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We show that the category of vector fields on a geometric stack has the structure of a Lie 2-algebra. This proves a conjecture of R. Hepworth. The construction uses a Lie groupoid that presents the geometric stack. We show that the category of vector fields on the Lie groupoid is equivalent to the category of vector fields on the stack. The category of vector fields on the Lie groupoid has a Lie 2-algebra structure built from known (ordinary) Lie brackets on multiplicative vector fields of Mackenzie and Xu and the global sections of the Lie algebroid of the Lie groupoid. After giving a precise formulation of Morita invariance of the construction, we verify that the Lie 2-algebra structure defined in this way is well-defined on the underlying stack.

1. Introduction

Vector fields on a Lie groupoid G form a category [Hepworth 2009]. We denote it by $\mathbb{X}(G)$. The *objects* of $\mathbb{X}(G)$ are the multiplicative vector fields of Mackenzie and Xu [1998]. These are functors $v : G \to TG$ satisfying $\pi_G \circ v = \mathrm{id}_G$, where TG denotes the tangent groupoid and $\pi_G : TG \to G$ is the projection functor. A *morphism* $\alpha : v \Rightarrow v'$ in this category is a natural transformation α such that $\pi_G(\alpha(x)) = \mathrm{id}_x$ for every object x of the groupoid G. The first result of this paper is:

Theorem 3.4. *The category of vector fields $\mathbb{X}(G)$ on a Lie groupoid G is a (strict) Lie 2-algebra. That is, $\mathbb{X}(G)$ is a category internal to the category of Lie algebras.*

Remark 1.1. When a manifold M is regarded as a discrete Lie groupoid, $\mathbb{X}(M)$ is the usual Lie algebra of vector fields on M regarded as a discrete Lie 2-algebra.

To every Lie groupoid G there corresponds the stack $\mathbb{B}G$ of principal G-bundles, and Morita equivalent Lie groupoids G and H correspond to isomorphic stacks $\mathbb{B}G$ and $\mathbb{B}H$. It is natural to wonder if the Lie 2-algebra $\mathbb{X}(G)$ lives on the stack $\mathbb{B}G$ in some appropriate sense. To start, we can ask whether Morita equivalent Lie groupoids G and H have "Morita equivalent" Lie 2-algebras $\mathbb{X}(G)$ and $\mathbb{X}(H)$. More precisely we could ask for the existence of a (2-)functor $\mathbb{X}$ from the bicategory Bi of Lie groupoids, bibundles and isomorphisms of bibundles to an appropriate

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bicategory of Lie 2-algebras that sends Morita equivalences to Morita equivalences. It turns out that such a functor is too much to ask for but there is a functor from a sub-bicategory of Bi.

The reasons behind this fact can already be seen in the case of manifolds. Recall that there is no naturally defined functor from the category of manifolds to the category of Lie algebras that assigns to each manifold its Lie algebra of vector fields. However if we restrict ourselves to the category $\mathsf{Man}_{\mathrm{iso}}$ whose objects are manifolds and whose morphisms are diffeomorphisms then there is a perfectly well defined functor with the desired properties.

Getting back to Lie groupoids, recall that there is a localization of the strict 2-category of Lie groupoids, internal functors, and internal natural transformations at the class of functors that are fully faithful and essentially surjective ; denote this localization by Bi. Lie 2-algebras, internal functors and internal natural transformations form the strict 2-category $\mathsf{Lie2Alg}_{\mathrm{strict}}$, and localizing at the essential equivalences produces a bicategory $\mathsf{Lie2Alg}$ (see Subsections 2B and 2C). Let $\mathsf{Bi}_{\mathrm{iso}}$ be the sub-bicategory of Bi whose objects are Lie groupoids, 1-morphisms are (weakly) *invertible* bibundles (i.e., Morita equivalences) and 2-morphisms are isomorphisms of bibundles. We recall that a bicategory with invertible 2-morphism is, by definition, a $(2,1)$-*bicategory*.

Theorem 4.1. *The map $G \mapsto \mathbb{X}(G)$ that assigns to each Lie groupoid its category of vector fields extends to a functor $\mathbb{X} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Lie2Alg}$. In particular, if $P : G \to H$ is a Morita equivalence of Lie groupoids then $\mathbb{X}(P) : \mathbb{X}(G) \to \mathbb{X}(H)$ is a (weakly) invertible 1-morphism of Lie 2-algebras in the bicategory* $\mathsf{Lie2Alg}$.

Remark 1.2. In the Lie groupoid literature there are two standard constructions that associate a Lie algebra to a Lie groupoid: global sections of its Lie algebroid and Mackenzie and Xu's multiplicative vector fields. The Lie 2-algebra structure on $\mathbb{X}(G)$ is built out of this pair of Lie algebras. At first pass this might seem surprising: neither multiplicative vector fields nor sections of Lie algebroids are well-behaved under Morita equivalence of Lie groupoids. Theorem 4.1 shows that combining this pair of Lie algebras into a Lie 2-algebra gives us an object that *is* preserved by Morita equivalence.

How does the existence of the functor in Theorem 4.1 imply that the Lie 2-algebra $\mathbb{X}(G)$ "lives" on the stack $\mathbb{B}G$? To answer this, we need to recall the relationship between the bicategory Bi and the 2-category Stack of stacks over the site of smooth manifolds. The assignment $G \mapsto \mathbb{B}G$ extends to a fully faithful functor $\mathbb{B} : \mathsf{Bi} \to \mathsf{Stack}$. The essential image of the functor $\mathbb{B}$ is the 2-category $\mathsf{GeomStack}$ of geometric stacks. Restricting the functor $\mathbb{B}$ to the bicategory $\mathsf{Bi}_{\mathrm{iso}}$ of groupoids and Morita equivalences gives us an equivalence of bicategories $\mathbb{B} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{GeomStack}_{\mathrm{iso}}$, where $\mathsf{GeomStack}_{\mathrm{iso}}$ is the $(2,1)$-category of geometric

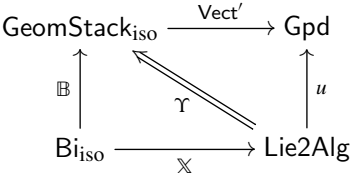
stacks, isomorphisms of stacks (that is, weakly invertible 1-morphisms of stacks) and 2-morphisms. By inverting this equivalence $\mathbb{B}$ and composing it with the functor $\mathbb{X}$ we get a functor

$$\mathsf{GeomStack}_{\mathsf{iso}} \xrightarrow{\mathbb{B}^{-1}} \mathsf{Bi}_{\mathsf{iso}} \xrightarrow{\mathbb{X}} \mathsf{Lie2Alg}. \tag{1-1}$$

So in particular we get a functorial assignment of a Lie 2-algebra to every geometric stack, with isomorphic stacks being assigned "isomorphic" Lie 2-algebras.

Hepworth [2009] introduced a category of vector fields $\mathsf{Vect}(\mathcal{A})$ on a stack $\mathcal{A}$, which is a groupoid. We introduce a groupoid $\mathsf{Vect}'(\mathcal{A})$ equivalent to $\mathsf{Vect}(\mathcal{A})$ which is more convenient for our purposes. In particular, the assignment $\mathcal{A} \mapsto \mathsf{Vect}'(\mathcal{A})$ easily extends to a functor $\mathsf{Vect}' : \mathsf{GeomStack}_{\mathsf{iso}} \to \mathsf{Gpd}$, where Gpd is the (2,1)-category of groupoids, functors and natural transformations (which are automatically natural isomorphisms). We show that the functor Vect' is compatible with the functors $\mathbb{B} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{GeomStack}_{\mathsf{iso}}$ and $\mathbb{X} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{Lie2Alg}$ in the following sense.

Theorem 6.1. *The diagram of* (2,1)*-bicategories and functors*



2*-commutes. Here as above* Gpd *denotes the* (2,1)*-category of groupoids, functors and natural isomorphisms, and* $u : \mathsf{Lie2Alg} \to \mathsf{Gpd}$ *denotes the functor that assigns to each Lie* 2*-algebra its underlying groupoid. The components of the transformation* Υ *are weakly invertible functors (i.e., equivalences of categories). In particular for a geometric stack* $\mathcal{A}$ *the category underlying the Lie* 2*-algebra* $(\mathbb{X} \circ \mathbb{B}^{-1})(\mathcal{A})$ *is equivalent to Hepworth's category* $\mathsf{Vect}(\mathcal{A})$ *of vector fields on the stack.*

Remark 1.3. Consider a geometric stack $\mathcal{A}$. The groupoids $\mathsf{Vect}(\mathcal{A})$ and $\mathsf{Vect}'(\mathcal{A})$ are equivalent. Let $G = \mathbb{B}^{-1}(\mathcal{A})$. Then the stacks $\mathbb{B}G$ and $\mathcal{A}$ are isomorphic and consequently the categories $\mathsf{Vect}'(\mathcal{A})$ and $\mathsf{Vect}'(\mathbb{B}G)$ are equivalent. Since the diagram above 2-commutes and the components of Υ are weakly invertible functors, $\mathsf{Vect}'(\mathcal{A})$ is equivalent to the groupoid $u(\mathbb{X}(G))$ underlying the Lie 2-algebra $\mathbb{X}(G)$. Consequently Hepworth's groupoid $\mathsf{Vect}(\mathcal{A})$ of vector fields on a stack is equivalent to the groupoid underlying the Lie 2-algebra $\mathbb{X}(\mathbb{B}^{-1}(\mathcal{A}))$.

A different choice $(\mathbb{B}^{-1})'$ of the inverse of $\mathbb{B}$ is isomorphic to $\mathbb{B}^{-1}$. Consequently the Lie 2-algebras $\mathbb{X}(\mathbb{B}^{-1}(\mathcal{A}))$ and $\mathbb{X}((\mathbb{B}^{-1})'(\mathcal{A}))$ are naturally weakly isomorphic and their underlying groupoids are equivalent.

Related work. The recent work of Cristian Ortiz and James Waldron [2019] lies in a similar circle of ideas. Recall that an $\mathcal{LA}$-groupoid is a groupoid object in Lie algebroids. Given an $\mathcal{LA}$-groupoid, Ortiz and Waldron introduced its category

of multiplicative sections and showed that it carries a natural strict Lie 2-algebra structure in the language of crossed modules of Lie algebras (which affords an equivalent description of the category of strict Lie 2-algebras). They showed that if two $\mathcal{LA}$-groupoids are Morita equivalent then the corresponding crossed modules of Lie algebras are connected by a zig-zag of equivalences. Furthermore, to every stack Ortiz and Waldron assigned an ordinary Lie algebra and showed that in the case of proper geometric stacks the set underlying this Lie algebra is in bijective correspondence with isomorphism classes of vector fields in Hepworth's definition.

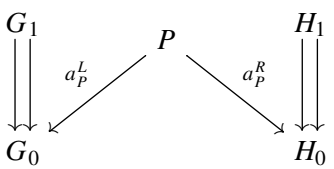
2. Background and notation

We assume that the reader is familiar with ordinary categories, strict 2-categories and bicategories (also known as weak 2-categories). We mostly work with (2,1)-bicategories, that is with bicategories whose 2-morphisms are invertible. Standard references for bicategories are [Bénabou 1967; Borceux 1994]. See also [Street 1996]. We assume familiarity with Lie groupoids, with a standard reference being [Moerdijk and Mrčun 2003]. We also assume that the reader is comfortable with stacks over the site of manifolds and the relationship with Lie groupoids, e.g., see [Behrend and Xu 2011]. This said, Sections 3, 4 and 7 do not use stacks.

Given a category C we denote its collection of objects by C_0 and the collection of arrows/morphisms by C_1.[1] We usually denote the source and target maps of C by s and t, respectively. We write $\mathsf{C} = \{\mathsf{C}_1 \rightrightarrows \mathsf{C}_0\}$ and suppress the other structure maps of the category C. The map $1 : \mathsf{C}_0 \to \mathsf{C}_1$ assigns the identity arrow 1_x to each object x of the category C. The composition/multiplication in the category C is defined on the collection C_2 of pairs of composable arrows. Our convention is that $\mathsf{C}_2 := \{(\gamma_2, \gamma_1) \in \mathsf{C}_1 \times \mathsf{C}_1 \mid s(\gamma_2) = t(\gamma_1)\} =: \mathsf{C}_1 \times_{s,\mathsf{C}_0,t} \mathsf{C}_1$. We denote the composition in the category C by m. In particular, we write the composition from right to left: $\gamma_2\gamma_1$ means γ_1 followed by γ_2. If the category C is a groupoid we denote the inversion map by i.

2A. *Bicategories of Lie groupoids.*

Definition 2.1. A *bibundle* $P : G \to H$ from a Lie groupoid G to a Lie groupoid H is a manifold P with two maps a_P^L and a_P^R:



along with a left action of G and a right action of H on P:

$$G_1 \times_{G_0} P \to P, \quad (g, p) \mapsto g \cdot p, \qquad P \times_{H_0} H_1 \to P, \quad (p, h) \mapsto p \cdot h.$$

[1] We use the words "arrow," "morphism" and "1-cell" interchangeably.

We refer to a_P^L as the *left anchor* and to a_P^R as the *right anchor*. We further require that

(1) the actions of G and H commute: for all $(g,\, p) \in G_1 \times_{G_0} P,\ (p, h) \in P \times_{H_0} H_1$

$$g \cdot (p \cdot h) = (g \cdot p) \cdot h;$$

(2) the map $a_L^P : P \to G_0$ is a surjective submersion and is H-invariant: $a_L^P(p \cdot h) = a_L^P(p)$ for all $(p, h) \in P \times_{H_0} H_1$;

(3) the map a_R^P is G-invariant: $a_R^P(g \cdot p) = a_R^P(p)$ for all $(g,\, p) \in G_1 \times_{G_0} P$;

(4) the map

$$P \times_{a_P^R, H_0, t} H_1 \to P \times_{a_P^L, G_0, a_P^L} P, \quad (p, h) \mapsto (p, p \cdot h)$$

is a diffeomorphism.

Definition 2.2. An *isomorphism* of two bibundles $P, P' : G \to H$ is a diffeomorphism $\alpha : P \to P'$ which is G- and H-equivariant.

Given two bibundles $P : G \to H$ and $Q : H \to K$ their composite $Q \circ P$ is defined to be the quotient of the fiber product $P_{a_R^P, H_0, a_L^Q} Q$ by the action of H: $Q \circ P := (P \times_{a_R^P, H_0, a_L^Q} Q)/H$. The composition of bibundles is not associative on the nose: given 3 composable bibundles P, Q and R the composites $(R \circ Q) \circ P$ and $R \circ (Q \circ P)$ are only isomorphic. Therefore Lie groupoids and bibundles do not form a category. One can show that Lie groupoids, bibundles and isomorphisms of bibundles form a bicategory which we denote by Bi.

Notation 2.3. An isomorphism of bibundles $\alpha : P \to Q$ is a smooth map and a 2-arrow (2-cell) in the bicategory Bi described above. For this reason we may sometimes write $\alpha : P \Rightarrow Q$ for an isomorphism of bibundles.

There is also the strict 2-category LieGpd of Lie groupoids, (smooth) functors and (smooth) natural transformations Note that both Bi and LieGpd are (2,1)-bicategories.

Notation 2.4. In the bicategories LieGpd and Bi, we write the horizontal composition of 2-arrows as $\star$. Given a 1-morphism f and a 2-morphism α we abuse notation by writing $f \star \alpha$ for the horizontal composition (whiskering) $1_f \star \alpha$, where 1_f is the identity 2-arrow on the 1-morphism f. The vertical composition of 2-morphisms is denoted by $\circ$. When convenient, we also use arrow notation to denote morphisms in groupoids with specified source or target, e.g., $x \xleftarrow{g} y$ for a morphism g with target x and source y.

Remark 2.5. There is a functor $U : \mathsf{LieGpd} \to \mathsf{Bi}$ that is the identity on objects. On 1-morphisms U sends a functor $f : G \to H$ to the bibundle

$$\begin{aligned} \langle f \rangle &:= G_0 \times_{f, H_0, t} H_1 \\ &:= \{(x, \gamma) \mid f(x) = t(\gamma)\} = \{(x,\, f(x) \xleftarrow{\gamma}) \mid x \in G_0, \gamma \in H_1\} \end{aligned} \tag{2-1}$$

whose left and right anchor maps are given respectively by

$$a^L_{\langle f\rangle}(x,\gamma) = x, \qquad a^R_{\langle f\rangle}(x,\gamma) = s(\gamma).$$

Here, as before, $s : H_1 \to H_0$ is the source map. The left action of the groupoid G on the manifold $\langle f\rangle$ is given by $(g,(x,\gamma)) \mapsto (t(g), f(g)\gamma)$. The right action of the groupoid H on $\langle f\rangle$ is given by $((x,\gamma),\nu) \mapsto (x,\gamma\nu)$. Note that $a^L_P : \langle f\rangle \to G_0$ has a canonical section $x \mapsto (x, 1_{f(x)})$. Given a pair of functors $f, k : G \to H$ and a natural isomorphism $\alpha : f \Rightarrow k$, we get an isomorphism of bibundles $\langle\alpha\rangle : \langle f\rangle \Rightarrow \langle k\rangle$. The isomorphism $\langle\alpha\rangle$ is defined by $\langle\alpha\rangle(x, f(x) \xleftarrow{\gamma}) = (x, k(x) \xleftarrow{\alpha(x)\gamma})$.

