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**A REMARK ON A TRACE PALEY–WIENER THEOREM**

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**We prove a version of a trace Paley–Wiener theorem for tempered representations of a reductive  $p$ -adic group. This is applied to complete certain investigations of Shahidi on the proof that a Plancherel measure is an invariant of an  $L$ -packet of discrete series.**

## 1. Introduction

Let  $G$  be a reductive  $p$ -adic group. Let  $\text{Rep}(G)$  be the category of smooth admissible complex representations of  $G$  of finite length, and let  $R(G)$  be the corresponding Grothendieck group. We write  $\Psi(G)$  (resp.,  $\Psi^u(G)$ ) for the group (resp., unitary group) of unramified characters of  $G$ . The group  $\Psi(G)$  has a structure of an algebraic variety (a complex torus). The corresponding algebra of regular functions  $\mathbb{C}[\Psi(G)]$  is generated by evaluations on elements of  $G$  as a  $\mathbb{C}$ -algebra. The subgroup  $\Psi^u(G)$  is Zariski dense in  $\Psi(G)$ . We say that a complex function is regular on  $\Psi^u(G)$  if it is a restriction of a regular function on  $\Psi(G)$ . We observe that the restriction map from  $\mathbb{C}[\Psi(G)]$  into functions on  $\Psi^u(G)$  is injective since  $\Psi^u(G)$  is Zariski dense in  $\Psi(G)$ .

We fix a minimal parabolic subgroup  $P_0$ , its Levi decomposition  $P_0 = M_0U_0$ , and, as usual related to these choices, we fix a set of standard parabolic subgroups  $P = MU$ , where  $M_0 \subset M$ ,  $P = MP_0$ . Since the standard parabolic subgroup is determined by the choice of Levi subgroup, the normalized parabolic induction  $\text{Ind}_P^G(\sigma)$ , where  $\sigma$  is a smooth representation of  $M$ , we write as usual  $i_{GM}(\sigma)$ .

In [Bernstein et al. 1986], Bernstein, Deligne, and Kazhdan proved a trace Paley–Wiener theorem for category  $\text{Rep}(G)$ . We consider a full subcategory  $\text{Rep}_t(G)$  of  $\text{Rep}(G)$  consisting of representations having all irreducible subquotients tempered. Let  $R_t(G)$  be the corresponding Grothendieck group. We write  $R_t^i(G)$  for the subgroup of  $R_t(G)$  generated by  $i_{GM}(\sigma)$ , where  $M$  ranges over all standard Levi subgroups of  $G$  (including  $G$ ), and  $\sigma$  ranges over a set of square-integrable modulo center irreducible representations of  $M$ . We warn the reader that this notion is not an analogue of the notion of strictly induced modules from

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[Bernstein et al. 1986, §3.1]. An analogue would be the subgroup of  $R_t(G)$  generated by  $i_{GM}(\tau)$ , where  $M$  ranges over all *proper* standard Levi subgroups of  $G$ , and  $\tau$  ranges over irreducible tempered representations of  $M$ . But this is not useful for us in the present paper.

The main result of the present paper is the following version of a trace Paley–Wiener theorem:

**Theorem 1.1.** *Let  $f : R_t(G) \rightarrow \mathbb{C}$  be a  $\mathbb{Z}$ -linear form such that the following hold:*

- (i) *There exists an open compact subgroup  $K \subset G$  which dominates  $f$  (i.e.,  $f$  is nonzero only on those irreducible tempered representations which have a nontrivial space of  $K$ -invariant vectors).*
- (ii) *For each standard maximal Levi subgroup  $M$ , or  $M = G$ , and a square-integrable modulo center representation  $\sigma$  of  $M$ , the function  $\psi \mapsto f(i_{GM}(\psi\sigma))$  is regular on  $\Psi^u(M)$ , and for any other proper standard Levi subgroup  $N$ , and a square-integrable modulo center representation  $\tau$  of  $N$ , we have  $f(i_{GN}(\tau)) = 0$ .*

*Then, there exists  $F \in C_c^\infty(G)$  such that*

$$f(\pi) = \text{tr}(\pi(F)) \quad \text{for all } \pi \in R_t^i(G).$$

Theorem 1.1 is proved by reduction to the main result of [Bernstein et al. 1986] using the Harish-Chandra theory of tempered representations [Waldspurger 2003] and some standard considerations related to the Langlands classification [Renard 2010, Chapter VII]. The proof is given in Section 3. It is a consequence of its effective version given by Proposition 3.4. Proposition 3.4 constructs a correct function needed in the proof of [Shahidi 1990, Proposition 9.3.2] in the case when  $M$  (see notation there) is a Levi subgroup of a maximal parabolic subgroup. We remark that since Plancherel factors are multiplicative, it is enough to prove [Shahidi 1990, Proposition 9.3.2] for a maximal Levi subgroup.

