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University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

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Department of Mathematics
University of California
Los Angeles, CA 90095-1555
matthias@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

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EXISTENCE OF STEADY MULTIPLE VORTEX PATCHES TO THE VORTEX-WAVE SYSTEM

DAOMIN CAO AND GUODONG WANG

We prove the existence of steady multiple vortex patch solutions to the vortex-wave system in a planar bounded domain. The construction is performed by solving a certain variational problem for the vorticity and studying the asymptotic behavior as the vorticity strength goes to infinity.

1. Introduction

The evolution of an incompressible inviscid fluid is described by the Euler equations

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla P, \\ \nabla\cdot\boldsymbol{v} = 0, \end{cases} \tag{1-1}$$

where $\boldsymbol{v}$ is the velocity field and P is the pressure. In the planar case $\boldsymbol{v} = (v_1, v_2)$ and we introduce the scalar vorticity of the fluid as follows:

$$\omega := \partial_1 v_2 - \partial_2 v_1. \tag{1-2}$$

By taking the curl on both sides of the first equation of (1-1), we obtain the following vorticity form of the Euler equations:

$$\partial_t \omega + \boldsymbol{v}\cdot\nabla\omega = 0. \tag{1-3}$$

In the whole plane, the velocity can be recovered from the vorticity via the Biot–Savart law, that is,

$$\boldsymbol{v}(x,t) = K * \omega(x,t) := -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{J(x-y)}{|x-y|^2}\omega(y,t)\,dy, \tag{1-4}$$

where $J(a,b) := (b,-a)$ denotes clockwise rotation through $\pi/2$ of the planar vector $(a,b)\in\mathbb{R}^2$, and $K(x) = -(1/(2\pi))(Jx/|x|^2)$ is called the Biot–Savart kernel. Equations (1-3) and (1-4) mean that the vorticity ω is transported by a divergence-free velocity field induced by itself.

In some cases the vorticity is sharply concentrated in N small disjoint regions, where N is a positive integer, then its time evolution can be approximately described

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by the point vortex model (see [Lin 1941] for example), an ODE system that can be written as

$$\frac{dx_i}{dt} = \sum_{j=1, j\neq i}^{N} \kappa_j K(x_i - x_j), \quad i = 1, \dots, N, \tag{1-5}$$

where x_i is the position of the i-th point vortex and κ_i is the corresponding vorticity strength. According to the point vortex model, for each point vortex located at x_i, the velocity it induces is $\kappa_i K(\cdot - x_i)$, and it moves in the velocity field induced by all the other $N - 1$ point vortices. The point vortex model and its relation with the Euler equations have been analyzed extensively; see [Cao et al. 2015; Marchioro and Pagani 1986; Smets and Van Schaftingen 2010; Turkington 1987] for example.

Now it is natural to consider the mixed problem, that is, the vorticity consists of a continuously distributed part and a finite number of concentrated vortices. Marchioro and Pulvirenti [1991] first studied this problem and they called it the vortex-wave system. In the whole plane the vortex-wave system can be written as

$$\begin{cases} \partial_t \omega + \boldsymbol{v} \cdot \nabla \omega = 0, \\ \dfrac{dx_i}{dt} = K * \omega(x_i, t) + \sum_{j=1, j\neq i}^{N} \kappa_j K(x_i - x_j), \quad i = 1, \dots, N, \\ \boldsymbol{v} = K * \omega + \sum_{j=1}^{k} \kappa_j K(\cdot - x_j). \end{cases} \tag{1-6}$$

Throughout this paper we call ω the background vorticity. System (1-6) means that the background vorticity is transported by the velocity field induced by itself (the term $K * \omega$) and the N point vortices (the term $\sum_{j=1}^{k} \kappa_j K(\cdot - x_j)$), and each point vortex moves by the velocity induced by the background vorticity (the term $K * \omega(x_i, t)$) and all the other $N - 1$ point vortices (the term $\sum_{j\neq i, j=1}^{N} \kappa_j K(\cdot - x_j)$). By constructing Lagrangian paths Marchioro and Pulvirenti [1991] proved a theorem of existence for initial background vorticity belonging to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. More results on the existence and uniqueness can be found in [Bjorland 2011; Lacave and Miot 2009; Lopes Filho et al. 2011; Miot 2012].

In this paper, we will be focusing on the steady vortex-wave system in a bounded domain, the precise form of which will be given in the next section. In this case the Kirchhoff–Routh function (defined by (2-2) in the next section) plays an essential role. On the one hand, it is easy to see that any critical point of the Kirchhoff–Routh function is a stationary point of the vortex model, and it has been shown in [Cao et al. 2015] that if this critical point is nondegenerate, then there exists a family of steady vortex patch solutions (that is, the vorticity is a piecewise constant function) of the Euler equations shrinking to this critical point. Similar results can also be found in [Cao et al. 2014; Smets and Van Schaftingen 2010; Turkington 1983a; Wan 1988]. On the other hand, it is proved in [Cao et al. 2019a] that if there is a family of steady vortex patch solutions of the Euler equations shrinking to a point, then this point must be in the interior of the domain and must be a critical point of the Kirchhoff–Routh function. In this way, the Kirchhoff–Routh function establishes connection between the Euler equations and the vortex model.

Our purpose in the present paper is to extend the result in [Cao et al. 2015] to the vortex-wave system. To be more precise, we prove that for any given strict local minimum point of the Kirchhoff–Routh function, there exists a family of steady vortex patch solutions to the vortex-wave system that shrinks to this point. Here by vortex patch solution of the vortex-wave system we mean that the background vorticity is a piecewise constant function.

The method we use in this paper to construct steady solutions is called the vorticity method, which was first established by Arnold [1978] and further developed by many authors [Burton 1989a; 1989b; Burton and McLeod 1991; Elcrat and Miller 1991; 1995; Turkington 1983a; 1983b]. Roughly speaking, the vorticity method is to maximize the kinetic energy of the fluid under some suitable constraints for the vorticity. For the Euler equations, the kinetic energy of the fluid with bounded vorticity is always finite, but for the vortex-wave system the kinetic energy is infinite due to the presence of point vortices. To overcome this difficulty, we drop the infinite self-energy term for each point vortex. We refer the interested reader to [Cao and Wang 2019] where the energy of the vortex-wave system with a single point vortex was calculated rigorously.

It is worth mentioning that the construction in [Cao et al. 2015] was based on a finite-dimensional reduction argument. The advantage of the method in [Cao et al. 2015] is that solutions concentrating at a given saddle point of the Kirchhoff–Routh function can be constructed. However, nondegeneracy of this saddle point is required in this situation. Using the vorticity method, we are able to construct solutions concentrating at a given strict local minimum point of the Kirchhoff–Routh function, even if the point is degenerate. Another advantage of the vorticity method is that we can analyze the energy of the solution, which is helpful to prove nonlinear stability; see [Burton 2005; Cao and Wang 2017] for example.

This paper is organized as follows. In Section 2, we give the formulation of the vortex-wave system in a bounded domain and state the main result. In Section 3, we solve a maximization problem for the vorticity and study the asymptotic behavior of the maximizers. In Section 4 we prove the main result.

2. Main result

Notation. Let $D \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary. Green's function for $-\Delta$ in D with zero Dirichlet data on ∂D can be written as

$$G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} - h(x, y), \qquad x, y \in D, \tag{2-1}$$

where h is called the regular part of Green's function. Define

$$H(x) := \frac{1}{2}h(x,x),$$

which is usually called the Robin function. Let k be a positive integer, $\kappa_i \in \mathbb{R} \setminus \{0\}$, $i = 1, \dots, k$. Define the Kirchhoff–Routh function as

$$\mathcal{K}_k(x_1, \dots, x_k) = -\sum_{i \neq j, 1 \leq i, j \leq k} \kappa_i \kappa_j G(x_i, x_j) + \sum_{i=1}^{k} \kappa_i^2 h(x_i, x_i), \tag{2-2}$$

where $x_i \in D$ and $x_i \neq x_j$ if $i \neq j$. Note that if $k = 1$, then $\mathcal{K}_1 = 2\kappa_1^2 H$.

Throughout we will use the following notation. For any function g, $\operatorname{supp}(g)$ denotes the support of g. For any real number a, $\operatorname{sgn}(a)$ denotes the sign of a, that is,

$$\operatorname{sgn}(a) := \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases} \tag{2-3}$$

For any Lebesgue measurable set $A \subset \mathbb{R}^2$, $|A|$ denotes the two-dimensional Lebesgue measure of A; I_A denotes the characteristic function of A, that is, $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ elsewhere; $\bar{A}$ denotes the closure of A in the Euclidean topology; and $\operatorname{diam}(A)$ denotes the diameter of A, that is,

$$\operatorname{diam}(A) = \sup_{x, y \in A} |x - y|. \tag{2-4}$$

Vortex-wave system in a bounded domain. We consider an incompressible steady flow confined in D with impermeability boundary condition. The evolution of the velocity field $\boldsymbol{v} = (v_1, v_2)$ and the pressure P is described by the Euler equations

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla P & \text{in } D \times (0, +\infty), \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in } D \times (0, +\infty), \\ \boldsymbol{v} \cdot \nu = 0 & \text{on } \partial D \times (0, +\infty), \end{cases} \tag{2-5}$$

where ν is the outward unit normal of ∂D.

Taking the curl on both sides of the first equation of (2-5), we get the vorticity equation

$$\partial_t \omega + \boldsymbol{v} \cdot \nabla \omega = 0. \tag{2-6}$$

Since $\boldsymbol{v}$ is divergence-free, there is a function ψ, called the stream function, such that

$$\boldsymbol{v} = J\nabla\psi = (\partial_2 \psi, -\partial_1 \psi). \tag{2-7}$$

By the definition of ω (recall (1-2)), it is easy to see that

$$-\Delta\psi = \omega. \tag{2-8}$$

The impermeability boundary condition given in the third equation of (2-5) implies that ψ is a constant on each connected component of ∂D. Since D is simply connected, after suitably adding a constant to ψ we can assume

$$\psi(x,t)=0, \quad x\in\partial D. \tag{2-9}$$

By (2-8) and (2-9), we have

$$\psi(x,t)=(-\Delta)^{-1}\omega(x,t):=\int_D G(x,y)\omega(y,t)\,dy, \quad x\in D. \tag{2-10}$$

For brevity we introduce the notation

$$\partial(f,g):=\nabla f\cdot J\nabla g=\partial_1 f\partial_2 g-\partial_2 f\partial_1 g,$$

then the vorticity equation (2-6) can be written as

$$\begin{cases}\partial_t\omega+\partial(\omega,\psi)=0,\\ \psi=(-\Delta)^{-1}\omega.\end{cases} \tag{2-11}$$

If the vorticity is a Dirac delta measure (also called a point vortex) located at $x\in D$, i.e., $\omega=\delta(x)$, then formally the velocity field it induces is

$$J\nabla(-\Delta)^{-1}\delta(x)=J\nabla G(x,\cdot)=\frac{1}{2\pi}\frac{J(x-\cdot)}{|x-\cdot|^2}-J\nabla h(x,\cdot).$$

Note that this velocity field is singular at x. Due to symmetry, we formally drop the term

$$\frac{1}{2\pi}\frac{J(x-\cdot)}{|x-\cdot|^2},$$

that is, we assume that the velocity at x is $-J\nabla h(x,\cdot)\big|_x=-J\nabla H(x)$, then the evolution of this point vortex is described by the following ODE:

$$\frac{dx}{dt}=-J\nabla H(x). \tag{2-12}$$

Similarly, the evolution of l point vortices can be described by the ODE system

$$\frac{dx_i}{dt}=-\kappa_i J\nabla H(x_i)+\sum_{j=1,j\neq i}^{l}\kappa_j J\nabla_{x_i}G(x_j,x_i), \quad i=1,\dots,l, \tag{2-13}$$

where κ_i is the vorticity strength of the i-th point vortex. System (2-13) is also called the Kirchhoff–Routh equation. It is easy to see that the Kirchhoff–Routh function $\mathcal{K}_l$ is exactly the Hamiltonian of the system.

Now we consider the mixed problem. That is the vorticity consists of a continuously distributed part ω and l point vortices x_i with strength κ_i, $i = 1, \dots, l$. Then the evolution of ω and x_i will obey the equations

$$\begin{cases} \partial_t \omega + \partial\big(\omega, \psi + \sum_{j=1}^{l} \kappa_j G(x_j, \cdot)\big) = 0, \\ \dfrac{dx_i}{dt} = J\nabla\big(\psi + \sum_{j=1, j\neq i}^{l} \kappa_j G(x_j, \cdot) - \kappa_i H\big)(x_i), \quad i = 1, \dots, l, \\ \psi = (-\Delta)^{-1}\omega, \end{cases} \tag{2-14}$$

which together are called the vortex-wave system in D.

In this paper, we confine ourselves to the stationary case, that is, we consider the following system of equations:

$$\begin{cases} \partial\big(\omega, \psi + \sum_{j=1}^{l} \kappa_j G(x_j, \cdot)\big) = 0, \\ \nabla\big(\psi + \sum_{j=1, j\neq i}^{l} \kappa_j G(x_j, \cdot) - \kappa_i H\big)(x_i) = 0, \quad i = 1, \dots, l, \\ \psi = (-\Delta)^{-1}\omega. \end{cases} \tag{2-15}$$

Since we are going to deal with vortex patch solutions which are discontinuous, it is necessary to give the weak formulation of (2-15).

Definition 2.1. Let $\omega \in L^\infty(D)$, $x_i \in D$, $i = 1, \dots, l$, then $(\omega, x_1, \dots, x_l)$ is called a weak solution to (2-15) if it satisfies

$$\begin{cases} \int_D \omega(x)\partial\big(\psi(x) + \sum_{j=1}^{l} \kappa_j G(x_j, x), \phi(x)\big)\,dx = 0, & \forall \phi \in C_c^\infty(D), \\ \nabla\big(\psi(x) + \sum_{j=1, j\neq i}^{l} \kappa_j G(x_j, x) - \kappa_i H(x)\big)\big|_{x=x_i} = 0, & i = 1, \dots, l, \end{cases} \tag{2-16}$$

where $\psi = (-\Delta)^{-1}\omega$.

Remark 2.2. Note that since $\omega \in L^\infty(D)$, by L^p estimate $\psi \in W^{2,p}(D)$ for any $1 < p < +\infty$, then by Sobolev embedding $\psi \in C^{1,\alpha}(\bar{D})$ for any $0 < \alpha < 1$.

Remark 2.3. Definition 2.1 can be derived formally from (2-15) by integration by parts; see [Cao and Wang 2019] for the detailed calculations.

Main result. Our main result in this paper is the following theorem:

Theorem 2.4. *Let k, p, l be positive integers such that $p+l = k$, and κ_i, $i = 1, \dots, k$, be k real numbers such that $\kappa_i \neq 0$. Suppose that $(\bar{x}_1, \dots, \bar{x}_k)$ is a strict local minimum point of $\mathcal{K}_k$ defined by (2-2), where $\bar{x}_i \in D$ and $\bar{x}_i \neq \bar{x}_j$ for $i \neq j$. Then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, (2-16) has a solution $(\omega^\lambda, x_{p+1}^\lambda, \dots, x_k^\lambda)$ satisfying*

$$\omega^\lambda = \sum_{i=1}^{p} \omega_i^\lambda, \quad \int_D \omega_i^\lambda(x)\,dx = \kappa_i, \quad \omega_i^\lambda = \operatorname{sgn}(\kappa_i)\lambda I_{A_i^\lambda}, \quad i = 1, \dots, p, \tag{2-17}$$

where A_i^λ has the form

$$A_i^\lambda = \left\{ x \in D \;\middle|\; \operatorname{sgn}(\kappa_i)\left(\psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x)\right) > c_i^\lambda \right\} \cap B_\delta(\bar{x}_i) \tag{2-18}$$

for some $c_i^\lambda > 0$ and $\delta > 0$ (δ does not depend on λ). Moreover,

$$\operatorname{diam}(A_i^\lambda) \le C\lambda^{-\frac{1}{2}}, \quad \lim_{\lambda\to+\infty}\left| \frac{1}{\kappa_i}\int_D x\omega_i^\lambda(x)\,dx - \bar{x}_i \right| = 0, \quad i = 1, \dots, p, \tag{2-19}$$

$$\lim_{\lambda\to+\infty} |x_j^\lambda - \bar{x}_j| = 0, \quad j = p+1, \dots, k, \tag{2-20}$$

where C is a positive number independent of λ.

Remark 2.5. By (2-19), we see that A_i^λ shrinks to $\bar{x}_i$ as $\lambda \to +\infty$, that is,

$$\lim_{\lambda\to+\infty} \sup_{x\in A_i^\lambda} |x - \bar{x}_i| = 0, \quad i = 1, \dots, p.$$

Consequently $\overline{A_i^\lambda} \subset B_\delta(\bar{x}_i)$ for sufficiently large λ.

3. Variational problem

Let $(\bar{x}_1, \dots, \bar{x}_k)$ be a strict local minimum point of $\mathcal{K}_k$, where $\bar{x}_i \in D$ and $\bar{x}_i \ne \bar{x}_j$ for $i \ne j$. Without loss of generality, we assume that $(\bar{x}_1, \dots, \bar{x}_k)$ is the unique minimum point of $\mathcal{K}_k$ on $\overline{B_{\delta_0}(\bar{x}_1)} \times \cdots \times \overline{B_{\delta_0}(\bar{x}_k)}$, where $\delta_0 > 0$ is a small positive number such that $\overline{B_{\delta_0}(\bar{x}_i)} \subset D$ and $\overline{B_{\delta_0}(\bar{x}_i)} \cap \overline{B_{\delta_0}(\bar{x}_j)} = \varnothing$ for all $i, j = 1, \dots, k$ and $i \ne j$.

Remark 3.1. To our knowledge, there is no general result that guarantees the existence of a strict local minimum point of $\mathcal{K}_k$ for $k \ge 2$. Some special cases are as follows: if $k = 1$ and D is convex, by [Caffarelli and Friedman 1985] $\mathcal{K}_1$ is a strictly convex function in D, thus has a unique minimum point; if $k \ge 2$, $\kappa_i > 0$ for each i and D is convex, by [Grossi and Takahashi 2010] there does not exist any critical point of $\mathcal{K}_k$; if $k = 2$, some examples of strict local minimum points of $\mathcal{K}_2$ are given computationally in [Elcrat and Miller 1995]. More related results can also be found in [Bartsch and Pistoia 2015; Bartsch et al. 2010].

Let λ be a positive real number. Define

$$N_p^\lambda = \left\{ \omega \in L^\infty(D) \;\middle|\; \omega = \sum_{i=1}^{p} \omega_i, \operatorname{supp}(\omega_i) \subset B_\delta(\bar{x}_i), \int_D \omega_i(x)\,dx = \kappa_i, 0 \le \operatorname{sgn}(\kappa_i)\omega_i \le \lambda \right\},$$

where $\delta < \frac{1}{2}\delta_0$ is a small positive number to be determined later. Hereafter we assume that λ is sufficiently large such that N_p^λ is not empty.

For $(\omega, x_{p+1}, \dots, x_k) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$, define

$$\mathcal{E}(\omega, x_{p+1}, \dots, x_k) := E(\omega) + \sum_{j=p+1}^{k} \kappa_j \psi(x_j) + \frac{1}{2} \sum_{\substack{i \neq j \\ p+1 \leq i, j \leq k}} \kappa_i \kappa_j G(x_i, x_j) - \sum_{j=p+1}^{k} \kappa_j^2 H(x_j),$$

where

$$E(\omega) := \frac{1}{2} \int_D \int_D G(x, y) \omega(x) \omega(y) \, dx \, dy,$$
$$\psi(x) = (-\Delta)^{-1} \omega(x) := \int_D G(x, y) \omega(y) \, dy.$$

Let us explain the definition of $\mathcal{E}$ briefly. The first term $E(\omega)$ represents the self-interacting energy of the background vorticity ω, the second term represents the mutual interaction energy between the background vorticity and the l point vortices, the third term represents the total interaction energy between any two different point vortices, and the fourth term represents the interaction energy between the l point vortices and the boundary of D. As we have mentioned in Section 1, we have dropped the infinite self-interacting energy for each point vortex.

Now we consider the maximization of $\mathcal{E}$ on $N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$.

Lemma 3.2. *For fixed λ, there exists*

$$(\omega^\lambda, x_{p+1}^\lambda, \dots, x_k^\lambda) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$$

such that

$$\text{(3-1)} \quad \mathcal{E}(\omega^\lambda, x_{p+1}^\lambda, \dots, x_k^\lambda) = \sup_{(\omega, x_{p+1}, \dots, x_k) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}} \mathcal{E}(\omega, x_{p+1}, \dots, x_k).$$

Proof. First, for any $(\omega, x_{p+1}, \dots, x_k) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$, since $G \in L^1(D \times D)$, $\mathrm{dist}(B_{\delta_0}(\bar{x}_i), B_{\delta_0}(\bar{x}_j)) > 0$ for any $i \neq j$, $p+1 \leq i, j \leq k$, and $\mathrm{dist}(B_{\delta_0}(\bar{x}_i), \partial D) > 0$, $p+1 \leq i \leq k$, we have

$$\text{(3-2)} \quad \begin{aligned} &\mathcal{E}(\omega, x_{p+1}, \dots, x_k) \\ &= \frac{1}{2} \int_D \int_D G(x, y) \omega(x) \omega(y) \, dx \, dy + \sum_{j=p+1}^{k} \kappa_j \psi(x_j) \\ &\qquad + \frac{1}{2} \sum_{\substack{i \neq j \\ p+1 \leq i, j \leq k}} \kappa_i \kappa_j G(x_i, x_j) - \sum_{j=p+1}^{k} \kappa_j^2 H(x_j) \\ &\leq \frac{\lambda^2}{2} \int_D \int_D |G(x, y)| \, dx \, dy + \sum_{j=p+1}^{k} |\kappa_j| |\psi|_{L^\infty(D)} + \sum_{j=p+1}^{k} \kappa_j^2 |H|_{L^\infty(B_\delta(\bar{x}_j))}. \end{aligned}$$

Thus

$$M := \sup_{(\omega, x_{p+1}, \dots, x_k) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}} \mathcal{E}(\omega, x_{p+1}, \dots, x_k) < +\infty.$$

Now we choose $(\omega^n, x_{p+1}^n, \dots, x_k^n) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$ such that

$$\lim_{n \to +\infty} \mathcal{E}(\omega^n, x_{p+1}^n, \dots, x_k^n) = M.$$

Since N_p^λ is weakly star closed in $L^\infty(D)$ (for a detailed proof of this fact, see Theorem 2.1 in [Cao et al. 2019b]) and $\overline{B_\delta(\bar{x}_j)}$ is closed in the Euclidean topology for $j = p+1, \dots, k$, there exists $(\omega^\lambda, x_{p+1}^\lambda, \dots, x_k^\lambda) \in N_p^\lambda \times \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$ such that (up to a subsequence)

$$\begin{aligned} \omega^n &\to \omega^\lambda, && \text{weakly star in } L^\infty(D), \\ x_j^n &\to x_j^\lambda, && j = p+1, \dots, k. \end{aligned}$$

Then obviously

$$\mathcal{E}(\omega^\lambda, x_{p+1}^\lambda, \dots, x_k^\lambda) = \lim_{n \to +\infty} \mathcal{E}(\omega^n, x_{p+1}^n, \dots, x_k^n) = M,$$

which completes the proof. □

Remark 3.3. It is easy to see that

$$E(\omega) + \sum_{j=p+1}^{k} \kappa_j \psi(x_j^\lambda) \le E(\omega^\lambda) + \sum_{j=p+1}^{k} \kappa_j \psi^\lambda(x_j^\lambda) \tag{3-3}$$

for any $\omega \in N_p^\lambda$, and

$$\begin{aligned} \sum_{j=p+1}^{k} \kappa_j \psi^\lambda(x_j) + \frac{1}{2} \sum_{\substack{i \ne j \\ p+1 \le i, j \le k}} \kappa_i \kappa_j G(x_i, x_j) - \sum_{j=p+1}^{k} \kappa_j^2 H(x_j) \\ \le \sum_{j=p+1}^{k} \kappa_j \psi^\lambda(x_j^\lambda) + \frac{1}{2} \sum_{\substack{i \ne j \\ p+1 \le i, j \le k}} \kappa_i \kappa_j G(x_i^\lambda, x_j^\lambda) - \sum_{j=p+1}^{k} \kappa_j^2 H(x_j^\lambda) \end{aligned} \tag{3-4}$$

for any $(x_{p+1}, \dots, x_k) \in \overline{B_\delta(\bar{x}_{p+1})} \times \cdots \times \overline{B_\delta(\bar{x}_k)}$, where $\psi^\lambda = (-\Delta)^{-1}\omega^\lambda$ and $\psi = (-\Delta)^{-1}\omega$.

Since $\omega^\lambda \in N_p^\lambda$, we can write $\omega^\lambda = \sum_{i=1}^{p} \omega_i^\lambda$, where $\int_D \omega_i^\lambda(x)\, dx = \kappa_i$, $\operatorname{supp}(\omega_i^\lambda) \subset B_\delta(\bar{x}_i)$ and $0 \le \operatorname{sgn}(\kappa_i)\omega_i^\lambda \le \lambda$. The profile of each ω_i^λ is as follows.

Lemma 3.4. *For* $i = 1, \dots, p$, ω_i^λ *has the form*

$$\omega_i^\lambda = \operatorname{sgn}(\kappa_i)\lambda I_{A_i^\lambda},$$

where

$$A_i^\lambda := \left\{ x \in D \;\middle|\; \operatorname{sgn}(\kappa_i)\left(\psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x)\right) > c_i^\lambda \right\} \cap B_\delta(\bar{x}_i)$$

for some $c_i^\lambda \in \mathbb{R}$ *depending on* λ *and* δ.

Proof. To make it clear, we divide the proof into two cases.

Case 1: $\kappa_i > 0$. For $s > 0$ we define a family of test functions $\omega_s^\lambda = \omega^\lambda + s(z_0 - z_1)$, where

$$\begin{cases} z_0, z_1 \in L^\infty(D),\, z_0, z_1 \geq 0,\, \int_D z_0(x)\,dx = \int_D z_1(x)\,dx, \\ \operatorname{supp}(z_0), \operatorname{supp}(z_1) \subset B_\delta(\bar{x}_i), \\ z_0 = 0 \text{ in } B_\delta(\bar{x}_i) \setminus \{\omega_i^\lambda \leq \lambda - \mu\}, \\ z_1 = 0 \text{ in } B_\delta(\bar{x}_i) \setminus \{\omega_i^\lambda \geq \mu\}, \end{cases} \tag{3-5}$$

where $\mu \in (0, \lambda)$. It is not hard to check that for fixed z_0, z_1 and μ, if s is sufficiently small (depending on z_0, z_1, μ), then $\omega_s^\lambda \in N_p^\lambda$. By (3-3) we have

$$\frac{d}{ds}\left(E(\omega_s^\lambda) + \sum_{j=p+1}^{k} \kappa_j \psi_s^\lambda(x_j^\lambda)\right)\Bigg|_{s=0^+} \leq 0,$$

where $\psi_s^\lambda = (-\Delta)^{-1}\omega_s^\lambda$. That is,

$$\int_D \left(\psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x)\right) z_0(x)\,dx \leq \int_D \left(\psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x)\right) z_1(x)\,dx.$$

Since z_0, z_1, μ are chosen arbitrarily as above we have

$$\sup_{\{\omega_i^\lambda < \lambda\} \cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) \leq \inf_{\{\omega_i^\lambda > 0\} \cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right).$$

But $\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)$ is a continuous function on $\overline{B_\delta(\bar{x}_i)}$ (notice that $x_j^\lambda \notin \overline{B_\delta(\bar{x}_i)}$ for $p + 1 \leq j \leq k$), so we obtain

$$\sup_{\{\omega_i^\lambda < \lambda\} \cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right) = \inf_{\{\omega_i^\lambda > 0\} \cap B_\delta(\bar{x}_i)} \left(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\right).$$

Define

$$c_i^\lambda := \sup_{\{\omega_i^\lambda<\lambda\}\cap B_\delta(\bar{x}_i)} \Big(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\Big) = \inf_{\{\omega_i^\lambda>0\}\cap B_\delta(\bar{x}_i)} \Big(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\Big). \tag{3-6}$$

It is easy to see that

$$\omega_i^\lambda \equiv \lambda \quad \text{a.e. on } \Big\{x \in D \,\Big|\, \psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) > c_i^\lambda\Big\} \cap B_\delta(\bar{x}_i), \tag{3-7}$$

$$\omega_i^\lambda \equiv 0 \quad \text{a.e. on } \Big\{x \in D \,\Big|\, \psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) < c_i^\lambda\Big\} \cap B_\delta(\bar{x}_i). \tag{3-8}$$

On the level set $\{x \in D \mid \psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) = c_i^\lambda\} \cap B_\delta(\bar{x}_i)$, we have

$$\nabla(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)) = 0, \quad \text{a.e.,} \tag{3-9}$$

which gives

$$\omega_i^\lambda = -\Delta(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)) = 0, \quad \text{a.e..} \tag{3-10}$$

Equations (3-7), (3-8) and (3-10) together give

$$\omega_i^\lambda = \lambda I_{\{\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot) > c_i^\lambda\}\cap B_\delta(\bar{x}_i)}, \tag{3-11}$$

which completes the proof of Case 1.

Case 2: $\kappa_i < 0$. The argument is similar. For $s > 0$, define $\omega_s^\lambda = \omega^\lambda + s(z_1 - z_0)$, where

$$\begin{cases} z_0, z_1 \in L^\infty(D),\, z_0, z_1 \geq 0,\, \int_D z_0(x)\,dx = \int_D z_1(x)\,dx, \\ \operatorname{supp}(z_0),\, \operatorname{supp}(z_1) \subset B_\delta(\bar{x}_i), \\ z_0 = 0 \text{ in } B_\delta(\bar{x}_i) \setminus \{\omega_i^\lambda \geq \mu - \lambda\}, \\ z_1 = 0 \text{ in } B_\delta(\bar{x}_i) \setminus \{\omega_i^\lambda \leq -\mu\}, \end{cases} \tag{3-12}$$

where $\mu \in (0, \lambda)$. Then

$$\frac{d}{ds}\Big(E(\omega_s^\lambda) + \sum_{j=p+1}^{k} \kappa_j \psi_s^\lambda(x_j^\lambda)\Big)\Big|_{s=0^+} \leq 0.$$

Repeating the argument in Case 1, we obtain

$$\sup_{\{\omega_i^\lambda<0\}\cap B_\delta(\bar{x}_i)}\Big(\psi^\lambda+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,\cdot)\Big)=\inf_{\{\omega_i^\lambda>-\lambda\}\cap B_\delta(\bar{x}_i)}\Big(\psi^\lambda+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,\cdot)\Big),$$

then we can define

$$\begin{aligned}c_i^\lambda &:= -\sup_{\{\omega_i^\lambda<0\}\cap B_\delta(\bar{x}_i)}\Big(\psi^\lambda+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,\cdot)\Big)\\ &= -\inf_{\{\omega_i^\lambda>-\lambda\}\cap B_\delta(\bar{x}_i)}\Big(\psi^\lambda+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,\cdot)\Big).\end{aligned}\tag{3-13}$$

Similarly we have

$$\omega_i^\lambda=-\lambda I_{\{\psi^\lambda+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,\cdot)<-c_i^\lambda\}\cap B_\delta(\bar{x}_i)}.$$

□

Lemma 3.5. *For the c_i^λ given in Lemma 3.4, we have the estimate*

$$c_i^\lambda>-\frac{|\kappa_i|}{2\pi}\ln\delta-C,$$

where $C>0$ is independent of λ and δ.

Proof. We only prove the case $\kappa_i>0$. The other case is similar. By the definition of c_i^λ (recall (3-6)), we have

$$\begin{aligned}c_i^\lambda &= \inf_{\{\omega_i^\lambda>0\}\cap B_\delta(\bar{x}_i)}\Big(\psi^\lambda+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,\cdot)\Big)\\ &\geq \inf_{B_\delta(\bar{x}_i)}\Big(\psi^\lambda+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,\cdot)\Big)\qquad\text{(by maximum principle)}\\ &\geq \inf_{\partial B_\delta(\bar{x}_i)}\Big(\psi^\lambda+\sum_{j=p+1}^{k}\kappa_j G(x_j^\lambda,\cdot)\Big)\\ &\geq \inf_{\partial B_\delta(\bar{x}_i)}\psi^\lambda-C\end{aligned}\tag{3-14}$$

for some $C>0$ not depending on λ and δ. But for any $x\in\partial B_\delta(\bar{x}_i)$,

$$\begin{aligned}\psi^\lambda(x) &= -\frac{1}{2\pi}\int_D\ln|x-y|\omega_i^\lambda(y)\,dy-\int_D h(x,y)\omega_i^\lambda(y)\,dy\\ &\qquad\qquad+\int_D G(x,y)\sum_{j=1,j\neq i}^{p}\omega_j^\lambda(y)\,dy\\ &\geq -\frac{\kappa_i}{2\pi}\ln|2\delta|-C\end{aligned}\tag{3-15}$$

for some $C > 0$ not depending on λ and δ. Here we use the fact that

$$\mathrm{dist}(B_{\delta_0}(\bar{x}_i), B_{\delta_0}(\bar{x}_j)) > 0$$

for any $i \neq j$, $1 \leq i, j \leq p$, and $\mathrm{dist}(B_{\delta_0}(\bar{x}_i), \partial D) > 0$, $1 \leq i \leq p$.

Combining (3-14) with (3-15) we get the desired result. □

Lemma 3.6. *For δ sufficiently small, not depending on λ, the following assertion holds true:*

$$\operatorname{sgn}(\kappa_i)\Bigl(\psi^\lambda + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, \cdot)\Bigr) - c_i^\lambda \leq 0 \quad \text{on } \partial B_{\delta_0}(\bar{x}_i) \tag{3-16}$$

for each $1 \leq i \leq p$.

Proof. Let $1 \leq i \leq p$ be fixed. By Lemma 3.5 it suffices to show that

$$\sup_{x \in \partial B_{\delta_0}(\bar{x}_i)} \Bigl| \psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) \Bigr| \leq C$$

for some $C > 0$ not depending on λ and δ. In fact, since for any $j \neq i$, $j = p+1, \dots, k$, $\mathrm{dist}(B_{\delta_0}(\bar{x}_i), B_{\delta_0}(\bar{x}_j)) > 0$, we have

$$\sup_{x \in \partial B_{\delta_0}(\bar{x}_i)} \Bigl| \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x) \Bigr| \leq C.$$

It remains to show that $\sup_{x \in \partial B_{\delta_0}(\bar{x}_i)} |\psi^\lambda(x)| \leq C$. For any $x \in \partial B_{\delta_0}(\bar{x}_i)$, we estimate $\psi^\lambda(x)$ as follows:

$$\begin{aligned}
|\psi^\lambda(x)| \leq{}& \Bigl| \int_D -\frac{1}{2\pi} \ln|x-y| \omega_i^\lambda(y)\,dy \Bigr| + \Bigl| \int_D h(x,y)\omega_i^\lambda(y)\,dy \Bigr| \\
&\qquad + \Bigl| \int_D G(x,y) \sum_{j \neq i, j=1}^{p} \omega_j^\lambda(y)\,dy \Bigr| \\
\leq{}& -\frac{|\kappa_i|}{2\pi} \ln(\delta_0 - \delta) + C \\
\leq{}& -\frac{|\kappa_i|}{2\pi} \ln\Bigl(\frac{\delta_0}{2}\Bigr) + C \qquad \Bigl(\text{recall that } \delta < \frac{\delta_0}{2}\Bigr).
\end{aligned}$$

Thus the proof is completed. □

From now on we fix δ such that (3-16) holds true.

Now we analyze the asymptotic behavior of the maximizer $(\omega^\lambda, x_{p+1}^\lambda, \dots, x_k^\lambda)$ as $\lambda \to +\infty$. We will show that as $\lambda \to +\infty$, the support of ω_i^λ shrinks to $\bar{x}_i$ for

$i = 1, \dots, p$ and $x_j^\lambda \to \bar{x}_j$ for $j = p+1, \dots, k$. To achieve this goal, we estimate the energy of the background vorticity first.

For simplicity, hereafter we will use C to denote various positive numbers independent of λ, and $o(1)$ to denote various quantities that go to zero as $\lambda \to +\infty$.

Lemma 3.7. $E(\omega^\lambda) \geq -\frac{1}{4\pi}\sum_{i=1}^{p} \kappa_i^2 \ln \varepsilon_i - C$, *where* ε_i *is the positive number such that* $\lambda\pi\varepsilon_i^2 = |\kappa_i|$.

Proof. Choose the test function $\omega = \sum_{i=1}^{p} \omega_i$, where $\omega_i = \operatorname{sgn}(\kappa_i)\lambda I_{B_{\varepsilon_i}(\bar{x}_i)}$. It is obvious that $\omega \in N_p^\lambda$, so by (3-3),

$$E(\omega) + \sum_{j=p+1}^{k} \kappa_j \psi(x_j^\lambda) \leq E(\omega^\lambda) + \sum_{j=p+1}^{k} \kappa_j \psi^\lambda(x_j^\lambda),$$

where $\psi = (-\Delta)^{-1}\omega$. It is easy to check that

$$\left|\sum_{j=p+1}^{k} \kappa_j \psi(x_j^\lambda)\right| \leq C, \qquad \left|\sum_{j=p+1}^{k} \kappa_j \psi^\lambda(x_j^\lambda)\right| \leq C,$$

so

$$E(\omega^\lambda) \geq E(\omega) - C. \tag{3-17}$$

On the other hand,

$$\begin{aligned} E(\omega) &= \frac{1}{2}\int_D \int_D G(x,y)\omega(x)\omega(y)\,dx\,dy \\ &= \sum_{i=1}^{p} E(\omega_i) + \sum_{1\leq i<j\leq p} \int_D \int_D G(x,y)\omega_i(x)\omega_j(y)\,dx\,dy \\ &\geq \sum_{i=1}^{p} E(\omega_i) - C. \end{aligned} \tag{3-18}$$

Each $E(\omega_i)$ can be calculated directly as follows

$$\begin{aligned} E(\omega_i) &= \frac{1}{2}\int_D \int_D G(x,y)\omega_i(x)\omega_i(y)\,dx\,dy \\ &= \frac{1}{2}\int_{B_{\varepsilon_i}(\bar{x}_i)} \int_{B_{\varepsilon_i}(\bar{x}_i)} -\frac{1}{2\pi}\ln|x-y|\omega_i(x)\omega_i(y)\,dx\,dy \\ &\qquad\qquad - \frac{1}{2}\int_{B_{\varepsilon_i}(\bar{x}_i)} \int_{B_{\varepsilon_i}(\bar{x}_i)} h(x,y)\omega_i(x)\omega_i(y)\,dx\,dy \\ &\geq -\frac{\kappa_i^2}{4\pi}\ln(2\varepsilon_i) - \sum_{i=1}^{p} \kappa_i^2 H(\bar{x}_i) + o(1). \end{aligned} \tag{3-19}$$

Now (3-17), (3-18) and (3-19) together give the desired result. □

Now define

$$T^\lambda:=\sum_{i=1}^{p}\frac{\operatorname{sgn}(\kappa_i)}{2}\int_D\omega_i^\lambda(x)\left(\operatorname{sgn}(\kappa_i)(\psi^\lambda(x)+\sum_{j=p+1}^{k}\kappa_jG(x_j^\lambda,x))-c_i^\lambda\right)dx, \tag{3-20}$$

which represents the total kinetic energy of the fluid on the vorticity set $\bigcup_{i=1}^{p}A_i^\lambda$.

Lemma 3.8. $T^\lambda\le C$.

Proof. For simplicity we denote

$$\zeta_i^\lambda:=\operatorname{sgn}(\kappa_i)\left(\psi^\lambda+\sum_{j=p+1}^{k}\kappa_jG(x_j^\lambda,\cdot)\right)-c_i^\lambda.$$

It suffices to prove that for each $1\le i\le p$,

$$\operatorname{sgn}(\kappa_i)\int_D\omega_i^\lambda\zeta_i^\lambda\,dx\le C.$$

On the one hand, by Hölder's inequality and Sobolev embedding $W^{1,1}(B_\delta(\bar{x}_i))\hookrightarrow L^2(B_\delta(\bar{x}_i))$, we have

$$\begin{aligned}\operatorname{sgn}(\kappa_i)\int_D\omega_i^\lambda\zeta_i^\lambda\,dx&=\lambda\int_{A_i^\lambda}\zeta_i^\lambda\,dx\\&\le\lambda|A_i^\lambda|^{\frac12}\left(\int_{A_i^\lambda}|\zeta_i^\lambda|^2\,dx\right)^{\frac12}\quad(\zeta_i^\lambda\ge0\text{ on }A_i^\lambda)\\&\le\lambda|A_i^\lambda|^{\frac12}\left(\int_{B_\delta(\bar{x}_i)}|(\zeta_i^\lambda)^+|^2\,dx\right)^{\frac12}\\&\le C\lambda|A_i^\lambda|^{\frac12}\left(\int_{B_\delta(\bar{x}_i)}(\zeta_i^\lambda)^+\,dx+\int_{B_\delta(\bar{x}_i)}|\nabla(\zeta_i^\lambda)^+|\,dx\right)\\&\le C\lambda|A_i^\lambda|^{\frac12}\left(\int_{A_i^\lambda}\zeta_i^\lambda\,dx+\int_{A_i^\lambda}|\nabla\zeta_i^\lambda|\,dx\right)\\&\le C\lambda|A_i^\lambda|^{\frac12}|A_i^\lambda|^{\frac12}\left(\int_{A_i^\lambda}|\nabla\zeta_i^\lambda|^2\,dx\right)^{\frac12}+C|A_i^\lambda|^{\frac12}\operatorname{sgn}(\kappa_i)\int_D\omega_i^\lambda\zeta_i^\lambda\,dx\\&\le C\left(\int_{A_i^\lambda}|\nabla\zeta_i^\lambda|^2\,dx\right)^{\frac12}+o(1)\operatorname{sgn}(\kappa_i)\int_D\omega_i^\lambda\zeta_i^\lambda\,dx,\end{aligned} \tag{3-21}$$

which implies

$$\operatorname{sgn}(\kappa_i)\int_D\omega_i^\lambda\zeta_i^\lambda\,dx\le C\left(\int_{A_i^\lambda}|\nabla\zeta_i^\lambda|^2\,dx\right)^{\frac12}. \tag{3-22}$$

On the other hand, since $\zeta_i^\lambda \leq 0$ on $\partial B_{\delta_0}(\bar{x}_i)$ (recall (3-16)), integration by parts gives

$$\begin{aligned} \operatorname{sgn}(\kappa_i) \int_D \omega_i^\lambda \zeta_i^\lambda \, dx &= \operatorname{sgn}(\kappa_i) \int_{B_{\delta_0}(\bar{x}_i)} \omega_i^\lambda (\zeta_i^\lambda)^+ \, dx \\ &= \int_{B_{\delta_0}(\bar{x}_i)} |\nabla(\zeta_i^\lambda)^+|^2 \, dx \geq \int_{A_i^\lambda} |\nabla \zeta_i^\lambda|^2 \, dx. \end{aligned} \tag{3-23}$$

Combining (3-22) with (3-23) we complete the proof. □

Lemma 3.9. $\sum_{i=1}^p c_i^\lambda |\kappa_i| \geq -\frac{1}{2\pi} \sum_{i=1}^p \kappa_i^2 \ln \varepsilon_i - C$.

Proof. By the definition of T^λ, the following identity holds true:

$$T^\lambda = E(\omega^\lambda) + \frac{1}{2} \sum_{i=1}^p \sum_{j=p+1}^k \int_D \omega_i^\lambda(x) \kappa_j G(x_j^\lambda, x) \, dx - \frac{1}{2} \sum_{i=1}^p c_i^\lambda |\kappa_i|.$$

Since $\operatorname{dist}(B_{\delta_0}(\bar{x}_i), B_{\delta_0}(\bar{x}_j)) > 0$ for any $i = 1, \ldots, p$ and $j = p+1, \ldots, k$, it is easy to see that

$$\left| \frac{1}{2} \sum_{i=1}^p \sum_{j=p+1}^k \int_D \omega_i^\lambda(x) \kappa_j G(x_j^\lambda, x) \, dx \right| \leq C,$$

then the desired result follows from Lemmas 3.7 and 3.8. □

Now we are ready to estimate the size of $\operatorname{supp}(\omega_i^\lambda)$.

Lemma 3.10. *There exists $R_0 > 0$ independent of λ such that for $i = 1, \ldots, p$,*

$$\operatorname{diam}(\operatorname{supp}(\omega_i^\lambda)) \leq R_0 \varepsilon_i.$$

Proof. For each $i = 1, \ldots, p$, any $x_i \in \operatorname{supp}(\omega_i^\lambda)$, we have

$$\operatorname{sgn}(\kappa_i) \Bigl(\psi^\lambda(x_i) + \sum_{j=p+1}^k \kappa_j G(x_j^\lambda, x_i) \Bigr) \geq c_i^\lambda.$$

It is easy to see that

$$\Bigl| \sum_{j=p+1}^k \kappa_j G(x_j^\lambda, x_i) \Bigr| \leq C,$$

so we have

$$\operatorname{sgn}(\kappa_i) \psi^\lambda(x_i) \geq c_i^\lambda - C,$$

which gives

$$\int_D -\frac{1}{2\pi} \ln |x_i - y| |\omega_i^\lambda(y)| \, dy \geq c_i^\lambda - C. \tag{3-24}$$

Combining (3-24) with Lemma 3.9 we obtain

$$\sum_{i=1}^{p}|\kappa_i|\int_D -\frac{1}{2\pi}\ln|x_i-y||\omega_i^\lambda(y)|\,dy \geq -\frac{1}{2\pi}\sum_{i=1}^{p}\kappa_i^2\ln\varepsilon_i - C,$$

or equivalently

$$\sum_{i=1}^{p}\frac{|\kappa_i|}{2\pi}\int_D \ln\frac{\varepsilon_i}{|x_i-y|}|\omega_i^\lambda(y)|\,dy \geq -C.$$

For any $R>1$ to be determined, we have

$$\sum_{i=1}^{p}\frac{|\kappa_i|}{2\pi}\int_{D\backslash B_{R\varepsilon_i}(x_i)} \ln\frac{\varepsilon_i}{|x_i-y|}|\omega_i^\lambda(y)|\,dy + \sum_{i=1}^{p}\frac{|\kappa_i|}{2\pi}\int_{B_{R\varepsilon_i}(x_i)} \ln\frac{\varepsilon_i}{|x_i-y|}|\omega_i^\lambda(y)|\,dy \geq -C. \tag{3-25}$$

The second integral in (3-25) is bounded (in fact, it can be calculated explicitly). That is,

$$\left|\sum_{i=1}^{p}\frac{|\kappa_i|}{2\pi}\int_{B_{R\varepsilon_i}(x_i)} \ln\frac{\varepsilon_i}{|x_i-y|}|\omega_i^\lambda(y)|\,dy\right| \leq C,$$

from which we deduce

$$\sum_{i=1}^{p}\frac{|\kappa_i|}{2\pi}\int_{D\backslash B_{R\varepsilon_i}(x_i)} \ln\frac{\varepsilon_i}{|x_i-y|}|\omega_i^\lambda(y)|\,dy \geq -C. \tag{3-26}$$

Therefore

$$\sum_{i=1}^{p}\frac{|\kappa_i|}{2\pi}\int_{D\backslash B_{R\varepsilon_i}(x_i)} |\omega_i^\lambda(y)|\,dy \leq \frac{C}{\ln R}, \tag{3-27}$$

and consequently for each $1\leq i\leq p$ there holds

$$\frac{|\kappa_i|}{2\pi}\int_{D\backslash B_{R\varepsilon_i}(x_i)} |\omega_i^\lambda(y)|\,dy \leq \frac{C}{\ln R}. \tag{3-28}$$

Since $\int_D|\omega_i^\lambda(y)|\,dy=|\kappa_i|$, we can choose R large enough such that

$$\int_{B_{R\varepsilon_i}(x_i)} |\omega_i^\lambda(y)|\,dy > \frac{|\kappa_i|}{2}, \qquad i=1,\dots,p. \tag{3-29}$$

Now we claim that

$$\operatorname{diam}(\operatorname{supp}(\omega_i^\lambda)) \leq 2R\varepsilon_i.$$

In fact, suppose that $\operatorname{diam}(\operatorname{supp}(\omega_i^\lambda)) > 2R\varepsilon_i$ Then we can choose $x_i, y_i \in \operatorname{supp}(\omega_i^\lambda)$ such that $|x_i - y_i| > 2R\varepsilon_i$, then by (3-29) (recall that in (3-29) $x_i \in \operatorname{supp}(\omega_i^\lambda)$ is arbitrary)

$$\int_D |\omega_i^\lambda(y)|\,dy \geq \int_{B_{R\varepsilon_i}(x_i)} |\omega_i^\lambda(y)|\,dy + \int_{B_{R\varepsilon_i}(y_i)} |\omega_i^\lambda(y)|\,dy > |\kappa_i|,$$

which is a contradiction.

Finally by choosing $R_0 = 2R$ we complete the proof. □

By now we have constructed $\omega_i^\lambda \in N_p^\lambda$, $i = 1, \dots, p$, and $x_j^\lambda \in \overline{B_\delta(\bar{x}_j)}$, $j = p+1, \dots, k$. Moreover, we have shown that the diameter of $\operatorname{supp}(\omega_i^\lambda)$ vanishes as $\lambda \to +\infty$. To analyze their limiting positions, we define the center of ω_i^λ as

$$z_i^\lambda := \frac{1}{\kappa_i} \int_D x\omega_i^\lambda\,dx, \qquad i = 1, \dots, p.$$

It is easy to see that $z_i^\lambda \in \overline{B_\delta(\bar{x}_i)}$.

Lemma 3.11. *$z_i^\lambda \to \bar{x}_i$ for $1 \leq i \leq p$ and $x_j^\lambda \to \bar{x}_j$ for $p+1 \leq j \leq k$ as $\lambda \to +\infty$.*

Proof. Up to a subsequence we assume that $z_i^\lambda \to z_i \in \overline{B_\delta(\bar{x}_i)}$, $1 \leq i \leq p$, and $x_j^\lambda \to z_j \in \overline{B_\delta(\bar{x}_j)}$, $p+1 \leq j \leq k$. It suffices to show that $(z_1, \dots, z_k) = (\bar{x}_1, \dots, \bar{x}_k)$.

Define $\omega = \sum_{i=1}^p \omega_i$, where $\omega_i = \operatorname{sgn}(\kappa_i)\lambda I_{B_{\varepsilon_i}(\bar{x}_i)}$. It is easy to see that $\omega \in N_p^\lambda$, so we have

$$\mathcal{E}(\omega, \bar{x}_{p+1}, \dots, \bar{x}_k) \leq \mathcal{E}(\omega^\lambda, x_{p+1}^\lambda, \dots, x_k^\lambda),$$

that is,

$$\begin{aligned} E(\omega) &+ \sum_{j=p+1}^k \kappa_j \psi(\bar{x}_j) + \frac{1}{2} \sum_{\substack{i \neq j \\ p+1 \leq i,j \leq k}} \kappa_i\kappa_j G(\bar{x}_i, \bar{x}_j) - \sum_{j=p+1}^k \kappa_j^2 H(\bar{x}_j) \\ &\leq E(\omega^\lambda) + \sum_{j=p+1}^k \kappa_j \psi^\lambda(x_j^\lambda) + \frac{1}{2} \sum_{\substack{i \neq j \\ p+1 \leq i,j \leq k}} \kappa_i\kappa_j G(x_i^\lambda, x_j^\lambda) - \sum_{j=p+1}^k \kappa_j^2 H(x_j^\lambda), \end{aligned} \tag{3-30}$$

where $\psi^\lambda = (-\Delta)^{-1}\omega^\lambda$ and $\psi = (-\Delta)^{-1}\omega$. It is easy to check that

$$\begin{aligned} E(\omega) = \frac{1}{2} \sum_{i=1}^p \int_D \int_D -\frac{1}{2\pi} \ln|x-y|\omega_i(x)\omega_i(y)\,dx\,dy - \sum_{i=1}^p \kappa_i^2 H(\bar{x}_i) \\ + \frac{1}{2} \sum_{\substack{i \neq j \\ 1 \leq i,j \leq p}} \kappa_i\kappa_j G(\bar{x}_i, \bar{x}_j) + o(1), \end{aligned} \tag{3-31}$$

and

$$\sum_{j=p+1}^k \kappa_j \psi(\bar{x}_j) = \sum_{i=1}^p \sum_{j=p+1}^k \kappa_i\kappa_j G(\bar{x}_i, \bar{x}_j) + o(1). \tag{3-32}$$

Similarly

$$E(\omega^\lambda)=\frac{1}{2}\sum_{i=1}^{p}\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i^\lambda(x)\omega_i^\lambda(y)\,dx\,dy-\sum_{i=1}^{p}\kappa_i^2H(z_i)+\frac{1}{2}\sum_{\substack{i\neq j\\1\le i,j\le p}}\kappa_i\kappa_jG(z_i,z_j)+o(1), \tag{3-33}$$

and

$$\sum_{j=p+1}^{k}\kappa_j\psi^\lambda(x_j^\lambda)=\sum_{i=1}^{p}\sum_{j=p+1}^{k}\kappa_i\kappa_jG(z_i,z_j)+o(1). \tag{3-34}$$

Hence from all the above we obtain

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{p}\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i(x)\omega_i(y)\,dx\,dy-\sum_{i=1}^{p}\kappa_i^2H(\bar{x}_i)\\&\quad+\frac{1}{2}\sum_{\substack{i\neq j\\1\le i,j\le p}}\kappa_i\kappa_jG(\bar{x}_i,\bar{x}_j)+\sum_{i=1}^{p}\sum_{j=p+1}^{k}\kappa_i\kappa_jG(\bar{x}_i,\bar{x}_j)\\&\quad+\frac{1}{2}\sum_{\substack{i\neq j\\p+1\le i,j\le k}}\kappa_i\kappa_jG(\bar{x}_i,\bar{x}_j)-\sum_{j=p+1}^{k}\kappa_j^2H(\bar{x}_j)\\&\le\frac{1}{2}\sum_{i=1}^{p}\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i^\lambda(x)\omega_i^\lambda(y)\,dx\,dy-\sum_{i=1}^{p}\kappa_i^2H(z_i)\\&\quad+\frac{1}{2}\sum_{\substack{i\neq j\\1\le i,j\le p}}\kappa_i\kappa_jG(z_i,z_j)+\sum_{i=1}^{p}\sum_{j=p+1}^{k}\kappa_i\kappa_jG(z_i,z_j)\\&\quad+\frac{1}{2}\sum_{\substack{i\neq j\\p+1\le i,j\le k}}\kappa_i\kappa_jG(z_i,z_j)-\sum_{j=p+1}^{k}\kappa_j^2H(z_j)+o(1).\end{aligned} \tag{3-35}$$

On the other hand, by Riesz's rearrangement inequality we have for each $1\le i\le p$

$$\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i^\lambda(x)\omega_i^\lambda(y)\,dx\,dy\le\int_D\int_D -\frac{1}{2\pi}\ln|x-y|\omega_i(x)\omega_i(y)\,dx\,dy. \tag{3-36}$$

Therefore by (3-35) and (3-36) we get

$$\frac{1}{2}\sum_{\substack{i\neq j\\1\le i,j\le k}}\kappa_i\kappa_j G(\bar{x}_i,\bar{x}_j)-\sum_{j=1}^{k}\kappa_j^2 H(\bar{x}_j) \le \frac{1}{2}\sum_{\substack{i\neq j\\1\le i,j\le k}}\kappa_i\kappa_j G(z_i,z_j)-\sum_{j=1}^{k}\kappa_j^2 H(z_j), \tag{3-37}$$

or equivalently

$$\mathcal{K}_k(z_1,\dots,z_k)\le\mathcal{K}_k(\bar{x}_1,\dots,\bar{x}_k). \tag{3-38}$$

Since $(\bar{x}_1,\dots,\bar{x}_k)$ is the unique minimum point for $\mathcal{K}_k$ on $\overline{B_\delta(\bar{x}_1)}\times\cdots\times\overline{B_\delta(\bar{x}_k)}$, we obtain

$$(z_1,\dots,z_k)=(\bar{x}_1,\dots,\bar{x}_k),$$

which completes the proof. □

Remark 3.12. It is easy to see that $x_j^\lambda\in B_\delta(\bar{x}_j)$ for $j=p+1,\dots,k$, and $\operatorname{dist}(\operatorname{supp}(\omega_i^\lambda),\partial B_\delta(\bar{x}_i))>0$ for $i=1,\dots,p$, provided that λ is sufficiently large.

4. Proof of Theorem 2.4

Having made all the preparations in the preceding sections, we are now ready to give the proof of Theorem 2.4.

Proof of Theorem 2.4. We only need to prove that $(\omega^\lambda,x_{p+1}^\lambda,\dots,x_k^\lambda)$ satisfies (2-16) if λ is sufficiently large, since the other assertions have been verified in Section 3.

First, by Lemma 3.11, $x_j^\lambda\in B_\delta(\bar{x}_j)$ for $j=p+1,\dots,k$, thus by (3-4) we have

$$\nabla\Bigg(\psi^\lambda(x)+\sum_{j=p+1,j\neq i}^{k}\kappa_j G(x_j^\lambda,x)-\kappa_i H(x)\Bigg)\Bigg|_{x=x_i^\lambda}=0,\qquad j=p+1,\dots,k.$$

Now for any given $\phi\in C_c^\infty(D)$, define a family of transformations $\Phi_t(x)$, $t\in\mathbb{R}$, from D to D by the following ordinary differential equation:

$$\begin{cases}\dfrac{d\Phi_t(x)}{dt}=J\nabla\phi(\Phi_t(x)), & t\in\mathbb{R},\\ \Phi_0(x)=x.\end{cases} \tag{4-1}$$

Since $J\nabla\phi$ is a smooth vector field with compact support in D, (4-1) is solvable for all $t\in\mathbb{R}$. It is also easy to see that $J\nabla\phi$ is divergence-free, so by the Liouville theorem (see [Marchioro and Pulvirenti 1994, Appendix 1.1]) $\Phi_t(x)$ is area-preserving, or equivalently, for any measurable set $A\subset D$

$$|\Phi_t(A)|=|A|\qquad\text{for all } t\in\mathbb{R}. \tag{4-2}$$

Now we define a family of test functions

$$\omega_t^\lambda(x) := \omega^\lambda(\Phi_{-t}(x)). \tag{4-3}$$

Since Φ_t is area-preserving and $\operatorname{dist}(\operatorname{supp}(\omega_i^\lambda), \partial B_\delta(\bar{x}_i)) > 0$ for each $1 \le i \le p$, we have $\omega_t^\lambda \in N_p^\lambda$ as long as $|t|$ is sufficiently small. Then by (3-3) we have

$$\frac{d}{dt}\left(E(\omega_t^\lambda) + \sum_{j=p+1}^{k} \kappa_j \psi_t^\lambda(x_j^\lambda) \right)\bigg|_{t=0} = 0,$$

where $\psi_t^\lambda = (-\Delta)^{-1}\omega_t^\lambda$. It is easy to check that (see [Turkington 1983a] for example)

$$E(\omega_t^\lambda) = E(\omega^\lambda) + t\int_D \omega^\lambda(x)\partial(\psi^\lambda(x), \phi(x))\,dx + o(t),$$

and

$$\psi_t^\lambda(x_j^\lambda) = \psi^\lambda(x_j^\lambda) + t\int_D \omega^\lambda(x)\partial(G(x_j^\lambda, x), \phi(x))\,dx + o(t),$$

where $o(t)/t \to 0$ as $t \to 0$. Therefore we get

$$\int_D \omega^\lambda(x)\partial\left(\psi^\lambda(x) + \sum_{j=p+1}^{k} \kappa_j G(x_j^\lambda, x), \phi(x)\right) dx = 0,$$

which completes the proof of Theorem 2.4. □

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DAOMIN CAO
INSTITUTE OF APPLIED MATHEMATICS
CHINESE ACADEMY OF SCIENCES
BEIJING,
CHINA

dmcao@amt.ac.cn

GUODONG WANG
INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS
HARBIN INSTITUTE OF TECHNOLOGY
HARBIN,
CHINA

wangguodong14@mails.ucas.ac.cn

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RELATIONS OF RATIONALITY FOR SPECIAL VALUES OF RANKIN–SELBERG L-FUNCTIONS OF $\mathrm{GL}_n \times \mathrm{GL}_m$ OVER CM-FIELDS

HARALD GROBNER AND GUNJA SACHDEVA

We establish an "automorphic version" of Deligne's conjecture for motivic L-functions in the case of Rankin–Selberg L-functions $L(s, \Pi \times \Pi')$ of $\mathrm{GL}_n \times \mathrm{GL}_m$ over arbitrary CM-fields F. Our main results are of two different kinds: Firstly, for arbitrary integers $1 \leq m < n$ and suitable pairs (Π, Π') of cohomological automorphic representations, we relate critical values of $L(s, \Pi \times \Pi')$ with a product of Whittaker periods attached to Π and Π', Blasius's CM-periods of Hecke-characters and certain nonzero values of standard L-functions. Secondly, these relations lead to quite broad generalizations of fundamental rationality-results of Waldspurger, Harder and Raghuram, and others.

Introduction

Motivated by conjectures of Deligne, Bellinson and Bloch and Kato, significant progress has been made in the study of special values of automorphic L-functions in recent decades. In this paper we continue this series of results on by treating the case of Rankin–Selberg L-functions over arbitrary CM-fields F. More precisely, let $1 \leq m < n$ be arbitrary integers and let Π and Π' be cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively. If the infinity types of Π and Π' are compatible (in a sense to be made precise below) we will prove relations of rationality for a certain string of special values of the attached Rankin–Selberg L-function $L(s, \Pi \times \Pi')$, which turn out to fit Deligne's prediction; as a particular example, the contributions of the archimedean components Π_∞ and Π'_∞ to our rationality-relations will be expressed by explicit powers of $(2\pi i)$, matching the ones conjectured by Deligne.

As compared to the extensive literature for the case $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ (for F a CM-field, we refer in particular to [Kurchanov 1978; 1979; Harder 1983; Hida

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1994; Lin 2015; Raghuram 2016; Grobner and Harris 2016; Grobner 2018; Grobner and Lin 2020; Januszewski 2019]), the rank m of our second GL-factor GL_m being in principle *any integer* $1 \leq m < n$ (for suitable pairs (Π, Π')) is arguably one of the most notable features of this article. In this regard, the results of this paper should not only be seen as an extension of the series of results mentioned above, but also of the approach taken in [Lin 2015] and in the second author's thesis [Sachdeva 2020]. We also refer to the very recent [Raghuram 2020], where an application of our Main Theorem in the special case of the standard L-function (i.e., $m = 1$) has been proven by a different approach.

Main theorem and applications. The main results of this article are Theorem 3.1 (reviewed as Main Theorem below) and its two corollaries (reviewed as Applications I and II). The main theorem is somewhat technical and turns out to be quite involved in its assumptions and assertions, while the two applications are much lighter statements, providing wide generalizations of important results of Waldspurger (Application I) and Harder and Raghuram (Application II), as well as of other people.

The main theorem. From now on F denotes an arbitrary CM-field with maximal totally real subfield F^+. The quadratic Hecke character associated with F/F^+ admits a unitary extension to $\mathbb{A}_F^\times$ which is denoted η. Then by construction $\eta\ \|\cdot\|^{-1/2}$ is algebraic. See Section 1A for details. For an integer $n \geq 2$ we let Π be an (irreducible) subrepresentation of the subspace of cuspidal functions in $L^2(\mathrm{GL}_n(F)\mathbb{R}_+\backslash\mathrm{GL}_n(\mathbb{A}_F))$. As above, we shall assume that Π is cohomological, i.e., there exists a finite-dimensional, irreducible algebraic representation $\mathcal{E}_\mu$ of the real Lie group $G_{n,\infty} := \mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$ such that Π_∞ has nontrivial relative Lie algebra cohomology with respect to $\mathcal{E}_\mu$. Here, μ stands for the highest weight of $\mathcal{E}_\mu$ (depending on a choice of a Borel subgroup $B_n \subset G_n$). Choosing coordinates one may indeed identify it with $\mu = (\mu_v)_{v \in S_\infty}$ where $\mu_v = (\mu_{v,1}, \ldots, \mu_{v,n}) \in \mathbb{Z}^n$ and $\mu_{v,1} \geq \cdots \geq \mu_{v,n}$.

Consider now another integer m, such that $1 \leq m < n$. We define Π' in analogy to Π above as an (irreducible) subrepresentation of the subspace of cuspidal functions in $L^2(\mathrm{GL}_m(F)\mathbb{R}_+\backslash\mathrm{GL}_m(\mathbb{A}_F))$ but which is now assumed to be conjugate self-dual with respect to the nontrivial Galois automorphisms of F/F^+ and with the property that $\Pi'^{\mathsf{alg}} := \Pi' \otimes \eta^e$ is cohomological. Here, $e \in \{0, 1\}$ and $e = 0$ if and only if $n \not\equiv m \mod 2$.

The reason for introducing the twist $\Pi'^{\mathsf{alg}} = \Pi' \otimes \eta^e$, i.e., for assuming different conditions on cohomology for n and m is explained by the following construction: In order to be able to use the main result of [Grobner 2018] (which is the starting-point of our proof), we choose any conjugate self-dual Hecke characters $\chi_1, \ldots, \chi_{n-m-1}$ of $\mathbb{A}_F^\times$ such that the isobaric sum $\Sigma := \Pi' \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-m-1}$ is cohomological with respect to algebraic coefficients $\mathcal{E}_{\mu'}$. It turns out that such a choice can be made if and only if Π'^{alg}, rather than Π' itself, is cohomological. We will say that the infinity types of Π and Π' are compatible (or, more metaphorically, satisfy the "piano-condition", see (1.8) and below), if a choice of Σ can be made such that $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ is nontrivial.

With these assumptions, nonzero *Whittaker periods* $p(\Pi)$ and $p(\Pi'^{\mathsf{alg}})$, and *CM-periods* $p(\chi_i \chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}})$, have been defined, respectively, in [Raghuram and Shahidi 2008; Grobner 2018] and [Blasius 1986]. By their very construction their product is well-defined up to multiplication by nonzero elements in a certain number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Pi'^{\mathsf{alg}})E^{\mathsf{cm}}$, where E^{cm} is an abbreviation for a number field, which depends on the chosen characters χ_i and contains a Galois closure $F^{\mathrm{Gal}} \subset \bar{\mathbb{Q}}$ of the extension $F/\mathbb{Q}$. We refer to Sections 1F and 2D for details.

We are now in the position to state our main theorem.

Main Theorem. *Assume that* $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ *is nontrivial and let* $s_0 = \frac{1}{2} + k$ *be any critical point of* $L(s, \Pi \times \Sigma)$. *If* $k \neq 0$, *then*

$$(0.1)\quad L^S(\tfrac{1}{2}+k, \Pi \times \Pi') \sim (2\pi i)^{[F^+:\mathbb{Q}]\left((n-1)\left(\left(k-\frac{1}{2}\right)n-1\right)\right)+\frac{1}{2}(n-m-1)(n-m-2))} p(\Pi)\, p(\Pi'^{\mathsf{alg}}) \cdot \prod_{1\le i<j\le n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1, \Pi' \otimes \chi_j^{-1})}{L^S\left(\frac{1}{2}+k, \Pi \otimes \chi_j\right)},$$

with the relation "$\sim$" being over the number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\eta\|\cdot\|^{-1/2})E^{\mathsf{cm}}$.

If $k = 0$, *i.e., if* $s_0 = \frac{1}{2}$ *denotes the central critical point, then the same relation holds under certain conditions of regularity on* Π_∞ *and* Σ_∞ *as well as a global nonvanishing hypothesis*; *see Theorem 3.1. Moreover*, *if* n *is even and* m *is odd*, *then all L-values* $L^S(\frac{1}{2}+k, \Pi \times \Pi')$, $L^S(1, \Pi' \otimes \chi_j^{-1})$ *and* $L^S(\frac{1}{2}+k, \Pi \otimes \chi_j)$ *in (0.1) are critical.*

The reader should observe that this result is the "best possible" since the individual quantities on the right-hand side are only well-defined up to multiplication by an element in the number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\eta\|\cdot\|^{-1/2})E^{\mathsf{cm}}$. We also remark that if $k \neq 0$, then the denominators $L^S\left(\frac{1}{2}+k, \Pi \otimes \chi_j\right)$ in (0.1) are nonzero, which is in turn part of the global nonvanishing hypothesis for the central case $k = 0$ mentioned in our Main Theorem.

If $m = n-1$, then our main theorem becomes Theorem 5.2 from [Grobner and Lin 2020] for cuspidal automorphic representations, which refined the main result of [Raghuram 2016] over CM-fields by giving an explicit power of $(2\pi i)$ instead of an abstract archimedean period. It is worth noting that this power is precisely what is predicted by Deligne's conjecture on critical values of motivic L-functions [1979], generalizing Euler's classical result on the nature of $\zeta(k)$ at even, positive integers. We refer to [Grobner and Lin 2020, Remark 5.8] for a more detailed exposition.

On the other extreme, if $m = 1$, i.e., if we look at the twisted standard L-function of Π, then we retrieve at once Theorem 3.9, Corollary 5.7 and Theorem 6.11 of [Grobner and Harris 2016], as well as a variant of the main result of [Raghuram 2020] over CM-fields. For general m our Main Theorem should hence be viewed as a theorem relating special values of $L(s, \Pi \times \Pi')$ with periods and quotients of special values of (other) standard L-functions. As already mentioned above, we refer to Theorem 3.1 for a proof.

Main applications. Our Main Theorem has the following two implications, which generalize important results of Waldspurger (see Application I) and Harder and Raghuram (see Application II). Indeed, Waldspurger [1985] established a rationality result for the quotient $L\bigl(\frac{1}{2}, \pi \otimes \alpha\bigr)/L\bigl(\frac{1}{2}, \pi \otimes \beta\bigr)$ of the standard L-functions attached to the twisted cohomological cuspidal automorphic representations $\pi \otimes \alpha$ and $\pi \otimes \beta$ of GL_2 over any number field at their joint critical value $s_0 = \frac{1}{2}$. More precisely, here α and β are assumed to be quadratic Hecke characters having the same archimedean component $\alpha_\infty = \beta_\infty$, π denotes a cohomological unitary cuspidal automorphic representation of GL_2 and $L\bigl(\frac{1}{2}, \pi \otimes \beta\bigr)$ is supposed to be nonzero. Under these assumptions, Waldspurger's rationality-relation is of the form

$$\frac{L\bigl(\frac{1}{2}, \pi \otimes \alpha\bigr)}{L\bigl(\frac{1}{2}, \pi \otimes \beta\bigr)} \sim_{\mathbb{Q}(\pi)} \frac{p(\alpha)}{p(\beta)},$$

with the two period-invariants $p(\alpha)$ and $p(\beta)$ depending only on α and β, respectively, and the archimedean component of the cuspidal representations π. See [Waldspurger 1985, p. 174].

In this paper we generalize Waldspurger's result to the case of quotients of standard L-functions of GL_n/F where $n \geq 2$ is arbitrary and $s_0 = \frac{1}{2} + k$ is a more general special value while F is any CM-field. More precisely, we let α and β be any conjugate self-dual Hecke characters of $\mathbb{A}_F^\times$ such that $\alpha_\infty = \beta_\infty$ and such that, writing $\alpha_v(z) = \beta_v(z) = z^{a_v}\bar{z}^{-a_v}$ at $v \in S_\infty$, the following two conditions are satisfied: $a_v \in \frac{n}{2} + \mathbb{Z}$ and $\mu_{v,1} \geq a_v \geq \mu_{v,n}$. This ensures that there is always a choice of conjugate self-dual Hecke characters $\chi_1, \dots, \chi_{n-2}$, such that the isobaric automorphic sums $\Sigma_\alpha = \alpha \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-2}$ and $\Sigma_\beta = \beta \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-2}$ are cohomological with respect to an algebraic coefficient module $\mathcal{E}_{\mu'}$ and such that $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ is nontrivial. We obtain the following application:

Application I. Choose any conjugate self-dual Hecke characters $\chi_1, \dots, \chi_{n-2}$, such that the isobaric automorphic sums Σ_α and Σ_β are cohomological and such that $\mathrm{Hom}_{G_{n-1,\infty}}[\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C}]$ is nontrivial. Let $s_0 = \frac{1}{2} + k$ be any critical point of $L(s, \Pi \times \Sigma_\alpha)$. If n is even, then all the $s_0 = \frac{1}{2} + k$ are indeed critical for $L(s, \Pi \otimes \alpha)$

and $L(s, \Pi \otimes \beta)$ and

$$\frac{L^S(\frac{1}{2}+k, \Pi \otimes \alpha)}{L^S(\frac{1}{2}+k, \Pi \otimes \beta)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_\alpha)\mathbb{Q}(\Sigma_\beta)\mathbb{Q}(\eta\|\cdot\|^{-1/2})E^{\mathsf{cm}}(\alpha;\beta;\chi_1;\cdots;\chi_{n-2})} \prod_{i=1}^{n-2} \frac{p(\alpha, \Psi_{\alpha\chi_i^{-1}})}{p(\beta, \Psi_{\beta\chi_i^{-1}})}.$$

If $k=0$, then we assume certain conditions of regularity on Π_∞ and Σ_∞ as well as a global nonvanishing hypothesis; see Theorem 4.1.

We point out that Application I should furthermore be viewed as a generalization as well as a certain refinement of a consequence of the main result of [Grobner and Raghuram 2014b, Theorem 7.1.2], and [Januszewski 2016, Theorem 8.2], established there for totally real fields F^+ and achieved here for general CM-fields F.

Our second application deals with an extension of the main result of [Harder and Raghuram 2020]. There, the authors achieved a fine relation of rationality between the quotients of consecutive critical values of Rankin–Selberg L-functions over totally real fields F^+ and so-called relative periods denoted $\Omega^{\epsilon'}(\pi_f)$: Let π and π' be cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{F^+})$ and $\mathrm{GL}_m(\mathbb{A}_{F^+})$, respectively, and let S be any finite set of nonarchimedean places, where π or π' are ramified. Suppose that both $-\frac{1}{2}(n+m)$ and $1-\frac{1}{2}(n+m)$ are critical for $L(s, \pi \times \pi'^{\vee})$ and that $L(1-\frac{1}{2}(n+m), \pi \times \pi'^{\vee})$ is nonzero. If n is even and m is odd, [Harder and Raghuram 2020, Theorem 7.40] shows that

$$\frac{L^S\big(-\frac{n+m}{2}, \pi \otimes \pi'^{\vee}\big)}{L^S\big(1-\frac{n+m}{2}, \pi \otimes \pi'^{\vee}\big)} \sim_{\mathbb{Q}(\pi)\mathbb{Q}(\pi')} \Omega^{\epsilon'}(\pi_f).$$

In [Grobner and Lin 2020] this result has recently been given a generalization and refinement for cohomological cusp forms of $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_{n-1}(\mathbb{A}_F)$, again with F denoting any CM-field.

Here we take up the CM-case for general even n and odd m. We obtain the following application:

Application II. Suppose that $1 \leq m < n$ are integers, m odd and n even. We assume that Π is obtained by weak base change from a unitary tempered cuspidal automorphic representation π of some rational similitude group $GU(V)/\mathbb{Q}$. Its infinite component π_∞ is supposed to belong to the antiholomorphic discrete series and to be cohomological with respect to an algebraic coefficient module of $GU(V)(\mathbb{R})$ which is defined over $\mathbb{Q}$. Let Π' be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$, satisfying the conditions of our Main Theorem and let S be any finite set of places of F, containing the archimedean ones, such that Π and Π' are unramified outside S.

Let $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ be two critical points of $L(s, \Pi \times \Sigma)$ different from $s_0 = \frac{1}{2}$, then $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are indeed critical for $L(s, \Pi \times \Pi')$ and the ratio of partial critical values satisfies

$$\frac{L^S\big(\frac{1}{2}+k,\,\Pi\times\Pi'\big)}{L^S\big(\frac{1}{2}+\ell,\,\Pi\times\Pi'\big)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{\mathrm{Gal}}} (2\pi i)^{[F^+:\mathbb{Q}](k-\ell)nm}.$$

See Corollary 4.4 for all details and a proof. Here we only remark that the appearance of base change is due to the fact that our proof uses the results of [Guerberoff 2016], which in turn proved a conjecture of Lin [2015]. As this already indicates, our Application II is hence a generalization of a consequence of Theorem 10.8.1 from [Lin 2015], but obtained by somewhat different techniques.

1. Notation

1A. ***Number fields and some particular Hecke characters.*** We let F be a CM-field, i.e., a totally imaginary quadratic extension of a totally real field F^+. Consequently, the degree of F over the rational numbers $\mathbb{Q}$ is even and we let $2d=[F:\mathbb{Q}]$. Abusing notation we identify the set of archimedean places S_∞ of F and F^+. More precisely, we fix a so-called CM-type of F first, i.e., we fix a choice of pairs of complex embeddings $(\iota_v,\bar{\iota}_v)$, and then we identify the so represented places $v=(\iota_v,\bar{\iota}_v)$ of F, with the places of F^+ through the first component ι_v. We will also fix a Galois closure F^{Gal} of $F/\mathbb{Q}$ in $\bar{\mathbb{Q}}$.

We denote by $\|\cdot\|$ the normalized absolute value on the ring of adèles $\mathbb{A}_F$. Let

$$\varepsilon:(F^+)^\times\backslash\mathbb{A}_{F^+}^\times\to\mathbb{C}^\times$$

be the quadratic Hecke character associated to F/F^+ via class field theory. As is well-known, (for what follows see, for instance, [Bellaïche and Chenevier 2009, §6.9.2]) it is possible to extend ε to a conjugate self-dual unitary Hecke character

$$\eta:F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times,$$

so that at $v\in S_\infty$ we have $\eta_v(z)=(z/|z|)^{2t}$ for $z\in F_v\cong\mathbb{C}$ and $t=t_v\in\frac{1}{2}+\mathbb{Z}$. As in [Bellaïche and Chenevier 2009, §6.9.2] we will abbreviate this by writing $\eta_v(z)=z^t\bar{z}^{-t}$, keeping in mind the possible sign ambiguities throughout. Furthermore, we may (and will) assume from now on that $t=0$, i.e., $\eta_v(z)=z^{1/2}\bar{z}^{-1/2}$; see [Bellaïche and Chenevier 2009, Lemma 6.9.2].

Finally, by letting

$$\phi:=\eta\,\|\cdot\|^{-1/2},$$

we may define a nonunitary, but algebraic Hecke character $\phi:F^\times\backslash\mathbb{A}_F^\times\to\mathbb{C}^\times$.

1B. ***Algebraic groups and real Lie groups.*** Let $n\geq 1$ be an integer. We will denote by $G_n:=\mathrm{GL}_n/F$ the linear algebraic, general linear group over F. Let $R_{F/\mathbb{Q}}$ be Weil's restriction of scalars. We will abbreviate $G_{n,\infty}:=R_{F/\mathbb{Q}}(G_n)(\mathbb{R})$:

It is important to notice that the group

$$G_{n,\infty} = \prod_{v \in S_\infty} \mathrm{GL}_n(F_v) \cong \prod_{v \in S_\infty} \mathrm{GL}_n(\mathbb{C})$$

— although in principle carrying a complex structure — is thought of as a *real* Lie group, namely as the archimedean factor of $G_n(\mathbb{A}_F)$. Furthermore, we let $K_{n,\infty}$ be the product of the center $Z_{n,\infty}$ of $G_{n,\infty}$ and a fixed maximal compact subgroup, i.e.,

$$K_\infty \cong \prod_{v \in S_\infty} \mathbb{C}^* U(n) \cong \prod_{v \in S_\infty} \mathbb{R}_+ U(n).$$

From this it is clear how to embed $\mathbb{R}_+$ as a subgroup of $Z_{n,\infty}$ (and hence of $K_{n,\infty}$ and of $G_{n,\infty}$): An $x \in \mathbb{R}_+$ is sent onto the d-tuple of diagonal matrices $(\mathrm{diag}(x, x, \dots, x))_{v \in S_\infty} \in G_{n,\infty}$. By $\mathfrak{g}_{n,\infty}$ we denote the real Lie algebra of $G_{n,\infty}$ and use the analogous notation for the Lie algebras of other Lie groups.

1C. *Equivalence relations and Galois equivariance.*

Definition 1.1 (i). Let $L \subseteq \mathbb{C}$ a subfield and let $x, y \in \mathbb{C}$. We write

$$x \sim_L y,$$

if there is an $\ell \in L$ such that $x = \ell y$ or $\ell x = y$.

(ii) Let $K, L \subset \mathbb{C}$ be subfields. Let $\underline{x} = \{x(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ and $\underline{y} = \{y(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ be two families of complex numbers. We write

$$\underline{x} \sim_L \underline{y}$$

and say that this relation *is equivariant under* $\mathrm{Aut}(\mathbb{C}/K)$, if either $y(\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, or if $y(\sigma) \neq 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ and the following two conditions are verified:

(1) $x(\sigma) \sim_{\sigma(L)} y(\sigma)$ for all σ.

(2) $\sigma\left(\dfrac{x(\tau)}{y(\tau)}\right) = \dfrac{x(\sigma\tau)}{y(\sigma\tau)}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and all $\tau \in \mathrm{Aut}(\mathbb{C})$.

Obviously, one may replace the first condition by requiring it only for all σ running through representatives of $\mathrm{Aut}(\mathbb{C})/\mathrm{Aut}(\mathbb{C}/K)$. In particular, if $K = \mathbb{Q}$, one only needs to verify it for the identity $id \in \mathrm{Aut}(\mathbb{C})$.

The following lemma is well known; see, for example, [Grobner and Lin 2020, Lemma 1.29]:

Lemma 1.2. *Let $L \subset \mathbb{C}$ be a number field, containing F^{Gal}. Let $x = \{x(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ and $y = \{y(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ be as in Definition 1.1 and suppose that $y(\sigma) \neq 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$. If the complex numbers $x(\sigma)$ and $y(\sigma)$ depend only on the restriction of σ to L, then the second condition of Definition 1.1 implies the first.*

Proof. Fix $\sigma_0 \in \mathrm{Aut}(\mathbb{C})$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ fixing $\sigma_0(L)$, one has $\sigma\sigma_0 |_L = \sigma_0 |_L$. Hence $x(\sigma\sigma_0) = x(\sigma_0)$ and $y(\sigma\sigma_0) = y(\sigma_0)$ by our assumptions. Moreover, since $L \supset F^{\mathrm{Gal}}$, we know $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$. By the second condition, we have

$$\sigma\left(\frac{x(\sigma_0)}{y(\sigma_0)}\right) = \frac{x(\sigma\sigma_0)}{y(\sigma\sigma_0)} = \frac{x(\sigma_0)}{y(\sigma_0)}.$$

Therefore, $x(\sigma_0)/y(\sigma_0) \in \sigma_0(L)$ for all σ_0 as claimed. ☐

1D. *Cohomological automorphic representations.* Let $1 \leq m < n$ be any integers. Throughout the paper, we will let Π be a unitary cuspidal automorphic representation of $G_n(\mathbb{A}_F) = \mathrm{GL}_n(\mathbb{A}_F)$ and let Π' be a unitary cuspidal automorphic representation of $G_m(\mathbb{A}_F) = \mathrm{GL}_m(\mathbb{A}_F)$, in the sense of [Borel and Jacquet 1979, §4.6]. However, for convenience, we will not distinguish between a cuspidal automorphic representation, its smooth automorphic LF-space completion or its (nonsmooth) Hilbert space completion in the L^2-spectrum; see [Grobner 2022] or [Grobner and Žunar $\geq$ 2020] for a detailed account. In this regard, we will now specify our standing assumptions on their archimedean components Π_∞ and Π'_∞.

1D1. *The representation* Π_∞. Unless otherwise stated, throughout the paper we always assume that Π_∞ is *cohomological*, i.e., there exists an irreducible finite-dimensional algebraic representation $\mathcal{E}_\mu$ of $G_{n,\infty}$ on a complex vector space, with respect to which Π_∞ has nontrivial $(\mathfrak{g}_{n,\infty}, K_{n,\infty})$-cohomology; see [Borel and Wallach 1980]. As Π_∞ is assumed to be unitary, $\mathcal{E}_\mu$ must be conjugate self-dual and hence breaks as $\mathcal{E}_\mu = E_\mu \otimes E_{\mu^\vee}$, where $E_\mu = \otimes_{v\in S_\infty} E_{\mu_v}$ and we view again each irreducible $\mathrm{GL}_n(F_v) = \mathrm{GL}_n(\mathbb{C})$-factor E_{μ_v} as being given by its highest weight μ_v. In terms of the standard choice of a maximal split torus in GL_n, positivity on the attached set of roots and standard coordinates, this highest weight is an n-tuple of integers $\mu_v = (\mu_{v,1}, \dots, \mu_{v,n}) \in \mathbb{Z}^n$ with $\mu_{v,1} \geq \cdots \geq \mu_{v,n}$. Let

$$\rho_n = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-3}{2}, -\frac{n-1}{2}\right)$$

be the half-sum of positive roots of GL_n with respect to the same conventions. As a consequence of classical results of Delorme and Enright (see [Enright 1979, Theorems 6.1 and 7.1]) we see that the condition that Π_∞ is cohomological with respect to $\mathcal{E}_\mu$ is equivalent to the much more explicit condition that

$$\Pi_v \cong \mathrm{Ind}_{B_n(\mathbb{C})}^{\mathrm{GL}_n(\mathbb{C})}[z_1^{\ell_{v,1}} \bar{z}_1^{-\ell_{v,1}} \otimes \cdots \otimes z_n^{\ell_{v,n}} \bar{z}_n^{-\ell_{v,n}}]$$

with

$$\ell_{v,i} = -\mu_{v,n-i+1} + \rho_{n,i} \tag{1.3}$$

at each $v \in S_\infty$. Here, B_n is the standard Borel subgroup of G_n (determined by our choice of positivity on the set of roots) and induction is normalized to preserve unitarity.

We will call a set of n real numbers $\{l_{v,i}\}_{1\le i\le n}$ an *infinity type at* $v\in S_\infty$, if

$$l_{v,1} > l_{v,2} > \cdots > l_{v,n},$$

i.e., if its members form a strictly decreasing string. As is obvious from (1.3), $\{\ell_{v,i}\}_{1\le i\le n}$ from above is such a set, called the infinity type of Π at $v\in S_\infty$. Recalling the well-known classification of irreducible unitary cohomological representations of $G_n(\mathbb{C})$ from [Enright 1979] (see also [Grobner and Raghuram 2014a, §5.5], for a presentation tailor-made for our purposes here), the following lemma is obvious:

Lemma 1.4. *There is a bijection, defined by* (1.3), *between the equivalence classes of irreducible unitary cohomological tempered representations of* $G_n(F_v)\cong G_n(\mathbb{C})$ *and the infinity types* $\{l_{v,i}\}_{1\le i\le n}$, *for which* $l_{v,i}\in\frac{1}{2}(n+1)+\mathbb{Z}$ *for all* $1\le i\le n$.

As a last ingredient we will call a highest weight μ as above *sufficiently regular*, if $\mu_{v,i}-\mu_{v,i+1}\ge 2$ for all $v\in S_\infty$ and $1\le i\le n-1$.

1D2. *The representation* Π'_∞. Similar to our assumptions on Π we suppose that the twisted representation $\Pi'_\infty\|\det\|^{(n-m-1)/2}$ is cohomological, or, equivalently, that

$$\Pi'^{\mathsf{alg}} := \begin{cases}\Pi' & \text{if } n-1\equiv m \mod 2,\\ \Pi'\otimes\eta & \text{otherwise,}\end{cases} \tag{1.5}$$

is cohomological. In terms of infinity types, this means that for each $v\in S_\infty$ and $1\le i\le m$, there are $a_{v,i}\in\frac{n}{2}+\mathbb{Z}$ with $a_{v,i}>a_{v,i+1}$, such that

$$\Pi'_v\cong \mathrm{Ind}_{B_m(\mathbb{C})}^{\mathrm{GL}_m(\mathbb{C})}[z_1^{a_{v,1}}\bar{z}_1^{-a_{v,1}}\otimes\cdots\otimes z_m^{a_{v,m}}\bar{z}_m^{-a_{v,m}}].$$

1D3. *An auxiliary representation* Σ *in piano-position.* We extend the infinity type $\{a_{v,i}\}_{1\le i\le m}$ of Π'_v, at each place $v\in S_\infty$, to an infinity type of length $n-1$, simply by choosing any distinct $b_{j,v}\in\frac{n}{2}+\mathbb{Z}$ for $1\le j\le n-m-1$, such that

$$\{a_{v,i}\}_{1\le i\le m}\cap\{b_{j,v}\}_{1\le j\le n-m-1}=\varnothing. \tag{1.6}$$

Denote this new infinity type by $\{\ell'_{v,i}\}_{1\le i\le n-1}$. As by construction $\ell'_{v,i}\in\frac{n}{2}+\mathbb{Z}$ for all $1\le i\le n-1$, this is the infinity type of a unique cohomological irreducible unitary tempered representation of $G_{n-1}(\mathbb{C})$ by Lemma 1.4.

Turning back to global representations, let $\chi_1,\dots,\chi_{n-m-1}$ be unitary Hecke characters with $\chi_{j,v}(z)=z^{b_{j,v}}\bar{z}^{-b_{j,v}}$ for all $v\in S_\infty$, i.e., such that $\chi_j\|\cdot\|^{n/2}$ (or χ_j^{alg}, if we specify $m=1$ in (1.5)) is algebraic. We note that such characters exist, as follows from [Weil 1956]. By its very construction the isobaric automorphic sum

$$\Sigma := \Pi'\boxplus\chi_1\boxplus\cdots\boxplus\chi_{n-m-1}$$

has our infinity type $\{\ell'_{v,i}\}_{1\le i\le n-1}$ from above and is therefore cohomological.

Let $\mathcal{E}_{\mu'}$ be the unique irreducible algebraic coefficients module of $G_{n-1,\infty}$ with respect to which Σ_∞ has nontrivial $(\mathfrak{g}_{n-1,\infty}, K_{n-1,\infty})$-cohomology. By the same reasoning as above, $\mathcal{E}_{\mu'} = E_{\mu'} \otimes E_{\mu'}^{\vee}$ and writing $\mu'_v = (\mu'_{v,1}, \dots, \mu'_{v,n-1}) \in \mathbb{Z}^{n-1}$ at $v \in S_\infty$ one has $\mu'_{v,1} \geq \cdots \geq \mu'_{v,n-1}$ and

$$\ell'_{v,i} = -\mu'_{v,n-i} + \rho_{n-1,i}. \tag{1.7}$$

Henceforth we will assume that Π_∞ and Σ_∞ satisfy the *piano-condition*, by which we mean that

$$\mu_{v,1} \geq -\mu'_{v,n-1} \geq \mu_{v,2} \geq -\mu'_{v,n-2} \geq \cdots \geq -\mu'_{v,1} \geq \mu_{v,n}. \tag{1.8}$$

Equivalently, $\mathrm{Hom}_{G_{n-1,\infty}}(\mathcal{E}_\mu \otimes \mathcal{E}_{\mu'}, \mathbb{C})$ is nonzero (and hence one-dimensional); see [Goodman and Wallach 2009, Theorem 8.1.1].

According to out previous definition, if $\mu'_{v,i} - \mu'_{v,i+1} \geq 2$ for all $v \in S_\infty$ and $1 \leq i \leq n-2$, we call μ' *sufficiently regular*.

1E. *Critical points of L-functions.* For a moment let $N, M \geq 1$ be any integers and let π be an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A}_F) \times \mathrm{GL}_M(\mathbb{A}_F)$ for which a completed standard L-function $L(s,\pi) = \prod_v L(s,\pi_v)$ is defined satisfying a global functional equation $L(s,\pi) = \varepsilon(s,\pi) \cdot L(1-s,\pi^{\mathsf{v}})$; see [Borel 1979, §IV]. The following definition is modeled after [Deligne 1979, Proposition-Définition 2.3]:

Definition 1.9. A complex number $s_0 \in \frac{1}{2}(N-M)+\mathbb{Z}$ is called *critical* for $L(s,\pi)$ if both $L(s,\pi_\infty)$ and $L(1-s,\pi_\infty^{\mathsf{v}})$ are holomorphic at $s = s_0$. We write $\mathrm{Crit}(\pi)$ for the set of critical points of $L(s,\pi)$.

We proceed with the following simple observation.

Observation 1.10. Recalling that $\Gamma(s)$ does not vanish, the set of holomorphic points of $L(s,\pi_\infty)$ coincides with the intersection of the sets of holomorphic points of the archimedean L-functions attached to the characters in the Langlands datum of π_∞; see [Knapp 1994, §4].

As a consequence we obtain the following lemma, which relates the critical points of $L(s,\Pi\times\Sigma)$ to the critical points of the isobaric summands of Σ.

Lemma 1.11. *The following hold*:

(i) *If* $n \not\equiv m \mod 2$, *then* $\mathrm{Crit}(\Pi\times\Sigma) \subseteq \mathrm{Crit}(\Pi\times\Pi')$.

(ii) *If n is even and m is odd*, $\mathrm{Crit}(\Pi\times\Sigma) = \mathrm{Crit}(\Pi\times\Pi') \cap \bigcap_{j=1}^{n-m-1} \mathrm{Crit}(\Pi\otimes\chi_j)$.

(iii) *If m is odd, then* $s_0 = 1 \in \mathrm{Crit}(\Pi'\otimes\chi_j^{-1})$ *for all* $1 \leq j \leq n-m-1$. *In any case*, $s_0 = \frac{1}{2} \in \mathrm{Crit}(\Pi\times\Sigma)$.

Proof. After our Observation 1.10 only (iii) needs a short argument. Writing down $L(s, \Pi'_\infty \chi_{j,\infty}^{-1})$ and $L(1-s, \Pi'^{\vee}_\infty \chi_{j,\infty})$, see [Knapp 1994, §4], we see that the behavior of holomorphy of these two L-factors is the same as of

$$\prod_{v\in S_\infty}\prod_{i=1}^{m}\Gamma(s+|a_{i,v}-b_{v,j}|) \quad\text{and}\quad \prod_{v\in S_\infty}\prod_{i=1}^{m}\Gamma(1-s+|-a_{i,v}+b_{v,j}|).$$

By (1.6), $a_{i,v}-b_{v,j}\neq 0$ for all $v\in S_\infty$, $1\le i\le m$ and $1\le j\le n-m-1$. Hence, $|a_{i,v}-b_{v,j}|=|-a_{i,v}+b_{v,j}|\ge 1$, and so all the above Γ-factors are holomorphic at $s_0=1$. Hence, $s_0=1\in \mathrm{Crit}(\Pi'\otimes\chi_j^{-1})$, if m is odd. The last assertion finally follows from the piano-hypothesis (1.8) and [Grobner 2018, §1.6.1.(4)] or [Raghuram 2016, Theorem 2.21]. □

1F. *Whittaker periods, σ-twists and fields of rationality.* Let Σ be as above. Unitarity of all isobaric summands implies that

$$\Sigma\cong \mathrm{Ind}_{P(\mathbb{A}_F)}^{\mathrm{GL}_{n-1}(\mathbb{A}_F)}[\Pi'\otimes\chi_1\otimes\cdots\otimes\chi_{n-1-m}]$$

is fully induced from the standard parabolic F-subgroup $P\subseteq \mathrm{GL}_{n-1}$ with Levi component $L_P\cong \mathrm{GL}_m\times\prod_{j=1}^{n-m-1}\mathrm{GL}_1$. Hence, a *Whittaker period* $p(\Sigma)\in\mathbb{C}^\times$ has been constructed in [Grobner and Lin 2020, Proposition 1.12 and Corollary 1.22]. It recovers the original construction of Raghuram and Shahidi for cuspidal representations. Hence, $p(\Pi)$, $p(\Pi'^{\mathsf{alg}})$ and $p(\chi_j^{\mathsf{alg}})$ are also all defined. We recall that in [Grobner and Lin 2020] the period $p(\chi)$ is normalized to $p(\chi)\sim_{\mathbb{Q}(\chi)} 1$ for all algebraic Hecke characters, which we also assume here.