The functor $U = \langle\ \rangle$ takes vertical and horizontal composition of natural transformations to the composition of isomorphisms of bibundles and horizontal composition of isomorphisms, respectively.

Remark 2.6. By construction of the functor $U = \langle\ \rangle$ the total space of the bibundle $\langle \mathrm{id}_G\rangle$ corresponding to the identity functor $\mathrm{id}_G : G \to G$ on a Lie groupoid G is the fiber product $G_0 \times_{G_0} G_1$. This fiber product is diffeomorphic to G_1. We therefore define the manifold G_1 together with the actions of G by left and right multiplication to be the identity bibundle for a Lie groupoid G.

The functor U is far from being an equivalence of 2-categories. The issue is that for almost all groupoids G and H the functor

$$U : \mathsf{Hom}_{\mathsf{LieGpd}}(G,H) \to \mathsf{Hom}_{\mathsf{Bi}}(G,H) \tag{2-2}$$

fails to be essentially surjective. The failure of essential surjectivity follows from the well-known fact:

Lemma 2.7. *A bibundle $P : G \to H$ is isomorphic to a bibundle $\langle f\rangle$ for some functor $f : G \to H$ if and only if the left anchor $a^L_P : P \to G_0$ has a section.*

We omit the proof.

Remark 2.8. Recall that a functor $f : G \to H$ between two Lie groupoids is an *essential equivalence* if the right anchor $a^R : \langle f\rangle \to H_0$ is a surjective submersion and the action of G on $\langle f\rangle$ is principal. The functor $U =: \mathsf{LieGpd} \to \mathsf{Bi}$ is a localization of the 2-category LieGpd at the class of all essential equivalences; see Example 2.12.

In contrast to failure of the functor U to be surjective on 1-morphisms, for 2-morphisms the following result holds. The result must be known but we are not aware of a reference.

Theorem 2.9 (folklore). *For any pair of functors $f, k : G \to H$ of Lie groupoids the map*

$$U : \mathsf{Hom}_{\mathsf{LieGpd}}(f,k) \to \mathsf{Hom}_{\mathsf{Bi}}(\langle f\rangle, \langle k\rangle), \quad \alpha \mapsto \langle\alpha\rangle$$

is a bijection.

Sketch of proof. Let $\delta : \langle f \rangle \to \langle k \rangle$ be an isomorphism of bibundles. The left anchor $a^L_{\langle f \rangle} : \langle f \rangle \to G_0$ has a natural section σ_f. It is defined by

$$\sigma_f(x) = (x,\, f(x) \xleftarrow{1_{f(x)}}).$$

Similarly we have a natural section $\sigma_k : \langle k \rangle \to G_0$ of the left anchor $a^L_{\langle k \rangle} : \langle k \rangle \to G_0$. Since $a^L_{\langle k \rangle} : \langle k \rangle \to G_0$ is a principal H bundle, for any $x \in G_0$ there is a unique arrow $\bar{\delta}(x) \in H_1$ so that

$$\delta(\sigma_f(x)) = \sigma_k(x) \cdot \bar{\delta}(x)$$

for all $x \in G_0$. By equivariance of δ, the map

$$\bar{\delta} : G_0 \to H_1, \quad x \mapsto \bar{\delta}(x)$$

is a natural isomorphism from f to k. □

2B. *Localizations of bicategories.*

Definition 2.10. Given a bicategory C and a class of 1-morphisms W in C we define a *localization* of C at the class W (if it exists) to be a pair $(\mathsf{C}[W^{-1}], U : \mathsf{C} \to \mathsf{C}[W^{-1}])$, where $\mathsf{C}[W^{-1}]$ is a bicategory and $U : \mathsf{C} \to \mathsf{C}[W^{-1}]$ is a functor satisfying the following universal property: For any bicategory D the precomposition with U induces an equivalence of bicategories

$$\mathsf{Hom}(\mathsf{C}[W^{-1}], \mathsf{D}) \xrightarrow{-\circ U} \mathsf{Hom}_W(\mathsf{C}, \mathsf{D}),$$

where $\mathsf{Hom}_W(\mathsf{C}, \mathsf{D})$ denotes the bicategory of functors sending elements of W to weakly invertible 1-morphisms in D.

In particular given any functor $F : \mathsf{C} \to \mathsf{D}$ mapping elements of W to invertible morphisms of D there exists a functor $\widetilde{F} : \mathsf{C}[W^{-1}] \to \mathsf{D}$ and a natural isomorphism

$$F \Rightarrow \widetilde{F} \circ U.$$

Following a common abuse of notation, we often denote the localizations of C at W simply as $\mathsf{C}[W^{-1}]$ (and omit the localization functor U).

The localization $\mathsf{C}[W^{-1}]$ is defined up to equivalence of bicategories, so it will be convenient to refer to any functor $F : \mathsf{C} \to \mathsf{C}'$ between bicategories as a localization of C at the class W if it has the same universal property as the localization functor $U : \mathsf{C} \to \mathsf{C}[W^{-1}]$. Namely we ask that for any bicategory D the precomposition with F induces an equivalence of bicategories $\mathsf{Hom}(\mathsf{C}', \mathsf{D}) \xrightarrow{-\circ F} \mathsf{Hom}_W(\mathsf{C}, \mathsf{D})$. Pronk [1996] gave a criterion for a functor $F : \mathsf{C} \to \mathsf{C}'$ to be "the" localization of C at the class W:

Proposition 2.11 [Pronk 1996, Proposition 24]. *A functor $F : \mathsf{C} \to \mathsf{C}'$ between bicategories is a localization of C at the class W if*

(1) *F sends the elements of W to (weakly) invertible 1-morphisms in C';*

(2) *F is essentially surjective on objects;*

(3) *for every 1-morphism f in C' there are 1-morphisms w in W and g in C with a 2-morphism $F(g) \Rightarrow f \circ F(w)$;*

(4) *F is fully faithful on 2-morphisms.*

Example 2.12. The functor $U := \langle\ \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ is the localization of the 2-categories of Lie groupoids, functors and natural transformations at the class of essential equivalences.

Localizations of bicategories will come up several times in this paper.

2C. *2-vector spaces and Lie 2-algebras.*

Definition 2.13. A (real) 2-*vector space* [Baez and Crans 2004] is a category V internal to the category of (real)vector spaces. Hence $V = \{V_1 \rightrightarrows V_0\}$, where V_0 a vector space of objects, V_1 a vector space of morphisms, and all the structure maps (source, target, unit, and composition) are linear.

There is a 2-category $\mathsf{2Vect}$ whose objects are 2-vector spaces, 1-morphisms are (linear) functors and 2-morphisms are (linear) natural transformations. There is a forgetful functor $\mathsf{2Vect} \to \mathsf{Cat}$ from the 2-category of 2-vector spaces to the 2-category Cat of categories that forgets the linear structure.

Remark 2.14. There is an equivalence of categories of 2-vector spaces and of 2-term chain complexes of vector spaces. See, for example, [Baez and Crans 2004]. A similar result was proved much earlier by Deligne [SGA 4_3 1973]. We remind the reader of how this equivalence is defined on objects. Given a 2-term complex $\partial : U \to W$ there is an action of the abelian group U on W given by

$$u \cdot w := \partial(u) + w \tag{2-3}$$

for all $u \in U, w \in W$. The corresponding action groupoid $\{U \times W \rightrightarrows W\}$ is a 2-vector space.

The converse is true as well: any 2-vector space $V = \{V_1 \rightrightarrows V_0\}$ is isomorphic to an action groupoid defined by the 2-term complex $\partial = t|_{\ker s} : \ker s \to V_0$. Here as before $s, t : V_1 \to V_0$ are the source and target map of the category V; see [Baez and Crans 2004] for a proof. In particular a category underlying a 2-vector space is a groupoid.

Definition 2.15. A strict *Lie 2-algebra* is a category internal to the category of Lie algebras (over the reals): the space of objects and morphisms of a Lie 2-algebra are ordinary Lie algebras and all the structure maps are maps of Lie algebras.

Notation 2.16. Categories internal to Lie algebras, internal functors and internal natural transformations form a strict 2-category which we denote by $\mathsf{Lie2Alg}_{\mathrm{strict}}$.

Definition 2.17. (see, for example, [Frégier and Wagemann 2011, Definition 15]) A *crossed module* of Lie algebras consists of a Lie algebra homomorphism $\partial : \mathfrak{m} \to \mathfrak{n}$ together with a Lie algebra homomorphism

$$D : \mathfrak{n} \to \mathsf{Der}(\mathfrak{m})$$

from $\mathfrak{n}$ to the Lie algebra $\mathsf{Der}(\mathfrak{m})$ of derivations of $\mathfrak{m}$ so that, for all $m, m' \in \mathfrak{m}$, $n \in \mathfrak{n}$,

(i) $\partial(D(n)m) = [n, \partial(m)]$ and

(ii) $D(\partial(m))m' = [m, m']$.

A crossed module of Lie algebras determines a Lie 2-algebra: see, for example, the proof of Theorem 3 in [Frégier and Wagemann 2011]. The converse is true as well: any Lie 2-algebra canonically defines a crossed module of Lie 2-algebras. In fact more is true: crossed modules form a strict 2-category, and the 2-categories of Lie 2-algebras and of crossed modules are equivalent (see [Frégier and Wagemann 2011, Theorem 3]). We need the following result:

Lemma 2.18. *Let $V = \{V_1 \rightrightarrows V_0\}$ be a 2-vector space. Suppose the corresponding 2-term complex $\partial = t|_{\ker s} : \ker s \to V_0$ is part of the data of a Lie algebra crossed module. That is, suppose that V_0, $\ker s$ are Lie algebras, ∂ is a Lie algebra map, and that there is an action $D : V_0 \to \mathsf{Der}(\ker s)$ of V_0 on $\ker s$ by derivations making $(\partial : \ker s \to V_0, D : V_0 \to \mathsf{Der}(\ker s))$ into a crossed module of Lie algebras. Then V is a Lie 2-algebra.*

Sketch of proof. Since $1 \circ s = \mathrm{id}_{V_0}$, we have $V_1 = \ker s \oplus V_0$. We define a bracket on $\ker s \oplus V_0$ by

$$[(x_1, y_1), (x_2, y_2)] := \big([x_1, x_2] + D(y_1)x_2 - D(y_2)x_1, [y_1, y_2]\big) \tag{2-4}$$

for all $(x_1, y_1), (x_2, y_2) \in \ker s \oplus V_0$. That is, we define the Lie algebra V_1 to be the semidirect product of V_0 and $\ker s$. Checking that source, target and unit maps of V are Lie algebra maps is easy. To check that the composition $m : V_1 \times_{V_0} V_1 \to V_1$ in the category V is a Lie algebra map we observe that m is given by

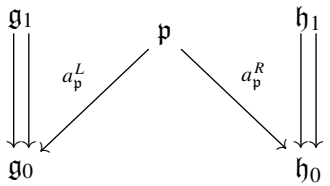
$$m((x_1, y_1), (x_2, y_2)) = (x_1 + x_2, y_2) \tag{2-5}$$

for all $(x_1, y_1), (x_2, y_2) \in \ker s \oplus V_0$ with $y_1 = t(x_2, y_2) = \partial x_2 + y_2$. This fact is not completely obvious. It lies in the heart of the correspondence between 2-vector spaces and 2-term chain complexes. See Remark 2.14 and [Baez and Crans 2004]. A computation now shows that the map m is a Lie algebra map. □

There is a problem with the 2-category $\mathsf{Lie2Alg}_{\text{strict}}$ of Lie 2-algebras. Namely, suppose $f : \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie 2-algebras which is fully faithful and essentially surjective, that is, an essential equivalence. Then f has a weak inverse (as a functor), but there is no reason for that inverse to be a morphism of Lie 2-algebras. In fact it is easy to come up with examples where such morphism of Lie 2-algebras does not exist.

Fortunately the problem has a universal solution: we localize the 2-category $\mathsf{Lie2Alg}_{\text{strict}}$ at the class of essential equivalences and obtain a bicategory $\mathsf{Lie2Alg}$ (see [Pronk 1996] and Section 2B above). This localization has a simple and explicit description: we define a morphism between Lie 2-algebras as a "bibundle internal to the category of Lie algebras." Here are the details.

Definition 2.19. A *bibundle* $\mathfrak{p} : \mathfrak{g} \to \mathfrak{h}$ from a Lie 2-algebra $\mathfrak{g}$ to a Lie 2-algebra $\mathfrak{h}$ is a Lie algebra $\mathfrak{p}$ with *left* and *right anchor maps* $a^L_{\mathfrak{p}}$ and $a^R_{\mathfrak{p}}$ (which are maps of Lie algebras),



along with a left action of the groupoid $\mathfrak{g}$ and right action of the groupoid $\mathfrak{h}$

$$\mathfrak{g}_1 \times_{s,\mathfrak{g}_0,a^L_{\mathfrak{p}}} \mathfrak{p} \to \mathfrak{p}, \quad (g, p) \mapsto g \cdot p, \qquad \mathfrak{p} \times_{a^R_{\mathfrak{p}},\mathfrak{h}_0,t} \mathfrak{h}_1 \to \mathfrak{p}, \quad (p, h) \mapsto p \cdot h.$$

We require that the actions are maps of Lie algebras and commute with each other. We require that $a^L_{\mathfrak{p}}$ is surjective. Finally, we require that the map

$$\mathfrak{p} \times_{a^R_{\mathfrak{p}},\mathfrak{h}_0,t} \mathfrak{h}_1 \to \mathfrak{p} \times_{a^L_{\mathfrak{p}},\mathfrak{g}_0,a^L_{\mathfrak{p}}} \mathfrak{p}, \quad (p, h) \mapsto (p, p \cdot h)$$

is an isomorphism of Lie algebras. Thus in particular we require that $a^L_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{g}_0$ is a principal $\mathfrak{h}$-bundle.

Remark 2.20. The composition of bibundles between Lie 2-algebras is defined in the same way as in the case of bibundles between Lie groupoids: it is the quotient of the appropriate fiber product. We will omit a proof that Lie 2-algebras, bibundles of Lie algebras and isomorphisms of bibundles form a bicategory. We denote this bicategory by $\mathsf{Lie2Alg}$. We note that biprincipal bibundles are weakly invertible in this bicategory.

As in the case of Lie groupoids there is a functor $\langle\ \rangle : \mathsf{Lie2Alg}_{\text{strict}} \to \mathsf{Lie2Alg}$. It sends a strict map $f : \mathfrak{g} \to \mathfrak{h}$ of Lie 2-algebras to the bibundle

$$\langle f \rangle := \mathfrak{g}_0 \times_{f_0,\mathfrak{h}_0,t} \mathfrak{h}_1 := \{(x, \gamma) \mid f_0(x) = t(\gamma)\}.$$

Lemma 2.21. *Suppose $f : \mathfrak{g} \to \mathfrak{h}$ is a strict map of Lie 2-algebras whose underlying functor is fully faithful and essentially surjective. Then the bibundle of Lie 2-algebras*

$$\langle f \rangle : \mathfrak{g} \to \mathfrak{h}$$

is weakly invertible.

Proof. It is enough to show that $a^R_{\langle f \rangle} : \langle f \rangle \to \mathfrak{h}$ is an $\mathfrak{g}$-principal bundle. That is, it is enough to show that $a^R_{\langle f \rangle}$ is surjective and that the map

$$\varphi : \mathfrak{p} \times_{a^R_{\mathfrak{p}}, \mathfrak{h}_0, t} \mathfrak{h}_1 \to \mathfrak{p} \times_{a^L_{\mathfrak{p}}, \mathfrak{g}_0, a^L_{\mathfrak{p}}} \mathfrak{p}, \quad \varphi(p, h) := (p, p \cdot h)$$

is an isomorphism of Lie algebras. Since $a^R_{\langle f \rangle}(x, \gamma) = s(\gamma)$ the surjectivity of $a^R_{\langle f \rangle}$ is equivalent to the essential surjectivity of the functor f. The fullness of f translates into φ being onto and faithfulness of f translates into φ being 1-1. □

We now apply Proposition 2.11 to conclude that $\langle\ \rangle : \mathsf{Lie2Alg}_{\mathrm{strict}} \to \mathsf{Lie2Alg}$ is the localization of $\mathsf{Lie2Alg}_{\mathrm{strict}}$ at the class of essential equivalences. See also Theorem 4.4 below for a similar argument.