## 2. Preliminaries

We continue with the notation introduced in the introduction. Let  $M$  be a standard Levi subgroup. Then, we write  $\Psi(M)^r$  for the group of all unramified characters  $\psi$  which are  $\mathbb{R}_{>0}$ -valued. As we stated in the introduction, every standard Levi subgroup  $M$  determines a unique standard parabolic subgroup, say  $P$ . We denote by  $\Psi(M)^{r,+}$  the set of all characters from  $\Psi(M)^r$  which correspond to the points of the (open) Weyl chamber determined by the roots of the split component of  $M$  which belong to the unipotent radical of  $P$  in the usual description of unramified characters (see, for example, [Muić 2008, Section 2]). If  $M = G$ , then  $\Psi(M)^{r,+} = \Psi(M)^r$ .

For a standard Levi subgroup  $M$ , an irreducible tempered representation  $\pi$  of  $M$ , and  $\psi \in \Psi(M)^{r,+}$ , the module  $i_{GM}(\psi\pi)$  is called a standard module; it has a

unique (Langlands quotient)  $L(i_{GM}(\psi\pi))$ . The condition is empty if  $M = G$ . By the Langlands classification [Renard 2010, Theorem VII.4.2], every irreducible representation can be expressed in the form  $L(i_{GM}(\psi\pi))$  for unique such datum  $(M, \pi, \psi)$ . The following standard result will be used in the proof:

**Lemma 2.1.** *The standard modules of  $G$  form a  $\mathbb{Z}$ -basis of  $R(G)$ .*

*Proof.* The proof is as in [Clozel 1986, Proposition 1]. □

In analogy with [Bernstein et al. 1986, §2.1], we make the following definitions.

Let  $\sigma \in \text{Irr}(M)$  where  $M$  is a standard Levi subgroup of  $G$ . We define the usual affine variety attached to  $\sigma$

$$\text{Irr}(M) \supset D(\sigma) = \Psi(M)\sigma = \Psi(M)/\text{Stab}_{\Psi(M)}(\sigma),$$

where  $\text{Stab}_{\Psi(M)}(\sigma)$  is a finite group consisting of all  $\psi \in \Psi(M)$  such that  $\psi\sigma \simeq \sigma$ .

If  $A$  is a maximal split torus in the center of  $M$ , the restriction map  $\Psi(M) \rightarrow \Psi(A)$  is surjective, and the kernel is a finite group. Therefore, by considering the restriction to  $A$  we find that

$$\text{Stab}_{\Psi^u(M)}(\sigma) = \text{Stab}_{\Psi(M)}(\sigma).$$

So, we may consider

$$D^u(\sigma) \stackrel{\text{def}}{=} \Psi^u(M)/\text{Stab}_{\Psi^u(M)}(\sigma) \subset D(\sigma).$$

It is easy to see that  $D^u(\sigma)$  is Zariski dense in  $D(\sigma)$ .

The action of the Weyl group

$$W(M) = N_G(M)/M$$

on  $\Psi(M)$  is algebraic. Furthermore,  $w \in W(M)$  transforms  $\text{Stab}_{\Psi(M)}(\sigma)$  onto  $\text{Stab}_{\Psi(M)}(w(\sigma))$ , so it maps  $D(\sigma)$  (resp.,  $D^u(\sigma)$ ) onto  $D(w(\sigma))$  (resp.,  $D^u(w(\sigma))$ ).

