

We decompose $\chi = \chi^r\chi^u$ into its real part $\chi^r \in \Psi^r(N)$ and unitary part $\chi^u \in \Psi^r(N)$. Let N'' be a standard Levi subgroup such that $N' \subset N'' \subset N$ obtained by adjoining all simple roots orthogonal to $\chi^r|_{N'}\chi'$ (see [Muić 2008, Section 2]). Then, $\chi^r|_{N'}\chi'$ is an unramified character of N'' which is not orthogonal to any simple root that determines a standard parabolic subgroup of N''. In particular, there exists $w \in G$ such that $N_1'' = wN''w^{-1}$ is a standard Levi subgroup, and

$$w(\chi^r|_{N'}\chi') \in \Psi^{r,+}(N_1'')$$

(see, for example, [Muić 2006, Section 1]). Also, we can write

$$i_{G,N'}(\chi|_{N'}\chi'\tau') = i_{G,N''}(\chi^r|_{N'}\chi'\, i_{N',N''}(\chi^u|_{N'}\tau')).$$

Obviously, $i_{N',N''}(\chi^u|_{N'}\tau')$ is an direct sum of irreducible tempered representations, say τ'' of N''. This implies that $i_{G,N'}(\chi|_{N'}\chi'\tau')$ is a direct sum induced by representations

$$i_{G,N''}(\chi|_{N'}\chi'\tau'').$$

By above, in $R(G)$, we have

$$i_{G,N''}(\chi|_{N'}\chi'\tau'') = i_{G,N_1''}(w(\chi|_{N'}\chi')w(\tau'')). \tag{3.9}$$

But the last induced representation is a standard module. Now, by the construction of f, $f = 0$ on all standard modules except those described in steps (1)–(5) above. This means that we have one of the following two cases:

(a) N'' is conjugate to G. In this case $N_1'' = N'' = N' = G$, τ' is a tempered irreducible representation of G, and $i_{G,N'}(\chi|_{N'}\chi'\tau') = \chi\chi'\tau'$. Thus, by the construction (1)–(4), $\chi \mapsto f(\chi\chi'\tau')$ is regular.

(b) N'' is conjugate to M. In this case, $N' = N''$, and τ' must be conjugate to an element of the orbit $\Psi^u(M)\sigma$ (see (5) above). The discussion in (5) implies that $\chi \mapsto f(i_{G,N'}(\chi|_{N'}\chi'\tau'))$ is regular.

This finally verifies (i) of [Bernstein et al. 1986, §1.2], and completes the proof of the proposition. □

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