Remark 2.22. A reader familiar with Noohi's butterflies [2013] should not have much trouble showing that the bicategory Lie2Alg of Lie 2-algebras defined above is equivalent to the bicategory of crossed modules of Lie algebras, butterflies and isomorphisms of butterflies.

2D. ***Tangent functors.*** Recall that we have a functor T from the category Man of C^∞ manifolds to itself which sends a manifold M to its tangent bundle $TM \xrightarrow{\pi_M} M$. We also have a natural transformation $\pi : T \Rightarrow \mathrm{id}_{\mathsf{Man}}$ whose components are the projections $\pi_M : TM \to M$.

Remark 2.23. Recall that if a point c is a regular value of a smooth map $f : M \to N$ between two manifolds then $(Tf)^{-1}(c, 0) = T(f^{-1}(c))$. Consequently the tangent functor T preserves transverse fiber products.

The following result is well-known. We omit the proof.

Lemma 2.24. *The tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ and the natural transformation $\pi : T \Rightarrow \mathrm{id}_M$ extend to a functor $T^{\mathsf{LieGpd}} : \mathsf{LieGpd} \to \mathsf{LieGpd}$ and to a natural transformation $\pi : T^{\mathsf{LieGpd}} \Rightarrow \mathrm{id}_{\mathsf{LieGpd}}$.*

Lemma 2.25. *The functor $T : \mathsf{Man} \to \mathsf{Man}$ extends to a functor $T^{\mathsf{Bi}} : \mathsf{Bi} \to \mathsf{Bi}$ from the bicategory* Bi *of Lie groupoids, bibundles and isomorphisms of bibundles to itself.*

Sketch of proof. For a Lie groupoid G we set $T^{\mathsf{Bi}}(G) = TG$. This defines T^{Bi} on 0-cells. Given a bibundle $P : G \to H$ the application of the tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ gives us the bibundle $TP : TG \to TH$. Given two bibundles

$P, Q : G \to H$ and an isomorphism $\alpha : P \to Q$ of bibundles its derivative $T\alpha : TP \to TQ$ is also an isomorphism of bibundles. It is not hard to check that for any two Lie groupoids G and H the map $T : \mathsf{Hom}_{\mathsf{Bi}}(G, H) \to \mathsf{Hom}_{\mathsf{Bi}}(TG, TH)$ defined above is a functor. Given a Lie groupoid G we defined the identity bibundle $\langle \mathrm{id}_G \rangle$ to be the manifold G_1 together with the source and target maps as left and right anchors and left and right multiplications as left and right actions of G on G_1. Then $T\langle \mathrm{id}_G \rangle = TG_1 = \langle \mathrm{id}_{TG} \rangle$. Hence the comparison 2-cells $\mu_G : \mathrm{id}_{T^{\mathsf{Bi}}(G)} \to T^{\mathsf{Bi}}(\mathrm{id}_G)$ are identity 2-cells. We also need the comparison 2-cells $\mu_{Q,P} : T^{\mathsf{Bi}}Q \circ T^{\mathsf{Bi}}P \to T^{\mathsf{Bi}}(Q \circ P)$. They are constructed as follows. Given a pair $G \xrightarrow{P} H \xrightarrow{Q} K$ of composable bibundles $T(P \times_{H_0} Q) = TQ \times_{TH_0} TQ$ (cf. Remark 2.23). Additionally $T(P \times_{H_0} Q)/H \simeq \big(T(P \times_{H_0} Q)\big)/TH$ since for any Lie groupoid H and any H-principal bundle $R \to B$, TR/TH is isomorphic to TB. Consequently

$$T(Q \circ P) = T((P \times_{H_0} Q)/H) \simeq T(P \times_{H_0} Q)/TH \simeq (TP \times_{TH_0} TQ)/TH = TQ \circ TP.$$

This diffeomorphism is the desired invertible 2-cell $\mu_{Q,P}$. □

Lemma 2.26. *The functors $U \circ T^{\mathsf{LieGpd}}$ and $T^{\mathsf{Bi}} \circ U$ are isomorphic. Here $U := \langle\ \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ is the localization functor.*

Proof. We construct a pseudonatural transformation $\sigma : U \circ T^{\mathsf{LieGpd}} \Rightarrow T^{\mathsf{Bi}} \circ U$ as follows. Since for a groupoid G $U \circ T^{\mathsf{LieGpd}}(G) = TG = T^{\mathsf{Bi}} \circ U(G)$ we set $\sigma_G := \langle \mathrm{id}_{TG} \rangle$. Given a functor $f : G \to H$ we define the 2-cell

$$\sigma_f : \sigma_H \circ T^{\mathsf{Bi}}(U(f)) \Rightarrow U(T^{\mathsf{LieGpd}}(f)) \circ \sigma_G$$

as composite of the isomorphisms of bibundles $\langle \mathrm{id}_{TH} \rangle \circ T\langle f \rangle \to T\langle f \rangle \to \langle Tf \rangle \to \langle Tf \rangle \circ \langle \mathrm{id}_{TG} \rangle$. Here the middle arrow is the diffeomorphism $T(G_0 \times_{f,H_0,t} H_1) \xrightarrow{\simeq} TG_0 \times_{Tf,TH_0,Tt} TH_1$. Given an isomorphism $\alpha : f \Rightarrow f'$ between two functors $f, f' : G \to H$ the diagram

$$\begin{array}{ccc} T(G_0 \times_{f,H_0,t} H_1) & \xrightarrow{\sim} & TG_0 \times_{Tf,TH_0,Tt} TH_1 \\ \Big\downarrow{\scriptstyle T\langle\alpha\rangle} & & \Big\downarrow{\scriptstyle \langle T\alpha\rangle} \\ T(G_0 \times_{f',H_0,t} H_1) & \xrightarrow[\sim]{} & TG_0 \times_{Tf',TH_0,Tt} TH_1 \end{array}$$

commutes. It follows that σ_f's are components of a natural transformation

$$\sigma_{G,H} : (U \circ T^{\mathsf{LieGpd}}) \circ \sigma_G \Rightarrow \sigma_H \circ (T^{\mathsf{Bi}} \circ U).$$ □

Lemma 2.27. *There exists a transformation $\tilde{\pi} : T^{\mathsf{Bi}} \Rightarrow \mathrm{id}_{\mathsf{Bi}}$ whose 1-cells $\tilde{\pi}_G : T^{\mathsf{Bi}}(G) \to G$ are (isomorphic to) the bibundles $\langle \pi_G \rangle$, where as before $\pi_G : TG \to G$ are the projection functors.*

Proof. Since $U = \langle\ \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ is a localization functor, the pullback by U defines an equivalence of bicategories

$$U^* := - \circ U : \mathsf{Hom}(\mathsf{Bi}, \mathsf{Bi}) \to \mathsf{Hom}_W(\mathsf{LieGpd}, \mathsf{Bi}).$$

Here as before $\mathsf{Hom}_W(\mathsf{LieGpd}, \mathsf{Bi})$ denotes the bicategory of functors that send essential equivalence in LieGpd to invertible bibundles. Consequently for any two functors $F, G : \mathsf{Bi} \to \mathsf{Bi}$ the functor

$$U^* : \mathsf{Hom}(F, G) \to \mathsf{Hom}(F \circ U, G \circ U)$$

is an equivalence of categories. Note that the objects of $\mathsf{Hom}(F, G)$ are pseudonatural transformations and morphisms are modifications. In Lemma 2.24 we constructed a natural transformation $\pi : T^{\mathsf{LieGpd}} \to \mathrm{id}$ and Lemma 2.26 we constructed a natural isomorphism $\sigma : U \circ T^{\mathsf{LieGpd}} \Rightarrow T^{\mathsf{Bi}} \circ U$. Therefore we have a natural transformation $U\pi \circ \sigma^{-1} : T^{\mathsf{Bi}} \circ U \Rightarrow U$. Since $U^* : \mathsf{Hom}(T^{\mathsf{Bi}}, \mathrm{id}) \to \mathsf{Hom}(T^{\mathsf{Bi}} \circ U, U)$ is essentially surjective, there exists a pseudonatural transformation $\tilde{\pi} : T^{\mathsf{Bi}} \to \mathrm{id}$ so that $\tilde{\pi} \circ U$ differs from $U\pi \circ \sigma^{-1}$ by a modification. It will be convenient to fix one such modification throughout the paper. It follows that for each Lie groupoid G we have chosen an isomorphism of bibundles $\tilde{\pi}_G \to \langle \pi_G \rangle$ where $\tilde{\pi}_G$ is the component of the transformation $\tilde{\pi}$ at G. □

3. The Lie 2-algebra $\mathbb{X}(G)$ of vector fields on a Lie groupoid G

In this section we prove Theorem 3.4: the category of multiplicative vector fields on a Lie groupoid underlies a strict Lie 2-algebra.

Definition 3.1 [Hepworth 2009]. Consider a Lie groupoid G with its tangent groupoid $\pi_G : TG \to G$. The *category* $\mathbb{X}(G)$ *of multiplicative vector fields* is defined as follows. The set of *objects* of $\mathbb{X}(G)$ is

$$\mathbb{X}(G)_0 := \{v : G \to TG \mid v \text{ is a functor and } \pi_G \circ v = \mathrm{id}_G\}.$$

This is the set of multiplicative vector fields of Mackenzie and Xu [1998]. A *morphism* in $\mathbb{X}(G)$ from a multiplicative vector field v to a multiplicative vector field w is a natural transformation $\alpha : v \Rightarrow w$ such that for every point $x \in G_0$

$$\pi_G(\alpha_x) = 1_x. \tag{3-1}$$

The composition of morphisms is the vertical composition of natural transformations. Note that the category $\mathbb{X}(G)$ is a groupoid.

Notation 3.2. We denote the source and target maps in the category $\mathbb{X}(G)$ by $\mathfrak{s}$ and $\mathfrak{t}$, respectively. The unit map is denoted by $\mathbf{1}$, the inversion by $(\ )^{-1}$ and the composition/multiplication of morphisms by $\circ$.

Lemma 3.3. *The category of multiplicative vector fields $\mathbb{X}(G)$ on a Lie groupoid G is a 2-vector space.*

Proof. Mackenzie and Xu [1998] proved that the set $\mathbb{X}(G)_0$ of multiplicative vector fields is a real vector space.

We next argue that the set of morphisms $\mathbb{X}(G)_1$ of the category $\mathbb{X}(G)$ is a vector space as well. Suppose $\alpha_1 : v_1 \Rightarrow w_1$ and $\alpha_2 : v_2 \Rightarrow w_2$ are morphisms between multiplicative vector fields. Equation (3-1) says that α_1 and α_2 are both sections of the vector bundle $TG_1|_{G_0} \to G_0$ where we have suppressed the unit map $1_G : G_0 \to G_1$. Clearly the linear combination $\lambda_1\alpha_1 + \lambda_2\alpha_2$ is again a section of the bundle $TG_1|_{G_0} \to G_0$ for any choice of scalars λ_1, λ_2. We need to check that it is actually a natural transformation from $\lambda_1 v_1 + \lambda_2 v_2$ to $\lambda_1 w_1 + \lambda_2 w_2$. That is, we need to check that for any arrow $y \xleftarrow{\gamma} x$ in the groupoid G

$$(\lambda_1\alpha_1 + \lambda_2\alpha_2)_y \bullet ((\lambda_1 v_1 + \lambda_2 v_2)(\gamma)) = ((\lambda_1 w_1 + \lambda_2 w_2)(\gamma)) \bullet (\lambda_1\alpha_1 + \lambda_2\alpha_2)_x.$$

Here and below $\bullet : TG_1 \times_{TG_0} TG_1 \to TG_1$ denotes the multiplication in the Lie groupoid TG.

Since $\bullet$ is the derivative of the multiplication $m : G_1 \times_{G_0} G_1 \to G_1$ in the groupoid G, it is fiberwise linear: for any $(\gamma_2, \gamma_1) \in G_1 \times_{G_0} G_1$ and $(a_1, a_2), (b_1, b_2) \in T_{\gamma_2}G_1 \times_{TG_0} T_{\gamma_1}G_1 = T_{(\gamma_1,\gamma_2)}(G_1 \times_{G_0} G_1)$ we have (in the prefix notation)

$$\bullet(\lambda(a_1, a_2) + \mu(b_1, b_2)) = \lambda(\bullet(a_2, a_1)) + \mu(\bullet(b_1, b_2)) \tag{3-2}$$

for all scalars λ, μ. In the infix notation (3-2) reads

$$(\lambda a_1 + \mu b_1) \bullet (\lambda a_2 + \mu b_2) = \lambda(a_1 \bullet a_2) + \mu(b_1 \bullet b_2). \tag{3-3}$$

Hence

$$\begin{aligned}(\lambda_1\alpha_1 + \lambda_2\alpha_2)_y \bullet ((\lambda_1 v_1 + \lambda_2 v_2)(\gamma)) &= \lambda_1((\alpha_1)_y \bullet v_1(\gamma)) + \lambda_2((\alpha_2)_y \bullet v_2(\gamma)) \\ &= \lambda_1(w_1(\gamma) \bullet (\alpha_1)_x) + \lambda_2(w_2(\gamma) \bullet (\alpha_2)_x) \\ &= ((\lambda_1 w_1 + \lambda_2 w_2)(\gamma)) \bullet (\lambda_1\alpha_1 + \lambda_2\alpha_2)_x.\end{aligned}$$

Here the first and third equalities hold by (3-3). In the second equality we used the fact that $\alpha_1 : v_1 \Rightarrow w_1$ and $\alpha_2 : v_2 \Rightarrow w_2$ are natural transformations. Therefore the space of morphisms $\mathbb{X}(G)_1$ is a vector space.

Moreover the computation above shows that for $\lambda_1, \lambda_2 \in \mathbb{R}$, $\alpha_1 : v_1 \Rightarrow w_1$, $\alpha_2 : v_2 \Rightarrow w_2 \in \mathbb{X}(G)_1$ the source of $\lambda_1\alpha_1 + \lambda_2\alpha_2$ is $\lambda_1 v_1 + \lambda_2 v_2$. That is, the source map $\mathfrak{s} : \mathbb{X}(G)_1 \to \mathbb{X}(G)_0$ of the category $\mathbb{X}(G)$ is linear. Similarly the target map $\mathfrak{t}$ is linear. It is also easy to see that the unit map $\mathbb{X}(G)_0 \to \mathbb{X}(G)_1$ is linear as well.

Finally we need to check that multiplication/composition $\circ$ in the category $\mathbb{X}(G)$, which is the vertical composition of natural transformations, is linear as a map from $\mathbb{X}(G)_1 \times_{\mathbb{X}(G)_0} \mathbb{X}(G)_1$ to $\mathbb{X}(G)_1$. That is, we need to check that

$$(\lambda\alpha_2 + \mu\beta_2) \circ (\lambda\alpha_1 + \mu\beta_1) = \lambda(\alpha_2 \circ \alpha_1) + \mu(\beta_2 \circ \beta_1) \tag{3-4}$$

for all $\lambda, \mu \in \mathbb{R}$, $(\alpha_2, \alpha_1), (\beta_2, \beta_1) \in \mathbb{X}(G)_1 \times_{\mathbb{X}(G)_0} \mathbb{X}(G)_1$. Recall that the vertical composition $\circ$ is computed pointwise: for any composable natural transformations δ_1, δ_1 and any point $x \in G_0$ $(\delta_2 \circ \delta_1)_x = (\delta_2)_x \bullet (\delta_1)_x$, where as before $\bullet$ is the multiplication in TG. Since $\bullet$ is fiberwise linear (3-4) follows. □

Theorem 3.4. *The category of vector fields $\mathbb{X}(G)$ on a Lie groupoid G is a (strict) Lie 2-algebra.*

Proof. Recall the notation: $s : G_1 \to G_0$ is the source map for the groupoid G, its differential $Ts : TG_1 \to TG_0$ is the source map for the tangent groupoid TG. We use $\mathfrak{s}, \mathfrak{t}$ to denote the source and target maps of the groupoid $\mathbb{X}(G)$, respectively.

By Lemma 2.18 it is enough to: give the vector spaces $\ker(\mathfrak{s} : \mathbb{X}(G)_1 \to \mathbb{X}(G)_0)$ and $\mathbb{X}(G)_0$ the structure of Lie algebras; check that $\partial := \mathfrak{t}|_{\ker \mathfrak{s}} : \ker \mathfrak{s} \to \mathbb{X}(G)_0$ is a Lie algebra map; define an action $D : \mathbb{X}(G)_0 \to \mathsf{Der}(\ker \mathfrak{s})$ on $\ker \mathfrak{s}$ by derivations; and check the compatibility of ∂ and D:

$$\partial(D(X)\alpha) = [X, \partial(\alpha)], \tag{3-5}$$

$$D(\partial\alpha_1)\alpha_2 = [\alpha_1, \alpha_2] \tag{3-6}$$

for all $\alpha, \alpha_1, \alpha_2 \in \ker \mathfrak{s}$ and all multiplicative vector fields X on the Lie groupoid G (compare with Definition 2.17).