Put  $D = D(\sigma)$  and  $D^u = D^u(\sigma)$ . As usual, we consider the group  $W(D)$  of all  $w \in W(M)$  such that there exists  $\psi_w \in \Psi(M)$  such that

$$(2.2) \quad w(\sigma) \simeq \psi_w\sigma.$$

The character  $\psi_w$  is determined uniquely modulo  $\text{Stab}_{\Psi(M)}(\sigma)$ . The group  $W(D)$  acts on the affine variety  $D = \Psi(M)/\text{Stab}_{\Psi(M)}(\sigma)$  as follows:

$$(2.3) \quad w \cdot \psi \text{Stab}_{\Psi(M)}(\sigma) = \psi_w w(\psi) \text{Stab}_{\Psi(M)}(\sigma).$$

The resulting orbit space

$$D/W(D)$$

is again an affine variety with algebra of regular functions given as usual,

$$\mathbb{C}[D/W(D)] = \mathbb{C}[D]^{W(D)}.$$

One can construct a regular function  $D/W(D)$  in the following way:

**Lemma 2.4.** *Let  $F \in C_c^\infty(G)$ . Then, the function  $\psi \mapsto \text{tr}(i_{GM}(\psi\sigma)(F))$  is a regular function on  $D/W(D)$ .*

*Proof.* It is standard that this function is regular on  $D$ . We show that it is  $W(D)$ -invariant. Let  $w \in W(D)$ . By [Bernstein et al. 1986, Lemma 5.4 (iii)], we have

$$\text{tr}(i_{GM}(\psi\sigma)(F)) = \text{tr}(i_{GM}(w(\psi\sigma))(F)),$$

which completes the proof.  $\square$

The above explicit description shows that the analogously defined group  $W(D^\mu)$  is a subgroup of  $W(D)$ . In fact, we have the following lemma:

**Lemma 2.5.** *Assume that the central character  $\omega_\sigma : A \rightarrow \mathbb{C}^\times$  of  $\sigma$  is unitary. Then,  $W(D^\mu) = W(D)$ . Moreover,  $D^\mu/W(D)$  is Zariski dense in  $D/W(D)$ .*

*Proof.* As we remarked above, it is always  $W(D^\mu) \subset W(D)$ . Conversely, if  $w \in W(D)$ , then  $w(\sigma) \simeq \psi_w\sigma$  by (2.2). Considering central characters, we find that

$$\omega_{w(\sigma)} = (\psi_w|_A)\omega_\sigma.$$

This implies that  $\psi_w|_A$  is a unitary character. By the standard description of unramified characters of  $M$ , and its relation to unramified characters of  $A$ , this implies that  $\psi_w \in \Psi^\mu(M)$  (see [Muić 2008, Section 2]). Hence,  $w \in W(D^\mu)$ . This completes the proof that  $W(D^\mu) = W(D)$ . The remaining claim is obvious from above considerations.  $\square$

The following lemma is a fundamental result of Harish-Chandra:

**Lemma 2.6.** *Assume that  $M$  and  $N$  are standard Levi subgroups of  $G$ , and  $\sigma$  and  $\tau$  are square-integrable modulo center representations of  $M$  and  $N$ , respectively. Then,  $i_{GM}(\sigma)$  and  $i_{GN}(\tau)$  have a common irreducible subrepresentation if and only if there exists  $w \in G$  such that  $N = wMw^{-1}$  and  $\tau \simeq w(\sigma)$ , where  $w(\sigma)$  is defined by  $w(\sigma)(n) = \sigma(w^{-1}nw)$ ,  $n \in N$ . Moreover, if there exists  $w \in G$  such that  $N = wMw^{-1}$ , then  $i_{GM}(\sigma)$  and  $i_{GM}(w(\sigma))$  are isomorphic, and in particular equal in  $R_t(G)$ .*

*Proof.* See [Waldspurger 2003].  $\square$

Motivated by [Bernstein et al. 1986, §2.1], we proceed as follows. By the standard theory of tempered irreducible representations due to Harish-Chandra (see [Waldspurger 2003]), for an irreducible tempered representation  $\pi \in \text{Irr}(G)$ , there exists a standard Levi subgroup  $M$  and a square-integrable modulo center representation  $\sigma$  of  $M$  such that  $\pi \hookrightarrow i_{GM}(\sigma)$ . The pair  $(M, \sigma)$  is unique up to a conjugation (see Lemma 2.6). We call the equivalence class  $[M, \sigma]$  under conjugation of the pair  $(M, \sigma)$  the  $t$ -infinitesimal character of  $\pi$ . The set of equivalence of such pairs we denote by  $\Theta_t(G)$ .