We remark that it is intrinsic to the construction of these Whittaker periods, that they are uniquely defined only up to multiplication by nonzero numbers in the respective *field of rationality*, i.e., if ν is any of the above representations, then $p(\nu)$ may be replaced by $q\cdot p(\nu)$ for any $q\in\mathbb{Q}(\nu)^\times := \mathbb{C}^{\mathfrak{S}(\nu)}$ where $\mathfrak{S}(\nu) := \{\sigma\in\mathrm{Aut}(\mathbb{C}) \mid \nu_f\cong {}^\sigma\nu_f\}$ and ${}^\sigma\nu_f := \nu_f\otimes_{\sigma^{-1}}\mathbb{C}$. For cohomological automorphic representations ν as above, the rationality fields $\mathbb{Q}(\nu)$ are number fields and ${}^\sigma\nu_f$ is the finite component of a uniquely determined, cohomological automorphic representation, denoted ${}^\sigma\nu$, justifying the notation $\mathbb{Q}(\nu)$.

As a last ingredient, for each critical point $s_0=\frac{1}{2}+k$ of $L(s,\Pi\times\Sigma)$, an *archimedean period* $p(k,\Pi_\infty,\Sigma_\infty)\in\mathbb{C}^\times$ has been defined in [Grobner 2018, 1.6.1.(6)] as the weighted sum of archimedean zeta-integrals. We do not repeat its precise definition here and rather refer to [Grobner 2018], because $p(k,\Pi_\infty,\Sigma_\infty)$ will not show up in the final results of this paper, but plays the role of an auxiliary quantity on the way there. Here we only point out that the normalization $p(\chi)\sim_{\mathbb{Q}(\chi)} 1$ for all algebraic Hecke characters, together with two conditions of compatibility in the construction of $p(\Sigma)$ (see [Grobner and Lin 2020, §1.5.3] for all details and further discussion), pin down $p(k,\Pi_\infty,\Sigma_\infty)$ uniquely.

2. Revisiting four results on period-relations

2A. ***Our starting point: the main result of*** [Grobner 2018]**.** We now recall the following algebraicity result for the critical points of the L-function attached to a pair (Π, Σ):

Proposition 2.1 [Grobner 2018, Theorem 1.8]. *Let Π and Σ be cohomological automorphic representations as in Section 1D. In particular, Π is a unitary cuspidal automorphic representation of $G_n(\mathbb{A}_F)$ and $\Sigma = \Pi' \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-m-1}$ is the isobaric sum of a unitary cuspidal automorphic representation Π' of $G_m(\mathbb{A}_F)$ and unitary Hecke characters χ_j, such that Π_∞ and Σ_∞ satisfy the piano-condition. Then, for every critical point $s_0 = \frac{1}{2} + k$ of $L(s, \Pi \times \Sigma)$,*

$$L^S\big(\tfrac{1}{2}+k, \Pi \times \Sigma\big) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)} p(k, \Pi_\infty, \Sigma_\infty)\, p(\Pi)\, p(\Sigma), \tag{2.2}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C})$.

In this result, we interpreted the left- and the right-hand sides in relation (2.2) as families $\underline{x} = \{x(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ and $\underline{y} = \{y(\sigma)\}_{\sigma \in \mathrm{Aut}(\mathbb{C})}$ in the obvious way and we will continue to do so henceforth in analogous situations. In the next three subsections we collect three additional results from the theory of special values — one of them achieved in [Blasius 1986], whereas the other two have only been quite recently established in [Grobner and Lin 2020]. These three results shall then be used in order to rewrite (2.2) in a much more refined way, being the key-step in the proof of our Main Theorem.

2B. ***Step I: The archimedean period as a power of $2\pi i$.*** Under certain conditions, J. Lin and Grobner [2020, Corollary 4.30] have computed the archimedean period-factor $p(k, \Pi_\infty, \Sigma_\infty)$ from (2.2) as a power of $(2\pi i)$.

In order to recall this result also for the archimedean period $p(0, \Pi_\infty, \Sigma_\infty)$ attached to the central critical point $s_0 = \frac{1}{2}$ of $L(s, \Pi \times \Sigma)$ (see Lemma 1.11), consider two cyclic extensions L and L' of F, of degree n (resp. $n-1$), which are still CM-fields. For an algebraic Hecke character χ of $\mathbb{A}_L^\times$ (resp. χ' of $\mathbb{A}_{L'}^\times$), let $\Pi(\chi)$ (resp. $\Pi(\chi')$) be the automorphic induction from χ to $G_n(\mathbb{A}_F)$ (resp. χ' to $G_{n-1}(\mathbb{A}_F)$); see Chapter 3 of [Arthur and Clozel 1989] and their Theorem 6.2 (as completed in [Henniart 2012, Theorem 3]. See also [Clozel 2017].) We denote

$$\Pi_\chi := \begin{cases} \Pi(\chi) & \text{if } n \text{ is odd,} \\ \Pi(\chi) \otimes \eta & \text{if } n \text{ is even,} \end{cases} \quad \text{and} \quad \Pi_{\chi'} := \begin{cases} \Pi(\chi') & \text{if } n \text{ is even,} \\ \Pi(\chi') \otimes \eta & \text{if } n \text{ is odd.} \end{cases} \tag{2.3}$$

It is argued in [Grobner and Lin 2020, §4.5] that, given Π and Σ as in Section 1D, one may always choose conjugate self-dual algebraic Hecke characters χ and χ' such that Π_χ and $\Pi_{\chi'}$ are cuspidal automorphic representations for which $\Pi_{\chi,\infty} \cong \Pi_\infty$ and $\Pi_{\chi',\infty} \cong \Sigma_\infty$. Whenever we use the symbols Π_χ and $\Pi_{\chi'}$ it is from now on silently assumed that such a choice has been made. Our definition now allows us to state the following proposition:

Proposition 2.4 [Grobner and Lin 2020, Corollary 4.30]. *Let* Π *and* Σ *be cohomological automorphic representations as in Section 1D and let* $s_0 = \frac{1}{2} + k$ *be a critical point of* $L(s, \Pi \times \Sigma)$. *Only if* $k = 0$, *i.e., if* $s_0 = \frac{1}{2}$ *denotes the central critical point of* $L(s, \Pi \times \Sigma)$, *we additionally assume that* μ *and* μ' *are both sufficiently regular and that there exists a choice of* χ, χ' *such that*

$$L^S\big(\tfrac{1}{2}, \Pi_\chi \times \Pi_{\chi'}\big) \neq 0.$$

With these assumptions,

$$p(k, \Pi_\infty, \Sigma_\infty) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)F^{\mathrm{Gal}}} (2\pi i)^{d(n-1)((k-\frac{1}{2})n+1)}, \tag{2.5}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$.

Remark 2.6. The reader who is interested in the nature of our nonvanishing hypothesis for the central critical value $L^S\big(\frac{1}{2}, \Pi_\chi \times \Pi_{\chi'}\big)$ of the Rankin–Selberg L-function attached to a suitable choice of twisted automorphically induced representations Π_χ and $\Pi_{\chi'}$, may find the following remark illuminating: Let $K \subset LL'$ be a subfield and let $N_{\mathbb{A}_{LL'}/\mathbb{A}_K}$ be the adelic extension of the norm attached to the field extensions LL'/K. Then, it follows from the very construction of Π_χ and $\Pi_{\chi'}$ (see [Arthur and Clozel 1989, Chapter 3]), that we have an equality of partial L-values

$$L^S\big(\tfrac{1}{2}, \Pi_\chi \times \Pi_{\chi'}\big) = L^S\big(\tfrac{1}{2}, (\chi \circ N_{\mathbb{A}_{LL'}/\mathbb{A}_L})(\chi' \circ N_{\mathbb{A}_{LL'}/\mathbb{A}_{L'}})(\eta \circ N_{\mathbb{A}_{LL'}/\mathbb{A}_F})\big).$$

Our nonvanishing assumption — made only in the case when we want to consider the archimedean period $p(0, \Pi_\infty, \Sigma_\infty)$ at the central critical value — hence reduces to a nonvanishing assumption for a Hecke L-function on LL', i.e., may be reduced from considering Rankin–Selberg L-functions of type $n \times (n-1)$ over F to standard L-functions of GL_1/LL'. In turn, the nonvanishing of the latter L-functions is studied in many sources in the literature: The results of Rohrlich [1989], Ginzburg, Jiang and Rallis [2004], Eischen [2017] and most recently Jiang and Zhang [2020] provide evidence that our nonvanishing assumption is indeed satisfied in all cases that we consider. For a formal argument and analysis of this latter assertion we refer to [Grobner and Lin 2020, §4.5.1].

2C. ***Step II: Breaking the period of* Σ.** As the next ingredient for rewriting (2.2), we will decompose the Whittaker period $p(\Sigma)$ in terms of the isobaric summands of Σ. The following is a special case of another result of the Grobner's recent work with J. Lin:

Proposition 2.7. *Let* $\Sigma = \Pi' \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-m-1}$ *be a cohomological isobaric automorphic representation as in Section 1D. Assume in addition that all summands*

Π' and χ_j are conjugate self-dual. Then

$$(2.8)\quad p(\Sigma) \sim_{\mathbb{Q}(\Sigma)\mathbb{Q}(\phi)F^{\mathrm{Gal}}} p(\Pi'^{\mathsf{alg}}) \prod_{j=1}^{n-m-1} L^S(1, \Pi' \otimes \chi_j^{-1}) \prod_{1 \le i < j \le n-m-1} L^S(1, \chi_i \chi_j^{-1}),$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$.

Proof. We recall that by our conventions (see Section 1F) $\prod_{j=1}^{n-m-1} p(\chi_j^{\mathsf{alg}}) \in \mathbb{Q}^\times$, whence the assertion follows from [Grobner and Lin 2020, Corollary 2.13]. □

2D. ***Step III: Relating $L^S(1, \chi_i \chi_j^{-1})$ to CM-periods.*** The last necessary ingredient for rewriting (2.2) has been established by Blasius [1986]. He described the critical L-values $L^S(1, \chi_i \chi_j^{-1})$ showing up in the formula (2.8) in terms of *CM-periods* $p(\chi_i \chi_j^{-1}, \Psi_{\chi_i \chi_j^{-1}})$.

2D1. *Review of CM-periods.* The reader familiar with Blasius's construction may skip this small subsection and proceed directly to Proposition 2.9.

As a first observation, for any pair (i, j) with $1 \le i < j \le n - m - 1$, the Hecke character $\xi := \chi_i \chi_j^{-1}$ is critical in the sense of [Deligne 1979]. That means that ξ is algebraic and has nontrivial archimedean components ξ_v for all $v \in S_\infty$. Clearly, the latter assertion follows from our definition of $\{b_{v,j}\}_{1 \le j \le n-m-1}$ being an infinity type for all $v \in S_\infty$, i.e., a set of strictly decreasing real numbers. Hence, one may define another CM-type Ψ_ξ of F by the rule

$$\Psi_\xi := \{v \in S_\infty \mid b_{v,i} < b_{v,j}\} \cup \overline{\{v \in S_\infty \mid b_{v,i} > b_{v,j}\}}.$$

Let now Ψ_F be any CM-type of F. Attached to (ξ, Ψ_F) one may define a CM-Shimura-datum as in [Harris 1993, Section 1.1] and a number field $E(\xi, \Psi_F)$, which contains $\mathbb{Q}(\xi)$ and the reflex field of the CM-Shimura-datum defined by Ψ_F. In particular, if $\Psi_F = \Psi_\xi$, one may associate a nonzero complex number $p(\xi, \Psi_\xi)$ to this datum, as explained in the appendix of [Harris and Kudla 1991]. This number $p(\xi, \Psi_\xi)$ is well-defined modulo $E(\xi, \Psi_\xi)^\times$ and called the *CM-period attached to* ξ. Let us abbreviate

$$E^{\mathsf{cm}}(\xi) := \prod_{\Psi_F} E(\xi, \Psi_F)$$

and, resuming the notation $\chi_i \chi_j^{-1}$,

$$E^{\mathsf{cm}}(\chi_1, \dots, \chi_{n-m-1}) := \prod_{1 \le i < j \le n-m-1} E^{\mathsf{cm}}(\chi_i \chi_j^{-1}).$$

This field is a number field by construction, which contains the finite compositum of the number fields $F^{\mathrm{Gal}} \prod_{1 \le i < j \le n-m-1} \mathbb{Q}(\chi_i \chi_j^{-1})$, as defined in Section 1F, but may be bigger than that.

The following result is proved in [Blasius 1986]. We also refer to Proposition 1.8.1 of [Harris 1993] (and the attached erratum [Harris 1997, p. 82]) or [Grobner and Lin 2020, Theorem 4.7] for a slightly more tailor-made presentation.

Proposition 2.9. *Let* $\chi_1, \dots, \chi_{n-m-1}$ *be conjugate self-dual Hecke characters, such that* $\chi_i \chi_j^{-1}$ *is algebraic and critical for all* $1 \leq i < j \leq n-m-1$. *Then*

$$\prod_{1 \leq i < j \leq n-m-1} L^S(1, \chi_i \chi_j^{-1}) \sim_{E^{\mathrm{cm}}(\chi_1,\dots,\chi_{n-m-1})} (2\pi i)^{d\frac{1}{2}(n-m-1)(n-m-2)} \prod_{1 \leq i < j \leq n-m-1} p(\chi_i \chi_j^{-1}, \Psi_{\chi_i \chi_j^{-1}}), \tag{2.10}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$.

3. Our main theorem

3A. ***Special values for*** $\mathbf{GL}_n \times \mathbf{GL}_m$, $1 \leq m < n$. We are now ready to prove our first new result. To this end, recall that $1 \leq m < n$ has been any pair of integers and that Π and Π' have been cohomological unitary cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively, the latter assumed to be conjugate self-dual as in Section 2. Our Main Theorem will relate special values of the partial Rankin–Selberg L-function $L(s, \Pi \times \Pi')$ (all of them indeed critical, if n and m are of different parity), to quantities only depending on Π, Π' and a suitable choice of auxiliary characters χ_j (as in Section 1D3).

Rendering this more precise, recall the Whittaker periods $p(\Pi)$, $p(\Pi'^{\mathrm{alg}})$ attached to Π and Π'^{alg} and the CM-periods $p(\chi_i \chi_j^{-1}, \Psi_{\chi_i \chi_j^{-1}})$ attached to a choice of auxiliary characters χ_j from Sections 1F and 2D. Recall that when n is even and m is odd, then $\Pi'^{\mathrm{alg}} = \Pi'$ and $\chi_j^{\mathrm{alg}} = \chi_j$ for all $1 \leq j \leq n-m-1$. Our main theorem may be written as:

Theorem 3.1. *We let* F *be any CM-field and let* $1 \leq m < n$ *be any integers. Let* Π *be a cohomological unitary cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$ *and let* Π' *be a conjugate self-dual cuspidal automorphic representation of* $\mathrm{GL}_m(\mathbb{A}_F)$. *Choose any conjugate self-dual Hecke characters* $\chi_1, \dots, \chi_{n-m-1}$, *such that the isobaric automorphic sum*

$$\Sigma = \Pi' \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-m-1}$$

is cohomological and assume that $(\Pi_\infty, \Sigma_\infty)$ *satisfies the piano-hypothesis*, (1.8). *Let* $s_0 = \frac{1}{2} + k \in \mathrm{Crit}(\Pi \times \Sigma)$ *be any critical point of* $L(s, \Pi \times \Sigma)$.

In the special case when $k = 0$ *only, i.e., if* $s_0 = \frac{1}{2}$ *denotes the central critical point, we additionally assume that the coefficient modules of* Π_∞ *and* Σ_∞ *are both sufficiently regular* (*see Sections 1D1 and 1D3*) *and that there exists a choice* *of Hecke characters* χ, χ' *such that* $L^S\big(\tfrac{1}{2}, \Pi_\chi \times \Pi_{\chi'}\big) \neq 0$ (*see Section 2B*) *and* $L^S\big(\tfrac{1}{2}, \Pi \otimes \chi_j\big) \neq 0$ *for all* $1 \leq j \leq n-m-1$. *Then*

$$L^S\big(\tfrac{1}{2}+k, \Pi \times \Pi'\big) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\phi)E^{\mathrm{cm}}(\chi_1,\dots,\chi_{n-m-1})} (2\pi i)^{d((n-1)((k-1/2)n-1))+1/2(n-m-1)(n-m-2))}\, p(\Pi)\, p(\Pi'^{\mathrm{alg}}) \cdot \prod_{1\leq i<j\leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1, \Pi'\otimes\chi_j^{-1})}{L^S\big(\tfrac{1}{2}+k, \Pi\otimes\chi_j\big)}, \tag{3.2}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$. *If n is even and m is odd, then all L-values* $L^S\big(\tfrac{1}{2}+k, \Pi \times \Pi'\big)$, $L^S(1, \Pi'\otimes\chi_j^{-1})$ *and* $L^S\big(\tfrac{1}{2}+k, \Pi\otimes\chi_j\big)$ *are critical.*

Proof. Putting our Steps I–III, i.e., equations (2.2), (2.5) and (2.8) together with (2.10), and observing the nonvanishing of $L^S(1, \Pi'\otimes\chi_j^{-1})$ (see [Shahidi 1981, Theorem 5.1]), we obtain

$$L^S\big(\tfrac{1}{2}+k, \Pi \times \Sigma\big) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\phi)E^{\mathrm{cm}}(\chi_1,\dots,\chi_{n-m-1})} (2\pi i)^{d((n-1)((k-1/2)n-1))+1/2(n-m-1)(n-m-2))}\, p(\Pi)\, p(\Pi'^{\mathrm{alg}}) \cdot \prod_{1\leq i<j\leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} L^S(1, \Pi'\otimes\chi_j^{-1}),$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$. As

$$L^S\big(\tfrac{1}{2}+k, \Pi \times \Sigma\big) = L^S\big(\tfrac{1}{2}+k, \Pi \times \Pi'\big) \cdot \prod_{j=1}^{n-m-1} L^S\big(\tfrac{1}{2}+k, \Pi\otimes\chi_j\big),$$

this yields

$$L^S\big(\tfrac{1}{2}+k, \Pi \times \Pi'\big) \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\phi)E^{\mathrm{cm}}(\chi_1,\dots,\chi_{n-m-1})} (2\pi i)^{d((n-1)((k-1/2)n-1))+1/2(n-m-1)(n-m-2))} p(\Pi)\, p(\Pi'^{\mathrm{alg}}) \cdot \prod_{1\leq i<j\leq n-m-1} p(\chi_i\chi_j^{-1}, \Psi_{\chi_i\chi_j^{-1}}) \prod_{j=1}^{n-m-1} \frac{L^S(1, \Pi'\otimes\chi_j^{-1})}{L^S\big(\tfrac{1}{2}+k, \Pi\otimes\chi_j\big)} \tag{3.3}$$

as a relation, which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$, implying the first assertion. Hence, assume now that n is even and m is odd. Criticality of $L^S\big(\tfrac{1}{2}+k, \Pi \times \Pi'\big)$, $L^S(1, \Pi'\otimes\chi_j^{-1})$ and $L^S\big(\tfrac{1}{2}+k, \Pi\otimes\chi_j\big)$ is then implied by Lemma 1.11. This completes the proof. □

Remark 3.4. Our Theorem 3.1 has an obvious corollary/consequence for critical values $L^S(k, \Pi\times\Pi')$ of Rankin–Selberg L-functions of type $n\times n$ — i.e., when both factors are of the same rank n — but when Π' is a (noncuspidal) isobaric sum. Let $\Pi' := (\Pi_1 \boxplus \Pi_2)\|\det\|^{1/2}$, where Π_1 and Π_2 are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_{m_1}(\mathbb{A}_F)$ and $\mathrm{GL}_{m_2}(\mathbb{A}_F)$ respectively, such that $m_1+m_2=n$. If we assume that both summands Π_i may be completed using conjugate self-dual algebraic Hecke characters to cohomological isobaric sums Σ_i on $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, which satisfy the piano-hypothesis with respect to Π_∞, then any $\frac{1}{2}+k\in\mathrm{Crit}(\Pi\times\Sigma_1)\cap\mathrm{Crit}(\Pi\times\Sigma_2)$ gives rise to a critical integer $k\in\mathrm{Crit}(\Pi\times\Pi')$. Hence, writing

$$L^S(k,\Pi\times\Pi')=L^S\big(\tfrac{1}{2}+k,\Pi\times(\Pi_1\boxplus\Pi_2)\big)=L^S\big(\tfrac{1}{2}+k,\Pi\times\Pi_1\big)\cdot L^S\big(\tfrac{1}{2}+k,\Pi\times\Pi_2\big),$$

we may apply Theorem 3.1 to both of the latter factors and derive a rationality result for the critical values $L^S(k,\Pi\times\Pi')$. We leave the obvious details to the reader. Finally, it is now also clear how one can obtain an analogous result if the representation Π' is the twisted isobaric sum of $r\geq 3$ conjugate self-conjugate cuspidal automorphic representations.

4. Main applications

In this section we provide a couple of applications of our Main Theorem, exemplifying the strength of period-relations such as the ones established in Theorem 3.1.

4A. ***Quotients of twisted standard L-functions at a joint special value.*** Application I concerns the twisted standard L-function and is a broad generalization of the main result of [Waldspurger 1985].

Waldspurger [1985] has shown a rationality result for the quotient

$$L\big(\tfrac{1}{2},\pi\otimes\alpha\big)/L\big(\tfrac{1}{2},\pi\otimes\beta\big)$$

of the standard L-functions attached to the twisted cohomological cuspidal automorphic representations $\pi\otimes\alpha$ and $\pi\otimes\beta$ of GL_2 over any number field at their joint critical value $s_0=\frac{1}{2}$. More precisely, here α and β are assumed to be quadratic Hecke characters having the same archimedean component $\alpha_\infty=\beta_\infty$, π denotes a cohomological unitary cuspidal automorphic representation of GL_2 and $L\big(\frac{1}{2},\pi\otimes\beta\big)$ is assumed to be nonzero. Under these assumptions, Waldspurger established a relation of the form

$$\frac{L\big(\tfrac{1}{2},\pi\otimes\alpha\big)}{L\big(\tfrac{1}{2},\pi\otimes\beta\big)}\sim_{\mathbb{Q}(\pi)}\frac{p(\alpha)}{p(\beta)},$$

the two period-invariants $p(\alpha)$ and $p(\beta)$ only depending on α and β, respectively, and the archimedean component of the cuspidal representations π. See [Waldspurger 1985, p. 174].

Here we generalize Waldspurger's result to the case of quotients of standard L-functions of GL_n/F where $n\geq 2$ is arbitrary, $s_0=\frac{1}{2}+k$ is a more general special value while F is any CM-field. Our result reads as follows.

Theorem 4.1. *Let F be any CM-field and let $n \geq 2$ be an integer. We assume that Π is a cohomological unitary cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$ *and let α and β be conjugate self-dual Hecke characters of $\mathbb{A}_F^\times$ having the same archimedean components $\alpha_v(z) = \beta_v(z) = z^{a_v}\bar{z}^{-a_v}$, $v \in S_\infty$. If $a_v \in \frac{n}{2} + \mathbb{Z}$ and $\mu_{v,1} \geq a_v \geq \mu_{n,v}$ for all $v \in S_\infty$, there is a choice of conjugate self-dual Hecke characters $\chi_1, \dots, \chi_{n-2}$, such that the isobaric automorphic sum $\Sigma = \alpha \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-2}$ is cohomological and such that $(\Pi_\infty, \Sigma_{\alpha,\infty})$ satisfies the piano-hypothesis* (1.8). *Fix any such choice and let $s_0 = \frac{1}{2} + k \in$* $\mathrm{Crit}(\Pi \times \Sigma_\alpha)$ *be any critical point of $L(s, \Pi \times \Sigma_\alpha)$.*

In the special case when $k = 0$ only, i.e., if $s_0 = \frac{1}{2}$ denotes the central critical point, we additionally assume that the coefficient modules of Π_∞ and $\Sigma_{\alpha,\infty}$ are both sufficiently regular (*see Sections 1D1 and 1D3*) *and that there exists a choice of Hecke characters χ, χ' such that $L^S\big(\frac{1}{2}, \Pi_\chi \times \Pi_{\chi'}\big) \neq 0$* (*see Section 2B*), *and that $L^S\big(\frac{1}{2}, \Pi \otimes \beta\big)$ and $L^S\big(\frac{1}{2}, \Pi \otimes \chi_j\big) \neq 0$ for all $1 \leq j \leq n-2$.*

We have

$$\frac{L^S\big(\frac{1}{2}+k, \Pi \otimes \alpha\big)}{L^S\big(\frac{1}{2}+k, \Pi \otimes \beta\big)} \sim \prod_{i=1}^{n-2} \frac{p(\alpha, \Psi_{\alpha\chi_i^{-1}})}{p(\beta, \Psi_{\beta\chi_i^{-1}})}. \tag{4.2}$$

where the relation "$\sim$" is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$ *and over the number field*

$$\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_\alpha)\mathbb{Q}(\Sigma_\beta)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\alpha, \beta, \chi_1, \dots, \chi_{n-2})E^{\mathsf{cm}}(\alpha)E^{\mathsf{cm}}(\beta)\prod_{i}^{n-2} E^{\mathsf{cm}}(\chi_i^{-1}).$$

If n is even, then all the $s_0 = \frac{1}{2} + k$ are indeed critical for $L(s, \Pi \otimes \alpha)$ and $L(s, \Pi \otimes \beta)$.

Proof. The discussion in Section 1D3, together with our assumption that $a_v \in \frac{n}{2} + \mathbb{Z}$ and $\mu_{v,1} \geq a_v \geq \mu_{n,v}$ for all $v \in S_\infty$, implies immediately that there is a choice of conjugate self-dual Hecke characters $\chi_1, \dots, \chi_{n-2}$, such that the isobaric automorphic sum

$$\Sigma = \alpha \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-2}$$

is cohomological and such that $(\Pi_\infty, \Sigma_{\alpha,\infty})$ satisfies the piano-hypothesis. Hence, putting $m = 1$ and $\Pi' = \alpha$, the pair (Π, α) of cuspidal representations on $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_1(\mathbb{A}_F)$ satisfies all the conditions of our Theorem 3.1 (with $\Sigma = \Sigma_\alpha$). As is again immediate, our assumption $\alpha_\infty = \beta_\infty$ implies that the isobaric sum

$$\Sigma_\beta := \beta \boxplus \chi_1 \boxplus \cdots \boxplus \chi_{n-2}$$

has the same archimedean component as Σ_α. Another short moment of thought convinces us that consequently Σ_α and Σ_β may be interchanged in the statement of Theorem 4.1 without changing any assertions. Otherwise put, the pair (Π, β)

automatically satisfies all the conditions of our Theorem 3.1, letting $\Sigma = \Sigma_\beta$. Hence, simply by inserting into the formula provided by Theorem 3.1 we obtain

$$\frac{L^S\big(\frac{1}{2}+k,\,\Pi\otimes\alpha\big)}{L^S\big(\frac{1}{2}+k,\,\Pi\otimes\beta\big)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_\alpha)\mathbb{Q}(\Sigma_\beta)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\chi_1,\dots,\chi_{n-2})} \frac{p(\alpha^{\mathsf{alg}})}{p(\beta^{\mathsf{alg}})}\prod_{i=1}^{n-2}\frac{L^S(1,\alpha\chi_i^{-1})}{L^S(1,\beta\chi_i^{-1})}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$. As the Whittaker periods $p(\alpha^{\mathsf{alg}})$ and $p(\beta^{\mathsf{alg}})$ are both chosen to be 1 (see Section 1F) and applying Blasius's result (see Proposition 2.9) once more to $\prod_{i=1}^{n-2} L^S(1,\alpha\chi_i^{-1})$ and $\prod_{i=1}^{n-2} L^S(1,\beta\chi_i^{-1})$, we get

$$\frac{L^S\big(\frac{1}{2}+k,\,\Pi\otimes\alpha\big)}{L^S\big(\frac{1}{2}+k,\,\Pi\otimes\beta\big)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma_\alpha)\mathbb{Q}(\Sigma_\beta)\mathbb{Q}(\phi)E^{\mathsf{cm}}(\alpha,\chi_1,\dots,\chi_{n-2})E^{\mathsf{cm}}(\beta,\chi_1,\dots,\chi_{n-2})} \prod_{i=1}^{n-2}\frac{p(\alpha\chi_i^{-1},\,\Psi_{\alpha\chi_i^{-1}})}{p(\beta\chi_i^{-1},\,\Psi_{\beta\chi_i^{-1}})}.$$

Obviously, we may harmlessly replace $E^{\mathsf{cm}}(\alpha\chi_1,\dots,\chi_{n-2})E^{\mathsf{cm}}(\beta,\chi_1,\dots,\chi_{n-2})$ by $E^{\mathsf{cm}}(\alpha,\beta,\chi_1,\dots,\chi_{n-2})$ in the latter relation. For each $1\le i\le n-2$ one has

$$p(\alpha\chi_i^{-1},\,\Psi_{\alpha\chi_i^{-1}}) \sim_{E^{\mathsf{cm}}(\alpha)E^{\mathsf{cm}}(\chi_i^{-1})} p(\alpha,\,\Psi_{\alpha\chi_i^{-1}})\; p(\chi_i^{-1},\,\Psi_{\alpha\chi_i^{-1}})$$

and likewise for β taking the role of α; see [Grobner and Lin 2020, Proposition 4.4]. As $\alpha_\infty = \beta_\infty$ by assumption, $\Psi_{\alpha\chi_i^{-1}} = \Psi_{\beta\chi_i^{-1}}$ by definition; see Section 2D. This implies the first assertion of the theorem. The second assertion follows applying Lemma 1.11. □

Remark 4.3 (further interpretations). Grobner and Raghuram [2014b] achieved a rationality result for the critical values of the twisted standard L-function $L(s,\Pi\otimes\chi)$ using unspecified archimedean periods. Here, Π denotes a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F^+})$, admitting a Shalika model and χ is a Hecke character of finite order. These assumptions necessarily imply that n is even as in the refined second assertion of our Theorem 4.1 above. In this regard, Theorem 4.1 provides a generalization as well as a certain refinement of a consequence of the main result of [Grobner and Raghuram 2014b] over general CM-fields, instead of totally real fields F^+.

4B. ***Quotients of a fixed Rankin–Selberg L-function at different critical values.*** Application II concerns quotients of a given Rankin–Selberg L-function $L^S(s,\Pi\times\Pi')$ of general type $n\times m$, $1\le m\le n$, at different critical values $s=\frac{1}{2}+k$ and $s=\frac{1}{2}+\ell$.

Our result may be viewed as a generalization of

(i) the main result of [Harder and Raghuram 2020] which was for Rankin–Selberg L-functions of general type $n\times m$, with nm even, but over totally real fields F^+;

(ii) Theorem 5.5 of [Grobner and Lin 2020] which was for general CM-fields F, but for Rankin–Selberg L-functions of type $n \times (n-1)$ only.

It should be pointed out though, that our result below is a rather mild generalization of a consequence of the main result of [Lin 2015]. Lin [2015, Theorem 10.8.1] has achieved a very general, fine rationality-result for Rankin–Selberg L-functions of type $n \times m$ under a list of additional local assumptions (and conjectures, but those were later proved in [Guerberoff 2016] and [Grobner and Lin 2020]). We hence do not claim much originality from our side, but rather include the following corollary of Theorem 3.1 for the sake of giving a new approach and an example of the use of our period-relations.

In order to explain our result, recall weak base change BC from an arbitrary rational unitary similitude group $GU(V)/\mathbb{Q}$ attached to a nondegenerate Hermitian space V of $\dim_{F^+} V = n$, as established in [Shin 2014]. Strictly speaking, the construction of BC in [Shin 2014] entails the claim that $F = \mathcal{K}F^+$ for some imaginary quadratic field $\mathcal{K}$, which we shall henceforth assume. The same assumption has been made in [Guerberoff 2016, §5], which we shall use in the proof of our Corollary 4.4. Then, for every cohomological cuspidal automorphic representation π of $GU(V)(\mathbb{A}_\mathbb{Q})$ a base change $BC(\pi) = \chi_\pi \otimes \Pi$ has been constructed in [Shin 2014]. Here, χ_π is a Hecke character of $\mathbb{A}_\mathcal{K}^\times$, while Π is a conjugate self-dual isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. By results of Delorme and Enright (see [Enright 1979]), Π_∞ is cohomological as well. See also [Labesse 2011, §5.1] and [Clozel 1991, §3.4].

Corollary 4.4. *Let $F = \mathcal{K}F^+$ be a CM-field and suppose that $1 \le m < n$ are integers, n even and m odd. We let $\Pi = BC(\pi)|_{\mathrm{GL}_n(\mathbb{A}_F)}$ be a cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$ *which we assume to be obtained by weak base change from a unitary tempered cuspidal automorphic representation π of some rational similitude group $GU(V)/\mathbb{Q}$. Its infinite component π_∞ is supposed to belong to the antiholomorphic discrete series and to be cohomological with respect to an algebraic coefficient module of $GU(V)(\mathbb{R})$ which is defined over $\mathbb{Q}$.*

Let Π' be a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$, *satisfying the conditions of Theorem 3.1.*

Let $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ be two critical points of $L(s, \Pi \times \Sigma)$ different from $s_0 = \frac{1}{2}$. Then $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are indeed critical for $L(s, \Pi \times \Pi')$ and the ratio of critical values satisfies

$$\frac{L^S\big(\frac{1}{2}+k, \Pi \times \Pi'\big)}{L^S\big(\frac{1}{2}+\ell, \Pi \times \Pi'\big)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{\mathrm{Gal}}} (2\pi i)^{d(k-\ell)nm},$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$.

Proof. By Theorem 3.1, the quotient of critical values satisfies

$$\frac{L^S\big(\tfrac{1}{2}+k,\Pi\times\Pi'\big)}{L^S\big(\tfrac{1}{2}+\ell,\Pi\times\Pi'\big)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)E^{\mathsf{cm}}(\chi_1,\dots,\chi_{n-m-1})} (2\pi i)^{d(k-\ell)(n-1)n}\cdot\prod_{j=1}^{n-m-1}\frac{L^S\big(\tfrac{1}{2}+\ell,\Pi\otimes\chi_j\big)}{L^S\big(\tfrac{1}{2}+k,\Pi\otimes\chi_j\big)}, \tag{4.5}$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$. Moreover, obviously, both sides of this relation are invariant under all $\sigma\in\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')\cap\mathfrak{S}(\{\chi_1,\dots,\chi_{n-m-1}\})$, where $\mathfrak{S}(\{\chi_1,\dots,\chi_{n-m-1}\})$ denotes the group of all $\sigma\in\mathrm{Aut}(\mathbb{C})$ such that

$$\{{}^\sigma\chi_1,\dots,{}^\sigma\chi_{n-m-1}\}=\{\chi_1,\dots,\chi_{n-m-1}\}.$$

Therefore, by Lemma 1.2, relation (4.5) holds over every field L, which contains F^{Gal} and the subfield of $\mathbb{C}$, fixed by $\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')\cap\mathfrak{S}(\{\chi_1,\dots,\chi_{n-m-1}\})$. In particular, (4.5) holds over the compositum of number fields

$$\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')\mathbb{Q}(\{\chi_1,\dots,\chi_{n-m-1}\})F^{\mathrm{Gal}}.$$

Now observe that, by Lemma 1.11, $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are both critical for all $L(s,\Pi\otimes\chi_j)$, $1\le j\le n-m-1$. Since $\frac{1}{2}+k$ and $\frac{1}{2}+\ell$ are also assumed to be different from the central critical value, our additional assumption on Π being obtained by base change from π hence allows us to use Guerberoff's theorem [2016, Theorem 4.5.1] on noncentral critical values of standard L-functions. (The careful reader may want to use §4.2 in [Grobner et al. 2018] in combination with [Kaletha et al. 2014, Theorem 1.7.1], which confirms Guerberoff's Hypothesis 4.5.1 for our representation π.) Hence, simply by inserting into Guerberoff's formula, we obtain

$$\prod_{j=1}^{n-m-1}\frac{L^S\big(\tfrac{1}{2}+\ell,\Pi\otimes\chi_j\big)}{L^S\big(\tfrac{1}{2}+k,\Pi\otimes\chi_j\big)} \sim_{\mathbb{Q}^{\mathsf{aux}}(\pi,\{\chi_j\})} (2\pi i)^{d(n-m-1)n(\ell-k)},$$

which is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$. Here, $\mathbb{Q}^{\mathsf{aux}}(\pi,\{\chi_j\})$ denotes any number field over which π_f and all characters $\chi_{j,f}$ are defined (such a field exists, e.g., by [Grobner and Sebastian 2017, Theorem A.2.4]). Collecting the powers of $(2\pi i)$ we hence obtain

$$\frac{L^S\big(\tfrac{1}{2}+k,\Pi\times\Pi'\big)}{L^S\big(\tfrac{1}{2}+\ell,\Pi\times\Pi'\big)} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')\mathbb{Q}(\{\chi_1,\dots,\chi_{n-m-1}\})\mathbb{Q}^{\mathsf{aux}}(\pi,\{\chi_j\})F^{\mathrm{Gal}}} (2\pi i)^{d(k-\ell)nm}. \tag{4.6}$$

As we have seen, relation (4.6) is equivariant under $\mathrm{Aut}(\mathbb{C}/F^{\mathrm{Gal}})$ and both sides are invariant under every $\sigma\in\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')$. Hence, applying Lemma 1.2 once more, we see that this relation actually holds over any field containing F^{Gal} and the field of rationality of $\mathfrak{S}(\Pi)\cap\mathfrak{S}(\Pi')$. In particular, (4.6) holds over the number field $\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{\mathrm{Gal}}$, which shows the claim. □

Remark 4.7 (further interpretations). Januszewski [2019]recently achieved a conditional rationality-result for Rankin–Selberg L-functions of type $n \times (n-1)$ with precise powers of $(2\pi i)$ as archimedean contributions, recovering the result of Harder and Raghuram [2020] in the case of $n \times (n-1)$ as a consequence (under the given hypotheses). Hence, our Corollary 4.4 may also be seen as an unconditional generalization of a consequence of the main result of [Januszewski 2019] for more general pairs $n \times m$ and over CM-fields $F = \mathcal{K}F^+$.

Most recently, Raghuram has presented a different approach to our corollary in the special case of $m = 1$ (i.e., the standard L-function) through automorphic induction; see Theorem 1 in [Raghuram 2020]

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HARALD GROBNER
FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT WIEN
WIEN
AUSTRIA

harald.grobner@univie.ac.at

GUNJA SACHDEVA
IISER TIRUPATI
TIRUPATI
INDIA

gunja@math.tifr.res.in

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A BOUND FOR THE CONDUCTOR OF AN OPEN SUBGROUP OF GL_2 ASSOCIATED TO AN ELLIPTIC CURVE

NATHAN JONES

Given an elliptic curve E without complex multiplication defined over a number field K, consider the image of the Galois representation defined by letting Galois act on the torsion of E. Serre's open image theorem implies that there is a positive integer m for which the Galois image is completely determined by its reduction modulo m. We prove a bound on the smallest such m in terms of standard invariants associated with E. The bound is sharp and improves upon previous results.

1. Introduction

Let K be a number field, let E/K be an elliptic curve and let E_{tors} denote its torsion subgroup. Denote by $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K and consider the Galois representation

$$\rho_{E,K} : G_K \to \mathrm{Aut}(E_{\mathrm{tors}}) \simeq \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

defined by letting G_K act on the torsion of E and choosing compatible bases thereof. A celebrated theorem of J.-P. Serre [1972] states that, if E has no complex multiplication, then the image of $\rho_{E,K}$ is open inside $\mathrm{GL}_2(\widehat{\mathbb{Z}})$, or equivalently that

$$[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : \rho_{E,K}(G_K)] < \infty. \tag{1}$$

Consequently, one may find a positive integer m with the property that

$$\ker\bigl(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})\bigr) \subseteq \rho_{E,K}(G_K).$$

Definition 1.1. Given an open subgroup $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$, we define the positive integer m_G by

$$m_G := \min\bigl\{m \in \mathbb{N} : \ker\bigl(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})\bigr) \subseteq G\bigr\}$$

and call it the *conductor of* G. In case $G = \rho_{E,K}(G_K)$ for an elliptic curve E defined over a number field K and without complex multiplication, we denote the conductor of G by $m_{E,K}$.

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The purpose of this note is to prove the following upper bound for $m_{E,K}$. In its statement, Δ_K denotes the absolute discriminant of the number field K, Δ_E denotes the minimal discriminant ideal attached to the elliptic curve E, $N_{K/\mathbb{Q}} : K^\times \to \mathbb{Q}^\times$ denotes the usual norm map and

$$\operatorname{rad}(m) := \prod_{\substack{\ell \mid m \\ \ell \text{ prime}}} \ell$$

denotes the radical of the positive integer m. Given a nonzero ideal $I \subseteq \mathcal{O}_K$, we identify the ideal $N_{K/\mathbb{Q}}(I) \subseteq \mathbb{Z}$ with the (unique) positive integer that generates it, and thus we may regard $N_{K/\mathbb{Q}}(\Delta_E) \in \mathbb{N}$.

Theorem 1.2. *Let K be a number field, let E be an elliptic curve over K without complex multiplication, and let $m_{E,K} \in \mathbb{N}$ be as in Definition 1.1. Then one has*

$$m_{E,K} \leq 2 \cdot [\mathrm{GL}_2(\widehat{\mathbb{Z}}) : \rho_{E,K}(G_K)] \cdot \operatorname{rad}(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|).$$

Remark 1.3. The bound in Theorem 1.2 both improves upon and generalizes a bound appearing in [Jones 2009] (see Corollary 1.5 below). Furthermore, using results in [Daniels 2015], we may see that there are infinitely many[1] elliptic curves E over $\mathbb{Q}$ satisfying

$$m_{E,\mathbb{Q}} = 2 \cdot [\mathrm{GL}_2(\widehat{\mathbb{Z}}) : \rho_{E,\mathbb{Q}}(G_{\mathbb{Q}})] \cdot \operatorname{rad}(|\Delta_E|). \tag{2}$$

Thus, our bound for $m_{E,K}$ is sharp when $K = \mathbb{Q}$.

Remark 1.4. Let

$$\rho_{E,m} : G_K \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z}), \quad \rho_{E,\ell^\infty} : G_K \to \mathrm{GL}_2(\mathbb{Z}_\ell)$$

denote the Galois representations defined by letting G_K act on $E[m]$ and on $E[\ell^\infty] := \bigcup_{n \geq 1} E[\ell^n]$ respectively, and let $K(E[m]) = \overline{K}^{\ker \rho_{E,m}(G_K)}$ denote the m-th division field of E. The conductor $m_{E,K}$ that we are considering should not be confused with “Serre’s constant,” defined for an elliptic curve E over $\mathbb{Q}$ in [Daniels and González-Jiménez 2018] (see also [Cojocaru 2005]) by

$$A(E) := \prod_{\substack{\ell^n \text{ a prime power} \\ \rho_{E,\ell^n}(G_{\mathbb{Q}}) \neq \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \\ \forall k<n,\ \rho_{E,\ell^k}(G_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Z}/\ell^k\mathbb{Z})}} \ell^n.$$

It is evident that $A(E)$ divides $m_{E,\mathbb{Q}}$, but $m_{E,\mathbb{Q}}$ is in general larger than $A(E)$. The main differences between these two constants are as follows:

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[1] Specifically, (2) holds for any Serre curve E with the property that Δ_E is square-free and $\Delta_E \not\equiv 1 \bmod 4$.

(1) A prime power ℓ^n divides $m_{E,\mathbb{Q}}$ whenever $\ker\big(\mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{n-1}\mathbb{Z})\big) \not\subseteq \rho_{E,\ell^\infty}(G_\mathbb{Q})$, whereas $A(E)$ is square-free, except possibly at the primes 2 and 3. In other words, for each prime ℓ, $m_{E,\mathbb{Q}}$ encodes the action of $G_\mathbb{Q}$ on the entire ℓ-adic Tate module, whereas, for $\ell \geq 5$, $A(E)$ only encodes the action of $G_\mathbb{Q}$ on the ℓ-torsion of E.

(2) It may happen that there is a nontrivial intersection $\mathbb{Q} \neq \mathbb{Q}(E[m_1]) \cap \mathbb{Q}(E[m_2])$ for some $m_1, m_2 \in \mathbb{N}$ with $\gcd(m_1, m_2) = 1$. The constant $m_{E,\mathbb{Q}}$ encodes such "entanglements," whereas $A(E)$ does not.

The general phenomenon of entanglements has come up in various recent papers; see for instance [Brau and Jones 2016], which studies elliptic curves E over $\mathbb{Q}$ satisfying $[\mathbb{Q}(E[2]) : \mathbb{Q}] = 6$ and $\mathbb{Q}(E[2]) \subseteq \mathbb{Q}(E[3])$, and also [Bourdon et al. 2019], in which potential entanglements come up in an analysis of sporadic points on the modular curve $X_1(N)$.

Given an elliptic curve E defined over a number field K, computing the positive integer $m_{E,K}$ is a step toward understanding the image $\rho_{E,K}(G_K) \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$. Following Serre's open image result, there has been much interest in the nature of $\rho_{E,K}(G_K)$, for instance regarding its mod ℓ reductions (see [Mazur 1978; Merel 1996; Bilu and Parent 2011; Bilu et al. 2013; Lozano-Robledo 2013; Zywina 2015a]) and also more recently its reductions at composite levels (see [Dokchitser and Dokchitser 2012; Sutherland and Zywina 2017; Daniels and González-Jiménez 2018; Morrow 2019]). In addition to this connection, Theorem 1.2 also has analytic relevance; for instance in [Bell et al. 2020] it is applied to the study averages of constants appearing in various elliptic curve conjectures.

Serre's open image result (1) implies that, for any E/K without complex multiplication (CM), there exists a bound $C_{E,K} > 0$ so that, for each prime $\ell > C_{E,K}$, we have $\rho_{E,\ell}(G_K) = \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$. Serre asked whether the constant $C_{E,K}$ may be chosen uniformly in E, i.e., whether

$$\begin{gathered}\text{there exists } C_K > 0 \text{ so that, for all } E/K \text{ without CM and all prime } \ell > C_K,\\ \rho_{E,\ell}(G_K) = \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}).\end{gathered} \tag{3}$$

This question is still open, even in the case $K = \mathbb{Q}$. An affirmative answer to it would imply that

$$[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : \rho_E(G_K)] \ll_K 1,$$

although the implied constant is ineffective, because of an appeal to Faltings' theorem (see [Zywina 2015b], which details this in the case $K = \mathbb{Q}$). Theorem 1.2 thus has the following corollary.

Corollary 1.5. *Assume that (3) holds. We then have*

$$m_{E,K} \ll_K \mathrm{rad}(N_{K/\mathbb{Q}}(\Delta_E)).$$

Theorem 1.2 is proved via the following two propositions, the first of which deals generally with open subgroups $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$. Because of group-theoretical differences[2] present for the prime 2, it will be convenient to introduce the following modified radical:

$$\mathrm{rad}'(m) := \begin{cases} \mathrm{rad}(m) & \text{if } 4 \nmid m \\ 2\,\mathrm{rad}(m) & \text{if } 4 \mid m. \end{cases} \tag{4}$$

We will also distinguish the following case involving the prime 3, in whose statement G_3 (resp. $G(3)$) denotes the image of G under the projection map $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}_3)$ (resp. under $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$). The analysis proceeds a bit differently according to whether or not the condition

$$9 \mid m_G, \quad \mathrm{SL}_2(\mathbb{Z}_3) \nsubseteq G_3 \quad \text{and} \quad G(3) = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \tag{5}$$

holds.

Proposition 1.6. *Let $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be an open subgroup and let m_G be as in Definition 1.1. We then have*

$$\frac{m_G}{\mathrm{rad}'(m_G)} \text{ divides } \left[\pi^{-1}(G(\mathrm{rad}'(m_G))) : G(m_G)\right],$$

where $\mathrm{rad}'(\,\cdot\,)$ is defined as in (4) and $\pi : \mathrm{GL}_2(\mathbb{Z}/m_G\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/\mathrm{rad}'(m_G)\mathbb{Z})$ denotes the canonical projection map. Assuming that (5) holds, we have

$$\frac{9m_G}{\mathrm{rad}'(m_G)} \text{ divides } \left[\pi^{-1}(G(\mathrm{rad}'(m_G))) : G(m_G)\right].$$

In contrast with Proposition 1.6, our second proposition is specific to the situation where $G = \rho_{E,K}(G_K)$, making use of facts about the Weil pairing on an elliptic curve, together with the Nerón–Ogg–Shafarevich criterion for ramification in division fields.

Proposition 1.7. *Let K be a number field and let E be an elliptic curve defined over K without complex multiplication. Let $G := \rho_{E,K}(G_K)$ be the image of the Galois representation associated to E and let m_G be as in Definition 1.1. Assuming that (5) does not hold, we have*

$$\mathrm{rad}'(m_G) \le 2\left[\mathrm{GL}_2(\mathbb{Z}/\mathrm{rad}'(m_G)\mathbb{Z}) : G(\mathrm{rad}'(m_G))\right] \mathrm{rad}(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|).$$

If (5) does hold, then

$$\frac{\mathrm{rad}'(m_G)}{3} \le 2\left[\mathrm{GL}_2(\mathbb{Z}/\mathrm{rad}'(m_G)\mathbb{Z}) : G(\mathrm{rad}'(m_G))\right] \mathrm{rad}(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|).$$

---

[2] See [Dokchitser and Dokchitser 2012] (resp. [Elkies 2006]), which concerns the Galois representation on the 2-adic (resp. on the 3-adic) Tate module, illustrating these differences.

Since the index of a subgroup is preserved under taking the full preimage, we have that

$$\big[\mathrm{GL}_2(\mathbb{Z}/\operatorname{rad}'(m_G)\mathbb{Z}) : G(\operatorname{rad}'(m_G))\big] = \big[\mathrm{GL}_2(\mathbb{Z}/m_G\mathbb{Z}) : \pi^{-1}(G(\operatorname{rad}'(m_G)))\big],$$

where $\pi : \mathrm{GL}_2(\mathbb{Z}/m_G\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/\operatorname{rad}'(m_G)\mathbb{Z})$ is the canonical projection map. Thus, Theorem 1.2 follows from Propositions 1.6 and 1.7.

Many of the ingredients that enter into the proof of Theorem 1.2 may be verified for algebraic groups other than GL_2. For instance, using these same techniques, one should be able to obtain a similar bound for the analogous integer $m_{A,K}$ associated to an abelian variety A defined over a number field K whose Galois representation has open image inside $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$.

2. Notation and preliminaries

Throughout the paper, p and ℓ will always denote prime numbers. As usual, $\mathbb{N}$ denotes the set of natural numbers (excluding zero) and $\mathbb{Z}$ denotes the set of integers. We will occasionally use the abbreviations

$$\mathbb{N}_{\geq\alpha} := \{n \in \mathbb{N} : n \geq \alpha\},$$
$$\mathbb{Z}_{\geq\alpha} := \{n \in \mathbb{Z} : n \geq \alpha\}.$$

We recall that

$$\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/m\mathbb{Z}$$

is the inverse limit of the rings $\mathbb{Z}/m\mathbb{Z}$ with respect to the canonical projection maps $\mathbb{Z}/nm\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. Under the isomorphism of the Chinese remainder theorem, we have that

$$\widehat{\mathbb{Z}} \simeq \prod_{\ell} \mathbb{Z}_\ell, \tag{6}$$

where $\mathbb{Z}_\ell$ as usual denotes the ring of ℓ-adic integers. More generally, for any $m \in \mathbb{N}_{\geq 2}$ we define $\mathbb{Z}_m$ and $\mathbb{Z}_{(m)}$ to be the quotients of $\widehat{\mathbb{Z}}$ corresponding under (6) to the following rings:

$$\mathbb{Z}_m \simeq \prod_{\ell \mid m} \mathbb{Z}_\ell, \qquad \mathbb{Z}_{(m)} \simeq \prod_{\ell \nmid m} \mathbb{Z}_\ell.$$

For any $m \in \mathbb{N}_{\geq 2}$ we have an isomorphism

$$\widehat{\mathbb{Z}} \simeq \mathbb{Z}_m \times \mathbb{Z}_{(m)},$$

and projection maps

$$\widehat{\mathbb{Z}} \to \mathbb{Z}_m, \quad \widehat{\mathbb{Z}} \to \mathbb{Z}_{(m)}.$$

We note that these observations may also be applied to points in an algebraic group; in particular we have

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \simeq \mathrm{GL}_2(\mathbb{Z}_m) \times \mathrm{GL}_2(\mathbb{Z}_{(m)}) \simeq \prod_{\ell} \mathrm{GL}_2(\mathbb{Z}_\ell)$$

and we have projection maps

$$\pi_m : \mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}_m), \quad \pi_{(m)} : \mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}_{(m)}). \tag{7}$$

In most cases we will denote any projection map simply by π, but on some occasions we will decorate it with subscripts, such as in (7) or

$$\pi_{m^\infty,m} : \mathrm{GL}_2(\mathbb{Z}_m) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z}), \quad \pi_{nm,n} : \mathrm{GL}_2(\mathbb{Z}/nm\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$

The ring $\widehat{\mathbb{Z}}$ is a topological ring under the profinite topology, and the group $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ inherits the structure of a profinite group. We recall that any open subgroup $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is a closed subgroup but not conversely. In general, given any closed subgroup $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$, we denote by $G_m \subseteq \mathrm{GL}_2(\mathbb{Z}_m)$ (resp. by $G_{(m)} \subseteq \mathrm{GL}_2(\mathbb{Z}_{(m)})$) its image under π_m (resp. under $\pi_{(m)}$) as in (7). We denote by $G(m)$ the image of G under the canonical projection

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z}).$$

For any $m \in \mathbb{N}$ and any d dividing m, we denote the prime-to-d part of m by

$$m_{(d)} := \frac{m}{\prod_{\ell | d} \ell^{\mathrm{ord}_\ell(m)}}.$$

Finally, we let

$$\mathrm{id}_m : \mathrm{GL}_2(\mathbb{Z}_m) \to \mathrm{GL}_2(\mathbb{Z}_m), \quad \mathrm{id}_{(m)} : \mathrm{GL}_2(\mathbb{Z}_{(m)}) \to \mathrm{GL}_2(\mathbb{Z}_{(m)})$$

denote the identity maps, and we let 1_m (resp. $1_{(m)}$) denote the identity element of $\mathrm{GL}_2(\mathbb{Z}_m)$ (resp. of $\mathrm{GL}_2(\mathbb{Z}_{(m)})$). We may also at times denote by 1_m the identity element of $\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$.

For an abelian group A and a positive integer n we as usual denote by $A[n]$ the n-torsion subgroup of A. For a prime number ℓ we define

$$A[\ell^\infty] := \bigcup_{n=0}^{\infty} A[\ell^n], \qquad A_{\mathrm{tors}} := \bigcup_{n=1}^{\infty} A[n], \qquad A_{\mathrm{tors},(\ell)} := \bigcup_{\substack{n=1 \\ \ell \nmid n}}^{\infty} A[n].$$

Note that, if $A[n]$ is finite for each $n \in \mathbb{N}$, we have

$$A_{\mathrm{tors}} \simeq A[\ell^\infty] \times A_{\mathrm{tors},(\ell)}.$$

For a number field K, we denote by $\mathcal{O}_K$ its ring of integers, by Δ_K its absolute discriminant and by

$$N_{K/\mathbb{Q}} : K \to \mathbb{Q}$$

the norm map. A critical issue that arises in the proof of Proposition 1.7 is that of *entanglement* of division fields, i.e., the possibility that the field extension $K \subseteq K(E[m_1]) \cap K(E[m_2])$ is a nontrivial extension, where m_1 and m_2 are relatively prime positive integers. Putting $F := K(E[m_1]) \cap K(E[m_2])$, we have by Galois theory that

$$\begin{aligned}&\mathrm{Gal}(K(E[m_1m_2])/K)\\ &\qquad \simeq \left\{(\sigma_1, \sigma_2) \in \mathrm{Gal}(K(E[m_1])/K) \times \mathrm{Gal}(K(E[m_2])/K) : \sigma_1|_F = \sigma_2|_F\right\}.\end{aligned}$$

More generally, if G_1, G_2 and H are groups and $\psi_1 : G_1 \to H$, $\psi_2 : G_2 \to H$ are surjective group homomorphisms, we introduce the following notation for the fibered product:

$$G_1 \times_\psi G_2 := \{(g_1, g_2) \in G_1 \times G_2 : \psi_1(g_1) = \psi_2(g_2)\}$$

(here ψ is an abbreviation for the ordered pair (ψ_1, ψ_2)). Evidently,

$$K \neq K(E[m_1]) \cap K(E[m_2])$$

if and only if the fibered product

$$\mathrm{Gal}(K(E[m_1])/K) \times_{\mathrm{res}} \mathrm{Gal}(K(E[m_2])/K)$$

is a fibered product over a nontrivial group, where

$$\mathrm{res}_i : \mathrm{Gal}(K(E[m_i])/K) \to \mathrm{Gal}(K(E[m_1]) \cap K(E[m_2])/K)$$

denotes the restriction map.

3. Proof of Proposition 1.6

In this section we prove Proposition 1.6, bounding $m_G/\operatorname{rad}'(m_G)$ in terms of the index of $G(m_G)$ in $\pi^{-1}(G(\operatorname{rad}'(m_G)))$, where $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is any open subgroup. We recall that, in the profinite topology, any open subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ is necessarily closed; we will establish some lemmas regarding closed subgroups of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ which thus apply to the open subgroup G.

We begin by giving a more precise description of the local exponents $\beta_\ell \geq 0$ occurring in

$$m_G =: \prod_{\ell} \ell^{\beta_\ell}. \tag{8}$$

In what follows we use the maps

$$\pi_{\ell^{\beta+1},\ell^{\beta}} \times \mathrm{id}_{(\ell)} : \mathrm{GL}_2(\mathbb{Z}/\ell^{\beta+1}\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}_{(\ell)}) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{\beta}\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}_{(\ell)}),$$
$$\pi_{\ell^{\infty},\ell^{\beta+1}} \times \mathrm{id}_{(\ell)} : \mathrm{GL}_2(\mathbb{Z}_{\ell}) \times \mathrm{GL}_2(\mathbb{Z}_{(\ell)}) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{\beta+1}\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}_{(\ell)})$$

defined by the obvious projection in the first factor and the identity map in the second factor. For any prime ℓ, we define

$$\alpha_{\ell} := \begin{cases} 2 & \text{if } \ell = 2, \\ 1 & \text{if } \ell \geq 3. \end{cases} \tag{9}$$

The next lemma follows from ideas in [Serre 1968, Lemma 3, IV-23]. In its statement and henceforth, we will interpret $\mathrm{GL}_2(\mathbb{Z}/\ell^0\mathbb{Z}) := \{1\}$ as the trivial group, so that $\ker \pi_{\ell,1} = \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$.

Lemma 3.1. *Let $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be a closed subgroup, let ℓ be a prime number, and let $\beta \in \mathbb{Z}_{\geq 0}$. Assume that*

$$\forall \gamma \in [\beta, \max\{\beta, \alpha_{\ell}\}] \cap \mathbb{Z}, \quad \ker(\pi_{\ell^{\gamma+1},\ell^{\gamma}}) \times \{1_{(\ell)}\} \subseteq (\pi_{\ell^{\infty},\ell^{\gamma+1}} \times \mathrm{id}_{(\ell)})(G),$$

where α_{ℓ} is as in (9). We then have

$$\ker(\pi_{\ell^{\infty},\ell^{\beta}}) \times \{1_{(\ell)}\} \subseteq G.$$

Proof. Since $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is closed, it suffices to prove that, for each $n \in \mathbb{Z}_{\geq \max\{\beta, \alpha_{\ell}\}}$,

$$\ker(\pi_{\ell^{n+1},\ell^{n}}) \times \{1_{(\ell)}\} \subseteq (\pi_{\ell^{\infty},\ell^{n+1}} \times \mathrm{id}_{(\ell)})(G). \tag{10}$$

We prove this by induction on n as follows (the base case $n = \max\{\beta, \alpha_{\ell}\}$ is true by hypothesis). First note that, for $n \geq 1$, we have

$$\ker(\pi_{\ell^{n+1},\ell^{n}}) = \{I + \ell^n \widetilde{X} \mod \ell^{n+1} : \widetilde{X} \in M_{2\times 2}(\mathbb{Z}_{\ell})\}. \tag{11}$$

Thus, (10) may be reformulated as saying

$$\begin{gathered} \text{for all } X \in M_{2\times 2}(\mathbb{F}_{\ell}), \text{ there exists } \widetilde{X} \in M_{2\times 2}(\mathbb{Z}_{\ell}) \\ \text{such that } \widetilde{X} \equiv X \bmod \ell \text{ and } g := (I + \ell^n \widetilde{X}, 1_{(\ell)}) \in G. \end{gathered} \tag{12}$$

Our goal is to deduce that (12) continues to hold when n is replaced by $n+1$. Since G is a group, $g^{\ell} \in G$, and one sees by considering the binomial expansion

$$(I+\ell^n\widetilde{X})^{\ell} = I + \binom{\ell}{1}\ell^n\widetilde{X} + \binom{\ell}{2}\ell^{2n}\widetilde{X}^2 + \cdots + \binom{\ell}{\ell-1}\ell^{(\ell-1)n}\widetilde{X}^{\ell-1} + \ell^{\ell n}\widetilde{X}^{\ell} \tag{13}$$

that

$$(\pi_{\ell^{\infty},\ell^{n+2}} \times \mathrm{id}_{(\ell)})(g^{\ell}) = \big(I + \ell^{n+1}\widetilde{X} \pmod{\ell^{n+2}}, 1_{(\ell)}\big).$$

Since X in (12) was arbitrary, it follows by (11) that

$$\ker(\pi_{\ell^{n+2},\ell^{n+1}}) \times \{1_{(\ell)}\} \subseteq (\pi_{\ell^{\infty},\ell^{n+2}} \times \mathrm{id}_{(\ell)})(G),$$

completing the induction and proving the lemma. □

Remark 3.2. The "purely ℓ-adic version" of Lemma 3.1 also follows by the same proof (without the $\mathrm{GL}_2(\mathbb{Z}_{(\ell)})$ factor). Precisely, for any prime ℓ and closed subgroup $G \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell)$, and any $\beta \in \mathbb{Z}_{\geq 0}$, one has

$$\forall \gamma \in [\beta, \max\{\beta, \alpha_\ell\}] \cap \mathbb{Z},\ \ker\bigl(\mathrm{GL}_2(\mathbb{Z}/\ell^{\gamma+1}\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{\gamma}\mathbb{Z})\bigr) \subseteq G(\ell^{\gamma+1}) \\ \Rightarrow \ker\bigl(\mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{\beta}\mathbb{Z})\bigr) \subseteq G, \tag{14}$$

where α_ℓ is as in (9).

Remark 3.3. The fact that the exponent α_ℓ in (14) is different for $\ell = 2$ and otherwise uniform for $\ell \geq 3$ stands in contrast with [Serre 1968, Lemma 3, IV-23], which breaks into cases according to whether $\ell \leq 3$ or $\ell \geq 5$. The underlying reason is that we are seeking to conclude that $\ker\bigl(\mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{\beta}\mathbb{Z})\bigr) \subseteq G$, rather than the weaker conclusion $\ker\bigl(\mathrm{SL}_2(\mathbb{Z}_\ell) \to \mathrm{SL}_2(\mathbb{Z}/\ell^{\beta}\mathbb{Z})\bigr) \subseteq G \cap \mathrm{SL}_2(\mathbb{Z}_\ell)$, the latter breaking into cases according to the condition $\bigl[\mathrm{SL}_2(\mathbb{Z}/\ell^{\beta}\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}/\ell^{\beta}\mathbb{Z})\bigr] = \mathrm{SL}_2(\mathbb{Z}/\ell^{\beta}\mathbb{Z})$, which happens if and only if $\ell \geq 5$; see Lemma 3.6 below.

Definition 3.4. We define the exponents $\beta'_\ell = \beta'_\ell(G)$ by

$$\beta'_\ell := \min\Bigl\{\beta \in \mathbb{Z}_{\geq 0} : \forall \gamma \in [\beta, \max\{\beta, \alpha_\ell\}] \cap \mathbb{Z}, \\ \ker(\pi_{\ell^{\gamma+1},\ell^{\gamma}}) \times \{1_{(\ell)}\} \subseteq (\pi_{\ell^\infty,\ell^{\gamma+1}} \times \mathrm{id}_{(\ell)})(G)\Bigr\},$$

where α_ℓ is as in (9).

Corollary 3.5. *We have $\beta_\ell = \beta'_\ell$, where β_ℓ is as in (8).*

Proof. By Lemma 3.1, for each prime ℓ we have

$$\ker(\pi_{\ell^\infty,\ell^{\beta'_\ell}}) \times \{1_{(\ell)}\} \subseteq G.$$

Since $\ker\bigl(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2\bigl(\mathbb{Z}/\prod_\ell \ell^{\beta'_\ell}\mathbb{Z}\bigr)\bigr)$ is equal to the subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ generated by $\ker(\pi_{\ell^\infty,\ell^{\beta'_\ell}}) \times \{1_{(\ell)}\}$ as ℓ varies over all primes, we then have

$$\ker\Bigl(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2\Bigl(\mathbb{Z}/\prod_\ell \ell^{\beta'_\ell}\mathbb{Z}\Bigr)\Bigr) \subseteq G.$$

Thus, by (8) and Definition 1.1, we see that $\beta_\ell \leq \beta'_\ell$.

Conversely, suppose for the sake of contradiction that $\beta_\ell < \beta'_\ell$. By definition of β_ℓ, we would then have

$$\ker(\pi_{\ell^\infty,\ell^{\beta'_\ell-1}}) \times \{1_{(\ell)}\} \subseteq \ker(\pi_{\ell^\infty,\ell^{\beta_\ell}}) \times \{1_{(\ell)}\} \subseteq G. \tag{15}$$

Furthermore, since $\pi_{\ell^\infty,\ell^{\beta'_\ell}}(\ker(\pi_{\ell^\infty,\ell^{\beta'_\ell-1}})) = \ker(\pi_{\ell^{\beta'_\ell},\ell^{\beta'_\ell-1}})$, we then see that (15) would then imply

$$\forall \gamma \in [\beta'_\ell - 1, \max\{\beta'_\ell - 1, \alpha_\ell\}] \cap \mathbb{Z},\ \ker(\pi_{\ell^{\gamma+1},\ell^{\gamma}}) \times \{1_{(\ell)}\} \subseteq (\pi_{\ell^\infty,\ell^{\gamma+1}} \times \mathrm{id}_{(\ell)})(G),$$

contradicting Definition 3.4. Thus, $\beta'_\ell \leq \beta_\ell$. □

We will find it useful to have sufficient conditions to conclude that $\mathrm{SL}_2(\mathbb{Z}_\ell) \subseteq G$, where $G \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell)$ is an arbitrary closed subgroup. The next lemma does so for ℓ odd, and gives us sufficient information to allow us to deal separately with the prime $\ell = 2$. As with Lemma 3.1, it can be largely deduced from arguments found in the proof of [Serre 1968, Lemma 3, IV-23]; we include the details here for the sake of completeness.

Lemma 3.6. *Let ℓ be a prime number and let $G \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell)$ be a closed subgroup. If $\ell \geq 5$, then we have*

$$\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \subseteq G(\ell) \;\Rightarrow\; \mathrm{SL}_2(\mathbb{Z}_\ell) \subseteq G.$$

If $\ell = 3$, we have

$$G(3) = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \;\textit{and}\; \mathrm{SL}_2(\mathbb{Z}/9\mathbb{Z}) \subseteq G(9) \;\Rightarrow\; \mathrm{SL}_2(\mathbb{Z}_3) \subseteq G.$$

Finally, if $\ell = 2$, we have

$$G(4) = \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \;\Rightarrow\; G = \mathrm{GL}_2(\mathbb{Z}_2) \;\textit{or}\; [\mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) : G(8)] = 2.$$

Proof. We first assume ℓ is odd. Under the stated hypotheses, we will show that $\mathrm{SL}_2(\mathbb{Z}_\ell) \subseteq G$ by establishing that

$$\mathrm{SL}_2(\mathbb{Z}_\ell) = [G, G], \tag{16}$$

where $[G, G]$ denotes the *closure* of the commutator subgroup of G. This amounts to showing that $\mathrm{SL}_2(\mathbb{Z}_\ell) \subseteq [G, G]$, since the reverse inclusion follows from the fact that every commutator has determinant 1. We begin by first showing, by induction on n, that

$$\ker\big(\mathrm{SL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/\ell^n\mathbb{Z})\big) \subseteq G(\ell^{n+1}) \quad \begin{pmatrix} \ell \geq 5 \text{ and } n \geq 0, \text{ or} \\ \ell = 3 \text{ and } n \geq 1 \end{pmatrix}. \tag{17}$$

The binomial expansion argument (13) of Lemma 3.1 shows this, except for the case $\ell \geq 5$ and $n = 0$. To establish this final case, we first observe that

$$\det(I + \ell^n \widetilde{X}) \equiv 1 + \ell^n \operatorname{tr} \widetilde{X} \mod \ell^{n+1} \quad (n \geq 1).$$

Thus, for $n \geq 1$, we have

$$\ker\big(\mathrm{SL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/\ell^n\mathbb{Z})\big) = \big\{I + \ell^n \widetilde{X} \mod \ell^{n+1} : \widetilde{X} \in M_{2\times 2}^{\mathrm{tr}\equiv 0}(\mathbb{Z}_\ell)\big\}, \tag{18}$$

where

$$M_{2\times 2}^{\mathrm{tr}\equiv 0}(\mathbb{Z}_\ell) := \{\widetilde{X} \in M_{2\times 2}(\mathbb{Z}_\ell) : \operatorname{tr} \widetilde{X} \equiv 0 \mod \ell\}.$$

In particular, $\ker\big(\mathrm{SL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/\ell^n\mathbb{Z})\big)$ is a 3-dimensional subspace of the 4-dimensional $\mathbb{Z}/\ell\mathbb{Z}$-vector space $\ker(\pi_{\ell^{n+1},\ell^n})$. It follows from this, together with the fact that $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ reduced modulo ℓ generate $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$, that the set

$$\mathcal{K} := \left\{ \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\-1&-1\end{pmatrix} \right\} \subseteq M_{2\times 2}(\mathbb{Z})$$

satisfies

$$\text{(19)}\quad \left\langle I+\ell^n\mathcal{K} \mod \ell^{n+1}\right\rangle = \ker\!\big(\mathrm{SL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/\ell^n\mathbb{Z})\big) \qquad (n \geq 0).$$

Fix $X \in \mathcal{K}$. Note that $I+\ell^0X \mod \ell \in \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$, which by hypothesis is contained in $G(\ell)$. Fix a lift $\widetilde{X} \in M_{2\times 2}(\mathbb{Z}_\ell)$ for which $I+\ell^0\widetilde{X} \in G$, and note that $\widetilde{X}^2 \equiv \mathbf{0} \mod \ell$, so $\widetilde{X}^4 \equiv \mathbf{0} \mod \ell^2$. Thus, since $\ell \geq 5$, we have

$$(I+\ell^0\widetilde{X})^\ell = I + \binom{\ell}{1}\widetilde{X} + \binom{\ell}{2}\widetilde{X}^2 + \cdots + \binom{\ell}{\ell-1}\widetilde{X}^{\ell-1} + \widetilde{X}^\ell \equiv I+\ell\widetilde{X} \mod \ell^2,$$

and more generally,

$$(I+\ell^n\widetilde{X})^\ell \equiv I+\ell^{n+1}\widetilde{X} \mod \ell^{n+2} \quad \big(\ell \geq 5 \text{ and } n \geq 0, \text{ or } \ell = 3 \text{ and } n \geq 1\big).$$

Therefore (17) is established by induction on n.

We now proceed to verify (16) for ℓ an odd prime. When $\ell \geq 5$, the group $\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z})$ is a nonabelian simple group (see, e.g., [Huppert 1967, Chapter II, Hauptsatz 6.13]), and the exact sequence

$$1 \to \{\pm I\} \to \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z}) \to 1$$

does not split (see, e.g., [Zywina 2010, Lemma 2.3]). From this and a computer calculation[3] for the prime $\ell = 3$, we then find that

$$\begin{aligned} \ell \geq 5 &\Rightarrow [\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})] = \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}), \\ \ell = 3 &\Rightarrow [\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}), \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})] = \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}), \end{aligned}$$

and so by the hypotheses of our lemma in this case, we have $[G(\ell), G(\ell)] = \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$. Note further that the commutator subgroup $[G,G] \subseteq G$ projects modulo ℓ onto the commutator subgroup $[G(\ell), G(\ell)]$. We will prove by induction on $n \in \mathbb{N}$ that

$$\text{(20)}\qquad [G(\ell^n), G(\ell^n)] = \mathrm{SL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \qquad (n \geq 1),$$

having just established the base case. Fix $n \geq 1$ and assume that (20) holds. Pick any $g \in G(\ell^{n+1})$ and $\widetilde{X} \in M_{2\times 2}^{\mathrm{tr}\equiv 0}(\mathbb{Z}_\ell)$, so that, by (17) and (18), we have $I+\ell^n\widetilde{X} \mod \ell^{n+1} \in G(\ell^{n+1})$. We then compute the commutator

$$\text{(21)}\qquad \begin{aligned} g(I+\ell^n\widetilde{X})g^{-1}(I+\ell^n\widetilde{X})^{-1} &\equiv g(I+\ell^n\widetilde{X})g^{-1}(I-\ell^n\widetilde{X}) \\ &\equiv I+\ell^n(g\widetilde{X}g^{-1}-\widetilde{X}) \mod \ell^{n+1}. \end{aligned}$$

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[3] For readers wishing to reproduce this or any other computer calculation mentioned in this paper, please find the appropriate Magma scripts listed in an appendix of the arXiv version [Jones 2019].

Consider the following computations in $M_{2\times 2}(\mathbb{Z}/\ell\mathbb{Z})$:

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}^{-1}=\begin{pmatrix}-1&1\\-1&1\end{pmatrix},$$
$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}=\begin{pmatrix}1&-1\\1&-1\end{pmatrix},$$
$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}^{-1}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

It follows that, inside the additive 3-dimensional $\mathbb{Z}/\ell\mathbb{Z}$-vector space

$$M_{2\times 2}^{\mathrm{tr}=0}(\mathbb{Z}/\ell\mathbb{Z}) := \{X \in M_{2\times 2}(\mathbb{Z}/\ell\mathbb{Z}) : \operatorname{tr} X = 0\},$$

we have

$$\ell \geq 3 \;\Rightarrow\; \big\langle \big\{ gXg^{-1} - X : g \in \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}),\, X \in M_{2\times 2}^{\mathrm{tr}=0}(\mathbb{Z}/\ell\mathbb{Z}) \big\} \big\rangle = M_{2\times 2}^{\mathrm{tr}=0}(\mathbb{Z}/\ell\mathbb{Z}).$$

Thus, varying g and $\widetilde{X}$ in (21), we see that

$$\ker\big(\mathrm{SL}_2(\mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/\ell^{n}\mathbb{Z})\big) \subseteq [G(\ell^{n+1}), G(\ell^{n+1})],$$

verifying that (20) holds with n replaced by $n+1$, thus completing the induction step. Since $[G, G] \subseteq \mathrm{SL}_2(\mathbb{Z}_\ell)$ is a closed subgroup, we have therefore verified (16), proving Lemma 3.6 in case ℓ is odd.

Now assume $\ell = 2$ and note that (19) is still valid. By the hypothesis that $G(4) = \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$, for each $X \in \mathcal{K}$ we may find a lift $\widetilde{X} \in M_{2\times 2}(\mathbb{Z}_2)$ for which $\widetilde{X} \equiv X \mod 2$ and $I + 2\widetilde{X} \in G$. Again computing

$$(I + 2\widetilde{X})^2 = I + 4\widetilde{X} + 4\widetilde{X}^2 \equiv I + 4\widetilde{X} \mod 8,$$

we see that $\ker(\mathrm{SL}_2(\mathbb{Z}/8\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})) \subseteq G(8)$. Hence $[\mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) : G(8)] \leq 2$. Finally, if $G(8) = \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z})$, then (14) with $\beta = 0$ implies that $G = \mathrm{GL}_2(\mathbb{Z}_2)$. □

Next we will employ the following group theoretical lemma.

Lemma 3.7. *Let G_1 and G_2 be finite groups and let $\pi : G_1 \to G_2$ be a surjective group homomorphism. Let $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be subgroups satisfying $\pi(H_1) = H_2$ and let $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ be normal subgroups satisfying $\pi(N_1) = N_2$. Assume that*

$$\gcd(\#N_1, \#\ker\pi) = 1 \quad \textit{and} \quad [N_1, \ker\pi] = \{1\}. \tag{22}$$

We then have

$$N_1 \subseteq H_1 \iff N_2 \subseteq H_2.$$

Proof. The implication $\Rightarrow$ is immediate and does not require (22). For the converse, suppose that $N_2 \subseteq H_2$ and let $n_1 \in N_1$. Since $\pi(N_1) = N_2 \subseteq H_2 = \pi(H_1)$, we see that there exists $h_1 \in H_1$ satisfying $\pi(n_1) = \pi(h_1)$, and we may thus find $k \in \ker \pi$ so that $n_1 k \in H_1$. Now by (22), we see that

$$(n_1 k)^{\# \ker \pi} = n_1^{\# \ker \pi} \in H_1,$$

which again by (22) implies that $n_1 \in H_1$. Thus, $N_1 \subseteq H_1$, proving the lemma. □

Applying Lemma 3.7 in a special case, we obtain

Lemma 3.8. *Let $G \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be an open subgroup, let m_G be as in Definition 1.1 and let* $\mathrm{rad}'(m_G)$ *be defined by* (4). *For any prime ℓ and $d \in \mathbb{N}$, one has*

$$\mathrm{rad}'(m_G) \mid d \mid d\ell \mid m_G \;\Rightarrow\; \ell \textit{ divides } [\pi_{\ell d, d}^{-1}(G(d)) : G(\ell d)].$$

Proof. We write $m := m_G$ and

$$d =: \ell^{\delta_\ell} \cdot d_{(\ell)}, \quad m =: \ell^{\beta_\ell} \cdot m_{(\ell)}$$

(where $\ell \nmid d_{(\ell)} m_{(\ell)}$), and note that, by hypothesis, $\alpha_\ell \le \delta_\ell < \beta_\ell$. Further observe that, since $\beta_\ell = \beta'_\ell$, by Definitions 1.1 and 3.4, we have

$$\ker(\pi_{\ell^{\delta_\ell + 1}, \ell^{\delta_\ell}}) \times \{1_{m_{(\ell)}}\} \not\subseteq G(\ell^{\delta_\ell + 1} m_{(\ell)}).$$

We now apply Lemma 3.7 with

$$G_1 := \mathrm{GL}_2(\mathbb{Z}/\ell^{\delta_\ell+1} m_{(\ell)}\mathbb{Z}), \quad H_1 := G(\ell^{\delta_\ell+1} m_{(\ell)}), \quad N_1 := \ker(\pi_{\ell^{\delta_\ell+1}, \ell^{\delta_\ell}}) \times \{1_{m_{(\ell)}}\},$$
$$G_2 := \mathrm{GL}_2(\mathbb{Z}/\ell^{\delta_\ell+1} d_{(\ell)}\mathbb{Z}), \quad H_2 := G(\ell^{\delta_\ell+1} d_{(\ell)}), \quad N_2 := \ker(\pi_{\ell^{\delta_\ell+1}, \ell^{\delta_\ell}}) \times \{1_{d_{(\ell)}}\},$$

and $\pi : \mathrm{GL}_2(\mathbb{Z}/\ell^{\delta_\ell+1} m_{(\ell)}\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/\ell^{\delta_\ell+1} d_{(\ell)}\mathbb{Z})$ the canonical projection map. The conclusion is that

$$\ker(\pi_{\ell^{\delta_\ell+1}, \ell^{\delta_\ell}}) \times \{1_{d_{(\ell)}}\} \not\subseteq G(\ell^{\delta_\ell+1} d_{(\ell)}).$$

Since $\ker(\pi_{\ell^{\delta_\ell+1}, \ell^{\delta_\ell}}) \times \{1_{d_{(\ell)}}\} \simeq \ker(\pi_{\ell d, d})$ is an ℓ-group, this proves the lemma. □

Applying Lemma 3.8 prime by prime, for each prime ℓ dividing $m_G / \mathrm{rad}'(m_G)$, we obtain

$$\frac{m_G}{\mathrm{rad}'(m_G)} \text{ divides } \left[\pi^{-1}\big(G(\mathrm{rad}'(m_G))\big) : G(m_G)\right],$$

proving Proposition 1.6 in the case that (5) does not hold. In case (5) does hold, we have $G(3) = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ and, by Lemma 3.6, we must also have $\mathrm{SL}_2(\mathbb{Z}/9\mathbb{Z}) \not\subseteq G(9)$. A computer search reveals that, up to conjugation in $\mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z})$, there are two subgroups $G_1, G_2 \subseteq \mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z})$ meeting these two criteria,[4] and $G_1 \subseteq$

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[4] The (genus zero) modular curve associated with G_2 has been considered by N. Elkies [2006], who exhibited an explicit map from it to the j-line.

G_2. Furthermore, $[\mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z}) : G_2] = 27$. From this it follows that 27 divides $[\mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z}) : G(9)]$, and so

$$9 \cdot 3 \ \text{ divides } \ \bigl[\pi^{-1}\bigl(G(\mathrm{rad}'(m_G))\bigr) : G(3\,\mathrm{rad}'(m_G))\bigr].$$

Now starting here and applying Lemma 3.8, prime by prime, we conclude the proof of Proposition 1.6 in the case that (5) holds.

4. Proof of Proposition 1.7

We now prove Proposition 1.7. The proof will rely, in part, on the following corollary to the Néron–Ogg–Shafarevich criterion (see for instance [Ogg 1967] or [Silverman 1986, Chapter VII, Theorem 7.1]).

Theorem 4.1. *Let K be a number field, let E be an elliptic curve over K and let $\mathcal{L} \subseteq \mathcal{O}_K$ be a prime ideal of K, lying over the rational prime ℓ of $\mathbb{Z}$. The following are equivalent*:

(a) *E has good reduction at $\mathcal{L}$.*

(b) *For each positive integer m that is not divisible by ℓ, the prime $\mathcal{L}$ is unramified in $K(E[m])$.*

(c) *The prime $\mathcal{L}$ is unramified in $K(E_{\mathrm{tors},(\ell)})$.*

We presently reduce the proof of Proposition 1.7 to the following four lemmas. The first lemma follows immediately from the classification subgroups of $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$.

Lemma 4.2. *Let ℓ be a prime number and let $G(\ell) \subseteq \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ be any subgroup. We have*

$$\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \nsubseteq G(\ell) \ \Rightarrow \ \ell \leq [\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : G(\ell)].$$

The second lemma is a consequence of the Weil pairing on an elliptic curve.

Lemma 4.3. *Let E be an elliptic curve defined over a number field K, let $G := \rho_{E,K}(G_K) \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$, and let ℓ be a prime number. For any positive integer n, we have*

$$\mathrm{SL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \subseteq G(\ell^n) \neq \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \ \Rightarrow \ \ell \mid \Delta_K.$$

Consequently,

$$\mathrm{SL}_2(\mathbb{Z}_\ell) \subseteq G_\ell \neq \mathrm{GL}_2(\mathbb{Z}_\ell) \ \Rightarrow \ \ell \mid \Delta_K.$$

Our third lemma utilizes the Néron–Ogg–Shafarevich criterion in the form of Theorem 4.1.

Lemma 4.4. *Let E be an elliptic curve defined over a number field K, let $G := \rho_{E,K}(G_K) \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$, let m_G be as in Definition 1.1 and let ℓ be an odd prime number dividing m_G. We then have*

$$G_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell) \Rightarrow \ell \mid \Delta_K N_{K/\mathbb{Q}}(\Delta_E).$$

For the prime $\ell = 2$ we must make a finer analysis, in the form of the next (and final) lemma. Let us make the abbreviation

$$r' := \mathrm{rad}'(m_G).$$

Lemma 4.5. *Let E be an elliptic curve defined over a number field K, let $G := \rho_{E,K}(G_K) \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$, let m_G be as in Definition 1.1 and assume that 4 divides m_G. We then have*

$$\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \times \{1_{r'_{(2)}}\} \not\subseteq G(r') \Rightarrow 4 \leq 2\big[\pi^{-1}(G(r'_{(2)})) : G(r')\big]$$

and

$$\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \times \{1_{r'_{(2)}}\} \subseteq G(r') \Rightarrow 2 \mid \Delta_K N_{K/\mathbb{Q}}(\Delta_E).$$

Let us now deduce Proposition 1.7 from Lemmas 4.2–4.5, postponing the proofs of those lemmas until later. First, combining Lemma 3.6 with Lemmas 4.2–4.4, one concludes the following implications, for any prime $\ell \geq 5$ that divides m_G:

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \not\subseteq G(\ell) &\Rightarrow \ell \leq [\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : G(\ell)], \\ \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \subseteq G(\ell) &\Rightarrow \ell \mid \Delta_K N_{K/\mathbb{Q}}(\Delta_E). \end{aligned}$$

This implies that

$$\begin{aligned} r'_{(6)} &\leq \prod_{\substack{\ell\geq 5,\, \ell \mid r' \\ \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \not\subseteq G(\ell)}} [\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : G(\ell)] \prod_{\substack{\ell \geq 5 \\ \ell \mid \Delta_K N_{K/\mathbb{Q}}(\Delta_E) \\ \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \subseteq G(\ell)}} \ell \\ &\leq [\mathrm{GL}_2(\mathbb{Z}/r'_{(6)}\mathbb{Z}) : G(r'_{(6)})]\, \mathrm{rad}\big(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|\big)_{(6)}. \end{aligned} \tag{23}$$

If the prime $\ell = 3$ divides m_G then either condition (5) holds or it does not hold. Let us first assume that (5) does not hold, i.e., we assume that it is *not* the case that 9 divides m_G, $G(3) = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}_3) \not\subseteq G_3$. We then use Lemmas 4.2–4.4, together with Lemma 3.6, to deduce the following implications:

$$\begin{aligned} &\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \not\subseteq G(3) \Rightarrow 3 \leq [\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) : G(3)], \\ &\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \subseteq G(3) \neq \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \Rightarrow 3 \mid \Delta_K, \\ &G(3) = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \text{ and } \mathrm{SL}_2(\mathbb{Z}_3) \subseteq G_3 \neq \mathrm{GL}_2(\mathbb{Z}_3) \Rightarrow 3 \mid \Delta_K, \\ &G(3) = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \text{ and } G_3 = \mathrm{GL}_2(\mathbb{Z}_3) \Rightarrow 3 \mid \Delta_K N_{K/\mathbb{Q}}(\Delta_E). \end{aligned}$$

Inserting this information into (23), we find that

$$r'_{(2)} \le [\mathrm{GL}_2(\mathbb{Z}/r'_{(2)}\mathbb{Z}) : G(r'_{(2)})]\,\mathrm{rad}\big(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|\big)_{(2)}. \tag{24}$$

On the other hand, in case (5) *does* hold, then we obviously have

$$\begin{aligned} \frac{r'_{(2)}}{3} = r'_{(6)} &\le [\mathrm{GL}_2(\mathbb{Z}/r'_{(6)}\mathbb{Z}) : G(r'_{(6)})]\,\mathrm{rad}\big(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|\big)_{(6)} \\ &\le [\mathrm{GL}_2(\mathbb{Z}/r'_{(2)}\mathbb{Z}) : G(r'_{(2)})]\,\mathrm{rad}\big(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|\big)_{(2)}. \end{aligned} \tag{25}$$

If $\ell = 2$ divides m_G, then either 4 divides m_G or does not. If $4 \nmid m_G$, then multiplying both sides of (24) and (25) by 2, we obtain the bounds of Proposition 1.7. Now assume that $4 \mid m_G$. In this case, when (5) does not hold, we insert the result of Lemma 4.5 into (24), concluding that

$$r' = 4r'_{(2)} \le 2[\mathrm{GL}_2(\mathbb{Z}/r'\mathbb{Z}) : G(r')]\,\mathrm{rad}\big(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|\big).$$

Likewise, in case condition (5) does hold, we insert these results into (25) and obtain

$$\frac{r'}{3} = \frac{4r'_{(2)}}{3} \le 2[\mathrm{GL}_2(\mathbb{Z}/r'\mathbb{Z}) : G(r')]\,\mathrm{rad}\big(|\Delta_K N_{K/\mathbb{Q}}(\Delta_E)|\big).$$

Thus we see that Lemmas 4.2–4.5 indeed imply Proposition 1.7.

We now prove each of these lemmas. First we state an auxiliary lemma that is used throughout and may be found in [Ribet 1976, Lemma (5.2.1)].

Lemma 4.6 (Goursat's lemma). *Let G_1, G_2 be groups and for $i \in \{1, 2\}$ denote by $\mathrm{pr}_i : G_1 \times G_2 \to G_i$ the projection map onto the i-th factor. Let $G \subseteq G_1 \times G_2$ be a subgroup and assume that*

$$\mathrm{pr}_1(G) = G_1, \quad \mathrm{pr}_2(G) = G_2.$$

Then there exists a group Γ together with a pair of surjective homomorphisms

$$\psi_1 : G_1 \to \Gamma, \quad \psi_2 : G_2 \to \Gamma$$

so that

$$G = G_1 \times_\psi G_2 := \{(g_1, g_2) \in G_1 \times G_2 : \psi_1(g_1) = \psi_2(g_2)\}.$$

Proof of Lemma 4.2. To prove Lemma 4.2, we will use the following classification of certain proper subgroups of GL_2.

Definition 4.7. Let ℓ be any prime number.

(i) A subgroup $G(\ell) \subseteq \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ is called a *Borel subgroup* if it is conjugate in $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ to the subgroup

$$\mathcal{B}(\ell) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \, b \in \mathbb{Z}/\ell\mathbb{Z}, \; a, d \in (\mathbb{Z}/\ell\mathbb{Z})^\times \right\}. \tag{26}$$

(ii) A subgroup $G(\ell) \subseteq \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ is called a *normalizer of a split Cartan subgroup* if it is conjugate in $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ to the subgroup

$$\mathcal{N}_{\mathrm{s}}(\ell) := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in (\mathbb{Z}/\ell\mathbb{Z})^{\times} \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in (\mathbb{Z}/\ell\mathbb{Z})^{\times} \right\}. \tag{27}$$

If ℓ is odd, then $G(\ell)$ is called a *normalizer of a nonsplit Cartan subgroup* if it is conjugate in $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ to the subgroup

$$\mathcal{N}_{\mathrm{ns}}(\ell) := \left\{ \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} : \begin{matrix} x, y \in \mathbb{Z}/\ell\mathbb{Z}, \\ x^2 - \varepsilon y^2 \neq 0 \end{matrix} \right\} \cup \left\{ \begin{pmatrix} x & -\varepsilon y \\ y & -x \end{pmatrix} : \begin{matrix} x, y \in \mathbb{Z}/\ell\mathbb{Z}, \\ x^2 - \varepsilon y^2 \neq 0 \end{matrix} \right\}, \tag{28}$$

where ε is any fixed nonsquare in $(\mathbb{Z}/\ell\mathbb{Z})^{\times}$. If $\ell = 2$, then $G(2)$ is called a normalizer of a nonsplit Cartan subgroup if $G(2) = \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.

(iii) A subgroup $G(\ell) \subseteq \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ is called an *exceptional group* if its image in $\mathrm{PGL}_2(\mathbb{Z}/\ell\mathbb{Z})$ is isomorphic to one of the groups A_4, S_4 or A_5 (the symmetric or alternating groups).

The following lemma may be deduced from Propositions 15, 16 and Section 2.6 of [Serre 1972].

Lemma 4.8. *Let $G(\ell) \subseteq \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ be a subgroup. Then one of the following must hold*:

(1) *$G(\ell)$ is contained in a Borel subgroup.*

(2) *$G(\ell)$ is contained in the normalizer of a split Cartan subgroup.*

(3) *$G(\ell)$ is contained in the normalizer of a nonsplit Cartan subgroup.*

(4) *$G(\ell)$ is an exceptional group.*

(5) $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \subseteq G(\ell)$.

We include the following table of indices $[\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : G(\ell)]$, for each of the proper subgroups $G(\ell)$ given in Lemma 4.8. In addition to the definitions (26), (27), and (28), we make the following abbreviations. For a prime ℓ for which $A_4 \subseteq \mathrm{PGL}_2(\mathbb{Z}/\ell\mathbb{Z})$, we define the exceptional subgroup $\mathcal{E}_{A_4}(\ell) \subseteq \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ by

$$\mathcal{E}_{A_4}(\ell) := \{g \in \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : \varpi(g) \in A_4\},$$

where $\varpi : \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Z}/\ell\mathbb{Z})$ denotes the usual projection. The exceptional subgroups $\mathcal{E}_{S_4}(\ell)$ and $\mathcal{E}_{A_5}(\ell)$ are defined similarly.

$G(\ell)$	$\mathcal{B}(\ell)$	$\mathcal{N}_{\mathrm{s}}(\ell)$	$\mathcal{N}_{\mathrm{ns}}(\ell)$	$\mathcal{E}_{A_4}(\ell)$	$\mathcal{E}_{S_4}(\ell)$	$\mathcal{E}_{A_5}(\ell)$
$[\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : G(\ell)]$	$\ell+1$	$\frac{\ell(\ell+1)}{2}$	$\frac{\ell(\ell-1)}{2}$	$\frac{\ell(\ell^2-1)}{12}$	$\frac{\ell(\ell^2-1)}{24}$	$\frac{\ell(\ell^2-1)}{60}$

We note that $\mathcal{N}_{\text{ns}}(2) = \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$, and also that each exceptional group only occurs for certain primes ℓ. In particular, if the expression given in the table is not a whole number, then the associated exceptional group does not occur as a subgroup of $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ for that prime ℓ. The conclusion of Lemma 4.2 follows immediately from this table. $\square$

Proof of Lemma 4.3. We will make use of the following commutative diagram, where

$$\mathrm{res} : \mathrm{Gal}(K(E_{\text{tors}})/K) \to \mathrm{Gal}(K(\mu_\infty)/K), \quad \mathrm{cyc} : \mathrm{Gal}(K(\mu_\infty)/K) \to \widehat{\mathbb{Z}}^\times$$

denote respectively the restriction map and the cyclotomic character (the containment $K(\mu_\infty) \subseteq K(E_{\text{tors}})$ follows from the Weil pairing [1940], see also [Silverman 1986, Chapter III, §8]).

$$\begin{tikzcd}
\mathrm{Gal}(K(E[\ell^n])/K) \arrow[r, "\rho_{E,K}"] \arrow[d, "\mathrm{res}"'] & \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \arrow[d, "\det"'] \\
\mathrm{Gal}(K(\mu_{\ell^n})/K) \arrow[r, "\mathrm{cyc}"] & (\mathbb{Z}/\ell^n\mathbb{Z})^\times
\end{tikzcd} \tag{29}$$

By considering the commutative diagram (29) and Galois theory, we see that

$$\mathrm{SL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \subseteq G(\ell^n) \neq \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \Rightarrow \det(G(\ell^n)) \neq (\mathbb{Z}/\ell^n\mathbb{Z})^\times \Rightarrow \mathbb{Q} \neq \mathbb{Q}(\mu_{\ell^n}) \cap K.$$

Since $\mathbb{Q}(\mu_{\ell^n})$ is totally ramified at ℓ, it follows that ℓ is then ramified in $\mathbb{Q}(\mu_{\ell^n}) \cap K$, so ℓ is ramified in K, and thus ℓ divides Δ_K. $\square$

Proof of Lemma 4.4. In order to prove Lemma 4.4, we will make use of the following definition and lemma, which allow us to understand in more detail the nature of the fibered products that may be present in G.