The fact that the vector space $\mathbb{X}(G)_0$ of multiplicative vector fields carries a Lie bracket is due to Mackenzie and Xu [1998]. We argue next that $\ker \mathfrak{s}$ is the space of sections of the Lie algebroid $A_G \to G_0$. By definition of the source map $\mathfrak{s}$, $\ker \mathfrak{s} = \{\alpha : X \Rightarrow Y \mid X = 0\}$. Therefore $\alpha \in \ker \mathfrak{s}$ if and only if there is a multiplicative vector field Y so that the diagram

$$\begin{array}{ccc} 0_x & \xrightarrow{0_\gamma} & 0_y \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ Y(x) & \xrightarrow[Y(\gamma)]{} & Y(y) \end{array} \tag{3-7}$$

commutes for all arrows $y \xleftarrow{\gamma} x$ in G_1. Hence if $\alpha \in \ker \mathfrak{s}$ then $Ts(\alpha_x) = 0_x$ for all $x \in G_0$. That is, α is a section of $A_G \to G_0$. Conversely if $\alpha : G_0 \to A_G$ is a section of the Lie algebroid we can define a multiplicative vector field $Y : G \to TG$ so that (3-7) commutes. Namely on objects we define $Y(x) := Tt(\alpha_x)$ for all $x \in G_0$.

And for $y \xleftarrow{\gamma} x$ in G_1 we set

$$Y(\gamma) = \alpha_y \bullet 0_\gamma \bullet (\alpha_x)^{-1}. \tag{3-8}$$

Here as before $\bullet$ is the multiplication in TG and $()^{-1} = Ti : TG_1 \to TG_1$ is the inverse map, which is the derivative of the inverse map i of the groupoid G. We conclude that $\ker(\mathfrak{s} : \mathbb{X}(G)_1 \to \mathbb{X}(G)_0) = \Gamma(A_G)$ and that $\partial := \mathfrak{t}|_{\ker \mathfrak{s}} : \ker \mathfrak{s} \to \mathbb{X}(G)_0$ is given by

$$(\partial\alpha)(\gamma) = \alpha_{t(\gamma)} \bullet 0_\gamma \bullet (\alpha_{s(\gamma)})^{-1} \tag{3-9}$$

for all $\gamma \in G_1$. Note that (3-9) can be written as

$$\partial\alpha = \overrightarrow{\alpha} + \overleftarrow{\alpha}. \tag{3-10}$$

where $\overrightarrow{\alpha}(\gamma) = TR_\gamma\, \alpha(t(\gamma))$ and $\overleftarrow{\alpha}(\gamma) = T(L_\gamma \circ i)\, \alpha(t(\gamma))$ for all $\gamma \in G_1$. Here R_γ and L_γ are right and left multiplications by γ, respectively. The bracket on the space of sections $\Gamma(A_G)$ of the Lie algebroid $A_G \to G_0$ is defined by requiring that the injective map $^{\rightarrow} : \Gamma(A_G) \to \Gamma(TG_1)$, $\alpha \mapsto \overrightarrow{\alpha}$ is a map of Lie algebras. Consequently $^{\leftarrow} : \Gamma(A_G) \to \Gamma(TG_1)$, $\alpha \mapsto \overleftarrow{\alpha}$ is also a map of Lie algebras. Since left- and right-invariant vector fields commute (cf. [Mackenzie 2005]) we conclude that $\partial = \mathfrak{t}|_{\ker \mathfrak{s}} : \ker \mathfrak{s} = \Gamma(A_G) \to \mathbb{X}(G)_0$ is a Lie algebra map.

Following Mackenzie and Xu we define the map D from the space $\mathbb{X}(G)_0$ of multiplicative vector fields to $\mathsf{Hom}(\Gamma(TG_1|_{G_0}), \Gamma(TG_1|_{G_0}))$ by setting $D(X)\alpha := [X, \overrightarrow{\alpha}]|_{G_0}$ for all multiplicative vector fields X and all sections $\alpha \in \Gamma(A_G)$. Mackenzie and Xu [1998, Proposition 3.7] proved that $[X, \overrightarrow{\alpha}]$ is tangent to the fibers of s and is right invariant. Hence $[X, \overrightarrow{\alpha}]|_{G_0}$ is a section of the Lie algebroid $A_G \to G_0$. They furthermore showed that $D(X)$ is a derivation of $\Gamma(A_G)$ and that $D : \mathbb{X}(G)_0 \to \mathsf{Der}(\Gamma(A_G))$ is a map of Lie algebras [Mackenzie and Xu 1998, Proposition 3.8]. Since left- and right-invariant vector fields commute, for any $\alpha_1, \alpha_2 \in \Gamma(A_G)$ we have $[\partial\alpha_1, \overrightarrow{\alpha}_2] = [\overrightarrow{\alpha}_1 + \overleftarrow{\alpha}_1, \overrightarrow{\alpha}_2] = [\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2]$ and (3-6) follows.

We end the proof by showing that (3-5) holds. On the right-hand side we have $[X, \partial\alpha] = [X, \overrightarrow{\alpha} + \overleftarrow{\alpha}] = [X, \overrightarrow{\alpha}] + [X, \overleftarrow{\alpha}]$ while on the left, $\partial(D(X)\,\alpha) = (D(X)\,\alpha)^{\rightarrow} + (D(X)\,\alpha)^{\leftarrow}$. By definition of D, $(D(X)\,\alpha)^{\rightarrow} = [X, \overrightarrow{\alpha}]$, so it remains to prove that $[X, \overleftarrow{\alpha}] = (D(X)\,\alpha)^{\leftarrow}$. Since X is a functor, $Ti \circ X = X \circ i$. The inversion map i relates right- and left-invariant vector fields. That is, $Ti \circ \overrightarrow{\alpha} = \overleftarrow{\alpha} \circ i$ for all α. Consequently

$$\begin{aligned}(D(X)\,\alpha)^{\leftarrow}(g) &= T(L_g \circ i)(D(X)\,\alpha)(1_{s(g)}) \\ &= T(L_g)Ti[X, \overrightarrow{\alpha}](1_{s(g)}) = TL_g[X, \overleftarrow{\alpha}](i(1_{s(g)})).\end{aligned}$$

Since $[X, \overleftarrow{\alpha}]$ is left-invariant, $TL_g[X, \overleftarrow{\alpha}](i(1_{s(g)})) = [X, \overleftarrow{\alpha}](g)$. It follows that $(D(X)\,\alpha)^{\leftarrow}(g) = [X, \overleftarrow{\alpha}](g)$ for all $g \in G_1$ and we are done. □

4. Morita invariance of the Lie 2-algebra of vector fields

The goal of this section is to prove:

Theorem 4.1. *The assignment $G \mapsto \mathbb{X}(G)$ of the category of vector fields to a Lie groupoid extends to a functor*

$$\mathbb{X} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Lie2Alg} \tag{4-1}$$

from the bicategory $\mathsf{Bi}_{\mathrm{iso}}$ *of Lie groupoids, invertible bibundles and isomorphisms of bibundles to the bicategory* Lie2Alg *of Lie* 2*-algebras. Hence, in particular, if $P : G \to H$ is a Morita equivalence of Lie groupoids then $\mathbb{X}(P) : \mathbb{X}(G) \to \mathbb{X}(H)$ is a* (*weakly*) *invertible* 1*-morphism of Lie* 2*-algebras in the bicategory* Lie2Alg.

Our strategy for constructing the functor $\mathbb{X}$ is to first construct it on a simpler category.

Definition 4.2. An *essentially surjective open embedding* of Lie groupoids is a functor $f : U \to G$ such that

(1) the maps on objects $f_0 : U_0 \to G_0$ and on morphisms $f_1 : U_1 \to G_1$ are open embeddings and

(2) the functor f is an essential equivalence, i.e., the corresponding bibundle

$$\langle f \rangle := U_0 \times_{f_0, G_0, t} G_1 : U \to G$$

is weakly invertible. Equivalently $\langle a^R \rangle : \langle f \rangle \to G_0$ is a principal U-bundle.

Remark 4.3. It is clear that Lie groupoids, essentially surjective open embeddings and natural transformations form a strict 2-category which we denote by $\mathscr{E}mb$.

Theorem 4.4. *The localization of the* 2*-category $\mathscr{E}mb$ at the class W of all* 1*-morphisms is the bicategory* $\mathsf{Bi}_{\mathrm{iso}}$ *of bicategory of Lie groupoids, invertible bibundles and isomorphisms of bibundles.*

Proof. We apply Proposition 2.11. Consider the localization functor $\langle\ \rangle : \mathsf{LieGpd} \to \mathsf{Bi}$ introduced in Remark 2.5. By definition of the 2-category $\mathscr{E}mb$ the restriction of the functor $\langle\ \rangle$ to $\mathscr{E}mb$ sends every 1-morphism $w : U \to G$ of $\mathscr{E}mb$ to an invertible bibundle $\langle w \rangle$ (and a 2-morphism to an isomorphism of bibundles). This gives us a functor

$$\langle\ \rangle : \mathscr{E}mb \to \mathsf{Bi}_{\mathrm{iso}}, \quad (G \xrightarrow{w} H) \mapsto (G \xrightarrow{\langle w \rangle} H). \tag{4-2}$$

The functor is surjective on objects. By Theorem 2.9 the functor is fully faithful on 2-morphisms.

It remains to check that given an invertible bibundle $P : G \to H$ there exist essentially surjective open embeddings w_G, w_H so that $\langle w_H \rangle \circ P$ is isomorphic to $\langle w_G \rangle$. Since the bibundle P is weakly invertible, it gives rise to the *linking*

groupoid [Weinstein 2009, Proposition 4.3], denoted $G *_P H$ and recalled presently. The manifold of objects $(G *_P H)_0$ is the disjoint union $G_0 \sqcup H_0$ of the objects of the groupoids G and H. The manifold of arrows $(G *_P H)_1$ is the disjoint union $G_1 \sqcup P \sqcup P^{-1} \sqcup H_1$. We think of the manifold P as the space of arrows from the points of H_0 to the points of G_0. We think of the elements of P^{-1} as the inverses of the elements of P. The multiplication in $G *_P H$ comes from the multiplications in the groupoids G and H and the actions of G and H on P and on P^{-1}. The inclusion $w_G : G \to G *_P H$ is given by the open embeddings

$$G_0 \hookrightarrow G_0 \sqcup H_0, \quad G_1 \hookrightarrow G_1 \sqcup P \sqcup P^{-1} \sqcup H_1.$$

It is easy to see that w_G is an essential equivalence, i.e., that the bibundle $\langle w_G \rangle$ is biprincipal, hence weakly invertible. Similarly we have the essentially surjective open embedding $w_H : H \hookrightarrow G *_P H$. A computation shows that the bibundles $\langle w_H \rangle \circ P$ and $\langle w_G \rangle$ are isomorphic. □

Proposition 4.5. *The assignment $G \mapsto \mathbb{X}(G)$ of the Lie 2-algebra of vector fields to a Lie groupoid extends to a contravariant functor*

$$(\mathscr{E}mb)^{op} \to \mathsf{Lie2Alg}_{\mathrm{strict}}, \quad (G \xrightarrow{w} H) \mapsto (\mathbb{X}(H) \xrightarrow{w^*} \mathbb{X}(G)) \tag{4-3}$$

from the 2-category $\mathscr{E}mb$ of Lie groupoids, essentially surjective open embeddings and natural isomorphism to the strict 2-category $\mathsf{Lie2Alg}_{\mathrm{strict}}$ of Lie 2-algebras.

Proof. Consider an essentially surjective open embedding $w : G \to H$. Then $w(G) \subset H$ is an open Lie subgroupoid and $w : G \to w(G)$ is an isomorphism of Lie groupoids. We now assume without any loss of generality that G is an open subgroupoid of H. Then the tangent bundle TG is an open subgroupoid of TH. Moreover, any multiplicative vector field $v : H \to TH$ restricts to a multiplicative vector field $v|_G : G \to TG$. Similarly, a morphism $\alpha : v \Rightarrow u$ of multiplicative vector fields restricts to a morphism $\alpha|_G : v|_G \Rightarrow u|_G$. This gives us a functor

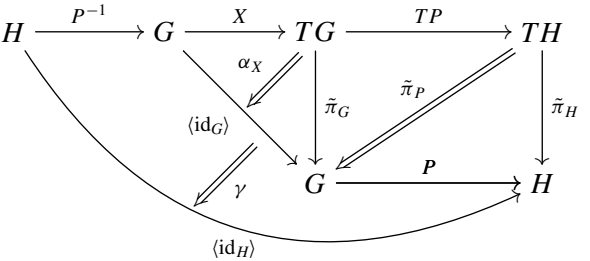
$$w^* : \mathbb{X}(H) \to \mathbb{X}(G), \quad w^*(\alpha : v \Rightarrow u) = (\alpha|_G : v|_G \Rightarrow u|_G). \tag{4-4}$$

The restriction to an open subgroupoid is a map of 2-vector spaces and preserves the brackets. Hence (4-4) is a map of Lie 2-algebras. □

Definition 4.6 (the category $\mathbb{X}_{\mathsf{gen}}(G)$ of *generalized vector fields on a Lie groupoid* G). Recall that there is a natural transformation $\tilde{\pi} : T^{\mathsf{Bi}} \Rightarrow \mathrm{id}_{\mathsf{Bi}}$ (Lemmas 2.25 and 2.27). An object of the category of generalized vector fields $\mathbb{X}_{\mathsf{gen}}(G)$ on a Lie groupoid G is a pair (P, α_P) where $P : G \to TG$ is a bibundle and $\alpha_P : \tilde{\pi}_G \circ P \Rightarrow \langle \mathrm{id}_G \rangle$ an isomorphism of bibundles. A morphism β in $\mathbb{X}_{\mathsf{gen}}(G)$ from (P, α_P) to (Q, α_Q) is a map of bibundles $\beta : P \Rightarrow Q$ so that $\alpha_Q = \alpha_P \circ (\tilde{\pi}_G \star \beta)$. Here as before $\star$ denotes whiskering in Bi, and $\circ$ is the composition of isomorphisms of bibundles.

Lemma 4.7. *A weakly invertible bibundle $P : G \to H$ between two Lie groupoids induces an equivalence of categories $P_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathbb{X}_{\mathsf{gen}}(H)$ between the corresponding categories of generalized vector fields.*

Proof. Since the 1-morphism P is (weakly) invertible, there is a 2-morphism $\gamma : (P \circ \langle \mathrm{id}_G \rangle) \circ P^{-1} \Rightarrow \langle \mathrm{id}_H \rangle$. Given an object (X, α_X) of $\mathbb{X}_{\mathsf{gen}}(G)$ we define $P_*X := TP \circ (X \circ P^{-1})$ The 2-morphism $\alpha_{P_*X} : \tilde{\pi}_H \circ P_*X \Rightarrow \langle \mathrm{id}_H \rangle$ comes from the 2-commutative diagram



We set

$$\alpha_{P_*X} := \gamma \circ (P \star \alpha_X \star P^{-1}) \circ (\tilde{\pi}_P \star (X \circ P^{-1})).$$

Given a morphism $\beta : (X, \alpha_X) \to (Y, \alpha_Y)$ in the category $\mathbb{X}_{\mathsf{gen}}(G)$ we define

$$P_*\beta := TP \star \beta \star P^{-1}.$$

A diagram chase ensures that $P_*\beta$ is a morphism in $\mathbb{X}_{\mathsf{gen}}(H)$ from (P_*X, α_{P_*X}) to (P_*Y, α_{P_*Y}). Finally one checks that the functor $(P^{-1})_* : \mathbb{X}(H) \to \mathbb{X}(G)$ is a weak inverse of P_*. Hence P_* is an equivalence of categories as claimed. □

Definition 4.8. The "inclusion" functor $\iota_G : \mathbb{X}(G) \hookrightarrow \mathbb{X}_{\mathsf{gen}}(G)$ is defined as follows. Suppose $v : G \to TG$ is a multiplicative vector field. Since $U : \mathsf{LieGpd} \to \mathsf{Bi}$ is a functor the composition $\langle \pi_G \rangle \circ \langle v \rangle$ is isomorphic to $\langle \pi_g \circ v \rangle = \langle \mathrm{id}_G \rangle$. Since we fixed the modification from $\tilde{\pi}$ to $\pi \circ U$ there is a canonical isomorphism of bibundles $\tilde{\pi}_G \to \langle \pi_G \rangle$. Consequently there is a canonical isomorphism $\alpha_{\langle v \rangle} : \tilde{\pi}_G \circ \langle v \rangle \Rightarrow \langle \mathrm{id}_G \rangle$. We define $\iota_G(v) := (\langle v \rangle, \alpha_{\langle v \rangle})$. Given a morphism $v \xrightarrow{\gamma} v'$ in $\mathbb{X}(G)$ we define $\iota_G(\gamma) = \langle \gamma \rangle$; it is a morphism in $\mathbb{X}_{\mathsf{gen}}(G)$ from $(\langle v \rangle, \alpha_{\langle v \rangle})$ to $(\langle v' \rangle, \alpha_{\langle v' \rangle})$.