For a pair  $(M, \sigma)$ , we define a natural map  $\Psi^u(M) \rightarrow \Theta_t(G)$  given by

$$\psi \mapsto [M, \psi\sigma].$$

The image is called a connected component of  $\Theta_t(G)$ . We denote it by  $\Theta_t(M, \sigma)$ . This map induces a bijection which enables us to identify

$$\Theta_t(M, \sigma) = D^u(\sigma)/W(D(\sigma)).$$

Thus, in view of [Lemma 2.5](#), we may consider

$$\Theta_t(M, \sigma) \subset D(\sigma)/W(D(\sigma)).$$

This realizes  $\Theta_t(M, \sigma)$  as a Zariski dense subset of the affine variety  $D(\sigma)/W(D(\sigma))$ .

As in [\[Bernstein et al. 1986, §2.1\]](#), we can decompose

$$(2.7) \quad R_t(G) = \bigoplus_{\theta} R_t(G)(\theta),$$

where  $\theta$  ranges over connected components of  $\Theta_t(G)$ . Here

$$R_t(G)(\theta)$$

is generated with all tempered irreducible representations with  $t$ -infinitesimal characters belonging to  $\theta$ . We denote by  $1_{\theta}$  the projector

$$R_t(G) \rightarrow R_t(G)(\theta),$$

for all  $\theta \in \Theta_t(G)$ .

We end this section with an analogue for  $\text{Rep}_t(G)$  of the decomposition theorem for the category of all smooth complex representations of  $G$  (see [\[Bernstein et al. 1986, §2.3\]](#); [\[Bernstein 1984, §2.10\]](#)).

**Lemma 2.8.** *Let  $K \subset G$  be an open compact subgroup. Then, there exists a finite set  $T_K$  consisting of connected components in  $\Theta_t(G)$  such that for each irreducible tempered representation  $\pi \in \text{Rep}_t(G)$ , having nonzero space of  $K$ -invariants, there exists  $\theta \in T_K$  such that  $\pi \in R_t(G)(\theta)$ .*

*Proof.* By the decomposition theorem (see [\[Bernstein et al. 1986, §2.3\]](#)), there exists a finite set, say  $S$ , of pairs  $(N, \rho)$ , where  $N$  is a standard Levi subgroup of  $G$ , and  $\rho$  are irreducible supercuspidal representations, such that for every irreducible representation  $\pi$  of  $G$ , having nonzero space of  $K$ -invariants, there exists  $(N, \rho) \in S$ , and an unramified character  $\chi$  such that  $\pi$  is a subquotient of  $i_{G,N}(\chi\rho)$ .

Now, assume that  $\pi$  is as in the statement of the lemma. Then, there exist a standard Levi subgroup  $M$  and a square-integrable modulo center  $\sigma$  of  $M$  such that  $\pi \hookrightarrow i_{GM}(\sigma)$ . Moreover, there exist a standard Levi subgroup  $M'$  of  $M$  (and of  $G$ ), and a supercuspidal irreducible representation  $\rho'$  such that  $\sigma$  is an irreducible subquotient of  $i_{M,M'}(\rho')$ . By induction in stages,  $\pi$  must be a subquotient of  $i_{G,M'}(\rho')$ .

By standard theory of induced representations [Bernstein and Zelevinsky 1977], the pair  $(M', \rho')$  must be  $G$ -conjugate to the one in  $S$ . Thus, we may assume that  $(M', \rho') \in S$  already.

Thus, it is enough to prove that given  $(N, \rho) \in S$  and given a standard Levi subgroup  $M$  of  $G$  such that  $N \subset M$ , there are finitely many  $\Psi^u(M)$ -orbits of square-integrable modulo center representations of  $M$  such they are subquotients of the induced representations in the family  $i_{M,N}(\chi\rho)$  parametrized by  $\chi \in \Psi(N)$ . But that is easy. We can select a sufficiently small open compact subgroup  $L \subset M$  such that every irreducible representation that appears as a subquotient of  $i_{M,N}(\chi\rho)$  for some  $\chi \in \Psi(N)$  has a nonzero space of  $L$ -invariants.