Definition 4.9. Let G be a profinite group and Σ a finite simple group. We say that Σ *occurs in* G if and only if there are closed subgroups G_1 and N_1 of G with $N_1 \subseteq G_1 \subseteq G$, N_1 normal in G_1 and $G_1/N_1 \simeq \Sigma$. We further define

$$\mathrm{Occ}(G) := \{\text{finite simple nonabelian groups } \Sigma : \Sigma \text{ occurs in } G\},$$

$$\mathrm{Occ}_{\mathrm{JH}}(G) := \{\text{finite simple nonabelian groups } \Sigma : \Sigma \text{ is a Jordan–Hölder factor of } G\}.$$

Note that any simple Jordan–Hölder factor of G occurs in G (but generally not vice versa), i.e., we have $\mathrm{Occ}_{\mathrm{JH}}(G) \subseteq \mathrm{Occ}(G)$. Also note that, if

$$1 \to G' \to G \to G'' \to 1$$

is an exact sequence of profinite groups, then

$$\mathrm{Occ}(G) = \mathrm{Occ}(G') \cup \mathrm{Occ}(G''), \quad \mathrm{Occ}_{\mathrm{JH}}(G) = \mathrm{Occ}_{\mathrm{JH}}(G') \cup \mathrm{Occ}_{\mathrm{JH}}(G''). \tag{30}$$

Finally, as observed in [Serre 1968, IV-25], one has that

$$\mathrm{Occ}(\mathrm{GL}_2(\mathbb{Z}_\ell)) = \begin{cases} \varnothing & \text{if } \ell \in \{2,3\}, \\ \{\mathrm{PSL}_2(\mathbb{Z}/5\mathbb{Z})\} = \{A_5\} & \text{if } \ell = 5, \\ \{\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z})\} & \text{if } \ell > 5,\ \ell \equiv \pm 2 \pmod 5, \\ \{\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z}),\ A_5\} & \text{if } \ell > 5,\ \ell \equiv \pm 1 \pmod 5. \end{cases}$$

Thus, by (30) we have

$$\mathrm{Occ}(\mathrm{GL}_2(\mathbb{Z}_{(\ell)})) = \{A_5\} \cup \{\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})\}_{p \neq \ell,\, p \geq 5}. \tag{31}$$

Lemma 4.10. *Let $\ell \geq 5$ be a prime and let $G \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell)$ be a closed subgroup satisfying $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \subseteq G(\ell)$. Suppose further that $\psi : G \to H$ is a surjective group homomorphism onto a finite group H. Then either*

1. $\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z}) \in \mathrm{Occ}_{\mathrm{JH}}(H)$, *or*
2. *H is abelian and* $\mathrm{SL}_2(\mathbb{Z}_\ell) \subseteq \ker \psi$.

Proof. As observed earlier, since $\ell \geq 5$, the group $\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z})$ is a simple nonabelian group, and we obviously have $\{\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z})\} \subseteq \mathrm{Occ}_{\mathrm{JH}}(\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}))$. Furthermore, by the hypothesis $\mathrm{SL}(\mathbb{Z}/\ell\mathbb{Z}) \subseteq G(\ell)$ together with (30), we see that $\{\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z})\} \subseteq \mathrm{Occ}_{\mathrm{JH}}(G(\ell))$. Thus, again by (30), we have

$$\{\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z})\} \subseteq \mathrm{Occ}_{\mathrm{JH}}(G). \tag{32}$$

Furthermore, we have that

$$\frac{\pm(\ker \psi)(\ell) \cap \mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})}{\{\pm I\}} = \begin{cases} \{1\} \ \text{ or} \\ \mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z}). \end{cases} \tag{33}$$

If the left side of (33) is trivial, then $\ker \psi$ is prosolvable (so that $\mathrm{Occ}_{\mathrm{JH}}(\ker \psi) = \varnothing$), and considering the exact sequence

$$1 \to \ker \psi \to G \to H \to 1,$$

we see by (30) and (32) that $\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z}) \in \mathrm{Occ}_{\mathrm{JH}}(H)$. If, on the other hand, we have $\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z})$ in (33), then $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z}) \subseteq (\ker \psi)(\ell)$, which by Lemma 3.6 applied to $G = \ker \psi$ implies that $\mathrm{SL}_2(\mathbb{Z}_\ell) \subseteq \ker \psi$. Thus H is abelian and ψ factors through the determinant map, as asserted. □

We now proceed with the proof of Lemma 4.4. By Lemma 4.6, the hypothesis that $G_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$ and that ℓ divides m_G imply that

$$G \simeq \mathrm{GL}_2(\mathbb{Z}_\ell) \times_\psi G_{(\ell)}, \tag{34}$$

where $\psi_\ell : \mathrm{GL}_2(\mathbb{Z}_\ell) \twoheadrightarrow H$ and $\psi_{(\ell)} : G_{(\ell)} \twoheadrightarrow H$ are surjective homomorphisms onto a common *nontrivial* group H. Under the Galois correspondence, we have $\mathrm{GL}_2(\mathbb{Z}_\ell) \simeq \mathrm{Gal}(K(E[\ell^\infty])/K)$, $G_{(\ell)} \simeq \mathrm{Gal}(K(E_{\mathrm{tors},(\ell)})/K)$ and $H \simeq \mathrm{Gal}(F/K)$, where $F := K(E[\ell^\infty]) \cap K(E_{\text{tors},(\ell)}) \neq K$. Thus, the corresponding field diagram is as follows.

$$\begin{tikzcd}
K(E[\ell^\infty]) \arrow[dr, no head] & & K(E_{\text{tors},(\ell)}) \arrow[dl, no head] \\
& F \arrow[d, no head] & \\
& K &
\end{tikzcd} \tag{35}$$

We first claim that

$$F \cap K(\mu_{\ell^\infty}) \neq K. \tag{36}$$

We separate the verification of (36) into cases.

Case: $\ell \geq 5$. By Lemma 4.10, we see that either $\mathrm{PSL}_2(\mathbb{Z}/\ell\mathbb{Z})$ occurs in H (and thus occurs in $G_{(\ell)}$), or H is abelian and $F \subseteq K(\mu_{\ell^\infty})$. If $\ell \geq 7$ then, by (31) we see that H must be abelian and $F \subseteq K(\mu_{\ell^\infty})$, verifying (36). If $\ell = 5$, we consider the further quotient induced by reduction modulo 5:

$$H \simeq \frac{\mathrm{GL}_2(\mathbb{Z}_5)}{\ker \psi_5} \to \frac{\mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})}{(\ker \psi_5)(5)} =: H(5).$$

Since the kernel of this quotient is pro-solvable, we see that if $\mathrm{PSL}_2(\mathbb{Z}/5\mathbb{Z}) \simeq A_5$ occurs in H, then it must occur in $H(5)$, and a computer calculation shows that we then must have

$$(\ker \psi_5)(5) \subseteq \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in (\mathbb{Z}/5\mathbb{Z})^\times \right\},$$

and thus

$$\langle \mathrm{SL}_2(\mathbb{Z}_5), \ker \psi_5 \rangle \subseteq \left\{ g \in \mathrm{GL}_2(\mathbb{Z}_5) : \left(\frac{\det(g) \mod 5}{5} \right) = 1 \right\}.$$

By the Galois correspondence, we then have

$$F \cap K(\mu_{5^\infty}) = K(E[5^\infty])^{\langle \mathrm{SL}_2(\mathbb{Z}_5), \ker \psi_5 \rangle} \supseteq K(\sqrt{5}) \neq K,$$

where we are using the fact that $G(5) = \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$, which precludes the possibility that $K(\sqrt{5}) = K$. Thus in any case, (36) also holds for $\ell = 5$.

Case: $\ell = 3$. As in the previous case, we have that (34) holds with $\ell = 3$. By Galois theory, we have that

$$F := K(E[3^\infty])^{\ker \psi_3} \supseteq K(E[3^\infty])^{\langle \mathrm{SL}_2(\mathbb{Z}_3), \ker \psi_3 \rangle} = K(\mu_{3^\infty}) \cap F.$$

As in the previous case, since $\ker \psi_3 \neq \mathrm{GL}_2(\mathbb{Z}_3)$, we have $F \neq K$. The following lemma will then imply that $K(\mu_{3^\infty}) \cap F \neq K$.

Lemma 4.11. *Let $N \trianglelefteq \mathrm{GL}_2(\mathbb{Z}_3)$ be a closed normal subgroup which satisfies $\langle \mathrm{SL}_2(\mathbb{Z}_3), N \rangle = \mathrm{GL}_2(\mathbb{Z}_3)$. Then $N = \mathrm{GL}_2(\mathbb{Z}_3)$.*

Proof. A computer calculation shows that, if $H \trianglelefteq \mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z})$ is a normal subgroup satisfying

$$\langle \mathrm{SL}_2(\mathbb{Z}/9\mathbb{Z}), H \rangle = \mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z}),$$

then $H = \mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z})$. Taking N as in the statement of the lemma and setting $H := N(9)$, we see that $N(9) = \mathrm{GL}_2(\mathbb{Z}/9\mathbb{Z})$, and applying (14) with $\beta = 1$, we conclude that $N = \mathrm{GL}_2(\mathbb{Z}_3)$. ☐

Applying Lemma 4.11 with $N = \ker \psi_3$, we find that $K(\mu_{3^\infty}) \cap F \neq K$, since $F \neq K$, verifying (36) in the $\ell = 3$ case as well.

Finally, we observe that (36) implies the conclusion of Lemma 4.4. Indeed, assume that $\ell \nmid \Delta_K$. Since $G_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$, we have $K \cap \mathbb{Q}(\mu_{\ell^\infty}) = \mathbb{Q}$, and so any prime $\mathfrak{L} \subseteq O_K$ over ℓ is totally ramified in $K(\mu_{\ell^\infty})$, hence ramified in $F \cap K(\mu_{\ell^\infty})$. Thus, by (35), $\mathfrak{L}$ is ramified in $K(E_{\mathrm{tors},(\ell)})$. By Theorem 4.1, we find that $\ell \mid N_{K/\mathbb{Q}}(\Delta_E)$, finishing the proof. ☐

Proof of Lemma 4.5. The proof of Lemma 4.5 will make use of the following sublemma.

Lemma 4.12. *Let K be a number field for which $2 \nmid \Delta_K$ and let $\mathfrak{p} \subseteq \mathcal{O}_K$ be a prime ideal lying over 2. Let $\alpha \in \mathcal{O}_K - \{0\}$ be any element for which $\mathfrak{p} \nmid \alpha\mathcal{O}_K$. Then 2α is not a square in $K^\times$, so the field $K(\sqrt{2\alpha})$ is a quadratic extension of K. Furthermore, $\mathfrak{p}$ ramifies in $K(\sqrt{2\alpha})$.*

Proof. Let $v_\mathfrak{p}$ denote the $\mathfrak{p}$-adic valuation on K, normalized so that $v_\mathfrak{p}(K^\times) = \mathbb{Z}$. Note that, since by assumption 2 is unramified in K and $v_\mathfrak{p}(\alpha) = 0$, we have

$$v_\mathfrak{p}(2\alpha) = v_\mathfrak{p}(2) + v_\mathfrak{p}(\alpha) = 1, \tag{37}$$

and so in particular 2α cannot be a square in $K^\times$, as asserted. Next, let $L := K(\sqrt{2\alpha})$, fix any prime $\mathfrak{P} \subseteq \mathcal{O}_L$ lying over $\mathfrak{p}$ and let $v_\mathfrak{P}$ be the $\mathfrak{P}$-adic valuation on L, normalized so that it extends $v_\mathfrak{p}$ on K. By (37), we then have

$$v_\mathfrak{P}((2\alpha)^{1/2}) = \tfrac{1}{2} v_\mathfrak{p}(2\alpha) = \tfrac{1}{2}.$$

It follows that L is ramified over K at $\mathfrak{p}$, as asserted. ☐

We now proceed with the proof of Lemma 4.5. Since we are assuming that 4 divides r', by Lemma 4.6 we may write $G(r')$ as a fibered product:

$$G(r') = G(4) \times_\psi G(r'_{(2)}). \tag{38}$$

Case: $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \times \{1_{r'_{(2)}}\} \not\subseteq G(r')$. In this case, either $G(4) \neq \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$ or $G(4) = \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$ and the common quotient $\psi_2(G(4)) = \psi_{(2)}(G(r'_{(2)}))$ in (38) is nontrivial. If $G(4) \neq \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$, we find that $2 \leq [\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) : G(4)] \leq [\pi^{-1}(G(r'_{(2)})) : G(r')]$, and so the result of the lemma follows. If on the other hand $G(4) = \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$, then the common quotient in (38) is nontrivial, and since

$$\pi^{-1}(G(r'_{(2)})) = \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \times G(r'_{(2)}),$$
$$G(r') = \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \times_{\psi} G(r'_{(2)}),$$

we thus have $2 \leq [\pi^{-1}(G(r'_{(2)})) : G(r')]$, proving the lemma in this subcase as well.

Case: $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \times \{1_{r'(2)}\} \subseteq G(r')$. In this case, (38) is a full product:

$$G(r') = \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \times G(r'_{(2)}). \tag{39}$$

By Lemma 3.6, either $G(8)$ is an index 2 subgroup of $\mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z})$ that surjects onto $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$, or $G_2 = \mathrm{GL}_2(\mathbb{Z}_2)$. Let us treat the former subcase first. A computer search reveals that there are exactly 4 index 2 subgroups of $\mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z})$ that map surjectively onto $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$, namely

$$\ker(\chi_8),\ \ker(\chi_8\chi_4),\ \ker(\chi_8\varepsilon),\ \ker(\chi_8\chi_4\varepsilon),$$

where $\chi_8 : \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \to \{\pm 1\}$ (resp. $\chi_4 : \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \to \{\pm 1\}$) denotes the Kronecker symbol associated to the quadratic field $\mathbb{Q}(\sqrt{2})$ (resp. to $\mathbb{Q}(\sqrt{-1})$), precomposed with the determinant map, and $\varepsilon : \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$ denotes the unique nontrivial character of order 2 on $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$, precomposed with reduction modulo 2. We have

$$\begin{aligned} K(E[8])^{\ker(\chi_8)} &= K(\sqrt{2}), & K(E[8])^{\ker(\chi_8\varepsilon)} &= K(\sqrt{2\Delta_E}), \\ K(E[8])^{\ker(\chi_8\chi_4)} &= K(\sqrt{-2}), & K(E[8])^{\ker(\chi_8\chi_4\varepsilon)} &= K(\sqrt{-2\Delta_E}). \end{aligned} \tag{40}$$

Here, by $K(\sqrt{\pm 2\Delta_E})$ we mean the quadratic field $K(\sqrt{\pm 2\Delta(E_{\mathrm{Weier}})})$, where E_{Weier} is any fixed Weierstrass model of E and $\Delta(E_{\mathrm{Weier}}) \in K^{\times}$ denotes its discriminant (note that although $\Delta(E_{\mathrm{Weier}})$ depends on the choice of E_{Weier}, the quadratic field $K(\sqrt{\pm 2\Delta(E_{\mathrm{Weier}})})$ depends only on E). By (40), we thus have

$$G(8) = \ker(\chi_8) \Rightarrow \sqrt{2} \in K \quad \text{and} \quad G(8) = \ker(\chi_8\chi_4) \Rightarrow \sqrt{-2} \in K,$$

either of which imply that $2 \mid \Delta_K$. On the other hand, for any Weierstrass model E_{Weier} of E, we have

$$\begin{aligned} G(8) = \ker(\chi_8\varepsilon) &\Rightarrow \sqrt{2\Delta(E_{\mathrm{Weier}})} \in K, \\ G(8) = \ker(\chi_8\chi_4\varepsilon) &\Rightarrow \sqrt{-2\Delta(E_{\mathrm{Weier}})} \in K. \end{aligned} \tag{41}$$

Let us suppose for the sake of contradiction that

$$2 \nmid \Delta_K N_{K/\mathbb{Q}}(\Delta_E). \tag{42}$$

Fix a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$ lying over 2. By (42), we must have $\mathfrak{p} \nmid \Delta_E$, and we may thus find a Weierstrass model E_{Weier} of E satisfying $\mathfrak{p} \nmid \Delta(E_{\text{Weier}})$. Applying Lemma 4.12 with $\alpha = \pm\Delta(E_{\text{Weier}})$, we see that

$$\sqrt{\pm 2\Delta(E_{\text{Weier}})} \notin K,$$

contradicting (41). Thus, we must have $2 \mid \Delta_K N_{K/\mathbb{Q}}(\Delta_E)$ whenever $G(8)$ has index 2 in $\mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z})$.

We now treat the second subcase, in which $G_2 = \mathrm{GL}_2(\mathbb{Z}_2)$. We evidently must have a nontrivial common quotient in

$$G_{r'} \simeq \mathrm{GL}_2(\mathbb{Z}_2) \times_\psi G_{r'_{(2)}}.$$

(If this fibered product were over a trivial quotient, then 2 would not divide m_G.) We note that any nontrivial finite quotient of $\mathrm{GL}_2(\mathbb{Z}_2)$ must have order divisible by 2 and that $\ker(G_{r'_{(2)}} \to G(r'_{(2)}))$ is a profinite group whose finite quotients each have order coprime with 2. It follows that the image of G under $\mathrm{id}_2 \times \pi_{(r'_{(2)})^\infty, r'_{(2)}}$ has the form

$$\mathrm{GL}_2(\mathbb{Z}_2) \times_\psi G(r'_{(2)}), \tag{43}$$

a fibered product with a common quotient of order divisible by 2 (and hence nontrivial). Consider the subgroup $N := \ker \psi_2 \subseteq \mathrm{GL}_2(\mathbb{Z}_2)$ where $\psi = (\psi_2, \psi_{(2)})$ in (43). The assumption $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \times \{1_{r'(2)}\} \subseteq G(r')$ then implies that $N(4) = \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$ (otherwise the mod r' image of (43) would have a nontrivial fibering between $G(4)$ and $G(r'_{(2)})$, contradicting (39)). By Lemma 3.6, we find that $[\mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) : N(8)] = 2$. By the same computation as mentioned in the previous subcase, we have

$$N(8) \in \{\ker(\chi_8),\ \ker(\chi_8\chi_4),\ \ker(\chi_8\varepsilon),\ \ker(\chi_8\chi_4\varepsilon)\},$$

and it follows by (40) and Galois theory that one of the fields $K(\sqrt{2})$, $K(\sqrt{-2})$, $K(\sqrt{2\Delta_E})$, or $K(\sqrt{-2\Delta_E})$ must be contained in $K(E_{\text{tors},(2)})$. By Lemma 4.12 and Theorem 4.1, it follows that, if $2 \nmid \Delta_K$ then 2 divides $N_{K/\mathbb{Q}}(\Delta_E)$. This finishes the proof of Lemma 4.5. □

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NATHAN JONES
DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, IL
UNITED STATES
ncjones@uic.edu

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TOPOLOGY OF COMPLEXITY ONE QUOTIENTS

YAEL KARSHON AND SUSAN TOLMAN

We describe of the topology of the geometric quotients of $2n$-dimensional compact connected symplectic manifolds with $(n-1)$-dimensional torus actions. When the isotropy weights at each fixed point are in general position, the quotient is homeomorphic to a sphere.

1. Introduction

This paper is a byproduct of our work on the classification of complexity one Hamiltonian torus actions [Karshon 1999; Karshon and Tolman 2001; 2003; 2014; $\geq$ 2020], but, in fact, it relies only on elementary aspects of such actions. It is motivated by a number of recent works by toric topologists (specifically, the papers by Buchstaber and Terzić [2016; 2019a; 2019b] and by Ayzenberg [2018]) that explore the topology of the geometric quotients of manifolds with certain torus actions. Our purpose in this paper is to highlight topological aspects of related works in equivariant symplectic geometry and to illustrate how equivariant symplectic methods reproduce some of the recent results in toric topology and yield new examples.

Similar results were recently obtained by Hendrik Süß [2018] from the point of view of algebraic geometry.

The examples Buchstaber and Terzić studied include the quotient of the Grassmannian of complex 2-planes in $\mathbb{C}^4$ by its standard torus action, which they showed is homeomorphic to a five-dimensional sphere, and the quotient of the manifold of complete flags in $\mathbb{C}^3$ by its standard torus action, which they showed is homeomorphic to a four-dimensional sphere. We exhibit these examples as special cases of a more general phenomenon: for any Hamiltonian action of a torus T on a compact symplectic manifold M, if the reduced spaces over the interior of the momentum polytope are two-dimensional and those over the boundary are single points — this condition holds if and only if the dimension of the torus is one less than the dimension of the manifold and at each fixed point the isotropy weights

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are in general position — then the geometric quotient M/T is homeomorphic to a sphere.

2. Background and main result

Let T be a torus and $\mathfrak{t}^*$ the dual to its Lie algebra.

Let (M, ω) be a symplectic manifold with a T action and with a momentum map $\mu\colon M \to \mathfrak{t}^*$. Such an action is called *Hamiltonian*. We recall the definitions and properties of Hamiltonian torus actions in Appendix A. In particular, the momentum map μ is constant on T orbits, so it induces a map, which is sometimes called the *orbital momentum map*, on the geometric quotient,

$$\bar{\mu}\colon M/T \to \mathfrak{t}^*.$$

In this paper we always assume that M is compact[1] and connected. Since M is compact, the fixed set M^T is not empty. To see this, fix a vector $\xi \in \mathfrak{t}$ that generates a dense one-parameter subgroup. Any point $p \in M$ on which the function $\langle \mu(\cdot), \xi \rangle\colon M \to \mathbb{R}$ achieves its minimal value is a fixed point for the one-parameter subgroup, and hence for T.

Local normal form and the convexity package. The local structure of a Hamiltonian torus action is governed by the local normal form, which describes a neighbourhood of an orbit up to an equivariant symplectomorphism that preserves momentum maps. We recall the statement of the local normal form in an appendix.

We denote

$$\Delta := \operatorname{image} \mu.$$

We will need the following theorem and corollary.

Theorem 2.1 (convexity package). *Δ is a rational[2] convex polytope, and the map $\mu\colon M \to \Delta$ is open and has connected fibres.*

Corollary 2.2. *For any convex subset C of $\mathfrak{t}^*$, the preimage $\mu^{-1}(C)$ is connected.*

The local normal form is due to Guillemin and Sternberg [1984] and Marle [1985].

The convexity package is due to Guillemin–Sternberg and Atiyah. Relevant references include the papers [Guillemin and Sternberg 1982; Atiyah 1982; Condevaux et al. 1988; Lerman and Tolman 1997; Hilgert et al. 1993; Lerman et al. 1998; Bjorndahl and Karshon 2010; Birtea et al. 2008; 2009]. The corollary follows from the (convexity of C and Δ, hence) connectedness of $C \cap \Delta$ by the following

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[1] Many of the results in this paper remain true when M is not necessarily compact but μ is proper as a map to some convex subset of $\mathfrak{t}^*$.

[2] "Rational" means that the facets have rational conormal vectors.

exercise in point set topology: Given an continuous open map with connected fibres, the preimage of any connected subset of the image is connected.

Principal orbit types over faces and in level sets; the complexity. We continue to assume that M is compact and connected. Let T_{eff} be the quotient of T by the kernel of the action. Because M is connected, it has a connected open dense subset where the action of T_{eff} is free. The formula for the momentum map implies that the affine span of the momentum image of M is a translation of the annihilator in $\mathfrak{t}^*$ of the Lie algebra of the kernel of the action. In particular,

$$\dim T_{\text{eff}} = \dim \Delta. \tag{2.3}$$

The action is *toric* if $\dim T_{\text{eff}} = \frac{1}{2} \dim M$. More generally, the *complexity* of the action is $\frac{1}{2} \dim M - \dim T_{\text{eff}}$; it measures how far the action is from being toric.

Lemma 2.4. *For every face*[3] *F of Δ, its preimage M_F in M, with the structures induced from M, is a compact connected symplectic manifold with a Hamiltonian T action.*

Proof. By the definition of "face", there exist $\xi \in \mathfrak{t}$ and $a \in \mathbb{R}$ such that $\langle \mu(p), \xi \rangle \geq a$ for all $p \in M$, with equality exactly if $p \in M_F$.

Given any $p \in M_F$, let $\mathfrak{h}$ be the Lie algebra of its stabilizer, and let $\eta_j \in \mathfrak{h}^*$ be the isotropy weights at p (see Appendix B). By the local normal form theorem, the fact that $\langle \mu(q), \xi \rangle \geq \langle \mu(p), \xi \rangle$ for all q near p implies that $\xi \in \mathfrak{h}$ and that $\langle \eta_j, \xi \rangle \geq 0$ for all j. The local normal form theorem then implies that the intersection of M_F with a neighbourhood of the orbit of p is a T invariant symplectic submanifold.

By Corollary 2.2, M_F is connected. □

Remark 2.5. Let K be the identity component of the kernel of the T action on M_F. By the definition of the momentum map, the affine span of F is a translation of the annihilator in $\mathfrak{t}^*$ of the Lie algebra of K. Moreover, M_F is a connected component of M^K, the set of points fixed by K, because the component of M^K containing M_F must lie in the preimage of the affine span of F. In particular, the preimage in M of any vertex of Δ is a component of the fixed point set M^T.

Lemma 2.6. *Given any face F and any fixed point p in the preimage M_F, the complexity of the T action on M_F is the number of isotropy weights at p that are parallel to F minus the dimension of F. Moreover, the linear span of the weights that are parallel to F is a translation of the affine span of F.*

Proof. By Lemma 2.4, the preimage M_F of F in M is a compact connected symplectic manifold with a Hamiltonian T action. Let K be the identity component

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[3] Because the convex set Δ is locally polyhedral, a subset F of Δ is a face if and only if it is equal either to Δ or to the intersection of Δ with a supporting hyperplane (a hyperplane that meets Δ and such that one of the two closed half-spaces that it bounds contains Δ).

of the kernel of the T action on M_F. By Remark 2.5, the affine span of F is a translation of the annihilator in $\mathfrak{t}^*$ of the Lie algebra of K, and M_F is a connected component of M^K. Hence, the dimension of T/K is the dimension of F, and the weights for the action on T_pM_F are those weights for the action on T_pM that annihilate the Lie algebra of K, or equivalently, are parallel to F. Therefore, the dimension of M_F is twice the number of such weights.

Finally, by the local normal form theorem, there is a neighbourhood of p in M_F that is equivariantly symplectomorphic to T_pM_F. Since K is the identity component of the stabilizer of an open dense set of points in M_F, the identity component of the kernel of the isotropy representation on T_pM_F is also K. Hence, the isotropy weights at p span the annihilator in $\mathfrak{t}^*$ of the Lie algebra of K. □

Corollary 2.7. *Let M_F and $M_{F'}$ be the preimage of faces F and F' of Δ, respectively. If $F \subseteq F'$, then the complexity of M_F is less than or equal to the complexity of $M_{F'}$.*

Proof. By Lemma 2.4, M_F and $M_{F'}$ are compact connected symplectic manifolds with Hamiltonian T actions. Consider a fixed point $p \in M_F$. Since the linear span of the isotropy weights at p that are parallel to F' is a translation of the affine span of F', the number of weights that are parallel to F' but not F must be greater than or equal to the codimension of F in F'. □

Given a point $\beta \in \mathfrak{t}^*$, let $M_\beta := \bar{\mu}^{-1}(\{\beta\}) = \mu^{-1}(\{\beta\})/T$ be the *reduced space* at β. If T_{eff} acts freely on $\mu^{-1}(\{\beta\})$, then M_β is naturally a manifold. More generally, the following holds.

Lemma 2.8. *Given a point β in the relative interior of Δ, the set of free orbits in the reduced space M_β is a connected open dense subset of M_β; moreover, it is naturally*[4] *a $2k$-dimensional manifold, where k is the complexity of the T action on M.*

Proof. This consequence of the local normal form theorem and the convexity package is proved by Sjamaar and Lerman [1991]. □

The *dimension* of a reduced space M_β is the dimension of an open dense subset of M_β that is a manifold; it is well defined, by Lemmas 2.4 and 2.8. For any nonnegative integer k, denote by Δ_k the set of points β in Δ such that $\dim M_\beta = 2k$, and denote $\Delta_{\leq k} := \Delta_0 \cup \cdots \cup \Delta_k$. By the connectedness of the momentum map fibres, Δ_0 is the set of points β in Δ such that the reduced space M_β consists of a single orbit.

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[4] Explicitly, there exists a unique manifold structure on the set of free orbits in M_β such that a real valued function on this set is smooth if and only if its pullback to the preimage in $\mu^{-1}(\{\beta\})$ extends to a smooth function on an open subset of M.

Lemma 2.9. *For any nonnegative integer k, the set $\Delta_{\leq k}$ is a union of faces of Δ. Consequently, there exists an open convex subset U of $\mathfrak{t}^*$ such that $\Delta \smallsetminus \Delta_{\leq k} = \Delta \cap U$.*

Proof. By Lemma 2.4, the preimage $M_F := \mu^{-1}(F)$ of each face F of Δ is a compact connected symplectic manifold with a Hamiltonian T action. Hence, by Lemma 2.8, each Δ_k is the union of the relative interiors of those faces F for which the complexity of M_F is equal to k. The first claim then follows from Corollary 2.7.

To prove the second claim, for each face F in $\Delta_{\leq k}$ choose a supporting hyperplane H_F of Δ such that $F = H_F \cap \Delta$. Then the intersection U of the appropriate open half-spaces bound by these hyperplanes is an open convex set. □

Remark 2.10 (toric manifolds). If we assume that the T action on M is toric, then the quotient M/T is homeomorphic to the disk D^n, where $n = \frac{1}{2}\dim M$. To see this, first note that Lemma 2.4 and Corollary 2.7 together show that the preimage $M_F := \mu^{-1}(F)$ of each face F of Δ is a symplectic toric manifold. Hence, by Lemma 2.8, the reduced space M_β is a point for all $\beta \in \Delta$, that is, $\Delta_0 = \Delta$. Thus, the orbital momentum map $\bar{\mu} \colon M/T \to \Delta$ is a bijection; since it is proper and continuous, this implies that it is a homeomorphism. Since Δ is a convex polytope, this proves the claim.

More generally, consider a complete unimodular fan in $\mathbb{R}^n$. Even if the fan does not correspond to any convex polytope, we can construct a complex toric manifold M from the fan, as described by Audin [2004]. The geometric quotient M/T is still homeomorphic to a sphere; see [Karshon and Tolman 1993, Lemma 3.2].

A collection of vectors in the vector space $\mathfrak{t}^*$ is *in general position* if every subcollection of size $< \dim \mathfrak{t}^*$ is linearly independent.

Lemma 2.11. *Assume that M is compact.*

(1) *Assume that there exists an isolated fixed point in M whose momentum image is a vertex of Δ; in particular, this holds if the fixed points in M are isolated. Then $\Delta_0 \neq \varnothing$.*

(2) *Assume that the T action on M has complexity ≥ 1 and that the isotropy weights at every fixed point are in general position. Then $\Delta_0 = \partial\Delta$.*

Proof. Part (1) is a consequence of the following two facts. First, since M is compact, its momentum image Δ has a vertex. Second, by Remark 2.5, the preimage of any vertex of Δ is a connected component of the fixed point set M^T.

We now prove Part (2). First, consider $\beta \in \partial\Delta$. Let $F \subsetneq \Delta$ be the face whose relative interior contains β. By Lemma 2.4, the preimage M_F of F in M is a compact connected symplectic T manifold with a Hamiltonian T action. So it has a fixed point p. Since $\dim F < \dim \mathfrak{t}^*$ and the isotropy weights at p are in general position, Lemma 2.6 implies that M_F is toric. Therefore, by Lemma 2.8, $\beta \in \Delta_0$.

In contrast, if β is in the relative interior of Δ then, since the action of T on M is not toric, Lemma 2.8 implies that β is not in Δ_0. ☐

Remark 2.12. In Part (2) of Lemma 2.11, if the complexity of the T action on M is *equal* to one, then the converse is true too, so $\Delta_0 = \partial\Delta$ if and only if the isotropy weights at every fixed point are in general position.

When the complexity of the Hamiltonian T action is equal to one, we denote by Δ_{short} the set of points in Δ whose reduced space contains a single orbit and by Δ_{tall} the set of points in Δ whose reduced space is two-dimensional. Thus,

$$\Delta_{\text{short}} = \Delta_0 \quad \text{and} \quad \Delta = \Delta_{\text{short}} \sqcup \Delta_{\text{tall}}.$$

By Lemma 2.9, Δ_{short} is closed,

Proposition 2.13. *Let T be a torus and $\mathfrak{t}^*$ the dual to its Lie algebra. Let M be a compact connected symplectic manifold with a T action and with a momentum map $\mu\colon M \to \mathfrak{t}^*$ with image Δ. Assume that the action has complexity one.*

Then there exists a connected closed oriented surface Σ and a homeomorphism

$$(M/T)_{\text{tall}} \to \Delta_{\text{tall}} \times \Sigma$$

that intertwines the orbital momentum map $\bar{\mu}$ with the projection map to Δ_{tall}.

If Δ_{short} is nonempty, then Σ is a two-sphere.

Proof. By Lemma 2.9, there exists a convex open subset U of $\mathfrak{t}^*$ such that $\Delta_{\text{tall}} = \Delta \cap U$. The first part of Proposition 2.2 of [Karshon and Tolman 2003] then implies that there is a homeomorphism $(M/T)_{\text{tall}} \to \Delta_{\text{tall}} \times \Sigma$ as required. By [Karshon and Tolman 2001, Lemma 5.7], if Δ_{short} is nonempty, then Σ is a sphere. ☐

We now state our main theorem.

Theorem 2.14. *Let T be a torus and $\mathfrak{t}^*$ the dual to its Lie algebra. Let M be a $2n$-dimensional compact connected symplectic manifold with a T action and with a momentum map $\mu\colon M \to \mathfrak{t}^*$ with image Δ. Assume that the action has complexity one. Then there exist a connected closed oriented surface Σ and a homeomorphism*

$$M/T \to (\Delta \times \Sigma)/\sim,$$

where $\sim$ is the finest equivalence relation with $(x, y) \sim (x, y')$ if $x \in \Delta_{\text{short}}$. Moreover,

(i) *If Δ_{short} is nonempty, then Σ is a two-sphere.*

(ii) *If $\Delta_{\text{short}} = \partial\Delta$, then M/T is homeomorphic to the $(n+1)$-sphere.*

Proof. By Proposition 2.13, there exists a connected closed oriented surface Σ and a homeomorphism

$$(M/T)_{\text{tall}} \to \Delta_{\text{tall}} \times \Sigma$$

that intertwines the orbital momentum map $\bar{\mu}$ and the projection map to Δ_{tall}. Since $\Delta = \Delta_{\text{short}} \sqcup \Delta_{\text{tall}}$ and Δ_{short} consists of those β such that M_β consists of a single orbit, this homeomorphism extends to a unique bijection

$$f \colon M/T \to (\Delta \times \Sigma)/{\sim}$$

that intertwines the orbital momentum map $\bar{\mu}$ with the map $\pi : (\Delta \times \Sigma)/{\sim} \to \mathfrak{t}^*$ induced by the projection to Δ. Since $(M/T)_{\text{tall}}$ is open in M/T and Δ_{tall} is open in Δ, the map f is continuous and open at every point of $(M/T)_{\text{tall}}$.

Since M and Σ are compact, the maps $\bar{\mu} \colon M/T \to \mathfrak{t}^*$ and $\pi : (\Delta \times \Sigma)/{\sim} \to \mathfrak{t}^*$ are proper. Since $\mathfrak{t}^*$ is a locally compact Hausdorff space, the proper maps $\bar{\mu}$ and π to $\mathfrak{t}^*$ are closed. Since π is closed, f is continuous at every point of $(M/T)_{\text{short}}$. Since $\bar{\mu}$ is closed and f is onto, f is open at every point of $(M/T)_{\text{short}}$.

Part (i) follows from the last claim of Proposition 2.13.

We now prove Part (ii). Since M is compact and connected, Δ is a convex polytope; hence, it is homeomorphic to D^{n-1}, where $\dim M = 2n$. Therefore, the map from $D^{n-1} \times S^2$ that sends (x, z) to $(x, \sqrt{1 - |x|^2}\, z)$ induces a continuous proper map from $(\Delta \times \Sigma)/{\sim}$ to S^{n+1}. If $\Delta_{\text{short}} = \partial\Delta$, this map is a bijection. Since S^{n+1} is a locally compact Hausdorff space, being a continuous proper bijection implies that this map is a homeomorphism. □

Remark 2.15. Part (ii) of Theorem 2.14 can be rephrased as follows: If $\Delta_{\text{short}} = \partial\Delta$, then M/T is homeomorphic to the join $\partial\Delta * S^2$. To see this, recall that the join $A * B$ of two topological spaces A and B is the quotient of $A \times B \times [0, 1]$ under the identifications $(a, b, 0) \sim (a', b, 0)$ and $(a, b, 1) \sim (a, b', 1)$ for all $a, a' \in A$ and $b, b' \in B$. We may assume without loss of generality that $0 \in \text{interior}\, \Delta$. Then, since Δ is convex, the map $\partial\Delta \times B \times [0, 1] \to \Delta \times B$ that is defined by $(a, b, t) \mapsto (ta, b)$ descends to a continuous proper bijection $\partial\Delta \times B \to (\Delta \times B)/\sim$, where here $\sim$ is the finest equivalence relation with $(x, y) \sim (x, y')$ if $x \in \partial\Delta$. When B is a locally compact Hausdorff space, this bijection is a homeomorphism.

Corollary 2.16. *Let T be a torus and $\mathfrak{t}^*$ the dual to its Lie algebra. Let M be a compact connected symplectic manifold with a T action and a momentum map $\mu \colon M \to \mathfrak{t}^*$ with image Δ. Assume that the action has complexity one.*

(a) *Assume that there exists an isolated fixed point in M whose momentum image is a vertex of Δ; in particular, this holds if the fixed points in M are isolated. Then M/T is homeomorphic to $(\Delta \times S^2)/{\sim}$, where $\sim$ is the finest equivalence relation with $(x, y) \sim (x, y')$ if $x \in \Delta_{\text{short}}$.*

(b) *Assume that the isotropy weights at every fixed point are in general position. Then M/T is homeomorphic to a sphere.*

Proof. Part (a) follows from Part (1) of Lemma 2.11 and Part (i) of Theorem 2.14. Part (b) follows from Part (2) of Lemma 2.11 and Part (ii) of Theorem 2.14. □

In Part (b) of Corollary 2.16, the fact that M/T is a topological manifold already follows from a result of Ayzenberg [2018]. Ayzenberg's work also implies that if the action extends to a toric action then M/T is homeomorphic to a sphere.

Corollary 2.17. *Let the circle S^1 act on a compact connected symplectic four-manifold (M, ω) with momentum map $\mu\colon M \to \mathbb{R}$. Then exactly one of the following is true.*

(1) *The fixed point set is finite and M/T is homeomorphic to a three-sphere.*

(2) *The fixed point set contains one surface, which is a sphere, and M/T is homeomorphic to a three-disk.*

(3) *The fixed point set contains two surfaces that have the same genus g, and M/T is homeomorphic to $[0, 1] \times \Sigma$, where Σ is a surface of genus g.*

Proof. By rescaling ω if necessary, we may assume that the momentum image is the interval $[0, 1]$. Since 0 and 1 are vertices of $[0, 1]$, Lemma 2.4 and Remark 2.5 imply that each of $\mu^{-1}(\{0\})$ and $\mu^{-1}(\{1\})$ is a connected component of the fixed point set that is either a single point or a fixed surface. By the local normal form theorem, a fixed point that is not isolated is a local minimum or local maximum of the momentum map; since by the convexity package the momentum map is open as a map to its image $[0, 1]$, such a fixed point must be mapped to 0 or to 1. Hence, there are at most two components of the fixed point set that are not isolated fixed points, and each of them is mapped to 0 or to 1.

Assume first that the fixed point set contains no surfaces. Then the fixed points are isolated, and so none of the isotropy weights at any fixed point are zero. Hence, M/T is homeomorphic to a three-sphere by Part (b) of Corollary 2.16.

Assume now that the fixed point set contains exactly one surface Σ. By replacing ω by $-\omega$ if necessary, we may assume that $\mu(\Sigma) = 1$. Since Σ is the only fixed surface, $\mu^{-1}(\{0\})$ is an isolated fixed point. Hence, $\Delta_{\text{short}} = \{0\}$. By Part (a) of Corollary 2.16, this implies the M/S^1 is homeomorphic to $[0, 1] \times S^2/{\sim}$, where $\sim$ is the finest equivalence relation such that $(0, x) \sim (0, x')$. Define a map

$$[0, 1] \times S^2 \to \mathbb{R}^3, \quad (t, x) \mapsto tx,$$

where we identify S^2 with the unit sphere in $\mathbb{R}^3$. This induces a homeomorphism from $[0, 1] \times S^2/{\sim}$ to the three-disk D^3, and hence from M/T to D^3.

Finally, assume that the fixed point set contains two surfaces, Σ and Σ'. By the first paragraph, we may assume that $\mu^{-1}(\{0\}) = \Sigma$ and $\mu^{-1}(\{1\}) = \Sigma'$. Hence, Δ_{short} is empty, and so Theorem 2.14 implies that M/T is homeomorphic to $[0, 1] \times \Sigma_g$ for some oriented surface Σ_g. □

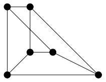
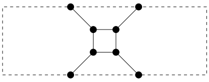
	M	$\dim_{\mathbb{R}} M$	T	complexity	$M/T \overset{\text{homeo}}{\cong}$
(1)	$G_2(\mathbb{C}^4) = \{E^2_{\mathbb{C}} \subset \mathbb{C}^4\}$	8	$(S^1)^4/\text{diag}$	1	S^5
(2)	$F_3 = \{L^1_{\mathbb{C}} \subset E^2_{\mathbb{C}} \subset \mathbb{C}^3\}$	6	$(S^1)^3/\text{diag}$	1	S^4
(3)	$G_2^+(\mathbb{R}^5) = \{E^2_{\text{oriented}} \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}\}$	6	$(S^1)^2$	1	S^4
(4)		6	$(S^1)^2$	1	S^4
(5)	$S^1 \circlearrowright (S^2)^2$ $a \cdot (u, v) = (a \cdot u, a \cdot v)$	4	S^1	1	S^3
(6)	$S^1 \circlearrowright \mathbb{CP}^2$ $a \cdot [z_0 : z_1 : z_2] = [a z_0 : z_1 : z_2]$	4	S^1	1	D^3
(7)	$S^1 \circlearrowright S^2 \times \Sigma_g$	4	S^1	1	$I \times \Sigma_g$
(8)	$S^1 \times S^1 \circlearrowright (S^2)^3$ $(a, b) \cdot (u, v, w) = (a \cdot u, a \cdot v, b \cdot w)$	6	$S^1 \times S^1$	1	$S^3 \times I$
(9)		6	$(S^1)^2$	1	$I \times I \times \Sigma_g$
(10)	$\mathbb{CP}^5 = \mathbb{P}\big(\bigwedge^2 \mathbb{C}^4\big)$	10	$(S^1)^4/\text{diag}$	2	$S^2 * \mathbb{CP}^2$ (†)
(11)	$G_2(\mathbb{C}^5) = \{E^2_{\mathbb{C}} \subset \mathbb{C}^5\}$	12	$(S^1)^5/\text{diag}$	2	(‡)

Table 1. Examples of geometric quotients. For (†), see [Buchstaber and Terzić 2016]. For (‡), see [Buchstaber and Terzić 2019b; Süß 2019].

3. Examples

In Table 1 we list some examples of symplectic torus actions and their geometric quotients.

We now discuss these examples and give some references.

(1) Let M be the Grassmannian of complex 2-planes in $\mathbb{C}^4$, with the three-dimensional torus action induced from the standard action of $(S^1)^4$ on $\mathbb{C}^4$. Then M/T is homeomorphic to the sphere S^5; this is shown in [Buchstaber and Terzić 2016] and revisited in [Buchstaber and Terzić 2019a, Section 10]. Alternatively, we can identify M equivariantly with a coadjoint orbit in SU(4), where T is a maximal torus acting through the coadjoint action. There is a natural symplectic structure on every coadjoint orbit of any Lie group, and the coadjoint action is Hamiltonian.

Hence, since the isotropy weights at each fixed point are in general position, we can apply Corollary 2.16 and conclude that M/T is homeomorphic to the sphere S^5.

(2) Let M be the manifold of complete complex flags in $\mathbb{C}^3$, with the two-dimensional torus action that is induced from standard action of $(S^1)^3$ on $\mathbb{C}^3$. Then M/T is homeomorphic to the sphere S^4; this is shown in [Buchstaber and Terzić 2019a]. Alternatively, M is a coadjoint orbit of SU(3), and so — as in the previous example — M/T is homeomorphic to the sphere S^4 by Corollary 2.16.

(3) Let M be the Grassmannian of oriented (real) 2-planes in $\mathbb{R}^5 \cong (\mathbb{R}^2)^2 \times \mathbb{R}$, with the two-dimensional torus action that is induced from the standard action of $(S^1)^2$ on the $(\mathbb{R}^2)^2$ factor. By identifying M with a coadjoint orbit of SO(5), we obtain a symplectic form such that the action is Hamiltonian. Since the isotropy weights at each fixed point are in general position, M/T is homeomorphic to the sphere S^4 by Corollary 2.16. For more details, see, e.g., [Karshon and Tolman 2001, Section 14].

(4) Let (M, ω, μ) be the compact symplectic six manifold with Hamiltonian $(S^1)^2$ action constructed in [Tolman 1998]. The picture drawn in the table shows the momentum map images of the orbit type strata. The solid dots are the images of isolated fixed points, and the segments are the images of 2-spheres with circle stabilizer. As the second author showed in [Tolman 1998], M does not admit any invariant Kähler structure. As in the previous examples, M/T is homeomorphic to the sphere S^4 by Corollary 2.16.

(5) Let M be the product of the two-sphere S^2 with itself. There is a standard area form on S^2; the height function is a momentum map for the circle action that rotates the sphere around the vertical axis. Take the product symplectic form on M; then the momentum map for the diagonal circle action sends (u, v) to the sum $u_3 + v_3$. By [Ayzenberg 2018], M/T is homeomorphic to the sphere S^3. Alternatively, this follows from Corollary 2.17 or from Corollary 2.16.

(6) Let $M = \mathbb{CP}^2$, with the Fubini–Study symplectic form, the circle action given by $a \cdot [z_0 : z_1 : z_2] = [az_0 : z_1 : z_2]$, and momentum map

$$[z_0 : z_1 : z_2] \mapsto \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

By Corollary 2.17, M/T is homeomorphic to the disc D^3.

(7) Let $M = \Sigma_g \times S^2$, where Σ_g is a surface of genus g, with the circle acting on the second factor, a product symplectic form, and momentum map $(u, v) \mapsto v_3$. Then M/T is homeomorphic to $I \times \Sigma_g$, where I is a closed interval. This follows from Corollary 2.17 and is also easy to see directly.

Let $\widehat{M}$ be an equivariant symplectic blowup of M at a fixed point. Then $\widehat{M}$ has one isolated fixed point and two fixed surfaces of genus g. The quotient $\widehat{M}/T$ is still homeomorphic to $I \times \Sigma_g$.

(8) Let $M = (S^2)^3$ with the product symplectic form, the $S^1 \times S^1$ action

$$(a, b) \cdot (u, v, w) = (a \cdot u, a \cdot v, b \cdot w),$$

and momentum map $(u, v, w) \mapsto (u_3 + v_3, w_3)$. The momentum image Δ is the rectangle $[-2, 2] \times [-1, 1]$, and $\Delta_{\text{short}} = \{-2, 2\} \times [-1, 1]$. By Theorem 2.14, this implies that M/T is homeomorphic to $S^3 \times I$. Alternatively, this follows from the facts that $S^2/S^1 \simeq I$ and, as we saw in (5), that $(S^2)^2/S^1 \simeq S^3$. Note that in this example $\varnothing \neq \Delta_{\text{short}} \subsetneq \partial\Delta$.

(9) Let Σ_g be a surface of genus g. Let (M, ω, μ) be any one of the compact symplectic six-manifolds with Hamiltonian $(S^1)^2$ action and reduced spaces homeomorphic to Σ_g that are described in [Karshon and Tolman 2014, Example 1.11]. (If $g > 0$, there is an infinite number of isomorphism classes of such manifolds even if we fix the Duistermaat–Heckman measure.) As is (4), the solid dots are the momentum map images of isolated fixed points, and the segments are the momentum map images of 2-spheres with circle stabilizer. The momentum image Δ is the closed rectangle whose boundary is marked by dashed lines, and Δ_{short} is empty. Theorem 2.14 implies that M/T is homeomorphic to $I \times I \times \Sigma_g$.

(10) Let M be the projective space $\mathbb{CP}^5$, with the three-dimensional torus action induced by the $(S^1)^4$ action on $\bigwedge^2\mathbb{C}^4 \cong \mathbb{C}^6$, which itself is induced by the standard action on $\mathbb{C}^4$. Corollary 12 in [Buchstaber and Terzić 2016, §10] states that the quotient M/T is homeomorphic to the join $S^2 * \mathbb{CP}^2$.

(11) Let M be the Grassmannian of two-planes in $\mathbb{C}^5$, with the four-dimensional torus action induced from the standard action of $(S^1)^5$ on $\mathbb{C}^5$. The quotient M/T was studied by Buchstaber and Terzić [2019b] and Süß [2019].

Appendix A: Hamiltonian T actions

A torus T is a Lie group that is isomorphic to $(S^1)^r$ for some nonnegative integer r. A symplectic manifold is a manifold M equipped with a differential two-form ω that is closed and nondegenerate. A momentum map is a map from the manifold to the dual of the Lie algebra of the torus such that, for every element ξ of the Lie algebra $\mathfrak{t}$ of the torus, the corresponding vector field ξ_M on M (whose value at a point $x \in M$ is $\xi_M|_x = \frac{d}{dt}|_{t=0} \exp(t\xi) \cdot x$) and the corresponding component of the momentum map $\mu^\xi : M \to \mathbb{R}$ (whose value at a point $x \in M$ is $\langle \mu(x), \xi \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$) are related by Hamilton's equations

$$d\mu^\xi = -\iota(\xi_M)\omega \quad \text{for all } \xi \in \mathfrak{t} \tag{A.1}$$

(where $\iota(\xi_M)\omega(v) = \omega(\xi_M, v)$ for any $v \in TM$). We then call the T action *Hamiltonian.*

If M is connected, then the affine span of the momentum image $\mu(M)$ is a translate of the annihilator of the Lie algebra of the kernel of the action. This is a consequence of Hamilton's equations (A.1).

The symplectic form is T invariant. We recall why. For any $\xi \in \mathfrak{t}$, the Lie derivative of ω along ξ_M satisfies $L_{\xi_M}\omega = d\iota(\xi_M)\omega + \iota(\xi_M)\, d\omega$; the first summand vanishes because (by Hamilton's equation) $\iota(\xi_M)\omega$ is exact; the second summand vanishes because (by assumption) ω is closed.

The momentum map is constant on orbits. We recall why. For any $\xi, \eta, \zeta \in \mathfrak{t}$ we have $L_{\xi_M}(\omega(\eta_M, \zeta_M)) = (L_{\xi_M}\omega)(\eta_M, \zeta_M) + \omega([\xi_M, \eta_M], \zeta_M) + \omega(\eta_M, [\xi_M, \zeta_M])$; the first summand vanishes because the symplectic form is T invariant; the second and third summands vanish because T is abelian. Hence, $\omega(\eta_M, \zeta_M)$ is constant along T orbits. By Hamilton's equation, $\omega(\eta_M, \zeta_M) = L_{\eta_M}\mu^{\zeta_M}$; because for each T orbit the right hand vanishes at the point on the (compact) orbit where μ^{ζ_M} attains its maximum, $L_{\eta_M}\mu^{\zeta_M} = 0$. Because $\eta \in \mathfrak{t}$ is arbitrary, μ^{ζ_M} is constant along T orbits; because $\zeta \in \mathfrak{t}$ is arbitrary, μ is constant along T orbits.

Appendix B: Local normal form

A *Hamiltonian T model* is a Hamiltonian T-manifold (Y, ω_Y, μ_Y) that is obtained by the following construction. Let a closed subgroup H of T act on $\mathbb{C}^\ell$ through a homomorphism $H \to (S^1)^\ell$ followed by the standard action of $(S^1)^\ell$ on $\mathbb{C}^\ell$; the corresponding quadratic momentum map $\mu_H : \mathbb{C}^\ell \to \mathfrak{h}^*$ is

$$z \mapsto \sum_{j=1}^{\ell} \frac{|z_j|}{2}\eta_j,$$

where $\eta_1, \dots, \eta_\ell \in \mathfrak{h}^*_{\mathbb{Z}}$ are the weights for the H action on $\mathbb{C}^\ell$. Take Y to be the manifold $T \times_H (\mathfrak{h}^0 \times \mathbb{C}^\ell)$, where $\mathfrak{h}^0$ is the annihilator in $\mathfrak{t}^*$ of the Lie algebra of H. Here, we quotient by the antidiagonal action of H, in which $a \in H$ acts on T by right multiplication by a^{-1}, it acts on $\mathfrak{h}^0$ trivially, and it acts on $\mathbb{C}^\ell$ through the given action. The torus T acts on Y by left multiplication on the T factor. The *central orbit* in the model Y is the orbit $[a, 0, 0]$.

Equip $(T \times \mathfrak{t}^*) \times \mathbb{C}^\ell$ with the product of the standard symplectic form on $T \times \mathfrak{t}^*$, viewed as the cotangent bundle of T, and the standard symplectic form $\sum_{j=1}^{\ell} dx_j \wedge dy_j$ on $\mathbb{C}^\ell$. The pullback of this symplectic form under the inclusion map from $T \times \mathfrak{h}^0 \times \mathbb{C}^\ell$ to $(T \times \mathfrak{t}^*) \times \mathbb{C}^\ell$ taking (a, ν, z) to $(a, \nu + \Phi_H(z), z)$ is equal to the pullback of the symplectic form ω_Y under the quotient map $T \times \mathfrak{h}^0 \times \mathbb{C}^\ell \to T \times_H (\mathfrak{h}^0 \times \mathbb{C}^\ell)$; this determines ω_Y. Here, we have identified $\mathfrak{t}^*$ with $\mathfrak{h}^0 \oplus \mathfrak{h}^*$.

The momentum map is $\mu_Y([a, \nu, z]) = \alpha + \nu + \mu_H(z)$ for some $\alpha \in \mathfrak{t}^*$, where μ_H is the quadratic momentum map for the linear H action.

Theorem B.1 (local normal form). *Let M be a symplectic manifold with a Hamiltonian T action. Then for every orbit in M there exists an equivariant symplectomorphism that preserves the momentum maps from a neighbourhood of the orbit in M to a neighbourhood of the central orbit in some Hamiltonian T model.*

In Theorem B.1, the H that appears in the Hamiltonian T model that corresponds to the orbit of a point p is the stabilizer of p. Moreover, the weights η_j that appear in the model are unique up to permutation; we call them the *isotropy weights* at p.

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YAEL KARSHON
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
TORONTO, ON
CANADA
karshon@math.toronto.edu

SUSAN TOLMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
URBANA, IL
UNITED STATES
tolman@illinois.edu

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FLAG BOTT MANIFOLDS AND THE TORIC CLOSURE OF A GENERIC ORBIT ASSOCIATED TO A GENERALIZED BOTT MANIFOLD

SHINTARÔ KUROKI, EUNJEONG LEE,
JONGBAEK SONG AND DONG YOUP SUH

To a direct sum of holomorphic line bundles, we can associate two fibrations, whose fibers are, respectively, the corresponding full flag manifold and the corresponding projective space. Iterating these procedures gives, respectively, a flag Bott tower and a generalized Bott tower. It is known that a generalized Bott tower is a toric manifold. However a flag Bott tower is not toric in general but we show that it is a GKM manifold, and we also show that for a given generalized Bott tower we can find the associated flag Bott tower so that the closure of a generic torus orbit in the latter is a blow-up of the former along certain invariant submanifolds. We use GKM theory together with toric geometric arguments.

1. Introduction

A Bott tower $M_\bullet = \{M_j \mid 0 \le j \le m\}$ is a sequence of $\mathbb{C}P^1$-fibrations $\mathbb{C}P^1 \hookrightarrow M_j \to M_{j-1}$ such that M_j is the projectivization of the sum of two complex line bundles over M_{j-1}, where M_0 is a point which is introduced in [Grossberg and Karshon 1994]. Then each M_j is a complex j-dimensional nonsingular algebraic variety called the *j-stage Bott manifold*. Each Bott manifold M_j has a $(\mathbb{C}^*)^j$-action with which M_j becomes a toric manifold, i.e., a nonsingular toric variety.

One of the important properties of Bott manifold is its relation with Bott–Samelson variety. A Bott–Samelson variety is a nonsingular algebraic variety that appeared in many areas of mathematics, for instance algebraic geometry and

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representation theory. For a given complex semisimple Lie group G and a Borel subgroup $B \subset G$, the set of sections of a holomorphic line bundle over a Bott–Samelson variety has a structure of B-module, called a *generalized Demazure module* (see [Lakshmibai et al. 2002]). This gives a fruitful connection between representation theory and algebraic geometry. It is shown in [Grossberg and Karshon 1994; Pasquier 2010] that every Bott–Samelson variety has a Bott manifold as its toric degeneration.[1] This relation between a Bott–Samelson variety and a Bott manifold gives interesting results on algebraic representations of G in [Grossberg and Karshon 1994].

Recently, a generalized notion of Bott–Samelson variety, called a *flag Bott–Samelson variety*, has been introduced in [Fujita et al. 2018]; it extends the rich connection between representation theory and algebraic geometry given by Bott–Samelson variety. Indeed, the set of sections of a holomorphic line bundle over a flag Bott–Samelson variety is also a generalized Demazure module. This result is applied to give polyhedral expressions for irreducible decompositions of tensor products of G-modules.

In this article, we define a flag Bott tower $F_\bullet = \{F_j \mid 0 \le j \le m\}$ to be a sequence of the full flag fibrations $\mathcal{F}\ell(n_j+1) \hookrightarrow F_j \xrightarrow{p_j} F_{j-1}$, where F_j is the flagification of a sum of n_j+1 many complex line bundles over F_{j-1}. We call each F_j a *flag Bott manifold*. In [Fujita et al. 2018], they construct a one-parameter family of complex structures on a flag Bott–Samelson variety which makes the flag Bott–Samelson variety into a flag Bott manifold, and this extends the known relation between Bott–Samelson varieties and Bott manifolds.

One of the goals of this article is to study torus actions on flag Bott manifolds. In fact, the complex dimension of F_m is $\sum_{j=1}^{m} n_j(n_j+1)/2$, but there is an effective action of complex torus $\boldsymbol{H}$ of dimension $\sum_{j=1}^{m} n_j$ on F_m. Hence a flag Bott manifold is not a toric manifold in general. With the restricted action of the compact torus $\boldsymbol{T}$ of dimension $\sum_{j=1}^{m} n_j$ on a flag Bott manifold F_m, we get the following result:

Theorem 1.1 (Theorem 3.6). *Let F_m be an m-stage flag Bott manifold with the effective action of $\boldsymbol{T}$. Then $(F_m, \boldsymbol{T})$ is a GKM manifold.*

Moreover the concrete information of the GKM graph of F_m is computed in Theorem 3.12.

On the other hand, Bott manifolds are an important family of toric manifolds because of the cohomological rigidity problem which asks whether toric manifolds are topologically classified by their cohomology rings. This question has the

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[1] More precisely, [Grossberg and Karshon 1994] provides a one-parameter family of complex structures on a Bott–Samelson variety which makes the Bott–Samelson variety into a Bott manifold. Besides, [Pasquier 2010] constructs a toric degeneration of a Bott–Samelson variety, i.e., there is a flat family $\mathcal{X}$ over $\mathbb{C}$ such that $\mathcal{X}(t)$ is isomorphic to the Bott–Samelson variety for all $t \in \mathbb{C} \setminus \{0\}$ and $\mathcal{X}(0)$ is a Bott manifold.

affirmative answers for some Bott manifolds (see [Choi and Masuda 2012; Ishida 2012; Choi 2015; Choi et al. 2015]). Moreover, it also has the affirmative answer for some generalized Bott manifolds (see [Masuda and Suh 2008; Choi et al. 2010b; 2012]). Here, a *generalized Bott tower* $B_\bullet = \{B_j \mid 0 \le j \le m\}$ is defined similarly to a Bott tower but the difference is that B_j is the projectivization of the sum of $n_j + 1$ many complex line bundles instead of two line bundles.

Even though generalized Bott towers and flag Bott towers are two different generalizations of Bott towers, there is an interesting relation between them. Namely, let $B_\bullet$ be a generalized Bott tower with bundle maps $\pi_j : B_j \to B_{j-1}$. Then we define the *associated flag Bott tower* $F_\bullet$ to $B_\bullet$ with bundle maps $p_j : F_j \to F_{j-1}$. Note that they satisfy $q_{j-1} \circ p_j = \pi_j \circ q_j$, where $q_j : F_j \to B_j$ is induced by the projection map

$$\mathcal{F}\ell(n_j + 1) \to \mathbb{C}P^{n_j}$$

on each fiber (see Section 4). Moreover we prove that a generalized Bott manifold and its associated flag Bott manifold have the following relation:

Theorem 1.2 (Theorem 5.7). *Let B_m be an m-stage generalized Bott manifold, and F_m its associated flag Bott manifold. Then the closure of a generic orbit of $\boldsymbol{H}$-action in F_m is the blow-up of B_m along certain invariant submanifolds.*

To obtain this result the GKM graph information of F_m from Theorem 3.12 is essentially used together with some toric topological arguments.

We remark that every flag Bott tower is a $\mathbb{C}P$-tower, i.e., a sequence of an iterated complex projective space fibrations. A $\mathbb{C}P$-tower is introduced in [Kuroki and Suh 2014; 2015] as a more generalized notion than a generalized Bott tower.

The paper is organized as follows. In Section 2, we give an alternative description of a flag Bott manifold as the orbit space of the product of general linear groups under the action of the product of their Borel subgroups defined in (2-4); see Proposition 2.7. In doing so, each complex line bundle appearing in the construction of a flag Bott tower can be described in terms of characters of maximal tori of general linear groups. Then we can associate a sequence of integer matrices defined by the weights of the above mentioned characters to a flag Bott manifold as in Theorem 2.10. We also give an explicit description of the tangent bundle of a flag Bott manifold in Proposition 2.16, which will be used in the GKM description of a flag Bott manifold in Section 3.

In Section 3, we define the canonical torus action on a flag Bott manifold, and find an explicit description of the tangential representation at a fixed point in Proposition 3.5. We then see easily that every flag Bott manifold is a GKM manifold. Moreover an explicit description of the GKM graph of a flag Bott manifold is given in Theorem 3.12.

In Section 4, we define the associated flag Bott tower to a given generalized Bott tower. Then Definition 4.6 gives the integer matrices corresponding to the associated flag Bott tower.

In Section 5, we study the relation between a generalized Bott manifold B_m and the closure X of a generic orbit of the associated flag Bott manifold F_m. This can be accomplished by calculating the fan of X in Theorem 5.4 using the axial functions of the GKM graph of F_m. Then we show that the toric variety X comes from a series of blow-ups of B_m in Theorem 5.7.

2. Flag Bott manifolds

2A. ***Definition of flag Bott manifolds.*** Let M be a complex manifold and E an n-dimensional holomorphic vector bundle over M. Recall from [Bott and Tu 1982, p. 282] that the *associated flag bundle* $\mathcal{F}\ell(E) \to M$ is obtained from E by replacing each fiber E_p by the full flag manifold $\mathcal{F}\ell(E_p)$.

Definition 2.1. A *flag Bott tower* $F_\bullet = \{F_j \mid 0 \le j \le m\}$ of height m, or an *m-stage flag Bott tower*, is a sequence,

$$F_m \xrightarrow{p_m} F_{m-1} \xrightarrow{p_{m-1}} \cdots \xrightarrow{p_2} F_1 \xrightarrow{p_1} F_0 = \{\text{a point}\},$$

of manifolds $F_j = \mathcal{F}\ell\big(\bigoplus_{k=1}^{n_j+1} \xi_k^{(j)}\big)$, where $\xi_k^{(j)}$ is a holomorphic line bundle over F_{j-1} for each $1 \le k \le n_j + 1$ and $1 \le j \le m$. We call F_j the *j-stage flag Bott manifold* of the flag Bott tower $F_\bullet$.

Here are some examples of flag Bott manifolds.

Example 2.2. (1) The flag manifold $\mathcal{F}\ell(\mathbb{C}^{n+1}) = \mathcal{F}\ell(n+1)$ is a flag Bott tower of height 1. In particular, $\mathcal{F}\ell(2) = \mathbb{C}P^1$ is a 1-stage flag Bott manifold.

(2) The product of flag manifolds $\mathcal{F}\ell(n_1+1) \times \cdots \times \mathcal{F}\ell(n_m+1)$ is a flag Bott manifold of height m.

(3) Recall from [Grossberg and Karshon 1994] that an *m-stage Bott manifold* is a sequence of $\mathbb{C}P^1$-fibrations such that each stage is the projective bundle of the sum of two line bundles. When $n_j = 1$ for $1 \le j \le m$, an m-stage flag Bott manifold is an m-stage Bott manifold.

Definition 2.3. Two flag Bott towers $F_\bullet$ and $F'_\bullet$ are *isomorphic* if there is a collection of (holomorphic) diffeomorphisms $\varphi_j : F_j \to F'_j$ which commute with the maps $p_j : F_j \to F_{j-1}$ and $p'_j : F'_j \to F'_{j-1}$.

Remark 2.4. (1) One can define F_j to be $\mathcal{F}\ell(E_j)$ for some holomorphic vector bundle E_j over F_{j-1}. However, since we want to consider torus actions on F_m, we assume E_j to be a sum of holomorphic line bundles in Definition 2.1.

(2) Even though we are concentrating on full flag fibrations in this paper, one can also study other kinds of induced fibrations such as partial flag fibrations, isotropic flag fibrations, etc., which require further works. In [Kaji et al. 2020], the authors study such iterated flag fibrations.

2B. ***Orbit space construction of flag Bott manifolds.*** In this subsection, we consider an orbit space construction of a flag Bott tower in Proposition 2.7 using the complex Lie groups $\mathrm{GL}(n) := \mathrm{GL}(n, \mathbb{C})$ in order to consider the canonical torus action on it (see Section 3A).

A flag Bott tower of height 1 is the flag manifold $\mathcal{F}\ell(n_1+1)$ which is the orbit space $\mathrm{GL}(n_1+1)/B_{\mathrm{GL}(n_1+1)}$, where $B_{\mathrm{GL}(n_1+1)}$ is the set of upper triangular matrices in $\mathrm{GL}(n_1+1)$. To describe flag Bott manifolds of higher stages, we begin with a matrix A of size $(n+1)\times(n'+1)$ whose row vectors are $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n+1} \in \mathbb{Z}^{n'+1}$, i.e.,

$$A = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1(1) & \boldsymbol{a}_1(2) & \cdots & \boldsymbol{a}_1(n'+1) \\ \boldsymbol{a}_2(1) & \boldsymbol{a}_2(2) & \cdots & \boldsymbol{a}_2(n'+1) \\ \vdots & \vdots & & \vdots \\ \boldsymbol{a}_{n+1}(1) & \boldsymbol{a}_{n+1}(2) & \cdots & \boldsymbol{a}_{n+1}(n'+1) \end{bmatrix} \in M_{n+1,n'+1}(\mathbb{Z}),$$

which encodes a $B_{\mathrm{GL}(n'+1)}$-action on $\mathrm{GL}(n+1)$ as follows. Let $H_{\mathrm{GL}(n+1)} \subset \mathrm{GL}(n+1)$, respectively $H_{\mathrm{GL}(n'+1)} \subset \mathrm{GL}(n'+1)$, be the set of diagonal matrices in $\mathrm{GL}(n+1)$, respectively $\mathrm{GL}(n'+1)$. Since the character group $\chi(H_{\mathrm{GL}(n'+1)})$ is isomorphic to $\mathbb{Z}^{n'+1}$, the matrix A gives a homomorphism $H_{\mathrm{GL}(n'+1)} \to H_{\mathrm{GL}(n+1)}$ defined by

$$h \mapsto \mathrm{diag}(h^{\boldsymbol{a}_1}, h^{\boldsymbol{a}_2}, \dots, h^{\boldsymbol{a}_{n+1}}) \in H_{\mathrm{GL}(n+1)}. \tag{2-1}$$

Here, for $h = \mathrm{diag}(h_1, \dots, h_{n'+1}) \in H_{\mathrm{GL}(n'+1)}$ and $\boldsymbol{a} = (\boldsymbol{a}(1), \dots, \boldsymbol{a}(n'+1)) \in \mathbb{Z}^{n'+1}$, $h^{\boldsymbol{a}} := h_1^{\boldsymbol{a}_1} \cdots h_{n'+1}^{\boldsymbol{a}(n'+1)}$. By composing the canonical projection $\Upsilon : B_{\mathrm{GL}(n'+1)} \to H_{\mathrm{GL}(n'+1)}$ with the homomorphism (2-1), we define the homomorphism

$$\Lambda(A) : B_{\mathrm{GL}(n'+1)} \to H_{\mathrm{GL}(n+1)}$$

associated to the matrix $A \in M_{n+1,n'+1}(\mathbb{Z})$:

$$\Lambda(A)(b) := \mathrm{diag}(\Upsilon(b)^{\boldsymbol{a}_1}, \Upsilon(b)^{\boldsymbol{a}_2}, \dots, \Upsilon(b)^{\boldsymbol{a}_{n+1}}) \in H_{\mathrm{GL}(n+1)} \quad \text{for } b \in B_{\mathrm{GL}(n'+1)}. \tag{2-2}$$

Now, let $n_1, \dots, n_m \in \mathbb{Z}_{>0}$. Then, for a given sequence of integer matrices

$$\mathcal{A} := (A_\ell^{(j)})_{1\le \ell < j \le m} \in \prod_{1\le \ell<j\le m} M_{n_j+1,n_\ell+1}(\mathbb{Z}), \tag{2-3}$$

we define a right action $\Phi_j^{\mathcal{A}}$ of $\prod_{\ell=1}^{j} B_{\mathrm{GL}(n_\ell+1)}$ on $\prod_{\ell=1}^{j} \mathrm{GL}(n_\ell+1)$ by

$$\begin{aligned}
\text{(2-4)}\quad &\Phi_j^{\mathcal{A}}((g_1, g_2, \dots, g_j), (b_1, b_2, \dots, b_j)) \\
&:= \Big(g_1 b_1, \big(\Lambda_1^{(2)}(b_1)\big)^{-1} g_2 b_2, \big(\Lambda_1^{(3)}(b_1)\big)^{-1}\big(\Lambda_2^{(3)}(b_2)\big)^{-1} g_3 b_3, \dots, \\
&\qquad \big(\Lambda_1^{(j)}(b_1)\big)^{-1}\big(\Lambda_2^{(j)}(b_2)\big)^{-1} \cdots \big(\Lambda_{j-1}^{(j)}(b_{j-1})\big)^{-1} g_j b_j\Big)
\end{aligned}$$

for $1 \le j \le m$, where $\Lambda_\ell^{(j)} := \Lambda(A_\ell^{(j)})$ is the homomorphism $B_{\mathrm{GL}(n_\ell+1)} \to H_{\mathrm{GL}(n_j+1)}$ associated to the matrix $A_\ell^{(j)}$ as defined in (2-2) for $1 \le \ell < j \le m$.

Example 2.5. For $n_1 = 2$, $n_2 = 1$, $n_3 = 1$, consider the following matrices:

$$A_1^{(2)} = \begin{bmatrix} c_1 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1^{(3)} = \begin{bmatrix} d_1 & d_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2^{(3)} = \begin{bmatrix} f_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the right action of $B_{\mathrm{GL}(3)} \times B_{\mathrm{GL}(2)} \times B_{\mathrm{GL}(2)}$ on $\mathrm{GL}(3) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$ defined in (2-4) is

$$\begin{aligned}
&(g_1, g_2, g_3) \cdot (b_1, b_2, b_3) \\
&\qquad = \big(g_1 b_1, \mathrm{diag}(b_1^{(-c_1,-c_2,0)}, 1) g_2 b_2, \mathrm{diag}(b_1^{(-d_1,-d_2,0)} b_2^{(-f_1,0)}, 1) g_3 b_3\big).
\end{aligned}$$

Lemma 2.6. *The right action $\Phi_j^{\mathcal{A}}$ in (2-4) is free and proper for $1 \le j \le m$.*

Proof. For $g := (g_1, \dots, g_j) \in \prod_{\ell=1}^{j} \mathrm{GL}(n_\ell+1)$ and $(b_1, \dots, b_j) \in \prod_{\ell=1}^{j} B_{\mathrm{GL}(n_\ell+1)}$, the equality $g_1 = g_1 b_1$ implies that b_1 is the identity matrix since g_1 is invertible. Similarly, the equation $g_2 = (\Lambda_1^{(2)}(b_1))^{-1} g_2 b_2 = g_2 b_2$ gives that b_2 is the identity. Continuing in this manner, we conclude that the isotropy subgroup at g is trivial, this shows that the action $\Phi_j^{\mathcal{A}}$ is free.

To prove the properness of the action, it is enough to show that for every sequence $(g^r) := (g_1^r, \dots, g_j^r)$ in $\prod_{\ell=1}^{j} \mathrm{GL}(n_\ell+1)$ and $(b^r) := (b_1^r, \dots, b_j^r)$ in $\prod_{\ell=1}^{j} B_{\mathrm{GL}(n_\ell+1)}$ such that both (g^r) and $(\Phi_j^{\mathcal{A}}(g^r, b^r))$ converge, a subsequence of (b^r) converges (see [Lee 2013, Proposition 21.5]). Note that for convergent sequences $(A^r) \to A$ and $(B^r) \to B$ in $\mathrm{GL}(n+1)$, the sequence $(A^r B^r)$ also converges to AB since the multiplication map is continuous. Also for a convergent sequence $(A^r) \to A$ in $\mathrm{GL}(n+1)$, we have that $A_{ij} = \lim_{r\to\infty} (A^r)_{ij}$. Since both sequences (g_1^r) and $(g_1^r b_1^r)$ converge, the sequence (b_1^r) also converges in $B_{\mathrm{GL}(n_1+1)}$. Similarly, sequences $\big((\Lambda_1^{(2)}(b_1^r))^{-1} g_2^r b_2^r\big)$, (g_2^r) and (b_1^r) converge so that the sequence (b_2^r) also converges. By continuing this process, we show that the action $\Phi_j^{\mathcal{A}}$ is proper. □

For a complex manifold M with a free and proper action of a group G, the orbit space M/G is a complex manifold (see [Huybrechts 2005, Proposition 2.1.13]).

Hence by Lemma 2.6, the orbit space

$$F_j^{\mathrm{quo}}(\mathcal{A}) := \prod_{\ell=1}^{j} \mathrm{GL}(n_\ell+1)/\Phi_j^{\mathcal{A}} \tag{2-5}$$

is a complex manifold, where $\Phi_j^{\mathcal{A}}$ is the action defined in (2-4).

For the remaining part of this subsection, we will prove that the orbit spaces $F_j^{\mathrm{quo}}(\mathcal{A})$ are flag Bott manifolds. Since $\chi\big(\prod_{\ell=1}^{j} H_{\mathrm{GL}(n_\ell+1)}\big) \cong \bigoplus_{\ell=1}^{j} \mathbb{Z}^{n_\ell+1}$, for each integer vector $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_j) \in \bigoplus_{\ell=1}^{j} \mathbb{Z}^{n_\ell+1}$ we can define a holomorphic line bundle over F_j^{quo} as follows:

$$\xi(\boldsymbol{a}_1, \dots, \boldsymbol{a}_j) := \Big(\prod_{\ell=1}^{j} \mathrm{GL}(n_\ell+1) \times \mathbb{C}\Big)\Big/\prod_{\ell=1}^{j} B_{\mathrm{GL}(n_\ell+1)}, \tag{2-6}$$

where the right action is

$$(g_1, \dots, g_j, v) \cdot (b_1, \dots, b_j) := \big(\Phi_j^{\mathcal{A}}((g_1, \dots, g_j), (b_1, \dots, b_j)), b_1^{-\boldsymbol{a}_1} \cdots b_j^{-\boldsymbol{a}_j} v\big).$$

Proposition 2.7. $F_\bullet^{\mathrm{quo}}(\mathcal{A}) := \{F_j^{\mathrm{quo}}(\mathcal{A}) \mid 0 \le j \le m\}$ *is a flag Bott tower of height* m.

Definition 2.8. We say that *a flag Bott tower* $F_\bullet$ *is determined by a sequence of matrices* $\mathcal{A} = (A_\ell^{(j)})_{1 \le \ell < j \le m} \in \prod_{1 \le \ell < j \le m} M_{n_j+1, n_\ell+1}(\mathbb{Z})$ if $F_\bullet$ is isomorphic to $F_\bullet^{\mathrm{quo}}(\mathcal{A}) = \{F_j^{\mathrm{quo}}(\mathcal{A}) \mid 0 \le j \le m\}$ as flag Bott towers.

Note that, in the next section, we will show that every flag Bott towers can be described as an orbit space, that is, every flag Bott tower is determined by a certain $\mathcal{A}$ (see Theorem 2.10).

Proof of Proposition 2.7. By the definition of the action $\Phi_j^{\mathcal{A}}$, we have the following fibration structure:

$$\mathrm{GL}(n_j+1)/B_{\mathrm{GL}(n_j+1)} \hookrightarrow F_j^{\mathrm{quo}} \to F_{j-1}^{\mathrm{quo}}.$$

Since $\mathrm{GL}(n_j+1)/B_{\mathrm{GL}(n_j+1)} \cong \mathcal{F}\ell(n_j+1)$, the manifold F_j^{quo} is a $\mathcal{F}\ell(n_j+1)$-fibration over F_{j-1}^{quo}. For simplicity, let $\xi^{(j)} := \bigoplus_{k=1}^{n_j+1} \xi(\boldsymbol{a}_{k,1}^{(j)}, \dots, \boldsymbol{a}_{k,j-1}^{(j)})$, where $\boldsymbol{a}_{k,\ell}^{(j)}$ is the k-th row vector of the matrix $A_\ell^{(j)}$ for $1 \le \ell \le j-1$. Consider the map $\varphi_j : F_j^{\mathrm{quo}} \to \mathcal{F}\ell(\xi^{(j)})$ defined by

$$[g_1, \dots, g_{j-1}, g_j] \mapsto ([g_1, \dots, g_{j-1}], V_\bullet).$$

Here $V_\bullet = \big(V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n_j} \subsetneq (\xi^{(j)})_{[g_1, \dots, g_{j-1}]}\big)$ is the full flag such that the vector space V_k is spanned by the first k columns of g_j. We claim that φ_j is a holomorphic diffeomorphism. First, we check that the map φ_j is well-defined. We observe that

$$\begin{aligned}&\bigl[\Phi_j^{\mathcal{A}}\bigl((g_1,\dots,g_{j-1},g_j),(b_1,\dots,b_{j-1},b_j)\bigr)\bigr]\\ &\qquad\mapsto \bigl([\Phi_{j-1}^{\mathcal{A}}\bigl((g_1,\dots,g_{j-1}),(b_1,\dots,b_{j-1})\bigr)], V'_\bullet\bigr) = ([g_1,\dots,g_{j-1}], V'_\bullet)\end{aligned}$$

for $(b_1,\dots,b_{j-1},b_j)\in\prod_{\ell=1}^{j-1}B_{\mathrm{GL}(n_\ell+1)}\times B_{\mathrm{GL}(n_j+1)}$. Here

$$V'_\bullet = \bigl(V'_1 \subsetneq V'_2 \subsetneq \cdots \subsetneq V'_{n_j} \subsetneq (\xi^{(j)})_{[g_1,\dots,g_{j-1}]}\bigr)$$

is the full flag whose vector space V'_k is spanned by the first k columns of the matrix $(\Lambda_1^{(j)}(b_1))^{-1}\cdots(\Lambda_{j-1}^{(j)}(b_{j-1}))^{-1}g_j$. Since we have

$$(\Lambda_1^{(j)}(b_1))^{-1}\cdots(\Lambda_{j-1}^{(j)}(b_{j-1}))^{-1}\boldsymbol{v}\sim\boldsymbol{v}\quad\text{for } \boldsymbol{v}\in(\xi^{(j)})_{[g_1,\dots,g_{j-1}]}, \tag{2-7}$$

the map φ_j is well-defined. Here the equivalence relation $\sim$ comes from the definition of the bundle $\xi^{(j)}$.

The inverse $\mathcal{F}\ell(\xi^{(j)})\to F_j^{\mathrm{quo}}$ of φ_j is given by

$$([g_1,\dots,g_j], V_\bullet)\mapsto[g_1,\dots,g_{j-1},g_j],$$

where g_j is the matrix such that the first k columns span the vector space V_k for $1\le k\le n_j+1$. Note that this map is again well-defined since $[g_1,\dots,g_{j-1},g_j]=[g_1,\dots,g_{j-1},g_jb_j]$ for $b_j\in B_{\mathrm{GL}(n_j+1)}$. Hence the map φ_j is a diffeomorphism, and the result follows since φ_j commutes with bundle projection maps. □

Example 2.9. For $n_1=2,\ n_2=1$, and $n_3=1$, let

$$A_1^{(2)}=\begin{bmatrix}c_1&c_2&0\\0&0&0\end{bmatrix},\quad A_1^{(3)}=\begin{bmatrix}d_1&d_2&0\\0&0&0\end{bmatrix},\quad A_2^{(3)}=\begin{bmatrix}f_1&0\\0&0\end{bmatrix}.$$

Let $\Phi_j^{\mathcal{A}}$ be the right action of $\prod_{\ell=1}^{j}B_{\mathrm{GL}(n_\ell+1)}$ on $\prod_{\ell=1}^{j}\mathrm{GL}(n_\ell+1)$ defined in (2-4) for $j=1,2,3$. Then, by Proposition 2.7, the following flag Bott tower $F_\bullet$ is isomorphic to $F_\bullet^{\mathrm{quo}}(\mathcal{A})$ as flag Bott towers:

$$\begin{tikzcd}
 & \xi((d_1,d_2,0),(f_1,0))\oplus\underline{\mathbb{C}} \arrow[d] & \xi(c_1,c_2,0)\oplus\underline{\mathbb{C}} \arrow[d] & \\
\mathcal{F}\ell\bigl(\xi((d_1,d_2,0),(f_1,0))\oplus\underline{\mathbb{C}}\bigr) \arrow[r] \arrow[d, equal] & \mathcal{F}\ell(\xi(c_1,c_2,0)\oplus\underline{\mathbb{C}}) \arrow[r] \arrow[d, equal] & \mathcal{F}\ell(3) \arrow[r] \arrow[d, equal] & \{\text{a point}\} \arrow[d, equal] \\
F_3 & F_2 & F_1 & F_0
\end{tikzcd}$$

The line bundle $\xi((d_1,d_2,0),(f_1,0))$ over F_2 is

$$(\mathrm{GL}(3)\times\mathrm{GL}(2)\times\mathbb{C})/(B_{\mathrm{GL}(3)}\times B_{\mathrm{GL}(2)}),$$

where the right action of $B_{\mathrm{GL}(3)}\times B_{\mathrm{GL}(2)}$ is

$$(g_1,g_2,v)\cdot(b_1,b_2):=\bigl(\Phi_2^{\mathcal{A}}((g_1,g_2),(b_1,b_2)),\,b_1^{-(d_1,d_2,0)}b_2^{-(f_1,0)}v\bigr).$$

2C. ***Tautological filtration over a flag Bott manifold.*** In this subsection, we prove the following theorem that every flag Bott tower $F_\bullet$ can be obtained by the orbit space construction as in Section 2B.

Theorem 2.10. *Let $F_\bullet$ be a flag Bott tower of height m. Then there is a sequence of integer matrices $\mathcal{A} = (A_\ell^{(j)})_{1\le\ell<j\le m} \in \prod_{1\le\ell<j\le m} M_{n_j+1,n_\ell+1}(\mathbb{Z})$ such that $F_\bullet$ is isomorphic to $F_\bullet^{\mathrm{quo}}(\mathcal{A}) := \{F_j^{\mathrm{quo}}(\mathcal{A}) \mid 0 \le j \le m\}$ as flag Bott towers.*

By the above theorem, for any flag Bott tower $F_\bullet$ there exists a set $\mathcal{A}$ satisfying that $F_\bullet$ *is determined by* $\mathcal{A}$ (see Definition 2.8). To give a proof of Theorem 2.10, we begin with studying holomorphic line bundles over a flag Bott manifold. For $1 \le j \le m$, there is a *universal* or *tautological* filtration of subbundles

$$0 = U_{j,0} \subset U_{j,1} \subset U_{j,2} \subset \cdots \subset U_{j,n_j} \subset U_{j,n_j+1} = p_j^*\xi^{(j)} \tag{2-8}$$

on $F_j = \mathcal{F}\ell(\xi^{(j)})$, where we put $\xi^{(j)} := \bigoplus_{k=1}^{n_j+1} \xi_k^{(j)}$ for simplicity. Over a point

$$(p, V_\bullet) = \bigl(p, (V_0 \subset V_1 \subset \cdots \subset V_{n_j} \subset (\xi^{(j)})_p)\bigr)$$

of F_j for $p \in F_{j-1}$, the fiber of the subbundle $U_{j,k}$ is the vector space V_k of the flag $V_\bullet$ for $1 \le k \le n_j + 1$. Hence we have the quotient line bundle $U_{j,k}/U_{j,k-1}$ over F_j for $1 \le k \le n_j + 1$. The following lemma states that using these line bundles, we can express any holomorphic line bundle over a flag Bott manifold.

Lemma 2.11. *Let $F_\bullet$ be a flag Bott tower. Then the set of line bundles*

$$\bigl\{U_{j,k}/U_{j,k-1} \mid 1 \le k \le n_j + 1\bigr\} \cup \bigcup_{\ell=1}^{j-1} \bigl\{p_j^* \circ \cdots \circ p_{\ell+1}^*(U_{\ell,k}/U_{\ell,k-1}) \mid 1 \le k \le n_\ell + 1\bigr\}$$

generates the Picard group $\mathrm{Pic}(F_j)$ *for* $1 \le j \le m$.

Proof. Using the result [Bott and Tu 1982, Remark 21.18] on the cohomology ring of the induced flag bundle and an induction on the stage of $F_\bullet$, one can see that the degree 2 cohomology group $H^2(F_j; \mathbb{Z})$ is generated by the first Chern classes of line bundles

$$\bigl\{U_{j,k}/U_{j,k-1} \mid 1 \le k \le n_j + 1\bigr\} \cup \bigcup_{\ell=1}^{j-1} \bigl\{p_j^* \circ \cdots \circ p_{\ell+1}^*(U_{\ell,k}/U_{\ell,k-1}) \mid 1 \le k \le n_\ell + 1\bigr\}$$

for $1 \le j \le m$. Therefore, any cohomology class of degree 2 can be obtained as the first Chern class of a tensor product of these line bundles. Hence it is enough to show that the cycle map $c_1 : \mathrm{Pic}(F_j) \to H^2(F_j; \mathbb{Z})$ is an isomorphism. We recall that the cycle map $\mathrm{Pic}(X) \to H^2(X; \mathbb{Z})$ is an isomorphism for a full flag manifold X. Also for the full flag bundle X over a smooth variety Y, if the cycle map for Y is an isomorphism, then the cycle map for X is also an isomorphism (see [Fulton 1998, Example 19.1.11]). This proves that the cycle map $c_1 : \mathrm{Pic}(F_j) \to H^2(F_j; \mathbb{Z})$ is an isomorphism for $1 \le j \le m$. □

Lemma 2.12. *For a sequence of integer matrices*

$$\mathcal{A} = (A_\ell^{(j)})_{1\le\ell<j\le m} \in \prod_{1\le\ell<j\le m} M_{n_j+1,n_\ell+1}(\mathbb{Z}),$$

let $F_\bullet^{\mathrm{quo}} := F_\bullet^{\mathrm{quo}}(\mathcal{A})$ *be the flag Bott tower defined as in* (2-5). *Then, the line bundle* $U_{j,k}/U_{j,k-1} \to F_j^{\mathrm{quo}}$ *is isomorphic to the bundle* $\xi(\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{e}_k) \to F_j^{\mathrm{quo}}$ *defined in* (2-6).

Proof. From Proposition 2.7 that the j-stage flag Bott manifold F_j^{quo} is the induced flag bundle $\mathcal{F}\ell(\xi^{(j)})$ over F_{j-1}^{quo}, where $\xi^{(j)} = \bigoplus_{k=1}^{n_j+1} \xi(\boldsymbol{a}_{k,1}^{(j)}, \dots, \boldsymbol{a}_{k,j-1}^{(j)})$ and $\boldsymbol{a}_{k,\ell}^{(j)}$ is the k-th row vector of the matrix $A_\ell^{(j)}$ for $1 \le \ell \le j-1$. Consider a point $g = [g_1, \dots, g_j]$ in F_j^{quo}. Because of the bundle structure $F_j^{\mathrm{quo}} = \mathcal{F}\ell(\xi^{(j)}) \xrightarrow{p_j} F_{j-1}^{\mathrm{quo}}$, this point g can be considered as a full flag $V_\bullet = \big(V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n_j} \subsetneq (\xi^{(j)})_{p_j(g)}\big)$, where $(\xi^{(j)})_{p_j(g)}$ is the fiber over $p_j(g)$. The fiber of $U_{j,k}$ at g is the vector space $V_k \subset (\xi^{(j)})_{p_j(g)}$ spanned by the first k column vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in (\xi^{(j)})_{p_j(g)}$ of $g_j \in \mathrm{GL}(n_j+1)$. Hence the fiber of $U_{j,k}/U_{j,k-1}$ at g is V_k/V_{k-1}, which is spanned by the vector $\boldsymbol{v}_k \in (\xi^{(j)})_{p_j(g)}$. For an element $b = (b_1, \dots, b_j) \in \prod_{\ell=1}^{j} B_{\mathrm{GL}(n_j)}$, the k-th column vector $\boldsymbol{v}_k'$ of the last coordinate of $\Phi_j^{\mathcal{A}}(g, b)$ is given by

$$\begin{aligned}\boldsymbol{v}_k' &= \big((\Lambda_1^{(j)}(b_1))^{-1} \cdots (\Lambda_{j-1}^{(j)}(b_{j-1}))^{-1}\boldsymbol{v}_k\big) b_j^{\boldsymbol{e}_k}\\ &= b_j^{\boldsymbol{e}_k}(\Lambda_1^{(j)}(b_1))^{-1} \cdots (\Lambda_{j-1}^{(j)}(b_{j-1}))^{-1}\boldsymbol{v}_k \sim b_j^{\boldsymbol{e}_k}\boldsymbol{v}_k.\end{aligned}$$

Here, the equivalence comes from (2-7). Hence the result follows. □

Proof of Theorem 2.10. Using the above two lemmas, we prove the theorem using the induction argument on the height of a flag Bott tower. When the height is 1, then it is obvious that any full flag manifold can be described as the orbit space $\mathrm{GL}(n_1+1)/B_{\mathrm{GL}(n_1+1)}$.

Assume that the theorem holds for flag Bott towers whose height is less than m. For a flag Bott tower $F_\bullet$ of height m, by the induction hypothesis, we have a sequence of integer matrices $(A_\ell^{(j)})_{1\le\ell<j\le m-1} \in \prod_{1\le\ell<j\le m-1} M_{n_j+1,n_\ell+1}(\mathbb{Z})$ such that $\{F_j \mid 0 \le j \le m-1\}$ is isomorphic to the orbit spaces $\{F_j^{\mathrm{quo}} \mid 0 \le j \le m-1\}$ as flag Bott towers. To prove the claim, it is enough to find suitable integer matrices $A_1^{(m)}, \dots, A_{m-1}^{(m)}$ such that $(A_\ell^{(j)})_{1\le\ell<j\le m}$ gives the flag Bott manifold F_m.

Let $F_m = \mathcal{F}\ell\big(\bigoplus_{k=1}^{n_m+1} \xi_k^{(m)}\big)$, where $\xi_k^{(m)}$ is a holomorphic line bundle over F_{m-1}. Then, by the induction hypothesis, the $(m-1)$-stage flag Bott manifold F_{m-1} can be expressed as the orbit $\prod_{\ell=1}^{m-1} \mathrm{GL}(n_\ell+1)/\Phi_{m-1}^{\mathcal{A}}$. Hence, by Lemmas 2.11 and 2.12, there exists a suitable integer vector $(\boldsymbol{a}_{k,1}^{(m)}, \dots, \boldsymbol{a}_{k,m-1}^{(m)}) \in \bigoplus_{\ell=1}^{m-1} \mathbb{Z}^{n_\ell+1}$ such that

$$\xi(\boldsymbol{a}_{k,1}^{(m)}, \dots, \boldsymbol{a}_{k,m-1}^{(m)}) = \xi_k^{(m)} \quad \text{for } 1 \le k \le n_m+1.$$

Consider the integer matrix $A_\ell^{(m)} \in M_{n_m+1,n_\ell+1}(\mathbb{Z})$ with row vectors $\boldsymbol{a}_{1,\ell}^{(m)}, \dots, \boldsymbol{a}_{n_m+1,\ell}^{(m)}$ for $1 \le \ell \le m-1$. Let F_m^{quo} be the flag Bott manifold determined by integer matrices $(A_\ell^{(j)})_{1\le \ell<j\le m}$. Then by Proposition 2.7, we have the following bundle map φ which is a holomorphic diffeomorphism:

$$\varphi : F_m^{\text{quo}} \to \mathcal{F}\ell\Bigg(\bigoplus_{k=1}^{n_m+1} \xi(\boldsymbol{a}_{k,1}^{(m)}, \dots, \boldsymbol{a}_{k,m-1}^{(m)})\Bigg) = F_m. \qquad \square$$

Remark 2.13 (description of F_m using compact Lie groups). Using the compact subgroups $\mathrm{U}(n_j+1) \subset \mathrm{GL}(n_j+1)$ and the compact maximal torus $T^{n_j+1} \subset H_{\mathrm{GL}(n_j+1)}$ for $1 \le j \le m$, consider the orbit space:

$$\prod_{j=1}^m U(n_j+1) \Big/ \prod_{j=1}^m T^{n_j+1},$$

where the right action is defined by

$$\begin{aligned}(2\text{-}9)\quad (g_1, \dots, g_m) \cdot (t_1, \dots, t_m) &\\ = \Big(g_1t_1, (\Lambda_1^{(2)}(t_1))^{-1}g_2t_2, (\Lambda_1^{(3)}(t_1))^{-1}&(\Lambda_2^{(3)}(t_2))^{-1}g_3t_3, \dots,\\ (\Lambda_1^{(m)}(t_1))^{-1}(\Lambda_2^{(m)}(t_2))^{-1} &\cdots (\Lambda_{m-1}^{(m)}(t_{m-1}))^{-1}g_mt_m\Big).\end{aligned}$$

Then the above manifold is a compact manifold which is diffeomorphic to F_m since $\mathrm{U}(n+1)/T^{n+1}$ is diffeomorphic to $\mathrm{GL}(n+1)/B_{\mathrm{GL}(n+1)}$. We will also use this description for F_m.

Remark 2.14. Let F_m be the m-stage flag Bott manifold defined by a sequence of integer matrices $\mathcal{A} = (A_\ell^{(j)})_{1\le\ell<j\le m-1} \in \prod_{1\le\ell<j\le m-1} M_{n_j+1,n_\ell+1}(\mathbb{Z})$. Every flag Bott manifold is a $\mathbb{C}P$-tower. Hence using Borel–Hirzebruch formula, the cohomology ring and the equivariant cohomology ring with respect to the torus action defined in Section 3A of F_m can be computed. The explicit formula is given in [Kaji et al. 2020] in terms of $\mathcal{A}$.

2D. *Tangent bundle of F_m.* In this subsection, we study the tangent bundle of a flag Bott manifold using a principal connection of a principal bundle. For more details, see [Spivak 1979, Chapter 8, Addendum 3]. For a principal H-bundle $\pi : P \to B$, the *vertical subbundle* $\mathcal{V}$ is defined to be $\mathcal{V} := \{v \in TP \mid \pi_* v = 0\} \subset TP$. If we let $o_p : H \to H(p)$ be the orbit map which maps H onto its orbit through $p \in P$, then we have

$$(2\text{-}10)\qquad \mathcal{V}_p = (o_p)_*\mathrm{Lie}(H).$$

A *principal connection* $\mathcal{H}$ is a subbundle of TP such that for $p \in P$,

- $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$,

- $(\Phi_h)_*\mathcal{H}_p = \mathcal{H}_{\Phi_h(p)}$, where Φ_h is the right action by $h \in H$, and
- $\mathcal{H}_p$ varies smoothly with respect to $p \in P$.

Because of the first property of principal connection, we have that $\pi_*(\mathcal{H}_p) = T_{\pi(p)}B$.