Theorem 4.9. *For any Lie groupoid G the inclusion functor $\iota_G : \mathbb{X}(G) \hookrightarrow \mathbb{X}_{\mathsf{gen}}(G)$ defined above is an equivalence of categories.*

In the case where the groupoid G is proper Theorem 4.9 can be deduced from [Hepworth 2009, Theorem 15]. The proof of Theorem 4.9 is technical; we carry it out in Section 7 below.

Lemma 4.10. *Let $w : G \to G'$ be an essentially surjective open embedding. Then (the 1-morphism of categories underlying) the pull-back/restriction functor $w^* : \mathbb{X}(G') \to \mathbb{X}(G)$, which is given by $w^*(v \xrightarrow{\gamma} v') = (v|_G \xrightarrow{\gamma|_G} v'|_G)$, is fully faithful and essentially surjective.*

Proof. We first argue that w^* is fully faithful. We want to show that given a morphism $\delta : v|_G \to v'|_G$ in $\mathbb{X}(G)$ there exists a unique arrow $\tilde{\delta} : v \to v'$ in $\mathbb{X}(G')$ so that $\tilde{\delta}|_G = \delta$.

We deal with uniqueness first. Suppose $\gamma, \gamma' : v \to v'$ are two morphisms in $\mathbb{X}(G')$ with $\gamma|_G = \gamma'|G$. Fix an object y of G'. Since $w : G \to G'$ is essentially surjective, for any object y of G' there is an arrow $\mu : y \to x$ with x an object of G. Then $\gamma_x = \gamma'_x$ and therefore $\gamma_y = v'(\mu^{-1}) \circ \gamma_x \circ v(\mu) = v'(\mu^{-1}) \circ \gamma'_x \circ v(\mu) = \gamma'_y$. We conclude that $\gamma = \gamma'$. Now given $\delta : v|_G \to v'|_G$ we define $\tilde{\delta} : v \to v'$ at an object y by choosing as above $\mu : y \to x$ and setting $\tilde{\delta}_y = v'(\mu^{-1}) \circ \delta_x \circ v(\mu)$. If $\nu : y \to x$ is another arrow then $v'(\nu \circ \mu^{-1}) \circ \delta_x = \delta_x \circ v(\nu \circ \mu^{-1})$. Therefore $v'(\mu^{-1}) \circ \delta_x \circ v(\mu) = v'(\nu^{-1}) \circ \delta_x \circ v(\nu)$. Moreover the dependence of $\tilde{\delta}$ on y is smooth: since $w : G \to G'$ is an essential equivalence the right anchor $a^R_{\langle w \rangle} : \langle w \rangle \to G'_0$ is a surjective submersion. Note that $\langle w \rangle = t^{-1}(G_0) \subset G'_1$ and $a^R_{\langle w \rangle}(\mu) = s(\mu)$. Choose a local section $\sigma : U \to \langle w \rangle$ of $a^R_{\langle w \rangle}$ with $y \in U$. Then for all points $z \in U$ $\tilde{\delta}(z) = v'((\sigma(z))^{-1}) \circ \delta_{t(\sigma(y))} \circ v(\sigma(z))$, which is smooth. We conclude that w^* is fully faithful.

To prove essential surjectivity we need to argue that for any multiplicative vector field $u : G \to TG$ there is a multiplicative vector field $\tilde{u} : G' \to TG'$ and an isomorphism $\tilde{u}|_G \xrightarrow{\sim} u$. The functor $\iota_{G'} : \mathbb{X}(G') \to \mathbb{X}_{\mathsf{gen}}(G')$ is an equivalence of categories by Theorem 4.9. The functor $(\langle w \rangle)_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathbb{X}_{\mathsf{gen}}(G')$ is an equivalence of categories by Lemma 4.7. Therefore for any $u \in \mathbb{X}(G)_0$ there is a vector field $\tilde{u} \in \mathbb{X}(G')$ and an isomorphism $\langle \tilde{u} \rangle \xrightarrow{\sim} \langle w \rangle_* \langle u \rangle$.

The diagram

$$\begin{array}{ccc} TG & \xrightarrow{Tw} & TG' \\ {\scriptstyle \tilde{u}|_G}\uparrow & & \uparrow{\scriptstyle \tilde{u}} \\ G & \xrightarrow[w]{} & G' \end{array}$$

commutes. This together with the fact that $U : \mathsf{LieGpd} \to \mathsf{Bi}$ is a functor and $T\langle w \rangle$ is isomorphic to $\langle Tw \rangle$ shows that there exists an isomorphism $T\langle w \rangle \circ \langle \tilde{u}|_G \rangle \xrightarrow{\sim} \langle \tilde{u} \rangle \circ \langle w \rangle$. Thus $\langle \tilde{u} \rangle$ is isomorphic to $T\langle w \rangle \circ (\langle \tilde{u}|_G \rangle \circ \langle w \rangle^{-1})$. Hence $T\langle w \rangle \circ (\langle \tilde{u}|_G \rangle \circ \langle w \rangle^{-1}) \xrightarrow{\sim} T\langle w \rangle \circ (\langle u \rangle \circ \langle w \rangle^{-1})$. Consequently there is an isomorphism $\beta : \langle \tilde{u}|_G \rangle \to \langle u \rangle$ of bibundles. Note that this *does not* yet imply that the generalized vector fields $(\langle \tilde{u}|_G \rangle, \alpha_{\langle \tilde{u}|_G \rangle})$ and $(\langle u \rangle, \alpha_{\langle u \rangle})$ are isomorphic in the category $\mathbb{X}_{\mathsf{gen}}(G)$. The issue is $(\langle \tilde{u}|_G \rangle, \alpha_{\langle \tilde{u}|_G \rangle})$ need not equal $\alpha_{\langle u \rangle} \circ (\tilde{\pi}_G \star \beta)$. But $\beta : (\langle \tilde{u}|_G \rangle, \tau) \to (\langle u \rangle, \alpha_{\langle u \rangle})$ is a morphism in $\mathbb{X}_{\mathsf{gen}}(G)$ if we set $\tau = \alpha_{\langle u \rangle} \circ (\tilde{\pi}_G \star \beta)$. By Lemma 7.4 $(\langle \tilde{u}|_G \rangle, \tau)$ is

isomorphic to $(\langle \tilde{u}|_G\rangle, \alpha_{\langle \tilde{u}|_G\rangle})$. Consequently $\iota_G(\tilde{u}|_G)$ is isomorphic to $\iota_G(u)$. Since ι_G is an equivalence of categories $\tilde{u}|_G$ is isomorphic to u and we are done. □

Lemma 4.11. *The functor (4-3) of Proposition 4.5 takes every essentially surjective open embedding to a weakly invertible* 1*-morphism of Lie* 2*-algebras.*

Proof. By Lemma 4.10 the functor $w^* : \mathbb{X}(G') \to \mathbb{X}(G)$ associated to an essentially surjective open embedding $w : G \to G'$ is fully faithful and essentially surjective, hence an essential equivalence of Lie 2-algebras. The localization functor $\langle\ \rangle : \mathsf{Lie2Alg}_{strict} \to \mathsf{Lie2Alg}$ takes all essential equivalences to weakly invertible 1-morphisms. □

We are now in position to extend the assignment $G \mapsto \mathbb{X}(G)$ to a (covariant) functor $\mathbb{X} : \mathscr{E}mb \to \mathsf{Lie2Alg}$.

Definition 4.12. We define the functor $\mathbb{X} : \mathscr{E}mb \to \mathsf{Lie2Alg}$ on objects to be the assignment $G \mapsto \mathbb{X}(G)$. Given an essentially surjective open embedding $G \xrightarrow{w} G'$, the bibundle $\langle w^*\rangle$ is weakly invertible in Lie2Alg by Lemma 4.11. We set $\mathbb{X}(w) := (\langle w^*\rangle)^{-1}$.

Proof of Theorem 4.1. Since $\langle\ \rangle : \mathscr{E}mb \to \mathsf{Bi}_{\mathrm{iso}}$ is a localization of the bicategory $\mathscr{E}mb$ at the class $\mathscr{E}mb_1$ of all 1-morphisms and since the functor $\mathbb{X} : \mathscr{E}mb \to \mathsf{Lie2Alg}$ sends every 1-morphism of $\mathscr{E}mb$ to an invertible morphism there exists by Proposition 2.11 a functor $\widetilde{\mathbb{X}} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Lie2Alg}$ (which is unique up to isomorphism) and an isomorphism $\widetilde{\mathbb{X}} \circ \langle\ \rangle \xRightarrow{\sim} \mathbb{X}$ of functors. It is no loss of generality to assume that $\widetilde{\mathbb{X}}(G) = \mathbb{X}(G)$ for every Lie groupoid G. We now drop the $\widetilde{\ }$ and obtain the desired functor $\mathbb{X} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Lie2Alg}$. □

We end the section with a result that we will need in Section 5.

Lemma 4.13. *Let $w : G \to G'$ be an essentially surjective open embedding, $w^* : \mathbb{X}(G') \to \mathbb{X}(G)$ the pull-back/restriction functor of Proposition 4.5 and $(\langle w\rangle)_* : \mathbb{X}_{\mathsf{gen}}(G) \to \mathbb{X}_{\mathsf{gen}}(G')$ the push-forward along the bibundle $\langle w\rangle$ constructed in Lemma 4.7. Then the diagram*

(4-5)
$$\begin{array}{ccc} \mathbb{X}(G) & \xrightarrow{\iota_G} & \mathbb{X}_{\mathsf{gen}}(G) \\ {\scriptstyle w^*}\uparrow & \Downarrow \sigma & \downarrow{\scriptstyle (\langle w\rangle)_*} \\ \mathbb{X}(G') & \xrightarrow[\iota_{G'}]{} & \mathbb{X}_{\mathsf{gen}}(G') \end{array}$$

2-commutes.

Proof. We will show that the functors $(\langle w\rangle)_* \circ \iota_G \circ w^*$ and $\iota_{G'}$ are isomorphic by directly constructing a natural isomorphism $\sigma : (\langle w\rangle)_* \circ \iota_G \circ w^* \Rightarrow \iota_{G'}$.

We have already seen in the proof of Lemma 4.10 that for any multiplicative vector field $u : G' \to TG'$ the bibundles $(\langle w\rangle)_*(\langle u|_G\rangle) = T\langle w\rangle \circ (\langle u|_G\rangle \circ \langle w\rangle^{-1})$ and $\langle u\rangle$ are isomorphic. We need to be more precise about choosing these isomorphisms: we

have to make sure that they are isomorphisms in $\mathbb{X}_{\mathsf{gen}}(G')$ from $(\langle w\rangle)_*(\langle u|_G\rangle, \alpha_{\langle u|_G\rangle})$ to $(\langle u\rangle, \alpha_{\langle u\rangle})$ and that they assemble into a natural transformation. As before we may assume that G is an open subgroupoid of G' and $w : G \to G'$ is an inclusion.

The category $\mathbb{X}_{\mathsf{gen}}(G')$ implicitly depends on the component $\tilde{\pi}_{G'}$ of the transformation $\tilde{\pi} : T^{\mathsf{Bi}} \Rightarrow \mathrm{id}_{\mathsf{Bi}}$. Recall that the bibundle $\tilde{\pi}_{G'}$ is isomorphic to the bibundle $\langle \pi_{G'}\rangle$. Replacing $\tilde{\pi}_{G'}$ in the definition of $\mathbb{X}_{\mathsf{gen}}(G')$ by $\langle \pi_{G'}\rangle$ and of $\tilde{\pi}_G$ by $\langle \pi_G\rangle$ in the definition of $\mathbb{X}_{\mathsf{gen}}(G)$ results in isomorphic categories. Consequently we may assume that $\tilde{\pi}_{G'} = \langle \pi_{G'}\rangle$ and $\tilde{\pi}_G = \langle \pi_G\rangle$.

By definition $\langle w\rangle$ is the fiber product $G_0 \times_{w,G'_0,t} G'_1$. Here as usual $t : G'_1 \to G'_0$ is the target map. Since the fiber product is defined by its universal property we may assume that $\langle w\rangle = t^{-1}(G_0) = \{\gamma \in G'_1 \mid t(\gamma) \in G_0\}$. Then the left anchor $a^L_{\langle w\rangle}$ is the restriction of the target map t and the right anchor is the restrictions of the source map s. The inverse of $\langle w\rangle$ is then $t^{-1}(G_0)$ with the left and right anchors reversed.

Some notation for elements of the composites of two bibundles is necessary. Given two composable bibundles $A \xrightarrow{P} B \xrightarrow{Q} C$ their composite is $Q \circ P = (P \times_{a^R_P, B_0, a^L_Q} Q)/B$. So an element of $Q \circ P$ is the B-orbit $[p, q]$ of an element (p, q) in the fiber product $P \times_{a^R_P, B_0, a^L_Q} Q \subset P \times Q$. For example given a multiplicative vector field $u : G' \to TG'$, an element of the composite $\langle \pi_{G'}\rangle \circ \langle u\rangle$ is of the form $[(x, u(x) \xleftarrow{\mu} \dot{y}), (\dot{y}, \pi_{G'}(\dot{y}) \xleftarrow{\nu} z)]$ for some $x \in G'_0$, $\mu \in TG_1$, $\dot{y} \in TG_0$, $\nu \in G'_1$ and $z \in G'_0$. The natural isomorphism $\alpha_{\langle u\rangle} : \langle \pi_{G'}\rangle \circ \langle u\rangle \to G'_1 = \langle \mathrm{id}_{G'}\rangle$ is then given by

$$\alpha_{\langle u\rangle}\big([(x, u(x) \xleftarrow{\mu} \dot{y}), (\dot{y}, \pi_{G'}(\dot{y}) \xleftarrow{\nu} z)]\big) = x \xleftarrow{\pi_{G'}(\mu)\circ\nu} z.$$

For any morphism $\beta : u \Rightarrow v$ in the category $\mathbb{X}(G')$ the isomorphism of bibundles $\langle \beta\rangle : \langle u\rangle \to \langle v\rangle$ is given by

$$\langle \beta\rangle(x, u(x) \xleftarrow{\Gamma} \dot{z}) = (x, v(x) \xleftarrow{\beta(x)\circ\Gamma} \dot{z}).$$

Given an element $[[y \xrightarrow{\gamma} x, (x, u(x) \xleftarrow{\dot{\mu}} \dot{z})], \dot{z} \xleftarrow{\dot{\nu}} \dot{d}]$ of

$$\big(\langle w\rangle^{-1} \times_{G_0} \langle u|_G\rangle/G\big) \times_{TG_0} T\langle w\rangle/TG = T\langle w\rangle \circ (\langle u|_G\rangle \circ \langle w\rangle^{-1}),$$

we define

$$\sigma_u\big([[y \xrightarrow{\gamma} x, (x, u(x) \xleftarrow{\dot{\mu}} \dot{z})], \dot{z} \xleftarrow{\dot{\nu}} \dot{d}]\big) := (y, u(y) \xleftarrow{u(\gamma^{-1})\circ\dot{\mu}\circ\dot{\nu}} \dot{d}) \in \langle u\rangle.$$

It is not very difficult to check that σ_u is well-defined isomorphism of the bibundles.