Hence, we need to prove that there are finitely many  $\Psi^u(M)$ -orbits of square-integrable modulo center representations of  $M$  having a nonzero space of  $L$ -invariants. This is proved in (iii) in the introduction of [Waldspurger 2003].  $\square$

### 3. Proof of Theorem 1.1

We begin the proof of Theorem 1.1 with the following lemma:

**Lemma 3.1.** *Let  $f$  be as in the statement of Theorem 1.1. Then, there exists a finite set  $T_f$  consisting of connected components in  $\Theta_t(G)$  such that for each irreducible tempered representation  $\pi \in \text{Rep}_t(G)$  such that  $f(\pi) \neq 0$  there exists  $\theta \in T_f$  such that  $\pi \in R_t(G)(\theta)$ .*

*Proof.* This follows from the assumption (i) in Theorem 1.1 combined with Lemma 2.8.  $\square$

By Lemma 3.1, we can decompose  $f$  into  $\mathbb{Z}$ -linear forms  $f_\theta : R_t(G) \rightarrow \mathbb{C}$ ,  $\theta \in T_f$ ,

$$f = \sum_{\theta \in T_f} f_\theta,$$

where  $f_\theta$  is defined as follows (see (2.7)):

$$f_\theta = f \circ 1_\theta.$$

Obviously, each  $f_\theta$  satisfies the assumptions analogous to (i) and (ii) in Theorem 1.1.

Hence, in what follows we may assume that  $f = f_\theta$  for some  $\theta \in \Theta_t(G)$ . By the assumption (ii) of Theorem 1.1, we may assume that  $\theta$  has the form  $\theta = \Theta_t(M, \sigma)$ , where  $M$  is a standard maximal Levi subgroup of  $G$ , or  $M = G$ , and  $\sigma$  is a square-integrable modulo center representation of  $M$ . We observe that

$$\psi \in \Psi^u(M) \mapsto f(i_{GM}(\psi\sigma))$$

is a regular function by the assumption (ii) of Theorem 1.1. Thus, by definition this means that it is a restriction of a regular function, say  $a$ , on the affine variety  $\Psi(M)$ .

By [Lemma 2.6](#), we have

$$(3.2) \quad a \in \mathbb{C}[D]^{W(D)},$$

where

$$(3.3) \quad D = \Psi(M) / \text{Stab}_{\Psi(M)}(\sigma).$$

We refer to previous section for the notation.

Now, the following proposition completes the proof of the theorem.

**Proposition 3.4.** *Let  $M$  be a standard maximal Levi subgroup of  $G$ , or  $M = G$ . Assume that  $\sigma$  is a square-integrable modulo center representation of  $M$ . We define  $D$  by (3.3), and let  $a$  be any function in  $\mathbb{C}[D]^{W(D)}$ . Then, there exists  $F \in C_c^\infty(G)$  such that*

$$\text{tr}(\pi(F)) = \begin{cases} a(\psi) & \text{for } \pi = i_{GM}(\psi\sigma), \psi \in \Psi^u(M), \\ 0 & \text{for } \pi = i_{GN}(\psi\tau), \psi \in \Psi^u(N), \end{cases}$$

for any other standard Levi subgroup  $N$  and a square-integrable modulo center representation  $\tau$  such that  $\Theta_t(N, \tau) \neq \Theta_t(M, \sigma)$ .

*Proof.* The proof of [Proposition 3.4](#) is a generalization of [[Clozel 1986](#), §4.2, Proposition 1] where the proof of existence of pseudocoefficients for semisimple  $G$  is given based also on [[Bernstein et al. 1986](#)]. We consider only the case where  $M$  is a standard maximal Levi subgroup of  $G$ . The case of  $M = G$  is about the construction of a specific pseudocoefficient of  $\sigma$ . The proof is on the same lines but considerably easier.

We remark that  $\Psi^u(G)$  acts on  $\Psi^u(M)$  in a usual way:

$$\psi \mapsto \chi|_M\psi, \quad \chi \in \Psi^u(G), \quad \psi \in \Psi^u(M).$$

For  $\psi \in \Psi^u(M)$ , the stabilizer

$$\text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$$

is the group of all  $\chi \in \Psi^u(G)$  such that

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

We remind the reader that for all  $\chi \in \Psi^u(G)$  we have

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\chi|_M\psi\sigma).$$

**Lemma 3.5.** *Assume that  $\chi \in \Psi^u(G)$  and  $\psi \in \Psi^u(M)$ . Then, for each irreducible constituent  $\pi$  of  $i_{GM}(\psi\sigma)$ , the multiplicity of  $\chi\pi$  in  $\chi i_{GM}(\psi\sigma)$  is the same as that of  $\pi$  in  $i_{GM}(\psi\sigma)$ .*

*Proof.* This is obvious. □

**Lemma 3.6.** *Assume that for  $\chi \in \Psi^u(G)$  and  $\psi \in \Psi^u(M)$  there exists an irreducible constituent  $\pi$  of  $i_{GM}(\psi\sigma)$  such that  $\chi\pi$  is an irreducible constituent of  $i_{GM}(\psi\sigma)$ . Then,  $\chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$ . In particular, we have*

$$\text{Stab}_{\Psi^u(G)}(\pi) \subset \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma)).$$