For convenience, let $\mathbb{T}$ denote the product of compact tori $\prod_{j=1}^m T^{n_j+1}$. By Remark 2.13, an m-stage flag Bott manifold F_m can be described as the orbit of the right action in (2-9), i.e., $F_m = \prod_{j=1}^m \mathrm{U}(n_j+1)/\mathbb{T}$. Since $\mathbb{T}$ acts freely on the space $\prod_{j=1}^m \mathrm{U}(n_j+1)$, we have the principal $\mathbb{T}$-bundle

$$\prod_{j=1}^m \mathrm{U}(n_j+1) \xrightarrow{\pi} F_m. \tag{2-11}$$

We describe the vertical subbundle $\mathcal{V}$ of the above principal bundle (2-11). For $1 \leq j \leq m$, let $\mathfrak{u}(n_j+1)$, respectively $\mathfrak{t}^{n_j+1}$, denote the Lie algebra of $\mathrm{U}(n_j+1)$, respectively $T^{n_j+1} \subset \mathrm{U}(n_j+1)$. For a point $g = (g_1, \dots, g_m) \in \prod_{j=1}^m \mathrm{U}(n_j+1)$, define

$$(L_{g^{-1}})_* := (L_{g_1^{-1}})_* \times \cdots \times (L_{g_m^{-1}})_* : T_g\Bigl(\prod_{j=1}^m \mathrm{U}(n_j+1)\Bigr) \to \bigoplus_{j=1}^m \mathfrak{u}(n_j+1),$$

where L_{g_j} is the left translation by g_j for $1 \leq j \leq m$. Then $(L_{g^{-1}})_*$ is an isomorphism, so that we have the trivialization

$$\prod_{j=1}^m \mathrm{U}(n_j+1) \times \bigoplus_{j=1}^m \mathfrak{u}(n_j+1) \cong T\Bigl(\prod_{j=1}^m \mathrm{U}(n_j+1)\Bigr).$$

For the principal bundle (2-11), it follows from (2-10) that $\mathcal{V}_g = (o_g)_*\bigl(\bigoplus_{j=1}^m \mathfrak{t}^{n_j+1}\bigr)$, where $o_g : \mathbb{T} \to \mathbb{T}(g)$ is the orbit map. For a given $\underline{t} = (\underline{t}_1, \dots, \underline{t}_m) \in \bigoplus_{j=1}^m \mathfrak{t}^{n_j+1}$, take a path

$$\gamma : (-\varepsilon, \varepsilon) \to \prod_{j=1}^m T^{n_j+1}, \quad s \mapsto (t_1(s), \dots, t_m(s))$$

such that $\gamma(0) = \mathbf{1}$, $t_j(s) \in T^{n_j+1}$ and $\frac{d}{ds}\gamma(s)|_{s=0} = \underline{t}$. For a point $g \in \prod_{j=1}^m \mathrm{U}(n_j+1)$ and $\boldsymbol{t} \in \mathbb{T}$, let $g \cdot \boldsymbol{t}$ denote the right action of $\mathbb{T}$ in (2-9). Then we have

$$\begin{aligned}
&(L_{g^{-1}})_*(o_g)_*\underline{t}\\
&= \frac{d}{ds} L_{g^{-1}}(g \cdot \gamma(s))|_{s=0}\\
&= \frac{d}{ds}\Bigl(t_1(s), g_2^{-1}\bigl(\Lambda_1^{(2)}(t_1(s))\bigr)^{-1} g_2 t_2(s),\\
&\qquad\qquad \dots, g_m^{-1}\bigl(\Lambda_1^{(m)}(t_1(s))\bigr)^{-1} \cdots \bigl(\Lambda_{m-1}^{(m)}(t_{m-1}(s))\bigr)^{-1} g_m t_m(s)\Bigr)\Big|_{s=0}\\
&= \bigl(\underline{t}_1, \underline{t}_2 - \mathrm{Ad}_{g_2^{-1}}(A_1^{(2)}(\underline{t}_1)), \dots, \underline{t}_m - \mathrm{Ad}_{g_m^{-1}}(A_1^{(m)}(\underline{t}_1) + \cdots + A_{m-1}^{(m)}(\underline{t}_{m-1}))\bigr).
\end{aligned}$$

Here $\mathrm{Ad}_g(X) = gXg^{-1}$, i.e., the usual adjoint representation of $\mathrm{U}(n_j+1)$ on $\mathfrak{u}(n_j+1)$. Therefore we see that the vertical subbundle $\mathcal{V}$ is the image of the injective map:

$$\prod_{j=1}^{m} \mathrm{U}(n_j+1) \times \bigoplus_{j=1}^{m} \mathfrak{t}^{n_j+1} \to \prod_{j=1}^{m} \mathrm{U}(n_j+1) \times \bigoplus_{j=1}^{m} \mathfrak{u}(n_j+1),$$

where $\big((g_1,\dots,g_m),(\underline{t}_1,\dots,\underline{t}_m)\big)$ maps to

$$\Big((g_1,\dots,g_m),\\ \big(\underline{t}_1, \underline{t}_2 - \mathrm{Ad}_{g_2^{-1}}(A_1^{(2)}(\underline{t}_1)),\dots,\underline{t}_m - \mathrm{Ad}(g_m^{-1})\big(A_1^{(m)}(\underline{t}_1)+\cdots+A_{m-1}^{(m)}(\underline{t}_{m-1})\big)\big)\Big).$$

Now we describe a principal connection. Let $\mathfrak{m}_j \subset \mathfrak{u}(n_j+1)$ be the subspace of matrices with the zeros along the diagonal. Then $\mathfrak{m}_j$ is invariant under the adjoint action of T^{n_j+1}, and $\mathfrak{m}_j \cap \mathfrak{t}^{n_j+1} = \{0\}$.

Proposition 2.15. *At the point* $e := (e,\dots,e) \in \prod_{j=1}^m \mathrm{U}(n_j+1)$, *choose the horizontal space* $\mathcal{H}_e := \bigoplus_{j=1}^m \mathfrak{m}_j \subset \bigoplus_{j=1}^m \mathfrak{u}(n_j+1)$. *For a point* $g = (g_1,\dots,g_m) \in \prod_{j=1}^m \mathrm{U}(n_j+1)$, *define* $\mathcal{H}_g \subset T_g\big(\prod_{j=1}^m \mathrm{U}(n_j+1)\big)$ *by*

$$\mathcal{H}_g := \bigoplus_{j=1}^{m} (L_{g_j})_* \mathfrak{m}_j.$$

Then $\mathcal{H}$ *is a connection.*

Proof. First we need to show that for each point $g \in \prod_{j=1}^m \mathrm{U}(n_j+1)$, we have that $\mathcal{H}_g \oplus \mathcal{V}_g = T_g\big(\prod_{j=1}^m \mathrm{U}(n_j+1)\big)$. We claim that $\mathcal{V}_g \cap \mathcal{H}_g = \{0\}$. Suppose that $(o_g)_*(\underline{t})$ is contained in $\mathcal{H}_g$ for some $\underline{t} = (\underline{t}_1,\dots,\underline{t}_m) \in \bigoplus_{j=1}^m \mathfrak{t}^{n_j+1}$. This implies that

$$\big(\underline{t}_1, \underline{t}_2 - \mathrm{Ad}_{g_2^{-1}}(A_1^{(2)}(\underline{t}_1)),\dots,\underline{t}_m - \mathrm{Ad}_{g_m^{-1}}\big(A_1^{(m)}(\underline{t}_1)+\cdots+A_{m-1}^{(m)}(\underline{t}_{m-1})\big)\big) \in \bigoplus_{j=1}^{m} \mathfrak{m}_j.$$

In particular, $\underline{t}_1 \in \mathfrak{m}_1$, but it is also contained in $\mathfrak{t}^{n_1+1}$. Since $\mathfrak{m}_1 \cap \mathfrak{t}^{n_1+1} = \{0\}$, we have that $\underline{t}_1 = 0$. Continuing in this manner we conclude that $\mathcal{V}_g \cap \mathcal{H}_g = \{0\}$, and hence by the dimension reason, we have $\mathcal{H}_g \oplus \mathcal{V}_g = T_g\big(\prod_{j=1}^m \mathrm{U}(n_j+1)\big)$.

Secondly, define the map

$$\Phi_{\boldsymbol{t}} : \prod_{j=1}^{m} \mathrm{U}(n_j+1) \to \prod_{j=1}^{m} \mathrm{U}(n_j+1)$$

as the right translation by $\boldsymbol{t}$ as defined in (2-9). For an element $\boldsymbol{t} = (t_1,\dots,t_m) \in \prod_{j=1}^m T^{n_j+1}$, we claim that $(\Phi_{\boldsymbol{t}})_*\mathcal{H}_g = \mathcal{H}_{\Phi_{\boldsymbol{t}}(g)}$. For any $(x_1,\dots,x_m) \in \prod_{j=1}^m \mathrm{U}(n_j+1)$, we have:

$$\begin{aligned}(2\text{-}12)\quad &(\Phi_{\boldsymbol{t}} \circ L_g)(x_1, \dots, x_m)\\ &= \Phi_{\boldsymbol{t}}(g_1x_1, \dots, g_mx_m)\\ &= \Big(g_1x_1t_1, (\Lambda_1^{(2)}(t_1))^{-1}g_2x_2t_2,\\ &\qquad\qquad \dots, (\Lambda_1^{(m)}(t_1))^{-1} \cdots (\Lambda_{m-1}^{(m)}(t_{m-1}))^{-1}g_mx_mt_m\Big)\\ &= \Big(g_1t_1(t_1^{-1}x_1t_1), (\Lambda_1^{(2)}(t_1))^{-1}g_2t_2(t_2^{-1}x_2t_2),\\ &\qquad\qquad \dots, (\Lambda_1^{(m)}(t_1))^{-1} \cdots (\Lambda_{m-1}^{(m)}(t_{m-1}))^{-1}g_mt_m(t_m^{-1}x_mt_m)\Big)\\ &= L_{\Phi_t(g)}(t_1^{-1}x_1t_1, \dots, t_m^{-1}x_mt_m).\end{aligned}$$

This gives $(\Phi_{\boldsymbol{t}})_*\mathcal{H}_g = \mathcal{H}_{\Phi_t(g)}$ since $\mathfrak{m}_j$ is invariant under the adjoint action of T^{n_j+1} for $1 \le j \le m$.

Finally since the left multiplication varies smoothly with

$$(g_1, \dots, g_m) \in \prod_{j=1}^{m} \mathrm{U}(n_j + 1),$$

this defines a connection. □

As a corollary of Proposition 2.15 we have the following description of the tangent bundle of F_m:

Proposition 2.16. *The tangent bundle of F_m is isomorphic to*

$$\prod_{j=1}^{m} \mathrm{U}(n_j + 1) \times_{\mathbb{T}} \bigoplus_{j=1}^{m} \mathfrak{m}_j,$$

where the following elements are identified:

$$\begin{aligned}(2\text{-}13)\quad &(g_1, \dots, g_m; X_1, \dots, X_m)\\ &\qquad \sim \big((g_1, \dots, g_m) \cdot (t_1, \dots, t_m); \mathrm{Ad}_{t_1^{-1}}X_1, \dots, \mathrm{Ad}_{t_m^{-1}}X_m\big)\end{aligned}$$

for $(t_1, \dots, t_m) \in \mathbb{T}$. Here $(g_1, \dots, g_m) \cdot (t_1, \dots, t_m)$ is as defined in (2-9).

Proof. Let

$$\varphi : \prod_{j=1}^{m} \mathrm{U}(n_j + 1) \times_{\mathbb{T}} \bigoplus_{j=1}^{m} \mathfrak{m}_j \to TF_m$$

be the map defined by $\varphi([g; X]) = ([g], (\pi_* \circ (L_g)_*(X)))$. We claim that the map φ is a bundle isomorphism. Because of the property of a principal connection and by the definition of $\mathcal{H}$, we have that $(\pi_* \circ (L_g)_*(X)) \in T_{[g]}F_m$. It is enough to check that the map φ is well-defined. For $\boldsymbol{t} = (t_1, \dots, t_m) \in \mathbb{T}$,

$$\varphi : \big[\Phi_{\boldsymbol{t}}(g); \operatorname{Ad}_{t_1^{-1}} X_1, \dots, \operatorname{Ad}_{t_m^{-1}} X_m\big] \\ \mapsto \big([\Phi_{\boldsymbol{t}}(g)], (\pi_* \circ (L_{\Phi_{\boldsymbol{t}}(g)})_*)(\operatorname{Ad}_{t_1^{-1}} X_1, \dots, \operatorname{Ad}_{t_m^{-1}} X_m)\big).$$

From (2-12), we can see that

$$(L_{\Phi_{\boldsymbol{t}}(g)})_*(\operatorname{Ad}_{t_1^{-1}} X_1, \dots, \operatorname{Ad}_{t_m^{-1}} X_m) = (\Phi_{\boldsymbol{t}})_* \circ (L_g)_*(X_1, \dots, X_m).$$

Because $\pi \circ \Phi_{\boldsymbol{t}} = \pi$, we have that $\pi_* \circ (\Phi_{\boldsymbol{t}})_* = \pi_*$. This implies that the map φ is well-defined. □

3. GKM descriptions of flag Bott manifolds

Let F_m be an m-stage flag Bott manifold. In Section 3A, we define the canonical torus action on F_m and by studying this action more carefully, we conclude that a flag Bott manifold F_m is a GKM manifold with the canonical action in Theorem 3.6.

3A. *Torus actions.* Let F_m be an m-stage flag Bott manifold. For $1 \le j \le m$, let $\mathbb{H} = \prod_{\ell=1}^{m} H_{\mathrm{GL}(n_\ell+1)}$ act on F_j by

$$(h_1, \dots, h_m) \cdot [g_1, \dots, g_j] := [h_1 g_1, \dots, h_j g_j]$$

for $(h_1, \dots, h_m) \in \mathbb{H}$ and $[g_1, \dots, g_j] \in F_j$. Then $F_j \to F_{j-1}$ is $\mathbb{H}$-equivariant fiber bundle. For notational convenience, we write

$$n := n_1 + \cdots + n_m. \tag{3-1}$$

Therefore $\sum_{j=1}^{m}(n_j + 1) = n + m$. Let $\mathbb{T} \subset \mathbb{H}$ be the compact torus of real dimension $n + m$. Note that the torus $\mathbb{H}$ acts holomorphically but does not act effectively on F_m. If we write $h_j = \operatorname{diag}(h_{j,1}, \dots, h_{j,n_j+1}) \in \mathrm{GL}(n_j + 1)$, the subtorus

$$\boldsymbol{H} := \{(h_1, \dots, h_m) \in \mathbb{H} \mid h_{1,n_1+1} = \cdots = h_{m,n_m+1} = 1\} \cong (\mathbb{C}^*)^n$$

acts effectively on F_m. Let $\boldsymbol{T} \subset \boldsymbol{H}$ denote the compact torus of real dimension n. In this paper, we call the action of these tori the *canonical* $\mathbb{H}$ ($\mathbb{T}$, $\boldsymbol{H}$ or $\boldsymbol{T}$)*-action* on F_m. For a space X with a G-action, we write (X, G) for this G-space X when we need to emphasize the acting group.

Remark 3.1. The complex dimension of an m-stage flag Bott manifold F_m is

$$\frac{n_1(n_1+1)}{2} + \cdots + \frac{n_m(n_m+1)}{2}$$

while the complex dimension of the torus $\boldsymbol{H}$, which acts effectively on the manifold F_m, is $n = n_1 + \cdots + n_m$. They are equal if and only if $n_1 = \cdots = n_m = 1$, which is the case when a flag Bott manifold is a Bott manifold (see Example 2.2(3)). The highest dimension of a torus which can act on F_m effectively is studied in [Kuroki 2017].

Example 3.2. A 1-stage flag Bott manifold is the flag manifold $\mathcal{F}\ell(n+1) = \mathrm{GL}(n+1)/B_{\mathrm{GL}(n+1)}$. Then the canonical torus action of $\mathbb{H} = H_{\mathrm{GL}(n+1)}$ on the flag manifold $\mathcal{F}\ell(n+1)$ is the left multiplication.

It is well known that the fixed point set $\mathcal{F}\ell(n+1)^{\mathbb{H}}$ can be identified with the symmetric group $\mathfrak{S}_{n+1}$ (see [Fulton 1997, Subsection 10.1]). For a given permutation $w \in \mathfrak{S}_{n+1}$, let $\dot{w}$ denote the column permutation matrix, i.e., $\dot{w}$ is an element in $\mathrm{GL}(n+1)$ whose $(w(k), k)$-entries are 1 for $1 \le k \le n+1$, and all others are zero. For instance, the permutation $w = (231) \in \mathfrak{S}_3$ corresponds to the matrix

$$\dot{w} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathrm{GL}(3). \tag{3-2}$$

Here we use the one-line notation, i.e., $w(1) = 2$, $w(2) = 3$, and $w(3) = 1$. Then the fixed point set is $\{[\dot{w}] \in \mathrm{GL}(n+1)/B_{\mathrm{GL}(n+1)} \mid w \in \mathfrak{S}_{n+1}\}$. This property can be extended to the canonical action of $\mathbb{H}$ on F_m.

Proposition 3.3. *Let F_m be an m-stage flag Bott manifold with the action of $\mathbb{H}$. Then the fixed point set is identified with the product of symmetric groups $\prod_{j=1}^{m} \mathfrak{S}_{n_j+1}$. More precisely, for an element $(w_1, \dots, w_m) \in \prod_{j=1}^{m} \mathfrak{S}_{n_j+1}$, the corresponding fixed point in F_m is $[\dot{w}_1, \dots, \dot{w}_m]$, where $\dot{w}_j \in \mathrm{GL}(n_j+1)$ is the column permutation matrix of w_j.*

3B. ***Tangential representations of flag Bott manifolds.*** In this subsection, we study the tangential representations of a flag Bott manifold F_m at the fixed points corresponding to the (noneffective) canonical action of $\mathbb{T}$ in Proposition 3.5. Recall the definition of GKM manifolds from [Goresky et al. 1998; Guillemin and Zara 2001].

Definition 3.4. Let T be the compact torus of dimension n, $\mathfrak{t}$ its Lie algebra, and M a compact manifold of real dimension $2d$ with an effective action of T. We say that a pair (M, T) is a *GKM manifold* if

(1) the fixed point set M^T is finite,

(2) M possesses a T-invariant almost-complex structure, and

(3) for every $p \in M^T$, the weights $\{\alpha_{i,p} \in \mathfrak{t}^* \mid 1 \le i \le d\}$ of the isotropy representation T_pM of T are pairwise linearly independent.

By considering the effective canonical action of $\boldsymbol{T}$ on F_m, we will see that $(F_m, \boldsymbol{T})$ is a GKM manifold in Theorem 3.6. For this, we first need to compute the tangential representations of a flag Bott manifold F_m at fixed points. From

Proposition 2.16, the tangent bundle TF_m of a flag Bott manifold F_m is isomorphic to

$$\prod_{j=1}^{m} \mathrm{U}(n_j+1) \times_{\mathbb{T}} \bigoplus_{j=1}^{m} \mathfrak{m}_j,$$

where $\mathfrak{m}_j \subset \mathfrak{u}(n_j+1)$ is the subspace of matrices with the zero diagonals for $1 \le j \le m$. For an element $(w_1, \ldots, w_m) \in \prod_{j=1}^{m} \mathfrak{S}_{n_j+1}$, the corresponding fixed point in the flag Bott manifold F_m is $\dot{w} := [\dot{w}_1, \ldots, \dot{w}_m]$. To describe the tangential representation $T_{\dot{w}}F_m$ of $\mathbb{T}$, it is enough to find homomorphisms $f_j : \mathbb{T} \to T^{n_j+1}$ satisfying that for $1 \le j \le m$

$$\begin{aligned} &[t_1\dot{w}_1, \ldots, t_m\dot{w}_m; X_1, \ldots, X_m] \\ &\qquad = \big[\dot{w}_1, \ldots, \dot{w}_m; \mathrm{Ad}_{f_1(t_1,\ldots,t_m)}X_1, \mathrm{Ad}_{f_2(t_1,\ldots,t_m)}X_2, \ldots, \mathrm{Ad}_{f_m(t_1,\ldots,t_m)}X_m\big]. \end{aligned}$$

Before computing the homomorphisms f_j, let us recall the adjoint action of $\mathbb{T}$ on $\mathfrak{m}_j$. Let $E_{(r,s)}$ be an element of $\mathfrak{gl}(n_j+1)$ whose (r,s)-entry is 1 and all others are zero. Now we have $\mathfrak{m}_j \cong \mathrm{span}_{\mathbb{C}}\{zE_{(r,s)} + (-\bar{z})E_{(s,r)} \mid z \in \mathbb{C}, 1 \le s < r \le n_j+1\}$. We denote the standard basis of $\mathrm{Lie}(\mathbb{T})^* \cong \mathbb{R}^{\sum_{j=1}^{m}(n_j+1)}$ by

$$\{\varepsilon^*_{1,1}, \ldots, \varepsilon^*_{1,n_1+1}, \ldots, \varepsilon^*_{m,1}, \ldots, \varepsilon^*_{m,n_m+1}\}. \tag{3-3}$$

With respect to this basis, let A be the integer matrix of size $(n_j+1) \times (n+m)$ whose row vectors $\boldsymbol{c}_{j,1}, \ldots, \boldsymbol{c}_{j,n_j+1}$ are weights of the homomorphism f_j, so that for an element $\boldsymbol{t} \in \mathbb{T}$,

$$f_j : \boldsymbol{t} \mapsto \mathrm{diag}(\boldsymbol{t}^{\boldsymbol{c}_{j,1}}, \ldots, \boldsymbol{t}^{\boldsymbol{c}_{j,n_j+1}}). \tag{3-4}$$

Since $\mathrm{Ad}_{f_j(\boldsymbol{t})}E_{(r,s)} = \boldsymbol{t}^{\boldsymbol{c}_{j,r}-\boldsymbol{c}_{j,s}}E_{(r,s)}$, using the weight vectors $\{\boldsymbol{c}_{j,k}\}$, we can describe that

$$\mathfrak{m}_j \cong \bigoplus_{1 \le s < r \le n_j+1} V(\boldsymbol{c}_{j,r} - \boldsymbol{c}_{j,s}),$$

where $V(\boldsymbol{c}_{j,r} - \boldsymbol{c}_{j,s})$ is the 1-dimensional $\mathbb{T}$-representation with the weight $\boldsymbol{c}_{j,r} - \boldsymbol{c}_{j,s} \in \bigoplus_{j=1}^{m} \mathbb{Z}^{n_j+1}$. For an integer matrix A, we define

$$V(A) := \bigoplus_{1 \le s < r \le n_j+1} V(\boldsymbol{c}_{j,r} - \boldsymbol{c}_{j,s}).$$

Using this notation, we have the following proposition whose proof will be given at the end of this subsection.

Proposition 3.5. *Let F_m be the m-stage flag Bott manifold determined by a set of integer matrices $(A^{(j)}_{\ell})_{1 \le \ell < j \le m-1} \in \prod_{1 \le \ell < j \le m-1} M_{n_j+1,n_\ell+1}(\mathbb{Z})$. Consider the (noneffective) canonical $\mathbb{T}$-action on F_m. For a fixed point $\dot{w} = [\dot{w}_1, \ldots, \dot{w}_m] \in F_m$,* *the tangential* $\mathbb{T}$*-representation is* $T_{\dot{w}}F_m \cong \bigoplus_{j=1}^m \mathfrak{m}_j$, *where*

$$\mathfrak{m}_j \cong V\big(\big[X_1^{(j)}\ \ X_2^{(j)}\ \ \cdots\ \ X_{j-1}^{(j)}\ \ B_j\ \ O\ \ \cdots\ \ O\big]\big). \tag{3-5}$$

Here $X_\ell^{(j)}$ *is the matrix of size* $(n_j{+}1)\times(n_\ell{+}1)$ *defined by*

$$X_\ell^{(j)} = \sum_{\ell<i_1<\cdots<i_r<j} (B_j A_{i_r}^{(j)})(B_{i_r} A_{i_{r-1}}^{(i_r)})\cdots(B_{i_1} A_\ell^{(i_1)})B_\ell + B_j A_\ell^{(j)} B_\ell \qquad \text{for } 1\le \ell < j \le m, \tag{3-6}$$

and B_j *is the row permutation matrix corresponding to* w_j, *i.e.*, $B_j = (\dot{w}_j)^T$. *Furthermore*, *the weights of the isotropy representation of* $\mathbb{T}$ *on* $T_{\dot{w}}F_m$ *are pairwise linearly independent.*

By considering the effective canonical action of $\boldsymbol{T}$ on F_m, the fixed point set is finite because of Proposition 3.3. Also the canonical action of $\boldsymbol{T}$ on F_m is holomorphic (see Section 3A). As a corollary of Proposition 3.5, we have the following theorem.

Theorem 3.6. *Let* F_m *be an* m*-stage flag Bott manifold with the effective canonical action of* $\boldsymbol{T}$. *Then* $(F_m, \boldsymbol{T})$ *is a GKM manifold.*

Example 3.7. Suppose that the flag Bott manifold F_1 is $\mathcal{F}\ell(3)$. With the canonical action of the torus $\mathbb{T} = (S^1)^3$, there are six fixed points $\{[\dot{w}] \mid w \in \mathfrak{S}_3\}$. Let $\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*\}$ be the standard basis of $\mathrm{Lie}((S^1)^3)^* \cong \mathbb{R}^3$. Consider an element $\dot{w}$ in $\mathrm{GL}(3)$ corresponding to the permutation $w = (231) \in \mathfrak{S}_3$. Then the row permutation matrix B is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

which is the transpose of the column permutation matrix $\dot{w}$ in (3-2). Then we have the following tangential representation:

$$T_{[\dot{w}]}F_1 = \mathfrak{m}_1 \cong V(B) = V(\varepsilon_3^* - \varepsilon_2^*) \oplus V(\varepsilon_1^* - \varepsilon_3^*) \oplus V(\varepsilon_1^* - \varepsilon_2^*).$$

Example 3.8. Consider a flag Bott tower F_2 of height 2 defined by the integer matrix

$$A_1^{(2)} = \begin{bmatrix} c_1 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then F_2 is a $\mathbb{C}P^1$-bundle over $\mathcal{F}\ell(3)$. The manifold F_2 has the action of $(S^1)^3\times(S^1)^2$, and there are 12 fixed points $\{[\dot{w}_1, \dot{w}_2] \mid w_1 \in \mathfrak{S}_3, w_2 \in \mathfrak{S}_2\}$. Let

$$\{\varepsilon_{1,1}^*, \varepsilon_{1,2}^*, \varepsilon_{1,3}^*, \varepsilon_{2,1}^*, \varepsilon_{2,2}^*\}$$

be the standard basis of $\mathrm{Lie}((S^1)^3 \times (S^1)^2)^* \cong \mathbb{R}^3 \oplus \mathbb{R}^2$. Consider the point $\dot{w} = [\dot{w}_1, \dot{w}_2]$, where $w_1 = e$ and $w_2 = (21)$. Then the corresponding row permutation matrices are

$$B_1 = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence the matrix $X_1^{(2)}$ is

$$X_1^{(2)} = B_2 A_1^{(2)} B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ c_1 & c_2 & 0 \end{bmatrix}.$$

The tangential representation at the point $\dot{w}$ can be computed as follows:

$$\begin{aligned} T_{\dot{w}} F_2 = \mathfrak{m}_1 \oplus \mathfrak{m}_2 &\cong V([I_3 \ O]) \oplus V([X_1^{(2)} \ B_2]) \\ &= V\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \right) \oplus V\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ c_1 & c_2 & 0 & 1 & 0 \end{bmatrix} \right) \\ &= V(\varepsilon_{1,2}^* - \varepsilon_{1,1}^*) \oplus V(\varepsilon_{1,3}^* - \varepsilon_{1,2}^*) \oplus V(\varepsilon_{1,3}^* - \varepsilon_{1,1}^*) \\ &\qquad \oplus V(c_1 \varepsilon_{1,1}^* + c_2 \varepsilon_{1,2}^* + \varepsilon_{2,1}^* - \varepsilon_{2,2}^*). \end{aligned}$$

Example 3.9. Consider a flag Bott tower of height 3 with $n_1 = 2$, $n_2 = 1$, and $n_3 = 1$ which is defined by

$$A_1^{(2)} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1^{(3)} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2^{(3)} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the flag Bott manifold F_3 has the action of $(S^1)^3 \times (S^1)^2 \times (S^1)^2$, and the set of fixed points is $\{[\dot{w}_1, \dot{w}_2, \dot{w}_3] \mid w_1 \in \mathfrak{S}_3,\ w_2, w_3 \in \mathfrak{S}_2\}$. Denote the standard basis of $\mathrm{Lie}((S^1)^3 \times (S^1)^2 \times (S^1)^2) \cong \mathbb{R}^3 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$ by $\{\varepsilon_{1,1}^*, \varepsilon_{1,2}^*, \varepsilon_{1,3}^*, \varepsilon_{2,1}^*, \varepsilon_{2,2}^*, \varepsilon_{3,1}^*, \varepsilon_{3,2}^*\}$. Consider the fixed point $\dot{w} = [\dot{w}_1, \dot{w}_2, \dot{w}_3]$, where $w_1 = (312)$, $w_2 = e$, and $w_3 = (21)$. The corresponding row permutation matrices are

$$B_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have the following computations of $X_1^{(2)}, X_1^{(3)}, X_2^{(3)}$:

$$\begin{aligned} X_1^{(2)} &= B_2 A_1^{(2)} B_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \\ X_1^{(3)} &= B_3 A_2^{(3)} B_2 A_1^{(2)} B_1 + B_3 A_1^{(3)} B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 14 & 0 & 8 \end{bmatrix}, \\ X_2^{(3)} &= B_3 A_2^{(3)} B_2 = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}. \end{aligned}$$

The tangential representation at the point $\dot{w}$ can be computed as follows:

$$\begin{aligned}T_{\dot{w}}F_3 &= \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \\ &\cong V([B_1\ O\ O]) \oplus V([X_1^{(2)}\ B_2\ O]) \oplus V([X_1^{(3)}\ X_2^{(3)}\ B_3]) \\ &= V\left(\begin{bmatrix} 0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \end{bmatrix}\right) \oplus V\left(\begin{bmatrix} 2&0&1&1&0&0&0 \\ 0&0&0&0&1&0&0 \end{bmatrix}\right) \\ &\qquad\qquad \oplus V\left(\begin{bmatrix} 0&0&0&0&0&0&1 \\ 14&0&8&5&0&1&0 \end{bmatrix}\right) \\ &= V(\varepsilon_{1,1}^*-\varepsilon_{1,3}^*) \oplus V(\varepsilon_{1,2}^*-\varepsilon_{1,3}^*) \oplus V(\varepsilon_{1,2}^*-\varepsilon_{1,1}^*) \oplus V(-2\varepsilon_{1,1}^*-\varepsilon_{1,3}^*-\varepsilon_{2,1}^*+\varepsilon_{2,2}^*) \\ &\qquad\qquad \oplus V(14\varepsilon_{1,1}^* + 8\varepsilon_{1,3}^* + 5\varepsilon_{2,1}^* + \varepsilon_{3,1}^* - \varepsilon_{3,2}^*).\end{aligned}$$

Before presenting the proof of Proposition 3.5, we give a lemma which is directly induced by the definition of $X_\ell^{(j)}$ in (3-6).

Lemma 3.10. *The matrix $X_\ell^{(j)}$ satisfies the following equality.*

$$X_\ell^{(j)} = B_j A_{j-1}^{(j)} X_\ell^{(j-1)} + B_j A_{j-2}^{(j)} X_\ell^{(j-2)} + \cdots + B_j A_{\ell+1}^{(j)} X_\ell^{(\ell+1)} + B_j A_\ell^{(j)} B_\ell.$$

Proof of Proposition 3.5. We first note that for any $t_j = \operatorname{diag}(t_{j,1}, \ldots, t_{j,n_j+1}) \in T^{n_j+1} \subset \mathrm{U}(n_j+1)$, we have that $\dot{w}_j^{-1} t_j \dot{w}_j = \operatorname{diag}(t_{j,w_j(1)}, t_{j,w_j(2)}, \ldots, t_{j,w_j(n_j+1)}) \in T^{n_j+1}$. Let $\widetilde{w}_j$ denote a homomorphism $T^{n_j+1} \to T^{n_j+1}$ define by $\widetilde{w}_j(t_j) := \dot{w}_j^{-1} t_j \dot{w}_j$. Then we have that

$$t_j \dot{w}_j = \dot{w}_j \dot{w}_j^{-1} t_j \dot{w}_j = \dot{w}_j \widetilde{w}_j(t_j). \tag{3-7}$$

For the row permutation matrix $B_j = (\dot{w})^T$, we have that $B_j(t_{j,1}, \ldots, t_{j,n_j+1})^T = (t_{j,w_j(1)}, \ldots, t_{j,w_j(n_j+1)})^T$. Hence B_j is the matrix for the homomorphism $\widetilde{w}_j : T^{n_j+1} \to T^{n_j+1}$.

Consider the case when $j = 1$. Then we can get

$$\begin{aligned}&[t_1\dot{w}_1, \ldots, t_m\dot{w}_m; X_1, \ldots, X_m] \\ &= [\dot{w}_1\widetilde{w}_1(t_1), t_2\dot{w}_2, \ldots, t_m\dot{w}_m; X_1, \ldots, X_m] && \text{(by (3-7))} \\ &= \Big[\big(\dot{w}_1, \Lambda_1^{(2)}(\widetilde{w}_1(t_1))t_2\dot{w}_2, \ldots, \Lambda_1^{(m)}(\widetilde{w}_1(t_1))t_m\dot{w}_m\big) \\ &\qquad \cdot (\widetilde{w}_1(t_1), 1, \ldots, 1); X_1, \ldots, X_m\Big] && \text{(by (2-9))} \\ &= \Big[\dot{w}_1, (\Lambda_1^{(2)} \circ \widetilde{w}_1)(t_1)t_2\dot{w}_2, \ldots, (\Lambda_1^{(m)} \circ \widetilde{w}_1)(t_1)t_m\dot{w}_m; \\ &\qquad \mathrm{Ad}_{\widetilde{w}_1(t_1)}X_1, X_2, \ldots, X_m\Big] && \text{(by (2-13)).}\end{aligned} \tag{3-8}$$

Therefore the homomorphism $f_1 : \mathbb{T} \to T^{n_1+1}$ in (3-4) is given by $(t_1, \ldots, t_m) \mapsto \widetilde{w}_1(t_1)$, and

$$\mathfrak{m}_1 \cong V\big([B_1\ \ O\ \ \cdots\ \ O]\big).$$

Hence the proposition holds for $j = 1$.

We continue the similar computation to (3-8) for the second coordinate as follows. For $\boldsymbol{t} = (t_1, \dots, t_m) \in \mathbb{T}$,

$$\begin{aligned}
&[t_1\dot{w}_1, t_2\dot{w}_2, t_3\dot{w}_3, \dots, t_m\dot{w}_m; X_1, \dots, X_m] \\
&= \big[\dot{w}_1, (\Lambda_1^{(2)} \circ \widetilde{w}_1)(t_1)t_2\dot{w}_2, (\Lambda_1^{(3)} \circ \widetilde{w}_1)(t_1)t_3\dot{w}_3, \\
&\qquad\qquad \dots, (\Lambda_1^{(m)} \circ \widetilde{w}_1)(t_1)t_m\dot{w}_m; \mathrm{Ad}_{\widetilde{w}_1(t_1)}X_1, X_2, \dots, X_m\big] \quad \text{(by (3-8))} \\
&= \big[\dot{w}_1, \Lambda_1^{(2)}(f_1(\boldsymbol{t}))t_2\dot{w}_2, \Lambda_1^{(3)}(f_1(\boldsymbol{t}))t_3\dot{w}_3, \\
&\qquad\qquad \dots, \Lambda_1^{(m)}(f_1(\boldsymbol{t}))t_m\dot{w}_m; \mathrm{Ad}_{f_1(\boldsymbol{t})}X_1, X_2, \dots, X_m\big] \\
&\qquad\qquad \text{(by substituting } \widetilde{w}_1(t_1) = f_1(\boldsymbol{t})) \\
&= \big[\dot{w}_1, \dot{w}_2\widetilde{w}_2(\Lambda_1^{(2)}(f_1(\boldsymbol{t}))t_2), \Lambda_1^{(3)}(f_1(\boldsymbol{t}))t_3\dot{w}_3, \\
&\qquad\qquad \dots, \Lambda_1^{(m)}(f_1(\boldsymbol{t}))t_m\dot{w}_m; \mathrm{Ad}_{f_1(\boldsymbol{t})}X_1, X_2, \dots, X_m\big] \quad \text{(by (3-7))} \\
&= \big[\dot{w}_1, \dot{w}_2 f_2(\boldsymbol{t}), \Lambda_1^{(3)}(f_1(\boldsymbol{t}))t_3\dot{w}_3, \dots, \Lambda_1^{(m)}(f_1(\boldsymbol{t}))t_m\dot{w}_m; \mathrm{Ad}_{f_1(\boldsymbol{t})}X_1, X_2, \dots, X_m\big] \\
&\qquad\qquad \text{(by letting } f_2(\boldsymbol{t}) = \widetilde{w}_2(\Lambda_1^{(2)}(f_1(\boldsymbol{t}))t_2)) \\
&= \big[\dot{w}_1, \dot{w}_2, \Lambda_2^{(3)}(f_2(\boldsymbol{t}))\Lambda_1^{(3)}(f_1(\boldsymbol{t}))t_3\dot{w}_3, \\
&\qquad \dots, \Lambda_2^{(m)}(f_2(\boldsymbol{t}))\Lambda_1^{(m)}(f_1(\boldsymbol{t}))t_m\dot{w}_m; \mathrm{Ad}_{f_1(\boldsymbol{t})}X_1, \mathrm{Ad}_{f_2(\boldsymbol{t})}X_2, X_3, \dots, X_m\big] \\
&\qquad\qquad \text{(by (2-13))}.
\end{aligned}$$

Continuing this process, we may assume that $f_1, \dots, f_{j-1}$ can be defined so that for $j > 1$,

$$\begin{aligned}
&[t_1\dot{w}_1, \dots, t_j\dot{w}_j, \dots; X_1, \dots, X_j, \dots] \\
&= \big[\dot{w}_1, \dots, \dot{w}_{j-1}, \Lambda_{j-1}^{(j)}(f_{j-1}(\boldsymbol{t}))\Lambda_{j-2}^{(j)}(f_{j-2}(\boldsymbol{t})) \cdots \Lambda_1^{(j)}(f_1(\boldsymbol{t}))t_j\dot{w}_j, \\
&\qquad\qquad \dots; \mathrm{Ad}_{f_1(\boldsymbol{t})}X_1, \dots, \mathrm{Ad}_{f_{j-1}(\boldsymbol{t})}X_{j-1}, X_j, \dots\big].
\end{aligned}$$

We now define f_j. By considering

$$\Lambda_{j-1}^{(j)}(f_{j-1}(\boldsymbol{t}))\Lambda_{j-2}^{(j)}(f_{j-2}(\boldsymbol{t})) \cdots \Lambda_1^{(j)}(f_1(\boldsymbol{t}))t_j\dot{w}_j,$$

we get the following:

$$\begin{aligned}
&\Lambda_{j-1}^{(j)}(f_{j-1}(\boldsymbol{t}))\Lambda_{j-2}^{(j)}(f_{j-2}(\boldsymbol{t})) \cdots \Lambda_1^{(j)}(f_1(\boldsymbol{t}))t_j\dot{w}_j \\
&= \dot{w}_j\widetilde{w}_j\big(\Lambda_{j-1}^{(j)}(f_{j-1}(\boldsymbol{t}))\Lambda_{j-2}^{(j)}(f_{j-2}(\boldsymbol{t})) \cdots \Lambda_1^{(j)}(f_1(\boldsymbol{t}))t_j\big) \quad \text{(by (3-7))} \\
&= \dot{w}_j\big(\widetilde{w}_j \circ \Lambda_{j-1}^{(j)} \circ f_{j-1}(\boldsymbol{t})\big)\big(\widetilde{w}_j \circ \Lambda_{j-2}^{(j)} \circ f_{j-2}(\boldsymbol{t})\big) \cdots \big(\widetilde{w}_j \circ \Lambda_1^{(j)} \circ f_1(\boldsymbol{t})\big)(\widetilde{w}_j(t_j)).
\end{aligned}$$

Therefore one can deduce that the map $f_j : \mathbb{T} \to T^{n_j+1}$ is given by

$$\begin{aligned} \boldsymbol{t} &= (t_1, \dots, t_m) \\ &\mapsto \big(\widetilde{w}_j \circ \Lambda^{(j)}_{j-1} \circ f_{j-1}(\boldsymbol{t})\big)\big(\widetilde{w}_j \circ \Lambda^{(j)}_{j-2} \circ f_{j-2}(\boldsymbol{t})\big) \cdots \big(\widetilde{w}_j \circ \Lambda^{(j)}_{1} \circ f_{1}(\boldsymbol{t})\big)(\widetilde{w}_j(t_j)). \end{aligned}$$

By considering the exponents of the map $\widetilde{w}_j \circ \Lambda^{(j)}_\ell \circ f_\ell : \mathbb{T} \to T^{n_j+1}$ for $\ell = 1, \dots, j-1$, we get the following matrix of size $(n_j+1) \times \big((n_1+1)+\cdots+(n_m+1)\big)$:

$$\begin{aligned} &\underbrace{B_j}_{(n_j+1)\times(n_j+1)} \cdot \underbrace{A^{(j)}_\ell}_{(n_j+1)\times(n_\ell+1)} \cdot \underbrace{\big[X^{(\ell)}_1 \;\; X^{(\ell)}_2 \;\; \cdots \;\; X^{(\ell)}_{\ell-1} \;\; B_\ell \;\; O \;\; \cdots \;\; O\big]}_{(n_\ell+1)\times((n_1+1)+\cdots+(n_m+1))} \\ &\qquad = \big[B_jA^{(j)}_\ell X^{(\ell)}_1 \;\; B_jA^{(j)}_\ell X^{(\ell)}_2 \;\; \cdots \;\; B_jA^{(j)}_\ell X^{(\ell)}_{\ell-1} \;\; B_jA^{(j)}_\ell B_\ell \;\; O \;\; \cdots \;\; O\big]. \end{aligned}$$

Therefore it is enough to show that

$$X^{(j)}_\ell = B_jA^{(j)}_{j-1}X^{(j-1)}_\ell + B_jA^{(j)}_{j-2}X^{(j-2)}_\ell + \cdots + B_jA^{(j)}_{\ell+1}X^{(\ell+1)}_\ell + B_jA^{(j)}_\ell B_\ell,$$

which comes from Lemma 3.10. Hence we have the tangential $\mathbb{T}$-representation as in the proposition.

Finally, we claim that the weights of the isotropy representation of $\mathbb{T}$ on T_wF_m are pairwise linearly independent. For a fixed point w, let $\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathbb{Z}^n$ be weights of the tangential $\mathbb{T}$-representation $T_wF_m \cong \bigoplus_{j=1}^m \mathfrak{m}_j$. Assume that the weight $\boldsymbol{c}_1$ comes from $\mathfrak{m}_{j_1}$ and $\boldsymbol{c}_2$ comes from $\mathfrak{m}_{j_2}$ for $j_1 < j_2$. Then by the description in (3-5), $\boldsymbol{c}_1$ is a linear combination of $\{\varepsilon^*_{j,k} \mid 1 \le j \le j_1,\, 1 \le k \le n_j+1\}$. Since $\boldsymbol{c}_2$ has nonzero coefficients in $\{\varepsilon^*_{j_2,k} \mid 1 \le k \le n_{j_2}+1\}$ and $j_1 < j_2$, two weights $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are linearly independent. Suppose that both of two weights $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ come from $\mathfrak{m}_j$. Then they have nonzero coefficients in $\{\varepsilon^*_{j,k} \mid 1 \le k \le n_j+1\}$ which are determined by the permutation matrix B_j by (3-5). Hence they are linearly independent, so the result follows. □

3C. *GKM graphs.* In the previous subsection, we showed that a flag Bott manifold $(F_m, \boldsymbol{T})$ is a GKM manifold. For a given GKM manifold (M, T), one can define the following labeled graph (Γ, α); see [Guillemin and Zara 2001] for more details.

Definition 3.11. Let (M, T) be a GKM manifold. The *GKM graph* (Γ, α) consists of

- vertices: $V(\Gamma) = M^T$,
- edges: $e : v \to w \in E(\Gamma)$ if and only if there exists a T-invariant embedded 2-sphere X_e containing $v, w \in M^T$, and
- axial function: for an edge $e : v \to w$, the *axial function* α maps an edge e to the weight of the isotropy representation T_vX_e of T.

For an oriented edge e we write $i(e)$, respectively $t(e)$, the initial, respectively terminal, vertex of e. Moreover we write $\bar{e}$ for the oriented edge e with the reversed orientation. For $v \in V(\Gamma)$ we set

$$E(\Gamma)_v = \{e \in E(\Gamma) \mid i(e) = v\}.$$

For the GKM graph (Γ, α) associated to a GKM manifold (M, T), a collection $\theta = \{\theta_e\}$ of bijections

$$\theta_e : E(\Gamma)_{i(e)} \to E(\Gamma)_{t(e)}, \quad e \in E(\Gamma)$$

satisfying the following conditions can be determined naturally:

(1) $(\theta_e)^{-1} = \theta_{\bar{e}}$ for $e \in E(\Gamma)$,

(2) θ_e maps e to $\bar{e}$ for $e \in E(\Gamma)$, and

(3) for $e \in E(\Gamma)$ and $e' \in E(\Gamma)_{i(e)}$, there exists $c \in \mathbb{Z}$ such that $\alpha(\theta_e(e')) = \alpha(e') + c\alpha(e)$.

The collection $\theta = \{\theta_e\}$ is called the *connection*.

In Section 3B, we considered F_m with the noneffective canonical $\mathbb{T}$-action, and expressed the tangential representation $T_{\dot{w}} F_m$ in terms of the weights using the standard basis $\{\varepsilon^*_{1,1}, \dots, \varepsilon^*_{1,n_1+1}, \dots, \varepsilon^*_{m,1}, \dots, \varepsilon^*_{m,n_m+1}\}$ in (3-3). But in the GKM description, we need to consider the effective canonical $\boldsymbol{T}$-action on F_m. Therefore to consider the axial function with respect to $\boldsymbol{T}$-action, we should put

$$\varepsilon^*_{1,n_1+1} = \cdots = \varepsilon^*_{m,n_m+1} = 0 \tag{3-9}$$

in the formula of Proposition 3.5.

Theorem 3.12. *Let F_m be a flag Bott manifold with the effective canonical $\boldsymbol{T}$-action. Then the GKM graph (Γ, α) of $(F_m, \boldsymbol{T})$ consists of*

- *vertices*: $V(\Gamma) = \prod_{j=1}^m \mathfrak{S}_{n_j+1}$,
- *edges*: *$E(\Gamma)$ is the set of elements $w = (w_1, \dots, w_m)$ and $w' = (w'_1, \dots, w'_m)$ in $V(\Gamma)$ such that $w' = (w_1, \dots, w_j(r, s), \dots, w_m)$ for some transposition $(r, s) \in \mathfrak{S}_{n_j+1}$, and*
- *axial function*: *for w and w' as above such that $r, s \in [n_j + 1]$, $r > s$, then*

  $$\alpha(ww') = \rho^{(j)}_r - \rho^{(j)}_s,$$

  where $\rho^{(j)}_k$ is the k-th row of the matrix $\begin{bmatrix} X^{(j)}_1 & X^{(j)}_2 & \cdots & X^{(j)}_{j-1} & B_j & O & \cdots & O \end{bmatrix}$ for $k \in [n_j + 1]$, the matrices $X^{(j)}_\ell$ are as in (3-6) with the modification according to (3-9).

Proof. To find the GKM graph Γ, we recall that the product $\Gamma_1 \times \Gamma_2$ of graphs Γ_1, Γ_2 consists of vertices $V(\Gamma_1 \times \Gamma_2) := V(\Gamma_1) \times V(\Gamma_2)$ and edges $E(\Gamma_1 \times \Gamma_2)$ such that $e : (w_1, w_2) \to (w_1', w_2') \in E(\Gamma_1 \times \Gamma_2)$ if and only if either $w_1 = w_1'$ and $w_2 \to w_2' \in E(\Gamma_2)$, or $w_2 = w_2'$ and $w_1 \to w_1' \in E(\Gamma_1)$. We claim that the GKM graph Γ of F_m is the product of graphs $\prod_{j=1}^{m} \Gamma_j$, where Γ_j is the GKM graph of $\mathcal{F}\ell(n_j + 1)$.

By Proposition 3.3, we know that $V(\Gamma) = V\big(\prod_{j=1}^{m} \Gamma_j\big)$. To find edges on the graph Γ, we use an induction argument on the stage. When the stage is 1, then our claim obviously holds. Assume that the GKM graph of F_j is the product $\prod_{\ell=1}^{j} \Gamma_\ell$ for $1 \leq j \leq m - 1$. For $w \in \mathfrak{S}_{n_m+1}$, let $s_w : F_{m-1} \to F_m$ be a section of the fibration $F_m \to F_{m-1}$ defined by $[g_1, \dots, g_{m-1}] \mapsto [g_1, \dots, g_{m-1}, \dot{w}]$. Since the section s_w is $\boldsymbol{T}$-equivariant, it produces the GKM graph of F_{m-1} in Γ. Hence the section s_w gives edges $(w_1, \dots, w_{m-1}, w) \to (w_1', \dots, w_{m-1}', w)$ in Γ such that $(w_1, \dots, w_{m-1}) \to (w_1', \dots, w_{m-1}') \in E\big(\prod_{j=1}^{m-1} \Gamma_j\big)$.

On the other hand, a fiber over each fixed point in F_{j-1} produces the GKM graph of $\mathcal{F}\ell(n_j + 1)$. Therefore for $(w_1, \dots, w_{m-1}) \in V\big(\prod_{j=1}^{m-1} \Gamma_j\big)$, we have edges $(w_1, \dots, w_{m-1}, w_m) \to (w_1, \dots, w_{m-1}, w_m')$ such that $w_m \to w_m' \in E(\Gamma_m)$. Let $2N$ be the real dimension of F_m. Then we have that $|E(\Gamma)_v| = N$ for every vertex $v \in V(\Gamma)$ by the definition of GKM graph. The above constructions give exactly N many edges starting from a vertex v, so we have that $\Gamma = \big(\prod_{j=1}^{m-1} \Gamma_j\big) \times \Gamma_m$. By Proposition 3.5 we have the axial function as stated in the theorem. □

As a direct consequence of Theorem 3.12, we get the following.

Corollary 3.13. *The GKM graph Γ of F_m is combinatorially equivalent to the product $\prod_{j=1}^{m} \Gamma_j$, where Γ_j is the GKM graph of $\mathcal{F}\ell(n_j + 1)$.*

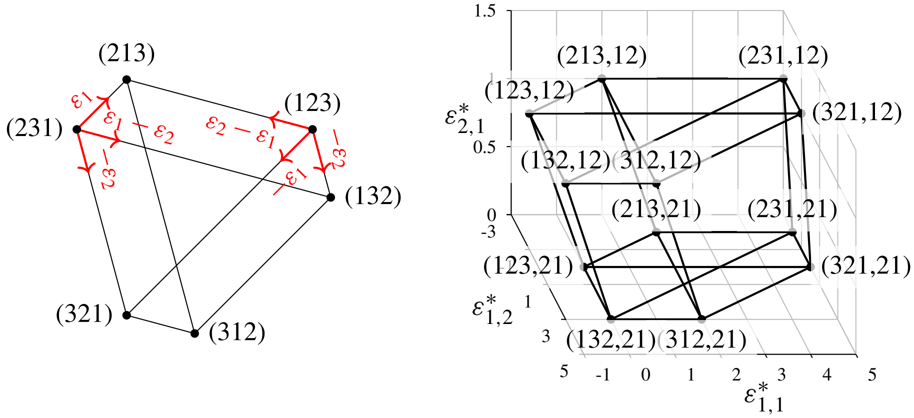
Example 3.14. Consider $F_1 = \mathcal{F}\ell(3)$ as in Example 3.7. At the point $[\dot{w}]$ determined by $w = (231) \in \mathfrak{S}_3$, we have that $T_{[\dot{w}]}F_1 \cong V(B)$, where

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

With the effective canonical torus action, the tangential representation is

$$T_{[\dot{w}]}F_1 \cong V(-\varepsilon_2^*) \oplus V(\varepsilon_1^*) \oplus V(\varepsilon_1^* - \varepsilon_2^*).$$

We have an edge (231) → (132) in the GKM graph since (132) = (231)(3, 1) for the transposition $(3, 1) \in \mathfrak{S}_3$. Hence the axial function for the edge (231) → (132) is $\varepsilon_1^* - \varepsilon_2^*$. One can do the similar computations for the other fixed points, and we have the GKM graph as in Figure 1, left. In the figure, parallel edges have the same axial functions.



GKM graph of $\mathcal{F}\ell(3)$. GKM graph of a $\mathbb{C}P^1$-bundle over $\mathcal{F}\ell(3)$.

Figure 1. GKM graphs.

Example 3.15. Let F_2 be the 2-stage flag Bott manifold defined by

$$A_1^{(2)} = \begin{bmatrix} c_1 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as in Example 3.8. The 3-dimensional compact torus acts effectively on F_2. Let $\{\varepsilon^*_{1,1}, \varepsilon^*_{1,2}, \varepsilon^*_{2,1}\}$ be the standard basis of $\mathrm{Lie}((S^1)^2 \times (S^1))^*$. Near the fixed point given by $(e, s_1) \in \mathfrak{S}_3 \times \mathfrak{S}_2$, we have the tangential representation as follows:

$$V(\varepsilon^*_{1,2} - \varepsilon^*_{1,1}) \oplus V(-\varepsilon^*_{1,2}) \oplus V(-\varepsilon^*_{1,1}) \oplus V(c_1\varepsilon^*_{1,1} + c_2\varepsilon^*_{1,2} + \varepsilon^*_{2,1}).$$

One can see that the induced subgraph Γ, respectively Γ', whose vertex set is $\mathfrak{S}_3 \times \{e\}$, respectively $\mathfrak{S}_3 \times \{s_1\}$, is the same as the GKM graph of $\mathcal{F}\ell(3)$ with the action of the torus T^2 in Example 3.14. Therefore it is enough to consider the axial functions of edges of the form $e_w := (w, e) \to (w, s_1)$ for $w \in \mathfrak{S}_3$. By a similar computation to Example 3.14, we get the GKM graph of F_2 as in Figure 1, right, whose axial function for vertical edges is listed as follows:

$$\begin{aligned}
\alpha(e_{(123)}) &= -c_1\varepsilon^*_{1,1} - c_2\varepsilon^*_{1,2} - \varepsilon^*_{2,1}, & \alpha(e_{(213)}) &= -c_2\varepsilon^*_{1,1} - c_1\varepsilon^*_{1,2} - \varepsilon^*_{2,1}, \\
\alpha(e_{(231)}) &= -c_1\varepsilon^*_{1,2} - \varepsilon^*_{2,1}, & \alpha(e_{(321)}) &= -c_2\varepsilon^*_{1,2} - \varepsilon^*_{2,1}, \\
\alpha(e_{(312)}) &= -c_2\varepsilon^*_{1,1} - \varepsilon^*_{2,1}, & \alpha(e_{(132)}) &= -c_1\varepsilon^*_{1,1} - \varepsilon^*_{2,1}.
\end{aligned}$$

Note that nontrivial coefficients of $\varepsilon^*_{1,1}$ and $\varepsilon^*_{1,2}$ shows that F_2 is a nontrivial $\mathbb{C}P^1$-bundle over $\mathcal{F}\ell(3)$.

Example 3.16. Consider the 3-stage flag Bott manifold F_3 as in Example 3.9. Let $\dot{w} = [\dot{w}_1, \dot{w}_2, \dot{w}_3]$ be a fixed point where $w_1 = (312) \in \mathfrak{S}_3$, $w_2 = e \in \mathfrak{S}_2$, and

$w_3 = (21) \in \mathfrak{S}_2$. For an edge $(w_1, w_2, w_3) \to (w_1, w_2, w_3(2, 1))$, the axial function is $\rho_2^{(3)} - \rho_1^{(3)}$, where $\rho_k^{(3)}$ is the k-th row of the matrix

$$\begin{bmatrix} X_1^{(3)} \; X_2^{(3)} \; B_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 14 & 0 & 8 & 5 & 0 & 1 & 0 \end{bmatrix}.$$

Hence with the modification according to (3-9), the axial function is

$$14\varepsilon_{1,1}^* + 5\varepsilon_{2,1}^* + \varepsilon_{3,1}^*.$$

Remark 3.17. Let $F_\bullet$ be a flag Bott tower, and (Γ_j, α_j) the GKM graph of j-stage flag Bott manifold F_j. Then $(\Gamma_j, \alpha_j) \to (\Gamma_{j-1}, \alpha_{j-1})$ is a GKM fiber bundle, see [Sabatini 2009, Definition 2.3.5], induced from the fibration $F_j \to F_{j-1}$ for $1 \le j \le m$. The module basis of GKM graph cohomology of GKM fiber bundle has been computed in [Sabatini 2009; Guillemin et al. 2012]. In the paper [Kaji et al. 2020], we compute the equivariant cohomology rings of flag Bott manifolds by using the Borel–Hirzebruch formula.

4. Generalized Bott manifolds and the associated flag Bott manifolds

We begin this section by reviewing *generalized Bott towers* studied in [Choi et al. 2010a; 2010b] and studying their fans based on [Cox et al. 2011, Section 7.3].

Definition 4.1 ([Choi et al. 2010a, Defintion 6.1]). A *generalized Bott tower* $B_\bullet = \{B_j \mid 0 \le j \le m\}$ of height m (or an *m-stage generalized Bott tower*) is a sequence,

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_3} B_2 \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

of manifolds $B_j = \mathbb{P}(E_1^j \oplus \cdots \oplus E_{n_j}^j \oplus \underline{\mathbb{C}})$, where E_k^j is a holomorphic line bundle over B_{j-1} for $1 \le k \le n_j$, $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}, and $\mathbb{P}(\cdot)$ stands for the projectivization of each fiber. We call B_j the *j-stage generalized Bott manifold* of a generalized Bott tower.

Example 4.2. (1) Every projective space $\mathbb{C}P^n$ is a generalized Bott tower of height 1.

(2) The product of projective spaces $\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_m}$ is an m-stage generalized Bott manifold.

(3) When $n_j = 1$ for $1 \le j \le m$, an m-stage generalized Bott manifold is an m-stage Bott manifold (see Example 2.2(3)).

Recall from [Hartshorne 1977, Exercise II.7.9] that for each $1 \le j \le m$, the set of isomorphic classes of holomorphic line bundles on B_{j-1} is isomorphic to $\mathbb{Z}^{j-1}$. More precisely, for $1 \le j \le m$, the homomorphism

$$\mathbb{Z}^{j-1} \to \operatorname{Pic}(B_{j-1}), \quad (a_1, \ldots, a_{j-1}) \mapsto (\eta_1^j)^{\otimes a_1} \otimes (\eta_2^j)^{\otimes a_2} \otimes \cdots \otimes (\eta_{j-1}^j)^{\otimes a_{j-1}}$$

is an isomorphism since B_j is an iterated sequence of projective space bundles. Here, η^j_{j-1} is the tautological line bundle over B_{j-1}, and $\eta^j_\ell = \pi_j^* \circ \cdots \circ \pi^*_{\ell+1}(\eta^{\ell+1}_\ell)$, for each $1 \le \ell \le j-2$. Therefore for each holomorphic line bundle E^j_k over B_{j-1}, there exist integers $a^j_{k,1}, \dots, a^j_{k,j-1}$ such that

$$E^j_k = (\eta^j_1)^{\otimes a^j_{k,1}} \otimes (\eta^j_2)^{\otimes a^j_{k,2}} \otimes \cdots \otimes (\eta^j_{j-1})^{\otimes a^j_{k,j-1}}.$$

Hence, we conclude that given a generalized Bott manifold B_{j-1}, the collection of integers

$$\{a^j_{k,\ell} \in \mathbb{Z} \mid 1 \le k \le n_j,\ 1 \le \ell \le j-1\}$$

determines B_j.

In general, a projectivization of the sum of holomorphic line bundles over a toric variety is again a toric variety (see [Cox et al. 2011, Section 7.3]).[2] Hence, so is a generalized Bott manifold B_m. To describe the fan of B_m, we prepare the following matrix Λ of size $n \times m$:

$$(4\text{-}1) \quad n := n_1 + \cdots + n_m \quad \text{and} \quad \Lambda := \left[\begin{array}{cccccc} -\mathbf{1} & \mathbf{0} & \cdots & & & \\ \boldsymbol{a}^2_1 & -\mathbf{1} & \mathbf{0} & \cdots & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \boldsymbol{a}^j_1 & \cdots & \boldsymbol{a}^j_{j-1} & -\mathbf{1} & \mathbf{0} & \cdots \\ \vdots & & & \ddots & \ddots & \\ \boldsymbol{a}^m_1 & \cdots & \cdots & & \boldsymbol{a}^m_{m-1} & -\mathbf{1} \end{array}\right] \begin{array}{l} \}\,n_1 \\ \}\,n_2 \\ \\ \}\,n_j \\ \\ \}\,n_m \end{array},$$

where we denote by $\mathbf{0}$, $\mathbf{1}$ and $\boldsymbol{a}^j_\ell$ the following vectors respectively:

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{a}^j_\ell = \begin{bmatrix} a^j_{1,\ell} \\ \vdots \\ a^j_{n_j,\ell} \end{bmatrix} \in \mathbb{Z}^{n_j} \quad \text{for } 1 \le \ell < j \le m.$$

Next, we define a set of vectors $\mathcal{U} := \{u^j_{k_j} \mid 1 \le j \le m,\ 1 \le k_j \le n_j + 1\}$ by

$$u^j_{k_j} = \begin{cases} \varepsilon_{j,k_j} & \text{if } 1 \le k_j \le n_j, \\ j\text{-th column of } \Lambda & \text{if } k_j = n_j + 1, \end{cases}$$

where $\varepsilon_{1,1}, \dots, \varepsilon_{1,n_1}, \dots, \varepsilon_{m,1}, \dots, \varepsilon_{m,n_m}$ is the standard basis vector in $\mathbb{R}^n = \mathbb{R}^{n_1+\cdots+n_m}$. Now, we consider the following cones

$$\sigma_{k_1,\dots,k_m} := \operatorname{Cone}\bigl(\mathcal{U} \setminus \{u^1_{k_1}, \dots, u^m_{k_m}\}\bigr) \subset \mathbb{R}^n,$$

---

[2] Note that [Cox et al. 2011] uses a different convention to construct iterated projective bundles. They put the trivial line bundle on the first, but we put it on the last when we sum up line bundles in the definition of generalized Bott manifolds.

and one can see that the vectors of $\mathcal{U} \setminus \{u^1_{k_1}, \dots, u^m_{k_m}\}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n \subset \mathbb{R}^n$. Hence $\sigma_{k_1,\dots,k_m}$ is a smooth cone of dimension n.

Proposition 4.3. *A fan Σ associated to B_m consists of the cones*

$$\left\{\sigma_{k_1,\dots,k_m} \mid (k_1, \dots, k_m) \in \textstyle\prod_{j=1}^m [n_j + 1]\right\} \tag{4-2}$$

and their faces.

Proof. We show the claim by the induction on the stage of a generalized Bott manifold. When $m = 1$, we have $u^1_k = \boldsymbol{e}_k$ for $1 \leq k \leq n_1$ and $u^1_{n_1+1} = -\mathbf{1}$. In this case, the fan Σ consists of the cones $\{\sigma_{k_1} \subset \mathbb{R}^{n_1} \mid 1 \leq k_1 \leq n_1 + 1\}$ and their faces, which yields $X_\Sigma \cong \mathbb{C}P^{n_1}$. Next, assuming that the claim holds for $(m-1)$-stage generalized Bott manifold B_{m-1}, a successively application of the result [Cox et al. 2011, Section 7.3], in particular [Cox et al. 2011, Proposition 7.3.3 and Example 7.3.5], establishes that the claim holds for the m-stage generalized Bott manifold B_m. □

Remark 4.4. The fan Σ defined above is a simplicial fan whose underlying simplicial complex is the dual complex of the product $P := \prod_{j=1}^m \Delta^{n_j}$ of simplices. As a quasitoric manifold [Davis and Januszkiewicz 1991; Buchstaber and Panov 2015], the polytope together with the set $\mathcal{U}$, where we assign a facet

$$\Delta^{n_1} \times \cdots \times \Delta^{n_{j-1}} \times f^j_{k_j} \times \Delta^{n_{j+1}} \times \cdots \times \Delta^{n_m}$$

for some facet $f^j_{k_j}$ of Δ^{n_j} to the vector $u^j_{k_j}$ for $1 \leq k_j \leq n_j + 1$, form a characteristic pair which determines the given generalized Bott manifold. We refer the readers to [Choi et al. 2010a; 2010b] for more details.

Example 4.5. Let $B_\bullet$ be a generalized Bott tower of height 3 with $n_1 = 2$, $n_2 = 1$, and $n_3 = 2$. The 2-stage generalized Bott manifold B_2 is a $\mathbb{C}P^1$-fiber bundle over $\mathbb{C}P^2$, and the 3-stage B_3 is a $\mathbb{C}P^2$-fiber bundle over the manifold B_2. More precisely,

$$\begin{tikzcd}
& E_1^3 \oplus E_2^3 \oplus \underline{\mathbb{C}} \arrow[d] & E_1^2 \oplus \underline{\mathbb{C}} \arrow[d] \\
\mathbb{P}(E_1^3 \oplus E_2^3 \oplus \underline{\mathbb{C}}) \arrow[r] \arrow[d, equal] & \mathbb{P}(E_1^2 \oplus \underline{\mathbb{C}}) \arrow[r] \arrow[d, equal] & \mathbb{C}P^2 \arrow[d, equal] \\
B_3 & B_2 & B_1
\end{tikzcd}$$

where $\underline{\mathbb{C}}$ is the trivial line bundle, and

$$E_1^2 = (\eta_1^2)^{\otimes a^2_{1,1}}, \quad E_1^3 = (\eta_1^3)^{\otimes a^3_{1,1}} \otimes (\eta_2^3)^{\otimes a^3_{1,2}}, \quad E_2^3 = (\eta_1^3)^{\otimes a^3_{2,1}} \otimes (\eta_2^3)^{\otimes a^3_{2,2}}$$

for some integers $a^2_{1,1}, a^3_{1,1}, a^3_{1,2}, a^3_{2,1}, a^3_{2,2}$. Hence the matrix Λ of B_3 is

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ a^2_{1,1} & -1 & 0 \\ a^3_{1,1} & a^3_{1,2} & -1 \\ a^3_{2,1} & a^3_{2,2} & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{a}^2_1 & \mathbf{-1} & \mathbf{0} \\ \boldsymbol{a}^3_1 & \boldsymbol{a}^3_2 & \mathbf{-1} \end{bmatrix} = [u^1_3 \; u^2_2 \; u^3_3],$$

where

$$\boldsymbol{a}^2_1 = a^2_{1,1} \in \mathbb{Z}, \quad \boldsymbol{a}^3_1 = (a^3_{1,1}, a^3_{2,1}) \in \mathbb{Z}^2, \quad \text{and} \quad \boldsymbol{a}^3_2 = (a^3_{2,1}, a^3_{2,2}) \in \mathbb{Z}^2.$$

Moreover the fan Σ associated to B_3 consists of cones

$$\begin{array}{lll}
\mathrm{Cone}(\varepsilon_{1,1}, \varepsilon_{1,2}, \varepsilon_{2,1}, \varepsilon_{3,1}, \varepsilon_{3,2}), & \mathrm{Cone}(\varepsilon_{1,1}, u^1_3, \varepsilon_{2,1}, \varepsilon_{3,1}, \varepsilon_{3,2}), & \mathrm{Cone}(\varepsilon_{1,2}, u^1_3, \varepsilon_{2,1}, \varepsilon_{3,1}, \varepsilon_{3,2}), \\
\mathrm{Cone}(\varepsilon_{1,1}, \varepsilon_{1,2}, u^2_2, \varepsilon_{3,1}, \varepsilon_{3,2}), & \mathrm{Cone}(\varepsilon_{1,1}, u^1_3, u^2_2, \varepsilon_{3,1}, \varepsilon_{3,2}), & \mathrm{Cone}(\varepsilon_{1,2}, u^1_3, u^2_2, \varepsilon_{3,1}, \varepsilon_{3,2}), \\
\mathrm{Cone}(\varepsilon_{1,1}, \varepsilon_{1,2}, \varepsilon_{2,1}, \varepsilon_{3,1}, u^3_3), & \mathrm{Cone}(\varepsilon_{1,1}, u^1_3, \varepsilon_{2,1}, \varepsilon_{3,1}, u^3_3), & \mathrm{Cone}(\varepsilon_{1,2}, u^1_3, \varepsilon_{2,1}, \varepsilon_{3,1}, u^3_3), \\
\mathrm{Cone}(\varepsilon_{1,1}, \varepsilon_{1,2}, u^2_2, \varepsilon_{3,1}, u^3_3), & \mathrm{Cone}(\varepsilon_{1,1}, u^1_3, u^2_2, \varepsilon_{3,1}, u^3_3), & \mathrm{Cone}(\varepsilon_{1,2}, u^1_3, u^2_2, \varepsilon_{3,1}, u^3_3), \\
\mathrm{Cone}(\varepsilon_{1,1}, \varepsilon_{1,2}, \varepsilon_{2,1}, \varepsilon_{3,2}, u^3_3), & \mathrm{Cone}(\varepsilon_{1,1}, u^1_3, \varepsilon_{2,1}, \varepsilon_{3,2}, u^3_3), & \mathrm{Cone}(\varepsilon_{1,2}, u^1_3, \varepsilon_{2,1}, \varepsilon_{3,2}, u^3_3), \\
\mathrm{Cone}(\varepsilon_{1,1}, \varepsilon_{1,2}, u^2_2, \varepsilon_{3,2}, u^3_3), & \mathrm{Cone}(\varepsilon_{1,1}, u^1_3, u^2_2, \varepsilon_{3,2}, u^3_3), & \mathrm{Cone}(\varepsilon_{1,2}, u^1_3, u^2_2, \varepsilon_{3,2}, u^3_3)
\end{array}$$

and their faces.

Definition 4.6. Let $B_\bullet$ be a generalized Bott tower determined by the block matrix Λ with entries $\boldsymbol{a}^j_\ell$ as in (4-1). We call a flag Bott tower $F_\bullet$ *is associated to* $B_\bullet$ if it is determined by the set of integer matrices

$$\left\{A^{(j)}_\ell \in M_{n_j+1, n_\ell+1}(\mathbb{Z}) \mid 1 \le \ell < j \le m\right\},$$

where

$$A^{(j)}_\ell = \begin{bmatrix} \boldsymbol{a}^j_\ell & \mathbf{0} & \cdots & \mathbf{0} \\ \underbrace{0}_{1} & \underbrace{0 & \cdots & 0}_{n_\ell} \end{bmatrix} \begin{matrix} \}\, n_j \\ \}\, 1 \end{matrix} .$$

Example 4.7. Let B_3 be the generalized Bott tower of height 3 in Example 4.5. The associated flag Bott manifold F_3 to B_3 is determined by the following integer matrices:

$$A^{(2)}_1 = \begin{bmatrix} \boldsymbol{a}^2_1 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2_{1,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_{2,3}(\mathbb{Z}),$$

$$A^{(3)}_1 = \begin{bmatrix} \boldsymbol{a}^3_1 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^3_{1,1} & 0 & 0 \\ a^3_{2,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_{3,3}(\mathbb{Z}), \quad A^{(3)}_2 = \begin{bmatrix} \boldsymbol{a}^3_2 & \mathbf{0} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^3_{1,2} & 0 \\ a^3_{2,2} & 0 \\ 0 & 0 \end{bmatrix} \in M_{3,2}(\mathbb{Z}).$$

For a generalized Bott tower $B_\bullet$ and its associated flag Bott tower $F_\bullet$, we have the following commutative diagram.

$$\begin{tikzcd}
 & & q_{m-1}^*E_m \arrow[dl] \arrow[dd] & & & q_1^*E_2 \arrow[dl] \arrow[dd] & & q_0^*E_1 \arrow[dl] \arrow[dd] \\
F_m \arrow[r, "p_m"] \arrow[dd, "q_m"] & F_{m-1} \arrow[rr, "p_{m-1}"] \arrow[dd, "q_{m-1}"] & & \cdots \arrow[r, "p_2"] & F_1 \arrow[rr, "p_1"] \arrow[dd, "q_1"] & & F_0 \arrow[dd, "q_0=\mathrm{id}"] & \\
 & & E_m \arrow[dl] & & & E_2 \arrow[dl] & & E_1 \arrow[dl] \\
B_m \arrow[r, "\pi_m"] & B_{m-1} \arrow[rr, "\pi_{m-1}"] & & \cdots \arrow[r, "\pi_2"] & B_1 \arrow[rr, "\pi_1"] & & B_0 &
\end{tikzcd} \tag{4-3}$$

Indeed, the associated flag Bott tower $F_\bullet$ can be constructed inductively as follows. For each $1 \leq j \leq m$, consider the following pull-back diagram.

$$\begin{tikzcd}
q_{j-1}^*E_j \arrow[r, "\widetilde{q}_{j-1}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E_j \arrow[d] \\
F_{j-1} \arrow[r, "q_{j-1}"] & B_{j-1}
\end{tikzcd}$$

By flagifying each fiber of the above bundles, we obtain the associated pull back diagram of flag bundles.

$$\begin{tikzcd}
F_j := \mathcal{F}\ell(q_{j-1}^*E_j) \arrow[r, "\widetilde{q}_{j-1}"] \arrow[rr, bend left, "q_j"] \arrow[d, "p_j"] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}\ell(E_j) \arrow[r, "s_j"] \arrow[d] & \mathbb{P}(E_j) = B_j \arrow[dl, "\pi_j"] \\
F_{j-1} \arrow[r, "q_{j-1}"] & B_{j-1} &
\end{tikzcd}$$

Then F_j is the total space of $\mathcal{F}\ell(q_{j-1}^*E_j)$, and $q_j := s_j \circ \widetilde{q}_{j-1}$. Here, the map $s_j : \mathcal{F}\ell(E_j) \to \mathbb{P}(E_j)$ sends each fiberwise full flag

$$V_\bullet = \big(V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n_j} \subsetneq (E_j)_p\big)$$

to the element V_1 in $\mathbb{P}((E_j)_p)$ for $p \in B_{j-1}$.

5. Generic orbit closures in the associated flag Bott manifolds

For an m-stage generalized Bott manifold B_m, let F_m be its associated flag Bott manifold with the effective canonical action of $\boldsymbol{H}$ defined in Section 3A. In this section, we study the closure of a generic orbit of the torus $\boldsymbol{H}$ in the associate flag Bott manifold F_m and its relation with B_m in Theorem 5.7. For this, we first review combinatorics of permutohedral varieties.

5A. ***Permutohedral varieties.*** The closure X_n of a generic orbit in the flag variety $\mathcal{F}\ell(n+1)$ with the effective action of $(\mathbb{C}^*)^n$ as in Example 3.2 is a toric variety called the *permutohedral variety*; see for instance [Klyachko 1985; Huh 2014]. In this subsection, we recall the fan $\Sigma_n \subset \mathbb{R}^n$ of the permutohedral variety. Note that the fan Σ_n is the normal fan of an n-dimensional permutohedron P_n with particular outward normal vectors. To be more precise, there is a bijection between the set $\Sigma_n(1)$ of rays and nonempty proper subsets of $[n+1]$:

$$\Sigma_n(1) \xleftrightarrow{1-1} \{A \mid \varnothing \subsetneq A \subsetneq [n+1]\}.$$

For a nonempty proper subset A of $[n+1]$, the corresponding ray ρ_A is generated by

$$u_A := \begin{cases} \sum_{x \in A} \varepsilon_x & \text{if } n+1 \notin A, \\ -\sum_{x \in [n+1] \setminus A} \varepsilon_x & \text{otherwise,} \end{cases} \tag{5-1}$$

where $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the standard basis vector of $\mathbb{R}^n$. Hence there are $2^{n+1}-2$ many rays in Σ_n. The minimal generator in the intersection of a ray and the underlying lattice is called the *ray generator*. We note that u_A defined in (5-1) is the ray generator of ρ_A.

The maximal cones are indexed by proper chains of n nonempty proper subsets of $[n+1]$. For a proper chain

$$A_\bullet : \varnothing \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n \subsetneq [n+1] \tag{5-2}$$

of nonempty proper subsets, we have the corresponding maximal cone

$$\operatorname{Cone}(u_{A_1}, u_{A_2}, \dots, u_{A_n}).$$

Therefore the number of maximal cones is $(n+1)!$.

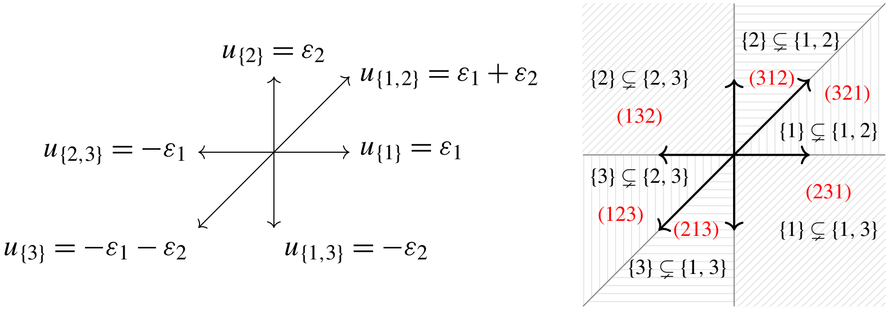
Moreover we have a correspondence between the maximal cones in Σ_n and the elements of the symmetric group $\mathfrak{S}_{n+1}$. For a permutation $w = (w(1) \ \cdots \ w(n+1))$ in $\mathfrak{S}_{n+1}$, we associate a maximal cone in Σ_n determined by the chain $A_\bullet$ where

$$A_k := \{w(n+2-k), \dots, w(n+1)\} \quad \text{for } 1 \le k \le n. \tag{5-3}$$

This description is sometimes much convenient to see the combinatorics of Σ_n. For instance, two maximal cones corresponding to permutations v and w in $\mathfrak{S}_{n+1}$ are adjacent if and only if there exists $i \in [n]$ such that $v = w \cdot s_i$, where s_i is the transposition $(i, i+1) \in \mathfrak{S}_{n+1}$.

Example 5.1. When $n = 2$, Figure 2, left, represents ray generators in Σ_2. Consider a permutation $(231) \in \mathfrak{S}_3$. Then the corresponding chain $A_\bullet$ defined in (5-3) is

$$A_\bullet : \varnothing \subsetneq \{1\} \subsetneq \{1,3\} \subsetneq \{1,2,3\}.$$



Ray generators in Σ_2. Maximal cones in Σ_2.

Figure 2. Fan Σ_2.

Hence the permutation (231) defines a maximal cone $\mathrm{Cone}(u_{\{1\}}, u_{\{1,3\}})$. As permutations (231) and (321) satisfy the relation $(231) = (321) \cdot s_1$, two maximal cones $\mathrm{Cone}(u_{\{1\}}, u_{\{1,3\}})$ and $\mathrm{Cone}(u_{\{1\}}, u_{\{1,2\}})$ are adjacent. Figure 2, right, describes the maximal cones in Σ_2.

Remark 5.2. Let $\Sigma'_n \subset \mathbb{R}^n$ be the fan of complex projective space $\mathbb{C}P^n$ whose ray generators $u_1, \dots, u_{n+1}$ are given by

$$u_k = \begin{cases} \varepsilon_k & \text{if } 1 \le k \le n, \\ -\varepsilon_1 - \cdots - \varepsilon_n & \text{if } k = n+1. \end{cases}$$

Then the set of cones in Σ'_n can be identified with the set of nonempty proper subsets of $[n+1]$. To be more precise, for any dimension d cone τ in Σ'_n, we have a subset $\{i_1, \dots, i_d\} \subset [n+1]$ such that

$$\tau = \mathrm{Cone}(u_{i_1}, \dots, u_{i_d}).$$

It is well known that the fan $\Sigma_n \subset \mathbb{R}^n$ of the permutohedral variety can be obtained from Σ'_n by star subdivisions of all cones of dimension grater than 0 in the decreasing order of the dimensions of the cones (see [Procesi 1990]). Hence, the set of rays in the fan Σ_n corresponds bijectively to the set of all cones of dimension grater than 0 in Σ'_n.

5B. ***The main result on generic orbit closures in F_m.*** Consider the canonical effective $\boldsymbol{H}$-action on F_m defined in Section 3A. In order to consider the closure of a generic $\boldsymbol{H}$-orbit in F_m, we first define a generic element in F_m. Let $g = (g_{ij})$ be an element in $\mathrm{GL}(n+1)$. For an ordered sequence $1 \le i_1 < i_2 < \cdots < i_k \le n+1$, we consider the Plücker coordinate

$$X_{i_1,\dots,i_k}(g) := \det((g_{i_p,p})_{1 \le p \le k}).$$

Definition 5.3. We call an element $g \in \mathrm{GL}(n+1)$ *generic* if $X_{i_1,\dots,i_k}(g)$ is nonzero for any $k \in [n+1]$ and ordered sequence $1 \le i_1 < i_2 < \cdots < i_k \le n+1$. We call a point $[g_1, \dots, g_m]$ in F_m is *generic* if $g_j \in \mathrm{GL}(n_j+1)$ is generic for $j = 1, \dots, m$.

For example, $g = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is not a generic element since $X_2(g) = 0$. But $g = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ is generic. The above definition of generic elements can be found in [Flaschka and Haine 1991; Klyachko 1995; Dabrowski 1996]. It is not difficult to show that the genericity of a point $[g_1, \dots, g_m]$ in F_m does not depend on the representative of a point.

A *generic orbit* in F_m is the $\boldsymbol{H}$-orbit of a generic point. In Theorem 5.7 we give a relation between a generalized Bott manifold B_m and the closure of a generic orbit of $\boldsymbol{H}$ in its associated flag Bott manifold F_m, which extends the relation between $\mathbb{C}P^n$, as an 1-stage generalized Bott manifold, and the n-dimensional permutohedral variety (see Remark 5.2).

Theorem 5.4. *Let B_m be an m-stage generalized Bott manifold determined by an integer matrix Λ as in (4-1) and let F_m be the associated m-stage flag Bott manifold. Then the closure of a generic orbit of $\boldsymbol{H}$ in the associated flag Bott manifold F_m is a nonsingular projective toric variety whose fan Σ is given as follows*:

(1) *The rays are parametrized by the set*

$$\{(\ell, A) \mid \varnothing \subsetneq A \subsetneq [n_\ell + 1],\ 1 \le \ell \le m\}.$$

For (ℓ, A) the corresponding ray is generated by the vector

$$u_A^\ell = \begin{cases} \sum_{x \in A} \varepsilon_{\ell,x} & \text{if } n_\ell + 1 \notin A, \\ -\sum_{x \in [n_\ell+1] \setminus A} \varepsilon_{\ell,x} + \sum_{j=\ell+1}^{m} \sum_{k=1}^{n_j} a_{k,\ell}^{j} \varepsilon_{j,k} & \text{otherwise}, \end{cases}$$

where $\{\varepsilon_{j,k}\}$ is the standard basis of the Lie algebra of the compact torus $\boldsymbol{T} \subset \boldsymbol{H}$ whose dual is the standard basis $\{\varepsilon_{j,k}^*\}$ of $\mathrm{Lie}(\boldsymbol{T})^*$.

(2) *The maximal cones are indexed by the sequences of proper chains of subsets*

$$\big\{(A_\bullet^1, \dots, A_\bullet^m) \mid A_\bullet^\ell = \big(\varnothing \subsetneq A_1^\ell \subsetneq A_2^\ell \subsetneq \cdots \subsetneq A_{n_\ell}^\ell \subsetneq [n_\ell + 1]\big),\ 1 \le \ell \le m\big\}.$$

For $(A_\bullet^1, \dots, A_\bullet^m)$, the corresponding maximal cone is defined to be

$$\mathrm{Cone}\left(\bigcup_{\ell=1}^{m} \{u_{A_1^\ell}^\ell, \dots, u_{A_{n_\ell}^\ell}^\ell\}\right).$$

The proof of Theorem 5.4 needs a series of lemmas, and will be given in the next subsection. The following corollary will play an important role in the proof of Theorem 5.7.

Corollary 5.5. *For each $1 \leq \ell \leq m$ and a nonempty proper subset $\varnothing \subsetneq A \subsetneq [n_\ell+1]$, we have the following relation*:

$$u^\ell_A = \sum_{x\in A} u^\ell_{\{x\}}. \tag{5-4}$$

Furthermore, for $x \in [n_\ell+1]$, the ray generator $u^\ell_{\{x\}}$ coincides with the ray generator u^ℓ_x in the fan Σ' of the generalized Bott manifold B_m.

Proof. First we notice that $u^\ell_{\{x\}} = \varepsilon_{\ell,x} = u^\ell_x$ if $x \neq n_\ell + 1$. Hence we get the equality (5-4) when $n_\ell + 1 \notin A$. On the other hand, we have that

$$u^\ell_{\{n_\ell+1\}} = -\sum_{x\in[n_\ell]} \varepsilon_{\ell,x} + \sum_{j=\ell+1}^{m} \sum_{k=1}^{n_j} a^j_{k,\ell} \varepsilon_{j,k} = u^\ell_{n_\ell+1}.$$

When $n_\ell + 1 \in A$, we get that

$$\begin{aligned}\sum_{x\in A} u^\ell_{\{x\}} &= u^\ell_{\{n_\ell+1\}} + \sum_{x\in A\setminus\{n_\ell+1\}} u^\ell_{\{x\}} \\ &= -\sum_{x\in[n_\ell]} \varepsilon_{\ell,x} + \sum_{j=\ell+1}^{m} \sum_{k=1}^{n_j} a^j_{k,\ell} \varepsilon_{j,k} + \sum_{x\in A\setminus\{n_\ell+1\}} \varepsilon_{\ell,x} \\ &= -\sum_{x\in[n_\ell+1]\setminus A} \varepsilon_{\ell,x} + \sum_{j=\ell+1}^{m} \sum_{k=1}^{n_j} a^j_{k,\ell} \varepsilon_{j,k} \\ &= u^\ell_A.\end{aligned}$$

□

Example 5.6. Let B_3 be a generalized Bott tower of height 3 as in Example 4.7 whose matrix Λ is given by

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ a^2_{1,1} & -1 & 0 \\ a^3_{1,1} & a^3_{1,2} & -1 \\ a^3_{2,1} & a^3_{2,2} & -1 \end{bmatrix}.$$

Let F_3 be the associated flag Bott manifold, and let X be the closure of a generic orbit of the torus $(\mathbb{C}^*)^5$. Then the fan $\widetilde{\Sigma}$ of X has 14 rays. Consider the ray generator $u^1_{\{3\}}$. Then by Theorem 5.4, the vector $u^1_{\{3\}}$ is

$$\sum_{x\in[3]\setminus\{3\}} -\varepsilon_{1,x} + \sum_{j=2}^{3} \sum_{k=1}^{n_j} a^j_{k,1} \varepsilon_{j,k} = -\varepsilon_{1,1} - \varepsilon_{1,2} + a^2_{1,1}\varepsilon_{2,1} + a^3_{1,1}\varepsilon_{3,1} + a^3_{2,1}\varepsilon_{3,2},$$

where $\{\varepsilon_{1,1}, \varepsilon_{1,2}, \varepsilon_{2,1}, \varepsilon_{3,1}, \varepsilon_{3,2}\}$ is the standard basis of the Lie algebra of the compact torus contained in $(\mathbb{C}^*)^5$. With this standard basis, we have the following ray generators:

$$\begin{array}{lll}
u^1_{\{1\}} = (1,0,0,0,0), & u^1_{\{2\}} = (0,1,0,0,0), & u^1_{\{3\}} = (-1,-1,a^2_{1,1},a^3_{1,1},a^3_{2,1}), \\
u^1_{\{1,2\}} = (1,1,0,0,0), & u^1_{\{1,3\}} = (0,-1,a^2_{1,1},a^3_{1,1},a^3_{2,1}), & u^1_{\{2,3\}} = (-1,0,a^2_{1,1},a^3_{1,1},a^3_{2,1}), \\
u^2_{\{1\}} = (0,0,1,0,0), & u^2_{\{2\}} = (0,0,-1,a^3_{1,2},a^3_{2,2}), & \\
u^3_{\{1\}} = (0,0,0,1,0), & u^3_{\{2\}} = (0,0,0,0,1), & u^3_{\{3\}} = (0,0,0,-1,-1), \\
u^3_{\{1,2\}} = (0,0,0,1,1), & u^3_{\{1,3\}} = (0,0,0,0,-1), & u^3_{\{2,3\}} = (0,0,0,-1,0).
\end{array}$$

For a subset $\{1, 3\} \subset [3]$, the ray generator $u^1_{\{1,3\}}$ is $(0, -1, a^2_{1,1}, a^3_{1,1}, a^3_{2,1})$. Also, we have the following:

$$u^1_{\{1,3\}} = (1, 0, 0, 0, 0) + (-1, -1, a^2_{1,1}, a^3_{1,1}, a^3_{2,1}) = u^1_{\{1\}} + u^1_{\{3\}}.$$

For a fan Σ and a cone $\tau \in \Sigma$, we recall from [Cox et al. 2011, Definition 3.3.17] the definition of star subdivision $\Sigma^*(\tau)$ of Σ along τ. Let $u_\tau = \sum_{\rho \in \tau(1)} u_\rho$, where u_ρ is the ray generator of a ray ρ. For each cone $\sigma \in \Sigma$ containing τ, set

$$\Sigma^*_\sigma(\tau) = \bigl\{\mathrm{Cone}(A) \mid A \subseteq \{u_\tau\} \cup \sigma(1),\ \tau(1) \nsubseteq A\bigr\}.$$

Then the *star subdivision* $\Sigma^*(\tau)$ is defined to be

$$\Sigma^*(\tau) = \{\sigma \in \Sigma \mid \tau \nsubseteq \sigma\} \cup \bigcup_{\tau \subseteq \sigma} \Sigma^*_\sigma(\tau).$$

Hence the fan $\Sigma^*(\tau)$ has one more ray generated by the vector u_τ.

Corollary 5.5 says that the set of ray generators

$$\bigcup_{\ell=1}^{m} \bigl\{u^\ell_{\{x\}} \mid x \in [n_\ell + 1]\bigr\}$$

can produce all other ray generators of the fan Σ, which yields the following property.

Theorem 5.7. *Let B_m be the m-stage generalized Bott manifold determined by the integer matrix Λ as in (4-1), and let Σ' be the fan of B_m. Let F_m be the associated m-stage flag Bott manifold to B_m. Then the fan Σ of the closure X of a generic orbit of the canonical $\boldsymbol{H}$-action in the associated flag Bott manifold F_m is the star subdivisions of Σ' along the following cones*

$$\bigl\{\mathrm{Cone}\bigl(\{u^\ell_x \mid x \in A\}\bigr) \mid \varnothing \subsetneq A \subsetneq [n_\ell + 1],\ 1 \le \ell \le m\bigr\} \subset \Sigma$$

in the increasing order of $1 \le \ell \le m$ *and in the decreasing order of* $|A|$.

Example 5.8. Let B_3 and F_3 be generalized Bott manifold and its associated flag Bott manifold given in Example 5.6. To obtain the fan Σ of the closure X of a generic torus orbit in F_3 from the fan Σ' of B_3, we consider the star subdivisions of Σ' along the following cones in the listed order:

$$\begin{aligned}
&\bigl\{\mathrm{Cone}(\{u_x^1 \mid x \in A\}) \mid \varnothing \subsetneq A \subsetneq [3], |A| = 2\bigr\}\\
&\qquad\qquad = \bigl\{\mathrm{Cone}(u_1^1, u_2^1), \mathrm{Cone}(u_1^1, u_3^1), \mathrm{Cone}(u_2^1, u_3^1)\bigr\},\\
&\bigl\{\mathrm{Cone}(\{u_x^1 \mid x \in A\}) \mid \varnothing \subsetneq A \subsetneq [3], |A| = 1\bigr\} = \bigl\{\mathrm{Cone}(u_1^1), \mathrm{Cone}(u_2^1), \mathrm{Cone}(u_3^1)\bigr\},\\
&\bigl\{\mathrm{Cone}(\{u_x^2 \mid x \in A\}) \mid \varnothing \subsetneq A \subsetneq [2], |A| = 1\bigr\} = \bigl\{\mathrm{Cone}(u_1^2), \mathrm{Cone}(u_2^2)\bigr\},\\
&\bigl\{\mathrm{Cone}(\{u_x^3 \mid x \in A\}) \mid \varnothing \subsetneq A \subsetneq [3], |A| = 2\bigr\}\\
&\qquad\qquad = \bigl\{\mathrm{Cone}(u_1^3, u_2^3), \mathrm{Cone}(u_1^3, u_2^3), \mathrm{Cone}(u_2^3, u_3^3)\bigr\},\\
&\bigl\{\mathrm{Cone}(\{u_x^3 \mid x \in A\}) \mid \varnothing \subsetneq A \subsetneq [3], |A| = 1\bigr\} = \bigl\{\mathrm{Cone}(u_1^3), \mathrm{Cone}(u_2^3), \mathrm{Cone}(u_3^3)\}.
\end{aligned}$$

To give a proof of Theorem 5.7, we first review the following classical result about a toric variety fibration over a toric variety. We refer to [Oda 1978, Proposition 7.3], as well as [Cox et al. 2011, Chapter 3.3; Ewald 1996, Chapter VI.6].

Proposition 5.9. *Let Σ and Σ' be complete fans in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $N'_{\mathbb{R}} := N' \otimes_{\mathbb{Z}} \mathbb{R}$ for some lattices N and N' respectively, which are compatible with a surjective $\mathbb{Z}$-linear map $\bar{\phi} : N \to N'$. Let Σ'' be a subfan of Σ consisting of the cones $\{\sigma \in \Sigma \mid \sigma \subset \ker \bar{\phi}_{\mathbb{R}}\}$ and $X_{\Sigma''}$ the corresponding toric variety. Then, the toric morphism $\phi : X_\Sigma \to X_{\Sigma'}$ induced from $\bar{\phi}$ is an equivariant fiber bundle with fiber $X_{\Sigma''}$ if and only if*

1. *there exists a lifting $\widetilde{\Sigma} \subseteq \Sigma$ of Σ' such that $\bar{\phi}_{\mathbb{R}} : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ maps $\tilde{\sigma} \in \widetilde{\Sigma}$ bijectively to a cone $\sigma' \in \Sigma'$,*
2. *Σ consists of cones $\{\tilde{\sigma} + \sigma'' \mid \tilde{\sigma} \in \widetilde{\Sigma},\ \sigma'' \in \Sigma''\}$.*

The fan Σ determined by the condition of Proposition 5.9 is called the *join* of $\widetilde{\Sigma}$ and Σ'' and denoted by $\Sigma = \widetilde{\Sigma} \bullet \Sigma''$. We refer to [Ewald 1996, Chapters III.1 and VI.6]. We need one more result to give a proof of Theorem 5.7.

Lemma 5.10. *Let Σ_1 and Σ_2 be fans such that $\Sigma_1(1) \cap \Sigma_2(1) = \varnothing$. Suppose that $\tau \in \Sigma_1$. Then*

$$\Sigma_1^*(\tau) \bullet \Sigma_2 = (\Sigma_1 \bullet \Sigma_2)^*(\tau).$$

Here we denote the cone $\tau + \{0\}$ in $\Sigma_1 \bullet \Sigma_2$ by τ.