It remains to check that the isomorphisms $\{\sigma_u\}$ are components of a natural isomorphism $\sigma : (\langle w\rangle_* \circ \iota_G \circ w^*) \Rightarrow \iota_{G'}$. Thus for a morphism $\beta : u \Rightarrow v$ in $\mathbb{X}(G')$ we need to check that $\iota_{G'}(\beta) \circ \sigma_u = \sigma_v \circ \big((\langle w\rangle_* \circ \iota_G \circ w^*)(\beta)\big)$. The isomorphism of bibundles

$$\big((\langle w\rangle_* \circ \iota_G \circ w^*)(\beta)\big) = \langle w\rangle_*(\langle \beta|_G\rangle) : \langle w\rangle_*(\langle w\rangle_*(\langle u|_G\rangle)) \to \langle w\rangle_*(\langle w\rangle_*(\langle v|_G\rangle))$$

is given by

$$\big([[y \xrightarrow{\gamma} x, (x, u(x) \xleftarrow{\dot{\mu}} \dot{z})], \dot{z} \xleftarrow{\dot{\nu}} \dot{d}]\big) \mapsto \big([[y \xrightarrow{\gamma} x, (x, v(x) \xleftarrow{\beta(x)\circ\dot{\mu}} \dot{z})], \dot{z} \xleftarrow{\dot{\nu}} \dot{d}]\big).$$

Hence

$$\sigma_v\Big(\langle w\rangle_*(\langle\beta|_G\rangle)\,\big([[y \xrightarrow{\gamma} x,\,(x,u(x) \xleftarrow{\dot\mu} \dot z)],\,\dot z \xleftarrow{\dot v} \dot d]\big)\Big) \\ = (y, v(y) \xleftarrow{v(\gamma^{-1})\circ\beta(x)\circ\dot\mu\circ\dot v} \dot d).$$

On the other hand

$$\langle\beta\rangle\Big(\sigma_u\big([[y \xrightarrow{\gamma} x,\,(x,u(x) \xleftarrow{\dot\mu} \dot z)],\,\dot z \xleftarrow{\dot v} \dot d]\big)\Big) = (y, v(y) \xleftarrow{\beta(y)\circ u(\gamma^{-1})\circ\dot\mu\circ\dot v} \dot d).$$

Since β is a natural transformation, $\beta(y)\circ u(\gamma^{-1}) = v(\gamma^{-1})\circ\beta(x)$ for any arrow $y\xrightarrow{\gamma}x$ of G'. Thus $\sigma_v\circ(\langle w\rangle_*(\langle\beta|_G\rangle))=\langle\beta\rangle\circ\sigma_u$ for any morphism $\beta : u\Rightarrow v$ in $\mathbb{X}(G')$. We conclude that $\{\sigma_u\}$ are components of the desired natural isomorphism. □

5. Categories of vector fields on stacks and Lie 2-algebras

We recall Hepworth's construction [2009] of the category of vector fields $\mathsf{Vect}(\mathcal{A})$ on a stack $\mathcal{A}$. The first step is to extend the tangent functor $T : \mathsf{Man}\to\mathsf{Man}$ on the category of manifolds to a functor $T^{\mathsf{Stack}} : \mathsf{Stack}\to\mathsf{Stack}$ on the 2-category of stacks over manifolds along the Yoneda embedding $y : \mathsf{Man}\to\mathsf{Stack}$. This results in a 2-commuting diagram

$$\begin{array}{ccc} \mathsf{Stack} & \xrightarrow{T^{\mathsf{Stack}}} & \mathsf{Stack} \\ {\scriptstyle y}\uparrow & \nearrow & \uparrow{\scriptstyle y} \\ \mathsf{Man} & \xrightarrow[T]{} & \mathsf{Man} \end{array}$$

and there is a natural transformation $\pi : T^{\mathsf{Stack}}\Rightarrow \mathrm{id}_{\mathsf{Stack}}$.

Definition 5.1 (Hepworth). The objects of the *category of vector fields* $\mathsf{Vect}(\mathcal{A})$ on a stack $\mathcal{A}$ are pairs (v,α_v) where $v : \mathcal{A}\to T^{\mathsf{Stack}}\mathcal{A}$ is a 1-morphism of stacks and $\alpha_v : \pi_{\mathcal{A}}\circ v\Rightarrow \mathrm{id}_{\mathcal{A}}$ is a 2-morphism. A morphism in $\mathsf{Vect}(\mathcal{A})$ from (v,α_v) to (u,α_u) is a 2-morphism $\beta : v\Rightarrow u$ so that $\alpha_u\circ(\pi_{\mathcal{A}}\star\beta)=\alpha_v$. Here $\circ$ is the vertical composition and $\star$ is whiskering.

Recall that for any Lie groupoid G there is a stack $\mathbb{B}G$ of principal G-bundles. The assignment $G\mapsto\mathbb{B}G$ can be promoted to a functor $\mathbb{B}$ in different ways depending on which source 2-category one chooses. Hepworth took the source to be the 2-category LieGpd of Lie groupoids, smooth functors and natural isomorphisms and considered the functor

$$\mathbb{B} : \mathsf{LieGpd}\to\mathsf{Stack}. \tag{5-1}$$

The essential image of this functor consists of the 2-category $\mathsf{GeomStack}$ of geometric stacks. The functor $\mathbb{B}$ is faithful but not full. In particular the functor $\mathbb{B}$ maps essential equivalences of Lie groupoids (which need not be invertible in LieGpd,

even weakly) to isomorphisms of stacks.[2] The tangent functor $T : \mathsf{Man} \to \mathsf{Man}$ is easily extended to a functor $T^{\mathsf{LieGpd}} : \mathsf{LieGpd} \to \mathsf{LieGpd}$. We have a natural transformation $\pi^{\mathsf{LieGpd}} : T^{\mathsf{LieGpd}} \Rightarrow \mathrm{id}_{\mathsf{LieGpd}}$.

Hepworth [2009, Theorem 3.11] proved that there is a natural isomorphism

$$\mathbb{B} \circ T^{\mathsf{LieGpd}} \Leftrightarrow T^{\mathsf{Stack}} \circ \mathbb{B}. \tag{5-2}$$

Consequently given a vector field $v : G \to TG$ on a Lie groupoid G we get a map of stacks $\mathbb{B}v : \mathbb{B}G \to \mathbb{B}TG$. Composing v with the isomorphism $\mathbb{B}TG \to T^{\mathsf{Stack}}(\mathbb{B}G)$ gives us a functor that we again denoted by $\mathbb{B}v : \mathbb{B}G \to T^{\mathsf{Stack}}(\mathbb{B}G)$. This determines an object $(\mathbb{B}v, a_{Bv})$ in the category $\mathsf{Vect}(\mathbb{B}G)$ of vector fields on the stack $\mathbb{B}G$. Hepworth showed that the assignment $v \mapsto (\mathbb{B}v, a_{Bv})$ can be promoted to a functor

$$\mathbb{X}(G) \to \mathsf{Vect}(\mathbb{B}G). \tag{5-3}$$

Here as before $\mathbb{X}(G)$ denotes the category of vector fields on a Lie groupoid G (see Definition 3.1). Hepworth [2009, Theorem 4.15] proved that if the groupoid G is proper then the functor (5-3) is an equivalence of categories.[3] Another important consequence of the existence of the isomorphism (5-2) is that for any geometric stack $\mathcal{A}$ the tangent stack $T^{\mathsf{Stack}}\mathcal{A}$ is geometric as well.

We can promote the assignment $G \to \mathbb{B}G$ to a functor out of a different bicategory, which at a slight risk of confusion we will again denote by $\mathbb{B}$. Namely we can choose as our source the bicategory Bi of Lie groupoids, bibundles and isomorphisms of bibundles. The advantage is that the functor $\mathbb{B} : \mathsf{Bi} \to \mathsf{Stack}$ is fully faithful: for Lie groupoids G and H, the functor $\mathbb{B} : \mathsf{Hom}_{\mathsf{Bi}}(G, H) \to \mathsf{Hom}_{\mathsf{Stack}}(\mathbb{B}G, \mathbb{B}H)$ is an equivalence of categories. Consequently the functor $\mathbb{B} : \mathsf{Bi} \to \mathsf{GeomStack}$ is an equivalence of bicategories. It is not hard to adapt [Hepworth 2009, Theorem 3.11] to this setting: the diagram

$$\begin{array}{ccc} \mathsf{Bi} & \xrightarrow{T^{\mathsf{Bi}}} & \mathsf{Bi} \\ {\scriptstyle\mathbb{B}}\downarrow & \swarrow & \downarrow{\scriptstyle\mathbb{B}} \\ \mathsf{GeomStack} & \xrightarrow[T^{\mathsf{Stack}}]{} & \mathsf{GeomStack} \end{array}$$

2-commutes. For convenience, we will choose a weak inverse $\mathbb{B}^{-1} : \mathsf{GeomStack} \to \mathsf{Bi}$ and consider the functor

$$T^{\mathsf{GeomStack}} : \mathsf{GeomStack} \to \mathsf{GeomStack}, \qquad T^{\mathsf{GeomStack}} := \mathbb{B} \circ T^{\mathsf{Bi}} \circ \mathbb{B}^{-1},$$

[2]Recall that by tradition a weakly invertible 1-morphism of stacks is called an *isomorphism.*

[3]The hypothesis that the groupoid G is proper is not explicit in the statement of [Hepworth 2009, Theorem 4.15]. However the proof depends on several lemmas: [Hepworth 2009, 4.11, 4.12, 2.11, 2.12]. In particular the proof uses the existence of partitions of unity and Weinstein–Zung linearization, both of which require properness.

which by construction is isomorphic to Hepworth's functor T^{Stack} restricted to geometric stacks. As in the case of T^{Stack} we have a transformation $\pi : T^{\mathsf{GeomStack}} \Rightarrow \mathrm{id}_{\mathsf{GeomStack}}$: namely $\pi := \mathbb{B} \star \tilde{\pi} \star \mathbb{B}^{-1}$. Given a geometric stack $\mathcal{A}$ we now define a category of vector fields $\mathsf{Vect}'(\mathcal{A})$ on $\mathcal{A}$ as follows (compare with Definition 5.1).

Definition 5.2. The *category of vector fields* $\mathsf{Vect}'(\mathcal{A})$ on a geometric stack $\mathcal{A}$, has as objects pairs (X, α_X) where $X : \mathcal{A} \to T^{\mathsf{GeomStack}}\mathcal{A}$ is a 1-morphism of stacks and $\alpha_X : \pi_{\mathcal{A}} \circ X \Rightarrow \mathrm{id}_{\mathcal{A}}$ is a 2-morphism. A morphism from (X, α_X) to (Y, α_Y) in $\mathsf{Vect}'(\mathcal{A})$ is a 2-morphism $\beta : X \Rightarrow Y$ so that $\alpha_Y \circ (\pi_{\mathcal{X}} \star \beta) = \alpha_X$.

It is easy to see that for a geometric stack $\mathcal{A}$ the categories $\mathsf{Vect}(\mathcal{A})$ and $\mathsf{Vect}'(\mathcal{A})$ are equivalent (and even isomorphic). For us there are several advantages in working with $\mathsf{Vect}'(\mathcal{A})$. First of all, the functor Vect' is more explicit than T^{Stack}: the latter involves 2-limits and stackification. Additionally the following result is easy to prove:

Lemma 5.3. *For a Lie groupoid G the classifying stack functor $\mathbb{B} : \mathsf{Bi} \to \mathsf{GeomStack}$ induces an equivalence of categories $(\mathbb{B}_*)_G : \mathbb{X}_{\mathsf{gen}}(G) \to \mathsf{Vect}'(\mathbb{B}G)$, where $\mathbb{X}_{\mathsf{gen}}(G)$ is the category of generalized vector fields (Definition 4.6).*

Proof. Consider a generalized vector field (P, α_P) on the Lie groupoid G. By definition we have an isomorphism $\alpha_P : \tilde{\pi}_G \circ P \Rightarrow \langle \mathrm{id}_G \rangle$ of bibundles. Apply the classifying stack functor $\mathbb{B}$ to the 2-morphism α_P. We get the 2-morphism of stacks $\mathbb{B}\alpha_P : \mathbb{B}(\tilde{\pi}_G \circ P) \Rightarrow \mathbb{B}\langle \mathrm{id}_G \rangle$. Since $\mathbb{B}$ is functor between bicategories, we have canonical 2-arrows $\mathbb{B}\langle \mathrm{id}_G \rangle \Rightarrow \mathrm{id}_{\mathbb{B}G}$ and $\mathbb{B}\langle \pi_G \rangle \circ \mathbb{B}P \Rightarrow \mathbb{B}(\tilde{\pi}_G \circ P)$. Note that these 2-morphisms are 2-isomorphisms since all 2-arrows in the 2-category of stacks are invertible. Composing the three 2-arrows we get a 2-arrow $\mathbb{B}\tilde{\pi}_G \circ \mathbb{B}P \Rightarrow \mathrm{id}_{\mathbb{B}G}$ which we denote by $\alpha_{\mathbb{B}P}$. By definition the pair $(\mathbb{B}P, \alpha_{\mathbb{B}P})$ is an object of $\mathsf{Vect}'(\mathbb{B}G)$.

Similarly a morphism $\beta : (P, \alpha_P) \to (Q, \alpha_Q)$ in $\mathbb{X}_{\mathsf{gen}}(G)$ gives rise to a morphism $\mathbb{B}\beta : \mathbb{B}P \Rightarrow \mathbb{B}Q$. One checks that $\alpha_{\mathbb{B}Q} \circ (\pi_{\mathbb{B}G} \star \mathbb{B}\beta) = \alpha_{\mathbb{B}Q}$. Consequently $\mathbb{B}\beta$ is a morphism in $\mathsf{Vect}'(\mathbb{B}G)$ from $(\mathbb{B}P, \alpha_{\mathbb{B}P})$ to $(\mathbb{B}Q, \alpha_{\mathbb{B}Q})$. We therefore get a functor $(\mathbb{B}_*)_G : \mathbb{X}_{\mathsf{gen}}(G) \to \mathsf{Vect}'(\mathbb{B}G)$. A weak inverse $\mathbb{B}^{-1} : \mathsf{GeomStack} \to \mathsf{Bi}$ gives rise to the functor $((\mathbb{B}^{-1})_*)_G : \mathsf{Vect}'(\mathbb{B}G) \to \mathbb{X}_{\mathsf{gen}}(G)$ in the other direction. The induced functors $(\mathbb{B}_*)_G$ and $((\mathbb{B}^{-1})_*)_G$ are weak inverses of each other. □

We now address the issue of giving the category of vector fields $\mathsf{Vect}'(\mathcal{A})$ on a geometric stack $\mathcal{A}$ the structure of a Lie 2-algebra. We study the functoriality of the assignment $\mathcal{A} \mapsto \mathbb{X}(G)$ of a Lie 2-algebra of vector fields to a geometric stack by a choice of an atlas $G_0 \to \mathcal{A}$. Consider the 2-category $\mathsf{GeomStack}_{\mathrm{iso}}$ of geometric stacks, *isomorphisms* of stacks and 2-morphisms of stacks. The classifying stack functor $\mathbb{B} : \mathsf{Bi} \to \mathsf{GeomStack}$ restricts to an equivalence of bicategories $\mathbb{B} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{GeomStack}_{\mathrm{iso}}$. A choice of a weak inverse $\mathbb{B}^{-1}$ of $\mathbb{B}$ amounts to choosing an atlas for each geometric stack. Once the inverse $\mathbb{B}^{-1}$ is chosen, we have the composite functor $\mathsf{GeomStack}_{\mathrm{iso}} \xrightarrow{\mathbb{B}^{-1}} \mathsf{Bi}_{\mathrm{iso}} \xrightarrow{\mathbb{X}} \mathsf{Lie2Alg}$. By construction, for a stack $\mathcal{A}$

the Lie 2-algebra $\mathbb{X}(\mathbb{B}^{-1}(\mathcal{A}))$ is the Lie 2-algebra of vector fields on the Lie groupoid $G = \mathbb{B}^{-1}(\mathcal{A})$. By the discussion above the category underlying the Lie 2-algebra $\mathbb{X}(\mathbb{B}^{-1}(\mathcal{A}))$ is equivalent to the category of vector fields $\mathsf{Vect}'(\mathcal{A})$ on the stack $\mathcal{A}$.

A different choice of a weak inverse $(\mathbb{B}^{-1})'$ of $\mathbb{B}$ amounts to choosing a possibly different atlas for each geometric stack. Once $(\mathbb{B}^{-1})'$ is chosen we have a natural isomorphism $\alpha : \mathbb{B}^{-1} \Rightarrow (\mathbb{B}^{-1})'$. For each geometric stack $\mathcal{A}$ the component $\alpha_{\mathcal{A}}$ of the natural transformation α is an invertible bibundle $\alpha_{\mathcal{A}} : \mathbb{B}^{-1}(\mathcal{A}) \to (\mathbb{B}^{-1})'(\mathcal{A})$. Applying the functor $\mathbb{X} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{Lie2Alg}$ to $\alpha_{\mathcal{A}}$ we get an invertible bibundle $\mathbb{X}(\alpha_{\mathcal{A}}) : \mathbb{X}(\mathbb{B}^{-1}(\mathcal{A})) \to \mathbb{X}((\mathbb{B}^{-1})'(\mathcal{A}))$ in the bicategory $\mathsf{Lie2Alg}$.