*Proof.* First,  $\chi\pi$  is a common constituent of  $i_{GM}(\psi\sigma)$  and  $i_{GM}(\chi|_M\psi\sigma)$ . So, by [Lemma 2.6](#), there exists  $w \in W(M)$  such that

$$\chi|_M\psi\sigma = w(\psi\sigma).$$

Then, again by [Lemma 2.6](#), we obtain

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\chi|_M\psi\sigma) \simeq i_{GM}(\psi\sigma). \quad \square$$

**Lemma 3.7.** *Let  $\psi \in \Psi^u(M)$ . Then, we have the following:*

- (i) *If  $\chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$ , then  $a(\chi|_M\psi) = a(\psi)$ .*
- (ii) *For each  $\eta \in \Psi(G)$  and  $\chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$ , we have*

$$a(\chi|_M\eta|_M\psi) = a(\eta|_M\psi).$$

*Proof.* We prove (i). Since  $\chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$ , we obtain

$$i_{GM}(\chi|_M\psi\sigma) \simeq \chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

So, by [Lemma 2.6](#), there exists  $w \in W(M)$  such that

$$\chi|_M\psi\sigma \simeq w(\psi\sigma) \simeq w(\psi)w(\sigma).$$

By definition of  $W(D)$  (see [\(2.2\)](#)), this implies  $w \in W(D)$ , and the above relation can be written as

$$\chi|_M\psi\sigma \simeq \psi_w w(\psi)\sigma,$$

where

$$\psi_w = w(\psi)^{-1}\chi|_M\psi.$$

Consequently, by the definition of the action of  $W(D)$  on  $D$  (see [\(2.3\)](#)) we obtain

$$\chi|_M\psi \text{Stab}_{\Psi(M)}(\sigma) = \psi_w w(\psi) \text{Stab}_{\Psi(M)}(\sigma) = w \cdot \psi \text{Stab}_{\Psi(M)}(\sigma).$$

This implies  $a(\chi|_M\psi) = a(\psi)$ . This proves (i).

To prove (ii), we may assume that  $\eta$  is unitary. Then, we obviously have

$$\text{Stab}_{\Psi^u(G)}(i_{GM}(\eta|_M\psi\sigma)) = \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma)).$$

Now, the claim follows from (i). □

Now, in order to complete the proof of [Proposition 3.4](#), we apply [\[Bernstein et al. 1986, Theorem 1.2\]](#). We define a  $\mathbb{Z}$ -linear form  $f : R(G) \rightarrow \mathbb{C}$  in several steps. We warn the reader that we use the same letter for a functional different than one from the statement of [Theorem 1.1](#).

(1) For each  $\Psi^u(G)$ -orbit  $\mathcal{O}$  in  $\Psi^u(M)$ , we fix a representative  $\psi_{\mathcal{O}} \in \mathcal{O}$  and an irreducible constituent  $\pi_{\mathcal{O}}$  in  $i_{GM}(\psi_{\mathcal{O}}\sigma)$ . By [Lemma 3.6](#), we have

$$(3.8) \quad \text{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}}) \subset \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)).$$

The quotient is finite and if  $\chi$  ranges over representatives of the quotient, then  $\chi\pi_{\mathcal{O}}$  ranges over the set of all mutually nonequivalent irreducible subrepresentations in  $i_{GM}(\psi_{\mathcal{O}}\sigma)$  which are  $\Psi^u(G)$ -equivalent to  $\pi_{\mathcal{O}}$ . Any of those representations have the same multiplicity in  $i_{GM}(\psi_{\mathcal{O}}\sigma)$ . Let  $m_{\mathcal{O}}$  be the sum of their multiplicities. We define

$$f(\chi\pi_{\mathcal{O}}) = \frac{a(\psi_{\mathcal{O}})}{m_{\mathcal{O}}}, \quad \chi \in \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)).$$

(2) For each  $\chi \in \Psi^u(G)$ , we obviously have

$$\text{Stab}_{\Psi^u(G)}(\chi\pi_{\mathcal{O}}) = \text{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}})$$

and

$$\text{Stab}_{\Psi^u(G)}(i_{GM}(\chi|_M\psi_{\mathcal{O}}\sigma)) = \text{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)).$$

By [Lemma 3.5](#) and these remarks, the sum of multiplicities of  $\Psi^u(G)$ -equivalent representations of  $\pi_{\mathcal{O}}$  which belong to  $i_{GM}(\chi|_M\psi_{\mathcal{O}}\sigma)$  is again  $m_{\mathcal{O}}$ . We let

$$f(\chi\pi_{\mathcal{O}}) = \frac{a(\chi|_M\psi_{\mathcal{O}})}{m_{\mathcal{O}}}, \quad \chi \in \Psi^u(G).$$

[Lemma 3.7](#) (ii) shows that this is well-defined.