Proof. For a cone $\tau \in \Sigma_1$, we have that

$$\Sigma_1^*(\tau) \bullet \Sigma_2 = \bigl(\{\sigma_1 \in \Sigma_1 \mid \tau \nsubseteq \sigma_1\} \bullet \Sigma_2\bigr) \cup \bigcup_{\substack{\tau \subseteq \sigma_1\\ \sigma_1 \in \Sigma_1}} ((\Sigma_1)^*_{\sigma_1}(\tau) \bullet \Sigma_2),$$

$$\begin{aligned}
(\Sigma_1 \bullet \Sigma_2)^*(\tau + \{0\}) = \bigl\{\sigma_1 + \sigma_2 \in \Sigma_1 + \Sigma_2 \mid \tau + \{0\} \nsubseteq \sigma_1 + \sigma_2\bigr\}&\\
\cup \bigcup_{\tau + \{0\} \subseteq \sigma_1 + \sigma_2} (\Sigma_1 \bullet \Sigma_2)^*_{\sigma_1 + \sigma_2}(\tau + \{0\}).&
\end{aligned}$$

We note that by the definition of join of fans, we get

$$\{\sigma_1 \in \Sigma_1 \mid \tau \nsubseteq \sigma_1\} \bullet \Sigma_2 = \bigl\{\sigma_1 + \sigma_2 \in \Sigma_1 + \Sigma_2 \mid \tau + \{0\} \nsubseteq \sigma_1 + \sigma_2\bigr\}.$$

Moreover, we have

$$\bigcup_{\tau+\{0\}\subseteq\sigma_1+\sigma_2} (\Sigma_1 \bullet \Sigma_2)^*_{\sigma_1+\sigma_2}(\tau + \{0\}) = \bigcup_{\substack{\tau\subseteq\sigma_1\\ \sigma_1\in\Sigma_1}} \bigcup_{\sigma_2\in\Sigma_2} (\Sigma_1 \bullet \Sigma_2)^*_{\sigma_1+\sigma_2}(\tau + \{0\}).$$

Therefore to prove the lemma, it is enough to show that for any $\sigma_1 \in \Sigma_1$ satisfying $\tau \subseteq \sigma_1$, the following equality holds:

$$(\Sigma_1)^*_{\sigma_1}(\tau) \bullet \Sigma_2 = \bigcup_{\sigma_2\in\Sigma_2} (\Sigma_1 \bullet \Sigma_2)^*_{\sigma_1+\sigma_2}(\tau + \{0\}). \tag{5-5}$$

We note that for $\sigma_2 \in \Sigma_2$,

$$(\Sigma_1 \bullet \Sigma_2)^*_{\sigma_1+\sigma_2}(\tau + \{0\}) = \bigl\{\mathrm{Cone}(B) \mid B \subseteq \{u_\tau\} \cup \sigma_1(1) \cup \sigma_2(1),\ \tau(1) \nsubseteq B\bigr\}. \tag{5-6}$$

Suppose that $A \subseteq \{u_\tau\} \cup \sigma_1(1)$ satisfying $\tau(1) \nsubseteq A$. Then for a cone $\sigma_2 \in \Sigma_2$, $\mathrm{Cone}(A)+\sigma_2$ is an element in $(\Sigma_1)^*_{\sigma_1}(\tau)\bullet\Sigma_2$. Since $\mathrm{Cone}(A)+\sigma_2 = \mathrm{Cone}(A\cup\sigma_2(1))$ and $\tau(1) \nsubseteq A\cup\sigma_2(1)$, the cone $\mathrm{Cone}(A)+\sigma_2$ is an element in $(\Sigma_1\bullet\Sigma_2)^*_{\sigma_1+\sigma_2}(\tau+\{0\})$ by (5-6).

Now, we consider $\mathrm{Cone}(B)$ in $(\Sigma_1 \bullet \Sigma_2)^*_{\sigma_1+\sigma_2}(\tau + \{0\})$ for some $\sigma_2 \in \Sigma_2$. We set $A := B\cap(\{u_\tau\}\cup\sigma_1(1))$ and $B' := B\cap\sigma_2(1)$. Since $B \subseteq \{u_\tau\}\cup\sigma_1(1)\cup\sigma_2(1)$, we have $\mathrm{Cone}(B) = \mathrm{Cone}(A)+\mathrm{Cone}(B')$. Moreover, $\mathrm{Cone}(A) \in (\Sigma_1)^*_{\sigma_1}(\tau)$, and $\mathrm{Cone}(B') \in \Sigma_2$ since $\mathrm{Cone}(B')$ is a face of the cone $\mathrm{Cone}(B)$. Hence the equality (5-5) holds, and we have proven the lemma. □

Proof of Theorem 5.7. By Proposition 5.9, there exist liftings $\widetilde{\Sigma}'_{n_1}, \dots, \widetilde{\Sigma}'_{n_{m-1}}$ of the fans $\Sigma'_{n_1}, \dots, \Sigma'_{n_{m-1}}$ of complex projective spaces such that

$$\Sigma' = \widetilde{\Sigma}'_{n_1} \bullet \cdots \bullet \widetilde{\Sigma}'_{n_{m-1}} \bullet \Sigma'_{n_m}.$$

More precisely, the lifting $\widetilde{\Sigma}'_{n_\ell} \subset \mathbb{R}^n$ consists of the cones

$$\mathrm{Cone}(u^\ell_1, \dots, \hat{u}^\ell_{k_\ell}, \dots, u^\ell_{n_\ell+1})$$

and their faces. On the other hand, the fan Σ of the closure of a generic orbit in the associated flag Bott manifold also can be written by

$$\Sigma = \widetilde{\Sigma}_{n_1} \bullet \cdots \bullet \widetilde{\Sigma}_{n_{m-1}} \bullet \Sigma_{n_m},$$

where $\widetilde{\Sigma}_{n_\ell}$ is a lifting of the fan Σ_{n_ℓ} of the permutohedral variety whose maximal cones are given by

$$\mathrm{Cone}(u^\ell_{A^\ell_1}, \dots, u^\ell_{A^\ell_{n_\ell}})$$

for a proper chain $\varnothing \subsetneq A^\ell_1 \subsetneq \cdots \subsetneq A^\ell_{n_\ell} \subsetneq [n_\ell + 1]$ of subsets.

By Lemma 5.10, the operations join and star subdivision commute each other. Hence it is enough to show that the star subdivisions of the fan $\widetilde{\Sigma}'_{n_\ell}$ along the cones $\{\mathrm{Cone}(\{u^\ell_x \mid x \in A\}) \mid \varnothing \subsetneq A \subsetneq [n_\ell + 1]\}$ in the decreasing order of dimensions of cones agrees with the fan $\widetilde{\Sigma}_{n_\ell}$. We note that the fan Σ_n of the permutohedral variety can be obtained by star subdivisions of all the cones of dimension grater than 0 of the fan Σ'_n of $\mathbb{C}P^n$ in the decreasing order of dimensions of cones (see Remark 5.2). Moreover, for $1 \le \ell \le m$ and any nonempty proper subset $\varnothing \subsetneq \{x_1, \dots, x_d\} \subsetneq [n_\ell+1]$, the following equalities hold by Corollary 5.5:

$$u^\ell_{\{x_1,\dots,x_d\}} = \sum_{i=1}^{d} u^\ell_{\{x_i\}} = \sum_{i=1}^{d} u^\ell_{x_i}.$$

Therefore the fan $\widetilde{\Sigma}_{n_\ell}$ is obtained from $\widetilde{\Sigma}'_{n_\ell}$ by star subdividing along the cones $\{\mathrm{Cone}(\{u^\ell_x \mid x \in A\}) \mid \varnothing \subsetneq A \subsetneq [n_\ell + 1]\}$ in the given order, so the result follows. □

Remark 5.11. In this paper, we concentrate on the closure of a generic torus orbit in the associated flag Bott manifold. Since the matrices for the associated flag Bott manifolds can have nonzero entries only on the first column, there are flag Bott manifolds which are not the associated flag Bott manifolds. The second and the fourth authors compute the fan of the closure of a generic torus orbit in any flag Bott manifold in [Lee and Suh 2019].

Remark 5.12. There are several studies on the closures of nongeneric torus orbits. For instance, Gelfand and Serganova [1987] studied torus orbit closures in homogeneous manifolds G/P in terms of matroids, and, recently, Lee and Masuda [2020] and Lee et al. [2019] study torus orbit closures associated to Schubert varieties and Richardson varieties, respectively.

5C. *Proof of Theorem 5.4.*

For an m-stage flag Bott manifold F_m, consider the effective canonical $\boldsymbol{H}$-action. Each fiber of a bundle $F_j \to F_{j-1}$ has the restricted $(\mathbb{C}^*)^{n_j}$-action, and its orbit closure of a generic point is the permutohedral variety X_{n_j}. Therefore the closure of a generic orbit of the torus $\boldsymbol{H}$ in F_m has the structure of iterated permutohedral variety bundles. Hence, the following lemma is straightforward from the successive application of Proposition 5.9.

Lemma 5.13. *Let F_m be the associated m-stage flag Bott manifold and X the closure of a generic orbit of the torus $\boldsymbol{H}$ in F_m. Let $\Sigma_{n_1}, \dots, \Sigma_{n_m}$ be fans of permutohedral varieties $X_{n_1}, \dots, X_{n_m}$, respectively. Then, there are liftings $\widetilde{\Sigma}_{n_1}, \dots, \widetilde{\Sigma}_{n_{m-1}}$ such that*

$$\Sigma = \widetilde{\Sigma}_{n_1} \bullet \cdots \bullet \widetilde{\Sigma}_{n_{m-1}} \bullet \Sigma_{n_m}.$$

It remains to compute the primitive generators of rays in Σ. In general, a toric variety can be regarded as a GKM manifold with respect to the action of compact torus in the algebraic torus.

Remark 5.14. Two combinatorial objects, a smooth complete fan Σ and a GKM graph (Γ, α), of a toric variety are related by associating maximal cones in Σ with vertices of Γ, and cones of codimension 1 in Σ with edges of Γ. In particular, if Σ is an n-dimensional smooth fan, then an n-dimensional cone σ has n facets, say $\tau_1, \dots, \tau_n$, which correspond to the outgoing edges, say $e_1, \dots, e_n$, in Γ from the vertex corresponding to σ. Let ρ be a 1-dimensional cone in Σ, then $(n-1)$ many facets of σ contains ρ except one facet.

Regarding Σ be a fan in $\mathrm{Lie}(\boldsymbol{T})$, the next Lemma 5.15 shows the relation between the ray generators of rays in Σ and the axial function $\alpha : E(\Gamma) \to \mathfrak{t}_{\mathbb{Z}}^*$.

Lemma 5.15 [Buchstaber and Panov 2015, Proposition 7.3.18]. *Let $e_1, \dots, e_n$ and ρ be as in Remark 5.14, and u_ρ the ray generator of ρ. Then the following system of equations holds*:

$$\langle \alpha(e_i), u_\rho \rangle = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } 2 \le i \le n. \end{cases} \tag{5-7}$$

In particular, given $\alpha(e_1), \dots, \alpha(e_n)$, the vector u_ρ is uniquely determined.

Lemma 5.15 says that the tangential representation at a fixed point determines the ray generator u_ρ of a 1-dimensional cone ρ contained in the a maximal dimensional cone σ corresponding to the given fixed point. The next lemma shows that u_ρ obtained in (5-7) is independent from the choice of a maximal dimensional cone containing ρ.

Lemma 5.16. *The primitive generator u_ρ of an 1-dimensional cone ρ obtained from (5-7) is well-defined, i.e., it is independent of the choice of a maximal dimensional cone σ containing ρ.*

Proof. Suppose that σ and σ' are two maximal cones containing ρ, whose facets are $\{\tau_i \mid 1 \le i \le n\}$ and $\{\tau_i' \mid 1 \le i \le n\}$, respectively. Here, we may assume that σ and σ' are adjacent, i.e., σ and σ' meet at a common facet, say $\tau_n = \tau_n'$, otherwise we choose a path of maximal cones connecting σ and σ', and apply the same argument.

By the correspondence between cones in a smooth complete fan and a GKM graph mentioned in Remark 5.14, we set up the following notation:

(1) τ_1 and τ_1': facets of σ and σ' which do not contain ρ, respectively;

(2) e_1 and e_1': edges in Γ corresponding to τ_1 and τ_1', respectively.

We refer to Figure 3 for a 3-dimensional example.

Now, it is enough to show that u_ρ satisfies the following relations:

$$\langle \alpha(e_i'), u_\rho \rangle = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } 2 \le i \le n. \end{cases} \tag{5-8}$$

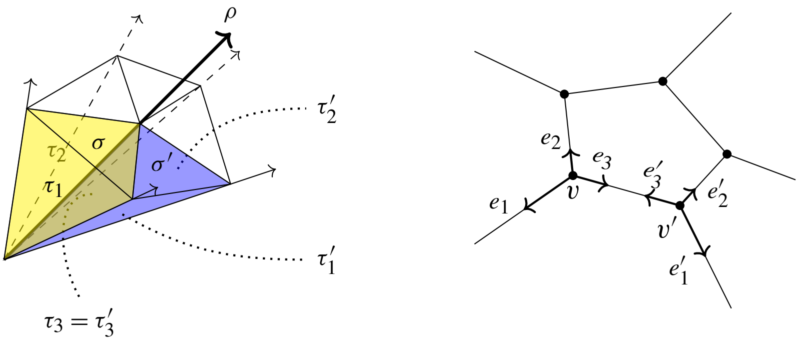


Figure 3. A 3-dimensional fan and corresponding GKM graph.

For the given GKM graph (Γ, α) and the connection $\theta = \{\theta_e \mid e \in E(\Gamma)\}$, consider

$$\theta_{e_n} : \{e_1, \dots, e_n\} \to \{e'_1, \dots, e'_n\}.$$

Since the closure $\overline{O(\rho)}$ of the orbit $O(\rho)$ is a toric subvariety of X_Σ, the subgraph by taking vertices corresponding to maximal cones containing ρ is indeed a GKM-subgraph, whose connection is inherited from the original one θ. Therefore θ_{e_n} maps $\{e_2, \dots, e_n\}$ bijectively to $\{e'_2, \dots, e'_n\}$. Hence we have that $\theta_{e_n}(e_1) = e'_1$. For convenience, we assume that $\theta_{e_n}(e_i) = e'_i$ for $i = 1, \dots, n$.

For $1 \le i \le n$, we have the relation

$$\alpha(e'_i) = \alpha(e_i) + c_i \alpha(e_n),$$

for some $c_i \in \mathbb{Z}$. Hence (5-7) becomes

$$\langle \alpha(e'_i) - c_i\alpha(e_n), u_\rho \rangle = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } 2 \le i \le n, \end{cases}$$

which turn out to be the relations (5-8), because $\langle \alpha(e_n), u_\rho \rangle = 0$. Hence the result follows. □

Now we give a proof of Theorem 5.4. By Lemma 5.13, we know that the combinatorial structure of the fan Σ is given as in Theorem 5.4(2). Now it is enough to show that the ray generators are given as in Theorem 5.4(1).

For a given $1 \le \ell \le m$ and a nonempty proper subset A of $[n_\ell + 1]$, consider a ray $\rho^\ell(A)$ of Σ. To compute the ray generator of $\rho^\ell(A)$, it is enough to consider only one maximal cone containing $\rho^\ell(A)$ because of Lemma 5.16.

We note that there is one-to-one correspondence between the set of maximal cones in $\widetilde{\Sigma}$ and $\prod_{j=1}^m \mathfrak{S}_{n_j+1}$ as in (5-3). More precisely, for $(v_1, \dots, v_m) \in \prod_{j=1}^m \mathfrak{S}_{n_j+1}$, we define

$$A^\ell_p := \{v(n_\ell + 2 - p), \dots, v(n_\ell + 1)\} \quad \text{for } 1 \le p \le n_\ell,\ 1 \le \ell \le m. \tag{5-9}$$

Moreover, for a given maximal cone indexed by $(v_1, \dots, v_m)$, the adjacent maximal cones σ_i^j are determined by permutations

$$(v_1, \dots, v_{j-1}, v_j \cdot s_i, v_{j+1}, \dots, v_m) \tag{5-10}$$

for $1 \le i \le n_j$ and $1 \le j \le m$.

From now on, set

$$A = \{x_1 < x_2 < \cdots < x_{n_\ell+1-d}\} \quad \text{and} \quad [n_\ell + 1] \setminus A = \{y_1 < y_2 < \cdots < y_d\}.$$

Define a permutation $v_{\ell,A}$ to be

$$v_{\ell,A} = (y_1\ y_2 \cdots y_d\ x_1\ x_2 \cdots x_{n_\ell+1-d}) \in \mathfrak{S}_{n_\ell+1}. \tag{5-11}$$

Also define $\boldsymbol{v} := (v_1, \dots, v_\ell, \dots, v_m) \in \prod_{j=1}^m \mathfrak{S}_{n_j+1}$ by setting $v_\ell = v_{\ell,A}$ and $v_j = e \in \mathfrak{S}_{n_j+1}$ for $j \neq \ell$. Then using (5-9), the maximal cone $\sigma_{\boldsymbol{v}}$ indexed by $\boldsymbol{v}$ contains the ray ρ_A^ℓ. We note that among adjacent maximal cones indexed by permutations in (5-10), the maximal cone σ_d^ℓ is the unique maximal cone which does not contain the ray ρ_A^ℓ, because

$$v_\ell \cdot s_d = v_{\ell,A}(d, d+1) = (y_1\ \cdots\ y_{d-1}\ x_1\ y_d\ x_2\ \cdots\ x_{n_\ell+1-d}).$$

Because of Lemmas 5.15 and 5.16, it is enough to show that the vector

$$u_A^\ell = \begin{cases} \sum_{x \in A} \varepsilon_{\ell,x} & \text{if } n_\ell + 1 \notin A, \\ \sum_{x \in [n_\ell+1] \setminus A} -\varepsilon_{\ell,x} + \sum_{j=\ell+1}^m \sum_{k=1}^{n_j} a_{k,\ell}^j \varepsilon_{j,k} & \text{otherwise} \end{cases}$$

in Theorem 5.4 satisfies the following equations:

$$\langle \alpha(e_i^j), u_A^\ell \rangle = \begin{cases} 1 & \text{if } j = \ell \text{ and } i = d, \\ 0 & \text{otherwise,} \end{cases}$$

where e_i^j is an edge of the GKM graph Γ of X corresponding to the facet $\sigma_{\boldsymbol{v}} \cap \sigma_i^j$ of the maximal cone $\sigma_{\boldsymbol{v}}$, and α is the axial function $\alpha : E(\Gamma) \to \mathfrak{t}_{\mathbb{Z}}^*$.

To prove the claim, we separate cases as $j < \ell$, $j = \ell$, and $j > \ell$.

Case 1: $j < \ell$. By Theorem 3.12, the axial functions of the edge $\alpha(e_i^j)$ is a linear combination of $\varepsilon_{1,1}^*, \dots, \varepsilon_{1,n_1}^*, \dots, \varepsilon_{j,1}^*, \dots, \varepsilon_{j,n_j}^*$. On the other hand, since u_A^ℓ is a linear combination of $\varepsilon_{\ell,1}, \dots, \varepsilon_{\ell,n_\ell}, \dots, \varepsilon_{m,1}, \dots, \varepsilon_{m,n_m}$ and $j < \ell$, their pairings always vanish.

Case 2: $j = \ell$. By Theorem 3.12, the axial functions of the edge $\alpha(e_i^\ell)$ is a linear combination of $\varepsilon_{1,1}^*, \dots, \varepsilon_{1,n_1}^*, \dots, \varepsilon_{\ell,1}^*, \dots, \varepsilon_{\ell,n_\ell}^*$. More precisely, we have that

$$\alpha(e_i^\ell) = (\varepsilon_{\ell, v_{\ell,A}(i+1)})^* - (\varepsilon_{\ell, v_{\ell,A}(i)})^* + \text{ other terms,}$$

where "other terms" are the terms of $\varepsilon_{p,k}^*$ for $p < \ell$ and $v_{\ell,A}$ is a permutation defined in (5-11). Since the vector u_A^ℓ is a linear combination of $\varepsilon_{\ell,1}, \dots, \varepsilon_{\ell,n_\ell}, \dots,$

$\varepsilon_{m,1}, \dots, \varepsilon_{m,n_m}$, we have

$$\langle \alpha(e_i^\ell), u_A^\ell \rangle = \big\langle (\varepsilon_{\ell, v_{\ell,A}(i+1)})^* - (\varepsilon_{\ell, v_{\ell,A}(i)})^*, u_A^\ell \big\rangle. \tag{5-12}$$

Because of the definition of the permutation $v_{\ell,A}$, we have that $v_{\ell,A}(i) \in A$ if and only if $i \geq d+1$. Therefore for the case when $n_\ell + 1 \notin A$, we have that the value $\langle (\varepsilon_{\ell, v_{\ell,A}(i)})^*, u_A^\ell \rangle$ equals 0 if $i \leq d$, and 1 otherwise. Also for the case when $n_\ell + 1 \in A$, we get that the pairing $\langle (\varepsilon_{\ell, v_{\ell,A}(i)})^*, u_A^\ell \rangle$ is -1 if $i \leq d$ and 0 otherwise.

By applying (5-12) for $n_\ell + 1 \notin A$, we have

$$\langle \alpha(e_i^\ell), u_A^\ell \rangle = \begin{cases} 0 - 0 = 0 & \text{for } 1 \leq i < d, \\ 1 - 0 = 1 & \text{for } i = d, \\ 1 - 1 = 0 & \text{for } d < i \leq n_\ell. \end{cases}$$

Similarly, when $n_\ell + 1 \in A$, we get

$$\langle \alpha(e_i^\ell), u_A^\ell \rangle = \begin{cases} -1 - (-1) = 0 & \text{for } 1 \leq i < d, \\ 0 - (-1) = 1 & \text{for } i = d, \\ 0 - 0 = 0 & \text{for } d < i \leq n_\ell. \end{cases}$$

<u>Case 3</u>: $j > \ell$. The matrix $X_j^{(\ell)}$ in Proposition 3.5 is

$$X_\ell^{(j)} = \sum_{\ell < i_1 < \cdots < i_r < j} (B_j A_{i_r}^{(j)})(B_{i_r} A_{i_{r-1}}^{(i_r)}) \cdots (B_{i_1} A_\ell^{(i_1)}) B_\ell + B_j A_\ell^{(j)} B_\ell.$$

Since $v_j = e$ for $j \neq \ell$, the matrix $X_\ell^{(j)}$ can be written as

$$X_\ell^{(j)} = \Bigg(\sum_{\ell < i_1 < \cdots < i_r < j} A_{i_r}^{(j)} A_{i_{r-1}}^{(i_r)} \cdots A_\ell^{(i_1)} + A_\ell^{(j)} \Bigg) B_\ell.$$

By Definition 4.6, the matrix $A_i^{(j)}$ has nonzero entries only on the first column. The matrix B_ℓ is the row permutation matrix corresponding to $v_{\ell,A}$, so that B_ℓ is the column permutation matrix corresponding to $v_{\ell,A}^{-1}$. Hence by multiplying the matrix B_ℓ on the right, the matrix $X_\ell^{(j)}$ has nonzero entries only on the y_1-th column.

<u>Subcase 1</u>: $n_\ell + 1 \notin A$. Since the matrix $X_\ell^{(j)}$ has nonzero entries only on the y_1-th column, we have that $\langle \alpha(e_i^j), u_A^\ell \rangle = 0$ for all $j > \ell$.

<u>Subcase 2</u>: $n_\ell + 1 \in A$. For a pair (p, j) such that $\ell < p < j \leq m$, the matrix $X_p^{(j)}$ has nonzero entries only on the first column. For simplicity, for $\ell < p < j$, denote the $(i, 1)$-entry of $X_p^{(j)}$ by $x_{p,i}^{(j)}$. Similarly, denote the (i, y_1)-entry of $X_\ell^{(j)}$ by $x_{\ell,i}^{(j)}$.

Then we have

$$
\begin{aligned}
&\langle \alpha(e_i^j), u_A^\ell \rangle \\
&= \left\langle (x^{(j)}_{\ell,i+1} - x^{(j)}_{\ell,i})(\varepsilon_{\ell,y_1})^* + \sum_{p=\ell+1}^{j-1} (x^{(j)}_{p,i+1} - x^{(j)}_{p,i})(\varepsilon_{p,1})^* + (\varepsilon_{j,i+1})^* - (\varepsilon_{j,i})^*, u_A^\ell \right\rangle \\
&= \big\langle (x^{(j)}_{\ell,i+1} - x^{(j)}_{\ell,i})(\varepsilon_{\ell,y_1})^*, -(\varepsilon_{\ell,y_1} + \cdots + \varepsilon_{\ell,y_d}) \big\rangle \\
&\qquad\qquad + \left\langle \sum_{p=\ell+1}^{j-1} (x^{(j)}_{p,i+1} - x^{(j)}_{p,i})(\varepsilon_{p,1})^* + (\varepsilon_{j,i+1})^* - (\varepsilon_{j,i})^*, \sum_{p=\ell+1}^{m} \sum_{k=1}^{n_p} a^p_{k,\ell} \varepsilon_{p,k} \right\rangle \\
&= (-1)(x^{(j)}_{\ell,i+1} - x^{(j)}_{\ell,i}) + \sum_{p=\ell+1}^{j-1} (x^{(j)}_{p,i+1} - x^{(j)}_{p,i})(a^p_{1,\ell}) + (a^j_{i+1,\ell} - a^j_{i,\ell}).
\end{aligned}
$$

To show the above pairing vanishes, it is enough to show that

$$
x^{(j)}_{\ell,i} = \sum_{p=\ell+1}^{j-1} x^{(j)}_{p,i} a^p_{1,\ell} + a^j_{i,\ell} \quad \text{for all } i,
$$

which comes from the definition of $X^{(j)}_\ell$:

$$
\begin{aligned}
X^{(j)}_\ell B_\ell^{-1} &= \sum_{\ell < i_1 < \cdots < i_r < j} A^{(j)}_{i_r} A^{(i_r)}_{i_{r-1}} \cdots A^{(i_1)}_\ell + A^{(j)}_\ell \\
&= X^{(j)}_{j-1} A^{(j-1)}_\ell + \cdots + X^{(j)}_{\ell+2} A^{(\ell+2)}_\ell + X^{(j)}_{\ell+1} A^{(\ell+1)}_\ell + A^{(j)}_\ell \\
&= \sum_{p=\ell+1}^{j-1} X^{(j)}_p A^{(p)}_\ell + A^{(j)}_\ell.
\end{aligned}
$$

Hence we have $\langle \alpha(e_i^j), u_A^\ell \rangle = 0$ for all $j > \ell$.

Now we prove the smoothness. Since the permutohedral variety X_n is nonsingular (see [Dabrowski 1996, Corollary of Theorem 3.3]), for a proper chain $\varnothing \subsetneq A_1 \subsetneq \cdots \subsetneq A_n \subsetneq [n+1]$ of nonempty proper subsets of $[n+1]$, we have that

$$
\det[u_{A_1}\ u_{A_2}\ \cdots\ u_{A_n}] = \pm 1. \tag{5-13}
$$

To show that a generic torus orbit closure is smooth, it is enough to show that every maximal cone in Σ is smooth. For a maximal cone indexed by $(A^1_\bullet, \ldots, A^m_\bullet)$, consider the matrix whose column vectors are the corresponding ray generators:

$$
\big[u^1_{A^1_1}\ \cdots\ u^1_{A^1_{n_1}}\ \cdots\ u^m_{A^m_1}\ \cdots\ u^m_{A^m_{n_m}}\big]. \tag{5-14}
$$

Then the matrix (5-14) is a block lower triangular matrix whose sizes of blocks are $n_1, \ldots, n_m$. Moreover, the determinant of the matrix in (5-14) is

$$
\det\big(\big[u_{A^1_1}\ \cdots\ u_{A^1_{n_1}}\big]\big) \cdot \det\big(\big[u_{A^2_1}\ \cdots\ u_{A^2_{n_2}}\big]\big) \cdots \det\big(\big[u_{A^m_1}\ \cdots\ u_{A^m_{n_m}}\big]\big) = \pm 1
$$

by (5-13). Here $\{u_{A_1^\ell}, \dots, u_{A_{n_\ell}^\ell}\}$ is the set of ray generators of the maximal cone in the fan of X_{n_ℓ} indexed by the proper chain $\varnothing \subsetneq A_1^\ell \subsetneq \cdots \subsetneq A_{n_\ell}^\ell \subsetneq [n_\ell + 1]$ for $1 \le \ell \le m$. This proves that the closure of a generic torus orbit in the associated flag Bott manifold is smooth.

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SHINTARÔ KUROKI
DEPARTMENT OF APPLIED MATHEMATICS
OKAYAMA UNIVERSITY OF SCIENCE
OKAYAMA-SHI
OKAYAMA
JAPAN
kuroki@xmath.ous.ac.jp

EUNJEONG LEE
CENTER FOR GEOMETRY AND PHYSICS
INSTITUTE FOR BASIC SCIENCE
POHANG
REPUBLIC OF KOREA
eunjeong.lee@ibs.re.kr

JONGBAEK SONG
SCHOOL OF MATHEMATICS
KIAS
SEOUL
REPUBLIC OF KOREA
jongbaek.song@gmail.com

DONG YOUP SUH
DEPARTMENT OF MATHEMATICAL SCIENCES
KAIST
DAEJEON
REPUBLIC OF KOREA
dysuh@kaist.ac.kr

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PROJECTIVE CASES FOR THE RESTRICTION OF THE OSCILLATOR REPRESENTATION TO DUAL PAIRS OF TYPE I

SABINE J. LANG

For all the irreducible dual pairs of type I (G, G'), we analyze the restriction of the oscillator representation as a $(\mathfrak{g}', K')$-module, when G' is the smaller group. For all (G, G') in the stable range, as well as one more case, the modules obtained are projective. We use the duality correspondence introduced by Howe to analyze these restrictions.

1. Introduction

A classical problem in representation theory is the understanding of the restriction of a representation Π of a group G to one of its subgroups H. This work focuses on $(\mathfrak{g}, K)$-modules, as defined by Harish-Chandra. In that setting, if Π is a $(\mathfrak{g}, K)$-module, it is useful to analyze $\mathrm{Hom}_{(\mathfrak{h}, H\cap K)}(\Pi, \pi)$, where π is an $(\mathfrak{h}, H\cap K)$-module. For this purpose, one may use the derived functors: calculating $\mathrm{Ext}^n_{(\mathfrak{h}, H\cap K)}(\Pi, \pi)$ is not necessarily easier than $\mathrm{Hom}_{(\mathfrak{h}, H\cap K)}(\Pi, \pi)$, but their Euler characteristic might be. This difficult part becomes much simpler when the restriction of Π is a projective $(\mathfrak{h}, H \cap K)$-module. In this case, $\mathrm{Ext}^n_{(\mathfrak{h}, H\cap K)}(\Pi, \pi)$ vanishes for every $n > 0$. It motivates this paper: the projectivity of a representation is an extremely powerful property. The link between Euler characteristic and projectivity is emphasized in [Adams et al. 2017], among others.

We focus on dual pairs, an approach introduced in the framework of the duality correspondence for the oscillator representation. A dual pair is a pair (G, G') of subgroups of a symplectic group $\mathrm{Sp}(V)$, such that G is the centralizer of G' in $\mathrm{Sp}(V)$, and vice versa. This work focuses on dual pairs of type I and uses the Fock model of the oscillator representation, ω. We prove:

Theorem. *Let G' be the smaller member of a dual pair (G, G') in a symplectic group $\mathrm{Sp}(V)$. Then the restriction of the Fock model of the oscillator representation ω of $\widetilde{\mathrm{Sp}}(V)$ to G' is a projective $(\mathfrak{g}', K')$-module under the condition $(*)$, as listed in Theorem 6.1. This condition includes the stable range but is slightly less restrictive.*

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It might seem unusual to focus on only one representation of one group. Due to the importance of the oscillator representation, this is however not surprising. This representation appears as (Segal–Shale)–Weil representation [Segal 1963; Shale 1962; Weil 1964] or metaplectic representation, among many other names. The theory of duality correspondence (or Theta correspondence) describes the representations that appear in the decomposition of the oscillator representation after restriction to a dual pair; see [Howe 1989] or [Kashiwara and Vergne 1978] for more details. The duality correspondence is one of the major tools used in this work.

2. Generalities

Let G be a Lie group with complexified Lie algebra $\mathfrak{g}$, and let K be a maximal compact subgroup in G, or its two-fold cover (as needed). We denote by $\mathfrak{k}$ the complexified Lie algebra of K, and we choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$.

Highest weight modules. We have the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. We let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Let $\mathfrak{b}_{\mathfrak{k}}$ is a Borel subalgebra for $\mathfrak{k}$ containing $\mathfrak{t}$, and $\mathfrak{b}$ be a Borel subalgebra for $\mathfrak{g}$ containing $\mathfrak{b}_{\mathfrak{k}}$. We denote the set of positive roots by Δ^+, and write Δ_c for the compact roots, which are the roots coming from $\mathfrak{k}$. The set of noncompact roots is defined as $\Delta_n = \Delta - \Delta_c$. By intersecting Δ^+, we can define the positive compact roots Δ_c^+ and the positive noncompact roots Δ_n^+.

We define $\mathfrak{p}_+$ as the irreducible $\mathfrak{k}$-module spanned by Δ_n^+, and $\mathfrak{p}_-$ as the irreducible $\mathfrak{k}$-module spanned by Δ_n^-. This gives a decomposition of $\mathfrak{p}$ as $\mathfrak{p}_+ + \mathfrak{p}_-$. We then write $\mathfrak{q} = \mathfrak{q}_+ = \mathfrak{k} + \mathfrak{p}_+$ and $\mathfrak{q}_- = \mathfrak{k} + \mathfrak{p}_-$. By definition of $\mathfrak{p}_+$ and $\mathfrak{p}_-$, $\mathfrak{q}_+$ and $\mathfrak{q}_-$ are subalgebras of $\mathfrak{g}$.

Finally, we write $\mathfrak{U}(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$. For a weight λ of $\mathfrak{g}$, F_λ is the irreducible $\mathfrak{k}$-module with highest weight λ, and E_λ is the irreducible $\mathfrak{g}$-module with highest weight λ, with respect to the Borel subalgebras chosen above. We use $N(\lambda)$ to denote the generalized Verma module $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q}_+)} F_\lambda$, which is a $\mathfrak{U}(\mathfrak{g})$-module.

Irreducibility criterion. For any $\alpha \in \Delta$ and $\lambda \in \mathfrak{t}^*$, we write $(\lambda)_\alpha = 2\langle \lambda, \alpha\rangle / \langle \alpha, \alpha\rangle$. The half sum of the positive roots is written ρ, and we use s_α for the reflection through the hyperplane determined by the root α. The following result about the irreducibility of $N(\lambda)$ appears in [Enright et al. 1983, Corollary 6.3 and Theorem 6.4], and the first part is originally due to Jantzen.

Proposition 2.1. *Assume that for any $\alpha \in \Delta_n^+$ with $(\lambda+\rho)_\alpha \in \mathbb{Z}_{>0}$, there is $\gamma \in \Delta_n$ with $(\lambda+\rho)_\gamma = 0$ and $s_\alpha(\gamma) \in \Delta_c$. Then $N(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{q})} F_\lambda$ is irreducible. Moreover, if $\mathfrak{g}$ is of type A_n, it is both a necessary and sufficient condition.*

($\mathfrak{g}$, K)-modules. To stay in an algebraic setting, this work takes place in the category of ($\mathfrak{g}$, K)-modules, defined below. It allows us to us K to denote a maximal compact subgroup in G or its two-fold cover, as this distinction does not affect $(\mathfrak{g}, K)$-modules.

Definition. A $(\mathfrak{g}, K)$-*module* is a complex vector space V with an action of $\mathfrak{g}$ and an action of K such that

(1) for all $v \in V$, $k \in K$, $X \in \mathfrak{g}$, we have $k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$;

(2) V is K-finite, i.e., for every $v \in V$, the space generated by $K \cdot v$ is a finite-dimensional vector space;

(3) for all $v \in V$, $Y \in \mathfrak{k}$, we have $\left(\frac{d}{dt} \exp(tY) \cdot v\right)\Big|_{t=0} = Y \cdot v$.

We recall the Frobenius reciprocity, together with one important corollary.

Proposition 2.2 (Frobenius reciprocity). *Let A, B be two rings with $A \subset B$. Let M be an A-module and N be a B-module. We have a vector space isomorphism* $\mathrm{Hom}_B(B \otimes_A M, N) \cong \mathrm{Hom}_A(M, N)$.

Corollary 2.3. *Let Q be an A-module, and let $P = B \otimes_A Q$. If Q is a projective A-module, then P is a projective B-module.*

As a consequence, we get the following result for $(\mathfrak{g}, K)$-modules:

Proposition 2.4. *Let V be a* $(\mathfrak{k}, K)$*-module. Then* $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{k})} V$ *is a projective* $(\mathfrak{g}, K)$*-module.*

Proof. By K-finiteness, every $(\mathfrak{k}, K)$-module is projective as a $(\mathfrak{k}, K)$-module. Now the result is a direct application of Corollary 2.3 restricted to $(\mathfrak{g}, K)$-modules. ☐

Oscillator representation. We are interested in a particular representation $\widetilde{\omega}$ of the metaplectic group $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$, a double cover of the symplectic group. This representation, called oscillator representation, was first introduced in [Segal 1963; Shale 1962], followed by [Weil 1964]. Several constructions and different models for the oscillator representation appear in [Li 2000; Adams 2007].

For a subgroup G of $\mathrm{Sp}(2N, \mathbb{R})$, we denote by $\widetilde{G}$ its preimage in $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$. We are only interested in algebraic $\widetilde{G}$-modules; hence we consider the category of $(\mathfrak{g}, K)$-modules, for K a maximal compact subgroup of G, or its two-fold cover. Therefore, we work with the Harish-Chandra module of the oscillator representation, a realization of $\widetilde{\omega}$ as an $(\mathfrak{sp}(2N, \mathbb{C}), \widetilde{U}(N))$-module. We still call it the oscillator representation but denote it by ω.

Since most of this work is done on the Lie algebra level, double covers do not play an important role. It is therefore enough to analyze subgroups G, G' in a symplectic group, and it is not necessary to focus on their preimage $\widetilde{G}$, $\widetilde{G}'$ in $\widetilde{\mathrm{Sp}}(2N, \mathbb{R})$.

Reductive dual pairs. To decompose the oscillator representation restricted to a subgroup, we use dual pairs, following Howe's approach.

Definition. A pair (G, G') of subgroups in a symplectic group $\mathrm{Sp}(2N, \mathbb{R})$ is a *reductive dual pair* if

(1) G and G' act reductively on $\mathbb{R}^{2N}$,

(2) G and G' are centralizers of each other inside $\mathrm{Sp}(2N, \mathbb{R})$.

Moreover, if G is compact, we say that (G, G') is a *compact dual pair*.

We assume that G' is the smaller member of the pair so that the duality correspondence holds. We also consider two dual pairs with a particular relation, as introduced in [Kudla 1984]:

Definition. Two dual pairs (G, G') and (H, H') form a *seesaw dual pair* if we have the inclusions $H \subset G$ and $G' \subset H'$. We denote it by $((G, G'), (H, H'))$.

Irreducible dual pairs, i.e., pairs that cannot be decomposed as a direct sum of two dual pairs, are classified in two types. Following [Howe 1989], each pair corresponds to either a division algebra (type II) or a division algebra with an involution (type I). We focus on pairs of type I, which come in four different types:

(1) $(O(p, q), \mathrm{Sp}(2n, \mathbb{R}))$, corresponding to $\mathbb{R}$ with the identity map,

(2) $(O(p, \mathbb{C}), \mathrm{Sp}(2n, \mathbb{C}))$, corresponding to $\mathbb{C}$ with the identity map,

(3) $(U(r, s), U(p, q))$, corresponding to $\mathbb{C}$ with the conjugation map,

(4) $(\mathrm{Sp}(p, q), O^*(2n))$, corresponding to $\mathbb{H}$ with the conjugation map.

This corresponds to seven different cases, depending which group of the pair is the smallest (except for (3), which is symmetric).

Duality correspondence. For a dual pair (G, G') with G compact, we decompose the oscillator representation ω of $\mathrm{Sp}(2N, \mathbb{R})$ under the action of G. We obtain

$$\omega = \bigoplus_{\sigma} (\mathrm{Hom}_G(\sigma, \omega) \otimes \sigma),$$

summing over all the irreducible representations σ of G. Indeed, if $T \in \mathrm{Hom}_G(\sigma, \omega)$ and $v \in \sigma$, then $T(v) \in \omega$ and we have a map $\mathrm{Hom}_G(\sigma, \omega) \times \sigma \to \omega$, $(T, v) \mapsto T(v)$. This map extends to $\mathrm{Hom}_G(\sigma, \omega) \otimes \sigma \to \omega$, which is injective when σ is irreducible. Since G is compact, ω is completely reducible, and $\omega = \bigoplus_{\sigma} (\mathrm{Hom}_G(\sigma, \omega) \otimes \sigma)$.

The duality correspondence gives an explicit description of $\theta(\sigma)$. By compactness of G, $\theta(\sigma)$ is a highest weight module, and we denote its highest weight by τ. We write E_τ for the irreducible $\mathfrak{g}'$-module with highest weight τ. Note that τ is also a dominant weight for $\mathfrak{k}'$, so τ is also the highest weight of a finite dimensional representation of $\mathfrak{k}'$. We use F_τ for the irreducible $\mathfrak{k}'$-module with highest weight τ. We list now the duality correspondence for the pairs of type I.

The explicit correspondence, originally due to Kashiwara and Vergne [1978], will be introduced in Section 4 when used.

3. Setup and method

This section defines the notation, for G, G' subgroups of a large symplectic group:

- G, G' are real Lie groups, with complexified Lie algebras $\mathfrak{g}$, $\mathfrak{g}'$, forming a dual pair (G, G') with G' the smaller member;
- K, K' are maximal compact subgroups of G, G' (or their two-fold covers), respectively, with complexified Lie algebras $\mathfrak{k}$, $\mathfrak{k}'$, and Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{r}'$ for $\mathfrak{g}'$;
- M' is the centralizer of K, so that $((K, M'), (G, G'))$ is a seesaw dual pair, with complexified Lie algebra $\mathfrak{m}'$;
- J' is a maximal compact subgroup of M' (or its two-fold cover) with complexified Lie algebra $\mathfrak{j}'$, and Cartan decomposition $\mathfrak{m}' = \mathfrak{j}' + \mathfrak{p}' = \mathfrak{j}' + \mathfrak{p}'_+ + \mathfrak{p}'_-$;
- $\mathfrak{t}'$ is a Cartan subalgebra of both $\mathfrak{m}'$ and $\mathfrak{j}'$;
- $\mathfrak{q}' = \mathfrak{q}'_+ = \mathfrak{j}' + \mathfrak{p}'_+$ and $\mathfrak{q}'_- = \mathfrak{j}' + \mathfrak{p}'_-$ are two parabolic subalgebras of $\mathfrak{m}'$.

To understand the restriction of ω to G', we encounter two different cases.

(1) $M' \ncong G' \times G'$: We let K act to get a decomposition $\omega = \bigoplus_\sigma (\sigma \otimes E_\tau)$, for σ an irreducible representation of K and E_τ an irreducible representation of M' with highest weight τ. We compute a condition $(*)$ so that $N(\tau)$ is irreducible, which forces $E_\tau = N(\tau) = \mathfrak{U}(\mathfrak{m}') \otimes_{\mathfrak{U}(\mathfrak{q}')} F_\tau$, a projective $(\mathfrak{m}', J')$-module. The restriction from M' to G' is computed to get

$$\omega \cong \bigoplus_\sigma \big(\sigma \otimes \big(\mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (F_\tau|_{\mathfrak{k}'})\big)\big),$$

where each summand is a projective $(\mathfrak{g}', K')$-module, under the condition $(*)$.

(2) $(K, M') = (K_1, G') \oplus (K_2, G')$ with K_1 and K_2 of the same type: This is the case where $K = K_1 \times K_2$, which can be $K = O(p) \times O(q)$, $K = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$, or $K = U(r) \times U(s)$. First, the action of $K_1 \times K_2$ decomposes $\omega = \omega_1 \otimes \omega_2^*$, with ω_1 a highest weight module for K_1, ω_2 a lowest weight module for K_2. Each piece is decomposed as above. We compute a condition $(*)$ so that ω_1 is a projective $(\mathfrak{g}', K')$-module, and another condition $(**)$ ensuring that ω_2^* is a projective $(\mathfrak{g}', K')$-module as well. The tensor product is computed, so that when both conditions $(*)$ and $(**)$ are met, we have

$$\omega = \omega_1 \otimes \omega_2^* = \bigoplus_{\sigma, \widetilde{\sigma}} \big((\sigma \otimes \widetilde{\sigma}) \otimes \big(\mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (F_\tau \otimes F_{\widetilde{\tau}})\big)\big),$$

and each summand is a projective $(\mathfrak{g}', K')$-module.

Section 4 explores conditions so that $N(\tau)$ is irreducible, for each compact dual pair. Section 5 analyzes the restriction from M' to G' in the first case, and the tensor product in the second case. Finally, the results are summarized in Section 6. This work follows a strategy from [Howe 1983], using seesaw pairs to reduce the problem to unitary highest weight modules. In that work, Howe gives a similar result [1983, Theorem 5.2] but from an L^2 perspective.

4. Irreducibility of $N(\tau)$

For each compact dual pair, we give a condition on the respective sizes of the groups so that the generalized Verma module $N(\tau)$ is irreducible. The stable range case is already known (see [Nishiyama et al. 2006], for example), but our results show that this irreducibility holds in one more case. We also show that our bound cannot be extended in a general case, by giving counterexamples.

Dual pair $(K, M') = (O(n,\mathbb{R}), \mathrm{Sp}(2p,\mathbb{R}))$. Since $O(n,\mathbb{R})$ is a disconnected group, we use the embedding

$$O(n,\mathbb{R}) = U(n) \cap \mathrm{GL}(n,\mathbb{R}).$$

Given a highest weight λ of $U(n)$ and a parameter $\epsilon = \pm 1$, the representation of $O(n,\mathbb{R})$ with highest weight $(\lambda; \epsilon)$ is the irreducible summand of the representation of $U(n)$ with highest weight λ containing the highest weight vector, tensored with the sign representation of $O(n,\mathbb{R})$ if $\epsilon = -1$.

The group $M' = \mathrm{Sp}(2p,\mathbb{R})$ has a maximal compact subgroup $J' = U(p)$. We have a correspondence between the highest weight σ for $O(n,\mathbb{R})$ and the highest weight τ for $\mathfrak{sp}(2p,\mathbb{C})$, which appears in [Kashiwara and Vergne 1978, Theorems 6.9 and 7.2, part II]. The correspondence is given by

$$\sigma = (a_1, \dots, a_k, 0, \dots, 0; \epsilon)$$
$$\mapsto \tau = \Bigl(-\frac{n}{2}, \dots, -\frac{n}{2}, \overbrace{-\frac{n}{2}-1, \dots, -\frac{n}{2}-1}^{\frac{1}{2}(1-\epsilon)(n-2k)}, -a_k - \frac{n}{2}, \dots, -a_1 - \frac{n}{2}\Bigr),$$

where σ defines an irreducible highest weight $O(n,\mathbb{R})$-module and τ defines an irreducible highest weight $\mathfrak{sp}(2p,\mathbb{C})$-module. All such weights occur, with the constraints $k \le \left[\frac{n}{2}\right]$ and $k + \frac{1}{2}(1-\epsilon)(n-2k) \le p$. Since we start with a highest weight σ for $O(n,\mathbb{R})$, we also have $a_1 \ge \dots \ge a_k \ge 0$.

The root system occurring here is given by

- $\Delta^+ = \{e_i - e_j \mid 1 \le i < j \le p\} \cup \{e_i + e_j \mid 1 \le i < j \le p\} \cup \{2e_i \mid 1 \le i \le p\}$,
- $\Delta_n^+ = \{e_i + e_j \mid 1 \le i < j \le p\} \cup \{2e_i \mid 1 \le i \le p\}$,
- $\rho = (p, \dots, p+1-i, \dots, 1)$, where $p+1-i$ is the i-th coordinate.

Case $\epsilon = 1$. The products between $\tau + \rho$ and a noncompact positive root are

$$(\tau+\rho)_{2e_i} = \begin{cases} p+1-i-\dfrac{n}{2} & \text{if } 1 \le i \le p-k, \\ p+1-i-\dfrac{n}{2}-a_{p+1-i} & \text{if } p-k < i \le p, \end{cases}$$

$$(\tau+\rho)_{e_i+e_j} = \begin{cases} 2p+2-i-j-n & \text{if } 1 \le i,\, j \le p-k, \\ 2p+2-i-j-n-a_{p+1-j} & \text{if } 1 \le i \le p-k < j \le p, \\ 2p+2-i-j-n-a_{p+1-i}-a_{p+1-j} & \text{if } p-k < i,\, j \le p. \end{cases}$$

If we take $n \geq 2p$, all these products are nonpositive, and by Proposition 2.1 $N(\tau)$ is irreducible. For $n = 2p-1$, we see that $p+1-i-\frac{n}{2} = \frac{3}{2}-i$ is not an integer. All the other products are nonpositive, so the criterion applies, and $N(\tau)$ is irreducible.

Case $\epsilon = -1$. The products of $\tau + \rho$ with noncompact positive roots are

$$(\tau+\rho)_{2e_i} = \begin{cases} p+1-i-\dfrac{n}{2} & \text{if } 1 \le i \le p+k-n, \\ p-i-\dfrac{n}{2} & \text{if } p+k-n < i \le p-k, \\ p+1-i-\dfrac{n}{2}-a_{p+1-i} & \text{if } p-k < i \le p, \end{cases}$$

$$(\tau+\rho)_{e_i+e_j} = \begin{cases} 2p+2-i-j-n & \text{if } 1 \le i,\, j \le p+k-n, \\ 2p-i-j-n & \text{if } p+k-n < i,\, j \le p-k, \\ 2p+2-i-j-n-a_{p+1-i}-a_{p+1-j} & \text{if } p-k < i,\, j \le p, \\ 2p+1-i-j-n & \text{if } 1 \le i \le p+k-n \\ & \quad \text{and } p+k-n < j \le p-k, \\ 2p+1-i-j-n-a_{p+1-j} & \text{if } p+k-n < i \le p-k \\ & \quad \text{and } p-k < j \le p, \\ 2p+2-i-j-n-a_{p+1-j} & \text{if } 1 \le i \le p+k-n \\ & \quad \text{and } p-k < j \le p. \end{cases}$$

For $n \geq 2p$, all these products are nonpositive; hence $N(\tau)$ is irreducible. For $n = 2p-1$, we have $p+1-i-\frac{n}{2} = \frac{3}{2}-i$, which is not an integer, and all the other products are nonpositive. We have proved:

Lemma 4.1. *If $n \geq 2p-1$, then the $(\mathfrak{sp}(2p, \mathbb{C}), \widetilde{U}(p))$-module $N(\tau)$ is irreducible for all the weights τ appearing in the restriction $\omega|_{\mathfrak{sp}(2p,\mathbb{C})}$.*

If $n = 2p-2$, we use Theorem 6.2 from [Enright and Joseph 1990]. Starting from

$$\sigma = (\overbrace{2, \dots, 2}^{p-1}, \overbrace{0, \dots, 0}^{p-1}; 1),$$

we get the highest weight $\tau = (-p+1, -p-1, \dots, -p-1)$, which we write as $\tau = (2, 0, \dots, 0) + (-p-1, \dots, -p-1) = (2, 0, \dots, 0) + (-p-1)\omega_\alpha$ following [Enright and Joseph 1990]. For the family $N(u\omega_\alpha + (2, 0, \dots, 0))$, the first reduction point of the family is given by $u = -p-1$. So $N(\tau)$ is reducible.

***Dual pair* $(K, M') = (U(p), U(m, n))$.** The duality correspondence for this case is expressed with $U(p)$-modules and $\mathfrak{gl}(m+n, \mathbb{C})$-modules, as explained in [Kashiwara and Vergne 1978, Theorem 6.3, part III]. Explicitly, the duality correspondence is given by the map $\sigma \mapsto \tau$ described below:

$$\sigma = \Big(a_1 + \frac{m-n}{2}, \dots, a_k + \frac{m-n}{2}, \frac{m-n}{2}, \dots, \frac{m-n}{2}, b_1 + \frac{m-n}{2}, \dots, b_l + \frac{m-n}{2}\Big)$$
$$\mapsto \tau = \Big(-\frac{p}{2}, \dots, -\frac{p}{2}, b_1 - \frac{p}{2}, \dots, b_l - \frac{p}{2}\Big) \oplus \Big(a_1 + \frac{p}{2}, \dots, a_k + \frac{p}{2}, \frac{p}{2}, \dots, \frac{p}{2}\Big),$$

where σ defines an irreducible highest weight $U(p)$-module and τ defines an irreducible highest weight $\mathfrak{gl}(m+n, \mathbb{C})$-module. All such weights occur, with the constraints $k+l \le p$, $k \le m$, $l \le n$. To make notation easier for the next step, we assume that a_i and b_j can be equal to zero, and change the numbering, so we rewrite τ as

$$\tau = \Big(b_1 - \frac{p}{2}, \dots, b_n - \frac{p}{2}\Big) \oplus \Big(a_{n+1} + \frac{p}{2}, \dots, a_{n+m} + \frac{p}{2}\Big)$$

with $b_n \le \cdots \le b_1 \le 0$ and $0 \le a_{n+m} \le \cdots \le a_{n+1}$.

We apply the irreducibility criterion given by Proposition 2.1. Our group $M' = U(n, m)$ contains a maximal compact subgroup $J' = U(n) \times U(m)$. The root system is of type A_n; hence this criterion is both necessary and sufficient for the irreducibility of $N(\tau)$. We have

- $\Delta^+ = \{e_i - e_j \mid 1 \le i < j \le n+m\}$,
- $\Delta_n^+ = \{e_i - e_j \mid 1 \le i \le n < j \le n+m\}$,
- $\rho = \Big(\frac{m+n-1}{2}, \dots, \frac{m+n-2i+1}{2}, \dots, \frac{-m-n+1}{2}\Big)$, where $\frac{m+n-2i+1}{2}$ is the i-th coordinate.

We obtain $(\tau+\rho)_{e_i - e_j} = b_i - a_j + j - i - p$ with $1 \le i \le n < j \le n+m$. Since $b_i - a_j \le 0$ for all i, j, we conclude that if $p \ge m+n-1$, then $(\tau+\rho)_{e_i-e_j}$ is nonpositive for all i, j, and $N(\tau)$ is irreducible. We deduce:

Lemma 4.2. *If $p \ge m+n-1$, then the $(\mathfrak{gl}(m+n, \mathbb{C}), \widetilde{U}(m) \times \widetilde{U}(n))$-module $N(\tau)$ is irreducible for all the weights τ appearing in the restriction $\omega|_{\mathfrak{gl}_{m+n}(\mathbb{C})}$.*

If $p = m+n-2$, with $m, n \ge 2$, and

$$\sigma = \Big(\overbrace{1 + \frac{m-n}{2}, \dots, 1 + \frac{m-n}{2}}^{n-1}, \overbrace{-1 + \frac{m-n}{2}, \dots, -1 + \frac{m-n}{2}}^{m-1}\Big),$$

we find the corresponding highest weight

$$\tau = \left(-\frac{p}{2}, -1-\frac{p}{2}, \dots, -1-\frac{p}{2}\right) \oplus \left(1+\frac{p}{2}, \dots, 1+\frac{p}{2}, \frac{p}{2}\right).$$

The products $(\tau+\rho)_{e_i-e_j}$ are strictly negative, except for $(\tau+\rho)_{e_1-e_{n+m}} = 1$. But there is no root $\gamma \in \Delta_n^+$ such that $(\tau+\rho)_\gamma = 0$. Since Proposition 2.1 becomes a necessary condition for type A_n, this $N(\tau)$ is reducible.

Dual pair $(K, M') = (\mathrm{Sp}(p), O^*(2n))$. We recall that $\mathrm{Sp}(p)$ can be seen either as the unitary quaternionic group, or as the intersection of $\mathrm{Sp}(2p, \mathbb{C})$ and $U(2p)$. Its complexified Lie algebra is given by $\mathfrak{sp}(2p, \mathbb{C})$. The group $O^*(2n) = \mathrm{SO}^*(2n)$ is the quaternionic orthogonal group. Its complexified Lie algebra is $\mathfrak{o}(2n, \mathbb{C})$.

The duality correspondence for the pair $(\mathrm{Sp}(p), O^*(2n))$ is given by the map $\sigma \mapsto \tau$ described below:

$$\sigma = (a_1, \dots, a_k, 0, \dots, 0) \mapsto \tau = (-p, \dots, -p, -p-a_k, \dots, -p-a_1),$$

where σ defines an irreducible highest weight $\mathrm{Sp}(p)$-module and τ defines an irreducible highest weight $\mathfrak{o}(2n, \mathbb{C})$-module. All such weights occur, with the constraints $k \le p$, $k \le n$. For this case, the correspondence can be deduced from [Howe 1995, Theorem 3.8.5.3]. Again, for notation purposes, we allow $a_i = 0$ and rewrite τ as

$$(-p-a_n, \dots, -p-a_{n-i+1}, \dots, -p-a_1)$$

with $a_1 \ge \dots \ge a_n \ge 0$.

A maximal compact subgroup of M' is $J' = U(n)$. The complexified Lie algebra of M' is of type D_n. Therefore the root system of M' is given by

- $\Delta^+ = \{e_i - e_j \mid 1 \le i < j \le n\} \cup \{e_i + e_j \mid 1 \le i < j \le n\}$,
- $\Delta_n^+ = \{e_i + e_j \mid 1 \le i < j \le n\}$,
- $\rho = (n-1, \dots, n-i, \dots, 0)$, where $n-i$ is the i-th coordinate.

To apply Proposition 2.1, we calculate

$$(\tau+\rho)_{e_i+e_j} = 2n - 2p - i - j - a_{n-i+1} - a_{n-j+1}$$

with $1 \le i < j \le n$. For all i, j, we know that $-a_{n-i+1} - a_{n-j+1} \le 0$. So we conclude that if $p \ge n - \frac{3}{2}$, then $(\tau+\rho)_{e_i-e_j}$ is nonpositive for all i and j, and $N(\tau)$ is irreducible. Since we only consider integral values of n and p, we rewrite the bound as $p \ge n-1$. We proved:

Lemma 4.3. *If* $p \ge n-1$, *then the* $(\mathfrak{o}(2n, \mathbb{C}), \widetilde{U}(n))$*-module* $N(\tau)$ *is irreducible for all the weights* τ *appearing in the restriction* $\omega|_{\mathfrak{o}(2n,\mathbb{C})}$.

When $p = n-2$, we use [Enright and Joseph 1990, Theorem 6.2] again to show that some modules $N(\tau)$ appearing in the restriction of ω are reducible. Choosing $\sigma = (1, \dots, 1, 0)$ gives a highest weight $\tau = (-p, -p-1, \dots, -p-1) = (-n+2, -n+1, \dots, -n+1)$, which is written as

$$(-n+1, \dots, -n+1) + (1, 0, \dots, 0) = (-n+1)\omega_\alpha + (1, 0, \dots, 0)$$

following the notation from [Enright and Joseph 1990]. The first reduction point of the family $N(u\omega_\alpha + (1, 0, \dots, 0))$ is at $u = -n+1$; hence $N(\tau)$ with τ given above is reducible.

5. Modules identifications

The results presented in this section are known; see [Loke and Ma 2015] when (G, G') is in the table range, for example. For readability and consistency of notation, we still include our approach in this paper.

Restriction from M' to G'. We start from M', with complexified Lie algebra $\mathfrak{m}'$ and maximal compact subgroup J'. We recall the Cartan decomposition $\mathfrak{m}' = \mathfrak{j}' + \mathfrak{p}'$, with $\mathfrak{p}' = \mathfrak{p}'_+ + \mathfrak{p}'_-$, and we write $\mathfrak{q}'$ for $\mathfrak{j}' + \mathfrak{p}'_+$. The group G' is a subgroup of M', with complexified Lie algebra $\mathfrak{g}'$. We have a maximal compact subgroup K' of G', and the Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{r}'$.

We consider a finite-dimensional $\mathfrak{j}'$-module E, so E is a $(\mathfrak{j}', J')$-module. By letting $\mathfrak{p}'_+$ act trivially, E becomes a $\mathfrak{q}'$-module and we form $W = \mathfrak{U}(\mathfrak{m}') \otimes_{\mathfrak{U}(\mathfrak{q}')} E$, which is a $(\mathfrak{m}', J')$-module. We analyze the restriction of W as a $(\mathfrak{g}', K')$-module. As vector spaces, we have $W \cong S(\mathfrak{p}'_-) \otimes E$, with $S(\mathfrak{p}'_-)$ the symmetric algebra on $\mathfrak{p}'_-$. From E, we also create a $(\mathfrak{g}', K')$-module: by restriction, we see E as a $\mathfrak{k}'$-module $E|_{\mathfrak{k}'}$, and form the tensor product $V = \mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (E|_{\mathfrak{k}'})$. Similarly, there is an isomorphism of vector spaces $V \cong S(\mathfrak{r}') \otimes (E|_{\mathfrak{k}'})$.

We define two filtrations, $V = \bigoplus_n V_n / V_{n-1}$ and $W = \bigoplus_n W_n / W_{n-1}$, by

$$V_n = \sum_{r \le n} S(\mathfrak{r}')[r] \mathfrak{U}(\mathfrak{k}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (E|_{\mathfrak{k}'}) \quad \text{and} \quad W_n = \sum_{r \le n} S(\mathfrak{p}'_-)[r] \mathfrak{U}(\mathfrak{j}') \otimes_{\mathfrak{U}(\mathfrak{q}')} E.$$

By Frobenius reciprocity, we have a map $T : V \to W$, $1 \otimes e \mapsto 1 \otimes e$ for any $e \in E$. Writing $\{x_1, \dots, x_r\}$ for a basis of $\mathfrak{r}'$, $\{y_1, \dots, y_r\}$ for a basis of $\mathfrak{p}'_+$ and $\{z_1, \dots, z_r\}$ for a basis of $\mathfrak{p}'_-$ such that $x_i = y_i + z_i$ in $\mathfrak{p}'$, we extend the map T linearly so that

$$T(x_1 \cdots x_n \otimes e) = (y_1 + z_1) \cdots (y_n + z_n) \otimes e$$

for any $e \in E$. The map $T : V \to W$ now preserves the filtrations, which proves:

Lemma 5.1. *The map $T : V \to W$ is an isomorphism of $\mathfrak{U}(\mathfrak{g}')$-modules, induced by an isomorphism of $S(\mathfrak{r}')$-modules on the graded spaces $T_G : \mathrm{Gr}(V) \to \mathrm{Gr}(W)$, through the isomorphism given by $\mathfrak{r}' \hookrightarrow \mathfrak{p}' \twoheadrightarrow \mathfrak{p}'_-$, as presented in* [Howe 1989].

This implies that

$$\big(\mathfrak{U}(\mathfrak{m}')\otimes_{\mathfrak{U}(\mathfrak{q}')} E\big)\big|_{\mathfrak{g}'} \cong \mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{k}')} (E|_{\mathfrak{k}'}).$$

This is applied to $E = F_\tau$ in this work.

Tensor product for $(K, M') = (K_1, G')\oplus(K_2, G')$. We start from the Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{r}'$. We decompose $\mathfrak{r}'$ further as $\mathfrak{r}'_+ + \mathfrak{r}'_-$ and note that $\mathfrak{r}'_+$ and $\mathfrak{r}'_-$ are commutative Lie algebras. We define $\mathfrak{q}'_+ = \mathfrak{k}' + \mathfrak{r}'_+$ and $\mathfrak{q}'_- = \mathfrak{k}' + \mathfrak{r}'_-$.

We consider two finite-dimensional $\mathfrak{k}'$-modules E and F; these are $(\mathfrak{k}', K')$-modules. We let $\mathfrak{r}'_+$ act on E by zero, so E becomes a $\mathfrak{q}'_+$-module. Similarly, we let $\mathfrak{r}'_-$ act on F by zero and obtain a $\mathfrak{q}'_-$-module. We define $V_E = \mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E$ and $V_F = \mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F$, which are $(\mathfrak{g}', K')$-modules, and $V = \mathfrak{U}(\mathfrak{g}')\otimes_{\mathfrak{U}(\mathfrak{k}')} (E\otimes F)$.

By the Poincaré–Birkhoff–Witt theorem, there exists a grading on both $\mathfrak{U}(\mathfrak{r}'_+)$ and $\mathfrak{U}(\mathfrak{r}'_-)$. Since $\mathfrak{r}'_+$ and $\mathfrak{r}'_-$ are commutative, $\mathfrak{U}(\mathfrak{r}'_+) = S(\mathfrak{r}'_+)$ and $\mathfrak{U}(\mathfrak{r}'_-) = S(\mathfrak{r}'_-)$. We identify the piece $\mathfrak{U}(\mathfrak{r}'_+)[n]$ of degree n with the space $S(\mathfrak{r}'_+)[n]$ of homogeneous polynomials of degree n (same for $\mathfrak{r}'_-$). We write M_n for the subspace of elements of degree less than or equal to n in $\mathfrak{U}(\mathfrak{r}')$:

$$M_n = \sum_{r+s\le n} \big(S(\mathfrak{r}'_-)[r]\otimes S(\mathfrak{r}'_+)[s]\big) \cong \bigoplus_{i\le n} S(\mathfrak{r}')[i].$$

From this, we define a filtration $V = \bigoplus_n V_n/V_{n-1}$ by $V_n = M_n\mathfrak{U}(\mathfrak{k}')\otimes_{\mathfrak{U}(\mathfrak{k}')} (E\otimes F)$. Note that $V_0 = E\otimes F$. By the description of M_n as $\bigoplus_{i\le n} S(\mathfrak{r}')[i]$, the quotient M_n/M_{n-1} is identified with $S(\mathfrak{r}')[n]$ and we get $V_n/V_{n+1} = S(\mathfrak{r}')[n]\otimes_{\mathfrak{U}(\mathfrak{k}')} (E\otimes F)$. We define a similar filtration on $V_E\otimes V_F$:

$$(V_E\otimes V_F)_n = \sum_{r+s\le n} \Big(\big(M_r\mathfrak{U}(\mathfrak{k})\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E\big)\otimes\big(M_s\mathfrak{U}(\mathfrak{k})\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F\big)\Big).$$

As vector spaces, this is equivalent to

$$(V_E\otimes V_F)_n = \sum_{r+s\le n} \big(S(\mathfrak{r}'_-)[r]\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E\big)\otimes\big(S(\mathfrak{r}'_+)[s]\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F\big).$$

Hence we have

$$(V_E\otimes V_F)_n/(V_E\otimes V_F)_{n-1} = \sum_{r+s=n} \big(S(\mathfrak{r}'_-)[r]\otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E\big)\otimes\big(S(\mathfrak{r}'_+)[s]\otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F\big).$$

Since $E\otimes F = V_0$ is naturally a subset of V, we use Frobenius reciprocity to extend this inclusion to a map

$$T : V \to V_E\otimes V_F, \quad 1\otimes(e\otimes f) \mapsto (1\otimes e)\otimes(1\otimes f),$$

for all $e \in E$ and $f \in F$. We extend this map so that it is compatible with the module structure. By using bases of $\mathfrak{r}'_+$, $\mathfrak{r}'_-$, it is a simple computation to show that T preserves the filtrations.

Lemma 5.2. *The map $T : V \to V_E \otimes V_F$, induced by an isomorphism of $S(\mathfrak{r}')$-modules on the graded spaces $T_G : \mathrm{Gr}(V) \to \mathrm{Gr}(V_E \otimes V_F)$, is an isomorphism of $\mathfrak{U}(\mathfrak{g}')$-modules.*

Proof. We know that $T(V_n) \subset (V_E \otimes V_F)_n$. Using computations with the action of basis elements of $\mathfrak{r}'_+$ and $\mathfrak{r}'_-$ and by tracking the degrees, we show that the map $T_G : \mathrm{Gr}(V) \to \mathrm{Gr}(V_E \otimes V_F)$ is surjective. Finally, as $\mathbb{C}$-vector spaces, we have

$$\begin{aligned}\dim(V_n) &= \dim(E)\dim(F)\Bigg(\sum_{r+s\le n} \dim(S(\mathfrak{p}'_-)[r])\dim(S(\mathfrak{p}'_+)[s])\Bigg)\\ &= \dim((V_E \otimes V_F)_n),\end{aligned}$$

which concludes the proof. □

Hence, we have proved that $V_E \otimes V_F \cong V$, i.e.,

$$\big(\mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{q}'_+)} E\big) \otimes \big(\mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{q}'_-)} F\big) \cong \mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\mathfrak{k}')} (E \otimes F).$$

6. Conclusion

Theorem 6.1. *Let G' be the smaller member of a dual pair (G, G') in a symplectic group $\mathrm{Sp}(V)$. Then the restriction of the Fock model of the oscillator representation ω of $\widetilde{\mathrm{Sp}}(V)$ to G' is a projective $(\mathfrak{g}', K')$-module under the condition $(*)$ listed in the table below*:

	(G, G')	(K, M')	$(*)$
(i)	$(\mathrm{Sp}(2n, \mathbb{R}), O(p, q))$	$(U(n), U(p, q))$	$n \ge p+q-1$
(ii)	$(O(p, q), \mathrm{Sp}(2n, \mathbb{R}))$	$(O(p), \mathrm{Sp}(2n, \mathbb{R}))$ $\oplus(O(q), \mathrm{Sp}(2n, \mathbb{R}))$	$p, q \ge 2n-1$
(iii)	$(O^*(2n), \mathrm{Sp}(p, q))$	$(U(n), U(2p, 2q))$	$n \ge 2(p+q)-1$
(iv)	$(\mathrm{Sp}(p, q), O^*(2n))$	$(\mathrm{Sp}(p), O^*(2n))$ $\oplus(\mathrm{Sp}(q), O^*(2n))$	$p, q \ge n-1$
(v)	$(\mathrm{Sp}(2n, \mathbb{C}), O(p, \mathbb{C}))$	$(\mathrm{Sp}(n), O^*(2p))$	$n \ge p-1$
(vi)	$(O(p, \mathbb{C}), \mathrm{Sp}(2n, \mathbb{C}))$	$(O(p), \mathrm{Sp}(4n, \mathbb{R}))$	$p \ge 4n-1$
(vii)	$(U(r, s), U(p, q))$	$(U(r), U(p, q))$ $\oplus(U(s), U(p, q))$	$r, s \ge p+q-1$

Proof. For cases (i), (iii), (v) and (vi), ω is decomposed under the action of K as $\omega = \bigoplus_\sigma (\sigma \otimes E_\tau)$. Section 4 shows that $(*)$ is a necessary condition for the equality $N(\tau) = E_\tau$. Finally, the restriction from M' to G' is computed in Lemma 5.1.

For cases (ii), (iv) and (vii), ω is decomposed under the action of $K_1 \times K_2$ as $\omega = \omega_1 \otimes \omega_2^*$. The condition $(*)$ is computed for each case to get $N(\tau) = E_\tau$ in

Section 4. The two pieces ω_1 and ω_2^* are put back together through the tensor product described in Lemma 5.2. □

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SABINE J. LANG
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
SALT LAKE CITY, UT
UNITED STATES

lang@math.utah.edu

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A REMARK ON A TRACE PALEY–WIENER THEOREM

GORAN MUIĆ

We prove a version of a trace Paley–Wiener theorem for tempered representations of a reductive p-adic group. This is applied to complete certain investigations of Shahidi on the proof that a Plancherel measure is an invariant of an L-packet of discrete series.

1. Introduction

Let G be a reductive p-adic group. Let $\mathrm{Rep}(G)$ be the category of smooth admissible complex representations of G of finite length, and let $R(G)$ be the corresponding Grothendieck group. We write $\Psi(G)$ (resp., $\Psi^u(G)$) for the group (resp., unitary group) of unramified characters of G. The group $\Psi(G)$ has a structure of an algebraic variety (a complex tours). The corresponding algebra of regular functions $\mathbb{C}[\Psi(G)]$ is generated by evaluations on elements of G as a $\mathbb{C}$-algebra. The subgroup $\Psi^u(G)$ is Zariski dense in $\Psi(G)$. We say that a complex function is regular on $\Psi^u(G)$ if it is a restriction of a regular function on $\Psi(G)$. We observe that the restriction map from $\mathbb{C}[\Psi(G)]$ into functions on $\Psi^u(G)$ is injective since $\Psi^u(G)$ is Zariski dense in $\Psi(G)$.

We fix a minimal parabolic subgroup P_0, its Levi decomposition $P_0 = M_0U_0$, and, as usual related to these choices, we fix a set of standard parabolic subgroups $P = MU$, where $M_0 \subset M,\ P = MP_0$. Since the standard parabolic subgroup is determined by the choice of Levi subgroup, the normalized parabolic induction $\mathrm{Ind}_P^G(\sigma)$, where σ is a smooth representation of M, we write as usual $i_{GM}(\sigma)$.

In [Bernstein et al. 1986], Bernstein, Deligne, and Kazhdan proved a trace Paley–Wiener theorem for category $\mathrm{Rep}(G)$. We consider a full subcategory $\mathrm{Rep}_t(G)$ of $\mathrm{Rep}(G)$ consisting of representations having all irreducible subquotients tempered. Let $R_t(G)$ be the corresponding Grothendieck group. We write $R_t^i(G)$ for the subgroup of $R_t(G)$ generated by $i_{GM}(\sigma)$, where M ranges over all standard Levi subgroups of G (including G), and σ ranges over a set of square-integrable modulo center irreducible representations of M. We warn the reader that this notion is not an analogue of the notion of strictly induced modules from

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[Bernstein et al. 1986, §3.1]. An analogue would be the subgroup of $R_t(G)$ generated by $i_{GM}(\tau)$, where M ranges over all *proper* standard Levi subgroups of G, and τ ranges over irreducible tempered representations of M. But this is not useful for us in the present paper.

The main result of the present paper is the following version of a trace Paley–Wiener theorem:

Theorem 1.1. *Let $f : R_t(G) \to \mathbb{C}$ be a $\mathbb{Z}$-linear form such that the following hold*:

(i) *There exists an open compact subgroup $K \subset G$ which dominates f (i.e., f is nonzero only on those irreducible tempered representations which have a nontrivial space of K-invariant vectors).*

(ii) *For each standard maximal Levi subgroup M, or $M = G$, and a square-integrable modulo center representation σ of M, the function $\psi \mapsto f(i_{GM}(\psi\sigma))$ is regular on $\Psi^u(M)$, and for any other proper standard Levi subgroup N, and a square-integrable modulo center representation τ of N, we have $f(i_{GN}(\tau)) = 0$.*

Then, there exists $F \in C_c^\infty(G)$ such that

$$f(\pi) = \operatorname{tr}(\pi(F)) \quad \textit{for all } \pi \in R_t^i(G).$$

Theorem 1.1 is a proved by reduction to the main result of [Bernstein et al. 1986] using the Harish-Chandra theory of tempered representations [Waldspurger 2003] and some standard considerations related to the Langlands classification [Renard 2010, Chapter VII]. The proof is given in Section 3. It is a consequence of its effective version given by Proposition 3.4. Proposition 3.4 constructs a correct function needed in the proof of [Shahidi 1990, Proposition 9.3 2] in the case when M (see notation there) is a Levi subgroup of a maximal parabolic subgroup. We remark that since Plancherel factors are multiplicative, it is enough to prove [Shahidi 1990, Proposition 9.3 2] for a maximal Levi subgroup.

2. Preliminaries

We continue with the notation introduced in the introduction. Let M be a standard Levi subgroup. Then, we write $\Psi(M)^r$ for the group of all unramified characters ψ which are $\mathbb{R}_{>0}$-valued. As we stated in the introduction, every standard Levi subgroup M determines a unique standard parabolic subgroup, say P. We denote by $\Psi(M)^{r,+}$ the set of all characters from $\Psi(M)^r$ which correspond to the points of the (open) Weyl chamber determined by the roots of the split component of M which belong to the unipotent radical of P in the usual description of unramified characters (see, for example, [Muić 2008, Section 2]). If $M = G$, then $\Psi(M)^{r,+} = \Psi(M)^r$.

For a standard Levi subgroup M, an irreducible tempered representation π of M, and $\psi \in \Psi(M)^{r,+}$, the module $i_{GM}(\psi\pi)$ is called a standard module; it has a unique (Langlands quotient) $L(i_{GM}(\psi\pi))$. The condition is empty if $M = G$. By the Langlands classification [Renard 2010, Theorem VII.4.2], every irreducible representation can be expressed in the form $L(i_{GM}(\psi\pi))$ for unique such datum (M, π, ψ). The following standard result will be used in the proof:

Lemma 2.1. *The standard modules of G form a $\mathbb{Z}$-basis of $R(G)$.*

Proof. The proof is as in [Clozel 1986, Proposition 1]. □

In analogy with [Bernstein et al. 1986, §2.1], we make the following definitions.

Let $\sigma \in \operatorname{Irr}(M)$ where M is a standard Levi subgroup of G. We define the usual affine variety attached to σ

$$\operatorname{Irr}(M) \supset D(\sigma) = \Psi(M)\sigma = \Psi(M)/\operatorname{Stab}_{\Psi(M)}(\sigma),$$

where $\operatorname{Stab}_{\Psi(M)}(\sigma)$ is a finite group consisting of all $\psi \in \Psi(M)$ such that $\psi\sigma \simeq \sigma$.

If A is a maximal split torus in the center of M, the restriction map $\Psi(M) \to \Psi(A)$ is surjective, and the kernel is a finite group. Therefore, by considering the restriction to A we find that

$$\operatorname{Stab}_{\Psi^u(M)}(\sigma) = \operatorname{Stab}_{\Psi(M)}(\sigma).$$

So, we may consider

$$D^u(\sigma) \overset{\text{def}}{=} \Psi^u(M)/\operatorname{Stab}_{\Psi^u(M)}(\sigma) \subset D(\sigma).$$

It is easy to see that $D^u(\sigma)$ is Zariski dense in $D(\sigma)$.

The action of the Weyl group

$$W(M) = N_G(M)/M$$

on $\Psi(M)$ is algebraic. Furthermore, $w \in W(M)$ transforms $\operatorname{Stab}_{\Psi(M)}(\sigma)$ onto $\operatorname{Stab}_{\Psi(M)}(w(\sigma))$, so it maps $D(\sigma)$ (resp., $D^u(\sigma)$) onto $D(w(\sigma))$ (resp., $D^u(w(\sigma))$).

Put $D = D(\sigma)$ and $D^u = D^u(\sigma)$. As usual, we consider the group $W(D)$ of all $w \in W(M)$ such that there exists $\psi_w \in \Psi(M)$ such that

$$w(\sigma) \simeq \psi_w\sigma. \tag{2.2}$$

The character ψ_w is determined uniquely modulo $\operatorname{Stab}_{\Psi(M)}(\sigma)$. The group $W(D)$ acts on the affine variety $D = \Psi(M)/\operatorname{Stab}_{\Psi(M)}(\sigma)$ as follows:

$$w.\psi \operatorname{Stab}_{\Psi(M)}(\sigma) = \psi_w w(\psi) \operatorname{Stab}_{\Psi(M)}(\sigma). \tag{2.3}$$

The resulting orbit space

$$D/W(D)$$

is again an affine variety with algebra of regular functions given as usual,

$$\mathbb{C}[D/W(D)] = \mathbb{C}[D]^{W(D)}.$$

One can construct a regular function $D/W(D)$ in the following way:

Lemma 2.4. *Let $F \in C_c^\infty(G)$. Then, the function $\psi \mapsto \mathrm{tr}(i_{GM}(\psi\sigma)(F))$ is a regular function on $D/W(D)$.*

Proof. It is standard that this function is regular on D. We show that it is $W(D)$-invariant. Let $w \in W(D)$. By [Bernstein et al. 1986, Lemma 5.4 (iii)], we have

$$\mathrm{tr}(i_{GM}(\psi\sigma)(F)) = \mathrm{tr}(i_{GM}(w(\psi\sigma))(F)),$$

which completes the proof. □

The above explicit description shows that the analogously defined group $W(D^u)$ is a subgroup of $W(D)$. In fact, we have the following lemma:

Lemma 2.5. *Assume that the central character $\omega_\sigma : A \to \mathbb{C}^\times$ of σ is unitary. Then, $W(D^u) = W(D)$. Moreover, $D^u/W(D)$ is Zariski dense in $D/W(D)$.*

Proof. As we remarked above, it is always $W(D^u) \subset W(D)$. Conversely, if $w \in W(D)$, then $w(\sigma) \simeq \psi_w\sigma$ by (2.2). Considering central characters, we find that

$$\omega_{w(\sigma)} = (\psi_w|_A)\omega_\sigma.$$

This implies that $\psi_w|_A$ is a unitary character. By the standard description of unramified characters of M, and its relation to unramified characters of A, this implies that $\psi_w \in \Psi^u(M)$ (see [Muić 2008, Section 2]). Hence, $w \in W(D^u)$. This completes the proof that $W(D^u) = W(D)$. The remaining claim is obvious from above considerations. □

The following lemma is a fundamental result of Harish-Chandra:

Lemma 2.6. *Assume that M and N are standard Levi subgroups of G, and σ and τ are square-integrable modulo center representations of M and N, respectively. Then, $i_{GM}(\sigma)$ and $i_{GN}(\tau)$ have a common irreducible subrepresentation if and only if there exists $w \in G$ such that $N = wMw^{-1}$ and $\tau \simeq w(\sigma)$, where $w(\sigma)$ is defined by $w(\sigma)(n) = \sigma(w^{-1}nw),\ n \in N$. Moreover, if there exists $w \in G$ such that $N = wMw^{-1}$, then $i_{GM}(\sigma)$ and $i_{GM}(w(\sigma))$ are isomorphic, and in particular equal in $R_t(G)$.*

Proof. See [Waldspurger 2003]. □

Motivated by [Bernstein et al. 1986, §2.1], we proceed as follows. By the standard theory of tempered irreducible representations due to Harish-Chandra (see [Waldspurger 2003]), for an irreducible tempered representation $\pi \in \mathrm{Irr}(G)$, there exists a standard Levi subgroup M and a square-integrable modulo center representation σ of M such that $\pi \hookrightarrow i_{GM}(\sigma)$. The pair (M, σ) is unique up to a conjugation (see Lemma 2.6). We call the equivalence class $[M, \sigma]$ under conjugation of the pair (M, σ) the t-infinitesimal character of π. The set of equivalence of such pairs we denote by $\Theta_t(G)$.

For a pair (M, σ), we define a natural map $\Psi^u(M) \to \Theta_t(G)$ given by

$$\psi \mapsto [M, \psi\sigma].$$

The image is called a connected component of $\Theta_t(G)$. We denote it by $\Theta_t(M, \sigma)$. This map induces a bijection which enables us to identify

$$\Theta_t(M, \sigma) = D^u(\sigma)/W(D(\sigma)).$$

Thus, in view of Lemma 2.5, we may consider

$$\Theta_t(M, \sigma) \subset D(\sigma)/W(D(\sigma)).$$

This realizes $\Theta_t(M, \sigma)$ as a Zariski dense subset of the affine variety $D(\sigma)/W(D(\sigma))$.

As in [Bernstein et al. 1986, §2.1], we can decompose

$$R_t(G) = \oplus_\theta R_t(G)(\theta), \tag{2.7}$$

where θ ranges over connected components of $\Theta_t(G)$. Here

$$R_t(G)(\theta)$$

is generated with all tempered irreducible representations with t-infinitesimal characters belonging to θ. We denote by 1_θ the projector

$$R_t(G) \to R_t(G)(\theta),$$

for all $\theta \in \Theta_t(G)$.

We end this section with an analogue for $\mathrm{Rep}_t(G)$ of the decomposition theorem for the category of all smooth complex representations of G (see [Bernstein et al. 1986, §2.3]; [Bernstein 1984, §2.10]).

Lemma 2.8. *Let $K \subset G$ be an open compact subgroup. Then, there exists a finite set T_K consisting of connected components in $\Theta_t(G)$ such that for each irreducible tempered representation $\pi \in \mathrm{Rep}_t(G)$, having nonzero space of K-invariants, there exists $\theta \in T_k$ such that $\pi \in R_t(G)(\theta)$.*

Proof. By the decomposition theorem (see [Bernstein et al. 1986, §2.3]), there exists a finite set, say S, of pairs (N, ρ), where N is a standard Levi subgroup of G, and ρ are irreducible supercuspidal representations, such that for every irreducible representation π of G, having nonzero space of K-invariants, there exists $(N, \rho) \in S$, and an unramified character χ such that π is a subquotient of $i_{G,N}(\chi\rho)$.

Now, assume that π is as in the statement of the lemma. Then, there exist a standard Levi subgroup M and a square-integrable modulo center σ of M such that $\pi \hookrightarrow i_{GM}(\sigma)$. Moreover, there exist a standard Levi subgroup M' of M (and of G), and a supercuspidal irreducible representation ρ' such that σ is an irreducible subquotient of $i_{M,M'}(\rho')$. By induction in stages, π must be a subquotient of $i_{G,M'}(\rho')$.

By standard theory of induced representations [Bernstein and Zelevinsky 1977], the pair (M', ρ') must be G-conjugate to the one in S. Thus, we may assume that $(M', \rho') \in S$ already.

Thus, it is enough to prove that given $(N, \ \rho) \in S$ and given a standard Levi subgroup M of G such that $N \subset M$, there are finitely many $\Psi^u(M)$-orbits of square-integrable modulo center representations of M such they are subquotients of the induced representations in the family $i_{M,N}(\chi\rho)$ parametrized by $\chi \in \Psi(N)$. But that is easy. We can select a sufficiently small open compact subgroup $L \subset M$ such that every irreducible representation that appears as a subquotient of $i_{M,N}(\chi\rho)$ for some $\chi \in \Psi(N)$ has a nonzero space of L-invariants.

Hence, we need to prove that there are finitely many $\Psi^u(M)$-orbits of square-integrable modulo center representations of M having a nonzero space of L-invariants. This is proved in (iii) in the introduction of [Waldspurger 2003]. ☐

3. Proof of Theorem 1.1

We begin the proof of Theorem 1.1 with the following lemma:

Lemma 3.1. *Let f be as in the statement of Theorem 1.1. Then, there exists a finite set T_f consisting of connected components in $\Theta_t(G)$ such that for each irreducible tempered representation $\pi \in \mathrm{Rep}_t(G)$ such that $f(\pi) \neq 0$ there exists $\theta \in T_f$ such that $\pi \in R_t(G)(\theta)$.*

Proof. This follows from the assumption (i) in Theorem 1.1 combined with Lemma 2.8. ☐

By Lemma 3.1, we can decompose f into $\mathbb{Z}$-linear forms $f_\Theta : R_t(G) \to \mathbb{C}$, $\theta \in T_f$,

$$f = \sum_{\theta \in T_f} f_\theta,$$

where f_θ is defined as follows (see (2.7)):

$$f_\theta = f \circ 1_\theta.$$

Obviously, each f_θ satisfies the assumptions analogous to (i) and (ii) in Theorem 1.1.

Hence, in what follows we may assume that $f = f_\theta$ for some $\theta \in \Theta_t(G)$. By the assumption (ii) of Theorem 1.1, we may assume that θ has the form $\theta = \Theta_t(M, \sigma)$, where M is a standard maximal Levi subgroup of G, or $M = G$, and σ is a σ is a square-integrable modulo center representation of M. We observe that

$$\psi \in \Psi^u(M) \mapsto f(i_{GM}(\psi\sigma))$$

is a regular function by the assumption (ii) of Theorem 1.1. Thus, by definition this means that it is a restriction of a regular function, say a, on the affine variety $\Psi(M)$.

By Lemma 2.6, we have

$$a \in \mathbb{C}[D]^{W(D)}, \tag{3.2}$$

where

$$D = \Psi(M)/\operatorname{Stab}_{\Psi(M)}(\sigma). \tag{3.3}$$

We refer to previous section for the notation.

Now, the following proposition completes the proof of the theorem.

Proposition 3.4. *Let M be a standard maximal Levi subgroup of G, or $M = G$. Assume that σ is a square-integrable modulo center representation of M. We define D by (3.3), and let a be any function in $\mathbb{C}[D]^{W(D)}$. Then, there exists $F \in C_c^\infty(G)$ such that*

$$tr(\pi(F)) = \begin{cases} a(\psi) & \textit{for } \pi = i_{GM}(\psi\sigma),\ \psi \in \Psi^u(M), \\ 0 & \textit{for } \pi = i_{GN}(\psi\tau),\ \psi \in \Psi^u(N), \end{cases}$$

for any other standard Levi subgroup N and a square-integrable modulo center representation τ such that $\Theta_t(N, \tau) \neq \Theta_t(M, \sigma)$.

Proof. The proof of Proposition 3.4 is a generalization of [Clozel 1986, §4.2, Proposition 1] where the proof of existence of pseudocoefficients for semisimple G is given based also on [Bernstein et al. 1986]. We consider only the case where M is a standard maximal Levi subgroup of G. The case of $M = G$ is about the construction of a specific pseudocoefficient of σ. The proof is on the same lines but considerably easier.

We remark that $\Psi^u(G)$ acts on $\Psi^u(M)$ in a usual way:

$$\psi \mapsto \chi|_M \psi, \quad \chi \in \Psi^u(G), \quad \psi \in \Psi^u(M).$$

For $\psi \in \Psi^u(M)$, the stabilizer

$$\operatorname{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$$

is the group of all $\chi \in \Psi^u(G)$ such that

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

We remind the reader that for all $\chi \in \Psi^u(G)$ we have

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\chi|_M \psi\sigma).$$

Lemma 3.5. *Assume that $\chi \in \Psi^u(G)$ and $\psi \in \Psi^u(M)$. Then, for each irreducible constituent π of $i_{GM}(\psi\sigma)$, the multiplicity of $\chi\pi$ in $\chi i_{GM}(\psi\sigma)$ is the same as that of π in $i_{GM}(\psi\sigma)$.*

Proof. This is obvious. □

Lemma 3.6. *Assume that for $\chi \in \Psi^u(G)$ and $\psi \in \Psi^u(M)$ there exists an irreducible constituent π of $i_{GM}(\psi\sigma)$ such that $\chi\pi$ is an irreducible constituent of $i_{GM}(\psi\sigma)$. Then, $\chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$. In particular, we have*

$$\mathrm{Stab}_{\Psi^u(G)}(\pi) \subset \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma)).$$

Proof. First, $\chi\pi$ is a common constituent of $i_{GM}(\psi\sigma)$ and $i_{GM}(\chi|_M\psi\sigma)$. So, by Lemma 2.6, there exists $w \in W(M)$ such that

$$\chi|_M\psi\sigma = w(\psi\sigma).$$

Then, again by Lemma 2.6, we obtain

$$\chi i_{GM}(\psi\sigma) \simeq i_{GM}(\chi|_M\psi\sigma) \simeq i_{GM}(\psi\sigma). \qquad \square$$

Lemma 3.7. *Let $\psi \in \Psi^u(M)$. Then, we have the following*:

(i) *If $\chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, then $a(\chi|_M\psi) = a(\psi)$.*

(ii) *For each $\eta \in \Psi(G)$ and $\chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, we have*

$$a(\chi|_M\eta|_M\psi) = a(\eta|_M\psi).$$

Proof. We prove (i). Since $\chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma))$, we obtain

$$i_{GM}(\chi|_M\psi\sigma) \simeq \chi i_{GM}(\psi\sigma) \simeq i_{GM}(\psi\sigma).$$

So, by Lemma 2.6, there exists $w \in W(M)$ such that

$$\chi|_M\psi\sigma \simeq w(\psi\sigma) \simeq w(\psi)w(\sigma).$$

By definition of $W(D)$ (see (2.2)), this implies $w \in W(D)$, and the above relation can be written as

$$\chi|_M\psi\sigma \simeq \psi_w w(\psi)\sigma,$$

where

$$\psi_w = w(\psi)^{-1}\chi|_M\psi.$$

Consequently, by the definition of the action of $W(D)$ on D (see (2.3)) we obtain

$$\chi|_M\psi\,\mathrm{Stab}_{\Psi(M)}(\sigma) = \psi_w w(\psi)\,\mathrm{Stab}_{\Psi(M)}(\sigma) = w.\psi\,\mathrm{Stab}_{\Psi(M)}(\sigma).$$

This implies $a(\chi|_M\psi) = a(\psi)$. This proves (i).

To prove (ii), we may assume that η is unitary. Then, we obviously have

$$\mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\eta|_M\psi\sigma)) = \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi\sigma)).$$

Now, the claim follows from (i). $\square$

Now, in order to complete the proof of Proposition 3.4, we apply [Bernstein et al. 1986, Theorem 1.2]. We define a $\mathbb{Z}$-linear form $f : R(G) \to \mathbb{C}$ in several steps. We warn the reader that we use the same letter for a functional different than one from the statement of Theorem 1.1.

(1) For each $\Psi^u(G)$-orbit $\mathcal{O}$ in $\Psi^u(M)$, we fix a representative $\psi_{\mathcal{O}} \in \mathcal{O}$ and an irreducible constituent $\pi_{\mathcal{O}}$ in $i_{GM}(\psi_{\mathcal{O}}\sigma)$. By Lemma 3.6, we have

$$\mathrm{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}}) \subset \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)). \tag{3.8}$$

The quotient is finite and if χ ranges over representatives of the quotient, then $\chi\pi_{\mathcal{O}}$ ranges over the set of all mutually nonequivalent irreducible subrepresentations in $i_{GM}(\psi_{\mathcal{O}}\sigma)$ which are $\Psi^u(G)$-equivalent to $\pi_{\mathcal{O}}$. Any of those representations have the same multiplicity in $i_{GM}(\psi_{\mathcal{O}}\sigma)$. Let $m_{\mathcal{O}}$ be the sum of their multiplicities. We define

$$f(\chi\pi_{\mathcal{O}}) = \frac{a(\psi_{\mathcal{O}})}{m_{\mathcal{O}}}, \quad \chi \in \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)).$$

(2) For each $\chi \in \Psi^u(G)$, we obviously have

$$\mathrm{Stab}_{\Psi^u(G)}(\chi\pi_{\mathcal{O}}) = \mathrm{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}})$$

and

$$\mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\chi|_M\psi_{\mathcal{O}}\sigma)) = \mathrm{Stab}_{\Psi^u(G)}(i_{GM}(\psi_{\mathcal{O}}\sigma)).$$

By, Lemma 3.5 and these remarks, the sum of multiplicities of $\Psi^u(G)$-equivalent representations of $\pi_{\mathcal{O}}$ which belong to $i_{GM}(\chi|_M\psi_{\mathcal{O}}\sigma)$ is again $m_{\mathcal{O}}$. We let

$$f(\chi\pi_{\mathcal{O}}) = \frac{a(\chi|_M\psi_{\mathcal{O}})}{m_{\mathcal{O}}}, \quad \chi \in \Psi^u(G).$$

Lemma 3.7 (ii) shows that this is well-defined.

(3) For any other tempered irreducible representation (and, in particular, square-integrable modulo center representation) π of G we let

$$f(\pi) = 0.$$

(4) For any quasitempered irreducible representation π of G, we can write $\pi = \chi\pi^u$, where $\chi \in \Psi^r(G)$ and π^u is tempered. We let

$$f(\pi) = 0,$$

if π^u is not in $\Psi^u(G)\pi_{\mathcal{O}}$ for any orbit $\mathcal{O}$ described in (1). But, if $\pi^u \in \Psi^u(G)\pi_{\mathcal{O}}$, for some $\mathcal{O}$, then we can write $\pi^u = \psi\pi_{\mathcal{O}}$, for some $\psi \in \Psi^u(G)$ uniquely determined modulo $\mathrm{Stab}_{\Psi^u(G)}(\pi_{\mathcal{O}})$. We let

$$f(\pi) = \frac{a(\chi|_M\psi|_M\psi_{\mathcal{O}})}{m_{\mathcal{O}}}.$$

Using (3.8) and Lemma 3.7(ii) we see that this is well-defined.

(5) Finally, we define f on nontempered Langlands quotients (see Lemma 2.1). Let f be equal to zero on all standard modules induced from proper parabolic subgroups except in the following two obvious cases:

(a) The standard module $i_{GM}(\chi\psi\sigma)$, where $\chi \in \Psi(M)^{r,+}$ and $\psi \in \Psi^u(M)$. In this case, we let

$$f(i_{GM}(\chi\psi\sigma)) = a(\chi\psi).$$

(b) It is also possible that $\chi \in \Psi(M)^r$ belongs to the positive Weyl chamber for the opposite parabolic $\overline{P}$ (see the beginning of the previous section). Then, there exists a unique standard maximal parabolic subgroup Q with standard Levi N, and $w \in G$ such that $N = wMw^{-1}$. Now, by [Bernstein et al. 1986, Lemma 5.3(iii)], we have

$$i_{GM}(\chi\psi\sigma) = i_{GN}(w(\chi)w(\psi)w(\sigma))$$

in $R(G)$. Also, $w(\chi) \in \Psi(N)^{r,+}$. On the standard module $i_{GN}(w(\chi)w(\psi)w(\sigma))$ we let

$$f(i_{GN}(w(\chi)w(\psi)w(\sigma))) = a(\chi\psi).$$

Thus, we have

$$f(i_{GM}(\chi\psi\sigma)) = f(i_{GN}(w(\chi)w(\psi)w(\sigma))) = a(\chi\psi),$$

for $\chi \in \Psi(M)^r$ such that $w(\chi) \in \Psi(N)^{r,+}$.