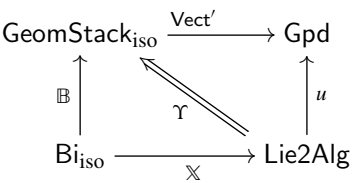
One can be fairly explicit as to what the bibundle $\mathbb{X}(\alpha_{\mathcal{A}})$ actually is. Namely let $G_0 \to \mathcal{A}$ be the atlas giving rise to the Lie groupoid $G = \mathbb{B}^{-1}(\mathcal{A})$ and $H_0 \to \mathcal{A}$ be the atlas giving rise to $H = (\mathbb{B}^{-1})'(\mathcal{A})$. Then the total space of the bibundle $\alpha_{\mathcal{A}} : G \to H = (\mathbb{B}^{-1})'(\mathcal{A})$ represents the fiber product $G_0 \times_{\mathcal{A}} H_0$. The linking groupoid $G *_{\alpha_{\mathcal{A}}} H$ is the groupoid corresponding to the atlas $G_0 \sqcup H_0 \to \mathcal{A}$. The linking groupoid comes with two canonical essentially surjective open embeddings $i_G : G \hookrightarrow G *_{\alpha_{\mathcal{A}}} H$ and $i_H : H \hookrightarrow G *_{\alpha_{\mathcal{A}}} H$. By Lemma 4.10 the pullback/restriction functors $i_G^* : \mathbb{X}(G *_{\alpha_{\mathcal{A}}} H) \to \mathbb{X}(G)$, $i_H^* : \mathbb{X}(G *_{\alpha_{\mathcal{A}}} H) \to \mathbb{X}(G)$ are 1-morphisms of Lie 2-algebras that are fully faithful and essentially surjective. Hence the bibundle $\langle i_G^* \rangle$ is invertible in the bicategory $\mathsf{Lie2Alg}$. On the other hand, as was noted in the proof of Theorem 4.4, the bibundles $\langle i_H \rangle \circ \alpha_{\mathcal{A}}$ and $\langle i_G \rangle$ are isomorphic. Hence $\mathbb{X}(\langle i_H \rangle) \circ \mathbb{X}(\alpha_{\mathcal{A}}) \simeq \mathbb{X}(\langle i_G \rangle)$. By construction of the functor $\mathbb{X} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{Lie2Alg}$ we have $\mathbb{X}(\langle i_G \rangle) = \langle i_G^* \rangle^{-1}$ and $\mathbb{X}(\langle i_H \rangle) = \langle i_H^* \rangle^{-1}$. Hence $\mathbb{X}(\alpha_{\mathcal{A}}) \simeq \langle i_H^* \rangle \circ \langle i_G^* \rangle^{-1}$.

6. Lie 2-algebras of vector fields on stacks

In the previous section we constructed a functor $\mathbb{X} \circ \mathbb{B}^{-1} : \mathsf{GeomStack}_{\mathsf{iso}} \to \mathsf{Lie2Alg}$. Recall that there is a forgetful functor $u : \mathsf{Lie2Alg} \to \mathsf{Gpd}$ that assigns to a Lie 2-algebra its underlying groupoid. Therefore for every geometric stack $\mathcal{A}$ we have the groupoid $(u \circ \mathbb{X} \circ \mathbb{B}^{-1})(\mathcal{A})$. We should make sure that this groupoid is equivalent to the groupoid of vector fields $\mathsf{Vect}'(\mathcal{A})$ (and hence to Hepworth's groupoid $\mathsf{Vect}(\mathcal{A})$ of vector fields on the stack $\mathcal{A}$).

We start by promoting the assignment $\mathcal{A} \mapsto \mathsf{Vect}'(\mathcal{A})$ to a functor $\mathsf{Vect}' : \mathsf{GeomStack}_{\mathsf{iso}} \to \mathsf{Gpd}$ whose source is the 2-category of geometric stacks and isomorphisms and whose target is the (2,1)-category Gpd of (small) groupoids. We then prove the following theorem:

Theorem 6.1. *The diagram of* (2,1)*-bicategories and functors*



2-*commutes. Here as above* Gpd *denotes the* (2,1)*-category of groupoids, functors and natural isomorphisms, and* $u : \mathsf{Lie2Alg} \to \mathsf{Gpd}$ *denotes the functor that assigns to each Lie* 2*-algebra its underlying groupoid. The* 1*-components of the transformation* Υ *are weakly invertible functors (i.e., equivalences of categories). In particular for a geometric stack* $\mathcal{A}$ *the category underlying the Lie* 2*-algebra* $(\mathbb{X} \circ \mathbb{B}^{-1})(\mathcal{A})$ *is equivalent to Hepworth's category* $\mathsf{Vect}(\mathcal{A})$ *of vector fields on the stack.*

We now construct the 2-functor $\mathsf{Vect}' : \mathsf{GeomStack} \to \mathsf{Gpd}$. An isomorphism $f : \mathcal{A}_1 \to \mathcal{A}_2$ of stacks induces an equivalence of categories $f_* : \mathsf{Vect}'(\mathcal{A}_1) \to \mathsf{Vect}'(\mathcal{A}_2)$: one adapts the proof of Lemma 4.7 to the setting of geometric stacks. Note that if $f = \mathrm{id}_{\mathcal{A}}$ we may take $f_* = \mathrm{id}_{\mathsf{Vect}'(\mathcal{A})}$.

Given isomorphisms $f : \mathcal{A}_1 \to \mathcal{A}_2$ and $g : \mathcal{A}_2 \to \mathcal{A}_3$ of stacks we get equivalences of categories: $(g \circ f)_*$ and $g_* \circ f_*$. We need to produce a natural transformation $\mu_{gf} : g_* \circ f_* \Rightarrow (g \circ f)_*$. So given an object (v, a_v) of $\mathsf{Vect}'(\mathcal{A}_1)$ we need to produce a 2-cell

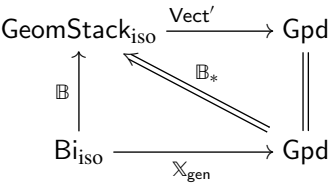
$$(\mu_{gf})_{(v,a_v)} : g_*(f_*(v, a_v)) \Rightarrow (g \circ f)_*(v, a_v)$$

in the category $\mathsf{Vect}'(\mathcal{A}_3)$. By the (adapted) proof of Lemma 4.7 $g_*(f_*(v)) = T^{\mathsf{GeomStack}} g \circ (T^{\mathsf{GeomStack}} f \circ v \circ f^{-1}) \circ g^{-1}$. Since $T^{\mathsf{GeomStack}}$ is a (pseudo-) functor, there is a natural isomorphism $T^{\mathsf{GeomStack}} g \circ T^{\mathsf{GeomStack}} f \Rightarrow T^{\mathsf{GeomStack}}(g \circ f)$. Consequently there is an isomorphism

$$T^{\mathsf{GeomStack}} g \circ (T^{\mathsf{GeomStack}} f \circ v \circ f^{-1}) \circ g^{-1} \Rightarrow T^{\mathsf{GeomStack}}(g \circ f) \circ v \circ (g \circ f)^{-1}.$$

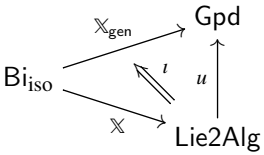
This isomorphism is the desired 2-cell $(\mu_{gf})_{(v,a_v)}$. We are now ready to describe the functor Vect'. To a geometric stack $\mathcal{A}$ it assigns the category $\mathsf{Vect}'(\mathcal{A})$. To an arrow $f : \mathcal{A}_1 \to \mathcal{A}_2$ it assigns the equivalence of categories $\mathsf{Vect}'(f) := f_*$. Additionally for each pair (g, f) we have a natural isomorphism $\mu_{gf} : g_* \circ f_* \Rightarrow (g \circ f)_*$ constructed above. Proceeding similarly (and keeping track of the coherence data) we can promote the assignment $\mathsf{Bi}_{\mathrm{iso}} \ni (G \xrightarrow{P} H) \mapsto (\mathbb{X}_{\mathsf{gen}}(G) \xrightarrow{P_*} \mathbb{X}_{\mathsf{gen}}(H))$ to a functor $\mathbb{X}_{\mathsf{gen}} : \mathsf{Bi}_{\mathrm{iso}} \to \mathsf{Gpd}$. Lemma 5.3 now generalizes as follows:

Lemma 6.2. *The equivalences of categories* $(\mathbb{B}_*)_G : \mathbb{X}_{\mathsf{gen}}(G) \to \mathsf{Vect}'(\mathbb{B}G)$ *(one for each Lie groupoid* G*) assemble into a transformation* $\mathbb{B}_* : \mathbb{X}_{\mathsf{gen}} \Rightarrow \mathbb{B} \circ \mathsf{Vect}'$*. That is, the diagram*



2-*commutes. Here as before* Gpd *is the* (2,1)*-category of groupoids, functors and natural isomorphisms.*

Lemma 6.3. *The diagram*



2*-commutes and the* 1*-components of* ι *are equivalences of categories.*

Proof. We have the underlying category functor $u_{\mathrm{strict}} : \mathsf{Lie2Alg}_{\mathrm{strict}} \to \mathsf{Gpd}$ which sends Lie 2-algebras to their underlying groupoids and morphisms of Lie 2-algebras to the underlying functors. The functor u_{strict} sends essential equivalences of Lie 2-algebras to weakly invertible functors. By the universal property of the localization $\langle\ \rangle : \mathsf{Lie2Alg}_{\mathrm{strict}} \to \mathsf{Lie2Alg}$ we get the underlying category functor $u : \mathsf{Lie2Alg} \to \mathsf{Gpd}$ with $u(\langle f \rangle)$ isomorphic to $u_{\mathrm{strict}}(f)$ for every essential equivalence of Lie 2-algebras. It follows that for any essential equivalence f in $\mathsf{Lie2Alg}_{\mathrm{strict}}$ the functor $u(\langle f \rangle^{-1})$ is a weak inverse of $u_{\mathrm{strict}}(f)$. We proved that for any essentially surjective open embedding $w : G \to G'$ of Lie groupoids the pullback functor $w^* : \mathbb{X}(G') \to \mathbb{X}(G)$ is an essential equivalence. We defined $\mathbb{X}(w) = \langle w^* \rangle^{-1}$. It follows that $u(\mathbb{X}(w))$ is a weak inverse of $u_{\mathrm{strict}}(w^*)$.

By Lemma 4.13 the diagram (4-5) 2-commutes for any 1-morphism $w : G \to G'$ in $\mathscr{E}mb$. Hence the diagram

$$\begin{array}{ccc} \mathbb{X}(G) & \xrightarrow{\iota_G} & \mathbb{X}_{\mathsf{gen}}(G) \\ {\scriptstyle u(\mathbb{X}(w))}\downarrow & & \downarrow{\scriptstyle \langle w \rangle_* = \mathbb{X}_{\mathsf{gen}}(w)} \\ \mathbb{X}(G') & \xrightarrow[\iota_{G'}]{} & \mathbb{X}_{\mathsf{gen}}(G') \end{array}$$

2-commutes as well. It follows that the functors $u \circ \mathbb{X}, \mathbb{X}_{\mathsf{gen}} \circ \langle\ \rangle \in \mathsf{Hom}_W(\mathscr{E}mb, \mathsf{Gpd})$ are isomorphic (i.e., differ by a transformation whose components are equivalences of categories). Here $\mathsf{Hom}_W(\mathscr{E}mb, \mathsf{Gpd})$ denotes the bicategory of functors that send the collection W of all 1-cells in $\mathscr{E}mb$ to weakly invertible functors.

By the definition of the functor $\mathbb{X} : \mathsf{Bi}_{\mathsf{iso}} \to \mathsf{Lie2Alg}$ its precomposition with the localization functor $\langle\ \rangle : \mathscr{E}mb \to \mathsf{Bi}_{\mathsf{iso}}$ is isomorphic to $\mathbb{X} : \mathscr{E}mb \to \mathsf{Lie2Alg}$. It follows that the functors $u \circ \mathbb{X} \circ \langle\ \rangle$ and $\mathbb{X}_{\mathsf{gen}} \circ \langle\ \rangle$ are isomorphic in $\mathsf{Hom}_W(\mathscr{E}mb, \mathsf{Gpd})$. By the universal property of the localization $\langle\ \rangle : \mathscr{E}mb \to \mathsf{Bi}_{\mathsf{iso}}$, the functors $u \circ \mathbb{X}$ and $\mathbb{X}_{\mathsf{gen}}$ are isomorphic in $\mathsf{Hom}(\mathsf{Bi}_{\mathsf{iso}}, \mathsf{Gpd})$. □

Theorem 6.1 now follows directly from Lemmas 6.2 and 6.3.

7. Generalized and multiplicative vector fields on a Lie groupoid

In this section we prove Theorem 4.9. In the case of *proper* Lie groupoids Theorem 4.9 follows from [Hepworth 2009, Theorem 4.15].

The fact that the functor $\iota_G : \mathbb{X}(G) \hookrightarrow \mathbb{X}_{\mathsf{gen}}(G)$ is fully faithful is an easy consequence of Theorem 2.9. We now address essential surjectivity. We first prove:

Lemma 7.1. *Let $V = \{V_1 \rightrightarrows V_0\}$ be a 2-vector space, $v_1, \dots, v_s \in V_0$ a finite collection of objects and $\{v_i \xleftarrow{w_{ij}} v_j\}_{i,j=1}^{s}$ a collection of morphisms satisfying the cocycle conditions: $w_{ii} = 1_{v_i}$ for all i; $w_{ji} = {w_{ij}}^{-1}$ for all i, j; $w_{ij}w_{jk} = w_{ik}$ for all i, j, k. Then for any $\lambda_1, \dots, \lambda_s \in [0, 1]$ with $\sum \lambda_k = 1$ there are morphisms $v_i \xleftarrow{z_i} \sum \lambda_k v_k$ $(i = 1, \dots, s)$ with $w_{ij} = z_i z_j^{-1}$ for all i, j.*

Proof. By Remark 2.14 the category V is isomorphic to the action groupoid $\{U \times V_0 \rightrightarrows V_0\}$ where $U_0 = \ker(s : V_1 \to V_0)$, $\partial : U \to V_0$ is $t|_U$ and the action of U on V_0 is given by $u \cdot v := v + \partial(u)$. Note that the multiplication/composition in $\{U \times V_0 \rightrightarrows V_0\}$ is given by $(u', v+\partial(u))(u, v) = (u'+u, v)$ for all $v \in V_0$, $u, u' \in U$. Consequently $(u, v)^{-1} = (-u, v+\partial(u))$. The isomorphism $f : V \to \{U \times V_0 \rightrightarrows V_0\}$ is given on morphisms by

$$f(w) = (w - 1_{s(w)}, s(w)) \in U \times V_0 \qquad \text{for all } w \in V_1.$$

The isomorphism f followed by the projection onto U sends the morphisms w_{ij} to vectors $u_{ij} \in U$. It is easy to see that the cocycle conditions translate into: $u_{ii} = 0$ for all i; $u_{ji} = -u_{ij}$ for all i, j and $u_{ik} - u_{jk} = u_{ij}$ for all i, j, k. Moreover $\partial(u_{ij}) = v_i - v_j$ for all i, j. Now consider $y_i = \left(\sum \lambda_k u_{ik}, \sum \lambda_k v_k\right) \in U \times V_0$ and set $z_i := f^{-1}(y_i) \in V_1$. We now verify that the z_i's are the desired morphisms. By definition the source of y_i is $\sum \lambda_k v_k$. The target of y_i is

$$\begin{aligned}\partial\left(\sum_k \lambda_k u_{ik}\right) + \sum_k \lambda_k v_k &= \sum_k \lambda_k \partial(u_{ik}) + \sum_k \lambda_k v_k \\ &= \sum_k \lambda_k (v_i - v_k) + \sum_k \lambda_k v_k = \sum_k \lambda_k v_i = v_i.\end{aligned}$$

Hence z_i is an arrow from $\sum \lambda_k v_k$ to v_i. Finally

$$\begin{aligned}y_i y_j^{-1} &= \left(\sum_k \lambda_k u_{ik}, \sum \lambda_k v_k\right)\left(-\sum_k \lambda_k u_{jk}, v_j\right) \\ &= \left(\sum_k \lambda_k (u_{ik} - u_{jk}), v_j\right) = \left(\sum_k \lambda_k u_{ij}, v_j\right) = (u_{ij}, v_j),\end{aligned}$$

and so $z_i z_j^{-1} = w_{ij}$ as desired. □

Proposition 7.2. *Let $G = \{G_1 \rightrightarrows G_0\}$ be a Lie groupoid, $U_0 \subset G_0$ an open submanifold and $U = \{U_1 \rightrightarrows U_0\}$ the restriction of G to U_0 (that is, U_1 consists of arrows of G with source and target in U_0). Given a functor $X : U \to TG$ together with a natural isomorphism $\alpha : (i : U \hookrightarrow G) \Rightarrow \pi_G \circ X$ there exists a functor $Y : U \to TU$ so that $\pi_U \circ Y = \mathrm{id}_U$ and a natural isomorphism $\beta : Ti \circ Y \Rightarrow X$.*