(3) For any other tempered irreducible representation (and, in particular, square-integrable modulo center representation)  $\pi$  of  $G$  we let

$$f(\pi) = 0.$$

(4) For any quasitempered irreducible representation  $\pi$  of  $G$ , we can write  $\pi = \chi\pi^u$ , where  $\chi \in \Psi^r(G)$  and  $\pi^u$  is tempered. We let

$$f(\pi) = 0,$$

if  $\pi^u$  is not in  $\Psi^u(G)\pi_{\mathcal{O}}$  for any orbit  $\mathcal{O}$  described in (1). But, if  $\pi^u \in \Psi^u(G)\pi_{\mathcal{O}}$ , for some  $\mathcal{O}$ , then we can write  $\pi^u = \psi\pi_{\mathcal{O}}$ , for some  $\psi \in \Psi^u(G)$  uniquely determined modulo  $\text{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}})$ . We let

$$f(\pi) = \frac{a(\chi|_M\psi|_M\psi_{\mathcal{O}})}{m_{\mathcal{O}}}.$$

Using (3.8) and Lemma 3.7(ii) we see that this is well-defined.

(5) Finally, we define  $f$  on nontempered Langlands quotients (see Lemma 2.1). Let  $f$  be equal to zero on all standard modules induced from proper parabolic subgroups except in the following two obvious cases:

- (a) The standard module  $i_{GM}(\chi\psi\sigma)$ , where  $\chi \in \Psi(M)^{r,+}$  and  $\psi \in \Psi^u(M)$ . In this case, we let

$$f(i_{GM}(\chi\psi\sigma)) = a(\chi\psi).$$

- (b) It is also possible that  $\chi \in \Psi(M)^r$  belongs to the positive Weyl chamber for the opposite parabolic  $\bar{P}$  (see the beginning of the previous section). Then, there exists a unique standard maximal parabolic subgroup  $Q$  with standard Levi  $N$ , and  $w \in G$  such that  $N = wMw^{-1}$ . Now, by [Bernstein et al. 1986, Lemma 5.3(iii)], we have

$$i_{GM}(\chi\psi\sigma) = i_{GN}(w(\chi)w(\psi)w(\sigma))$$

in  $R(G)$ . Also,  $w(\chi) \in \Psi(N)^{r,+}$ . On the standard module  $i_{GN}(w(\chi)w(\psi)w(\sigma))$  we let

$$f(i_{GN}(w(\chi)w(\psi)w(\sigma))) = a(\chi\psi).$$

Thus, we have

$$f(i_{GM}(\chi\psi\sigma)) = f(i_{GN}(w(\chi)w(\psi)w(\sigma))) = a(\chi\psi),$$

for  $\chi \in \Psi(M)^r$  such that  $w(\chi) \in \Psi(N)^{r,+}$ .

The third case is that  $\chi \in \Psi(M)^r$  is in neither chamber. Then,  $\chi \in \Psi(G)^r$ , by standard description of unramified characters [Muić 2008, Section 2]. In this case

$$i_{GM}(\chi\psi\sigma) = \chi i_{GM}(\psi\sigma)$$

is a quasitempered representation, and, by

$$f(i_{GM}(\chi\psi\sigma)) = f(\chi i_{GM}(\psi\sigma)) = a(\chi\psi),$$

by (1)–(4).