The third case is that $\chi \in \Psi(M)^r$ is in neither chamber. Then, $\chi \in \Psi(G)^r$, by standard description of unramified characters [Muić 2008, Section 2]. In this case

$$i_{GM}(\chi\psi\sigma) = \chi i_{GM}(\psi\sigma)$$

is a quasitempered representation, and, by

$$f(i_{GM}(\chi\psi\sigma)) = f(\chi i_{GM}(\psi\sigma)) = a(\chi\psi),$$

by (1)–(4).

This completes the construction of $\mathbb{Z}$-linear form $f : R(G) \to \mathbb{C}$. In order to complete the proof of Proposition 3.4, we just need to check that it satisfies the assumptions of [Bernstein et al. 1986, Theorem 1.2]. First, let N be a standard Levi subgroup of G contained in M, and ρ an irreducible supercuspidal representation of N such that σ is an irreducible subquotient of $i_{M,N}(\rho)$. Then, by construction, f is zero on irreducible representations which are not irreducible subquotients of members of the family $i_{M,N}(\chi\rho)$ parametrized by $\chi \in \Psi(N)$. Then, as in the proof of Lemma 2.8, there exists an open compact subgroup K such that f is zero on all irreducible representations which do not have a nonzero K-invariant vector. This is (ii) in [Bernstein et al. 1986, §1.2]. It remains to check (i) in [Bernstein et al. 1986, §1.2]. We need to check that for an arbitrary standard Levi subgroup N of G and an irreducible representation τ of N, the function $\chi \mapsto f(i_{G,N}(\chi\tau))$ is regular on $\Psi(N)$. By Lemma 2.1 applied to N, τ is a $\mathbb{Z}$-linear combination of standard modules for N. So, instead of being irreducible, we may assume that τ is a standard module for N, i.e.,

$$\tau = i_{NN'}(\chi'\tau'),$$

N' is a standard Levi subgroup, τ' is an irreducible tempered representation of N' and $\chi' \in \Psi^{r,+}(N', N)$. Here, by definition $\Psi^{r,+}(N', N)$ is an analogue of $\Psi^{r,+}(N', G) \overset{\text{def}}{=} \Psi^{r,+}(N')$ defined in the previous section. Now, by induction in stages, we have

$$i_{G,N}(\chi\tau) = i_{G,N'}(\chi|_{N'}\chi'\tau').$$

We decompose $\chi = \chi^r\chi^u$ into its real part $\chi^r \in \Psi^r(N)$ and unitary part $\chi^u \in \Psi^r(N)$. Let N'' be a standard Levi subgroup such that $N' \subset N'' \subset N$ obtained by adjoining all simple roots orthogonal to $\chi^r|_{N'}\chi'$ (see [Muić 2008, Section 2]). Then, $\chi^r|_{N'}\chi'$ is an unramified character of N'' which is not orthogonal to any simple root that determines a standard parabolic subgroup of N''. In particular, there exists $w \in G$ such that ${N_1}'' = wN''w^{-1}$ is a standard Levi subgroup, and

$$w(\chi^r|_{N'}\chi') \in \Psi^{r,+}({N_1}'')$$

(see, for example, [Muić 2006, Section 1]). Also, we can write

$$i_{G,N'}(\chi|_{N'}\chi'\tau') = i_{G,N''}(\chi^r|_{N'}\chi'\ i_{N',N''}(\chi^u|_{N'}\tau')).$$

Obviously, $i_{N',N''}(\chi^u|_{N'}\tau')$ is an direct sum of irreducible tempered representations, say τ'' of N''. This implies that $i_{G,N'}(\chi|_{N'}\chi'\tau')$ is a direct sum induced by representations

$$i_{G,N''}(\chi|_{N'}\chi'\tau'').$$

By above, in $R(G)$, we have

$$i_{G,N''}(\chi|_{N'}\chi'\tau'') = i_{G,{N_1}''}(w(\chi|_{N'}\chi')w(\tau'')). \tag{3.9}$$

But the last induced representation is a standard module. Now, by the construction of f, $f = 0$ on all standard modules except those described in steps (1)–(5) above. This means that we have one of the following two cases:

(a) N'' is conjugate to G. In this case ${N_1}'' = N'' = N' = G$, τ' is a tempered irreducible representation of G, and $i_{G,N'}(\chi|_{N'}\chi'\tau') = \chi\chi'\tau'$. Thus, by the construction (1)–(4), $\chi \mapsto f(\chi\chi'\tau')$ is regular.

(b) N'' is conjugate to M. In this case, $N' = N''$, and τ' must be conjugate to an element of the orbit $\Psi^u(M)\sigma$ (see (5) above). The discussion in (5) implies that $\chi \mapsto f(i_{G,N'}(\chi|_{N'}\chi'\tau'))$ is regular.

This finally verifies (i) of [Bernstein et al. 1986, §1.2], and completes the proof of the proposition. ☐

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GORAN MUIĆ
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
UNIVERSITY OF ZAGREB
ZAGREB
CROATIA

gmuic@math.hr

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SPECTRUM OF THE LAPLACIAN AND THE JACOBI OPERATOR ON ROTATIONAL CMC HYPERSURFACES OF SPHERES

Oscar M. Perdomo

Let $M \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be a compact CMC rotational hypersurface of the $(n+1)$-dimensional Euclidean unit sphere. Denote by $|A|^2$ the square of the norm of the second fundamental form and $J(f) = -\Delta f - nf - |A|^2 f$ the stability or Jacobi operator. In this paper we compute the spectra of their Laplace and Jacobi operators in terms of eigenvalues of second order Hill's equations. For the minimal rotational examples, we prove that the stability index — the numbers of negative eigenvalues of the Jacobi operator counted with multiplicity — is greater than $3n+4$ and we also prove that there are at least 2 positive eigenvalues of the Laplacian of M smaller than n. When H is not zero, we have that every nonflat CMC rotational immersion is generated by rotating a planar profile curve along a geodesic called the axis of rotation. We assume that the coordinates of this plane has been set up so that the axis of rotation goes through the origin. The planar profile curve is made up of m copies, each one of them is a is rigid motion of a single curve that we will call the fundamental piece. For this reason every nonflat rotational CMC hypersurface has Z_m in its group of isometries. If θ denotes the change of the angle of the fundamental piece when written in polar coordinates, then $l = \frac{m\theta}{2\pi}$ is a nonnegative integer. For unduloids (a subfamily of the rotational CMC hypersurfaces that include all the known embedded examples), we show that the number of negative eigenvalues of the operator J counted with multiplicity is at least $(2l-1)n + (2m-1)$.

1. Introduction

Rotational constant mean curvature hypersurfaces of spheres provide a variety of examples that can be used to understand the nature of CMC hypersurfaces in general. In this paper we derive a formula for the Laplacian on these hypersurfaces that allows us to compute the spectra of their Jacobi and Laplace operators to any desired degree of accuracy. Results on the stability index are relevant in both cases, the case $H = 0$ and the case $H \neq 0$. Let us denote by $\mathcal{M}^n$ any minimal, not necessarily

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rotational, compact n-dimensional minimal hypersurface of the sphere. The spectra of the Laplacian and the stability operators on compact minimal hypersurfaces $\mathcal{M}^n \subset \mathbb{S}^{n+1}$ have been one of the central topics in differential geometry. Let us denote by $\lambda_1(\mathcal{M}^n)$, the first nonzero eigenvalue of the Laplace operator of $\mathcal{M}$. For minimal hypersurfaces of spheres, the spectrum of the Laplacian is known only when $\mathcal{M}^n$ is an Euclidean sphere, $\mathcal{M}^n$ is the product of Euclidean spheres, or $\mathcal{M}^n$ is a cubic isoparametric hypersurface, [Solomon 1990a, 1990b]. It is known that for any minimal hypersurface of the sphere, n is one of the eigenvalues of the Laplacian. Yau [1982] has conjectured that if $\mathcal{M}^n$ is compact and embedded, then $\lambda_1(\mathcal{M}^n) = n$. A positive partial result of this conjecture was given by Choi and Wang [1983]. They showed that if $\mathcal{M}^n$ is compact and embedded, then $\lambda_1(\mathcal{M}^n) \geq \frac{n}{2}$. In this paper we prove that if $\mathcal{M}^n$ is rotational, then $\lambda_1(\mathcal{M}^n) < n$. Moreover we show that there are at least two positive eigenvalues of the Laplace operator smaller n.

Regarding the spectrum of the Jacobi operator, we have that the number of negative eigenvalues of the operator J counted with multiplicity is known as the stability index of $\mathcal{M}^n$ and it is denoted as $\operatorname{ind}(\mathcal{M}^n)$. It has been conjectured that if $\mathcal{M}^n$ is not an Euclidean sphere, then $\operatorname{ind}(\mathcal{M}^n) = n+3$ implies that $\mathcal{M}^n$ is a product of Euclidean spheres. This conjecture was proven by Urbano [1990], when $n = 2$. For general n only partial results are known. In [Perdomo 2004a], the author shows that if $\operatorname{ind}(\mathcal{M}^n) = n+3$, then $\int_M |A|^2 \leq \int_M (n-1)$ with equality only if $\mathcal{M}^n$ is a product of Euclidean spheres. On the other hand in [Perdomo 2004b], the author shows that if $\mathcal{M}^n$ is rotational, then, $\int_M |A|^2 \leq \int_M (n-1)$. Therefore, it may seem that the rotational minimal hypersurfaces are good candidates for a counterexample of the conjecture. This is not the case. Due to their symmetries, their stability index must be greater than $n+3$. See [Perdomo 2001].

In this paper we show that there is a big jump in the stability index among the minimal rotational examples. We prove that if $\mathcal{M}^n$ is rotational and minimal and it is not a product of spheres then $\operatorname{ind}(\mathcal{M}^n) \geq 3n+5$. Some other partial results on this conjecture are found in [Simons 1968; Perdomo 2002; Savo 2010]. One of the most important applications of this type of estimate on the stability index was given by Marques and Neves [2014], where, among other tools, they used Urbano's result to prove Willmore's conjecture.

To motivate the study of the stability index when H is a nonzero constant, we would like to point out that any time we have a one-parametric family of CMC hypersurfaces, a jump in the stability index as a function of the parameter indicates a potential bifurcation point, a hypersurface that is part of another one-parametric family of CMC hypersurfaces. As an example of this fact, the families of embedded rotational CMC hypersurfaces on the sphere explained by the author in [Perdomo 2010] can be viewed as families emerging from bifurcation points of the family $S^1(r) \times S^{n-1}(\sqrt{1-r^2}) \subset \mathbb{S}^n$. See [Alías and Piccione 2013].

To illustrate the way we can use our results to estimate the eigenvalues, we pick a 3-dimensional rotational minimal hypersurface in $\mathbb{S}^4$ and prove that the first three eigenvalues of the Laplace operator are: 0, a number near 0.4404 with multiplicity 2, and 3 with multiplicity 5. We also show that the negative eigenvalues of the Jacobi operator are: a number near -8.6534 with multiplicity 1, a number near -8.52078 with multiplicity 2, -3 with multiplicity 5, a number near -2.5596 with multiplicity 6, and a number near -1.17496 with multiplicity 1. The stability index of this hypersurface is thus 15.

When H is not zero, we prove (see Theorem 3.7) a lower bound for the number of negative eigenvalues of the Jacobi operator that generalizes to any dimension the result on the stability index of constant mean curvature tori of revolution in the 3-sphere proven by Rossman and Sultana [2007].

We will be using the *oscillation theorem* for the periodic problem on the Hill's equation (a proof can be found in [Magnus and Winkler 1966]).

Theorem 1.1. *Consider the differential equation*

$$z''(t)+(\lambda+Q(t))z(t)=0 \tag{1-1}$$

where Q is a smooth T-periodic function. For any λ let us define

$$\delta(\lambda):=z_1(T,\lambda)+z_2'(T,\lambda),$$

where $z_1(t,\lambda)$ and $z_2(t,\lambda)$ are solutions of (1-1) such that $z_1(0,\lambda)=1,\ z_1'(0,\lambda)=0$ and $z_2(0,\lambda)=0,\ z_2'(0,\lambda)=1$. There exists an increasing infinite sequence of real numbers $\lambda_1,\lambda_2\dots$ such the differential equation (1-1) has a T-periodic solution if and only if $\lambda=\lambda_j$. Moreover the λ_j are the roots of the equation $\delta(\lambda)=2$. The function δ is called the discriminant function of the operator $K[z]=z''(t)+Q(t)z(t)$.

We will also be using the following theorem proven by Haupt [1914] from the Hill's equation theory. The next presentation of the Haupt theorem can be found in [Magnus and Winkler 1966].

Theorem 1.2 [Haupt 1914]. *Let us denote by $\lambda_1<\lambda_2\le\lambda_3\le\lambda_4\cdots$ the sequence of eigenvalues of the Hill's equation presented in Theorem 1.1. If $z(t)$ is a nonzero T-periodic solution of (1-1) with $\lambda=\lambda_i$ then, the number of zeros of $z(t)$ in the interval $[0,T)$ is $2\lfloor\frac{i}{2}\rfloor$.*

We would like to point out that Beekmann and Lökes [1990] have used the Hill equation to find bounds on the eigenvalues of the Laplacian on toroidal surfaces.

2. Describing rotational CMC hypersufaces of spheres

Any compact CMC rotational hypersurface of $\mathbb{S}^{n+1}$ is given by an immersion $\phi : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{S}^{n+1}$ where,

$$\phi(y,t) = \big(r(t)\,y, \sqrt{1-r(t)^2}\cos(\theta(t)), \sqrt{1-r(t)^2}\sin(\theta(t))\big) \tag{2-1}$$

and $r(t)$ is a positive T-periodic function that satisfies the conditions

$$(r')^2 + r^2(1+\lambda^2) = 1, \tag{2-2}$$

with

$$\lambda = H + c^{-n/2} r^{-n}, \quad \theta(t) = \int_0^t \frac{r(\tau)\lambda(\tau)}{1-r^2(\tau)}\,d\tau, \tag{2-3}$$

and c is a positive real number that satisfies

$$\theta(T) = 2\pi\frac{l}{m} \quad \text{where } l \text{ and } m \text{ are relative prime integers.} \tag{2-4}$$

The condition on c in (2-4) guarantees that the immersion ϕ satisfies

$$\phi(y, t+mT) = \phi(y,t)$$

and makes M compact. Recall that the function $r(t)$ depends on c since $\lambda(t)$ depends on c. Also, since the function $\theta(t)$ depends on $r(t)$ and $\lambda(t)$, it follows that $\theta(T)$ depends on c as well.

Remark 2.1. When M is minimal, Otsuki [1972; 1993], showed that the expression

$$\theta(T) = K(c) = \int_0^{T/2} \frac{c^{-1/n} r^{1-n}(\tau)}{1-r^2(\tau)}\,d\tau$$

lies between lies π and $\sqrt{2}\pi$. As a consequence the equation $\theta(T) = 2\pi\frac{l}{m}$ cannot be solved with $l=1$ and therefore no minimal rotational hypersurface is embedded.

The principal curvatures of M are λ with multiplicity $(n-1)$ and

$$\mu = H - (n-1)c^{-n/2}r^{-n}$$

with multiplicity one. Differentiating (2-2) we obtain

$$r'' + r + r\lambda\mu = 0. \tag{2-5}$$

The next expression explicitly provides the Gauss map ν of the immersion (2-1),

$$\nu(y,t) = \left(-r\lambda y, \frac{r^2\lambda\cos\theta - r'\sin\theta}{\sqrt{1-r(t)^2}}, \frac{r^2\lambda\sin\theta + r'\cos\theta}{\sqrt{1-r(t)^2}}\right) \tag{2-6}$$

All the details of the construction of these hypersurfaces can be found in [Perdomo 2010].

3. Main theorems

Before stating the following theorem, we recall that m is the integer given in (2-4), T is the period of the function $r(t)$, and mT is the period of the immersion ϕ.

Theorem 3.1. *Let M be the rotational symmetric hypersurface defined in (2-1). For any function $\bar{f}:\mathbb{S}^{n-1}\to\mathbb{R}$ we define $f:M\to\mathbb{R}$ as $f(\phi(t,y))=\bar{f}(y)$. Likewise, for any mT-periodic function $\bar{g}:\mathbb{R}\to\mathbb{R}$ we define $g:M\to\mathbb{R}$ as $g(\phi(t,y))=\bar{g}(t)$. We will denote by $\bar{\Delta}$ the Laplacian operator on $\mathbb{S}^{n-1}$. With this notation we have*

$$\Delta(fg)=f\Big(\bar{g}''+(n-1)\frac{r'}{r}\bar{g}'\Big)+\frac{\bar{\Delta}(\bar{f})}{r^2}\,g\,. \tag{3-1}$$

Proof. The proof is a direct computation using the fact that $\nabla(fg)=\frac{g}{r}\bar{\nabla}\bar{f}+f\bar{g}'\frac{\partial}{\partial t}$ and that $\operatorname{div}\big(\frac{\partial}{\partial t}\big)=(n-1)\frac{r'}{r}$. ☐

Remark 3.2. We would like to point out that we are using the fact that the ambient space is $\mathbb{R}^{n+2}$ in the argument above. In particular we are using the natural identification of all tangent spaces $T_x\mathbb{R}^{n+2}$ with $\mathbb{R}^{n+2}$. If we decide to work intrinsically we will notice that the formula that compares the gradients will have an additional factor of r. From the point of view of intrinsic differential geometry, the formula in Theorem 3.1 can be generalized to warped products. See [Marrocos and Gomes 2019].

Theorem 3.3. *Let $\alpha_1=0,\alpha_2=(n-1),\dots,\alpha_k=(k-1)(n+k-3),\dots$ denote the spectrum of $\mathbb{S}^{n-1}$. The spectrum of the Laplace operator Δ on M is given by $\bigcup_{k=1}^{\infty}\Gamma_k$, where,*

$$\Gamma_k=\{\lambda(k,1),\lambda(k,2),\dots\}$$

is the ordered spectrum of the operator (defined on the set of mT periodic functions)

$$K_{\Delta,k}[z]=z''+(n-1)\frac{r'}{r}z'-\frac{\alpha_k}{r^2}z$$

The spectrum of the Jacobi operator J on M is given by $\bigcup_{k=1}^{\infty}\mathbb{F}_k$, where

$$\mathbb{F}_k=\{\tilde{\lambda}(k,1),\tilde{\lambda}(k,2),\dots\}$$

is the ordered spectrum of the operator (defined on the set of mT periodic functions)

$$K_{J,k}[z]=z''+(n-1)\frac{r'}{r}z'+\bigg(n+nH^2+n(n-1)c^{-n}r^{-2n}-\frac{\alpha_k}{r^2}\bigg)z$$

Proof. Let us prove the case of the Laplacian. By Theorem 3.1 we have that if $\bar{f}_k$ is an eigenfunction of the Laplacian on $\mathbb{S}^{n-1}$ with eigenvalue α_k and $\bar{g}_l(t)$ is an eigenfunction of the operator $K_{\Delta,k}$ with eigenvalue $\lambda(k,l)$, then $h_{k,l}: M \to \mathbb{R}$ given by $h_{kl} = f_k g_l$ is an eigenfunction of the Laplacian with eigenvalue $\lambda(k,l)$. Therefore $\bigcup_{k=1}^{\infty} \Gamma_k$ is contained in the spectrum of Δ. To prove the reverse inclusion, we only need to point out that every smooth function $h: M \to \mathbb{R}$ can be written as a series of eigenfunctions of the form h_{kl}. There exists a basis

$$\bar{f}_{1,1}, \bar{f}_{2,1}, \bar{f}_{2,2}, \dots, \bar{f}_{2,n}, \bar{f}_{3,1}, \bar{f}_{3,2}, \dots, \bar{f}_{3,2n}, \bar{f}_{4,1}, \dots$$

for $L^2(\mathbb{S}^{n-1})$ with $\bar{\Delta}\bar{f}_{k,j} + \alpha_k \bar{f}_{k,j} = 0$. So any $h: M \to \mathbb{R}$ can be written as a sum of the form

$$a_{1,1}(t)\bar{f}_{1,1} + a_{2,1}(t)\bar{f}_{2,1} + \cdots + a_{2,n}(t)\bar{f}_{2,n} + a_{3,1}(t)\bar{f}_{3,1} + \cdots$$

We obtain the desired expression for the function $h: M \to \mathbb{R}$, by noticing that each $a_{k,l}(t)$ can now be expanded in eigenfunctions of the operator $K_{\Delta,k}$. The proof for the Jacobi operator is similar and uses the following expression for $|A|^2$:

$$|A|^2 = n\big(H^2 + (n-1)c^{-n}r^{-2n}\big). \qquad \square$$

The following lemma allows us to use the oscillation theorem to compute the eigenvalues for the second order differential equations on Theorem 3.3.

Lemma 3.4. *Let us denote by* $\lambda = H + c^{-n/2}r^{-n}$ *and* $\mu = H - (n-1)c^{-n/2}r^{-n}$. *The change of variables* $u = r^{(n-1)/2}z$ *gives us,*

$$\begin{aligned}K_{\Delta,k} &= z'' + (n-1)\frac{r'}{r}z' - \frac{\alpha_k}{r^2}z \\ &= \frac{1}{r^{(n-1)/2}}\Bigg(u'' + \bigg(\tfrac{1}{4}\lambda^2((n-1)(n-3)) + \tfrac{1}{2}\lambda\mu(n-1) - \frac{4\alpha_k + (n-3)(n-1)}{4r^2} \\ &\qquad\qquad + \tfrac{1}{4}(n-1)^2\bigg)u\Bigg)\end{aligned}$$

and

$$\begin{aligned}K_{J,k} &= z'' + (n-1)\frac{r'}{r}z' + \left(n + nH^2 + n(n-1)c^{-n}r^{-2n} - \frac{\alpha_k}{r^2}\right)z \\ &= \frac{1}{r^{(n-1)/2}}\Bigg(u'' + \tfrac{1}{4}\bigg(\lambda^2(n^2-1) + 2\lambda\mu(n-1) + 4\mu^2 + (n+1)^2 \\ &\qquad\qquad - \frac{4\alpha_k + (n-3)(n-1)}{r^2}\bigg)u\Bigg)\end{aligned}$$

Theorem 3.5. *If* $\mathcal{M} \subset \mathbb{S}^{n+1}$ *is a rotational minimal compact hypersurface, then* $\operatorname{ind}(\mathcal{M}^n) \geq 3n + 5$ *and there are at least two positive eigenvalues of the Laplace operator smaller than n. More precisely, if m and l are the relative prime integers in (2-4), then*

- $l \geq 2$, $m \geq 3$,

- $\operatorname{ind}(\mathcal{M}^n) \geq (2l-1)n + (2m-1)$ *and,*
- *there are exactly* $2l-2$ *positive eigenvalues of the Laplacian smaller than* n.

Proof. As pointed out in Remark 2.1, the positive integer l that satisfies the equation $\theta(T) = 2\pi \frac{l}{m}$ must be greater than 1. We also have that $m \geq 3$. Recall that Otsuki has shown that, for any c, the integral that defines $\theta(T)$ is a number between π and $\sqrt{2}\pi$. In this proof we will be using the notation for $\lambda(k, j)$ and $\tilde{\lambda}(k, j)$ introduced in Theorem 3.3. More precisely, the eigenvalues of the operator $K_{\Delta,k}$ are

$$\lambda(k, 1) < \lambda(k, 2) \leq \lambda(k, 3) \leq \cdots$$

and the eigenvalues of the operator $K_{J,k}$ are

$$\tilde{\lambda}(k, 1) < \tilde{\lambda}(k, 2) \leq \tilde{\lambda}(k, 3) \leq \cdots$$

A direct verification shows that the functions $f_1(t) = \sqrt{1-r^2(t)}\cos(\theta)$ and $f_2(t) = \sqrt{1-r^2(t)}\sin(\theta)$ satisfy the equation $K_{\Delta,1}(f_i) + nf_i = 0$, for $i = 1, 2$. Since the functions f_1 and f_2 have $2l$ zeroes in the interval $[0, mT)$ then, by Theorem 1.2 we conclude that $n = \lambda(1, 2l) = \lambda(1, 2l+1)$. Therefore, the eigenvalues of $K_{\Delta,1}$ are

$$\lambda(1, 1) = 0 < \lambda(1, 2) \leq \lambda(1, 3) < \cdots < \lambda(1, 2l) = \lambda(1, 2l+1) = n < \cdots .$$

This shows the existence of $2l-2$ positive eigenvalues (counted with multiplicity) of the Laplace operator smaller than n. A direct verification shows that $r(t)$ satisfies the equation $K_{\Delta,2}(r) + nr = 0$. Since $r(t)$ is positive, then $\lambda(2, 1) = n$. Therefore all the eigenvalues of the Laplace operator smaller than n comes from the operators $K_{\Delta,1}$. We conclude that there are exactly $2l-2$ positive eigenvalues of the Laplacian smaller than n. Let us show the inequality for the stability index. A direct verification shows that the function r' satisfy the equation $K_{J,1}(r') = 0$. The numbers of zeroes of r' in the interval $[0, mT)$ is $2m$. By Theorem 1.2 we conclude that either $0 = \tilde{\lambda}(1, 2m)$ or $0 = \tilde{\lambda}(1, 2m+1)$ Therefore the operator $K_{J,1}$ has at least $2m-1$ negative eigenvalues. Let us analyze the spectrum of the operator $K_{J,2}$. A direct verification shows that the functions

$$f_3(t) = \frac{-c^{-n/2}r^{1-n}\cos(\theta) + rr'\sin(\theta)}{\sqrt{1-r^2(t)}} \quad \text{and} \quad f_4(t) = \frac{c^{-n/2}r^{1-n}\sin(\theta) + rr'\cos(\theta)}{\sqrt{1-r^2(t)}}$$

satisfy the equation $K_{J,2}(f_i) = 0$ for $i = 3, 4$. The numbers of zeroes f_3 and f_4 in the interval $[0, mT)$ is at least $2l$. Let us check this property for the function $f_3(t)$. We have that $\theta(t)$ changes from 0 to $2\pi l$ when t moves from 0 mT. Let $t_1, t_2, \ldots, t_{2l-1}$ be increasing values of t such that $\theta(t_i) = i\pi$. Recall that $\theta(0) = 0$ and $\theta(mT) = 2\pi l$.

Since

$$f_3(0) = -\frac{c^{-n/2}r(0)^{1-n}}{\sqrt{1-r(0)^2}} < 0 \quad \text{and} \quad f_3(t_1) = \frac{c^{-n/2}r(0)^{1-n}}{\sqrt{1-r(0)^2}} > 0,$$

there is at least one zero of f_3 between 0 and t_1. The same argument shows that there must be at least one zero of $f_3(t)$ between t_1 and t_2. By continuing with this argument we conclude that f_3 must have at least $2l$ zeroes. A similar argument shows that f_4 has also at least $2l$ zeroes. By Theorem 1.2 we conclude that $0 = \tilde{\lambda}(2, 2k) = \tilde{\lambda}(2, 2k+1)$, where $2k$ is the number of zeroes of f_4 and f_5 on the interval $[0, mT)$. Recall that $k \geq l$. Therefore the operator $K_{J,2}$ has at least $2l-1$ negative eigenvalues. Since the multiplicity of the eigenvalue $\alpha_2 = (n-1)$ of the Laplace operator on S^{n-1} is n, we conclude that $\operatorname{ind}(\mathcal{M}) \geq (2l-1)n + (2m-1)$. □

Remark 3.6. There is a geometrical reason for the functions r' and f_i, $i = 1, 2, 3, 4$ to satisfy the respective Hill's equation mentioned in the proof of the previous theorem. If we write the immersion ϕ in (2-1) as $(\phi_1, \dots, \phi_{n+2})$ and the Gauss map ν in (2-6) as $(\nu_1, \dots, \nu_{n+2})$ then, the functions f_1 and f_2 are the last two coordinates of the immersion ϕ and, in general, all the coordinates of an immersion of a minimal hypersurface of $\mathbb{S}^{n+1}$ are eigenfunctions of the Laplacian associated with the eigenvalue n. The function r' is the simplification of the expression $\phi_{n+2}\nu_{n+1} - \phi_{n+1}\nu_{n+2}$ and in general all the functions of the form $\nu_i\phi_j - \nu_j\phi_i$ are either the zero function or they are eigenfunctions of the stability operator associated with the eigenvalue 0. Finally, we have that f_4 and f_5 are factors of the expression $\phi_1\nu_{n+1} - \phi_{n+1}\nu_1$ and $\phi_1\nu_{n+2} - \phi_{n+2}\nu_1$.

Theorem 3.7. *Let $M \subset \mathbb{S}^{n+1}$ be a rotational CMC compact hypersurface described in* (2-1) *with $H \geq 0$. If m and l are the relative prime integers in* (2-4) *then, the number of negative eigenvalues of the Jacobi operator counted with multiplicity is at least $(2l-1)n + (2m-1)$*

Proof. The proof is similar to the one presented for the minimal case. A direct verification shows that $J_{J,0}[r']$ vanishes and the numbers of zeroes of r' in the interval $[0, mT)$ is $2m$. By Theorem 1.2 we conclude that either $0 = \tilde{\lambda}(1, 2m)$ or $0 = \tilde{\lambda}(1, 2m+1)$. Therefore the operator $K_{J,1}$ has at least $2m-1$ negative eigenvalues. Let us analyze the spectrum of the operator $K_{J,2}$. A direct verification shows that the functions

$$f_1(t) = \frac{c^{-n/2}r(t)\big(c^{n/2}\big(H\cos(\theta(t)) - r'(t)\sin(\theta(t))\big) + r(t)^{-n}\cos(\theta(t))\big)}{\sqrt{1-r(t)^2}}$$

and

$$f_2(t) = \frac{c^{-n/2}r(t)\big(c^{n/2}\big(r'(t)\cos(\theta(t)) + H\sin(\theta(t))\big) + r(t)^{-n}\sin(\theta(t))\big)}{\sqrt{1-r(t)^2}}$$

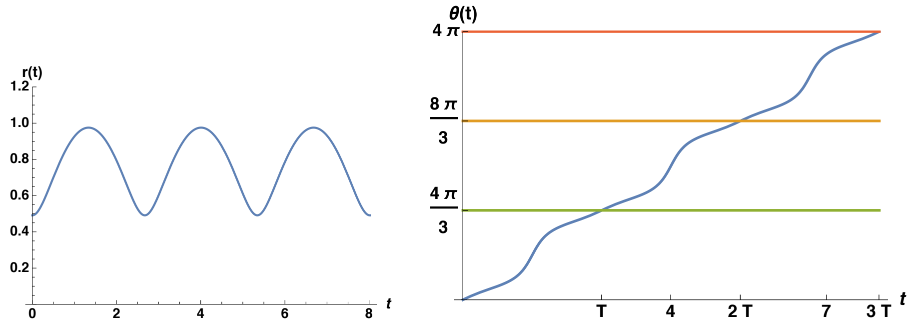

Figure 1. On the left we have the graph of the solution $r(t)$ for the value $ac = 2.8284247911397589$. On the right we have the graph of the function $\theta(t)$ defined in (2-3).

satisfy the equation $K_{J,2}(f_i) = 0$ for $i = 1, 2$. The numbers of zeroes f_1 and f_2 in the interval $[0, mT)$ is at least $2l$. By Theorem 1.2 we conclude that $0 = \tilde{\lambda}(2, 2k) = \tilde{\lambda}(2, 2k+1)$, where $2k$ is the number of zeroes of f_1 and f_2 on the interval $[0, mT)$. Recall that $k \geq l$. Therefore the operator $K_{J,2}$ has at least $2l-1$ negative eigenvalues. Since the multiplicity of the eigenvalue $\alpha_2 = (n-1)$ of the Laplace operator on S^{n-1} is n, we conclude that the number of negative eigenvalue of the operator J is greater than $(2l-1)n + (2m-2)$. ☐

4. An example that illustrates the method

In this section we pick an explicit rotational 3-dimensional minimal hypersurface $M \subset \mathbb{S}^4$ and we compute the first three eigenvalues of the Laplacian and its stability index.

Construction of the particular example. By the intermediate value theorem, it is easy to see that there is a value of c near $ac = 2.82842479911$ such that the function $r(t)$ has period T near $aT = 2.6722005616$ and $\theta(T) = \frac{4\pi}{3}$. See Equation (2-4). In this case $l = 2$, $m = 3$ and our manifold M is defined by this choice of c. Recall that the immersion $\phi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^4$ is given by

$$\phi(y, t) = \big(r(t)y, \sqrt{1 - r(t)^2}\cos(\theta(t)), \sqrt{1 - r(t)^2}\sin(\theta(t))\big). \tag{4-1}$$

Figure 1 shows the solution $r(t)$ that produces the manifold M, and Figure 2 shows the profile curve of the rotational manifold M.

Computing the first three eigenvalues of the Laplacian. In order to use the oscillation theorem (Theorem 1.1) we notice that making $u = rz$ we obtain (see also Lemma 3.4)

$$K_{\Delta,k}[z] = z'' + 2\frac{r'}{r}z' - \frac{\alpha_k}{r^2}z = \frac{1}{r}\left(u'' + \left(1 - \frac{2}{c^3 r^6} - \frac{\alpha_k}{r^2}\right)u\right).$$

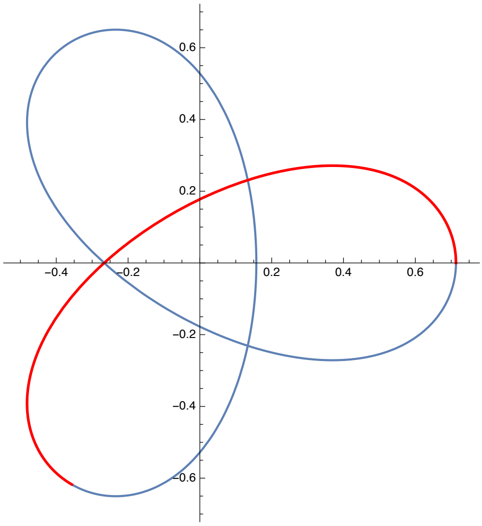

Figure 2. Profile curve of the M. This curve is parametrized by $\left(\sqrt{1-r(t)^2}\cos(\theta(t)),\sqrt{1-r(t)^2}\sin(\theta(t))\right)$. The red piece represent the curve when t moves from 0 to T. Recall that the period of the immersion is $3T$.

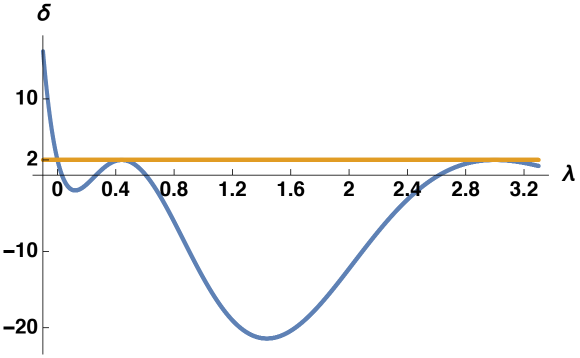

Figure 3. Graph of the function $\delta(\lambda)$. The roots of the equation $\delta(\lambda)=2$ give us eigenvalues of the Laplacian of the form $\lambda(1,j)$ defined on Theorem 3.3.

Therefore $\lambda(k,i)$ is an eigenvalue of the operator $K_{\Delta,k}$ if and only if $\lambda(k,i)$ is an eigenvalue of the operator

$$\bar{K}_{\Delta,k}[u]=u''+\left(1-\frac{2}{c^3r^6}-\frac{\alpha_k}{r^2}\right)u.$$

For $\alpha_1=0$, Figure 3 shows the discriminant function $\delta(\lambda)$ for the operator $\bar{K}_{\Delta,1}$.

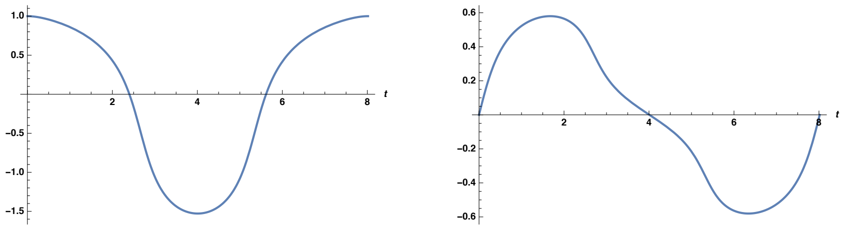

Figure 4. Graph of two solutions ξ_1 and ξ_2 of the equation $K_{\Delta,1}[z]+0.4404z=0$.

Figure 3 was made by taking 3400 values of λ between -0.1 and 3.3, one every 0.001. For each value of λ we solve two differential equations to find the functions $z_1(t,\lambda)$ and $z_2(t,\lambda)$ defined in Theorem 1.1. Once we have $z_1(t,\lambda)$ and $z_2(t,\lambda)$ we computed $\delta(\lambda)$. The crossing of the graph of $\delta(\lambda)$ with the horizontal line $y=2$ at $\lambda(1,1)=0$ was expected since $z(t)=1$ is an eigenfunction and the crossing at $\lambda=3$ with multiplicity 2 was expected because the last two coordinates of the immersion, the functions $\sqrt{1-r^2}\cos(\theta)$ and $\sqrt{1-r^2}\sin(\theta)$, are eigenfunctions of the Laplacian of M, see [Simons 1968]. Regarding the crossing near 0.44 we can check that $|\delta(0.4404)-2|$ is smaller than 10^{-6}. Figure 4 shows two linearly independent solutions ξ_1 and ξ_2 of the equation $K_{\Delta,1}[z]+0.4404z=0$. All together we have 3 eigenvalues of $K_{\Delta,1}$ smaller than 3, which agrees with Theorem 3.5.

We will move now to study the operator $K_{\Delta,2}$. Since the coordinates of the immersion ϕ are eigenfunction of the Laplacian we have that the function $r(t)$ satisfies the equation $K_{\Delta,2}(r(t))=-3r(t)$. The previous equation also follows from (2-5). Since $r(t)$ is positive, $\lambda(2,1)=3$ is the first eigenvalue of $K_{\Delta,2}$ and it has multiplicity 1.

Remark 4.1. Since the sequence α_k is increasing, the sequence $\lambda(k,1)$ is also increasing.

From the previous remark we deduce that all other eigenvalues of the Laplacian of M are greater than 3.

Computing the negative eigenvalues of the Jacobi operator. Once again we use the oscillation theorem (Theorem 1.1). The change of variables $u=rz$ gives us (see also Lemma 3.4)

$$K_{J,k}[z]=z''+2\frac{r'}{r}z'+\left(\frac{6}{c^3r^6}+3-\frac{\alpha_k}{r^2}\right)z=\frac{1}{r}\left(u''+\left(4+\frac{4}{c^3r^6}-\frac{\alpha_k}{r^2}\right)u\right).$$

Similar to the case of the Laplacian operator, we can compute the eigenvalues of the Jacobi operator by computing the eigenvalues of the operator

$$\overline{K}_{J,k}[u]=u''+\left(4+\frac{4}{c^3r^6}-\frac{\alpha_k}{r^2}\right)u.$$

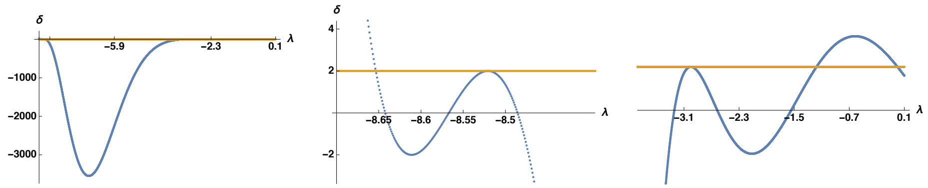

Figure 5. Graph of the function $\delta(\lambda)$ for the operator $\bar{K}_{J,1}$. The roots of the equation $\delta(\lambda) = 2$ give us eigenvalues of the Jacobi operator of the form $\lambda(1, j)$. The graph on the center and on the right shows the function δ on smaller intervals

Figure 5 shows the discriminant δ for the operator $\bar{K}_{J,1}$. A closer look at the function tell us that the negative solutions of the equation $\delta(\lambda) = 2$ are on the intervals $[-8.7, -8.5]$ and $[-3.1, 0]$.

For the first eigenvalue of the Jacobi operator, it is easy to use the intermediate value theorem to show that $\tilde{\lambda}(1, 1) = -8.6534$ within an error of 10^{-4}. This eigenvalue has multiplicity one and Figure 6 shows a nonzero periodic eigenfunction of the operator $K_{J,1}$. For the next value we have that $|\delta(-8.53078) - 2| < 10^{-5}$. For this value of λ the two fundamental solutions of the equation $K_{J,1}[z] + \lambda z = 0$ are shown in Figure 7. The next eigenvalue is -3 with multiplicity 2, this eigenvalue was expected due to the fact that the coordinate functions of the Gauss map are eigenfunctions of the Jacobi operator. For the next eigenvalue we have that $|\delta(-1.1749673) - 2| < 10^{-5}$. The existence of an eigenvalue near -1.17496 with multiplicity one is given by the Intermediate Value Theorem, see Figure 5. We know that this is the last negative eigenvalue because 0 is known to be an eigenvalue of $K_{J,1}$.

We now study the operator $K_{J,2}$. Figure 8 shows the discriminant δ for the operator $\bar{K}_{J,2}$. We can directly check that $K_{J,2}(r^{-2}) = 3r^{-2}$. Since $r(t)$ is positive, then we have that -3 is the first eigenvalue of $K_{J,2}$ with multiplicity one. Since we have that $|\delta(-2.5596) - 2| < 10^{-5}$, then there is an eigenvalue of $K_{J,2}$ with multiplicity 2 near -2.5596. We can check that the first eigenvalue of the operator $K_{J,3}$ is close to 4.3484453. Therefore we have gotten all negative eigenvalues of the Jacobi operator, in summary we have

Remark 4.2. Since the first two eigenvalues of $\mathbb{S}^2$ are 0 with multiplicity 1 and 2 with multiplicity 3, then the stability index of M is 15. Counting with multiplicity we have that the first eigenvalue of the stability operator J on M is near -8.6534 and has multiplicity one. We have two eigenvalues near -8.52078, it could be only one with multiplicity 2. We have -3 with multiplicity 5. Even though this was known, it is interesting to point out that the multiplicity is 5 because -3 is an eigenvalue with multiplicity 2 of $K_{J,1}$ and -3 is an eigenvalue of multiplicity 1 of

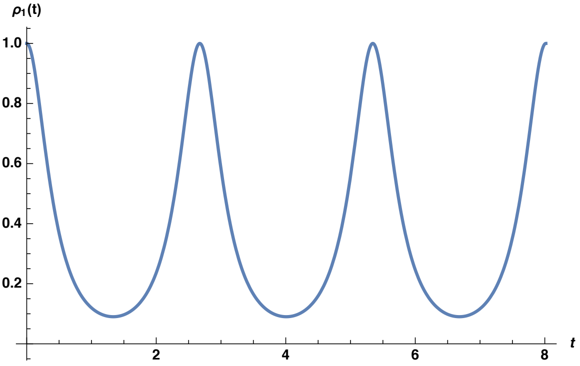

Figure 6. Graph of an eigenfunction associated with $\tilde{\lambda}(1,1) = -8.65\ldots$. This function also represents the first eigenfunction of the stability operator.

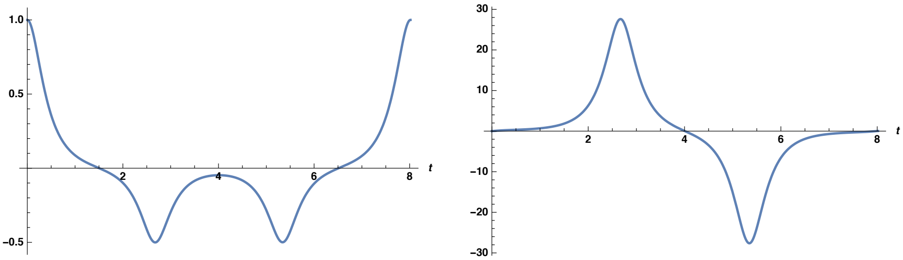

Figure 7. Two solutions of the equation $K_{J,1}[z] - 8.53078z = 0$.

$K_{J,2}$, this multiplicity one needs to be multiply by 3 because the eigenvalue $\alpha_2 = 2$ of the Laplace operator on $\mathbb{S}^2$ has multiplicity 3. The next eigenvalues are six near -2.5596, they are either two with multiplicity 3 or one with multiplicity 6. The last negative eigenvalue is one near -1.17496.

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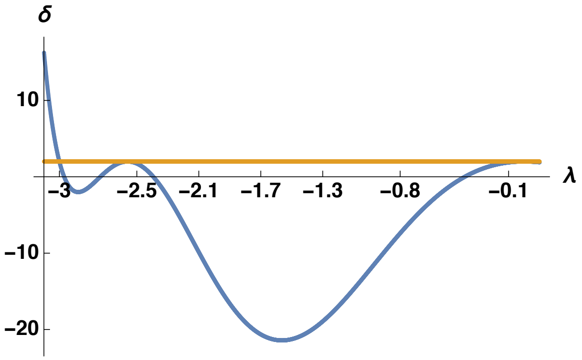

Figure 8. Graph of the discriminant of the operator $K_{J,2}$.

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OSCAR M. PERDOMO
DEPARTMENT OF MATHEMATICAL SCIENCES
CENTRAL CONNECTICUT STATE UNIVERSITY
NEW BRITAIN, CT
UNITED STATES

perdomoosm@ccsu.edu

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MEAN CURVATURE FLOW IN A RIEMANNIAN MANIFOLD ENDOWED WITH A KILLING VECTOR FIELD

LIANGJUN WENG

We consider the Killing graphs over a bounded regular domain M in an integral distribution orthogonal to a Killing vector field with prescribed variable contact angle. Under some appropriate condition between the geometry of the domain and the contact angle, based on the maximum principle and the approximation method, we show that the solutions to the mean curvature flow of Killing graphs with capillarity type boundary condition converge to a translating solution.

1. Introduction

In this paper, we are interested in the study of the evolution of graphs defined over a Riemannian manifold by the nonparametric mean curvature flow, whose speed in the direction of their normal is equal to their mean curvature and with a prescribed variable contact angle at the boundary. Throughout this paper, let (N^{n+1}, g) be an $(n+1)$-dimensional Riemannian manifold endowed with a Killing vector field $\mathcal{V}$. Suppose that the distribution orthogonal to $\mathcal{V}$ is of constant rank and integrable. Given an integral leaf $\mathbb{L}^n$ of that distribution, let $M \subset \mathbb{L}^n$ be a bounded domain with boundary $\partial M \in C^3$. We suppose for simplicity that $\mathbb{L}^n$ is complete. In this case, let $\zeta : \overline{M} \times \mathbb{R} \to N$ be the flow generated by $\mathcal{V}$ with initial values in N. In geometric terms, the ambient manifold N is a warped product manifold as

$$N = \mathbb{L}^n \times_{\frac{1}{\sqrt{\gamma}}} \mathbb{R},$$

with the metric given by $g = \sigma + \frac{1}{\gamma}\, ds^2$, where $\gamma := 1/g(\mathcal{V}, \mathcal{V})^2$ and σ is the metric on $\mathbb{L}^n$. Since we have $\mathcal{V}(\gamma) = 0$ by the Killing equation, γ can be viewed as a function on $\mathbb{L}^n$.

The Killing graph of a differentiable function $u : \overline{M} \to \mathbb{R}$ is the hypersurface $\Sigma_u \subset N$ parametrized by the map $X(x) = \zeta(x, u(x))$, $x \in \overline{M}$. The Killing cylinder

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MSC2010: primary 53C44; secondary 35J66.

Keywords: Killing vector field, warped product manifold, mean curvature flow, uniform gradient estimate, variable contact angle.

$\mathcal{K}$ over ∂M is defined by

$$\mathcal{K} := \{\zeta(x, s) : x \in \partial M,\ s \in \mathbb{R}\}.$$

We want to study the asymptotic behavior of solutions $u(x, t)$ to the Killing graphs dominated by its mean curvature in the direction of the unit normal μ with prescribed variable contact angle along the Killing cylinder, that is,

$$\begin{cases} \frac{\partial X}{\partial t} = \mathcal{H}\mu & \text{in } M^n \times (0, \infty), \\ \langle \mu \circ X, \nu \circ X \rangle = -\varphi & \text{on } \partial M \times (0, \infty), \\ X(\cdot, 0) = \zeta(\cdot, u_0(\cdot)) & \text{on } M, \end{cases} \tag{1-1}$$

where $M \subset \mathbb{L}^n$, $n \geq 2$, is a compact domain with smooth boundary ∂M, μ is the upward normal for the Killing graph $\mathcal{K}$, which is given by

$$\mu := \frac{\gamma \mathcal{V} - \zeta_*(\nabla u)}{v},$$

with $v := \sqrt{e^{2\rho} + |\nabla u|^2}$ and $\rho := \log \sqrt{\gamma}$ throughout this paper for convenience. And φ is the cosine of the contact angle, that is, $\varphi := \cos\theta$ for some $\theta : \partial M \to \mathbb{R}$ being the variable contact angle on the intersection of the Killing cylinder and the graph, which is given by $\langle \mu, \nu \rangle = -\cos\theta$. Then the boundary value condition in (1-1) is equivalent to $\nabla_\nu u = \cos\theta \sqrt{\gamma + |\nabla u|^2}$, where ν is the unit inner normal of ∂M, one may extend φ to $\overline{M}$ with $\varphi \in C^2(\overline{M})$. And $u_0(x)$ is a smooth function and satisfying the compatible condition

$$\nabla_\nu u_0 = \varphi \sqrt{\gamma + |\nabla u_0|^2} \quad \text{on } \partial M,$$

while $\mathcal{H}$ in (1-1) is the Killing mean curvature operator with the expression (see [Dajczer et al. 2008]) that

$$\mathcal{H} := \operatorname{div}\left(\frac{\nabla u}{\sqrt{\gamma + |\nabla u|^2}} \right) - \left\langle \frac{\nabla \gamma}{2\gamma}, \frac{\nabla u}{v} \right\rangle,$$

where the differential operators div and ∇ are respectively the divergence and gradient in $\mathbb{L}$ with respect to the metric σ.

For the prescribed contact angle boundary value problem, a more general type problem with an extra term is to study the following equation,

$$u_t = \sqrt{e^{2\rho} + |\nabla u|^2}(\mathcal{H} - \psi) \quad \text{in } M \times [0, T). \tag{1-2}$$

Such types of evolution problem were studied by de Lira and Wanderley [2015] previously, where they proved the long time existence for $\psi : \overline{M} \to \mathbb{R}$ and prescribed contact angle function $\varphi : \partial M \to \mathbb{R}$ with $|\varphi| \leq b_0 < 1$. Nevertheless, they obtained the convergence results only for the vertical angle case, which corresponds to $\varphi = 0$ on the boundary. One could also refer to [Huisken 1989] or [Zhou 2018] for similar results in the vertical angle case. Besides, when $\mathcal{V} := \frac{\partial}{\partial s}$, the long time existence of flow (1-2) was obtained for product ambient manifolds by Zhou [2018] recently, which is a generalization of the previous result in Euclidean space by Guan [1996].

However, when it goes to study the asymptotic behavior of $u(x,t)$ in (1-2) for the not perpendicular contact angle case, to the author's best knowledge, only a few results are known. For instance, when $n = 2$, Altschuler and Wu [1994] showed that $u(x,t)$ of Euclidean nonparametric mean curvature flow with contact angle boundary condition will converge to the translating surface. For $n \geq 2$, recently, [Gao et al. $\geq$ 2020] proved the same results under the condition that the contact angle is the small perturbation of $\frac{\pi}{2}$. See also [Zhou 2018] for the same convergence conclusion under the condition that the ambient Riemannian surface requires carrying the metric with nonnegative Gauss curvature. One could also refer to [Ma et al. 2018; Oliker and Uraltseva 1993; Schnürer 2002] for various studies on the asymptotic behavior of geometric curvature flow with Dirichlet, Neumann and second boundary value condition problems.

We only focus on the capillarity type boundary value condition in this paper. Based on above discussions, we may rewrite (1-1) in the following expression,

$$\begin{cases} u_t = \sum_{i,j=1}^n a^{ij}\nabla_{ij}u - \frac{1}{2}(1/\gamma + 1/v^2)\langle\nabla\gamma, \nabla u\rangle & \text{in } M \times [0, +\infty), \\ \nabla_\nu u = \varphi(x)\sqrt{\gamma + |\nabla u|^2} & \text{on } \partial M \times [0, +\infty), \\ u(x,0) = u_0(x) & \text{on } \overline{M} \times \{0\}, \end{cases} \tag{1-3}$$

where

$$a^{ij} := \sigma^{ij} - \frac{\nabla^i u \cdot \nabla^j u}{\gamma + |\nabla u|^2}$$

and ν is the unit inner normal of ∂M.

Our first main result about the asymptotic behavior of a solution to (1-3) can be stated as follows.

Theorem 1.1. *Let $M \subset \mathbb{L}^n$ be a bounded strictly convex domain with $\partial M \in C^3$. If $\mathrm{Ric}_{\mathbb{L}} + \nabla^2\rho \geq k_0\sigma$ for some positive constant k_0, where $\mathrm{Ric}_{\mathbb{L}}$ denotes the Ricci curvature tensor of $\mathbb{L}$, then there exists $\delta_0 > 0$ such that*

$$\|\varphi\|_{C^2(\overline{M})} \leq \delta_0, \tag{1-4}$$

and it holds that the unique smooth solution $u(x,t)$ of (1-3) uniformly converges to $\hat{u}(x) + \tau t$ as $t \to \infty$, which means that

$$\lim_{t\to+\infty} \big\| u(\cdot, t) - (\hat{u}(\cdot) + \tau t) \big\|_{C^0(\overline{M})} = 0,$$

where $(\tau, \hat{u})$ is a solution satisfying

$$\begin{cases} \sum_{i,j=1}^n a^{ij}\nabla_{ij}u - \frac{1}{2}(1/\gamma + 1/v^2)\langle\nabla\gamma, \nabla u\rangle = \tau & \text{in } M, \\ \nabla_\nu u = \varphi\sqrt{\gamma + |\nabla u|^2} & \text{on } \partial M. \end{cases} \tag{1-5}$$

In particular, if $\int_{\partial M}\varphi/\sqrt{\gamma}\,d\sigma=0$, *then* $\tau=0$, *hence* $\Sigma_{\hat{u}}$ *is a minimal hypersurface in* (N^{n+1},g).

The main ingredients and approach to get such convergence results are as follows. Firstly, by using the standard comparison principle, one shows that

$$\frac{u(x,t)}{t}\to\tau\quad\text{uniformly as } t\to+\infty,$$

which indicates that the so called ergodic constant τ governs the asymptotic behavior of the evolution equation. This part is somewhat relatively easy. Secondly, by establishing the uniform gradient estimate, we get the existence of solutions to the stationary equations (1-5) and show the more precise asymptotic behavior

$$u(x,t)-\tau t\to\hat{u}(x)\quad\text{uniformly as } t\to+\infty.$$

We will achieve this by firstly deriving an a priori estimate for the C^1-norm of $u(x,t)$ to (1-3), which is time-independent. This will be achieved by choosing an appropriate auxiliary function and combining with the maximum principle.

We mainly use the methods in [Altschuler and Wu 1994; Ma and Xu 2016; Schnürer 2002], but with the necessary technical tricks for choosing the right functions to get estimates, which take control of the complications introduced by the terms containing the warped function γ and the curvature tensor of $\mathbb{L}$. We want to point out that the existence of solutions to stationary equations (1-5) are closely related to the "ergodic control problems," which consist in solving the following type of fully nonlinear elliptic equations associated with nonlinear oblique type boundary conditions:

$$\begin{cases}F(x,\nabla u,\nabla^2u)=\mu_1 & \text{in } M,\\ L(x,\nabla u)=\mu_2 & \text{on } \partial M.\end{cases}$$

Such types of problem not only have a close relation with the asymptotic behavior of solutions to parabolic equations, which are the case we study here, but also have a strong impact and application in ergodic control problems, homogenization of elliptic and parabolic PDEs, etc. We recommend [Barles 1993; Barles and Da Lio 2005; Barles and Souganidis 2001; Ishii 2013] for more details and interesting results.

Thus, we adapted the idea used there to prove the existence of a translating solution to our flow equation. That is, we have the following existence results for stationary equations (1-5) (called the solvability problem of the translating soliton equation with capillary boundary condition in some literature).

Theorem 1.2. *Under the assumption of Theorem 1.1, there exist a unique* $\tau\in\mathbb{R}$ *and a solution* $\hat{u}\in C^{2,\alpha}(\overline{M})$ *satisfying* (1-5), $0<\alpha<1$. *In particular, the solution* $\hat{u}$ *is unique up to an additive constant.*

Moreover, for the dimension $n = 2$, we could release the condition on the range of the variable contact angle and the compatible relation for the Ricci curvature on leafs with the product function γ in Theorem 1.1. Indeed, we have the following theorem.

Theorem 1.3. *Let M be a strictly convex domain in $\mathbb{L}^2$ with κ the geodesic curvature of ∂M, satisfying*

$$\kappa - \left(\frac{|\nabla^T \varphi|}{\sqrt{1-\varphi^2}} + |\varphi| \cdot |\nabla \rho| \right) \geq \delta_1, \quad \text{on } \partial M,$$

for some positive constant $\delta_1 > 0$, where $\nabla^T \varphi$ is the tangential part of $\nabla \varphi$ restricted on the boundary. If γ satisfies the compatible condition

$$K + \Delta \log \sqrt{\gamma} \geq 0 \quad \text{and} \quad \Delta\gamma - \frac{|\nabla \gamma|^2}{2\gamma} \geq 0, \qquad \text{in } M,$$

where K is the Gauss curvature of M. Then the unique smooth solution $u(x, t)$ of (1-3) uniformly converges to $\hat{u}(x) + \tau t$ as $t \to \infty$, where $(\tau, \hat{u})$ is a solution satisfying (1-5).

In particular, if $\int_{\partial M} \varphi/\sqrt{\gamma} d\sigma = 0$, then $\tau = 0$, hence $\Sigma_{\hat{u}}$ is a minimal surface in (N^3, g).

We want to emphasize the fact that the above conditions are very well adapted for applications to some special cases, say [Altschuler and Wu 1994] or [Zhou 2018] for example. Let us recall that for the Euclidean graph case, such results were first proved in [Altschuler and Wu 1994], later for the product Riemannian manifold case $\mathbb{L} \times \mathbb{R}$, see [Zhou 2018], which both correspond to the special Killing vector field $\mathcal{V} = \frac{\partial}{\partial s}$ in our setting.

Outline of the paper. The article is organized as follows. In Section 2, by using the maximum principle, the key uniform gradient estimate is established for flow (1-3). Theorem 1.2 is proved in Section 3, which is complied with the approach that has been used in [Altschuler and Wu 1994] or [Ma et al. 2018], once one gets the uniform gradient estimate. In Section 4, we turn to discuss the $n = 2$ case and prove the uniform gradient estimate being stated as Theorem 4.2. The last section is devoted to showing the asymptotic behavior of solutions to (1-3) and proving Theorem 1.1, which used the method in [Altschuler and Wu 1994; Schnürer 2002]. Also, the same idea can be utilized to verify Theorem 1.3, after obtaining the key uniform gradient estimate for a Riemann surface in Section 4.

2. Priori estimates

In this section, for studying the asymptotic behavior of the nonparametric mean curvature with prescribed contact angle condition, we establish the uniformly gradient estimate for (1-3) under condition (1-4).

Firstly, we describe the evolution problem in local coordinates and compute some geometric quantities induced by embedding of the hypersurface into (N, g). One could also find those results in references [Dajczer et al. 2008], [Impera et al. 2018] or [de Lira and Wanderley 2015]. Assume that $\{e_i\}_{i=1}^n$ is the local frame on $\mathbb{L}^n$, and s is the flow parameter of the Killing vector field $\mathcal{V}$. We use the notation $\sigma_{ij} = \sigma(e_i, e_j) := \langle e_i, e_j\rangle$ and $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j$. Then the tangent vector of Σ_u at the point $X(x)$ are

$$X_*(e_j) = \zeta_*(e_j) + \zeta_*\Big(\frac{\partial}{\partial s}\Big)\nabla_j u := e_j|_X + \partial_s|_X \nabla_j u,$$

then the induced metric $\hat{g}$ on Σ_u is given by

$$\hat{g}_{ij} := X^* g(e_i, e_j) = \sigma_{ij} + \frac{1}{\gamma}\nabla_i u \nabla_j u,$$

where we note that $\gamma := 1/g(\mathcal{V}, \mathcal{V})^2$ and σ is the metric on $\mathbb{L}$. For a differentiable function u defined on M^n. We lifted the indices with respect to the metric on $\mathbb{L}$, i.e., $\nabla^i u := \sigma^{ij}\nabla_j u$ and $|\nabla u|^2 := \sigma(\nabla u, \nabla u) = \sigma^{ij}\nabla_i u \nabla_j u$, where ∇ denotes the Riemannian connection in $\mathbb{L}$ and $\nabla u = \sigma^{ij}\nabla_i u e_j$ is the gradient relatively to $\mathbb{L}$. And $\nabla^2 u$ is the Hessian which is given by

$$\nabla_{ij} u = \nabla_i(\nabla_j u) - (\nabla_i e_j)u.$$

Recall that $\nabla_{ij} u = \nabla_{ji} u$ and by using the Ricci identities for the third covariant derivative of u, we have

$$\nabla_{kji} u = \nabla_{jik} u + \sum_{l=1}^{n} \nabla_l u R^l_{ijk},$$

where R is the Riemannian curvature tensor

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

with $R_{ijkl} := \sigma(R(e_i, e_j)e_l, e_k)$ and $R^i_{jkl} := \sum_{s=1}^n \sigma^{is} R_{sjkl}$.

We will use the distance function to construct the auxiliary function to get the uniform gradient estimate, this kind of idea has been widely used previously, see for instance [Guan 1996; Korevaar 1988; de Lira and Wanderley 2015; Ma and Xu 2016; Xu 2014]. We note that $d(x) := \operatorname{dist}(x, \partial M)$ is a smooth function for x close to the boundary, then extend it to the whole manifold M, and $d = 0$, $\nabla_\nu d = 1$ on ∂M. For convenience, we denote $L_1 := \sup_{\overline{M}} |\nabla^2 d|$, $L_2 := \sup_{\overline{M}} |\nabla^3 d|$, and define the big O notation $O(s)$, which means that there exists a constant $C > 0$, such that $|O(s)| \leq Cs$. In particular, we have the positive constant C only depending on (M, σ), γ, L_1, L_2 and n throughout this paper.

The following result has been proved previous by de Lira and Wanderley [2015, Proposition 1], which established an a priori bound for $\dot{u} := \partial_t u$ for a general domain $M \subset \mathbb{L}$ and $\varphi \in C^1$ with $\|\varphi\|_{C^0} < 1$.

Lemma 2.1. *If $u(x,t)$ is a smooth solution to (1-3), then*

$$\sup_{M\times[0,T]} \dot{u}^2 = \sup_{M} \dot{u}^2\big|_{t=0},$$

that is, there exists $C = C(u_0) > 0$ such that

$$\sup_{M\times[0,T]} |\dot{u}| \leq C. \tag{2-1}$$

Next we obtain the uniform gradient estimate for problem (1-3), which turns the quasilinear evolution equation (1-3) into a uniformly parabolic equation and the infinite time existence of smooth solutions follows by standard regularity theory.

Theorem 2.2. *Under the assumption of Theorem 1.1, there exists a positive constant C depending on n, M, u_0, φ, γ such that*

$$\sup_{M\times[0,T]} |\nabla u| \leq C.$$

Remark 2.3. The constant δ_0 in (1-4) depends only on the geometry of domain M. In fact, even for 2-dimensional Euclidean space, see [Altschuler and Wu 1994], under the condition that $M \subset \mathbb{R}^2$ is strictly convex and $\kappa - |\nabla^T \varphi|/\sqrt{1-\varphi^2} \geq c_0 > 0$, where κ is the geodesic curvature of the curve ∂M, then they deduce that the solutions to the nonparametric mean curvature flow with capillarity type boundary condition converge to translating solution. This means that the contact angle will be affected by the geometry of domain along the flow.

Proof. Choosing the auxiliary function

$$w(x,t) := v - \sigma(\nabla u, \nabla d)\varphi,$$

where $v := \sqrt{\gamma + |\nabla u|^2}$. We want to get the uniform bound of $|\nabla u|$ in $M_{T'} := M \times [0, T']$, which is independent of T' $(0 < T' < T)$.

Assume that $w(x,t)$ attains its maximum value at $(x_0, t_0) \in \overline{M}_{T'}$. We split it into three cases to discuss. And all the computation below are done at this maximum value point.

Case 1: $(x_0, t_0) \in \partial M \times [0, T']$. In order to do the calculation, we choose an orthonormal frame $\{e_i\}_{i=1}^n$ at x_0 such that e_n be the inner normal vector field of ∂M, which is exactly equal to ν.

First we notice that $w = v - \varphi\sigma(\nabla u, \nabla d) = v - \varphi u_n$ on the boundary ∂M. Denote $\nabla' u$ and u_n be the tangential and normal part of ∇u on the boundary by our choice of frame above. From the boundary condition $u_n = \varphi v$, we deduce that

$$u_n^2 = \varphi^2 v^2 = \varphi^2(\gamma + |\nabla' u|^2 + u_n^2),$$

so it directly follows that

$$u_n^2 = \frac{\varphi^2}{1-\varphi^2}(\gamma + |\nabla' u|^2), \tag{2-2}$$

and in particular, we have

$$w = (1-\varphi^2)v = \sqrt{(\gamma + |\nabla' u|^2)\cdot(1-\varphi^2)}.$$

By using the Gauss–Weingarten equation, we get

$$\begin{aligned}\nabla_n v &= \frac{1}{2v}\left(\nabla_n \gamma + 2\sum_{\alpha=1}^{n-1} u_\alpha \nabla_{n\alpha} u + 2u_n \nabla_{nn} u\right)\\ &= \frac{\nabla_n \gamma}{2v} + \frac{1}{v}\sum_{\alpha=1}^{n-1} u_\alpha \cdot \left(u_{n\alpha} + \sum_{\beta=1}^{n-1} b_{\alpha\beta} u_\beta\right) + \varphi \nabla_{nn} u,\end{aligned}$$

where $(b_{\alpha\beta})$ is the second fundamental form of ∂M with respect to the inner normal ν and satisfies $(b_{\alpha\beta}) \geq \kappa(\delta_{\alpha\beta})$ for some $\kappa > 0$, if ∂M is strictly convex. Using the Hopf lemma, it gives us that

$$\begin{aligned}0 &\geq \nabla_n w(x_0, t_0) = \nabla_n v - \nabla_n(\langle \nabla u, \nabla d\rangle)\varphi\\ &= \nabla_n v - \nabla_{nn} u\varphi - \sum_{k=1}^{n} \nabla_k u \nabla_{kn} d\varphi\\ &= \frac{\nabla_n \gamma}{2v} + \frac{1}{v}\sum_{\alpha=1}^{n-1}\left(u_\alpha u_{n\alpha} + \sum_{\beta=1}^{n-1} u_\alpha b_{\alpha\beta} u_\beta\right) - \sum_{k=1}^{n} \nabla_k u \nabla_{kn} d\varphi.\end{aligned} \tag{2-3}$$

Since $\{e_\alpha\}$ are the tangential vector fields for all $1 \leq \alpha \leq n-1$, we obtain

$$0 = \nabla'_\alpha w(x_0, t_0) = v_\alpha - u_{n\alpha}\varphi - \varphi\varphi_\alpha v. \tag{2-4}$$

On the other hand, by taking the tangential derivative to the boundary value condition in (1-3) and combining with (2-4), it yields that

$$u_{n\alpha} = \nabla'_\alpha(\varphi v) = \varphi v_\alpha + \varphi_\alpha v = \varphi^2 u_{n\alpha} + \varphi^2 \varphi_\alpha v + \varphi_\alpha v,$$

that is,

$$u_{n\alpha} = \frac{1+\varphi^2}{1-\varphi^2}\varphi_\alpha v. \tag{2-5}$$

Substituting (2-5) into (2-3), and using the assumption in Theorem 1.1, it follows that

$$\begin{aligned}0 \geq \nabla_n w(x_0, t_0) &= \frac{\nabla_n \gamma}{2v} + \frac{1}{v}\sum_{\alpha=1}^{n-1}\biggl(u_\alpha u_{n\alpha} + \sum_{\beta=1}^{n-1} u_\alpha b_{\alpha\beta} u_\beta\biggr) - \sum_{k=1}^{n} u_k \nabla_{kn} d\varphi \\ &= \frac{\nabla_n \gamma}{2v} + \frac{1+\varphi^2}{1-\varphi^2}\sigma(\nabla'\varphi, \nabla' u) + \sum_{\alpha,\beta=1}^{n-1} \frac{u_\alpha u_\beta b_{\alpha\beta}}{v} - \varphi^2 v \nabla_{nn} d - \sum_{\alpha=1}^{n-1} u_\alpha \nabla_{\alpha n} d\varphi \\ &\geq \frac{\kappa |\nabla' u|^2}{v} - 2L_1 |\varphi| v - \frac{2|\nabla'\varphi|}{1-\varphi^2}|\nabla' u| - \frac{C}{v} \\ &\geq \frac{1}{v}\biggl(\kappa - \frac{2L_1\delta_0}{1-\delta_0^2} - \frac{2\delta_0}{(1-\delta_0^2)^{3/2}}\biggr)|\nabla' u|^2 - \frac{C}{v},\end{aligned}$$

where C is a positive constant, only depending on $\|\nabla\gamma\|_{C^0(M)}$. By choosing δ_0 with $\|\varphi\|_{C^1} \leq \delta_0$ such that

$$0 < \delta_0 \leq \min\biggl\{\frac{\sqrt{3}}{2}, \frac{\kappa}{16L_1 + 32}\biggr\}, \tag{2-6}$$

we obtain from above inequality that $|\nabla' u| \leq C$, which also gives us $|\nabla u| \leq C$.

<u>Case 2</u>: $(x_0, t_0) \in \overline{M} \times \{t = 0\}$. Then it directly yields that

$$\begin{aligned} w(x,t) &\leq w(x_0, 0) = \sqrt{\gamma + |\nabla u_0|^2} - \sigma(\nabla u_0, \nabla d)\varphi \\ &\leq C(u_0, \varphi, d, \gamma).\end{aligned} \tag{2-7}$$

Note that $|\varphi| \leq b_0 < 1$, thus it follows from above that

$$\sup_{M\times[0,T']} |\nabla u| \leq C(u_0, \gamma, d). \tag{2-8}$$

<u>Case 3</u>: $(x_0, t_0) \in M \times [0, T']$. Firstly, by choosing the local orthonormal frame $\{e_i\}_{i=1}^n$ on M such that at x_0, it holds that

$$e_1(x_0) = \frac{\nabla u}{|\nabla u|}(x_0), \quad \text{and} \quad \{\nabla_{\alpha\beta} u\}_{2\leq\alpha,\beta\leq n} \quad \text{is diagonal.} \tag{2-9}$$

Then it follows that,

$$a^{ij}|_{(x_0,t_0)} = \begin{cases} \frac{\gamma}{\gamma + (\nabla_1 u)^2}, & \text{for } i = j = 1, \\ 1, & \text{for } i = j \geq 2, \\ 0, & \text{otherwise.}\end{cases} \tag{2-10}$$

We always assume $\nabla_1 u(x_0, t_0)$ large enough in the below computation, such that $\nabla_1 u$, $v = \sqrt{\gamma + (\nabla_1 u)^2}$, and $w = v - \nabla_1 u \cdot \nabla_1 d\varphi$ (since we assume $|\varphi| \leq b_0 < 1$) are equivalent to each other at (x_0, t_0). Otherwise, we have completed the proof.

Due to the direct computation, it gives us that

$$\begin{aligned}0 &= \nabla_i w(x_0, t_0)\\ &= \nabla_i v - \sum_{k=1}^{n} \nabla_{ik} u \nabla_k d\varphi - \sum_{k=1}^{n} \nabla_k u \nabla_{ik} d\varphi - \sigma(\nabla u, \nabla d)\nabla_i \varphi\end{aligned} \tag{2-11}$$

for all $1 \le i \le n$. Equivalently, we have

$$\sum_{l=1}^{n}\left(\frac{\nabla_l u}{v} - \nabla_l d\varphi\right)\nabla_{il} u = \sum_{l=1}^{n} \nabla_l u \nabla_{il} d\varphi + \sigma(\nabla u, \nabla d)\nabla_i \varphi - \frac{\nabla_i \gamma}{2v}, \tag{2-12}$$

Let $S := \nabla_1 u/v - \nabla_1 d\varphi$, then we obtain

$$\nabla_{11} u = \sum_{\alpha=2}^{n} \frac{\nabla_\alpha d\varphi}{S}\nabla_{1\alpha} u + \frac{\nabla_{11} d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d \nabla_1 \varphi}{S}\nabla_1 u - \frac{\nabla_1 \gamma}{2vS}, \tag{2-13}$$

and for $2 \le \alpha \le n$, it holds that

$$\nabla_{1\alpha} u = \frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha} u + \frac{\nabla_{1\alpha} d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d \nabla_\alpha \varphi}{S}\nabla_1 u - \frac{\nabla_\alpha \gamma}{2vS}. \tag{2-14}$$

Substituting (2-14) into (2-13), we conclude that

$$\begin{aligned}\nabla_{11} u = \sum_{\alpha=2}^{n} \frac{\nabla_\alpha d\varphi}{S}&\left(\frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha} u + \frac{\nabla_1 u \nabla_{1\alpha} d\varphi}{S} + \frac{\nabla_1 u \nabla_1 d \nabla_\alpha \varphi}{S} - \frac{\nabla_\alpha \gamma}{2vS}\right)\\ &\qquad + \frac{\nabla_{11} d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d \nabla_1 \varphi}{S}\nabla_1 u - \frac{\nabla_1 \gamma}{2vS}\\ = \sum_{\alpha=2}^{n}&\left[\frac{(\nabla_\alpha d)^2\varphi^2}{S^2}\nabla_{\alpha\alpha} u + \frac{\nabla_\alpha d \nabla_{1\alpha} d\varphi^2}{S^2}\nabla_1 u + \frac{\nabla_1 d \nabla_\alpha d\varphi \nabla_\alpha \varphi}{S^2}\nabla_1 u - \frac{\nabla_\alpha d \nabla_\alpha \gamma\varphi}{2vS^2}\right]\\ &\qquad + \frac{\nabla_{11} d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d \nabla_1 \varphi}{S}\nabla_1 u - \frac{\nabla_1 \gamma}{2vS},\end{aligned}$$

then it follows that

$$\nabla_{11} u = \frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n} (\nabla_\alpha d)^2 \nabla_{\alpha\alpha} u + \big[O(|\varphi|) + O(|\nabla\varphi|)\big]\nabla_1 u. \tag{2-15}$$

For convenience, we denote

$$F(x, \nabla u) := -\tfrac{1}{2}\left(\frac{1}{\gamma} + \frac{1}{v^2}\right)\langle \nabla\gamma, \nabla u\rangle.$$

By taking the covariant derivative to the first equation in (1-3), for $1 \leq k \leq n$, we get

$$
\begin{aligned}
\text{(2-16)} \quad \nabla_k u_t &= \sum_{i,j=1}^{n} \nabla_k a^{ij} \cdot \nabla_{ij} u + \sum_{i,j=1}^{n} a^{ij} \nabla_{kij} u + \sum_{i=1}^{n} F_{p_i} \nabla_{ki} u + F_k \\
&= \sum_{i,j=1}^{n} a^{ij} \nabla_{kij} u + \sum_{i=1}^{n} F_{p_i} \nabla_{ik} u - \sum_{i,j=1}^{n} \frac{2\nabla_{ij} u \nabla_{ik} u \nabla_j u}{v^2} \\
&\quad + \sum_{i,j=1}^{n} \frac{\nabla_{ij} u \nabla_i u \nabla_j u}{v^4} \Big(\nabla_k \gamma + \sum_{l=1}^{n} 2\nabla_l u \nabla_{lk} u \Big) \\
&\quad + \tfrac{1}{2} \Big(\frac{\nabla_k \gamma}{\gamma^2} + \frac{\nabla_k \gamma}{v^4} \Big) \langle \nabla \gamma, \nabla u \rangle - \tfrac{1}{2} \Big(\frac{1}{\gamma} + \frac{1}{v^2} \Big) \sum_{i=1}^{n} \nabla_{ik} \gamma \nabla_i u,
\end{aligned}
$$

where

$$
F_{p_i} := -\frac{\nabla_i \gamma}{2} \Big(\frac{1}{\gamma} + \frac{1}{v^2} \Big) + \langle \nabla \gamma, \nabla u \rangle \frac{\nabla_i u}{v^4},
$$

and

$$
F_k := -\sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2\gamma} + \frac{\nabla_k \gamma}{2\gamma^2} \langle \nabla \gamma, \nabla u \rangle - \sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2v^2} + \langle \nabla \gamma, \nabla u \rangle \frac{\nabla_k \gamma}{2v^4}.
$$

Therefore, by substituting above equations into (2-16) and rearranging them, we obtain

$$
\begin{aligned}
\text{(2-17)} \quad & \nabla_k u_t - \sum_{i,j=1}^{n} a^{ij} \nabla_{kij} u - \sum_{i=1}^{n} F_{p_i} \nabla_{ki} u \\
&= -\frac{2}{v^4} \Big(\nabla_1 u \nabla_{11} u \nabla_{1k} u \gamma + \sum_{\alpha=2}^{n} \nabla_{1\alpha} u \nabla_{\alpha k} u \nabla_1 u v^2 \Big) + \frac{\nabla_{11} u (\nabla_1 u)^2 \nabla_k \gamma}{v^4} \\
&\quad - \sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2\gamma} + \frac{\nabla_k \gamma}{2\gamma^2} \langle \nabla \gamma, \nabla u \rangle - \sum_{i=1}^{n} \frac{\nabla_{ik} \gamma \nabla_i u}{2v^2} + \langle \nabla \gamma, \nabla u \rangle \frac{\nabla_k \gamma}{2v^4}
\end{aligned}
$$

On the other hand, by direct computation, we see that

$$
\begin{aligned}
0 &\leq \Big(\partial_t - \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} - \sum_{i=1}^{n} F_{p_i} \nabla_i \Big) w \\
&= \Big(\partial_t v - \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} v - \sum_{i=1}^{n} F_{p_i} \nabla_i v \Big) \\
&\quad + \Big(\sum_{i,j,k=1}^{n} a^{ij} \nabla_{jik} u \nabla_k d\varphi - \sum_{k=1}^{n} \nabla_k u_t \nabla_k d\varphi + \sum_{i,k=1}^{n} F_{p_i} \nabla_{ki} u \nabla_k d\varphi \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad +2\sum_{i,j,k=1}^{n}\bigl(\varphi a^{ij}\nabla_{ki}u\nabla_{kj}d+a^{ij}\nabla_{ki}u\nabla_{k}d\nabla_{j}\varphi\bigr)\\
&+\sum_{i,j,k=1}^{n}\bigl(2a^{ij}\nabla_{ki}d\nabla_{j}\varphi\nabla_{k}u+\nabla_{k}ua^{ij}\nabla_{jik}d\varphi+a^{ij}\nabla_{ij}\varphi\nabla_{k}u\nabla_{k}d\bigr)\\
&\qquad\qquad+\sum_{i,k=1}^{n}F_{p_i}\bigl(\nabla_{k}u\nabla_{ki}d\varphi+\nabla_{k}u\nabla_{k}d\nabla_{i}\varphi\bigr)\\
&:=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5.
\end{aligned}
$$

Next we handle the above five terms one by one. Firstly, by using the Ricci identities for the third covariant derivative of u, it follows that

$$
\begin{aligned}
\mathrm{I}_1 &:= \partial_t v-\sum_{i,j=1}^{n}a^{ij}\nabla_{ij}v-\sum_{i=1}^{n}F_{p_i}\nabla_i v\\
&=\sum_{k=1}^{n}\frac{\nabla_k u\nabla_k u_t}{v}-\frac{1}{2v}\sum_{i,j=1}^{n}\biggl(a^{ij}\nabla_{ij}\gamma+2a^{ij}\sum_{k=1}^{n}\nabla_{ki}u\nabla_{kj}u+2\sum_{k=1}^{n}a^{ij}\nabla_k u\nabla_{jik}u\biggr)\\
&\qquad+\frac{1}{4v^3}\sum_{i,j=1}^{n}a^{ij}\biggl(\nabla_i\gamma+2\sum_{k=1}^{n}\nabla_k u\nabla_{ki}u\biggr)\biggl(\nabla_j\gamma+2\sum_{l=1}^{n}\nabla_l u\nabla_{lj}u\biggr)\\
&\qquad\qquad-\sum_{i=1}^{n}F_{p_i}\frac{1}{2v}\biggl(\nabla_i\gamma+2\sum_{k=1}^{n}\nabla_k u\nabla_{ki}u\biggr)\\
&=\sum_{k=1}^{n}\frac{\nabla_k u}{v}\biggl(\nabla_k u_t-\sum_{i,j=1}^{n}a^{ij}\nabla_{kji}u-\sum_{i=1}^{n}F_{p_i}\nabla_{ki}u\biggr)\\
&\qquad+\sum_{i,j,k=1}^{n}\frac{1}{v}\biggl(\frac{1}{v^2}\sum_{l=1}^{n}a^{ij}\nabla_k u\nabla_l u\nabla_{ki}u\nabla_{lj}u-a^{ij}\nabla_{ki}u\nabla_{kj}u\biggr)\\
&\qquad\qquad+\frac{1}{v^3}\sum_{i,j,l=1}^{n}a^{ij}\nabla_i\gamma\nabla_l u\nabla_{lj}u\\
&\qquad+\biggl(\frac{1}{4v^3}\sum_{i,j=1}^{n}a^{ij}\nabla_i\gamma\nabla_j\gamma-\frac{1}{2v}\sum_{i=1}^{n}F_{p_i}\nabla_i\gamma-\frac{1}{2v}\sum_{i,j=1}^{n}a^{ij}\nabla_{ij}\gamma\biggr)\\
&\qquad\qquad-\sum_{i,j,k,l=1}^{n}\frac{a^{ij}\nabla_k u\nabla_l uR_{likj}}{v}\\
&:=\mathrm{I}_{11}+\mathrm{I}_{12}+\mathrm{I}_{13}+\mathrm{I}_{14}+\mathrm{I}_{15}.
\end{aligned}
$$

Now we switch to handle those terms one by one. By substituting (2-17) into term I_{11}, we have

$$
\mathrm{I}_{11}:=\sum_{k=1}^{n}\frac{\nabla_k u}{v}\biggl(\nabla_k u_t-\sum_{i,j=1}^{n}a^{ij}\nabla_{kji}u-\sum_{i=1}^{n}F_{p_i}\nabla_{ki}u\biggr)
$$

$$\begin{aligned}
&=\sum_{k=1}^{n}\frac{\nabla_k u}{v}\bigg[-\frac{2}{v^4}\bigg(\nabla_1 u\nabla_{11}u\nabla_{1k}u\gamma+\sum_{\alpha=2}^{n}\nabla_{1\alpha}u\nabla_{\alpha k}u\nabla_1 u v^2\bigg)+\frac{\nabla_{11}u(\nabla_1 u)^2\nabla_k\gamma}{v^4}\\
&\qquad\qquad-\sum_{i=1}^{n}\frac{\nabla_{ik}\gamma\nabla_i u}{2\gamma}+\frac{\nabla_k\gamma}{2\gamma^2}\langle\nabla\gamma,\nabla u\rangle-\sum_{i=1}^{n}\frac{\nabla_{ik}\gamma\nabla_i u}{2v^2}+\langle\nabla\gamma,\nabla u\rangle\frac{\nabla_k\gamma}{2v^4}\bigg]\\
&=\bigg[-\frac{2(\nabla_1 u)^2\gamma}{v^5}(\nabla_{11}u)^2-\frac{2(\nabla_1 u)^2}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2+\frac{(\nabla_1 u)^3\gamma_1}{v^5}\nabla_{11}u\bigg]\\
&\qquad\qquad+\bigg[\frac{\langle\nabla\gamma,\nabla u\rangle^2}{2\gamma^2 v}-\frac{\nabla^2\gamma(\nabla u,\nabla u)}{2\gamma v}-\frac{\nabla^2\gamma(\nabla u,\nabla u)}{2v^3}+\frac{(\nabla_1 u)^2(\nabla_1\gamma)^2}{2v^5}\bigg]\\
&:=\mathrm{I}_{111}+\mathrm{I}_{112}.
\end{aligned}$$

We note that

$$\begin{aligned}
\mathrm{I}_{112}&:=\frac{\langle\nabla\gamma,\nabla u\rangle^2}{2\gamma^2 v}-\frac{\nabla^2\gamma(\nabla u,\nabla u)}{2\gamma v}-\frac{\nabla^2\gamma(\nabla u,\nabla u)}{2v^3}+\frac{(\nabla_1 u)^2(\nabla_1\gamma)^2}{2v^5}\\
&=\frac{\langle\nabla\gamma,\nabla u\rangle^2}{2\gamma^2 v}-\frac{\nabla^2\gamma(\nabla u,\nabla u)}{2\gamma v}+O\Big(\frac{1}{v}\Big).
\end{aligned}$$

Using (2-10), we have

$$\begin{aligned}
\mathrm{I}_{12}&:=\frac{1}{v^3}\sum_{i,j,k,l=1}^{n}a^{ij}\nabla_k u\nabla_l u\nabla_{ki}u\nabla_{lj}u-\frac{1}{v}\sum_{i,j,k=1}^{n}a^{ij}\nabla_{ki}u\nabla_{kj}u\\
&=\frac{\gamma(\nabla_1 u)^2}{v^5}(\nabla_{11}u)^2+\frac{(\nabla_1 u)^2}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2-\frac{\gamma}{v^3}\sum_{k=1}^{n}(\nabla_{k1}u)^2-\frac{1}{v}\sum_{k=1}^{n}\sum_{\alpha=2}^{n}(\nabla_{k\alpha}u)^2\\
&=-\frac{\gamma^2}{v^5}(\nabla_{11}u)^2-\frac{2\gamma}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2-\frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2,
\end{aligned}$$

and similarly, combining with (2-14) and (2-15), we obtain

$$\begin{aligned}
\mathrm{I}_{13}&:=-\frac{1}{v^3}\sum_{i,j,l=1}^{n}a^{ij}\nabla_i\gamma\nabla_l u\nabla_{lj}u=-\frac{\gamma\nabla_1\gamma\nabla_1 u}{v^5}\nabla_{11}u-\sum_{\alpha=2}^{n}\frac{\nabla_\alpha\gamma\nabla_1 u}{v^3}\nabla_{1\alpha}u\\
&=-\frac{\gamma\nabla_1\gamma\nabla_1 u}{v^5}\bigg[\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u+\big(O(|\varphi|)+O(|\nabla\varphi|)\big)\nabla_1 u\bigg]\\
&\qquad\qquad-\sum_{\alpha=2}^{n}\frac{\nabla_\alpha\gamma\nabla_1 u}{v^3}\bigg(\frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha}u+\frac{\nabla_{1\alpha}d\varphi}{S}\nabla_1 u+\frac{\nabla_1 d\nabla_\alpha\varphi}{S}\nabla_1 u-\frac{\nabla_\alpha\gamma}{2vS}\bigg)\\
&=-\frac{\gamma\nabla_1\gamma\nabla_1 u}{v^5}\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u-\frac{\nabla_1 u\varphi}{v^3 S}\sum_{\alpha=2}^{n}\nabla_\alpha\gamma\nabla_\alpha d\nabla_{\alpha\alpha}u+O\Big(\frac{1}{v}\Big)
\end{aligned}$$

$$= O\Big(\frac{1}{v^2}\Big)\sum_{\alpha=2}^{n} |\nabla_{\alpha\alpha}u| + O\Big(\frac{1}{v}\Big),$$

also

$$\begin{aligned}
\mathrm{I}_{14} &:= \frac{1}{4v^3}\sum_{i,j=1}^{n} a^{ij}\nabla_i\gamma\nabla_j\gamma - \frac{1}{2v}\sum_{i=1}^{n} F_{p_i}\nabla_i\gamma - \frac{1}{2v}\sum_{i,j=1}^{n} a^{ij}\nabla_{ij}\gamma \\
&= \frac{|\nabla\gamma|^2}{4v}\Big(\frac{1}{\gamma}+\frac{1}{v^2}\Big) - \frac{\langle\nabla\gamma,\,\nabla u\rangle^2}{2v^5} - \frac{\gamma\nabla_{11}\gamma}{2v^3} - \frac{\nabla_{\alpha\alpha}\gamma}{2v} + \frac{\gamma(\nabla_1\gamma)^2}{4v^5} + \frac{(\nabla_\alpha\gamma)^2}{4v^3} \\
&= O\Big(\frac{1}{v}\Big).
\end{aligned}$$

And the term I_{15}, which relates to the curvature, is

$$\mathrm{I}_{15} := -\sum_{i,j,k,l=1}^{n} \frac{a^{ij}\nabla_k u\nabla_l u R_{likj}}{v} = -\frac{1}{v}\,\mathrm{Ric}_{\mathbb{L}}(\nabla u,\,\nabla u).$$

Secondly, we are going to handle term I_2, by using (2-17) again, it yields that

$$\begin{aligned}
\mathrm{I}_2 &:= \sum_{i,j,k=1}^{n} a^{ij}\nabla_{jik}u\nabla_k d\varphi - \sum_{k=1}^{n}\nabla_k u_t\nabla_k d\varphi + \sum_{i,k=1}^{n} F_{p_i}\nabla_{ki}u\nabla_k d\varphi \\
&= \sum_{k=1}^{n}\nabla_k d\varphi\cdot\Big(\sum_{i,j=1}^{n} a^{ij}\nabla_{kji}u - \nabla_k u_t + \sum_{i=1}^{n} F_{p_i}\nabla_{ki}u + \sum_{i,j,l=1}^{n} a^{ij}\nabla_l u R^{l}_{ikj}\Big) \\
&= \sum_{k=1}^{n}\nabla_k d\varphi\Big[-\frac{2}{v^4}\Big(\nabla_1 u\nabla_{11}u\nabla_{1k}u\gamma + \sum_{\alpha=2}^{n}\nabla_{1\alpha}u\nabla_{\alpha k}u\nabla_1 u v^2\Big) \\
&\qquad\qquad + \frac{\nabla_{11}u(\nabla_1 u)^2\nabla_k\gamma}{v^4} - \sum_{i=1}^{n}\frac{\nabla_{ik}\gamma\nabla_i u}{2\gamma} + \frac{\nabla_k\gamma}{2\gamma^2}\langle\nabla\gamma,\,\nabla u\rangle \\
&\qquad\qquad\qquad - \sum_{i=1}^{n}\frac{\nabla_{ik}\gamma\nabla_i u}{2v^2} + \langle\nabla\gamma,\,\nabla u\rangle\frac{\nabla_k\gamma}{2v^4}\Big] + \mathrm{Ric}_{\mathbb{L}}(\nabla u,\,\nabla d)\varphi \\
&= \sum_{k=1}^{n}\Big(\frac{2\nabla_k d\gamma\nabla_1 u}{v^4}\nabla_{11}u\nabla_{1k}u\varphi - \sum_{\alpha=2}^{n}\frac{2\nabla_k d\varphi\nabla_1 u}{v^2}\nabla_{1\alpha}u\nabla_{k\alpha}u \\
&\qquad - \frac{\nabla_k d\nabla_k\gamma(\nabla_1 u)^2}{v^4}\nabla_{11}u - \frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u}{2\gamma}\varphi + \frac{\nabla_k d\nabla_k\gamma\nabla_1\gamma}{2\gamma^2}\nabla_1 u\varphi \\
&\qquad\qquad - \frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u\varphi}{2v^2} + \frac{\nabla_k d\nabla_k\gamma\nabla_1\gamma\nabla_1 u}{2v^4}\varphi\Big) + \mathrm{Ric}_{\mathbb{L}}(\nabla u,\,\nabla d)\varphi \\
&= \Big(\frac{2\nabla_1 d\gamma\nabla_1 u}{v^4}(\nabla_{11}u)^2\varphi - \frac{2\nabla_1 d\varphi\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2\Big)
\end{aligned}$$

$$\begin{aligned}
&+\bigg(\frac{2\gamma\nabla_1 u}{v^4}\nabla_{11}u\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{1\alpha}u\varphi-\frac{2\varphi\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{1\alpha}u\nabla_{\alpha\alpha}u\\
&\qquad\qquad-\frac{(\nabla_1 u)^2}{v^4}\nabla_{11}u\langle\nabla d,\nabla\gamma\rangle\bigg)\\
&+\bigg[-\sum_{k=1}^{n}\frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u}{2\gamma}\varphi+\frac{\varphi\nabla_1\gamma}{2\gamma^2}\nabla_1 u\langle\nabla d,\nabla\gamma\rangle\\
&\qquad-\sum_{k=1}^{n}\frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u\varphi}{2v^2}+\frac{\varphi\nabla_1\gamma\nabla_1 u}{2v^4}\langle\nabla d,\nabla\gamma\rangle+\mathrm{Ric}_{\mathbb{L}}(\nabla u,\nabla d)\varphi\bigg]\\
&:=\mathrm{I}_{21}+\mathrm{I}_{22}+\mathrm{I}_{23}.
\end{aligned}$$

Combining terms I_{111}, I_{12} and I_{21} together, and using (2-15), it yields that

$$\begin{aligned}
&\mathrm{I}_{111}+\mathrm{I}_{12}+\mathrm{I}_{21}\\
&:=-\frac{2(\nabla_1 u)^2\gamma}{v^5}(\nabla_{11}u)^2-\frac{2(\nabla_1 u)^2}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2+\frac{(\nabla_1 u)^3\nabla_1\gamma}{v^5}\nabla_{11}u-\frac{\gamma^2}{v^5}(\nabla_{11}u)^2\\
&\qquad-\frac{2\gamma}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2-\frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2\\
&\qquad+\bigg[\frac{2\nabla_1 d\gamma\nabla_1 u}{v^4}(\nabla_{11}u)^2\varphi-\frac{2\nabla_1 d\varphi\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2\bigg]\\
&=-\frac{2(\nabla_1 u)^2\gamma}{v^5}\Big(1+\frac{v}{\nabla_1 u}\nabla_1 d\varphi\Big)(\nabla_{11}u)^2-\frac{\gamma^2}{v^5}(\nabla_{11}u)^2-\frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2\\
&\qquad-\frac{2}{v}\bigg(1+\frac{\nabla_1 u}{v}\nabla_1 d\varphi\bigg)\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2\\
&\qquad+\frac{(\nabla_1 u)^3\nabla_1\gamma}{v^5}\bigg[\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u+\big(O(|\varphi|)+O(|\nabla\varphi|)\big)\nabla_1 u\bigg].
\end{aligned}$$

By choosing δ_0 small enough with $\|\varphi\|_{C^0(M)}\le\delta_0$, say $\delta_0\le\frac{1}{2}$, such that

$$1+\frac{v}{\nabla_1 u}d_1\varphi<0\quad\text{and}\quad 1+\frac{\nabla_1 u}{v}d_1\varphi<0.$$

Then it follows that

$$\begin{aligned}
&\mathrm{I}_{111}+\mathrm{I}_{12}+\mathrm{I}_{21}\\
&\le-\frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2+\frac{(\nabla_1 u)^3\nabla_1\gamma}{v^5}\bigg[\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u+\big(O(|\varphi|)+O(|\nabla\varphi|)\big)\nabla_1 u\bigg]
\end{aligned}$$

$$= -\frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2 + O\Big(\frac{1}{v^2}\Big)\sum_{\alpha=2}^{n}|\nabla_{\alpha\alpha}u| + \big[O(|\varphi|)+O(|\nabla\varphi|)\big]\nabla_1 u.$$

Secondly, we are going to handle term I_{22}, by substituting (2-14) and (2-15) into term I_{22}; we obtain that

$$\begin{aligned}
\mathrm{I}_{22} := {} & \frac{2\gamma\nabla_1 u}{v^4}\varphi\nabla_{11}u\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{1\alpha}u - \frac{2\varphi\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{1\alpha}u\nabla_{\alpha\alpha}u \\
& \qquad - \frac{(\nabla_1 u)^2}{v^4}\nabla_{11}u\langle\nabla d, \nabla\gamma\rangle \\
= {} & \frac{2\gamma\nabla_1 u}{v^4}\varphi\bigg[\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u + \big(O(|\varphi|)+O(|\nabla\varphi|)\big)\nabla_1 u\bigg] \\
& \cdot\sum_{\alpha=2}^{n}\nabla_\alpha d\cdot\bigg[\frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha}u + \frac{\nabla_{1\alpha}d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d\nabla_\alpha\varphi}{S}\nabla_1 u - \frac{\nabla_\alpha\gamma}{2vS}\bigg] \\
& - \frac{2\varphi\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}\nabla_\alpha d\cdot\bigg[\frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha}u + \frac{\nabla_{1\alpha}d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d\nabla_\alpha\varphi}{S}\nabla_1 u - \frac{\nabla_\alpha\gamma}{2vS}\bigg]\cdot\nabla_{\alpha\alpha}u \\
& - \frac{(\nabla_1 u)^2}{v^4}\cdot\bigg[\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u + \big(O(|\varphi|)+O(|\nabla\varphi|)\big)\nabla_1 u\bigg]\langle\nabla d, \nabla\gamma\rangle \\
= {} & O\Big(\frac{1}{v^3}\Big)\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2 + O\Big(\frac{1}{v}\Big)\sum_{\alpha=2}^{n}|\nabla_{\alpha\alpha}u| + O\Big(\frac{1}{v}\Big).
\end{aligned}$$

And we notice that for term I_{23}, we have

$$\begin{aligned}
\mathrm{I}_{23} := {} & -\sum_{k=1}^{n}\frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u}{2\gamma}\varphi + \frac{\varphi\nabla_1\gamma}{2\gamma^2}\nabla_1 u\langle\nabla d, \nabla\gamma\rangle - \sum_{k=1}^{n}\frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u\varphi}{2v^2} \\
& \qquad + \frac{\varphi\nabla_1\gamma\nabla_1 u}{2v^4}\langle\nabla d, \nabla\gamma\rangle + \mathrm{Ric}_{\mathbb{L}}(\nabla u, \nabla d)\varphi \\
= {} & O(|\varphi|)\nabla_1 u.
\end{aligned}$$

Substituting (2-14) and (2-15) into term I_3, we deduce that

$$\begin{aligned}
\mathrm{I}_3 := {} & \sum_{i,j,k=1}^{n}\big(2\varphi a^{ij}\nabla_{ki}u\nabla_{kj}d + 2a^{ij}\nabla_{ki}u\nabla_k d\nabla_j\varphi\big) \\
= {} & 2\sum_{k=1}^{n}\bigg(\sum_{\alpha=2}^{n}\nabla_{k\alpha}u\nabla_{k\alpha}d\varphi + \sum_{\alpha=2}^{n}\nabla_{k\alpha}u\nabla_k d\nabla_\alpha\varphi + \frac{\nabla_{k1}u\nabla_{k1}d}{v^2}\gamma\varphi + \frac{\nabla_{k1}u\nabla_k d\nabla_1\varphi}{v^2}\gamma\bigg) \\
= {} & 2\bigg[\frac{\nabla_{11}u}{v^2}\gamma\big(\nabla_{11}d\varphi + \nabla_1 d\nabla_1\varphi\big) + \sum_{\alpha=2}^{n}\nabla_{1\alpha}u\big(\nabla_{1\alpha}d\varphi + \nabla_1 d\nabla_\alpha\varphi\big)
\end{aligned}$$

$$\begin{aligned}
&\qquad\qquad + \sum_{\alpha=2}^{n} \frac{\nabla_{1\alpha}u}{v^2}\gamma\big(\nabla_{1\alpha}d\varphi + \nabla_\alpha d\nabla_1\varphi\big) + \sum_{\alpha=2}^{n} \nabla_{\alpha\alpha}u\big(\nabla_{\alpha\alpha}d\varphi + \nabla_\alpha d\nabla_\alpha\varphi\big)\Bigg]\\
&= \frac{2\gamma}{v^2}\left[\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u + \big(O(|\varphi|) + O(|\nabla\varphi|)\big)\nabla_1 u\right]\cdot\big(\nabla_{11}d\varphi + \nabla_1 d\nabla_1\varphi\big)\\
&\quad + 2\sum_{\alpha=2}^{n}\left(\frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha}u + \frac{\nabla_{1\alpha}d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d\nabla_\alpha\varphi}{S}\nabla_1 u - \frac{\nabla_\alpha\gamma}{2vS}\right)\cdot\big(\nabla_{1\alpha}d\varphi + \nabla_1 d\nabla_\alpha\varphi\big)\\
&\quad + \sum_{\alpha=2}^{n}\frac{4\gamma}{v^2}\left(\frac{\nabla_\alpha d\varphi}{S}\nabla_{\alpha\alpha}u + \frac{\nabla_{1\alpha}d\varphi}{S}\nabla_1 u + \frac{\nabla_1 d\nabla_\alpha\varphi}{S}\nabla_1 u - \frac{\nabla_\alpha\gamma}{2vS}\right)\cdot\big(\nabla_{1\alpha}d\varphi + \nabla_\alpha d\nabla_1\varphi\big)\\
&\qquad\qquad\qquad + 2\sum_{\alpha=2}^{n}\nabla_{\alpha\alpha}u\big(\nabla_{\alpha\alpha}d\varphi + \nabla_\alpha d\nabla_\alpha\varphi\big)\\
&= \frac{2\gamma}{v^2}\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u\big(\nabla_{11}d\varphi + \nabla_1 d\nabla_1\varphi\big)\\
&\quad + \frac{2\varphi}{S}\sum_{\alpha=2}^{n}\nabla_{\alpha\alpha}u\nabla_\alpha d\big(\nabla_{1\alpha}d\varphi + \nabla_1 d\nabla_\alpha\varphi\big) + \frac{4\gamma\varphi}{v^2 S}\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{\alpha\alpha}u\big(\nabla_{1\alpha}d\varphi + \nabla_\alpha d\nabla_1\varphi\big)\\
&\qquad\qquad + 2\sum_{\alpha=2}^{n}\nabla_{\alpha\alpha}u\big(\nabla_{\alpha\alpha}d\varphi + \nabla_\alpha d\nabla_\alpha\varphi\big) + \big(O(|\varphi|) + O(|\nabla\varphi|)\big)\nabla_1 u\\
&= O(|\varphi| + |\nabla\varphi|)\sum_{\alpha=2}^{n}|\nabla_{\alpha\alpha}u| + \big(O(|\varphi|) + O(|\nabla\varphi|)\big)\nabla_1 u.
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
\mathrm{I}_4 &:= \sum_{i,j,k=1}^{n}\big(2a^{ij}\nabla_{ki}d\nabla_j\varphi\nabla_k u + a^{ij}\nabla_k u\nabla_{jik}d\varphi + a^{ij}\nabla_{ij}\varphi\nabla_k u\nabla_k d\big)\\
&= \sum_{\alpha=2}^{n}\big(\nabla_1 u\nabla_{\alpha\alpha}\varphi\nabla_1 d + 2\nabla_1 u\nabla_\alpha\varphi\nabla_{1\alpha}d + \nabla_1 u\nabla_{\alpha\alpha 1}d\varphi\big)\\
&\qquad\qquad + \gamma\left(\frac{\nabla_1 u}{v^2}\nabla_{111}d\varphi + \frac{\nabla_1 u}{v^2}\nabla_{11}\varphi\nabla_1 d + \frac{2\nabla_1 u}{v^2}\nabla_{11}d\nabla_1\varphi\right)\\
&= \big[O(|\varphi|) + O(|\nabla\varphi|) + O(|\nabla^2\varphi|)\big]\nabla_1 u + O\Big(\frac{1}{v}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_5 &:= \sum_{i,k=1}^{n} F_{p_i}\big(\nabla_k u\nabla_{ik}d\varphi + \nabla_k u\nabla_k d\nabla_i\varphi\big)\\
&= \nabla_1 u\cdot\sum_{i=1}^{n}\left[-\frac{\nabla_i\gamma}{2}\Big(\frac{1}{\gamma} + \frac{1}{v^2}\Big) + \langle\nabla\gamma, \nabla u\rangle\frac{\nabla_i u}{v^4}\right]\cdot\big(\nabla_{1i}d\varphi + \nabla_1 d\nabla_i\varphi\big)\\
&= \big[O(|\varphi|) + O(|\nabla\varphi|)\big]\nabla_1 u + O\Big(\frac{1}{v}\Big).
\end{aligned}$$

By adding terms I_3, I_4, and I_5 together, we obtain

$$\begin{aligned}
&\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5\\
&:=\frac{2\gamma}{v^2}\frac{\varphi^2}{S^2}\sum_{\alpha=2}^{n}(\nabla_\alpha d)^2\nabla_{\alpha\alpha}u\big(\nabla_{11}d\varphi+\nabla_1 d\nabla_1\varphi\big)\\
&\quad+\frac{2\varphi}{S}\sum_{\alpha=2}^{n}\nabla_{\alpha\alpha}u\nabla_\alpha d\big(\nabla_{1\alpha}d\varphi+\nabla_1 d\nabla_\alpha\varphi\big)+\frac{4\gamma\varphi}{v^2 S}\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{\alpha\alpha}u\big(\nabla_{1\alpha}d\varphi+\nabla_\alpha d\nabla_1\varphi\big)\\
&\quad+2\sum_{\alpha=2}^{n}\nabla_{\alpha\alpha}u\big(\nabla_{\alpha\alpha}d\varphi+\nabla_\alpha d\nabla_\alpha\varphi\big)+\big[O(|\varphi|)+O(|\nabla\varphi|)+O(|\nabla^2\varphi|)\big]\nabla_1 u+O\Big(\frac{1}{v}\Big)\\
&=\big[O(|\varphi|)+O(|\nabla\varphi|)\big]\sum_{\alpha=2}^{n}|\nabla_{\alpha\alpha}u|+O\big(|\varphi|+|\nabla\varphi|+|\nabla^2\varphi|\big)\nabla_1 u+O\Big(\frac{1}{v}\Big).
\end{aligned}$$

Finally, by adding all above terms together and using the assumption in Theorem 1.1, we get

$$\begin{aligned}
0&\le\Big(\partial_t-\sum_{i,j=1}^{n}a^{ij}\nabla_{ij}-\sum_{i=1}^{n}F_{p_i}\nabla_i\Big)w=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5\\
&\le\frac{\langle\nabla\gamma,\nabla u\rangle^2}{2\gamma^2 v}-\frac{\nabla^2\gamma(\nabla u,\nabla u)}{2\gamma v}-\frac{1}{v}\operatorname{Ric}_{\mathbb{L}}(\nabla u,\nabla u)\\
&\qquad-\frac{1}{v}\sum_{\alpha=2}^{n}(\nabla_{\alpha\alpha}u)^2+\big[O(|\varphi|)+O(|\nabla\varphi|)\big]\sum_{\alpha=2}^{n}|\nabla_{\alpha\alpha}u|\\
&\qquad\qquad+O\big(|\varphi|+|\nabla\varphi|+|\nabla^2\varphi|\big)\nabla_1 u+O\Big(\frac{1}{v}\Big)\\
&\le -k_0\frac{|\nabla u|^2}{v}+C_1|\varphi\|_{C^2}\nabla_1 u+\frac{C_2}{v},
\end{aligned}$$

where we have used the inequality that $-\alpha s^2+\beta s\le\beta^2/(4\alpha)$ holds for any $\alpha>0$ in the last equality above. And C_1, C_2 are the positive constants which only depend on the L_1, L_2, n, and M.