Proof. By definition of α the diagram

$$\begin{array}{ccc} U_0 & \xrightarrow{X} & TG_0 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\pi_G} \\ G_1 & \xrightarrow[t]{} & G_0 \end{array}$$

commutes. Hence there is a smooth map $(\alpha, X) : U_0 \to G_1 \times_{t,G_0,\pi} TG_0 = t^*TG_0$. Since the target map $t : G_1 \to G_0$ is a submersion, its differential $Tt_\gamma : T_\gamma G_1 \to T_{t(\gamma)}G_0$ is a surjective linear map for each $\gamma \in G_1$. Consequently the map $\Phi : TG_1 \to t^*TG_0$, $\Phi(\gamma, v) = (\gamma, Tt_\gamma v)$ is a surjective map of vector bundles over G_1. Choose a smooth section $\sigma : t^*TG_0 \to TG_1$ of Tt and consider the composite $\beta := \sigma \circ (\alpha, X) : U_0 \to TG_1$. By construction of β,

$$\beta(x) \in T_{\alpha(x)}G_1 \quad \text{and} \quad Tt_{\alpha(x)}\beta(x) = X(x)$$

for any $x \in U_0$. We now define a functor $Y : U \to TU$. On objects we set

$$Y(x) = Ts(\beta(x)).$$

For an arrow $x \xrightarrow{\gamma} y \in U_1$ we set

$$Y(\gamma) = \beta(y)^{-1}X(\gamma)\beta(x).$$

It is easy to check that Y is indeed functor, $\beta : Ti \circ Y \Rightarrow X$ is a natural transformation and $\pi_U \circ Y = \mathrm{id}_U$. □

Proposition 7.3. *Let G be a Lie groupoid and*

$$\begin{array}{ccccc} G_1 & & P & & TG_1 \\ \Downarrow & \overset{a_P^L}{\swarrow} & & \overset{a_P^R}{\searrow} & \Downarrow \\ G_0 & & & & TG_0 \end{array}$$

be a bibundle from G to the tangent groupoid TG such that the composite $\langle\pi\rangle \circ P$ is isomorphic to $\langle \mathrm{id}_G\rangle$ by way of a bibundle isomorphism $\mathbf{a} : \langle\pi\rangle \circ P \Rightarrow \langle \mathrm{id}_G\rangle$. Then the left anchor $a_P^L : P \to G_0$ has a global section $\tau : G_0 \to P$. Moreover we may choose τ so that the corresponding functor $X_\tau : G \to TG$ is a multiplicative vector field (i.e., $\pi_G \circ X_\tau = \mathrm{id}_G$). Consequently the functor $\iota_G : \mathbb{X}(G) \to \mathbb{X}_{\mathsf{gen}}(G)$ of Definition 4.8 is essentially surjective.

Proof. Since $a_P^L : P \to G_0$ is a surjective submersion, it has local sections. Choose a collection of local sections $\{\sigma_i : U_0^{(i)} \to P\}$ of a_P^L so that $\{U_0^{(i)}\}$ is an open cover of G_0. It is no loss of generality to assume that the cover is locally finite. Denote the restriction of the groupoid G to $U_0^{(i)}$ by $U^{(i)}$. That is, the manifold of objects

of $U^{(i)}$ is $U_0^{(i)}$ and the manifold of arrows $U_1^{(i)}$ consists of all arrows of G with source and target in $U_0^{(i)}$, so $U_1^{(i)} := s^{-1}(U_0^{(i)}) \bigcap t^{-1}(U_0^{(i)})$.

For each section σ_i we get a functor $X_i : U^{(i)} \to TG$ whose value on objects is $X_i(x) = a_P^R(\sigma_i(x))$. The value of X_i on an arrow $y \xleftarrow{\gamma} x \in U_1^{(i)}$ is uniquely defined by the equation $\gamma \cdot \sigma_i(x) = \sigma_i(y) \cdot X_i(\gamma)$ (see Lemma 2.7). We next observe that the isomorphism $\mathbf{a} : \langle \pi_G \rangle \circ P \to \langle \mathrm{id}_G \rangle$ gives rise to natural isomorphisms $\alpha_j : \pi_G \circ X_j \Rightarrow (\iota_j : U^{(j)} \hookrightarrow G)$ where $\iota_j : U^{(j)} \hookrightarrow G$ is the inclusion functor. This can be seen as follows.

Recall that the composite $Q \circ P$ of bibundles $P : K \to L$ and $Q : L \to M$ is the quotient of the fiber product $P \times_{a_P^R, L_0, a_Q^L} Q$ by the action of L. We denote by $[p, q]$ the orbit of $(p, q) \in P \times_{a_P^R, L_0, a_Q^L} Q$ in $Q \circ P = (P \times_{a_P^R, L_0, a_Q^L} Q)/L$. The bibundle $\langle \pi_G \rangle$ is the fiber product $TG_0 \times_{\pi_G, G_0, t} G_1$ with the anchor maps $a^R_{\langle \pi_G \rangle}(v, \gamma) = v$, $a^L_{\langle \pi_G \rangle}(v, \gamma) = s(\gamma)$. Consequently in our case

$$\langle \pi_G \rangle \circ P = (P \times_{a_P^R, TG_0, a^L_{\langle \pi_G \rangle}} (TG_0 \times_{\pi_G, G_0, s} G_1))/TG.$$

It is convenient to identify $P \times_{a_P^R, TG_0, a^L_{\langle \pi_G \rangle}} (TG_0 \times_{\pi_G, G_0, s} G_1)$ with $P \times_{\pi_G \circ a_P^R, G_0, s} G_1$ by way of the TG-equivariant isomorphism $(p, (\pi_G \circ a_P^R)(p), \gamma) \mapsto (p, \gamma)$. We then have a $G \times G$ equivariant diffeomorphism

$$\mathbf{a} : (P \times_{\pi_G \circ a_P^R, G_0, s} G_1)/TG \to G_1, \quad [p, \gamma] \mapsto \mathbf{a}([p, \gamma])$$

with

$$s(\mathbf{a}([p, \gamma])) = s(\gamma) \quad \text{and} \quad t(\mathbf{a}([p, \gamma])) = a_L^P(p).$$

A local section $\sigma_i : U_0^{(i)} \to P$ also defines a local section $\bar{\sigma}_i : U_0^{(i)} \to (P \times_{G_0} G_1)/TG$ of $a^L_{\langle \pi_G \rangle \circ P} : (P \times_{G_0} G_1)/TG \to G_0$. It is given by

$$\bar{\sigma}_i(x) = [\sigma_i(x), 1_{(\pi_G \circ a_P^R \circ \sigma_i)(x)}] (= [\sigma_i(x), 1_{\pi_G \circ X_i(x)}]).$$

The arrow $\mathbf{a}(\bar{\sigma}_i(x)) \in G_1 = \langle \mathrm{id}_G \rangle$ is an arrow with the target $a_P^L(\sigma_i(x)) = x$ and the source $s(1_{\pi_G \circ X_i(x)}) = \pi_G \circ X_i(x)$. We define the desired natural isomorphism α_i by setting $\alpha_i(x) = (\mathbf{a}(\bar{\sigma}_i(x)))^{-1}$. By Proposition 7.2 there are smooth maps $\beta_i : U_0^{(i)} \to TG_1$ so that $\pi_G \circ \beta_i = \alpha_i$ and $Tt \circ \beta_i = X_i$. Moreover the functors $Y_i : U^{(i)} \to TG$ given by $Y_i = Ts \circ \beta_i$ define multiplicative vector fields on each groupoid $U^{(i)}$. This is because their images land in $TU^{(i)} \subset TG$. In particular $\pi_G(Y_i(x)) = x$ for all $x \in U_0^{(i)}$.

Define the local sections $\nu_i : U_0^{(i)} \to P$ of a_P^L by $\nu_i(x) := \sigma_i(x) \cdot \beta_i(x)$ for all $x \in U_0^{(i)}$. Then by definition

$$a_P^R(\nu_i(x)) = Y_i(x) \quad \text{and} \quad \gamma \cdot \nu_i(x) = \nu_i(y) \cdot Y_i(x)$$

for all arrows $y \xleftarrow{\gamma} x$. For all i and all $x \in U_0^{(i)}$, we have

$$\begin{aligned}\mathbf{a}\big([\nu_i(x), 1_{\pi_G \circ a_P^R \circ \nu_i(x)}]\big) &= \mathbf{a}\big([\sigma_i(x)\beta_i(x), 1_{\pi_G \circ Y_i(x)}]\big) = \mathbf{a}\big([\sigma_i(x), \pi_G(\beta(x))]\big) \\ &= \mathbf{a}\big([\sigma_i(x), 1_{\pi_G \circ X_i(x)}]\big)\pi_G(\beta(x)) = \mathbf{a}(\overline{\sigma}_i(x))\alpha_i(x) = 1_x.\end{aligned}$$

Hence $\mathbf{a}([\nu_i(x), 1_{\pi_G \circ Y_i(x)}] = 1_x$. Finally, we construct a global section $\tau : G_0 \to P$ of a_P^L and the corresponding global multiplicative vector fields $X_\tau : G \to TG$ using a partition of unity argument. Choose a partition of unity $\{\lambda_i\}$ on G_0 subordinate to the cover $\{U_0^{(i)}\}$. Since the cover is locally finite it is no loss of generality to assume that the cover is in fact finite.

Consider a point $x \in U_0^{(i)} \cap U_0^{(j)}$. Then

$$\pi_G \circ a_P^R \circ \nu_i(x) = x = \pi_G \circ a_P^R \circ \nu_j(x).$$

Moreover

$$\mathbf{a}\big([\nu_i(x), 1_x]\big) = \mathbf{a}\big([\nu_i(x), 1_{\pi_G \circ a_R \circ \nu_i(x)}]\big) = 1_x.$$

Similarly $\mathbf{a}([\nu_j(x), 1_x]) = 1_x$. It follows that $[\nu_j(x), 1_x] = [\nu_i(x), 1_x]$ in the orbit space $(P \times_{G_0} G_1)/TG$ since $\mathbf{a}$ is a diffeomorphism. Therefore there is an arrow $w_{ij} \in TG_1$ so that

$$(\nu_i(x)w_{ij}(x), 1_x) = (\nu_j(x), \pi_G(w_{ij}(x))1_x).$$

Consequently

$$\nu_i(x)w_{ij}(x) = \nu_j(x) \quad \text{and} \quad \pi_G(w_{ij}(x)) = 1_x,$$

that is, $w_{ij}(x) \in T_{1_x}G_1$. Moreover since $a_P^L : P \to G_0$ is a principal TG_1 bundle, the arrow $w_{ij}(x)$ with this property is unique and depends smoothly on x. Note that the source of w_{ij} is $Y_j(x)$ and the target is $Y_i(x)$. The uniqueness of the $w_{ij}(x)$'s implies that the collection $\{w_{ij}(x)\}$ satisfies the cocycle conditions of Lemma 7.1. Therefore there exist arrows $Y_i(x) \xleftarrow{z_i(x)} \sum_k \lambda_k Y_k(x)$ with $z_i(x)z_j(x)^{-1} = w_{ij}(x)$. A quick look at the proof of Lemma 7.1 should convince the reader that $z_i(x)$'s depend smoothly on x.

For $x \in U_0^{(i)}$ we set $\tau(x) = \nu_i(x) \cdot z_i(x)$. Note that for $x \in U_0^{(i)} \cap U_0^{(j)}$, we have

$$\nu_j(x) = \nu_i(x)w_{ij}(x) = \nu_i(x)z_i(x)z_j(x)^{-1}.$$

Therefore $\nu_j(x) \cdot z_j(x) = \nu_i(x) \cdot z_i(x)$. It follows that τ is a globally defined section of $a_P^L : P \to G_0$. It remains to show that the corresponding functor $X_\tau : G \to TG$ is a multiplicative vector field. By construction for each index i we have a natural isomorphism $z_i : Y_i \Rightarrow X_\tau|_{U^{(i)}}$. Since $z_i(x) \in T_{1_x}G_1$ and since Y_i is a multiplicative vector field, the restriction $X_\tau|_{U^{(i)}}$ is also a multiplicative vector field. We conclude that X_τ is a multiplicative vector field globally.

We now argue that $\iota_G : \mathbb{X}(G) \to \mathbb{X}_{\mathsf{gen}}(G)$ is essentially surjective. Given a generalized vector field (P, α_P) the isomorphism $\alpha_P : \tilde{\pi}_G \circ P \to \langle \mathrm{id}_G \rangle$ defines an isomorphism $\mathbf{a} : \langle \pi_G \rangle \circ P \to \langle \mathrm{id}_G \rangle$ (remember that we fixed a 2-cell $\tilde{\pi}_G \to \langle \pi_G \rangle$ for every Lie groupoid G). By the above argument we have a multiplicative vector field $X : G \to TG$ and an isomorphism of bibundles $\gamma : P \to \langle X \rangle$. It remains to show that the vector fields (P, α_P) and $(\langle X \rangle, \alpha_{\langle X \rangle})$ are isomorphic. This is not entirely obvious since $\alpha_{\langle X \rangle} \circ (\tilde{\pi}_G \star \gamma)$ need not equal to α_P. Nonetheless, γ can be modified to a new isomorphism $\beta : P \to \langle X \rangle$ so that $\alpha_{\langle X \rangle} \circ (\tilde{\pi}_G \star \beta) = \alpha_{\langle X \rangle}$. This follows from Lemma 7.4 below. □

Lemma 7.4. *Let G be a Lie groupoid, $P : G \to TG$ a bibundle, $\alpha, \alpha' : \widetilde{P} \circ P \to \langle \mathrm{id}_G \rangle$ two isomorphisms of bibundles. Then there exists an isomorphism $\beta : P \to P$ of bibundles so that $\alpha \circ (\tilde{\pi}_G \star \beta) = \alpha'$. Moreover we may take $\beta = P \star (\alpha' \circ \alpha^{-1})$.*

Proof. For any two isomorphisms $\gamma, \delta : \langle \mathrm{id}_G \rangle \to \langle \mathrm{id}_G \rangle$ of bibundles, we have

$$\gamma \circ \delta = \gamma \star \delta.$$

Consequently for any $\gamma : \langle \mathrm{id}_G \rangle \to \langle \mathrm{id}_G \rangle$ we have

$$\begin{aligned}
(\alpha \circ (\alpha')^{-1}) \circ \gamma &= (\alpha \circ (\alpha')^{-1}) \star \gamma \\
&= (\alpha \circ 1_{\tilde{\pi}_G \circ P} \circ (\alpha')^{-1}) \star (1_{\langle \mathrm{id}_G \rangle} \circ \gamma \circ 1_{\langle \mathrm{id}_G \rangle}) \\
&= (\alpha \star 1_{\langle \mathrm{id}_G \rangle}) \circ ((\tilde{\pi_G} \circ P) \star \gamma) \circ ((\alpha')^{-1} \star 1_{\langle \mathrm{id}_G \rangle}) \\
&= \alpha \circ (\tilde{\pi}_G \star (P \star \gamma)) \circ (\alpha')^{-1}.
\end{aligned}$$

Hence

$$1_{\langle \mathrm{id}_G \rangle} = \alpha \circ (\tilde{\pi}_G \star (P \star (\alpha' \circ \alpha^{-1}))) \circ (\alpha')^{-1}$$

if $\gamma = \alpha' \circ \alpha^{-1}$. Therefore

$$\alpha' = \alpha \circ (\tilde{\pi}_G \star (P \star (\alpha' \circ \alpha^{-1}))).$$ □

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Volume 309 No. 1 November 2020

Lie 2-algebras of vector fields 1
DANIEL BERWICK-EVANS and EUGENE LERMAN

Lower regularity solutions of the biharmonic Schrödinger equation in a quarter plane 35
ROBERTO DE A. CAPISTRANO-FILHO, MÁRCIO CAVALCANTE and FERNANDO A. GALLEGO

The arithmetic Hodge index theorem and rigidity of dynamical systems over function fields 71
ALEXANDER CARNEY

On the vanishing of the theta invariant and a conjecture of Huneke and Wiegand 103
OLGUR CELIKBAS

Algebraic and geometric properties of flag Bott–Samelson varieties and applications to representations 145
NAOKI FUJITA, EUNJEONG LEE and DONG YOUP SUH

On a modular form of Zaremba's conjecture 195
NIKOLAY G. MOSHCHEVITIN and ILYA D. SHKREDOV

The first nonzero eigenvalue of the p-Laplacian on differential forms 213
SHOO SETO

Global regularity of the Navier–Stokes equations on 3D periodic thin domain with large data 223
NA ZHAO