This completes the construction of  $\mathbb{Z}$ -linear form  $f : R(G) \rightarrow \mathbb{C}$ . In order to complete the proof of Proposition 3.4, we just need to check that it satisfies the assumptions of [Bernstein et al. 1986, Theorem 1.2]. First, let  $N$  be a standard Levi subgroup of  $G$  contained in  $M$ , and  $\rho$  an irreducible supercuspidal representation of  $N$  such that  $\sigma$  is an irreducible subquotient of  $i_{M,N}(\rho)$ . Then, by construction,  $f$  is zero on irreducible representations which are not irreducible subquotients of members of the family  $i_{M,N}(\chi\rho)$  parametrized by  $\chi \in \Psi(N)$ . Then, as in the proof of Lemma 2.8, there exists an open compact subgroup  $K$  such that  $f$  is zero on all irreducible representations which do not have a nonzero  $K$ -invariant vector. This

is (ii) in [Bernstein et al. 1986, §1.2]. It remains to check (i) in [Bernstein et al. 1986, §1.2]. We need to check that for an arbitrary standard Levi subgroup  $N$  of  $G$  and an irreducible representation  $\tau$  of  $N$ , the function  $\chi \mapsto f(i_{G,N}(\chi\tau))$  is regular on  $\Psi(N)$ . By Lemma 2.1 applied to  $N$ ,  $\tau$  is a  $\mathbb{Z}$ -linear combination of standard modules for  $N$ . So, instead of being irreducible, we may assume that  $\tau$  is a standard module for  $N$ , i.e.,

$$\tau = i_{NN'}(\chi'\tau'),$$

$N'$  is a standard Levi subgroup,  $\tau'$  is an irreducible tempered representation of  $N'$  and  $\chi' \in \Psi^{r,+}(N', N)$ . Here, by definition  $\Psi^{r,+}(N', N)$  is an analogue of  $\Psi^{r,+}(N', G) \stackrel{\text{def}}{=} \Psi^{r,+}(N')$  defined in the previous section. Now, by induction in stages, we have

$$i_{G,N}(\chi\tau) = i_{G,N'}(\chi|_{N'}\chi'\tau').$$

We decompose  $\chi = \chi^r\chi^u$  into its real part  $\chi^r \in \Psi^r(N)$  and unitary part  $\chi^u \in \Psi^r(N)$ . Let  $N''$  be a standard Levi subgroup such that  $N' \subset N'' \subset N$  obtained by adjoining all simple roots orthogonal to  $\chi^r|_{N'}\chi'$  (see [Muić 2008, Section 2]). Then,  $\chi^r|_{N'}\chi'$  is an unramified character of  $N''$  which is not orthogonal to any simple root that determines a standard parabolic subgroup of  $N''$ . In particular, there exists  $w \in G$  such that  $N_1'' = wN''w^{-1}$  is a standard Levi subgroup, and

$$w(\chi^r|_{N'}\chi') \in \Psi^{r,+}(N_1'')$$

(see, for example, [Muić 2006, Section 1]). Also, we can write

$$i_{G,N'}(\chi|_{N'}\chi'\tau') = i_{G,N''}(\chi^r|_{N'}\chi' i_{N',N''}(\chi^u|_{N'}\tau')).$$

Obviously,  $i_{N',N''}(\chi^u|_{N'}\tau')$  is a direct sum of irreducible tempered representations, say  $\tau''$  of  $N''$ . This implies that  $i_{G,N'}(\chi|_{N'}\chi'\tau')$  is a direct sum induced by representations

$$i_{G,N''}(\chi|_{N'}\chi'\tau'').$$

By above, in  $R(G)$ , we have

$$(3.9) \quad i_{G,N''}(\chi|_{N'}\chi'\tau'') = i_{G,N_1''}(w(\chi|_{N'}\chi')w(\tau'')).$$

But the last induced representation is a standard module. Now, by the construction of  $f$ ,  $f = 0$  on all standard modules except those described in steps (1)–(5) above. This means that we have one of the following two cases:

(a)  $N''$  is conjugate to  $G$ . In this case  $N_1'' = N'' = N' = G$ ,  $\tau'$  is a tempered irreducible representation of  $G$ , and  $i_{G,N'}(\chi|_{N'}\chi'\tau') = \chi\chi'\tau'$ . Thus, by the construction (1)–(4),  $\chi \mapsto f(\chi\chi'\tau')$  is regular.

(b)  $N''$  is conjugate to  $M$ . In this case,  $N' = N''$ , and  $\tau'$  must be conjugate to an element of the orbit  $\Psi''(M)\sigma$  (see (5) above). The discussion in (5) implies that  $\chi \mapsto f(i_{G,N'}(\chi|_{N'}\chi'\tau'))$  is regular.

This finally verifies (i) of [Bernstein et al. 1986, §1.2], and completes the proof of the proposition.  $\square$

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