Hence, by choosing $\delta_0\le k_0/(2C_1+1)$ for $\|\varphi\|_{C^2}\le\delta_0<1$, we obtain the gradient estimate

$$v(x_0,t_0)\le C.$$

Therefore, combining all above three cases together, by choosing

$$0<\delta_0\le\min\Big\{\frac{1}{2},\frac{\kappa}{16L_1+32},\frac{k_0}{2C_1+1}\Big\},\tag{2-18}$$

we conclude that $v(x_0,t_0)\le C$, where C is a positive constant which is independent of T. This finishes the proof of Theorem 2.2. □

We conclude this section by giving some particular examples to illustrate the Ricci compatible condition in Theorem 1.1. In particular, if the induced metric on $\mathbb{L}$ is rotational invariant. That is,

$$\sigma = dr^2 + h^2(r) g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard metric on $\mathbb{S}^{n-1}$. We can write the Ricci curvature condition with respect to γ explicitly. In fact, let $\{\hat{\omega}_\alpha\}_{\alpha=2}^n$ be an orthonormal coframe on $\mathbb{S}^{n-1}$ with respect to $g_{\mathbb{S}^{n-1}}$, then we define $\omega_1 = dr$ and $\omega_\alpha = h(r)\hat{\omega}_\alpha$ for $2 \le \alpha \le n$. Then the coframe $\{\omega_i\}_{i=1}^n$ forms an orthonormal coframe of $\mathbb{L}$ with respect to σ. The Ricci curvature of σ is then given by (see for example the Appendix A in the monograph [Li 2012])

$$R_{1j} = -(n-1)\big\{(\log h(r))'' + [(\log h(r))']^2\big\}\delta_{1j},$$

and

$$\begin{aligned} R_{\alpha\beta} &= h^{-2}(r) R_{\alpha\beta}^{\mathbb{S}^{n-1}} - \big\{(\log h(r))'' + (n-1)[(\log h)']^2\big\}\delta_{\alpha\beta} \\ &= \big\{(n-1)(n-2)h^{-2}(r) - (\log h(r))'' + (n-1)[(\log h)']^2\big\}\delta_{\alpha\beta}. \end{aligned}$$

We also assume that $\gamma = \gamma(r)$, which means the norm of the Killing vector field only depends on r. Thus the ambient space metric can be written as

$$g = \varrho^2(r) ds^2 + dr^2 + h^2(r) g_{\mathbb{S}^{n-1}},$$

where $\varrho(r) := 1/\sqrt{\gamma(r)} = e^{-\rho}$. In this case, the ambient space N can be viewed as a doubly warped product manifold with the warping functions depending only on r. Therefore, the Ricci curvature compatible condition

$$\operatorname{Ric}_{\mathbb{L}} + \nabla^2\rho = \operatorname{Ric}_{\mathbb{L}} + \nabla^2 \log \varrho^{-1}(r) \ge k_0\sigma,$$

is corresponding to that h and ϱ satisfy

$$-(n-1)\big\{(\log h(r))'' + [(\log h(r))']^2\big\} - \frac{\varrho''}{\varrho} + \frac{(\varrho')^2}{\varrho^2} \ge k_0,$$

and

$$(n-1)(n-2)h^{-2}(r) - (\log h(r))'' + (n-1)[(\log h)']^2 \ge k_0,$$

for any positive constant k_0. One can check directly that specific examples like

(1) $h(r) := r,\quad \varrho(r) := e^{-(c/2)r^2}$ for any $c > 0$,

(2) $h(r) := \sin r,\quad \varrho(r) := e^{-(c/2)r^2}$ for any $c > 1 - n$,

(3) $h(r) := \sinh r,\quad \varrho(r) := e^{-(c/2)r^2}$ for any $c > n - 1$

are included in the above conditions.

3. Existence for the approximating problems

The aim of this section is to show the existence of Equations (1-5), and then prove Theorem 1.2. One can see that if $\hat{u}$ is a solution of (1-5), then $\hat{u}+c$ is also a solution of (1-5). Hence one can not expect to obtain the C^0 estimate of $\hat{u}$. We use the approximation scheme to handle this problem. The main idea is that the limit of $\hat{u}_\varepsilon$ to the approximating problems (3-1) will solve problem (1-5). Firstly, we need to get the uniform gradient estimate of Equations (3-1), which does not depend on the C^0 norm of the solution and ε. To be precise, we get the following results.

Lemma 3.1. *Under the assumption of Theorem 1.1, if u solves the equations*

$$\begin{cases} \sum_{i,j=1}^n a^{ij}\nabla_{ij}u - \frac{1}{2}(1/\gamma + 1/v^2)\langle \nabla\gamma, \nabla u\rangle = \varepsilon u & \text{in } M, \\ \nabla_\nu u = \varphi\sqrt{\gamma + |\nabla u|^2} & \text{on } \partial M. \end{cases} \tag{3-1}$$

Then we have

$$\sup_{\overline{M}} |\nabla u| \leq C,$$

where C is a positive constant depending only on n, M, and φ, but not on ε and $\|u\|_{C^0}$.

Proof. As before, we use the same auxiliary function

$$w(x) := v - \sigma(\nabla u, \nabla d)\varphi,$$

where $v := \sqrt{\gamma + |\nabla u|^2}$. We want to get the uniform bound of $|\nabla u|$ in $\overline{M}$, which is independent of ε and $\|u\|_{C^0}$.

Assume $w(x)$ attains its maximum value at $x_0 \in \overline{M}$. We split it into two cases to discuss and finish the proof.

Case 1: If $x_0 \in \partial M$. This case is the same as in Case 1 in Theorem 2.2, since we retain the same boundary value condition. By choosing δ_0 as in (2-6), we obtain the estimate for $|\nabla u|$.

Case 2: If $x_0 \in M$. As the idea is same as in Case 3 in Theorem 2.2, we mainly focus only on the difference when we replace u_t there with εu here. Firstly, we have

$$\begin{aligned} \varepsilon\nabla_k u = \sum_{i,j=1}^n a^{ij}\nabla_{kij}u + \sum_{i=1}^n F_{p_i}\nabla_{ki}u - \sum_{i,j=1}^n \frac{2\nabla_{ij}u\nabla_{ik}u\nabla_j u}{v^2} \\ + \sum_{i,j=1}^n \frac{\nabla_{ij}u\nabla_i u\nabla_j u}{v^4}\cdot\left(\nabla_k\gamma + 2\sum_{l=1}^n \nabla_l u\nabla_{kl}u\right) \\ + \frac{1}{2}\left(\frac{\nabla_k\gamma}{\gamma^2} + \frac{\nabla_k\gamma}{v^4}\right)\langle\nabla\gamma, \nabla u\rangle - \frac{1}{2}\left(\frac{1}{\gamma} + \frac{1}{v^2}\right)\sum_{i=1}^n \nabla_{ik}\gamma\nabla_i u. \end{aligned}$$

By choosing the local orthonormal frame $\{e_i\}_{i=1}^n$ on M such that at x_0, it holds that

$$e_1(x_0)=\frac{\nabla u}{|\nabla u|}(x_0),\quad \text{and}\quad \{\nabla_{\alpha\beta}u\}_{2\le\alpha,\beta\le n}\quad \text{is diagonal.} \tag{3-2}$$

Then it follows from the above equation that at x_0,

$$\begin{aligned}
&-\Big(\sum_{i,j=1}^n a^{ij}\nabla_{kji}u+\sum_{i=1}^n F_{p_i}\nabla_{ki}u\Big)\\
&\quad=-\frac{2}{v^4}\Big(\nabla_1u\nabla_{11}u\nabla_{1k}u\gamma+\sum_{\alpha=2}^n\nabla_{1\alpha}u\nabla_{\alpha k}u\nabla_1uv^2\Big)+\frac{\nabla_{11}u(\nabla_1u)^2\nabla_k\gamma}{v^4}\\
&\qquad-\sum_{i=1}^n\frac{\nabla_{ik}\gamma\nabla_iu}{2\gamma}+\frac{\nabla_k\gamma}{2\gamma^2}\langle\nabla\gamma,\nabla u\rangle-\sum_{i=1}^n\frac{\nabla_{ik}\gamma\nabla_iu}{2v^2}+\langle\nabla\gamma,\nabla u\rangle\frac{\nabla_k\gamma}{2v^4}-\varepsilon\nabla_ku.
\end{aligned}\tag{3-3}$$

On the other hand, we have

$$\begin{aligned}
0&\le-\Big(\sum_{i,j=1}^n a^{ij}\nabla_{ij}+\sum_{i=1}^n F_{p_i}\nabla_i\Big)w(x_0)\\
&=-\Big(\sum_{i,j=1}^n a^{ij}\nabla_{ij}v+\sum_{i=1}^n F_{p_i}\nabla_iv\Big)+\varphi\sum_{k=1}^n\nabla_kd\cdot\Big(\sum_{i,j=1}^n a^{ij}\nabla_{jik}u+\sum_{i=1}^n F_{p_i}\nabla_{ik}u\Big)\\
&\qquad+\sum_{i,j,k=1}^n\big(2\varphi a^{ij}\nabla_{ki}u\nabla_{kj}d+2a^{ij}\nabla_{ki}u\nabla_kd\nabla_j\varphi\big)\\
&\qquad+\sum_{i,j,k=1}^n\big(2a^{ij}\nabla_{ki}d\nabla_j\varphi\nabla_ku+\nabla_kua^{ij}\nabla_{jik}d\varphi+a^{ij}\nabla_{ij}\varphi\nabla_ku\nabla_kd\big)\\
&\qquad+\sum_{i,k=1}^n F_{p_i}\big(\nabla_ku\nabla_{ki}d\varphi+\nabla_ku\nabla_kd\nabla_i\varphi\big)\\
&:=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5.
\end{aligned}$$

From direct computation as before, we have

$$\begin{aligned}
\mathrm{I}_1&:=-\Big(\sum_{i,j=1}^n a^{ij}\nabla_{ij}v+\sum_{i=1}^n F_{p_i}\nabla_iv\Big)\\
&=-\frac{1}{2v}\sum_{i,j=1}^n\Big(a^{ij}\nabla_{ij}\gamma+2a^{ij}\sum_{k=1}^n\nabla_{ki}u\nabla_{kj}u+2\sum_{k=1}^n a^{ij}\nabla_ku\nabla_{jik}u\Big)\\
&\qquad+\frac{1}{4v^3}\sum_{i,j=1}^n a^{ij}\Big(\nabla_i\gamma+2\sum_{k=1}^n\nabla_ku\nabla_{ki}u\Big)\Big(\nabla_j\gamma+2\sum_{l=1}^n\nabla_lu\nabla_{lj}u\Big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad -\sum_{i=1}^{n} F_{p_i}\frac{1}{2v}\Big(\nabla_i\gamma+2\sum_{k=1}^{n}\nabla_k u\nabla_{ki}u\Big)\\
&=-\sum_{k=1}^{n}\frac{\nabla_k u}{v}\cdot\Big(\sum_{i,j=1}^{n}a^{ij}\nabla_{kji}u+\sum_{i=1}^{n}F_{p_i}\nabla_{ik}u\Big)\\
&\qquad+\sum_{i,j,k=1}^{n}\frac{1}{v}\Big(\frac{1}{v^2}\sum_{l=1}^{n}a^{ij}\nabla_k u\nabla_l u\nabla_{ki}u\nabla_{lj}u-a^{ij}\nabla_{ki}u\nabla_{kj}u\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\frac{1}{v^3}\sum_{i,j,l=1}^{n}a^{ij}\nabla_i\gamma\nabla_l u\nabla_{lj}u\\
&\qquad+\Big(\frac{1}{4v^3}\sum_{i,j=1}^{n}a^{ij}\nabla_i\gamma\nabla_j\gamma-\frac{1}{2v}\sum_{i=1}^{n}F_{p_i}\nabla_i\gamma-\frac{1}{2v}\sum_{i,j=1}^{n}a^{ij}\nabla_{ij}\gamma\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\sum_{i,j,k,l=1}^{n}\frac{a^{ij}}{v}\nabla_k u\nabla_l uR_{likj}\\
&:=\mathrm{I}_{11}+\mathrm{I}_{12}+\mathrm{I}_{13}+\mathrm{I}_{14}+\mathrm{I}_{15}.
\end{aligned}
$$

Therefore, by substituting (3-3) into term I_{11}, we get

$$
\begin{aligned}
\mathrm{I}_{11}:=&-\sum_{k=1}^{n}\frac{\nabla_k u}{v}\Big(\sum_{i,j=1}^{n}a^{ij}\nabla_{kji}u+\sum_{i=1}^{n}F_{p_i}\nabla_{ik}u\Big)\\
=&\sum_{k=1}^{n}\frac{\nabla_k u}{v}\Bigg[-\frac{2}{v^4}\Big(\nabla_1 u\nabla_{11}u\nabla_{1k}u\gamma+\sum_{\alpha=2}^{n}\nabla_{1\alpha}u\nabla_{\alpha k}u\nabla_1 uv^2\Big)+\frac{\nabla_{11}u(\nabla_1 u)^2\nabla_k\gamma}{v^4}\\
&-\sum_{i=1}^{n}\frac{\nabla_{ik}\gamma\nabla_i u}{2\gamma}+\frac{\nabla_k\gamma}{2\gamma^2}\langle\nabla\gamma,\nabla u\rangle-\sum_{i=1}^{n}\frac{\nabla_{ik}\gamma\nabla_i u}{2v^2}+\langle\nabla\gamma,\nabla u\rangle\frac{\nabla_k\gamma}{2v^4}\Bigg]-\varepsilon\frac{|\nabla u|^2}{v}\\
=&\Bigg[-\frac{2(\nabla_1 u)^2\gamma}{v^5}(\nabla_{11}u)^2-\frac{2(\nabla_1 u)^2}{v^3}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2+\frac{(\nabla_1 u)^3\nabla_1\gamma}{v^5}\nabla_{11}u\Bigg]\\
&+\Bigg[\frac{\langle\nabla\gamma,\nabla u\rangle^2}{2\gamma^2 v}-\frac{\nabla^2\gamma(\nabla u,\nabla u)}{2\gamma v}-\frac{\nabla^2\gamma(\nabla u,\nabla u)}{2v^3}+\frac{(\nabla_1 u)^2(\nabla_1\gamma)^2}{2v^5}\Bigg]-\varepsilon\frac{|\nabla u|^2}{v}\\
:=&\,\mathrm{I}_{111}+\mathrm{I}_{112}+\mathrm{I}_{113}.
\end{aligned}
$$

While for term I_2, by using (3-3) again, we obtain

$$
\begin{aligned}
\mathrm{I}_2:=&\sum_{k=1}^{n}\nabla_k d\varphi\cdot\Big(\sum_{i,j=1}^{n}a^{ij}\nabla_{jik}u+\sum_{i=1}^{n}F_{p_i}\nabla_{ik}u\Big)\\
=&\sum_{k=1}^{n}\nabla_k d\varphi\cdot\Big(\sum_{i,j=1}^{n}a^{ij}\nabla_{kji}u+\sum_{i=1}^{n}F_{p_i}\nabla_{ik}u+\sum_{i,j,l=1}^{n}a^{ij}\nabla_l uR_{ikj}^{l}\Big)
\end{aligned}
$$

$$\begin{aligned}
&= \left(\frac{2\gamma\nabla_1 d\nabla_1 u}{v^4}(\nabla_{11}u)^2\varphi - \frac{2\varphi\nabla_1 d\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}(\nabla_{1\alpha}u)^2\right) \\
&\quad + \left(\frac{2\gamma\nabla_1 u}{v^4}\nabla_{11}u\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{1\alpha}u\varphi - \frac{2\varphi\nabla_1 u}{v^2}\sum_{\alpha=2}^{n}\nabla_\alpha d\nabla_{1\alpha}u\nabla_{\alpha\alpha}u \right. \\
&\qquad\qquad\qquad \left. - \frac{(\nabla_1 u)^2}{v^4}\nabla_{11}u\langle\nabla d, \nabla\gamma\rangle\right) \\
&\quad + \left[-\sum_{k=1}^{n}\frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u}{2\gamma}\varphi + \frac{\varphi\nabla_1\gamma}{2\gamma^2}\nabla_1 u\langle\nabla d, \nabla\gamma\rangle - \sum_{k=1}^{n}\frac{\nabla_k d\nabla_{1k}\gamma\nabla_1 u\varphi}{2v^2}\right. \\
&\qquad\qquad \left. + \frac{\varphi\nabla_1\gamma\nabla_1 u}{2v^4}\langle\nabla d, \nabla\gamma\rangle + \mathrm{Ric}_{\mathbb{L}}(\nabla u, \nabla d)\varphi\right] + \varepsilon\langle\nabla u, \nabla d\rangle\varphi \\
&:= \mathrm{I}_{21} + \mathrm{I}_{22} + \mathrm{I}_{23} + \mathrm{I}_{24}.
\end{aligned}$$

We assume that $\nabla_1 u(x_0) \geq \sup_M \sqrt{\gamma}$, otherwise we have completed the proof. Due to a simple observation about the extra terms I_{113} and I_{24}, we find that

$$\begin{aligned}
\mathrm{I}_{113} + \mathrm{I}_{24} &:= -\varepsilon\frac{|\nabla u|^2}{v} + \varepsilon\langle\nabla u, \nabla d\rangle\varphi \\
&= -\varepsilon u_1\left(\frac{u_1}{v} - d_1\varphi\right) \leq 0,
\end{aligned}$$

where the last inequality follows from taking $\delta_0 \leq 1/\sqrt{2}$ with $\|\varphi\|_{C^0(M)} \leq \delta_0$. While the rest all terms can be handled as same as in Case 3 in the proof of Theorem 2.2.

Hence, eventually, by choosing

$$0 < \delta_0 \leq \min\left\{\frac{1}{2}, \frac{\kappa}{16L_1 + 32}, \frac{k_0}{2C_1 + 1}\right\}, \tag{3-4}$$

we conclude that $v(x_0) \leq C$, where C is a positive constant which is independent of ε and $\|u\|_{C^0(M)}$. So we have finished the proof of the Lemma 3.1. □

Now we are going to give the proof of Theorem 1.2.

Proof of Theorem 1.2. First of all, we show the existence of solution u_ε to problem (3-1) for any fixed $\varepsilon \in (0, 1)$. Let $\phi \in C^2(\overline{M})$ be the smooth function satisfying

$$\nabla_\nu\phi \leq \varphi\sqrt{\gamma + |\nabla\phi|^2} \quad \text{on } \partial M,$$

and $\phi \in C^2(\overline{M})$. In fact, the existence of function ϕ can be constructed as follows. Define $d(x) := \mathrm{dist}_g(x, \partial M)$ for x in the near neighborhood of ∂M, afterward, smoothly extends it to $\overline{M}$, which we still denote as $d(x)$. Let α be a positive constant such that $\alpha \leq \inf_M(\varphi)\sqrt{\inf_M \gamma + \alpha^2}$. Then $\phi := \alpha d(x)$ would satisfy our requirement. Assume $\phi - u_\varepsilon$ attains its minimum value at $x_0 \in \overline{M}$. If $x_0 \in \partial M$, we get $\nabla'(\phi - u_\varepsilon)(x_0) = 0$ and $\nabla_\nu(\phi - u_\varepsilon)(x_0) > 0$, that is, $\nabla' u_\varepsilon(x_0) = \nabla'\phi(x_0) := q$ and $\nabla_\nu u_\varepsilon(x_0) < \nabla_\nu\phi(x_0)$, where we denote ∇' and ν as the tangential and normal part of ∇ on boundary ∂M. On the other hand, from the boundary value condition in (3-1), it follows that

$$\frac{\nabla_\nu u_\varepsilon}{\sqrt{\gamma+q^2+|\nabla_\nu u_\varepsilon|^2}}=\varphi(x_0)\geq\frac{\nabla_\nu\phi}{\sqrt{\gamma+q^2+|\nabla_\nu\phi|^2}},$$

which is a contradiction with the fact that function $s/\sqrt{\gamma(x_0)+q^2+s^2}$ is strictly increasing with respect to $s\in\mathbb{R}$ and $\nabla_\nu u_\varepsilon(x_0)<\nabla_\nu\phi(x_0)$. Hence $x_0\in M$, and it follows that $\nabla(\phi-u_\varepsilon)(x_0)=0$ and $\nabla^2(\phi-u_\varepsilon)(x_0)\geq 0$. Thus at $x=x_0$, we get

$$\begin{aligned}C&\geq\sum_{i,j=1}^{n}a^{ij}(x_0,\nabla\phi)\nabla_{ij}\phi-\tfrac{1}{2}\Big(\frac{1}{\gamma}+\frac{1}{\gamma+|\nabla\phi|^2}\Big)\langle\nabla\gamma,\nabla\phi\rangle\\&\geq\sum_{i,j=1}^{n}a^{ij}(x_0,\nabla u_\varepsilon)\nabla_{ij}u_\varepsilon-\tfrac{1}{2}\Big(\frac{1}{\gamma}+\frac{1}{\gamma+|\nabla u_\varepsilon|^2}\Big)\langle\nabla\gamma,\nabla u_\varepsilon\rangle\\&=\varepsilon u_\varepsilon(x_0).\end{aligned}$$

It yields that, for all $x\in\overline{M}$,

$$\varepsilon u_\varepsilon(x)\leq\varepsilon\phi(x)-\varepsilon\big(\phi(x_0)-u_\varepsilon(x_0)\big)\leq C.$$

Similarly, we can also get the lower bound of $\varepsilon u_\varepsilon$, that is $\varepsilon u_\varepsilon(x)\geq -C$ for all $x\in\overline{M}$. Therefore, we obtain

$$\sup_M|\varepsilon u_\varepsilon|\leq C,$$

where C is a positive constant depending only on n, γ and ϕ. Hence the existence of solutions u_ε to problem (3-1) follows from the Schauder estimates and the continuity method, see [Gilbarg and Trudinger 1977] or [de Lira and Wanderley 2014] for example.

Second of all, we show the existence of the limit solution when $\varepsilon\to 0^+$. To begin with, we introduce the normalized function

$$\hat{u}_\varepsilon:=u_\varepsilon-\frac{1}{|M|}\int_M u_\varepsilon\,dV.$$

It is easy to check that $\hat{u}_\varepsilon$ satisfies

$$\text{(3-5)}\quad\begin{cases}\sum_{i,j=1}^{n}a^{ij}(x,\nabla\hat{u}_\varepsilon)\nabla_{ij}\hat{u}_\varepsilon-\frac{1}{2}\big(\frac{1}{\gamma}+1/(\gamma+|\nabla\hat{u}_\varepsilon|^2)\big)\langle\nabla\gamma,\nabla\hat{u}_\varepsilon\rangle\\ \qquad\qquad=\varepsilon\hat{u}_\varepsilon+\frac{1}{|M|}\int_M\varepsilon u_\varepsilon\,dV & \text{in } M,\\ \nabla_\nu\hat{u}_\varepsilon=\varphi\sqrt{\gamma+|\nabla\hat{u}_\varepsilon|^2} & \text{on } \partial M.\end{cases}$$

Since there exists some $x_0\in\overline{M}$ such that $\hat{u}_\varepsilon(x_0)=0$, by combining this with the gradient estimate from Lemma 3.1,

$$|\nabla\hat{u}_\varepsilon|=|\nabla u_\varepsilon|\leq C.$$

Thus we know $\sup_M |\hat{u}_\varepsilon| \le C$ and note that $\left|\frac{1}{|M|}\int_M \varepsilon u_\varepsilon \, dV\right| \le C$. Using the Schauder theory, we obtain that for some $\alpha \in (0, 1)$ such that

$$\|\hat{u}_\varepsilon\|_{C^{2,\alpha}(\overline{M})} \le C,$$

where C is a positive constant, independent of ε. By taking $\varepsilon \to 0$, we know that $\hat{u}_\varepsilon$ converges to some $\hat{u} \in C^2(\overline{M})$ and $\varepsilon\hat{u}_\varepsilon + \frac{1}{|M|}\int_M \varepsilon\hat{u}_\varepsilon \, dV \to \tau$ for some $\tau \in [-2C, 2C]$, which yields that $(\tau, \hat{u})$ solves (1-5).

Lastly, we show the uniqueness in Theorem 1.2. Assume that $(\tau_i, \hat{u}_i)$ for $i = 1, 2$ are solutions to (1-5). Without loss of generality, we assume that $\tau_2 \le \tau_1$. Then it follows that

$$\mathcal{L}(\hat{u}_1 - \hat{u}_2) := \tau_1 - \tau_2 \ge 0 \quad \text{in } M,$$

where $\mathcal{L}$ denotes the elliptic operator as

$$\mathcal{L}h := \sum_{i,j=1}^{n} A^{ij}\nabla_{ij}h + \sum_{i=1}^{n} b^i \cdot \nabla_i h,$$

with

$$A^{ij}(\nabla\hat{u}_1, \nabla\hat{u}_2) := \int_0^1 a^{ij}\big(x, s\nabla\hat{u}_1 + (1-s)\nabla\hat{u}_2\big)\, ds,$$

and

$$\begin{aligned} b^i(\nabla\hat{u}_1, \nabla\hat{u}_2) := -\frac{\nabla_i\gamma}{2\gamma} - \int_0^1 &\frac{\nabla_i\gamma}{2\big(\gamma + |s\nabla\hat{u}_1 + (1-s)\nabla\hat{u}_2|^2\big)}\, ds \\ &+ \int_0^1 \frac{\big\langle\nabla\gamma, s\nabla\hat{u}_1 + (1-s)\nabla\hat{u}_2\big\rangle}{\big(\gamma + |s\hat{u}_1 + (1-s)\nabla\hat{u}_2|^2\big)^2} \cdot \big[(1-s)\nabla_i\hat{u}_2 + s\nabla\hat{u}_1\big]\, ds \\ &+ \sum_{k,l=1}^{n}\int_0^1 a^{kl}_{,p_i}\big(x, s\nabla\hat{u}_1 + (1-s)\nabla\hat{u}_2\big) \cdot \nabla_{kl}\big(s\hat{u}_1 + (1-s)\hat{u}_2\big)\, ds. \end{aligned}$$

Hence, it follows that $\hat{u}_1 - \hat{u}_2$ attains the maximum value at ∂M, say x_0. we get $\nabla'(\hat{u}_1 - \hat{u}_2)(x_0) = 0$ and $\nabla_\nu(\hat{u}_1 - \hat{u}_2)(x_0) < 0$, that is, $\nabla'\hat{u}_1(x_0) = \nabla'\hat{u}_2(x_0) := q$ and $\nabla_\nu\hat{u}_1(x_0) < \nabla_\nu\hat{u}_2(x_0)$, where we denote ∇' and ν as the tangential and normal part of ∇ on boundary ∂M. On the other hand, from the boundary value condition in (1-5), it yields that

$$\frac{\nabla_\nu\hat{u}_1}{\sqrt{\gamma + q^2 + |\nabla_\nu\hat{u}_1|^2}} = \varphi(x_0) = \frac{\nabla_\nu\hat{u}_2}{\sqrt{\gamma + q^2 + |\nabla_\nu\hat{u}_2|^2}},$$

which is a contradiction with the fact that function $s/\sqrt{\gamma(x_0) + q^2 + s^2}$ is strictly increasing with respect to $s \in \mathbb{R}$ and $\nabla_\nu\hat{u}_1(x_0) < \nabla_\nu\hat{u}_2(x_0)$. Therefore, we get $\hat{u}_1 - \hat{u}_2 = \text{const}$. Combining this with the first equation in (1-5), it gives us that $\tau_1 = \tau_2$. Hence we have completed the proof. □

4. Translating surfaces

In this section, we switch to the dimension $n = 2$ case, and we could release the range of the variable contact angle. To be more precise, assume that (M, σ) is a bounded domain with smooth boundary in $\mathbb{L}^2$, we denote $f := u|_{\partial M}$, $\Gamma = \gamma|_{\partial M}$, $\Phi := \varphi|_{\partial M}$, and $\chi := \nabla_\nu u|_{\partial M}$. We use the arclength s to parametrize the boundary ∂M, thus $\left\{e_1 := \frac{\partial}{\partial s}, e_2 := \nu\right\}$ forms an orthonormal frame near the boundary ∂M. Then

$$v(s) := v|_{\partial M} = \sqrt{\Gamma(s) + f'(s)^2 + \chi^2(s)},$$

and combining with boundary value condition $\chi = \Phi v$ on ∂M, it follows that

$$\chi^2 = \frac{\Phi^2}{1-\Phi^2}(\Gamma + f'(s)^2), \tag{4-1}$$

$$f'(s)^2 = (1 - \Phi^2)v^2 - \Gamma. \tag{4-2}$$

In particular, on ∂M, we have the following identities.

$$\begin{aligned} a^{11} &= 1 - \frac{f'(s)^2}{v^2} = \frac{\Gamma + \chi^2(s)}{v^2}, \\ a^{12} &= -\frac{f'(s)\chi}{v^2} = -\frac{f'(s)\Phi}{v} = a^{21}, \\ a^{22} &= 1 - \frac{\chi(s)^2}{v^2} = \frac{\Gamma + f'(s)^2}{v^2}. \end{aligned}$$

Now we are going to show the gradient estimate, which will be divided into two parts. Firstly, we get the boundary gradient estimate under the appropriate condition of geodesic curvature of ∂M, and secondly, we adopt the maximum principle to get the global gradient estimate.

Lemma 4.1. *Let M be a strictly convex domain in $\mathbb{L}^2$ with κ the geodesic curvature of ∂M, satisfying*

$$\kappa \geq \left(\frac{|\nabla^T \varphi|}{\sqrt{1-\varphi^2}} + |\varphi| \cdot |\nabla \log \sqrt{\gamma}| \right) + \delta_1,$$

for some positive constant $\delta_1 > 0$, where $\nabla^T \varphi$ is the tangential part of $\nabla \varphi$ restricted to the boundary. Suppose $u \in C^4(\overline{M})$ such that

$$\begin{cases} C_0 = \sum_{i,j=1}^{2} a^{ij} \nabla_{ij} u - \frac{1}{2}\left(\frac{1}{\gamma} + 1/(\gamma + |\nabla u|^2)\right)\langle \nabla \gamma, \nabla u\rangle & \textit{in } M, \\ \nabla_\nu u = \varphi(x)\sqrt{\gamma + |\nabla u|^2} & \textit{on } \partial M. \end{cases} \tag{4-3}$$

If v attains its maximum at somewhere on the boundary, then we have

$$\sup_{\partial M} v \leq C,$$

where C is a constant only depending on φ, M, γ, n and δ_1.

Proof. Firstly, we notice that

$$\begin{aligned}\nabla^2 u\Big(\frac{\partial}{\partial s}, \nu\Big) &= \langle \nabla_{\partial/\partial s}\nabla u, \nu\rangle = \frac{\partial}{\partial s}\langle \nabla u, \nu\rangle - \langle \nabla u, \nabla_{\partial/\partial s}\nu\rangle \\ &= \chi'(s) - f'(s)\Big\langle \frac{\partial}{\partial s}, \nabla_{\partial/\partial s}\nu\Big\rangle \\ &= \Phi'(s)v + \Phi v'(s) + f'(s)\kappa,\end{aligned}$$

where the last equality follows from combining with the boundary value condition $\chi = \Phi v$. Hence by direct computation, it yields that

$$\begin{aligned}\frac{\partial v}{\partial \nu} &= \frac{1}{2v}\bigg(\frac{\partial \gamma}{\partial \nu} + 2\nabla^2 u(\nabla u, \nu)\bigg) \\ &= \frac{1}{2v}\bigg(\frac{\partial \gamma}{\partial \nu} + 2\nabla^2 u\Big(\frac{\partial}{\partial s}, \nu\Big)f'(s) + 2\nabla^2 u(\nu, \nu)\chi\bigg) \\ &= \frac{\chi}{v}\nabla^2 u(\nu, \nu) + \frac{1}{2v}\frac{\partial \gamma}{\partial \nu} + \frac{f'(s)}{v}\big(\Phi'(s)v + \Phi v'(s) + f'(s)\kappa\big).\end{aligned} \tag{4-4}$$

Similarly, it holds that

$$\begin{aligned}\nabla^2 u\Big(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\Big) &= \Big\langle \nabla_{\partial/\partial s}\nabla u, \frac{\partial}{\partial s}\Big\rangle = \frac{\partial}{\partial s}\Big\langle \nabla u, \frac{\partial}{\partial s}\Big\rangle - \Big\langle \nabla u, \nabla_{\partial/\partial s}\frac{\partial}{\partial s}\Big\rangle \\ &= f''(s) - \chi(s)\kappa.\end{aligned} \tag{4-5}$$

Hence, from the first equation in (4-3), which follows that on ∂M,

$$\begin{aligned}C_0 &= \sum_{i,j=1}^{2} a^{ij}\nabla_{ij}u - \tfrac{1}{2}\Big(\frac{1}{\Gamma} + \frac{1}{v^2}\Big)\langle \nabla\gamma, \nabla u\rangle \\ &= \frac{\Gamma + \chi^2(s)}{v^2}\cdot\big(f''(s) - \chi(s)\kappa\big) - 2\frac{f'(s)\Phi}{v}\big(\chi'(s) + f'(s)\kappa\big) \\ &\quad + \frac{\Gamma + f'(s)^2}{v^2}\nabla^2 u(\nu, \nu) - \tfrac{1}{2}\Big(\frac{1}{\Gamma} + \frac{1}{\Gamma + f'(s)^2 + \chi^2(s)}\Big)\cdot f'(s)\Gamma'(s) \\ &\quad - \tfrac{1}{2}\Big(\frac{1}{\Gamma} + \frac{1}{\Gamma + f'(s)^2 + \chi^2(s)}\Big)\chi\frac{\partial \gamma}{\partial \nu}.\end{aligned} \tag{4-6}$$

Since v attains its maximum value somewhere on the boundary, say $x_0 \in \partial M$ whose local parameter is s_0. We may assume that $|f'(s_0)| \geq 1$ in the below, otherwise we have done the gradient estimate for v, due to Equation (4-2). In the sequel, we do all the computation at $s = s_0$. Now we have

$$v'(s_0) = 0, \quad \text{and} \quad \frac{\partial v}{\partial \nu}(s_0) \leq 0, \tag{4-7}$$

that is,

$$\Gamma' + 2f'f'' + 2\chi\chi' = 0, \tag{4-8}$$

and respectively,

$$0 \geq \frac{\partial v}{\partial \nu} = \frac{\chi}{v}\nabla^2 u(\nu,\nu) + \frac{1}{2v}\frac{\partial \gamma}{\partial \nu} + f'\Phi' + \frac{f'(s)^2}{v}\kappa. \tag{4-9}$$

Note that from boundary value equation in (4-3), we get

$$\chi'(s_0) = \Phi' v + \Phi v' = \Phi' v. \tag{4-10}$$

Substituting above equation into (4-8) gives us

$$f''(s_0) = -\frac{1}{2f'(s)}\big[2\Phi\Phi' v^2 - (1-\Phi^2)2vv' + \Gamma'\big] = -\frac{\Phi\Phi'}{f'}v^2 - \frac{\Gamma'}{2f'}. \tag{4-11}$$

We rewrite (4-9) in the following expression,

$$\chi \cdot \nabla^2 u(\nu,\nu) \leq -f'(s)^2\kappa - f'\Phi' v - \tfrac{1}{2}\frac{\partial \gamma}{\partial \nu}. \tag{4-12}$$

By multiplying χ into (4-6) first, then substituting (4-10), (4-11) and (4-12) into it, we obtain

$$\begin{aligned}
\chi \cdot C_0 = {} & \chi\frac{\Gamma+\chi^2}{v^2}\cdot\Big(-\frac{\Phi\Phi'}{f'}v^2 - \frac{\Gamma'}{2f'} - \chi\kappa\Big) - 2\frac{\chi(s)f'(s)\Phi}{v}\big(\Phi' v + f'(s)\kappa\big)\\
& + \frac{\Gamma+f'(s)^2}{v^2}\chi\cdot\nabla^2 u(\nu,\nu) - \frac{\chi}{2}\Big(\frac{1}{\Gamma} + \frac{1}{\Gamma+f'(s)^2+\chi^2(s)}\Big)\\
& \cdot\Big[f'(s)\Gamma'(s) + \chi\frac{\partial\gamma}{\partial\nu}\Big]\\
\leq {} & \Big[-\frac{v}{f'}\Phi^2\Phi'(\Gamma+\Phi^2v^2) - \frac{\Phi\Gamma'(\Gamma+\Phi^2v^2)}{2f'v} - \kappa\Phi^2(\Gamma+\Phi^2v^2)\Big]\\
& - 2f'(s)v\Phi^2\Phi' - 2\kappa f'(s)^2\Phi^2\\
& - \frac{\Gamma+f'(s)^2}{v^2}\Big(f'(s)^2\kappa + f'\Phi' v + \tfrac{1}{2}\frac{\partial\gamma}{\partial\nu}\Big) - \frac{\Phi\Gamma'}{2\Gamma}vf' - \frac{\Phi^2}{2\Gamma}\frac{\partial\gamma}{\partial\nu}\cdot v^2\\
& - \frac{\Gamma'\Phi}{2}\frac{f'v}{\Gamma+f'^2+\chi^2} - \frac{\Phi^2}{2}\frac{v^2}{\Gamma+f'(s)^2+\chi^2(s)}\frac{\partial\gamma}{\partial\nu},
\end{aligned} \tag{4-13}$$

which is equivalent to

$$\begin{aligned}
& \kappa\underbrace{\big[(1-\Phi^2)f'^2 + \Phi^4v^2 + \Phi^2\Gamma + 2f'^2\Phi^2\big]}_{:=\mathrm{J}}\\
& + \Big[\frac{v}{f'}\Phi^2\Phi'(\Gamma+\Phi^2v^2) + 2f'v\Phi^2\Phi' + \frac{\Gamma+f'(s)^2}{v}f'\Phi' + \frac{\Phi^2}{2\Gamma}\frac{\partial\gamma}{\partial\nu}\cdot v^2 + \frac{\Phi\Gamma'}{2\Gamma}vf'\Big]\\
& \leq -\Phi v C_0 - \frac{\Phi\Gamma'(\Gamma+\Phi^2v^2)}{2f'v} - \frac{\Gamma+f'(s)^2}{2v^2}\frac{\partial\gamma}{\partial\nu} - \frac{\Gamma'\Phi}{2}\frac{f'v}{\Gamma+f'^2+\chi^2}\\
& \qquad - \frac{\Phi^2}{2}\frac{v^2}{\Gamma+f'(s)^2+\chi^2(s)}\frac{\partial\gamma}{\partial\nu}.
\end{aligned} \tag{4-14}$$

Note that the right-hand side of above inequality are at most the linear terms of v or f', while the left-hand side contains all the possible quadratic terms of v or f'. Firstly, we tackle term J in the left-hand side of above inequality, which can be written as

$$\begin{aligned} \mathrm{J} &:= (1-\Phi^2)f'^2 + \Phi^4 v^2 + \Phi^2\Gamma + 2f'^2\Phi^2 \\ &= f'^2 + \Phi^2 v^2 = v^2 - \Gamma, \end{aligned}$$

where we have used that on boundary, it holds that

$$f'(s) = \sqrt{(1-\Phi^2)v^2 - \Gamma}.$$

On the other hand, the rest terms of the left-hand side of above inequality gives us

$$\begin{aligned} &\left|\frac{v}{f'}\Phi^2\Phi'(\Gamma+\Phi^2v^2) + 2f'v\Phi^2\Phi' + \frac{\Gamma+f'(s)^2}{v}f'\Phi' + \frac{\Phi^2}{2\Gamma}\frac{\partial\gamma}{\partial\nu}\cdot v^2 + \frac{\Phi\Gamma'}{2\Gamma}vf'\right| \\ &\qquad = \left|\frac{\Phi'(v^2-\Gamma)}{f'}v + \frac{\Phi^2}{2\Gamma}\frac{\partial\gamma}{\partial\nu}\cdot v^2 + \frac{\Phi\Gamma'}{2\Gamma}v\sqrt{(1-\Phi^2)v^2-\Gamma}\right| \\ &\qquad \le v^2\left[\frac{|v\Phi'|}{\sqrt{(1-\Phi^2)v^2-\Gamma}} + \frac{\Phi^2}{2\Gamma}\left|\frac{\partial\gamma}{\partial\nu}\right| + \frac{|\Phi\Gamma'|\sqrt{1-\Phi^2}}{2\Gamma}\right] + |\Phi'\Gamma|\frac{v}{|f'|}. \end{aligned}$$

Notice that, by using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left(\frac{\Phi^2}{2\Gamma}\left|\frac{\partial\gamma}{\partial\nu}\right| + \frac{|\Phi\Gamma'|\sqrt{1-\Phi^2}}{2\Gamma}\right) &\le \left[\left(\Phi^4+\Phi^2(1-\Phi^2)\right)\cdot\left(\frac{1}{4\Gamma^2}\left|\frac{\partial\gamma}{\partial\nu}\right|^2 + \frac{\Gamma'^2}{4\Gamma^2}\right)\right]^{1/2} \\ &= |\Phi|\cdot|\nabla\log\sqrt{\gamma}|. \end{aligned}$$

Substituting above inequality into (4-14) yields that

$$\begin{aligned} &\left[v^2\left(\kappa - \frac{|v\Phi'|}{\sqrt{(1-\Phi^2)v^2-\Gamma}} - |\Phi|\cdot|\nabla\log\sqrt{\gamma}|\right)\right] \\ &\le \kappa\Gamma - \Phi v u_t - \frac{\Phi\Gamma'(\Gamma+\Phi^2v^2)}{2f'v} - \frac{\Gamma+f'(s)^2}{2v^2}\frac{\partial\gamma}{\partial\nu} - \frac{\Gamma'\Phi}{2}\frac{f'v}{\Gamma+f'^2+\chi^2} \\ &\qquad\qquad - \frac{\Phi^2}{2}\frac{v^2}{\Gamma+f'(s)^2+\chi^2(s)}\frac{\partial\gamma}{\partial\nu} \\ &\le C_1 + C_2 v, \end{aligned}$$

where C_1, C_2 are positive constants only depending on Γ, Φ, n, and M. Under the assumption that

$$\kappa - \left(\frac{|\nabla^T\varphi|}{\sqrt{1-\varphi^2}} + |\varphi|\cdot|\nabla\log\sqrt{\gamma}|\right) \ge \delta_1,$$

finally note that

$$\lim_{v\to+\infty}\frac{|v\Phi'|}{\sqrt{(1-\Phi^2)v^2-\Gamma}}=\frac{|\Phi'|}{\sqrt{1-\Phi^2}}.$$

Hence, we can obtain the gradient estimate for v from above, that is

$$v(s_0)\le C,$$

where C is a positive constant only depending on $\gamma,\varphi,n,\delta_1$ and M. □

Theorem 4.2. *Let M be a strictly convex domain in $\mathbb{L}^2$ with κ the geodesic curvature of ∂M, satisfying*

$$\kappa-\left(\frac{|\nabla^T\varphi|}{\sqrt{1-\varphi^2}}+|\varphi|\cdot|\nabla\log\sqrt{\gamma}|\right)\ge\delta_1,$$

for some positive constant $\delta_1>0$ and $\nabla^T\varphi$ is the tangential part of $\nabla\varphi$ restricted on the boundary. If K and γ satisfies the compatible condition

$$K+\lambda_1(\nabla^2\rho)\ge 0\quad\textit{and}\quad\Delta\gamma-\frac{|\nabla\gamma|^2}{2\gamma}\ge 0,\quad\textit{in } M,$$

where K is the Gauss curvature of M and λ_1 is the minimum eigenvalue of the Hessian $\nabla^2\rho$ with $\rho:=\log\sqrt{\gamma}$. Suppose $u\in C^4(\overline{M})$ such that

$$\begin{cases}u_t=\sum_{i,j=1}^2 a^{ij}\nabla_{ij}u\\ \qquad\qquad -\frac{1}{2}\big(\frac{1}{\gamma}+1/(\gamma+|\nabla u|^2)\big)\langle\nabla\gamma,\nabla u\rangle & \textit{in } M\times[0,T),\\ \nabla_\nu u=\varphi(x)\sqrt{\gamma+|\nabla u|^2} & \textit{on } \partial M\times[0,T),\\ u(x,0)=u_0(x) & \textit{on } M.\end{cases}\tag{4-15}$$

Then we have the gradient estimate,

$$\sup_M|\nabla u|\le C,$$

where C is a constant only depending on φ,M,γ,n,u_0 and δ_1.

Proof. From the computation in the previous section, we recall that

$$\mathcal{L}v:=\left(\partial_t-\sum_{i,j=1}^{2}a^{ij}\nabla_{ij}-\sum_{i=1}^{2}F_{p_i}\nabla_i\right)v\tag{4-16}$$

$$\begin{aligned}
\text{(4-16 cont.)}\qquad &= \sum_{k=1}^{2} \frac{\nabla_k u}{v}\Big(\nabla_k u_t - \sum_{i,j=1}^{2} a^{ij}\nabla_{kji}u - \sum_{i=1}^{2} F_{p_i}\nabla_{ik}u\Big)\\
&\quad + \frac{1}{v}\sum_{i,j,k=1}^{2}\Big(-a^{ij}\nabla_{ki}u\nabla_{kj}u + \frac{1}{v^2}\sum_{l=1}^{2} a^{ij}\nabla_k u\nabla_l u\nabla_{ki}u\nabla_{lj}u\Big)\\
&\quad + \Big(\frac{1}{v^3}\sum_{i,j,l=1}^{2} a^{ij}\nabla_i\gamma\nabla_l u\nabla_{lj}u - \frac{1}{2v}\sum_{i=1}^{2} F_{p_i}\nabla_i\gamma - \frac{1}{2v}\sum_{i,j=1}^{2} a^{ij}\nabla_{ij}\gamma\\
&\qquad + \frac{1}{4v^3}\sum_{i,j=1}^{2} a^{ij}\nabla_i\gamma\nabla_j\gamma - \sum_{i,j,k,l=1}^{2}\frac{a^{ij}\nabla_k u\nabla_l u R_{likj}}{v}\Big)\\
&= \mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3.
\end{aligned}$$

Notice that

$$2v\nabla_i v = \nabla_i\gamma + 2\sum_{l=1}^{2}\nabla_l u\nabla_{li}u,$$

by using (2-16), it follows that

$$\begin{aligned}
\mathrm{P}_1 &= \sum_{k=1}^{2}\frac{\nabla_k u}{v}\Big(\nabla_k u_t - \sum_{i,j=1}^{2} a^{ij}\nabla_{kji}u - \sum_{i=1}^{2} F_{p_i}\nabla_{ik}u\Big)\\
&= \sum_{k=1}^{2}\frac{\nabla_k u}{v}\Big[-\sum_{i,j=1}^{2}\frac{2\nabla_{ij}u\nabla_{ki}u\nabla_j u}{v^2} + \sum_{i,j=1}^{2}\frac{\nabla_{ij}u\nabla_i u\nabla_j u}{v^4}\Big(\nabla_k\gamma + 2\sum_{l=1}^{2}\nabla_l u\nabla_{kl}u\Big)\\
&\qquad + \tfrac{1}{2}\Big(\frac{\nabla_k\gamma}{\gamma^2} + \frac{\nabla_k\gamma}{v^4}\Big)\langle\nabla u, \nabla\gamma\rangle - \tfrac{1}{2}\Big(\frac{1}{\gamma} + \frac{1}{v^2}\Big)\sum_{i=1}^{2}\nabla_{ik}\gamma\nabla_i u\Big]\\
&= -\frac{1}{2v^3}\sum_{i=1}^{2}(2v\nabla_i v - \nabla_i\gamma)^2 + \frac{2\nabla^2 u(\nabla u, \nabla u)}{v^4}\langle\nabla u, \nabla v\rangle + \frac{\langle\nabla u, \nabla\gamma\rangle^2}{2v}\Big(\frac{1}{\gamma^2} + \frac{1}{v^4}\Big)\\
&\qquad - \frac{\nabla^2\gamma(\nabla u, \nabla u)}{2v}\Big(\frac{1}{\gamma} + \frac{1}{v^2}\Big)\\
&= -\frac{|\nabla\gamma|^2}{2v^3} + \frac{\langle\nabla u, \nabla\gamma\rangle^2}{2v}\Big(\frac{1}{\gamma^2} + \frac{1}{v^4}\Big) - \frac{\nabla^2\gamma(\nabla u, \nabla u)}{2v}\Big(\frac{1}{\gamma} + \frac{1}{v^2}\Big) \quad \operatorname{mod} \nabla v.
\end{aligned}$$

Since $[a^{ij}]$ is positive definite, $A^{kl} := \sum_{i,j=1}^{n} a^{ij}\nabla_{ki}u\nabla_{lj}u$ is also positive definite. Using the Cauchy–Schwarz inequality yields that

$$\begin{aligned}
\mathrm{P}_2 &:= \frac{1}{v^3}\sum_{i,j,k,l=1}^{2} a^{ij}\nabla_k u\nabla_l u\nabla_{ki}u\nabla_{lj}u - \frac{1}{v}\sum_{i,j,k=1}^{2} a^{ij}\nabla_{ki}u\nabla_{kj}u\\
&= \frac{1}{v^3}\Big(\sum_{k,l=1}^{2} A^{kl}\nabla_k u\nabla_l u - v^2\sum_{k=1}^{2} A^{kk}\Big) \le 0.
\end{aligned}$$

Besides, we have

$$\begin{aligned}
\mathrm{P}_3 := \frac{1}{v^3}\sum_{i,j,l=1}^{2} a^{ij}\nabla_i\gamma\nabla_l u\nabla_{lj}u + \frac{1}{2v}\sum_{i,j=1}^{2}\Big(\frac{1}{2v^2}a^{ij}\nabla_i\gamma\nabla_j\gamma - a^{ij}\nabla_{ij}\gamma\Big)\\
-\frac{1}{2v}\sum_{i=1}^{2}F_{p_i}\nabla_i\gamma - \sum_{i,j,k,l=1}^{2}\frac{a^{ij}\nabla_k u\nabla_l u R_{likj}}{v}\\
= \frac{|\nabla\gamma|^2}{4v\gamma} - \frac{\Delta\gamma}{2v} + \frac{\nabla^2\gamma(\nabla u,\nabla u)}{2v^3} - \frac{\langle\nabla\gamma,\nabla u\rangle^2}{4v^5} - \frac{|\nabla u|^2}{v}K \quad \mathrm{mod}\ \nabla v.
\end{aligned}$$

Thus we add the above three terms P_1, P_2 and P_3 together, and take advantage of the assumption in Theorem 4.2, to conclude that

$$\begin{aligned}
\text{(4-17)}\quad \mathcal{L}v &:= \Big(\partial_t - \sum_{i,j=1}^{2}a^{ij}\nabla_{ij} - \sum_{i=1}^{2}F_{p_i}\nabla_i\Big)v := \mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3\\
&\le \frac{1}{v}\sum_{i,j=1}^{2}\Big(\frac{\nabla_i\gamma\nabla_j\gamma}{2\gamma^2} - \frac{\nabla_{ij}\gamma}{2\gamma} - K\sigma_{ij}\Big)\cdot\nabla_i u\nabla_j u + \frac{1}{2v}\Big(\frac{|\nabla\gamma|^2}{2\gamma} - \Delta\gamma\Big)\\
&\qquad - \frac{|\nabla\gamma|^2}{2v^3} + \frac{\langle\nabla\gamma,\nabla u\rangle^2}{4v^5}\\
&\le 0 \quad \mathrm{mod}\ \nabla v.
\end{aligned}$$

Hence the maximum principle implies that v attains its maximum value at (x_0, t_0) for either $x_0 \in \partial M$ or $t_0 = 0$. If $x_0 \in \partial M$, since we have the estimate for $|u_t| \le C_0$ from Lemma 2.1, then combining with Lemma 4.1 we have $v \le C$. If $t_0 = 0$, then we get $v \le \sup_M \sqrt{\gamma + |\nabla u_0|^2}$.

Therefore, we have

$$v \le C,$$

where C is a constant only depending on φ, M, γ, n, u_0 and δ_1. ☐

5. Stationary equation and asymptotic behavior

In this section, we use the approach and argument in [Altschuler and Wu 1994] to prove Theorem 1.1. Firstly, we use an equivalent way to rewrite the first equation in (1-5), which has a nice weighted divergence structure. Recall that $\rho := \log\sqrt{\gamma}$, thus first equation in (1-5) turns out to be

$$\text{(5-1)}\qquad \mathrm{div}_\rho\Big(\frac{\nabla u}{\sqrt{\gamma+|\nabla u|^2}}\Big) = \frac{\tau}{\sqrt{\gamma+|\nabla u|^2}} \quad \text{in } M,$$

where the weighted divergence operator div_ρ is defined as

$$\operatorname{div}_\rho(X) = e^\rho \operatorname{div}(e^{-\rho} X),$$

for any vector field X in TM. From this, one can see that the constant $\tau \in \mathbb{R}$ in (5-1) is a uniquely determined constant. In fact, by using integration by parts over (5-1), yields that

$$\tau = \frac{\int_{\partial M} \varphi/\sqrt{\gamma}\, d\sigma}{\int_M \sqrt{\gamma(\gamma + |\nabla u|^2)}\, dV}. \tag{5-2}$$

Moreover, we see that if $\hat{u} := \hat{u}(x)$ solves (5-1), then $\tilde{u}(x,t) := \hat{u}(x) + \tau t$ satisfies

$$\begin{cases} \tilde{u}_t = \sum_{i,j=1}^n a^{ij}(x, \nabla\tilde{u})\nabla_{ij}\tilde{u} & \\ \qquad\qquad -\frac{1}{2}\big(\frac{1}{\gamma} + 1/(\gamma + |\nabla\tilde{u}|^2)\big)\langle\nabla\gamma, \nabla\tilde{u}\rangle & \text{in } M \times [0, +\infty), \\ \nabla_\nu \tilde{u} = \varphi(x)\sqrt{\gamma + |\nabla\tilde{u}|^2} & \text{on } \partial M \times [0, +\infty), \\ \tilde{u}(x, 0) = \hat{u}(x) & \text{on } \overline{M} \times \{0\}. \end{cases} \tag{5-3}$$

Therefore, based on the maximum principle and boundary value condition similar to the one used in Theorem 1.2, we can obtain the following oscillation bound on the solutions to the parabolic problem (1-3) as in [Altschuler and Wu 1994] (see Corollary 2.7 there).

Corollary 5.1. *For a solution $u(x,t)$ to (1-3), there exists a positive constant C independent of t such that*

$$|u(x,t) - \tau t| \le C. \tag{5-4}$$

In particular, one has

$$\frac{u(x,t)}{t} \to \tau \quad \textit{uniformly as } t \to \infty.$$

Proof. Define the new function $U(x,t) = u(x,t) - \tilde{u}(x,t)$, by direct calculation, we find that U satisfies

$$\partial_t U = \sum_{i,j=1}^n A^{ij}(\nabla u, \nabla\tilde{u}) \cdot \nabla_{ij} U + \sum_{i=1}^n b^i(\nabla u, \nabla\tilde{u}) \cdot \nabla_i U,$$

where A^{ij}, b^i are defined as previously in the proof of Theorem 1.2. From the maximum principle, we know $U(x,t)$ attains its maximum value at the point (x_0, t_0) with either $x_0 \in \partial M$ or $t_0 = 0$. If $x_0 \in \partial M$, using a similar argument to that in the proof of Theorem 1.2, we reach a contradiction. So $t_0 = 0$, hence we have

$$|u(x,t) - \tau t| \le \hat{u}(x) + \sup_M |u_0(x) - \hat{u}(x)| \le 2\|\hat{u}\|_{C^0(M)} + \|u_0\|_{C^0(M)},$$

for all $(x,t) \in M \times [0, +\infty)$. Thus we have completed the proof. □

Eventually, based on the above preparation, combining Lemma 2.1, Theorem 2.2 and Corollary 5.1 together, one can follow an argument of Schnürer used in [Schnürer 2002, Section 6] to show the asymptotic behavior of solutions to (1-3), that is, u converges to the translating solution as $t \to +\infty$. For completeness, we give the proof of Theorem 1.1 with a few minor modifications of [Schnürer 2002] to our situation here.

Proof of Theorem 1.1. Recall that

$$U(x,t) := u(x,t) - \tilde{u}(x,t),$$

from the proof of Corollary 5.1, and

$$\partial_t U = \sum_{i,j=1}^{n} A^{ij}(\nabla u, \nabla \tilde{u}) \nabla_{ij} U + \sum_{i=1}^{n} b^i(\nabla u, \nabla \tilde{u}) \nabla_i U \quad \text{in } M.$$

Firstly, we let

$$h(p) := \frac{\langle p, \nu \rangle}{\sqrt{\gamma + |p|^2}} - \varphi \quad \text{for } p \in \mathbb{R}^n.$$

By direct computation, we find that

$$h_{p_i} = \frac{\nu^i}{\sqrt{\gamma + |p|^2}} - \frac{\langle p, \nu \rangle}{(\gamma + |p|^2)^{\frac{3}{2}}} p_i,$$

and

$$h(p) - h(q) = \sum_{i=1}^{n} \int_0^1 h_{p_i}(sp + (1-s)q) ds \cdot (p-q)_i := \big\langle \beta(p,q), p-q \big\rangle.$$

Combining this with the boundary value condition in (1-3), we have

$$\big\langle \nabla U, \beta(\nabla u(x,t), \nabla \tilde{u}(x,t)) \big\rangle = 0 \quad \text{on } \partial M.$$

Note that this is a uniformly strictly oblique boundary condition, since

$$\begin{aligned}\langle \beta, \nu \rangle = \int_0^1 & \frac{1}{\sqrt{\gamma + |s\nabla u(x,t) + (1-s)\nabla \tilde{u}(x,t)|^2}} ds \\ & - \int_0^1 \frac{\langle s\nabla u(x,t) + (1-s)\nabla \tilde{u}(x,t), \nu \rangle^2}{\big(\gamma + |s\nabla u(x,t) + (1-s)\nabla \tilde{u}(x,t)|^2\big)^{3/2}} ds \\ \geq & \int_0^1 \frac{\gamma}{(\gamma + |s\nabla u(x,t) + (1-s)\nabla \tilde{u}(x,t)|^2)^{3/2}} ds \\ > & \, 0,\end{aligned}$$

where the last inequality in above follows from the uniform gradient estimate.

Next we define the oscillation of function U as

$$\operatorname{osc}(U)(t) := \sup_M U - \inf_M U.$$

Then by the strong maximum principle and the Hopf lemma, we know that $\operatorname{osc}(U)(t)$ is either a strictly decreasing function in t or a constant function.

Claim. $$\operatorname{osc}(U)(t) \to 0 \quad \text{as } t \to \infty.$$

In fact, we can verify this by contradiction. If $\operatorname{osc}(U)(t) \to \alpha_0$ as $t \to \infty$ for some $\alpha_0 > 0$, we can choose a sequence $t_k \to +\infty$ as $k \to \infty$. Consider $(x, t) \in \overline{M} \times [-t_k, \infty)$ and for fixed $x_0 \in M$, we consider the function

$$\tilde{u}^{k,1}(x, t) := u(x, t + t_k) - \tau t_k, \quad \tilde{u}^{k,2}(x, t) := \tilde{u}(x, t + t_k) - \tau t_k, \tag{5-5}$$

both of which satisfies (1-3) in $M \times [-t_k, \infty)$.

And for any $t_k > T$ and $(x, t) \in \overline{M} \times [-T, T]$, using Corollary 5.1 and Lemma 2.1, we obtain that

$$\begin{aligned} |\tilde{u}^{k,1}(x, t)| &\le |u(x, t + t_k) - u(x, t_k)| + |u(x, t_k) - \tau t_k| \\ &\le T \cdot \sup_M |\dot{u}| + C, \end{aligned}$$

and

$$|\tilde{u}^{k,2}(x, t)| = |\hat{u}(x) + \tau t| \le C.$$

Combining Lemma 2.1, Theorem 2.2, Corollary 5.1 and Schauder theory together, we get the locally uniform bounds for any C^l-norm ($l \ge 0$) for both sequences $\{\tilde{u}^{k,1}\}_{k\ge1}$ and $\{\tilde{u}^{k,2}\}_{k\ge1}$. By applying the Arzelà–Ascoli theorem to both sequences in (5-5) we can extract a subsequence of t_k (still denoted as t_k) such that the limits of both subsequences are $\tilde{u}^{\infty,1}$ and $\tilde{u}^{\infty,2}$, and satisfy the Equations (1-3) in $\overline{M} \times [-T, T]$.

Now we define the new function $\tilde{u} := \tilde{u}^{\infty,1} - \tilde{u}^{\infty,2}$, due to the uniform convergence of above two sequences, it follows that, for any fixed $t \in \mathbb{R}$,

$$\begin{aligned} \operatorname{osc}(\tilde{u})(t) &= \operatorname{osc}\bigl[\lim_{k\to\infty} \bigl(\tilde{u}^{k,1}(x, t) - \tilde{u}^{k,2}(x, t)\bigr)\bigr] \\ &= \operatorname{osc}\bigl[\lim_{k\to\infty} \bigl(u(x, t + t_k) - \tilde{u}(x, t + t_k)\bigr)\bigr] \\ &= \lim_{k\to\infty} \operatorname{osc}\bigl[\bigl(u(x, t + t_k) - \tilde{u}(x, t + t_k)\bigr)\bigr] \\ &= \lim_{s\to\infty} \operatorname{osc}(U)(s) = \alpha_0 > 0. \end{aligned}$$

However, this is a contradiction with the fact that $\tilde{u} \equiv \text{const}$, which follows from the strong maximum principle applied to the function $\tilde{u}$, as $\tilde{u}$ satisfies

$$\begin{cases} \partial_t \tilde{u} = \sum_{i,j=1}^n A^{ij}(\nabla \tilde{u}^{\infty,1}, \nabla \tilde{u}^{\infty,2}) \nabla_{ij} \tilde{u} \\ \qquad\qquad + \sum_{i=1}^n b^i(\nabla \tilde{u}^{\infty,1}, \nabla \tilde{u}^{\infty,2}) \nabla_i \tilde{u} & \text{in } M \times \mathbb{R}, \\ \langle \nabla \tilde{u}, \beta(\nabla \tilde{u}^{\infty,1}, \nabla \tilde{u}^{\infty,2}) \rangle = 0 & \text{on } \partial M \times \mathbb{R}. \end{cases}$$

Hence we have proved the Claim.

The Claim yields that

$$\lim_{t\to\infty} \sup_M U = \lim_{t\to\infty} \inf_M U = c,$$

for some constant c. Finally, combining with Theorem 1.2, up to an additive constant,

$$\tilde{u}(x,t) := \hat{u}(x) + \tau t$$

is the only translating solution of (1-5), we finish the proof that any solution of flow (1-3) tends smoothly to a translating solution. □

Similarly, we can also prove Theorem 1.3 for $n = 2$. Since we have established the gradient estimate in Theorem 4.2, and under the assumption of Theorem 1.3, one can get the existence of the translating solution firstly, then adopt the same above argument and approach to show Theorem 1.3. For conciseness, we omit it here.

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LIANGJUN WENG
SCHOOL OF MATHEMATICAL SCIENCES
SHANGHAI JIAO TONG UNIVERSITY
SHANGHAI
CHINA
ljweng08@sjtu.edu.cn

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GREEN CORRESPONDENCE AND RELATIVE PROJECTIVITY FOR PAIRS OF ADJOINT FUNCTORS BETWEEN TRIANGULATED CATEGORIES

ALEXANDER ZIMMERMANN

Auslander and Kleiner proved in 1994 an abstract version of Green correspondence for pairs of adjoint functors between three categories. They produced additive quotients of certain subcategories giving the classical Green correspondence in the special setting of modular representation theory. Carlson, Peng and Wheeler showed in 1998 that Green correspondence in the classical setting of modular representation theory is actually an equivalence between triangulated categories with respect to a nonstandard triangulated structure. We first define and study versions of relative projectivity and relative injectivity with respect to pairs of adjoint functors. We then modify Auslander and Kleiner's construction such that the correspondence holds in the setting of triangulated categories.

Introduction

Green correspondence is a very classical and highly important tool in modular representation theory of finite groups. For a finite group G and a field k of finite characteristic p, we associate to every indecomposable kG-module M a p-subgroup D, called its vertex. Simplifying slightly, Green correspondence then says that for H being a subgroup of G containing $N_G(D)$, restriction and induction give a mutually inverse bijection between the indecomposable kH-modules with vertex D and the indecomposable kG-modules with vertex D. It was known for a long time that this is actually a categorical correspondence, and in case of trivial intersection Sylow p-subgroups it was known to be more precisely actually an equivalence between the triangulated stable categories. Carlson, Peng and Wheeler [Carlson et al. 1998] showed that it is possible to define triangulated structures also in the general case, and again the Green correspondence is an equivalence between triangulated categories.

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Auslander and Kleiner [1994] showed that Green correspondence has a vast generalisation, and actually is a property of pairs of adjoint functors between three categories

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "S'"] & \mathcal{H} \arrow[l, bend left, "T'"] \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"] \end{tikzcd}$$

such that (S, T) and (S', T') are adjoint pairs and an additional mild hypothesis on the unit of the adjunction (S, T). Auslander and Kleiner showed that then there is an equivalence between certain additive quotient categories mimicking the classical Green correspondence. For more details we recall the precise statement as Theorem 1.2 and Corollary 1.3 below.

Auslander and Kleiner did not study the question of whether their abstract Green correspondence provides an equivalence between triangulated categories. The present paper aims to fill this gap. Starting with triangulated categories $\mathcal{D}$, $\mathcal{H}$, $\mathcal{G}$ and pairs of adjoint triangle functors (S', T') and (S, T) as above, we replace the additive quotient construction by Verdier localisation modulo the thick subcategories generated by the subcategories for which Auslander and Kleiner took the additive quotient. We obtain this way triangulated quotient categories and we show the precise analogue of Theorem 1.2 for the Verdier localisations instead of the additive quotient categories. In case S is left and right adjoint to T, and if in addition the unit of the adjunction is a monomorphism and the counit is an epimorphism our result shows that the additive quotient category is actually already triangulated, and that therefore the Verdier localisation and the additive quotient coincide. This way we directly generalise the result of Carlson, Peng and Wheeler [Carlson et al. 1998].

In recent years classification results of thick subcategories of various triangulated categories were obtained mainly by parametrisations with subvarieties of support varieties. However, most results use those thick subcategories which form an ideal in an additional monoidal structure, so-called tensor triangulated categories. Many examples, such as stable or derived categories of nonprincipal blocks of group rings, actually are not quite tensor triangulated since a unit is missing. So we study more generally a semigroup tensor structure, which is basically the same as a monoidal structure, but without a unit object. We study properties of our triangulated Green correspondence in this setting.

We further recall the classical situation and explain how we can recover parts of the results of [Benson and Wheeler 2001; Wang and Zhang 2018] using our approach.

The paper is organised as follows. In Section 1 we recall the main result of Auslander and Kleiner. Generalising the case of relative projective with respect to subgroups in the case of module categories, Section 2 then introduces the notion of T-relative projective (resp. T-relative injective) objects in categories for functors T, and characterises this property in case of T having a left (resp. right) adjoint S. Here we push further and generalise a result due to Broué [2009, Theorem 6.8]. We illustrate our constructions in the case of group algebras. Section 3 then compares Verdier localisation and the additive quotient categories. We prove there as well our first main result Theorem 3.17, generalising Auslander and Kleiner's theorem to triangulated categories using Verdier localisations. In Section 4 we revisit tensor triangulated categories and study their behaviour within our setting. In particular in Section 4C we compare our results to existing results in the literature in the case of group rings, their stable and derived categories, generalising various situations in this context.

1. Summary of Auslander-Kleiner's theory

Let $\mathcal{D}$, $\mathcal{H}$, $\mathcal{G}$ be three additive categories and S, S', T, T' be additive functors

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "S'"] & \mathcal{H} \arrow[l, bend left, "T'"] \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"]
\end{tikzcd}$$

such that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \mathrm{id}_{\mathcal{H}} \to TS$ be the unit of the adjunction (S, T). Assume that there is an endofunctor U of $\mathcal{H}$ such that $TS = \mathrm{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \to \mathrm{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism. If ϵ is a split monomorphism, then this is satisfied, but the condition is slightly weaker. Auslander and Kleiner [1994] proved a Green correspondence result for this situation.

Notation 1.1.

- For a functor $F : \mathcal{A} \to \mathcal{B}$ and a full subcategory $\mathcal{V}$ of $\mathcal{B}$ denote for short $F^{-1}(\mathcal{V})$ the full subcategory of $\mathcal{A}$ consisting of objects A such that $F(A) \in \mathrm{add}(\mathcal{V})$.
- For an additive category $\mathcal{W}$ and an additive subcategory $\mathcal{V}$ denote by $\mathcal{W}/\mathcal{V}$ the category whose objects are the same objects as those of $\mathcal{W}$, and for any two objects X, Y of $\mathcal{W}$ we put

$$(\mathcal{W}/\mathcal{V})(X, Y) := \mathcal{W}(X, Y)/I_{\mathcal{V}}^{\mathcal{W}}(X, Y),$$

where

$$I_{\mathcal{V}}^{\mathcal{W}}(X, Y) := \{f \in \mathcal{W}(X, Y) \mid \exists V \in \mathrm{obj}(\mathcal{V}),\, g \in \mathcal{W}(V, Y),\, h \in \mathcal{W}(X, V) : f = g \circ h\}.$$

- If $\mathcal{S}$ and $\mathcal{R}$ are subcategories of a Krull–Schmidt category $\mathcal{W}$, then $\mathcal{R} - \mathcal{S}$ denotes the full subcategory of $\mathcal{R}$ consisting of those objects X of $\mathcal{R}$ such that no direct factor of X is an object of $\mathcal{S}$.
- Recall that a full triangulated subcategory $\mathcal{U}$ of a triangulated category is thick (épaisse), if it is in addition closed under taking direct summands (and a fortiori under isomorphisms) in $\mathcal{T}$.

- Let $\mathcal{U}$ be a thick (épaisse) subcategory of a triangulated category $\mathcal{T}$. Then the Verdier localisation $\mathcal{T}_{\mathcal{U}}$ (see [SGA 4½ 1977, Chapitre I, §2, nos. 1, 3, 4; Verdier 1996]) is the category formed by the same objects as the objects of $\mathcal{T}$ and morphisms in $\mathcal{T}_{\mathcal{U}}$ are limits of diagrams
$$X \xleftarrow{s} Z \xrightarrow{f} Y,$$
where f and s are morphisms in $\mathcal{T}$, and where cone(s) is an object in $\mathcal{U}$. The notation we use for the Verdier localisation is not quite standard, however in order to distinguish from the additive quotient above we decided to use this notation (see Remark 3.4).

Theorem 1.2 [Auslander and Kleiner 1994, Theorem 1.10]. *Assume the hypotheses at the beginning of the section. Let $\mathcal{Y}$ be a full additive subcategory of $\mathcal{H}$ and let $\mathcal{Z} := (US')^{-1}(\mathcal{Y})$. Then the following two conditions* (†) *are equivalent*:

- *Each object of $S'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$ and of an object of $U^{-1}(\mathcal{Y})$.*
- *Each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$.*

Suppose that the above conditions hold for $\mathcal{Y}$. Then

(1) *S and T induce functors*
$$\mathcal{H}/S'T'\mathcal{Y} \xrightarrow{S} \mathcal{G}/SS'T'\mathcal{Y} \quad \textit{and} \quad \mathcal{G}/SS'T'\mathcal{Y} \xrightarrow{T} \mathcal{H}/\mathcal{Y},$$

(2) *for any object L of $\mathcal{D}$ and any object B of $U^{-1}(\mathcal{Y})$ the functor S induces an isomorphism*
$$\mathcal{H}/S'T'\mathcal{Y}(S'L, B) \to \mathcal{G}/SS'T'\mathcal{Y}(SS'L, SB),$$

(3) *for any object L of $(US')^{-1}\mathcal{Y}$ and any object A of $\mathcal{G}$ the functor T induces an isomorphism*
$$\mathcal{G}/SS'T'\mathcal{Y}(SS'L, B) \to \mathcal{H}/\mathcal{Y}(TSS'L, TA),$$

(4) *the restrictions of S*
$$(\text{add}\, S'\mathcal{Z})/S'T'\mathcal{Y} \xrightarrow{S} (\text{add}\, SS'\mathcal{Z})/SS'T'\mathcal{Y}$$
and T
$$(\text{add}\, SS'\mathcal{Z})/SS'T'\mathcal{Y} \xrightarrow{T} (\text{add}\, TSS'\mathcal{Z})/\mathcal{Y}$$
are equivalences of categories, *and*
$$(\text{add}\, S'\mathcal{Z})/S'T'\mathcal{Y} \xrightarrow{TS} (\text{add}\, TSS'\mathcal{Z})/\mathcal{Y}$$
is isomorphic to the natural projection,

(5) *if each object of $S'T'US'\mathcal{D}$ is a direct factor of $US'\mathcal{D}$*, *then $\mathcal{Y} = US'\mathcal{D}$ satisfies the hypothesis of the theorem.*

A main consequence is:

Corollary 1.3 [Auslander and Kleiner 1994, Corollary 1.12]. *Let $\mathcal{Y}$ be a full additive subcategory of $\mathcal{H}$ satisfying* (†) *of Theorem 1.2 and suppose that $\mathcal{H}$ and $\mathcal{G}$ are both Krull–Schmidt categories. Using the notations of Theorem 1.2, the following hold.*

1. *For each indecomposable object N of $(\mathrm{add}(S'\mathcal{Z})) - S'T'\mathcal{Y}$ the object SN has a unique indecomposable direct factor $g(N)$ which is not a direct factor of an object in $SS'T'\mathcal{Y}$.*
2. *For each indecomposable object M of $(\mathrm{add}(SS'\mathcal{Z})) - SS'T'\mathcal{Y}$ the object TM has a unique indecomposable direct factor $f(M)$ which is not a direct factor of an object in $\mathcal{Y}$.*
3. $f(g(N)) = N$.
4. $g(f(M)) = M$.

2. Relative projectivity and injectivity with respect to pairs of adjoint functors

2A. ***Relative homological algebra revisited.*** We shall need to revise some facts from relative homological algebra, following [Beligiannis and Marmaridis 1994]. Recall that a full subcategory $\mathcal{X}$ of an additive category $\mathcal{S}$ is contravariantly finite if for any object S of $\mathcal{S}$ there is an object X of $\mathrm{add}\,\mathcal{X}$ and a morphism $f \in \mathcal{S}(X, S)$ such that for any X' in $\mathcal{X}$ the induced map

$$\mathcal{S}(X', f) : \mathcal{S}(X', X) \to \mathcal{S}(X', S)$$

is surjective. We call such an object X of $\mathcal{S}$ a right $\mathcal{X}$-approximation of S. The dual notion, using the covariant Hom-functor, leads to the notion of a covariantly finite subcategory. With this notion in mind we shall have the following

Lemma 2.1 [Auslander and Reiten 1992, Proposition 1.2]. *If the additive functor $T : \mathcal{S} \to \mathcal{T}$ between the additive categories $\mathcal{S}$ and $\mathcal{T}$ admits a left adjoint S_ℓ, then* $\mathrm{add}(\mathrm{im}(S_\ell))$ *is a contravariantly finite subcategory of $\mathcal{S}$. If T admits a right adjoint S_r, then* $\mathrm{add}(\mathrm{im}(S_r))$ *is a covariantly finite subcategory of $\mathcal{S}$.*

Proof. Let $\mathcal{X} := \mathrm{add}(\mathrm{im}(S_\ell))$. Consider the counit

$$\eta : S_\ell T \to \mathrm{id}_{\mathcal{S}}$$

of the adjoint pair (S_ℓ, T). Evaluation on any object Q of $\mathcal{S}$ gives a morphism

$$\eta_Q : S_\ell T Q \to Q.$$

Now, given an object $S_\ell P$ in $\mathrm{im}(S_\ell)$, we have

$$\begin{tikzcd}
\mathcal{S}(S_\ell P, S_\ell T Q) \arrow[r, "\mathcal{S}(S_\ell P, \eta_Q)"] & \mathcal{S}(S_\ell P, Q) \\
\mathcal{T}(P, TS_\ell T Q) \arrow[u, "\simeq"'] & \mathcal{T}(P, T Q) \arrow[u, "\simeq"]
\end{tikzcd}$$

and the composite map is a split epimorphism by [Mac Lane 1971, IV, Theorem 1(ii)]. Since the property holds true for direct factors of an object $S_\ell P$ as well, we have shown that $\mathcal{X}$ is a contravariantly finite subcategory of $\mathcal{S}$.

By the dual argument, if T admits a right adjoint S_r, then $\mathrm{add}(\mathrm{im}\, S_r)$ is a covariantly subcategory of $\mathcal{S}$. □

Recall from [Beligiannis and Marmaridis 1994] that we may produce from contravariantly finite subcategories a relative homological algebra. Let $\mathcal{X}$ be a contravariantly finite subcategory of an additive category $\mathcal{S}$. Then a morphism $g \in \mathcal{S}(A, B)$ is said to be $\mathcal{X}$-epic if for any object X of $\mathcal{X}$ the morphism

$$\mathcal{S}(X, g) : \mathcal{S}(X, A) \to \mathcal{S}(X, B)$$

is surjective. By the very definition, a right $\mathcal{X}$-approximation is an $\mathcal{X}$-epic. If $\mathcal{X}$ is contravariantly finite and if each $\mathcal{X}$-epic has a kernel, [Beligiannis and Marmaridis 1994, Theorem 2.12] shows that for any object S of $\mathcal{S}$ the choice of a right $\mathcal{X}$-approximation $X_S \to S$ induces a left triangulation on the stable category $\mathcal{S}/\mathcal{X}$. Moreover, two such choices give equivalent left triangulated categories. Hence, a contravariantly finite subcategory $\mathcal{X}$ such that each $\mathcal{X}$-epic has a kernel gives rise to the relative Ext^n-group with respect to $\mathcal{X}$, denoted by $\mathrm{Ext}^n_{\mathcal{X}}(A, B)$, namely the evaluation on the object B, of the n-th derived functor of $\mathcal{S}(-, B)$, obtained by a $\mathcal{X}$-resolution of A.

Lemma 2.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be additive categories, let $T : \mathcal{S} \to \mathcal{T}$ be an additive functor admitting a left adjoint S_ℓ. Then $g \in \mathcal{S}(A, B)$ is* $\mathrm{add}(\mathrm{im}(S_\ell))$*-epic if and only if $T(g)$ is a split epimorphism.*

Proof. Let $g \in \mathcal{S}(A, B)$ be $\mathrm{add}(\mathrm{im}(S_\ell))$-epic. Then for any object C of $\mathcal{T}$ we get

$$\mathcal{S}(S_\ell(C), g) : \mathcal{S}(S_\ell(C), A) \to \mathcal{S}(S_\ell(C), B)$$

is surjective. Hence, for any object C of $\mathcal{T}$ we get

$$\mathcal{T}(C, Tg) : \mathcal{T}(C, TA) \to \mathcal{T}(C, TB)$$

is surjective. In particular, for $C = TB$ there is $f \in \mathcal{T}(TB, TA)$ with $Tg \circ f = \mathrm{id}_{TB}$. Hence Tg is a split epimorphism.

Let $g \in \mathcal{S}(A, B)$ be such that Tg is a split epimorphism. Then there is $f \in \mathcal{T}(TB, TA)$ with $Tg \circ f = \mathrm{id}_{TB}$. Let C be an object of $\mathcal{T}$ and let $h \in \mathcal{S}(S_\ell(C), B)$. We need to show that there is $k \in \mathcal{S}(S_\ell C, A)$ such that $\mathcal{S}(S_\ell(C), g)(k) = g \circ k = h$, where

$$\mathcal{S}(S_\ell(C), g) : \mathcal{S}(S_\ell(C), A) \to \mathcal{S}(S_\ell(C), B).$$

Since T is right adjoint to S_ℓ, this is equivalent to

$$\mathcal{T}(C, Tg) : \mathcal{T}(C, TA) \to \mathcal{T}(C, TB)$$

being surjective. For $h \in \mathcal{T}(C, TB)$ we get

$$h = (Tg \circ f) \circ h = Tg \circ (f \circ h) = \mathcal{T}(C, Tg)(f \circ h) = \mathcal{S}(S_\ell C, g)(f \circ h).$$

Clearly we can pass to direct factors of $S_\ell C$. Therefore, g is add(im(S_ℓ))-epic. □

Note that Lemma 2.2 has a dual version for functors T admitting right adjoint functors S_r.

Again, in the setting of [Beligiannis and Marmaridis 1994] the add(im(S_ℓ))-relative projectives are those objects Q with

$$\mathrm{Ext}^n_{\mathrm{add}(\mathrm{im}\, S_\ell)}(Q, B) = 0$$

for all objects B and $n > 0$. By definition, this coincides with the objects Q for which the counit of the adjunction (S_ℓ, T) splits. These are precisely the objects in add(im(S_ℓ)).

The dual statement applies in case of T having a right adjoint S_r, and considering covariantly finite subcategories and add(im(S_r))-coresolutions instead of contravariantly finite subcategories and add(im(S_ℓ))-resolutions.

This motivates the following definition.

Definition 2.3. Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories, and let $T : \mathcal{S} \to \mathcal{T}$ be a triangle functor.

- Suppose T has a left adjoint. Then an object Q of $\mathcal{S}$ is *T-relative projective* if the natural transformation

  $$\mathcal{S}(Q, -) \to \mathcal{T}(TQ, T-)$$

  induced by T is injective.

- Suppose T has a right adjoint. Then an object Q of $\mathcal{S}$ is *T-relative injective* if the natural transformation

  $$\mathcal{S}(-, Q) \to \mathcal{T}(T-, TQ)$$

  induced by T is injective.

Remark 2.4. Recall that for a field k of finite characteristic $p > 0$ and a finite group G with a subgroup H, an indecomposable kG-module M is called relatively H-projective if each epimorphism $N \twoheadrightarrow M$ of kG-modules, which is known to be split as kH-module morphism, splits as kG-module morphism. This definition of relative projectivity was developed by Hochschild [1956] in the situation of a ring R, a subring S of R. Hochschild declares an R-module M to be (S, R)-projective if any short exact sequence

$$0 \to X \to Y \to M \to 0,$$

which is known to be split as short exact sequence of S-modules, is automatically split as short exact sequence of R-modules. Denoting by $\mathrm{res}_S^R : R-\mathrm{Mod} \to S-\mathrm{Mod}$ the exact functor given by restriction to the subring S, this translates, in slightly more modern terms, into the statement that M is (S, R)-projective if and only if

$$\mathrm{Ext}^1_R(M, X) \to \mathrm{Ext}^1_S(\mathrm{res}_S^R(M), \mathrm{res}_S^R(X))$$

is injective for any R-module X. Hence, since $\mathrm{res}_S^R(X)[1] \simeq \mathrm{res}_S^R(X[1])$ we get that M is (S, R)-projective if and only if

$$\mathrm{Hom}_{D^b(R)}(M, X[1]) \to \mathrm{Hom}_{D^b(S)}(\mathrm{res}_S^R(M), \mathrm{res}_S^R(X[1]))$$

is injective for all objects X. Since each object X can be seen as an object $X = Y[-1]$, Definition 2.3 could make sense in a broader context. We will not elaborate on this here (see [Zimmermann 2020]).

Remark 2.5. Grime [2008] defined an object to be relative projective with respect to a functor F admitting a left adjoint L as those which are direct factors of an object in the image of L. This is a direct generalisation of Green's definition [1958], whereas our definition is closer to Hochschild's definition [1956]. However, the concepts coincide, as will be shown in Proposition 2.10 below.

2B. ***Relative projectivity for triangulated categories.*** We shall study the concept of T-relative projectivity/injectivity from Definition 2.3 for triangle functors T between triangulated categories admitting a left adjoint S_ℓ and a right adjoint S_r. Then the concept has a very nice interpretation.

Lemma 2.6. *Let $\mathcal{S}$ and $\mathcal{T}$ be additive categories and let $T : \mathcal{S} \to \mathcal{T}$ be an additive functor.*

- *If T has a left adjoint S_ℓ, then an object Q is T-relative projective if and only if the evaluation on Q of the counit η of the adjunction $\eta_Q : S_\ell T Q \to Q$ is an epimorphism. Any object in* $\mathrm{add}(\mathrm{im}(S_\ell))$ *is T-relative projective.*

- *If T has a right adjoint S_r, then an object Q is T-relative injective if and only if the evaluation on Q of the unit η of the adjunction $\epsilon_Q : Q \to S_rTQ$ is a monomorphism. Any object in* $\operatorname{add}(\operatorname{im}(S_r))$ *is T-relative injective.*

Proof. Suppose that T has a left adjoint S_ℓ. Then the counit $\eta_Q : S_\ell TQ \to Q$ is an epimorphism if and only if for any object A the morphism

$$\mathcal{S}(\eta_Q, A) : \mathcal{S}(Q, A) \to \mathcal{S}(S_\ell TQ, A)$$

is a monomorphism. This in turn is equivalent to the statement that the natural transformation of functors $\mathcal{S} \to \mathbb{Z} - \mathrm{Mod}$

$$\mathcal{S}(\eta_Q, -) : \mathcal{S}(Q, -) \to \mathcal{S}(S_\ell TQ, -)$$

is a monomorphism. Using the defining property of (S_ℓ, T) being an adjoint pair, this is equivalent to

$$\mathcal{S}(\eta_Q, -) : \mathcal{S}(Q, -) \to \mathcal{T}(TQ, T-)$$

being a monomorphism. Hence, the statement is equivalent to Q being T-relative projective. Now $\eta_{S_\ell Q'}$ is a split epimorphism for any object Q' of $\mathcal{T}$ by [Mac Lane 1971, IV, Theorem 1(ii)].

Suppose that T has a right adjoint S_r. Then the unit $\epsilon_Q : Q \to S_rTQ$ is a monomorphism if and only if

$$\mathcal{S}(A, \epsilon_Q) : \mathcal{S}(A, Q) \to \mathcal{S}(A, S_rTQ)$$

is a monomorphism. This is equivalent to

$$\mathcal{S}(-, \epsilon_Q) : \mathcal{S}(-, Q) \to \mathcal{S}(-, S_rTQ) = \mathcal{T}(T-, TQ)$$

being a monomorphism, which is equivalent to Q being T-relative injective. Now $\epsilon_{S_r Q'}$ is a split monomorphism for any object Q' of $\mathcal{T}$ by [Mac Lane 1971, IV, Theorem 1(ii)]. □

Remark 2.7. Note that in a triangulated category $\mathcal{S}$ the notions of epimorphism (resp. monomorphism) and split epimorphism (resp. split monomorphism) coincide.

Proposition 2.8. *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories and let $T : \mathcal{S} \to \mathcal{T}$ be a triangle functor. Suppose that T has a left* (*resp. right*) *adjoint S. Then an object Q is T-relative projective* (*resp. injective*) *if and only if Q is in* $\operatorname{add}(\operatorname{im}(S))$.

Proof. By Lemma 2.6 and Remark 2.7 Q is T-relative projective (resp. T-relative injective) if and only if Q is in $\operatorname{add}(\operatorname{im} S)$. □

Corollary 2.9. *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories and let $T : \mathcal{S} \to \mathcal{T}$ be a triangle functor. Suppose that T has a left* (*resp. right*) *adjoint S, and let $\eta : ST \to \mathrm{id}$ be the counit* (*resp. $\tilde{\epsilon} : \mathrm{id} \to ST$ the unit*) *of the adjunction. Then Q is T-relative* *projective (resp. injective) if and only if* η_Q *is a split epimorphism (resp.* $\tilde{\epsilon}_Q$ *is a split monomorphism).*

Proof. This is precisely Proposition 2.8 in connection with Lemma 2.6 and Remark 2.7. □

We summarise the situation to an analogue of Higman's lemma for pairs of adjoint functors between triangulated categories.

Proposition 2.10. *Let* $\mathcal{T}$ *and* $\mathcal{S}$ *be triangulated categories and let* $T : \mathcal{S} \to \mathcal{T}$ *be a triangle functor. Suppose that* T *has a left (resp. right) adjoint* S. *Let* M *be an object of* $\mathcal{T}$. *Then the following are equivalent:*

(1) M *is* T*-relative projective (resp. injective).*

(2) M *is in* $\operatorname{add}(\operatorname{im} S)$.

(3) M *is a direct factor of some* $S(L)$ *for some* L *in* $\mathcal{S}$.

(4) M *is a direct factor of some* $ST(M)$.

Proof. (1) $\Leftrightarrow$ (2) by Proposition 2.8.

(2) $\Leftrightarrow$ (3) by the definition of $\operatorname{add}(\operatorname{im} S)$.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) by Corollary 2.9. □

Remark 2.11. Note that Corollary 2.9 generalises [Zimmermann 2014, Propositions 2.1.6, 2.1.8] to this more general situation.

Remark 2.12. In case T has a left adjoint S, which is also assumed to be a right adjoint, and $\mathcal{T}$ is an abelian or triangulated category Broué [2009, Theorem 6.8] defined T-relative projective and T-relative injective objects. In this situation he showed a version of Higman's lemma as Proposition 2.10 by completely different means.

Corollary 2.13. *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be triangulated categories and let* $T : \mathcal{S} \to \mathcal{T}$ *be a triangle functor admitting a left (resp. right) adjoint* S. *Then all objects of* $\mathcal{S}$ *are* T*-relative projective (resp. injective) if and only if* $\mathcal{S} = \operatorname{add}(\operatorname{im}(S))$.

Note that all objects of $\mathcal{S}$ are T-relative projective (resp. injective) if and only if the global dimension of the relative homological algebra described in Section 2A is 0.

Example 2.14. Let G be a group, and let H be a subgroup of finite index n. Denote by $\downarrow_H^G$ the functor given by restriction of the G-action to the H-action, and by $\uparrow_H^G$ the functor $kG \otimes_{kH} -$ given by induction. If n is invertible in the field k, then every object M in $D^b(kG)$ is $\downarrow_H^G$-relative projective. Indeed, [Zimmermann 2014, Proposition 2.1.10] shows that the multiplication $kG \otimes_{kH} kG \to kG$ splits as morphism of $kG-kG$-bimodules. The counit of the adjunction $(\uparrow_H^G, \downarrow_H^G)$ is $kG \otimes_{kH} kG \otimes_{kG} - \to kG \otimes_{kG} -$, and by hypothesis this map splits.

2C. ***Revisiting the case studied by Carlson, Peng and Wheeler.*** The purpose of this section is to give a structural explanation of an argument in the proof of [Carlson et al. 1998, p. 304; Theorem 6.2] for the statement that the relative stable category is triangulated. Note that Grime [2008, Example 3.6] gave a slightly less general structural explanation.

Remark 2.15. Carlson, Peng and Wheeler considered the classical case of group rings, namely they let k be a field of characteristic $p > 0$, let G be a finite group, let D be a p-subgroup of G and let H be a subgroup of G containing the normaliser of D in G. They considered the additive quotient of the module category modulo the morphisms which factor through $\downarrow_E^G$-projective modules, for some $E \in \mathfrak{Y}$, where $\mathfrak{Y} = \{E \leq H \cap D^g \mid g \in G \setminus H\}$, and showed that this produces a triangulated category. Carlson, Peng and Wheeler used the general approach given by Happel [1988, Theorem I.2.6] which shows that the additive quotient of any Frobenius category modulo relative injective-projectives is triangulated. However, Carlson, Peng and Wheeler mentioned that Happel's proof for Frobenius categories having triangulated stable categories carries over to this more general situation. The purpose of this section is to show that the fact that the proof carries over has a structural reason, and uses more precisely the properties from Sections 2A and 2B.

Note that group rings are symmetric; hence the module category is Frobenius. Moreover, the functors considered in classical Green correspondence, namely restriction and induction, are left and right adjoint to each other. We note that in our general abstract situation relative injectives and relative projectives do not coincide in general. The situation changes in case S is at the same time left and right adjoint to T and the categories are already Frobenius categories.

Recall from [Bühler 2010, Definition 2.1] the concept of an exact category. Let $\mathcal{A}$ be an additive category. Given three objects A_1, A_2, A_3 in $\mathcal{A}$ and $f \in \mathcal{A}(A_1, A_2)$ and $g \in \mathcal{A}(A_2, A_3)$, we have that (f, g) is a short exact sequence, denoted by

$$0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0,$$

(or occasionally by $A_1 \overset{f}{\hookrightarrow} A_2 \overset{g}{\twoheadrightarrow} A_3$), if $\ker(g) = f$ and $g = \operatorname{coker}(f)$.

An exact structure on the additive category $\mathcal{A}$ is given by a class $E_{\mathcal{A}}$ of short exact sequences, called admissible short exact sequences, satisfying the following axioms below. If

$$0 \to A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \to 0$$

is a short exact sequence in $E_{\mathcal{A}}$, then we say that f is an admissible monomorphism and g is an admissible epimorphism.

- For all objects A the identity on A is admissible monomorphism and admissible epimorphism.

- Admissible monomorphisms are closed under composition, and admissible epimorphisms are closed under composition.
- If $\alpha : X \to Y$ is an admissible monomorphism, and $f : X \to Z$ is any morphism, then the pushout
$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & Y \arrow[d, "\check{f}"] \\ Z \arrow[r, "\check{\alpha}"] & U \end{tikzcd}$$
exists and $\check{\alpha}$ is an admissible monomorphism.
- If $\alpha : Y \to X$ is an admissible epimorphism, and $f : Z \to X$ is any morphism, then the pullback
$$\begin{tikzcd} Y \arrow[r, "\alpha"] & X \\ U \arrow[u, "\hat{f}"] \arrow[r, "\hat{\alpha}"] & Z \arrow[u, "f"'] \end{tikzcd}$$
exists and $\hat{\alpha}$ is an admissible epimorphism.

An exact category is an additive category $\mathcal{A}$ with a class $E_{\mathcal{A}}$ of short exact sequences, stable under isomorphism and satisfying the above axioms. See [Bühler 2010] for an exhaustive development of exact categories.

Proposition 2.16. *Let* $(\mathcal{S}, E_{\mathcal{S}})$ *and* $(\mathcal{T}, E_{\mathcal{T}})$ *be exact categories with* $E_{\mathcal{S}}$ *and* $E_{\mathcal{T}}$ *being the class of admissible exact sequences, respectively. If* $T : \mathcal{S} \to \mathcal{T}$ *is a functor with a left adjoint* S_ℓ *and a right adjoint* S_r,

- *then*
$$E_T := \left\{ (X \xhookrightarrow{f} Y \xrightarrow{g} Z) \in E_{\mathcal{S}} \mid (TX \xhookrightarrow{Tf} TY \xrightarrow{Tg} TZ) \in E_{\mathcal{T}} \right\}$$
defines an exact structure on $\mathcal{S}$.
- *If moreover the unit* $\epsilon : \mathrm{id}_{\mathcal{S}} \to S_r T$ *is an admissible monomorphism in* $E_{\mathcal{S}}$ *and if the counit* $\eta : S_\ell T \to \mathrm{id}_{\mathcal{S}}$ *is an admissible epimorphism in* $E_{\mathcal{S}}$,
  - *then* $(\mathcal{S}, E_T)$ *has enough* T*-relative projectives and enough* T*-relative injectives.*
  - *Suppose now in addition that* $\mathcal{S}$ *and* $\mathcal{T}$ *are abelian Frobenius* (*i.e., an abelian category which is Frobenius with respect to the class of all exact sequences*). *Then the class of* T*-relative projectives coincides with the class of objects in* $\mathrm{add}(\mathrm{im}\, S_\ell)$ *and the class of* T*-relative injectives coincides with the class of objects in* $\mathrm{add}(\mathrm{im}\, S_r)$.

Proof. We first show that E_T is an exact structure. $T(\mathrm{id}_A) = \mathrm{id}_{TA}$, which implies the first condition. T maps compositions to compositions, and hence compositions of admissible monics/epics are admissible monics/epics. Sequences are closed under isomorphisms, as T is exact and hence maps isomorphisms to isomorphisms. Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be an exact sequence in $E_{\mathcal{S}}$ and let $X \xrightarrow{f} X'$ be any morphism. Then, since $E_{\mathcal{S}}$ is an exact structure, we may form the pushout

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[d, equal] \\
X' \arrow[r, "\check{\alpha}"] & Y' \arrow[r, "\check{\beta}"] & Z
\end{tikzcd}$$

As $E_{\mathcal{S}}$ is an exact structure, the lower row is an element of $E_{\mathcal{S}}$. The sequence

$$0 \to X \xrightarrow{\left(\begin{smallmatrix} f \\ -\alpha \end{smallmatrix}\right)} X' \oplus Y \xrightarrow{(\check{\alpha}, g)} Y' \to 0$$

is exact, since the above is a pushout and α, $\check{\alpha}$ are monomorphisms. Since T is exact,

$$0 \xrightarrow{\left(\begin{smallmatrix} Tf \\ -T\alpha \end{smallmatrix}\right)} TX' \oplus TY \xrightarrow{(T\check{\alpha}, Tg)} TY' \to 0$$

is exact. Therefore

$$\begin{tikzcd}
TX \arrow[r, "T\alpha"] \arrow[d, "Tf"] & TY \arrow[r, "T\beta"] \arrow[d, "Tg"] & TZ \arrow[d, equal] \\
TX' \arrow[r, "T\check{\alpha}"] & TY' \arrow[r, "T\check{\beta}"] & TZ
\end{tikzcd}$$

is a pushout diagram. Since the above row is in $E_{\mathcal{T}}$, and since $E_{\mathcal{T}}$ is an exact structure, also the lower row is in $E_{\mathcal{T}}$. This shows the third axiom. Dually also the fourth axiom holds.

We now assume the additional condition on the unit and the counit. The fact that $\operatorname{add}(\operatorname{im} S_\ell)$ are T-relative injective objects and $\operatorname{add}(\operatorname{im} S_r)$ are T-relative projective objects is Lemma 2.6. The fact that we then get enough T-relative projective objects follows from the hypothesis on the counit, and the fact that we then get enough T-relative injective objects follows from the hypothesis on the unit.

The hypothesis on $\mathcal{S}$ and $\mathcal{T}$ being Frobenius with respect to all short exact sequences implies that the stable categories modulo projective-injective objects $\underline{\mathcal{S}}$ and $\underline{\mathcal{T}}$ are triangulated (see [Happel 1988, Theorem I.2.6]). Proposition 2.8 applied to this triangulated category shows that $\operatorname{add}(\operatorname{im} S_\ell)$ are precisely the T-relative projective objects and $\operatorname{add}(\operatorname{im} S_r)$ are precisely the T-relative injective objects of this new exact structure. □

Remark 2.17. Note that the hypothesis of ϵ being a monomorphism and η being an epimorphism for the adjunctions involved is very strong. For an abelian category $\mathcal{S}$, if $T : \mathcal{S} \to \mathcal{T}$ has left and right adjoints S_ℓ and S_r, then T is exact. Further, Eilenberg and Moore [1965, Proposition 1.5] (see also [Grime 2008, Lemma 2.1]) showed that the unit $\mathrm{id} \to S_r T$, as in Proposition 2.16, is a monomorphism if and only if $TX = 0$ implies $X = 0$, if and only if the counit $S_\ell T \to \mathrm{id}$ is an epimorphism. The counit $S_\ell T \to \mathrm{id}$ is an epimorphism if and only if T is faithful.

In order to assure all quotient categories in Theorem 1.2 being triangulated, using Proposition 2.16 we need to assume the hypothesis for all the functors S, S', T, T', and hence get quite a few restrictions on these functors.

Remark 2.18. The first item in Proposition 2.16 should be compared with the statement by Eilenberg and Moore [1965, Theorem II.2.1].

Remark 2.19. Let $(\mathcal{S}, E_{\mathcal{S}})$ and $(\mathcal{T}, E_{\mathcal{T}})$ be exact categories, let $T : \mathcal{S} \to \mathcal{T}$ be an exact functor admitting a left adjoint S_ℓ and a right adjoint S_r, and suppose the unit $\epsilon : \mathrm{id}_{\mathcal{S}} \to S_r T$ of the adjoint property (T, S_r) is a monomorphism, and the counit $\eta : S_\ell T \to \mathrm{id}_{\mathcal{S}}$ of the adjoint property (S_ℓ, T) is an epimorphism. Suppose moreover that $\mathrm{add}(\mathrm{im}(S_r)) = \mathrm{add}(\mathrm{im}(S_\ell))$. If in addition $\mathcal{S}$ and $\mathcal{T}$ are abelian Frobenius categories, i.e., abelian categories which are Frobenius with respect to the exact structures given by all exact sequences, then Proposition 2.16 shows that

$$E_T := \left\{(X \xhookrightarrow{f} Y \xrightarrow{g}\!\!\!\!\rightarrow Z) \in E_{\mathcal{S}} \mid (TX \xhookrightarrow{Tf} TY \xrightarrow{Tg}\!\!\!\!\rightarrow TZ) \in E_{\mathcal{T}}\right\}$$

is a Frobenius structure on $\mathcal{S}$. We call this the T-relative Frobenius structure. Following [Happel 1988, Theorem I.2.6] the stable category $\underline{\mathcal{S}}^T$ of $\mathcal{S}$ modulo the T-relative projectives is in this case a triangulated category. The distinguished triangles are constructed as follows. Given two objects M and N in $\mathcal{S}$ and $f \in \mathcal{S}(M, N)$. Then we may form the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\tilde{\epsilon}_M"] \arrow[d, "f"'] & S_r TM \arrow[r] \arrow[d] & \Omega_T^{-}-1(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "c_1(f)"] & C(f) \arrow[r, "c_2(f)"] & \Omega_T^{-1}(M) \arrow[r] & 0
\end{tikzcd}$$

(or analogously the pullback diagram along $\eta_N : S_\ell TN \twoheadrightarrow N$). Then $\underline{\mathcal{S}}^T$ is a triangulated category with distinguished triangles being isomorphic to triangles of the form

$$M \xrightarrow{c_1(f)} C(f) \xrightarrow{c_2(f)} \Omega_T^{-1}(M)$$

for any $f \in \mathcal{S}(M, N)$.

We recall a result implicit in [Grime 2008].

Proposition 2.20 [Grime 2008, Theorem 3.3]. *Let $(\mathcal{S}, E_{\mathcal{S}})$ and $(\mathcal{T}, E_{\mathcal{T}})$ be exact categories and let $T : \mathcal{S} \to \mathcal{T}$ be a functor which admits a left adjoint S_ℓ and a right adjoint S_r. Assume that the counit $S_\ell T \to \mathrm{id}_{\mathcal{S}}$ of the adjoint pair (S_ℓ, T) is an admissible epimorphism in the exact category $(\mathcal{S}, E_{\mathcal{S}})$ and that the unit* $\mathrm{id}_{\mathcal{S}} \to S_r T$

of the adjoint pair (T, S_r) *is an admissible monomorphism in the exact category* $(\mathcal{S}, E_{\mathcal{S}})$. *Putting*

$$E_T := \left\{(X \overset{f}{\hookrightarrow} Y \overset{g}{\twoheadrightarrow} Z) \in E_{\mathcal{S}} \mid (TX \overset{Tf}{\hookrightarrow} TY \overset{Tg}{\twoheadrightarrow} TZ) \in E_{\mathcal{T}} \text{ is split exact in } \mathcal{T}\right\}$$

then $(\mathcal{S}, E_T)$ *is an exact category with enough projective and enough injective objects. The full subcategory of projective objects coincides with the full subcategory* $\mathrm{add}(\mathrm{im}(S_\ell))$ *and the full subcategory of injective objects coincides with the full subcategory* $\mathrm{add}(\mathrm{im}(S_r))$.

Grime's proposition follows from Proposition 2.16 when it is applied to the case of the split exact structure on $\mathcal{T}$.

2D. ***Relative projectivity for derived categories of group rings.*** We shall apply our concept of relative projectivity to the special case of the derived category of a block of a group ring kG. We first note that if A is a finite-dimensional k-algebra over a field k, then $D^b(A)$ is a Krull–Schmidt category. Let G be a finite group, let H be a subgroup of G, let k be a field of characteristic $p > 0$. Then we consider the functors $\uparrow_H^G$ and $\downarrow_H^G$. Note that both functors are exact functors between $kG-\mathrm{mod}$ and $kH-\mathrm{mod}$. These functors form an adjoint pair, in the sense that $(\uparrow_H^G, \downarrow_H^G)$ and $(\downarrow_H^G, \uparrow_H^G)$ are both adjoint pairs. Note that since both functors are exact, they provide functors

$$S :=\uparrow_H^G: D^b(kH) \to D^b(kG) \quad \text{and} \quad T :=\downarrow_H^G: D^b(kG) \to D^b(kH).$$

We define $\mathcal{G} := D^b(kG)$ and $\mathcal{H} := D^b(kH)$. Moreover, $(\uparrow_H^G, \downarrow_H^G)$ and $(\downarrow_H^G, \uparrow_H^G)$ are both adjoint pairs also between the derived categories. As for its restriction to the module categories we have

Lemma 2.21. *Let* $K \leq H \leq G$ *be an increasing sequence of groups. Then for the functors*

$$\begin{aligned} \uparrow_H^G &: D^b(kH) \to D^b(kG), & \uparrow_K^H &: D^b(kK) \to D^b(kH), \\ \downarrow_H^G &: D^b(kG) \to D^b(kH), & \downarrow_K^H &: D^b(kH) \to D^b(kK), \end{aligned}$$

we get

$$\uparrow_H^G \circ \uparrow_K^H = \uparrow_K^G \quad \textit{and} \quad \downarrow_K^H \circ \downarrow_H^G = \downarrow_K^G .$$

Proof. This follows trivially from the module case. □

Note that the notion of $\downarrow_H^G$-relative projectivity in $D^b(kG)$ corresponds to the similar concept of relative projectivity with respect to a subalgebra as developed in [Zimmermann 2014, Section 2.1.1]. We shall need to extend the statements from there to our more general situation.

Lemma 2.22. *Let G be a finite group, and let k be a field of characteristic $p > 0$. Let D be a minimal subgroup of G such that the bounded complex of kG-modules M is $\downarrow_D^G$-relative projective. Then D is a p-group.*

Proof. Let $D \in \mathrm{Syl}_p(G)$. By Example 2.14 every object M of $D^b(kG)$ is $\downarrow_D^G$-relative projective since $|G : D|$ is prime to p by the definition of a Sylow subgroup. If M is $\downarrow_H^G$-relative projective, by Proposition 2.8 it is in $\mathrm{add}(\mathrm{im} \uparrow_H^G)$ and if $D' \in \mathrm{Syl}_p(H)$, then M is also $\downarrow_{D'}^H$-relative projective, whence in $\mathrm{add}(\mathrm{im} \uparrow_{D'}^H)$ by Proposition 2.8 again. Therefore M is in $\mathrm{add}(\mathrm{im} \uparrow_{D'}^G)$, and therefore $\downarrow_{D'}^G$-relatively projective, again by Proposition 2.8. ☐

Definition 2.23. Let G be a finite group, and let k be a field of characteristic $p > 0$. Then, an indecomposable object M of $D^b(kG)$ has vertex D if M is relatively kD-projective, and if D is minimal with this property.

Proposition 2.24. *The vertex D of an indecomposable object M of $D^b(kG)$ is a p-subgroup of G, and D is unique up to conjugacy.*

Proof. Using Lemma 2.22 we only need to show unicity up to conjugation.

The unicity part up to conjugation can be shown completely analogous to the classical case. Suppose that M is a direct summand of $L \uparrow_K^G$ and of $N \uparrow_H^G$ for two subgroups H and K of G and two indecomposable objects L in $D^b(kK)$ and N in $D^b(kH)$. By Proposition 2.10 we may suppose $L = M \downarrow_K^G$ and $N = M \downarrow_H^G$. Then M is a direct factor of

$$M \downarrow_H^G \uparrow_H^G \downarrow_K^G \uparrow_K^G = \bigoplus_{KgH \in K\backslash G/H} {}^gM \downarrow_H^G \downarrow_{{}^gH\cap K}^H \uparrow_{{}^gH\cap K}^G$$
$$= \bigoplus_{KgH \in K\backslash G/H} {}^gM \downarrow_{{}^gH\cap K}^G \uparrow_{{}^gH\cap K}^G .$$

Using the Krull–Schmidt property, M is a direct factor of ${}^gM \downarrow_{{}^gH\cap K}^G \uparrow_{{}^gH\cap K}^G$ for some g, and since K is minimal, there is $g \in G$ such that ${}^gH = K$. ☐

Remark 2.25. The statements of the above results should remain true when we replace this quite specific setting by a Mackey functor with values in the functor category between triangulated categories.

Lemma 2.26. *Let G be a finite group, and let k be a field of characteristic $p > 0$. Let M be an indecomposable object of $D^b(kG)$. If M is kH-projective, then each indecomposable direct factor of $H^n(M)$ for all $n \in \mathbb{N}$ is relatively H-projective.*

Proof. By Proposition 2.8 we see that M is relatively $D^b(kH)$-projective if and only if M is a direct factor of $L \uparrow_H^G$ for some L in $D^b(kH)$. Since $\uparrow_H^G$ is exact, also $H^n(L \uparrow_H^G)$ has a direct factor $H^n(M)$. However, $H^n(L \uparrow_H^G) = H^n(L) \uparrow_H^G$ and hence $H^n(M)$ is a direct factor of $H^n(L) \uparrow_H^G$. Hence, by Higman's lemma

[Zimmermann 2014, Proposition 2.1.15] from modular representation theory, each direct factor of $H^n(M)$ is relatively H-projective. □

3. Localising on triangulated subcategories

As in [Auslander and Kleiner 1994] we consider the situation of three triangulated categories with functors

$$\begin{tikzcd}\mathcal{D} \arrow[r, bend left, "S'"] & \mathcal{H} \arrow[l, bend left, "T'"] \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"]\end{tikzcd}$$

such that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \mathrm{id}_{\mathcal{H}} \to TS$ be the unit of the adjunction (S, T) and suppose that the unit is a split monomorphism. Hence

$$\mathrm{id}_{\mathcal{H}} \xrightarrow{\epsilon} TS \xrightarrow{\pi} U \xrightarrow{0} \mathrm{id}_{\mathcal{H}}[1]$$

is a distinguished triangle of functors. In particular, $TS = \mathrm{id}_{\mathcal{H}} \oplus U$.

Remark 3.1. Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories, and let $F : \mathcal{S} \to \mathcal{T}$ be a triangle functor. Then with the convention of Notation 1.1 for any full triangulated subcategory $\mathcal{U}$ of $\mathcal{T}$ the category $F^{-1}(\mathcal{U})$ is not necessarily triangulated. However, if $\mathcal{U}$ is in addition closed under direct summands, then this is true, as is shown in Proposition 3.2 below.

Proposition 3.2. *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories, and let $F : \mathcal{S} \to \mathcal{T}$ be a triangle functor. Then for any thick subcategory $\mathcal{U}$ of $\mathcal{T}$ the category $F^{-1}(\mathcal{U})$ is a triangulated subcategory of $\mathcal{S}$.*

Proof. Let X be an object of $\mathcal{S}$ such that $F(X)$ is a direct factor of the object U of $\mathcal{U}$. Hence, $F(X) \oplus U' = U$ for some object U' of $\mathcal{T}$. Since $\mathcal{U}$ is closed under direct factors, $F(X)$ and U' are actually already objects of $\mathcal{U}$. Let X_1 and X_2 be two objects of $\mathcal{S}$ such that $F(X_1)$ and $F(X_2)$ are objects of $\mathcal{U}$. If

$$X_1 \xrightarrow{\alpha} X_2 \to C(\alpha) \to X_1[1]$$

is a distinguished triangle in $\mathcal{S}$, since F is a triangle functor, also

$$FX_1 \xrightarrow{F\alpha} FX_2 \to FC(\alpha) \to FX_1[1]$$

is a distinguished triangle in $\mathcal{T}$, and hence $F(C(\alpha)) \simeq C(F(\alpha))$. Since $\mathcal{U}$ is triangulated $C(F(\alpha))$ is an object in $\mathcal{U}$, and since $\mathcal{U}$ is closed under isomorphisms, $F(C(\alpha))$ is an object of $\mathcal{U}$. Hence $C(\alpha)$ is an object of $F^{-1}(\mathcal{U})$. Therefore, $F^{-1}(\mathcal{U})$ is a triangulated subcategory of $\mathcal{S}$. □

Lemma 3.3. *Let $\mathcal{Y}$ be a full triangulated subcategory of $\mathcal{H}$. Then $\mathcal{Z} := (US')^{-1}(\mathcal{Y})$ satisfies $S'(\mathcal{Z}) = S'(\mathcal{D}) \cap U^{-1}(\mathcal{Y})$. Moreover, $\mathcal{Z}$ is triangulated if $\mathcal{Y}$ is thick.*

Proof. By definition $S'(\mathcal{Z})$ is the full subcategory of $\mathcal{H}$ formed by objects $S'M$ such that $US'M \in \mathrm{add}(\mathcal{Y})$. Hence $S'(\mathcal{Z})$ is contained in $S'(\mathcal{D}) \cap U^{-1}(\mathcal{Y})$. Moreover, an object X in $S'(\mathcal{D}) \cap U^{-1}(\mathcal{Y})$ is an object of the form $S'M$, since $X \in S'(\mathcal{D})$ and such that $US'M \in \mathrm{add}(\mathcal{Y})$ since $X \in U^{-1}(\mathcal{Y})$. The rest follows from Proposition 3.2. □

Remark 3.4. We remind the reader that we have two different localisation or quotient constructions of a triangulated category $\mathcal{T}$ by a triangulated subcategory $\mathcal{U}$ (see Notation 1.1).

- First we have the additive quotient, denoted traditionally $\mathcal{S}/\mathcal{U}$ having the same objects as $\mathcal{S}$ but we consider morphisms between two objects as residue classes of morphisms in $\mathcal{T}$ modulo those factoring through an object of $\mathcal{U}$.
- Second, the Verdier localisation [SGA 4½ 1977; Verdier 1996] which we denote by $\mathcal{S}_{\mathcal{U}}$. In the literature the Verdier localisation is often denoted by $\mathcal{S}/\mathcal{U}$. In order to distinguish from the additive quotient we decided to use the symbol $\mathcal{S}_{\mathcal{U}}$, contrary to the established convention in the literature.

Lemma 3.5. *Let $\mathcal{S}$ be a triangulated category, and let $\mathcal{U}$ be a thick subcategory of $\mathcal{S}$. Then there is a unique and natural functor $\mathcal{S}/\mathcal{U} \xrightarrow{L_{\mathcal{U}}} \mathcal{S}_{\mathcal{U}}$ making the diagram*

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl, "Q_{\mathcal{U}}"'] \arrow[dr, "V_{\mathcal{U}}"] & \\
\mathcal{S}/\mathcal{U} \arrow[rr, dashed, "L_{\mathcal{U}}"'] & & \mathcal{S}_{\mathcal{U}}
\end{tikzcd}$$

commutative. Here we denote $\mathcal{S} \xrightarrow{Q_{\mathcal{U}}} \mathcal{S}/\mathcal{U}$ and $\mathcal{S} \xrightarrow{V_{\mathcal{U}}} \mathcal{S}_{\mathcal{U}}$ the canonical functors given by the respective universal properties.

Proof. The proof is implicit in [Rickard 1989a, Proposition 1.3]. The statement follows from the well-known fact that for any additive category $\mathcal{A}$ and any additive functor $F : \mathcal{S} \to \mathcal{A}$ such that $F(U) = 0$ for any object U of $\mathcal{U}$, there is a unique additive functor $F^* : \mathcal{S}/\mathcal{U} \to \mathcal{A}$ with $F = F^* \circ Q_{\mathcal{U}}$. For $\mathcal{A} = \mathcal{S}_{\mathcal{U}}$, we observe that $F = V_{\mathcal{U}} : \mathcal{S} \to \mathcal{S}_{\mathcal{U}}$ is additive with $F(U) = 0$ for any object U of $\mathcal{U}$. Indeed, a morphism in the localisation becomes invertible if its cone is in $\mathcal{U}$. Hence, for an object U of $\mathcal{U}$ the cone of the zero morphism on U is in $\mathcal{U}$. Therefore the image $V_{\mathcal{U}}(0_U)$ of the zero morphism 0_U on U in the localisation is invertible in $\mathcal{S}_{\mathcal{U}}$. The only object with invertible zero endomorphism is the zero object in $\mathcal{S}_{\mathcal{U}}$. This proves the statement. □

Remark 3.6. We need to recall from Verdier [1996, Chapitre II, Sections 2.1, 2.2], or alternatively from the Stacks project [Stacks, Part 1, Chapter 13, Section 13.6], some properties of Verdier localisation. If $F : \mathcal{S} \to \mathcal{T}$ is a triangle functor between triangulated categories, then the full subcategory $\ker(F)$ of $\mathcal{S}$ generated by those objects X of $\mathcal{S}$ such that $F(X) = 0$ is thick. If $\mathcal{U}$ is a full triangulated subcategory of some triangulated category $\mathcal{T}$, then the Verdier localisation defined by inverting all morphisms f in $\mathcal{T}$ with cone in $\mathcal{U}$ is triangulated, and there is a canonical functor $V_{\mathcal{U}} : \mathcal{T} \to \mathcal{T}_{\mathcal{U}}$ with $\mathcal{U}$ is a full triangulated subcategory of $\ker(V_{\mathcal{U}})$. Moreover, $\ker(V_{\mathcal{U}})$ is thick, namely the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{U}$, the thickening $\mathrm{thick}(\mathcal{U})$.

Remark 3.7. We see that even if $\mathcal{Y}$ is a thick subcategory of the triangulated category $\mathcal{C}$ and if H is a triangle functor $\mathcal{C} \to \mathcal{D}$ for some triangulated category $\mathcal{D}$, then $H(\mathcal{Y})$ is triangulated, but is not thick anymore in general. The Verdier localisation of $\mathcal{D}$ at $H(\mathcal{Y})$ has good properties with respect to thick subcategories. Since $\ker(V_{H(\mathcal{Y})}) = \mathrm{thick}(H(\mathcal{Y}))$, we need to consider $\mathrm{thick}(H(\mathcal{Y}))$. Then

$$\ker(\mathcal{T} \xrightarrow{V_{\mathrm{thick}(H\mathcal{Y})}} \mathcal{T}_{\mathrm{thick}(H(\mathcal{Y}))}) = \mathrm{thick}(H(\mathcal{Y})) = \ker(\mathcal{T} \xrightarrow{V_{H\mathcal{Y}}} \mathcal{T}_{H\mathcal{Y}}).$$

Lemma 3.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories, let $\mathcal{Y}$ be a subcategory of $\mathcal{C}$, and let $H : \mathcal{C} \to \mathcal{D}$ be a triangle functor. Then H extends to a unique functor $\mathcal{C}/\mathcal{Y} \to \mathcal{D}_{\mathrm{thick}\,H(\mathcal{Y})}$, also denoted by H, such that $H \circ Q_{\mathcal{Y}} = V_{\mathrm{thick}\,H\mathcal{Y}} \circ H$, i.e., making the diagram*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "H"] \arrow[d, "Q_{\mathcal{Y}}"'] & & \mathcal{D} \arrow[dl, "Q_{H\mathcal{Y}}"'] \arrow[d, "V_{\mathrm{thick}\,H\mathcal{Y}}"] \\
\mathcal{C}/\mathcal{Y} \arrow[r, "H"] \arrow[rr, bend right, "H"'] & \mathcal{D}/H\mathcal{Y} \arrow[r] & \mathcal{D}_{\mathrm{thick}\,H\mathcal{Y}}
\end{tikzcd}$$

commutative. The functor $\mathcal{D}/H\mathcal{Y} \to \mathcal{D}/\mathrm{thick}\,H\mathcal{Y}$ combined with the functor $L_{\mathrm{thick}\,H\mathcal{Y}} : \mathcal{D}/\mathrm{thick}\,H\mathcal{Y} \to \mathcal{D}_{\mathrm{thick}\,H\mathcal{Y}}$ make the right triangle of the above diagram commutative.

Proof. Consider

$$\mathcal{C} \xrightarrow{H} \mathcal{D} \xrightarrow{V_{\mathrm{thick}\,H\mathcal{Y}}} \mathcal{D}_{\mathrm{thick}\,H\mathcal{Y}}.$$

Then for all objects X of $\mathcal{Y}$ we get $(V_{\mathrm{thick}\,H\mathcal{Y}} \circ H)(X) = 0$. Likewise consider

$$\mathcal{C} \xrightarrow{H} \mathcal{D} \xrightarrow{Q_{H\mathcal{Y}}} \mathcal{D}/H\mathcal{Y}.$$

Again, for all objects X of $\mathcal{Y}$ we get $(Q_{H\mathcal{Y}} \circ H)(X) = 0$. Hence there is a unique functor $\mathcal{C}/\mathcal{Y} \xrightarrow{H_1} \mathcal{D}_{\mathrm{thick}\,H(\mathcal{Y})}$ satisfying $H_1 \circ Q_{\mathcal{Y}} = V_{\mathrm{thick}\,H\mathcal{Y}} \circ H$ and a unique functor $\mathcal{C}/\mathcal{Y} \xrightarrow{H_2} \mathcal{D}/H(\mathcal{Y})$ satisfying $H_2 \circ Q_{\mathcal{Y}} = Q_{H\mathcal{Y}} \circ H$. Moreover, by the universal property of $\mathcal{D}/H\mathcal{Y}$ the functor H_1 factors through H_2 and through $L_{\mathrm{thick}\,H\mathcal{Y}}$. □

Lemma 3.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangle functor between triangulated categories, let $\mathcal{X}$ be a full subcategory of $\mathcal{C}$, and let $\mathcal{Y}$ be a full subcategory of $\mathcal{D}$. If $F(\mathcal{X}) \subseteq \mathcal{Y}$,* *then there exists a unique additive functor* $\mathcal{C}_{\mathrm{thick}\,\mathcal{X}} \to \mathcal{D}_{\mathrm{thick}\,\mathcal{Y}}$, *still denoted by* F, *making the following diagram commutative.*

$$\begin{tikzcd}
\mathcal{C}/\mathcal{X} \arrow[r] \arrow[d, "F"'] & \mathcal{C}/\mathrm{thick}\,\mathcal{X} \arrow[r, "L_{\mathrm{thick}\,\mathcal{X}}"] \arrow[d, "F"'] & \mathcal{C}_{\mathrm{thick}\,\mathcal{X}} \arrow[d, "F"'] \\
\mathcal{D}/\mathcal{Y} \arrow[r] & \mathcal{D}/\mathrm{thick}\,\mathcal{Y} \arrow[r, "L_{\mathrm{thick}\,\mathcal{Y}}"'] & \mathcal{D}_{\mathrm{thick}\,\mathcal{Y}}
\end{tikzcd}$$

Proof. Since F is a triangle functor,

$$F(\mathrm{thick}\,\mathcal{X}) \subseteq \mathrm{thick}\,F\mathcal{X} \subseteq \mathrm{thick}\,\mathcal{Y}$$

and the left square is commutative. By Lemma 3.8 the right square is commutative as well. □

For a triangulated category $\mathcal{T}$ and a full subcategory $\mathcal{X}$ there is a natural functor $\mathcal{T}/\mathcal{X} \to \mathcal{T}/\,\mathrm{thick}\,\mathcal{X}$. For simplicity the composition

$$\mathcal{T}/\mathcal{X} \to \mathcal{T}/\,\mathrm{thick}\,\mathcal{X} \xrightarrow{L_{\mathrm{thick}\,\mathcal{X}}} \mathcal{T}_{\mathrm{thick}\,\mathcal{X}}$$

is also denote by $L_{\mathrm{thick}\,\mathcal{X}}$.

Proposition 3.10. *Let* $\mathcal{D}$, $\mathcal{H}$, $\mathcal{G}$ *be three triangulated categories and triangle functors* S, S', T, T'

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "S'"] & \mathcal{H} \arrow[l, bend left, "T'"] \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"]
\end{tikzcd}$$

so that (S,T) *and* (S',T') *are adjoint pairs. Let* $\epsilon : \mathrm{id}_{\mathcal{H}} \to TS$ *be the unit of the adjunction* (S,T). *Assume that there is an endofunctor* U *of* $\mathcal{H}$ *such that* $TS = \mathrm{id}_{\mathcal{H}} \oplus U$, *denote by* $p_1 : TS \to \mathrm{id}_{\mathcal{H}}$ *the projection, and suppose that* $p_1 \circ \epsilon$ *is an isomorphism.*

Let $\mathcal{Y}$ *be a thick subcategory of* $\mathcal{H}$, *and suppose that each object of* $TSS'T'\mathcal{Y}$ *is a direct factor of an object of* $\mathcal{Y}$. *Then* S *and* T *induce triangle functors*

$$\mathcal{H}_{\mathrm{thick}\,S'T'\mathcal{Y}} \xrightarrow{S} \mathcal{G}_{\mathrm{thick}\,SS'T'\mathcal{Y}} \quad \text{and} \quad \mathcal{G}_{\mathrm{thick}\,SS'T'\mathcal{Y}} \xrightarrow{T} \mathcal{H}_{\mathcal{Y}}$$

making the diagram

$$\begin{tikzcd}
\mathcal{H}/\mathrm{thick}\,S'T'\mathcal{Y} \arrow[r, "S"] \arrow[d, "L_{\mathrm{thick}\,S'T'\mathcal{Y}}"'] & \mathcal{G}/\mathrm{thick}\,SS'T'\mathcal{Y} \arrow[r, "T"] \arrow[d, "L_{\mathrm{thick}\,SS'T'\mathcal{Y}}"] & \mathcal{H}/\mathcal{Y} \arrow[d, "L_{\mathcal{Y}}"] \\
\mathcal{H}_{\mathrm{thick}\,S'T'\mathcal{Y}} \arrow[r, "S"] & \mathcal{G}_{\mathrm{thick}\,SS'T'\mathcal{Y}} \arrow[r, "T"] & \mathcal{H}_{\mathcal{Y}}
\end{tikzcd}$$

commutative.

Proof. The existence of the functors in the left square and the commutativity of the left square follow from Lemma 3.8. From Lemma 3.8 we get natural functors giving a commutative diagram

$$\begin{tikzcd}
\mathcal{H}/S'T'\mathcal{Y} \arrow[r, "S"] \arrow[d, "L_{\text{thick}\,S'T'\mathcal{Y}}"'] & \mathcal{G}/SS'T'\mathcal{Y} \arrow[r, "T"] \arrow[d, "L_{\text{thick}\,SS'T'\mathcal{Y}}"] & \mathcal{H}/TSS'T'\mathcal{Y} \arrow[d, "L_{\text{thick}\,SS'T'\mathcal{Y}}"] \\
\mathcal{H}_{\text{thick}\,S'T'\mathcal{Y}} \arrow[r, "S"] & \mathcal{G}_{\text{thick}\,SS'T'\mathcal{Y}} \arrow[r, "T"] & \mathcal{H}_{\text{thick}\,TSS'T'\mathcal{Y}}
\end{tikzcd}$$

For $\mathcal{X} = SS'T'\mathcal{Y}$, Theorem 1.2 shows that $T(\mathcal{X}) \subseteq \mathcal{Y}$. Using Lemma 3.9 and the fact that $\mathcal{Y}$ is thick, and therefore that $\text{thick}\,\mathcal{Y} = \mathcal{Y}$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{H}/S'T'\mathcal{Y} \arrow[r, "S"] \arrow[d, "L_{\text{thick}\,S'T'\mathcal{Y}}"'] & \mathcal{G}/SS'T'\mathcal{Y} \arrow[r, "T"] \arrow[d, "L_{\text{thick}\,SS'T'\mathcal{Y}}"] & \mathcal{H}/\mathcal{Y} \arrow[d, "L_{\mathcal{Y}}"] \\
\mathcal{H}_{\text{thick}\,S'T'\mathcal{Y}} \arrow[r, "S"] & \mathcal{G}_{\text{thick}\,SS'T'\mathcal{Y}} \arrow[r, "T"] & \mathcal{H}_{\mathcal{Y}}
\end{tikzcd}$$

as requested.

The fact that

$$\mathcal{H}_{\text{thick}\,S'T'\mathcal{Y}} \xrightarrow{S} \mathcal{G}_{\text{thick}\,SS'T'\mathcal{Y}} \quad \text{and} \quad \mathcal{G}_{\text{thick}\,SS'T'\mathcal{Y}} \xrightarrow{T} \mathcal{H}_{\mathcal{Y}}$$

are triangle functors comes from the universal property of the Verdier localisation [1996, Chapitre II, Théorème 2.2.6]. □

Corollary 3.11. *Let $\mathcal{D}$, $\mathcal{H}$, $\mathcal{G}$ be three triangulated categories and triangle functors S, S', T, T'*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "S'"] & \mathcal{H} \arrow[l, bend left, "T'"] \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"]
\end{tikzcd}$$

so that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \mathrm{id}_{\mathcal{H}} \to TS$ be the unit of the adjunction (S, T). Assume that there is an endofunctor U of $\mathcal{H}$ such that $TS = \mathrm{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \to \mathrm{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism.

Let $\mathcal{Y}$ be a thick subcategory of $\mathcal{H}$, and suppose that each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}_{\text{thick}\,S'T'\mathcal{Y}} \arrow[r, "S"] \arrow[d, "\text{can}"'] & \mathcal{G}_{\text{thick}\,SS'T'\mathcal{Y}} \arrow[dl, "T"] & & \\
\mathcal{H}_{\mathcal{Y}} & & & \\
& & \mathcal{H}/S'T'\mathcal{Y} \arrow[r, "S"] \arrow[d, "\text{can}"'] \arrow[uull, "L_{\text{thick}\,S'T'\mathcal{Y}}"] & \mathcal{G}/SS'T'\mathcal{Y} \arrow[dl, "T"] \arrow[uull, "L_{\text{thick}\,SS'T'\mathcal{Y}}"'] \\
& & \mathcal{H}/\mathcal{Y} \arrow[uull, "L_{\mathcal{Y}}"] &
\end{tikzcd}$$

Proof. Indeed, since each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$, each object of $S'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$. Hence, there is a natural functor can as indicated. The rest of the statement is an immediate consequence of Proposition 3.10. □

Corollary 3.12. *Let $\mathcal{D}$, $\mathcal{H}$, $\mathcal{G}$ be three triangulated categories and triangle functors S, S', T, T'*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "S'"] & \mathcal{H} \arrow[l, bend left, "T'"] \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"]
\end{tikzcd}$$

so that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \mathrm{id}_{\mathcal{H}} \to TS$ be the unit of the adjunction (S, T). Assume that there is an endofunctor U of $\mathcal{H}$ such that $TS = \mathrm{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \to \mathrm{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism.

Let $\mathcal{Y}$ be a thick subcategory of $\mathcal{H}$, put $\mathcal{Z} := (US')^{-1}(\mathcal{Y})$, and suppose that each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$. Then the restriction of S to the subcategory $\operatorname{add} S'\mathcal{Z}/S'T'\mathcal{Y}$ *and the restriction of T to the subcategory* $\operatorname{add} SS'\mathcal{Z}/SS'T'\mathcal{Y}$ *are equivalences and gives a commutative diagram*

$$\begin{tikzcd}
\operatorname{thick} S'\mathcal{Z}_{\operatorname{thick} S'T'\mathcal{Y}} \arrow[r, "S"] \arrow[d, "\mathrm{can}"'] & \operatorname{thick} SS'\mathcal{Z}_{\operatorname{thick} SS'T'\mathcal{Y}} \arrow[dl, "T"] & & \\
\operatorname{thick} S'\mathcal{Z}_{\mathcal{Y}} & & & \\
& & \operatorname{add} S'\mathcal{Z}/S'T'\mathcal{Y} \arrow[r, "S"] \arrow[d, "\mathrm{can}"'] \arrow[uull, "L_{\operatorname{thick} S'T'\mathcal{Y}}"] & \operatorname{add} SS'\mathcal{Z}/SS'T'\mathcal{Y} \arrow[dl, "T"] \arrow[uull, "L_{\operatorname{thick} SS'T'\mathcal{Y}}"] \\
& & \operatorname{add} TSS'\mathcal{Z}/\mathcal{Y} \arrow[uull, "L_{\mathcal{Y}}"] &
\end{tikzcd}$$

where the lower triangle consists of equivalences.

Proof. The fact that the lower triangle exists and is commutative follows from Theorem 1.2. Since for any subcategory $\mathcal{X}$ of $\mathcal{T}$ we get that $\operatorname{add}\mathcal{X}$ is a full subcategory of $\operatorname{thick}\mathcal{X}$, we have a commutative diagram

$$\begin{tikzcd}
(\operatorname{thick} S'\mathcal{Z})/S'T'\mathcal{Y} \arrow[r, "S"] \arrow[d, "\mathrm{can}"'] & (\operatorname{thick} SS'\mathcal{Z})/SS'T'\mathcal{Y} \arrow[dl, "T"] & & \\
(\operatorname{thick} S'\mathcal{Z})/\mathcal{Y} & & & \\
& & \operatorname{add} S'\mathcal{Z}/S'T'\mathcal{Y} \arrow[r, "S"] \arrow[d, "\mathrm{can}"'] \arrow[uull] & \operatorname{add} SS'\mathcal{Z}/SS'T'\mathcal{Y} \arrow[dl, "T"] \arrow[uull] \\
& & \operatorname{add} TSS'\mathcal{Z}/\mathcal{Y} \arrow[uull] &
\end{tikzcd}$$

By Lemma 3.8 we obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{thick} S'\mathcal{Z}_{\operatorname{thick} S'T'\mathcal{Y}} \arrow[r, "S"] \arrow[d, "\mathrm{can}"'] & \operatorname{thick} SS'\mathcal{Z}_{\operatorname{thick} SS'T'\mathcal{Y}} \arrow[dl, "T"] & & \\
\operatorname{thick} S'\mathcal{Z}_{\mathcal{Y}} & & & \\
& & (\operatorname{thick} S'\mathcal{Z})/S'T'\mathcal{Y} \arrow[r, "S"] \arrow[d, "\mathrm{can}"'] \arrow[uull, "L_{\operatorname{thick} S'T'\mathcal{Y}}"] & (\operatorname{thick} SS'\mathcal{Z})/SS'T'\mathcal{Y} \arrow[dl, "T"] \arrow[uull, "L_{\operatorname{thick} SS'T'\mathcal{Y}}"'] \\
& & (\operatorname{thick} S'\mathcal{Z})/\mathcal{Y} \arrow[uull, "L_{\mathcal{Y}}"] &
\end{tikzcd}$$

Composition of the two diagrams yields the statement. ☐

Proposition 3.13. *Let $\mathcal{T}$ and $\mathcal{U}$ be two triangulated categories, let $F : \mathcal{T} \to \mathcal{U}$ be a triangle functor, let $\mathcal{X}$ be a full additive subcategory of $\mathcal{T}$ and let $\mathcal{Y}$ be a full additive subcategory of $\mathcal{X}$. Then the restriction of F to $\mathcal{X}/\mathcal{Y} \xrightarrow{F_{\mathcal{X}}} (\operatorname{add} F\mathcal{X})/F\mathcal{Y}$ extends to a functor $(\operatorname{thick} \mathcal{X})/\mathcal{Y} \xrightarrow{F_{\operatorname{thick}\mathcal{X}}} (\operatorname{thick} F\mathcal{X})/F\mathcal{Y}$ such that $F_{\operatorname{thick}\mathcal{X}}$ coincides with $F_{\mathcal{X}}$ on the subcategory $\mathcal{X}/\mathcal{Y}$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be full subcategories of a triangulated category $\mathcal{V}$, then as in [Beilinson et al. 1982] we denote by $\mathcal{A} * \mathcal{B}$ the full subcategory of $\mathcal{V}$ generated by $C(t)[-1]$ where

$$A \xrightarrow{f} C(t)[-1] \xrightarrow{s} B \xrightarrow{t} A[1]$$

is a distinguished triangle, and where A is an object of $\mathcal{A}$, B is an object of $\mathcal{B}$, and $t \in \mathcal{T}(B, A[1])$. Since $F : \mathcal{T} \to \mathcal{U}$ is a triangle functor, F sends distinguished triangles to distinguished triangle. Therefore $F(\mathcal{A} * \mathcal{B})$ is a subcategory of $F(\mathcal{A}) * F(\mathcal{B})$. Hence F induces a functor

$$\mathcal{A} * \mathcal{B} \xrightarrow{F} (F\mathcal{A}) * (F\mathcal{B}).$$

Let $X_1 \in \operatorname{add} F\mathcal{A}$ and $X_2 \in \operatorname{add} F\mathcal{B}$. Then for any $t \in \mathcal{U}(X_2, X_1[1])$ we get

$$C(t)[-1] \in \operatorname{add}(F(\mathcal{A}) * F(\mathcal{B})).$$

Indeed, denote by

$$X_1 \xrightarrow{f} C(t)[-1] \xrightarrow{s} X_2 \xrightarrow{t} X_1[1]$$

the distinguished triangle given by t. Let X_1' and X_2' be objects of $\mathcal{U}$ such that $X_1 \oplus X_1' \in F(\mathcal{A})$ and $X_2 \oplus X_2' \in F(\mathcal{B})$. Then

$$X_1 \oplus X_1' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & \mathrm{id}_{X_1'} \\ 0 & 0 \end{pmatrix}} C(t)[-1] \oplus X_1' \oplus X_2' \xrightarrow{\begin{pmatrix} s & 0 & 0 \\ 0 & 0 & \mathrm{id}_{X_2'} \end{pmatrix}} X_2 \oplus X_2' \xrightarrow{\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}} (X_1 \oplus X_1')[1]$$

is a distinguished triangle. Hence $C(t)[-1] \in \mathrm{add}(F(\mathcal{A}) * F(\mathcal{B}))$. This shows

$$\mathrm{add}(F(\mathcal{A})) * \mathrm{add}(F(\mathcal{B})) \subseteq \mathrm{add}(F(\mathcal{A}) * F(\mathcal{B})),$$

and therefore

$$\mathrm{add}\big(\mathrm{add}(F(\mathcal{A})) * \mathrm{add}(F(\mathcal{B}))\big) = \mathrm{add}(F(\mathcal{A}) * F(\mathcal{B})).$$

If we define $(\mathcal{Z})_n := (\mathcal{Z})_{n-1} * \mathcal{Z}$ for any subcategory $\mathcal{Z}$ of $\mathcal{U}$, and $(\mathcal{Z})_1 := \mathcal{Z}$, then

$$\mathrm{thick}(F(\mathcal{X})) = \bigcup_{n\in\mathbb{N}} \mathrm{add}((\mathrm{add}(F(\mathcal{X})))_n) = \bigcup_{n\in\mathbb{N}} \mathrm{add}((F(\mathcal{X}))_n) = \mathrm{add}\Big(\bigcup_{n\in\mathbb{N}} (F(\mathcal{X}))_n\Big).$$

Now,

$$\mathrm{thick}\,\mathcal{X} = \bigcup_{n\in\mathbb{N}} \mathrm{add}\,\mathcal{X}_n = \mathrm{add}\Big(\bigcup_{n\in\mathbb{N}} \mathcal{X}_n\Big)$$

and $F(\mathcal{X}_n) \subseteq (F\mathcal{X})_n$. Hence

$$\begin{aligned} F(\mathrm{thick}\,\mathcal{X})/F\mathcal{Y} &= F\Big(\mathrm{add}\Big(\bigcup_{n\in\mathbb{N}} \mathcal{X}_n\Big)\Big)/F\mathcal{Y} \subseteq \mathrm{add}\Big(\bigcup_{n\in\mathbb{N}} F(\mathcal{X}_n)\Big)/F\mathcal{Y} \\ &\subseteq \mathrm{add}\Big(\bigcup_{n\in\mathbb{N}} (F(\mathcal{X})_n)\Big)/F\mathcal{Y} = \mathrm{thick}(F(\mathcal{X}))/F\mathcal{Y} \end{aligned}$$

Therefore F extends to a functor

$$\mathrm{thick}(\mathcal{X})/\mathcal{Y} \xrightarrow{F_{\mathrm{thick}\,\mathcal{X}}} \mathrm{thick}(F(\mathcal{X}))/F\mathcal{Y}.$$

By construction the restriction of $F_{\mathrm{thick}\,\mathcal{X}}$ to $\mathcal{X}/\mathcal{Y}$ coincides with $F_{\mathcal{X}}$. ☐

Remark 3.14. If in Proposition 3.13 the functor F induces an equivalence $\mathcal{X}/\mathcal{Y} \to (\mathrm{add}\,F\mathcal{X})/F\mathcal{Y}$, then there is no reason why this should imply an equivalence

$$\mathrm{thick}(\mathcal{X})/\mathcal{Y} \to \mathrm{thick}(F(\mathcal{X}))/F\mathcal{Y}.$$

Lemma 3.15. *Let $\mathcal{Y}$ be a subcategory of $\mathcal{X}$ admitting finite direct sums. Then the natural projection $\mathcal{X}/\mathcal{Y} \to \mathcal{X}/(\mathrm{add}\,\mathcal{Y})$ is an equivalence of categories.*

Proof. Since $\mathcal{Y}$ is a subcategory of $\mathrm{add}\,\mathcal{Y}$, if a morphism f factors through an object of $\mathcal{Y}$, it factors also through an object of $\mathrm{add}\,\mathcal{Y}$. Hence, the natural projection is well-defined and full. If f factors through an object X of $\mathrm{add}\,\mathcal{Y}$, then there is an object X' of $\mathrm{add}\,\mathcal{Y}$, such that $X \oplus X'$ is an object of $\mathcal{Y}$. Extending by the zero morphism to and from X', it follows that f factors also through the object $X \oplus X'$ of $\mathcal{Y}$. This shows that the natural projection is faithful as well.

From the above it also follows that the natural projection is dense, since the objects of both quotient categories coincide, and the natural projection is the identity on objects. ☐

Proposition 3.16. *Let $\mathcal{T}$ and $\mathcal{U}$ be triangulated categories, let $\mathcal{Y}$ be a subcategory of $\mathcal{T}$, and let $F:\mathcal{T}\to\mathcal{U}$ be a triangle functor. Suppose that F induces an equivalence*

$$F_Q:\mathcal{T}/(\operatorname{thick}\mathcal{Y})\to\mathcal{U}/(\operatorname{thick}F(\mathcal{Y})).$$

Then F induces a dense and full triangle functor

$$F_V:\mathcal{T}_{(\operatorname{thick}\mathcal{Y})}\to\mathcal{U}_{(\operatorname{thick}F(\mathcal{Y}))}.$$

If in addition $F(\operatorname{thick}\mathcal{Y})$ is thick in $\mathcal{U}$ then F_V is a triangle equivalence.

Proof. The functor F_V exists by the universal property of the Verdier localisation [1996, Chapitre II, Corollaire 2.2.11.c].

We shall now show that F_V is dense. The objects of $\mathcal{U}/(\operatorname{thick}F(\mathcal{Y}))$ coincide with the objects of $\mathcal{U}_{(\operatorname{thick}F(\mathcal{Y}))}$, since they both coincide with the objects of $\mathcal{U}$. By hypothesis, for every object U of $\mathcal{U}$ there is an object T of $\mathcal{T}$, and $f\in\mathcal{U}(FT,U)$ as well as $g\in\mathcal{U}(U,FT)$ such that $g\circ f-\mathrm{id}_{FT}$ factors through an object Y' of $\operatorname{thick}(F\mathcal{Y})$ and $f\circ g-\mathrm{id}_U$ factors through an object Y of $\operatorname{thick}(F\mathcal{Y})$. Hence, applying $L_{\operatorname{thick}F\mathcal{Y}}$ to these equations, and observing that $L_{\operatorname{thick}F\mathcal{Y}}(Y)=0$, respectively $L_{\operatorname{thick}F\mathcal{Y}}(Y')=0$ for all objects Y in $\operatorname{thick}(F\mathcal{Y})$, respectively all objects Y' in $\operatorname{thick}(F\mathcal{Y})$, we get that the image of f in the Verdier localisation is an isomorphism. Hence F_V is dense.

We will show now that F_V is full.

First step: Let $f\in\mathcal{U}(FZ,FX)$. Since F_Q is full, there is $f'\in\mathcal{T}(Z,X)$ such that $f-Ff'$ factors through an object M of $\operatorname{thick}(F\mathcal{Y})$. Hence there is $g\in\mathcal{U}(M,X)$ and $h\in\mathcal{U}(Z,M)$ with $f-Ff'=g\circ h$ in $\mathcal{U}$. We denote by $(1,f)$ the morphism represented by the diagram $FZ\xrightarrow{\mathrm{id}_{FZ}}FZ\xrightarrow{f}FX$. Then $(1,f-Ff')=(1,g)\circ(1,h)$ in $\mathcal{U}_{(\operatorname{thick}F(\mathcal{Y}))}$. Since $M\simeq 0$ in $\mathcal{U}_{(\operatorname{thick}F(\mathcal{Y}))}$, we get

$$(1,f)-(1,Ff')=(1,f-Ff')=0$$

in $\mathcal{U}_{(\operatorname{thick}F(\mathcal{Y}))}$ and therefore

$$(1,f)=(1,Ff')=F_V(1,f').$$

Hence $f=F_Qf'$ in $\mathcal{U}/\operatorname{thick}F\mathcal{Y}$ implies $(1,f)=F_V(1,f')$ in $\mathcal{U}_{(\operatorname{thick}F(\mathcal{Y}))}$.

Second step: Let $FX\xleftarrow{s}\widehat{Z}\xrightarrow{f}FY$ represent a morphism in $\mathcal{U}_{(\operatorname{thick}F(\mathcal{Y}))}(F_VX,F_VY)$. Since F_V is dense, we may suppose that $\widehat{Z}=FZ$ for some object Z of $\mathcal{U}$. Then $s=F_Qs'$ for some $s'\in\mathcal{T}(Z,X)$, and $f=F_Qf'$ for some $f'\in\mathcal{T}(Z,Y)$, giving $(1,s)=F_V(1,s')$, and $(1,f)=F_V(1,f')$ by the first step. Now,

$$F(\operatorname{cone}(s'))\simeq\operatorname{cone}(F(s'))\simeq\operatorname{cone}(s)\in\operatorname{thick}(F(\mathcal{Y}))$$

and therefore $F(\operatorname{cone}(s'))=0$ in $\mathcal{U}/(\operatorname{thick}F\mathcal{Y})$. Since F induces an equivalence

$$\mathcal{U}/(\operatorname{thick}F\mathcal{Y})\simeq\mathcal{T}/(\operatorname{thick}\mathcal{Y}),$$

we get $\mathrm{cone}(s') = 0$ in $\mathcal{T}/(\mathrm{thick}\,\mathcal{Y})$, which shows that $\mathrm{cone}(s') \in \mathrm{thick}\,\mathcal{Y}$. Hence $X \xleftarrow{s'} Z \xrightarrow{f'} Y$ maps to $FX \xleftarrow{s} FZ \xrightarrow{f} FY$. Therefore F_V is full.

We now assume that in addition $F(\mathrm{thick}\,\mathcal{Y})$ is thick in $\mathcal{U}$.

We need to show that F_V is faithful. Since $F(\mathrm{thick}\,\mathcal{Y})$ is thick in $\mathcal{U}$, we get $\mathrm{thick}(F(\mathcal{Y})) = F(\mathrm{thick}\,\mathcal{Y})$. Hence, using the notation from [Ringel and Zhang 2015], and using that F_Q is an equivalence, $\ker(F_V) = 0$. By a result of Rickard [1989b, p. 446, first paragraph] or Ringel and Zhang [2015, Propositions 3.1, 3.3, Theorem 1.1] we see that F_V is faithful. This finishes the proof. ☐

Theorem 3.17 (Green correspondence for triangulated categories). *Let $\mathcal{D}$, $\mathcal{H}$, $\mathcal{G}$ be three triangulated categories and let S, S', T, T' be triangle functors*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left, "S'"] & \mathcal{H} \arrow[l, bend left, "T'"] \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"]
\end{tikzcd}$$

such that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \mathrm{id}_{\mathcal{H}} \to TS$ be the unit of the adjunction (S, T). Assume that there is an endofunctor U of $\mathcal{H}$ such that $TS = \mathrm{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \to \mathrm{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism.

Let $\mathcal{Y}$ be a thick subcategory of $\mathcal{H}$, put $\mathcal{Z} := (US')^{-1}(\mathcal{Y})$, and suppose that each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$.

(1) *Then S and T induce triangle functors S_Z and T_Z fitting into the commutative diagram*

$$\begin{tikzcd}
(\mathrm{thick}(S'\mathcal{Z}))_{(\mathrm{thick}(S'T'\mathcal{Y}))} \arrow[r, "S_Z"] \arrow[d, "\mathrm{can}"'] & (\mathrm{thick}(SS'\mathcal{Z}))_{(\mathrm{thick}(SS'T'\mathcal{Y}))} \arrow[dl, "T_Z"] \\
(\mathrm{thick}(S'\mathcal{Z}))_{\mathcal{Y}} &
\end{tikzcd}$$

of Verdier localisations.

(2) *There is an additive functor S_{thick}, induced by S, and an additive functor T_{thick} induced by T, making the diagram*

$$\begin{tikzcd}
(S'\mathcal{Z})/(S'T'\mathcal{Y}) \arrow[r, "\pi_1"] \arrow[d, "S"] & (\mathrm{thick}(S'\mathcal{Z}))/\mathrm{thick}(S'T'\mathcal{Y}) \arrow[d, "S_{\mathrm{thick}}"] \\
(SS'\mathcal{Z})/(SS'T'\mathcal{Y}) \arrow[r, "\pi_2"] \arrow[d, "T"] & (\mathrm{thick}(SS'\mathcal{Z}))/\mathrm{thick}(SS'T'\mathcal{Y}) \arrow[d, "T_{\mathrm{thick}}"] \\
S'\mathcal{Z}/\mathcal{Y} \arrow[r, "\pi_3"] & \mathrm{thick}(S'\mathcal{Z})/\,\mathrm{thick}(\mathcal{Y})
\end{tikzcd}$$

commutative. Moreover, the restriction to the respective images of π_1, π_2, and π_3 of functors S_{thick} and T_{thick} on the right is an equivalence.

(3) *S and T induce equivalences S_L and T_L of additive categories fitting into the commutative diagram*

$$\begin{tikzcd}
(\mathrm{thick}(S'\mathcal{Z}))_{(\mathrm{thick}(S'T'\mathcal{Y}))} \arrow[r, "S_Z"] \arrow[ddd, bend right=60, "\mathrm{can}"'] & (\mathrm{thick}(SS'\mathcal{Z}))_{(\mathrm{thick}(SS'T'\mathcal{Y}))} \arrow[dddl, bend left, "T_Z"] \\
L_{S'T'\mathcal{Y}}((S'\mathcal{Z})/(S'T'\mathcal{Y})) \arrow[u, hook] \arrow[r, "S_L"] \arrow[d, "\mathrm{can}"'] & L_{SS'T'\mathcal{Y}}((SS'\mathcal{Z})/(SS'T'\mathcal{Y})) \arrow[u, hook] \arrow[dl, "T_L"] \\
L_{\mathcal{Y}}((S'\mathcal{Z})/\mathcal{Y}) \arrow[d, hook] & \\
(\mathrm{thick}(S'\mathcal{Z}))_{\mathcal{Y}} &
\end{tikzcd}$$

where the outer triangle consists of triangulated categories and triangle functors, and the inner triangle are full additive subcategories.

(4) *If S and T induce equivalences of additive categories*

$$\begin{tikzcd}
(\mathrm{thick}(S'\mathcal{Z}))/\mathrm{thick}(S'T'\mathcal{Y}) \arrow[r, "S_{\mathrm{thick}}"] \arrow[d, "\mathrm{can}"'] & (\mathrm{thick}(SS'\mathcal{Z}))/\mathrm{thick}(SS'T'\mathcal{Y}) \arrow[dl, "T_{\mathrm{thick}}"] \\
(\mathrm{thick}(S'\mathcal{Z}))/\mathrm{thick}\,\mathcal{Y} &
\end{tikzcd}$$

then the restriction S_Z of S to the triangulated category $(\mathrm{thick}(S'\mathcal{Z}))_{\mathrm{thick}(S'T'\mathcal{Y})}$ *and the restriction T_Z of T to the triangulated category* $(\mathrm{thick}(SS'\mathcal{Z}))_{\mathrm{thick}(SS'T'\mathcal{Y})}$ *are equivalences of triangulated categories, making the diagram*

$$\begin{tikzcd}
(\mathrm{thick}(S'\mathcal{Z}))_{(\mathrm{thick}(S'T'\mathcal{Y}))} \arrow[r, "S_Z"] \arrow[d, "\mathrm{can}"'] & (\mathrm{thick}(SS'\mathcal{Z}))_{(\mathrm{thick}(SS'T'\mathcal{Y}))} \arrow[dl, "T_Z"] \\
(\mathrm{thick}(S'\mathcal{Z}))_{\mathcal{Y}} &
\end{tikzcd}$$

commutative.

Proof. We first recall from Lemma 3.3 that $\mathcal{Z}$ is triangulated. By Corollary 3.12 the functors coming from Theorem 1.2 extend to functors on the localisations. Now S and T are equivalences on the additive quotient constructions. Using Propositions 3.10 and 3.13, the functors extend to triangle functors

$$\begin{tikzcd}
(\mathrm{thick}(S'\mathcal{Z}))_{(\mathrm{thick}(S'T'\mathcal{Y}))} \arrow[r, "S_Z"] \arrow[d, "\mathrm{can}"'] & (\mathrm{thick}(SS'\mathcal{Z}))_{(\mathrm{thick}(SS'T'\mathcal{Y}))} \arrow[dl, "T_Z"] \\
(\mathrm{thick}(S'\mathcal{Z}))_{(\mathcal{Y})} &
\end{tikzcd}$$

The functors S and T are triangle functors on the ambient categories, and hence they induce functors S_{thick} and T_{thick} as required.

Since Theorem 1.2 shows that S is an equivalence with quasi-inverse T on the above subcategories, the functor T_{thick} is also a quasi-inverse to S_{thick} on the images under π_1, π_2 and π_3. Corollary 3.12 shows item (3)enumz.

Suppose now that S and T induce equivalences

$$\begin{tikzcd}
(\mathrm{thick}(S'\mathcal{Z}))/(\mathrm{thick}(S'T'\mathcal{Y})) \arrow[r, "S_{\mathrm{thick}}"] \arrow[d, "\mathrm{can}"'] & (\mathrm{thick}(SS'\mathcal{Z}))/(\mathrm{thick}(SS'T'\mathcal{Y})) \arrow[dl, "T_{\mathrm{thick}}"] \\
(\mathrm{thick}(S'\mathcal{Z}))/\mathrm{thick}\,\mathcal{Y} &
\end{tikzcd}$$

Since by hypothesis each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$, the right vertical functor can is the identity. The restriction of the functors S and T in the statement of item (4)enumz are full and dense by Proposition 3.16. Since their composition TS is the identity, the functors are also faithful. The statement follows. □

Remark 3.18. In Theorem 3.17 (1)enumz and (3)enumz the functor T maps from the localisation at the thick subcategory of images under S to the localisation at the thick subcategory of images under T. By Proposition 2.10 we get a functor from the localisation at the thick subcategory generated by T-relative injective objects to the localisation at the thick subcategory generated by S-relative projective objects.

Remark 3.19. Consider the special situation when G is a finite group and k is a field of characteristic $p > 0$. Then, following Carlson, Peng and Wheeler [Carlson et al. 1998] the classical Green correspondence is an equivalence of full additive subcategories of triangulated categories.

More precisely, let D be a p-subgroup of G and let H be a subgroup of G containing $N_G(D)$, the normaliser of D in G.

Consider $\mathcal{G} = kG - \underline{\mathrm{mod}}$, $\mathcal{H} = kN_G(D) - \underline{\mathrm{mod}}$, and $\mathcal{D} = kD - \underline{\mathrm{mod}}$, the stable categories of kG-modules, kH-modules, and kD-modules. Here the stable categories are taken modulo morphisms factoring through projective modules. Let

$$S = kG \otimes_{kH} - = \mathrm{ind}_H^G : kH - \underline{\mathrm{mod}} \to kG - \underline{\mathrm{mod}}$$

and

$$S' = kH \otimes_{kD} - = \mathrm{ind}_D^H : kD - \underline{\mathrm{mod}} \to kH - \underline{\mathrm{mod}}$$

be the induction functors. These have left and right adjoints, namely the restriction

$$T := \mathrm{Hom}_{kH}(kG, -) = \mathrm{res}_H^G : kG - \underline{\mathrm{mod}} \to kH - \underline{\mathrm{mod}}$$

is left and right adjoint to S. Similarly,

$$T' := \mathrm{Hom}_{kH}(kG, -) = \mathrm{res}_H^G : kH - \underline{\mathrm{mod}} \to kD - \underline{\mathrm{mod}}$$

is left and right adjoint to S'.

Since group algebras are symmetric, following Remark 2.19 the stable categories $\mathcal{G} = kG - \underline{\text{mod}}$, $\mathcal{H} = kN_G(D) - \underline{\text{mod}}$, and $\mathcal{D} = kD - \underline{\text{mod}}$ are triangulated and moreover, the functors $\text{ind}_H^G, \text{ind}_D^H, \text{res}_H^G, \text{res}_D^H$ come from exact functors of the corresponding module categories, and hence are triangle functors. Further,

$$\text{res}_H^G \,\text{ind}_H^G = \text{id}_{kH-\text{mod}} \oplus U$$

for

$$U = \bigoplus_{HgH \in H\backslash G/H \backslash \{H\}} kHgH \otimes_{kH} -.$$

Therefore Theorem 3.17 applies for appropriate choices of $\mathcal{Y}$. Following [Auslander and Kleiner 1994, p. 311, Section 3] we fix a collection $\mathfrak{Y}$ of subgroups of H, closed under H-conjugation and under taking subgroups, we consider $\mathcal{Y}$ the full subcategory of $\mathcal{H}$ given by $\text{ind}_{\mathfrak{Y}}^H$, i.e., those kH-modules induced from kY-modules for some $Y \in \mathfrak{Y}$. By [Auslander and Kleiner 1994, Corollary 3.4(a) and (b)] we may put $\mathfrak{Y} := \{Y \mid Y \leq H \cap gDg^{-1} \;;\; g \in G \setminus H\}$ which satisfies the hypotheses of Theorem 3.17. Moreover, the functors S_L and T_L in item (3) of Theorem 3.17 implies the classical Green correspondence. Furthermore, the bijection of indecomposable kG-modules and kH-modules with vertex D is the restriction of a triangle functor between triangulated categories, namely the Verdier localisation of triangulated subcategories.

However, if D is TI, i.e., $D \cap D^g \in \{1, D\}$ for all $g \in G$, the stable categories involved in the theorem are the usual stable categories modulo projectives, which are already triangulated, and by the universal property of the Verdier localisation (see [SGA 4½ 1977, §2, no. 3] or [Verdier 1996, Chapitre II, Corollaire 2.2.11.c]) there is an inverse functor to L (which was introduced in Lemma 3.5).

By the same argument, for general D, the Verdier localisation in item (3) of Theorem 3.17 is the W-stable category from Carlson, Peng and Wheeler [Carlson et al. 1998] (see also [Grime 2008, Example 3.6]).

4. Tensor triangulated categories — Green correspondence abstractly and for group rings

We had to deal with thick subcategories of triangulated categories. Our main model was the case of versions of derived or stable categories of group rings. Classification results are known in this case, but mainly in presence of an additional monoidal structure.

4A. ***Recalling Balmer's results.*** We first recall some results from [Balmer 2005].

- A tensor triangulated category $\mathcal{K}$ is an essentially small triangulated category $\mathcal{K}$ together with a symmetric monoidal structure $(\mathcal{K}, \otimes, 1)$, such that the functor $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is assumed to be exact in each variable.

- A tensor triangulated functor is an exact functor between tensor triangulated categories sending the identity object to the identity object and respecting the monoidal structures.
- A $\otimes$-ideal $\mathcal{P}$ of $\mathcal{K}$ is a thick triangulated subcategory such that if an object M is in $\mathcal{P}$ and X is an object in $\mathcal{K}$, then $M \otimes X$ is in $\mathcal{P}$.
- An ideal $\mathcal{P}$ is prime if $A \otimes B$ being an object in $\mathcal{P}$ if and only if A is an object in $\mathcal{P}$ or B is an object in $\mathcal{P}$.
- The spectrum $\mathrm{Spec}(\mathcal{K})$ is defined to be the set (!) of prime ideals of $\mathcal{K}$.
- The support of an object M of $\mathcal{K}$ is
$$\mathrm{supp}_{\mathcal{K}}(M) := \{\mathcal{P} \in \mathrm{Spec}(\mathcal{K}) \mid M \text{ is not an object of } \mathcal{P}\}.$$
- For any family of objects $\mathcal{S}$ of $\mathcal{K}$ let $Z(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spec}(\mathcal{K}) \mid \mathcal{S} \cap \mathcal{P} = \varnothing\}$. The sets $Z(\mathcal{S})$ form the closed sets of a topology, the Zariski topology on $\mathrm{Spec}(\mathcal{K})$.
- The radical $\sqrt{\mathcal{P}}$ of an ideal $\mathcal{P}$ is the class of objects M in $\mathcal{K}$ such that there is $n \in \mathbb{N}$ so that $M^{\otimes n}$ is an object of $\mathcal{P}$. An ideal $\mathcal{P}$ is called radical if $\sqrt{\mathcal{P}} = \mathcal{P}$.

One of the main results of [Balmer 2005] is:

Theorem 4.1 [Balmer 2005, Theorem 4.10]. *Let $\mathfrak{S}(\mathcal{K})$ denote the subsets $Y \subseteq$* $\mathrm{Spec}(\mathcal{K})$ *such that $Y = \bigcup_{i\in I} Y_i$ with all Y_i closed and* $\mathrm{Spec}(\mathcal{K}) \setminus Y_i$ *quasicompact. Let $\mathfrak{R}(\mathcal{K})$ be the set of radical thick $\otimes$-ideals of $\mathcal{K}$. Then the following maps are mutually inverse bijections*:

$$\mathfrak{S}(\mathcal{K}) \longleftrightarrow \mathfrak{R}(\mathcal{K}), \quad Y \mapsto \mathcal{K}_Y := \{M \in \mathcal{K} \mid \mathrm{supp}(M) \subseteq Y\},$$
$$\bigcup_{M\in\mathcal{J}} \mathrm{supp}(M) =: \mathrm{supp}(\mathcal{J}) \leftarrow\!\shortmid \mathcal{J}.$$

4B. ***Green correspondence of the spectrum in a tensor triangulated category.*** Recall that, following [Etingof et al. 2015], a tensor subcategory of a tensor category still has a unit element. We shall need to define a concept without this restriction since for our natural examples we do not necessarily have a unit element. Note that a semigroup is a set with a binary associative structure, and a monoid is a semigroup with a unit. We transport this vocabulary to the world of tensor categories under the name of semigroup category (see [Boyarchenko 2011]).

Definition 4.2.
- A semigroup category is a category $\mathcal{C}$ with a symmetric binary operation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the associative pentagon axiom.
- A triangulated semigroup category is an essentially small triangulated category $\mathcal{C}$, which is in addition a semigroup category $(\mathcal{C}, \otimes)$ such that $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in each variable.

- A $\otimes$-ideal $\mathcal{P}$ of a triangulated semigroup category $\mathcal{C}$ is a thick triangulated subcategory such that if an object M is in $\mathcal{P}$ and X is an object in $\mathcal{C}$, then $M \otimes X$ is in $\mathcal{P}$.
- Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ be semigroup categories.
  - A functor $F : \mathcal{C} \to \mathcal{D}$ is called semitensor functor if F allows a natural equivalence $J : F(V \otimes W) \to F(V) \otimes F(W)$ which satisfies the associahedron diagram [Etingof et al. 2015, Diagram 2.23].
  - If $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ are triangulated semigroup categories, then a semitensor functor $F : \mathcal{C} \to \mathcal{D}$ is called triangle semitensor functor if F is in addition a triangle functor.

Lemma 4.3. *If $\mathcal{Y}$ is a thick semigroup triangulated subcategory (resp. $\otimes$-ideal) of a triangulated semigroup category $\mathcal{H}$, and if $F : \mathcal{D} \to \mathcal{H}$ is a triangle semitensor functor, then $\mathcal{Z} := F^{-1}(\mathcal{Y})$ is again a thick semigroup triangulated subcategory (resp. $\otimes$-ideal) of $\mathcal{D}$.*

Proof. By Lemma 3.3 $\mathcal{Z}$ is triangulated. By definition, $\mathcal{Z}$ is thick. We need to show that $\mathcal{Z}$ is a semigroup tensor category. Let X be an object of $\mathcal{Z}$ and Y be an object of $\mathcal{Z}$ (resp. $\mathcal{H}$). Then there are objects X' and Y' such that $F(X) \oplus X'$ and $F(Y) \oplus Y'$ are objects of $\mathcal{Y}$. But then

$$\begin{aligned}
&(F(X) \oplus X') \otimes (F(Y) \oplus Y') \\
&\qquad \simeq (F(X) \otimes F(Y)) \oplus (X' \otimes F(Y)) \oplus (F(X) \otimes Y') \oplus (X' \otimes Y') \\
&\qquad \simeq F(X \otimes Y) \oplus (X' \otimes F(Y)) \oplus (F(X) \otimes Y') \oplus (X' \otimes Y').
\end{aligned}$$

Since $\mathcal{Y}$ is tensor triangulated, $F(X \otimes Y)$ is a direct factor of the object $(F(X) \oplus X') \otimes (F(Y) \oplus Y')$ of $\mathcal{Y}$. Therefore $X \otimes Y$ is an object of $\mathcal{Z}$. ☐

Lemma 4.4. *If $\mathcal{H}$ is a tensor triangulated category, if $\mathcal{Y}$ is a $\otimes$-ideal in $\mathcal{H}$, then the tensor triangulated structure on $\mathcal{H}$ induces a tensor triangulated structure on $\mathcal{H}_{\mathcal{Y}}$. Moreover, the natural functor $\nu : \mathcal{H} \to \mathcal{H}_{\mathcal{Y}}$ is a functor of tensor triangulated categories.*

Proof. The objects of $\mathcal{H}$ coincides with the objects of $\mathcal{H}_{\mathcal{Y}}$. We need to define a tensor product $\bar{\otimes}$ on $\mathcal{H}_{\mathcal{Y}}$. Denote by $\nu : \mathcal{H} \to \mathcal{H}_{\mathcal{Y}}$ the natural functor. We define for any two objects M, N in $\mathcal{H}_{\mathcal{Y}}$ the object $M \bar{\otimes} N := \nu(M \otimes N)$ in $\mathcal{H}_{\mathcal{Y}}$.

Since $\mathcal{Y}$ is an ideal, this construction is also well-defined on morphisms. Since $\mathrm{id}_{\mathcal{H}}$ is the neutral element of $\otimes$, we get $\nu(\mathrm{id}_{\mathcal{H}})$ is the neutral element of $\bar{\otimes}$. Since $\otimes$ is monoidal symmetric, also $\bar{\otimes}$ is monoidal symmetric. The functor is tensor triangulated by construction. ☐

Recall that every thick subcategory is a full triangulated subcategory, but a full triangulated subcategory is thick only if it is in addition closed under taking direct summands. A full triangulated subcategory $\mathcal{A}$ of $\mathcal{D}$ is strict if any object of $\mathcal{D}$ which is isomorphic in $\mathcal{D}$ to an object in $\mathcal{A}$ is also an object of $\mathcal{A}$.

Proposition 4.5. *Let* $(\mathcal{T}, \otimes, 1)$ *be a tensor triangulated category and let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be* $\otimes$*-ideals of* $\mathcal{T}$*. Suppose moreover that* $\mathcal{Q}$ *is a full triangulated subcategory of* $\mathcal{P}$*. Suppose that* $\mathcal{P}$ *is strictly full in* $\mathcal{T}$*.*

Then the following hold.

- *The tensor triangulated structure on* $\mathcal{T}$ *induces a tensor triangulated structure* $\bar{\otimes}$ *on the Verdier localisation* $\mathcal{T}_{\mathcal{Q}}$*.*
- *Furthermore, consider the natural functor* $\nu : \mathcal{T} \to \mathcal{T}_{\mathcal{Q}}$*. Let* ν' *be the restriction of* ν *to* $\mathcal{P}$*, as indicated in the commutative diagram*

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\nu"] & \mathcal{T}_{\mathcal{Q}} \\
\mathcal{P} \arrow[u, hook] \arrow[r, "\nu'"] & \nu(\mathcal{P}) \arrow[u, hook]
\end{tikzcd}$$

  Denote by $\mathcal{P}_{(\mathcal{Q})}$ *the image* $\nu(\mathcal{P})$ *of* $\mathcal{P}$ *in* $\mathcal{T}_{\mathcal{Q}}$ *under* ν*, and denote by* $\mathcal{P}_{\mathcal{Q}}$ *the Verdier localisation of* $\mathcal{P}$ *at* $\mathcal{Q}$*. Then*

$$\mathcal{P}_{(\mathcal{Q})} = \mathcal{P}_{\mathcal{Q}}.$$

- $\mathcal{P}_{\mathcal{Q}}$ *is a* $\bar{\otimes}$*-ideal of* $\mathcal{T}_{\mathcal{Q}}$*.*

Proof. Lemma 4.4 is precisely the first statement.

Denote by $\iota : \mathcal{P} \to \mathcal{T}$ the inclusion functor. As for the second statement we have the Verdier localisation $\mathcal{P}_{\mathcal{Q}}$ of $\mathcal{P}$ at $\mathcal{Q}$. Denote by $\mu : \mathcal{P} \to \mathcal{P}_{\mathcal{Q}}$ the natural functor. Then, the universal property of Verdier localisations ([SGA 4½ 1977, §2, no. 3] or [Verdier 1996, Chapitre II, Corollaire 2.2.11.c]) induces a unique functor $\sigma : \mathcal{P}_{\mathcal{Q}} \to \mathcal{P}_{(\mathcal{Q})}$ such that $\sigma \circ \mu = \nu \circ \iota$. This shows that the functor σ is dense since μ, ν, ι are the identity on objects.

We need to show that σ is fully faithful. Let Z be an object of $\mathcal{T}$, let P_1 and P_2 be objects of $\mathcal{P}$, and a diagram of morphisms of $\mathcal{T}$

$$\begin{tikzcd}
 & Z \arrow[dl, "\gamma"'] \arrow[dr, "\alpha"] & \\
P_1 & & P_2
\end{tikzcd}$$

representing a morphism ω in $\mathcal{T}_{\mathcal{Q}}(P_1, P_2)$. If γ has cone in $\mathcal{Q}$, since $\mathcal{P}$ is triangulated, and since $\mathcal{Q}$ is a triangulated subcategory of $\mathcal{P}$, also Z is isomorphic to an object of $\mathcal{P}$, and since $\mathcal{P}$ is strictly full in $\mathcal{T}$, the object Z is actually an object of $\mathcal{P}$. Hence σ is full.

If λ is represented by

$$\begin{tikzcd} & Z \arrow[dl, "\gamma"'] \arrow[dr, "\alpha"] & \\ P_1 & & P_2 \end{tikzcd}$$

for some object Z of $\mathcal{P}$, and if $\sigma(\lambda) = 0$ in $\mathcal{P}_{\mathcal{Q}}$, then there is an object Z' of $\mathcal{T}$ and a morphism $\delta : Z' \to Z$ with cone(δ) in $\mathcal{Q}$, and with $\alpha \circ \delta = 0$. But, again $\mathcal{Q}$ is a triangulated subcategory of $\mathcal{P}$, and $\mathcal{P}$ being strictly full triangulated subcategory of $\mathcal{T}$ implies Z' is an object of $\mathcal{P}$. Since $\mathcal{P}$ is a full subcategory of $\mathcal{T}$, the morphism δ is actually already in $\mathcal{P}$. Hence $\lambda = 0$. This shows that σ is faithful. Altogether we get the second statement.

Since $\mathcal{P}$ is a $\otimes$-ideal, for any P in $\mathcal{P}$, and any X in $\mathcal{T}$ we get $P \otimes X$ is in $\mathcal{P}$. Hence

$$\nu(P)\bar{\otimes}\nu(X) = \nu(P \otimes X)$$

is an object of $\mathcal{P}_{(\mathcal{Q})} = \mathcal{P}_{\mathcal{Q}}$. This proves the third statement. ☐

4C. ***Thick tensor triangulated categories and tensor ideals in the special case of group rings.*** Various results are known for classification of thick subcategories of various triangulated categories (see, e.g., [Benson et al. 1997; 2011; Carlson and Iyengar 2015; Friedlander and Pevtsova 2007; Thomason 1997]), giving mostly a parametrisation with certain subsets of support varieties. For a fixed, essentially small triangulated category $\mathcal{D}$ a general result describing the relation between full triangulated essentially small subcategories $\mathcal{A}$ and thick$(\mathcal{A}) = \mathcal{D}$ is given by Thomason.

Theorem 4.6 [Thomason 1997, Theorem 2.1]. *Let $\mathcal{D}$ be an essentially small triangulated category. Consider the set $\mathfrak{U}$ of strictly full triangulated subcategories $\mathcal{A}$ in $\mathcal{D}$, having the property that each object of $\mathcal{D}$ is isomorphic to a direct summand of an object in $\mathcal{A}$. Then $\mathfrak{U}$ is in bijection with the set of subgroups of the Grothendieck group $K_0(\mathcal{D})$. The isomorphism is given by mapping $\mathcal{A}$ to the subgroup $K_0(\mathcal{A})$ of $K_0(\mathcal{D})$.*

Thomason [1997, Theorem 3.15] also gave a classification of tensor triangulated thick subcategories of the derived category of perfect complexes over a quasicompact quasiseparated scheme.

We focus on those dealing with group rings. Let k be an algebraically closed field of characteristic $p > 0$ and let G be a finite group with order divisible by p. Let $H^\bullet(G)$ be $\bigoplus_{i\geq 0} H^{2i}(G, k)$ if p is odd, and $H^\bullet(G) = H^*(G, k)$ if $p = 2$. Then $H^\bullet(G)$ is a graded commutative algebra, and $\mathrm{Ext}^*_{kG}(M, M)$ is a finitely generated $H^\bullet(G)$-module. Let $V_G(k)$ be the maximal ideal spectrum of $H^\bullet(G)$. A set $\mathcal{X}$ of closed subvarieties of $V_G(k)$ is said to be closed under specialisation if whenever $W \in \mathcal{X}$ and $W' \subseteq W$, then we also get $W' \in \mathcal{X}$. For a set $\mathcal{X}$ of closed subvarieties of $V_G(k)$ which is closed under specialisation we let $\mathcal{C}(\mathcal{X})$ be the thick subcategory of $kG-\underline{\text{mod}}$ consisting of modules M with

$$V_G(M) := \left\{\mathfrak{m} \in V_G(k) \mid \text{Ann}_{H^\bullet(G)}(\text{Ext}^*_{kG}(M, M)) \subseteq \mathfrak{m}\right\} \in \mathcal{X}.$$

Benson, Carlson and Rickard [Benson et al. 1997] showed the following.

Theorem 4.7 [Benson et al. 1997, Theorem 3.4]. *Let k be an algebraically closed field, and let G be a finite group. Let $V_G(k)$ be the maximal ideal spectrum of $H^\bullet(G)$. Then the thick tensor ideals I in $kG-\underline{\text{mod}}$ are of the form $\mathcal{C}(\mathcal{X})$ for some nonempty set $\mathcal{X}$ of homogeneous subvarieties of $V_G(k)$, closed under specialisation and finite unions.*

Carlson and Iyengar [2015] determined the thick subcategories of the derived category of the group algebra of a finite group. For each object M of $D^b(kG)$ there is a morphism of k-algebras $H^*(G, k) \to \text{Ext}^*_{kG}(M, M)$. Again $\text{Ext}^*_{kG}(M, M)$ is a finitely generated $H^\bullet(G)$-module. Then

$$V_{D^b(kG)}(M) := \text{Supp}_{H^\bullet(G)}(M) := \left\{\wp \in \text{Spec}(H^\bullet(G)) \mid H(M_\wp) \neq 0\right\} \subseteq \text{Spec}(H^\bullet(G)).$$

Theorem 4.8 [Carlson and Iyengar 2015, Theorem 6.6 and Corollary 6.7]. *For an algebraically closed field k of characteristic $p > 0$ and a finite group G with order divisible by p, and two objects M and N in $D^b(kG)$ with $V_{D^b(kG)}(M) \subseteq V_{D^b(kG)}(N)$ we have that M is in the thick tensor ideal generated by N. In particular, if $\boldsymbol{C}$ is a thick tensor ideal of $D^b(kG)$, then there is a specialisation closed subset V of $V_{D^b(kG)}(k)$ such that $\boldsymbol{C}$ equals the subcategory obtained by all those M in $D^b(kG)$ with $V_{D^b(kG)}(M) \subseteq V$.*

Carlson [2018] studied thick subcategories of what he calls relatively stable categories of group rings. Let $\mathfrak{H}$ be a set of subgroups of G. A kG-module M is called $\mathfrak{H}$-projective if M is $\downarrow_H^G$-relative projective for all $H \in \mathfrak{H}$. It is classical that a module M is $\mathfrak{H}$-projective if and only if M is a direct summand of modules which are induced from modules over elements of $\mathfrak{H}$. The category $kG-\underline{\text{mod}}_{\mathfrak{H}}$ has the same objects as $kG-\text{mod}$. However, the set of morphisms from M to N is the set of equivalence classes of kG-module morphisms modulo those factoring through $\mathfrak{H}$-projective modules.

Carlson, Peng and Wheeler [Carlson et al. 1998, Theorem 6.2] showed that $kG-\underline{\text{mod}}_{\mathfrak{H}}$ is actually a triangulated category. Moreover, an immediate consequence is that Green correspondence is the restriction of a functor between triangulated categories to certain subcategories, namely those full subcategories generated by indecomposable modules with a specific vertex. This restriction is then an equivalence of categories.

Benson and Wheeler extended the concept to infinitely generated modules, and showed in [Benson and Wheeler 2001, Proposition 2.3] that we get again triangulated categories and a Green correspondence, which is an equivalence between triangulated categories.

These results are special cases of our more general approach, when applied to bounded derived categories of modules over the respective group rings and appropriate choices for $\mathcal{Y}$, as is shown by the following.

Proposition 4.9. *Let D be a p-subgroup of G, let H be a subgroup of G containing $N_G(D)$, let*

$$\mathfrak{Y} := \{S \leq H \cap {}^gD \mid g \in G \setminus H\} \quad \textit{and} \quad \mathfrak{X} := \{S \leq D \cap {}^gD \mid g \in G \setminus H\}.$$

Then for $\mathcal{Y}$ being the class of complexes having indecomposable factors with vertex in $\mathfrak{Y}$, the natural functors

$$L_{SS'T'Y} : kG - \underline{\text{mod}}_{\mathfrak{X}} \to D^b(kG)_{\text{thick}(SS'T'\mathcal{Y})}$$

and

$$L_{\mathcal{Y}} : kH - \underline{\text{mod}}_{\mathfrak{Y}} \to D^b(kH)_{\text{thick}(\mathcal{Y})}$$

are equivalences of triangulated categories.

Proof. Using [Wang and Zhang 2018, Lemma 4.1], Lemma 3.5 defines $L_{SS'T'\mathcal{Y}}$ and $L_{\mathcal{Y}}$. Since the subcategory of bounded complexes of finitely generated projectives is a subcategory of $\mathcal{Y}$, the category $D^b(kG)_{\text{thick}(SS'T'\mathcal{Y})}$ is a localisation of the singularity category $D_{sg}(kG)$ of kG. The singularity category of a self-injective algebra is just the stable category of the algebra modulo projective-injectives (see [Buchweitz 1986; Keller and Vossieck 1987; Rickard 1989a]). Likewise $D^b(kH)_{\text{thick}(\mathcal{Y})}$ is a localisation of the stable category of kH. Since the categories $kG - \underline{\text{mod}}_{\mathfrak{X}}$ and $kH - \underline{\text{mod}}_{\mathfrak{Y}}$ are triangulated, the universal property of the Verdier localisation ([SGA 4½ 1977, §2, no. 3] or [Verdier 1996, Chapitre II, Corollaire 2.2.11.c]) gives the quasi-inverse functors to $L_{SS'T'\mathcal{Y}}$ and $L_{\mathcal{Y}}$ respectively. □

Observe that now Theorem 3.17(3) gives Carlson, Peng and Wheeler's theorem.

Moreover, Harris [2014] and independently Wang and Zhang [2018] gave a blockwise version of the Green correspondence.

Localising subcategories are a vast variety in this setting. Carlson [2018] showed for example that for $p = 2$ and a collection $\mathcal{C}$ of subgroups H of G all of which containing an elementary abelian subgroup of rank at least 2, then the spectrum of the relatively $\mathcal{C}$-stable category is not Noetherian.

Note that for a nonprincipal block we do not get a monoidal category but only a semigroup category in the sense of Definition 4.2. Indeed, the unit element is the trivial module, which belongs to the principal block.

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ALEXANDER ZIMMERMANN
DÉPARTEMENT DE MATHÉMATIQUES ET LAMFA (UMR 7352 DU CNRS)
UNIVERSITÉ DE PICARDIE
AMIENS
FRANCE
alexander.zimmermann@u-picardie.fr

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